POINCARE DUALITY SPACES

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ABSTRACT

POINCARÉ DUALITY SPACES

by Oliver Costich

This thesis is a study of algebraic conditions which will guarantee that a quotient space of a manifold is itself a manifold.

Suppose X is a locally compact Hausdorff space and A is a compact, connected subspace of X. Moreover assume that there is a "Poincaré Duality" isomorphism Δ of the cohomology of X onto the homology of X, $\Delta : H^q(X) \to H_{n-q}(X)$. A is said to be a divisor of X if the homomorphism

$$H_q(A) \longrightarrow H_q(X) \xrightarrow{\Delta^{-1}} H^{n-q}(X) \longrightarrow H^{n-q}(A)$$

is an isomorphism for $q \neq 0,n$.

In Chapter I it is shown that if X is an orientable, compact, polyhedral homology n-manifold, then A is a divisor of X for singular homology and cohomology if and only if the quotient X/A is an orientable, compact, polyhedral homology n-manifold.

Chapter II demonstrates that if X is an orientable n-dimensional cohomology manifold, then A is a divisor of X for Alexander-Spanier cohomology and Borel-Moore homology if and only if X/A is an orientable, n-dimensional cohomology manifold. In addition, the following generalization of R. L. Wilder's theorem on monotone mappings of manifolds is

given.

Theorem: Let $f : X \to Y$ be a surjection of a compact, orientable, n-dimensional cohomology manifold X to a locally compact Hausdorff space Y. If $f^{-1}(y)$ is a divisor for each $y \in Y$, then Y is also an n-dimensional cohomology manifold.

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INTRODUCTION

Various authors studying mappings of 2-manifolds discovered that certain "monotoneity" conditions imposed on the counter-images of points guaranteed that the image was also a 2-manifold. In the study of mappings of higher dimensional manifolds, as might be expected, similar conditions were imposed.

Definition: A mapping $f : X \to Y$ is said to be n-monotone if $\check{H}^r(f^{-1}(y)) = 0$ for all $y \in Y$ and $r \leq n$.

However, the identification mapping of the 3-sphere onto the space obtained by collapsing a suitable "wild" arc [8, EX 1.1] to a point is n-monotone for all n, but the image is not a manifold.

There is a class of spaces for which such "monotoneity" conditions are sufficient. R. L. Wilder has demonstrated that a monotone mapping of a generalized manifold yields a generalized manifold as its image [16], [17].

In this thesis we prove that conditions closely related to Poincaré duality imposed on counter-images of points give similar results for mappings of generalized manifolds.

CHAPTER 0

PRELIMINARIES

Section 1. Notation and Terminology

Throughout this dissertation X will denote a connected, locally compact Hausdorff space. In fact, all spaces will be locally compact and Hausdorff. We take L to be a principal ideal domain and all homomorphisms to be L-module homomorphisms.

If A is a compact subspace of X, we denote by Y the quotient space X/A, by $c : X \to Y$ the canonical identification, and by $*$ the point $c(A) \in Y$. Notice that the compactness of A is essential in order that Y be locally compact.

A *homology theory* (H,∂) consists of a covariant functor H from a category of locally compact pairs to the category of graded L-modules and homomorphisms of degree 0, and a natural transformation ∂ of degree -1 from the functor H on (X,A) to the functor H on $(A,\emptyset)$. The domain of H need not contain all locally compact pairs nor all continuous maps. Indeed, different theories may have different domains. We do require that the domain contain all proper maps. ($g : S \to T$ is proper if $g^{-1}(K)$ is compact for all compact K in T.) In addition, we insist that (H,∂) satisfy

(1) For $A \xrightarrow{i} X \xrightarrow{j} (X,A)$, there is an exact sequence

$$\cdots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots$$

where $i_* = H(i)$ whenever these homomorphisms are defined.

(2) On the full subcategory of connected spaces H_o behaves as follows:

If $g : S \to T$ is in the domain of H and S and T are connected, then

a) If S and T are compact, $g_* : H_o(S) \to H_o(T)$ is an isomorphism and $H_o(S) \approx L$.

b) If S and T are non-compact, $g_* : H_o(S) \to H_o(T)$ is an isomorphism.

Analogously, a <u>cohomology theory</u> (H^*, δ) consists of a contravariant functor H^* from a category of locally compact Hausdorff pairs to the category of graded L-modules, and a natural transformation δ of degree $+1$ from the functor H^* on A to the functor H^* on (X,A) satisfying

(1) For $A \xrightarrow{i} X \xrightarrow{j} (X,A)$, there is an exact sequence

$$\cdots \xrightarrow{\delta} H^q(X,A) \xrightarrow{j^*} H^q(X) \xrightarrow{i^*} H^q(A) \xrightarrow{\delta} H^{q+1}(X,A) \to \cdots$$

(2) On the full subcategory of connected spaces H^o behaves as follows

If $g : S \to T$ is in the domain of H^* and S and T are connected, then

a) If S and T are compact, $g^* : H^o(T) \to H^o(S)$

is an isomorphism and $H^o(S) \approx L$.

b) If S and T are non-compact, $g^* : H^o(T) \to H^o(S)$ is an isomorphism and $H^o(S) = 0$.

The remarks made about the domains of homology theories also apply to cohomology theories.

Suppose H'_*, H_* are homology theories and H^* is a cohomology theory. By a "cap product" we mean a homomorphism

$$\cap : H'_m(X) \otimes H^q(X) \longrightarrow H_{m-q}(X)$$

which is functorial in the following sense

Let $f : X \to Y$ which induces maps $f'_* : H'_*(X) \to H'_*(Y)$, $f : H_*(X) \to H_*(Y)$, and $f^* : H^*(Y) \to H^*(X)$. Then for $\alpha \in H'_m(X)$ the diagram

$$\begin{array}{ccc} H^p(Y) & \xrightarrow{f_*\alpha\cap} & H_{m-p}(Y) \\ \downarrow f^* & & \downarrow f_* \\ H^p(X) & \xrightarrow{\alpha\cap} & H_{m-p}(X) \end{array}$$

is commutative for all p, where $f_*\alpha\cap$ and $\alpha\cap$ are induced by the "cap product".

X is called a <u>Poincaré Duality space</u> of formal dimension n (n-PD) if there is a $\gamma \in H'_n(X)$ such that $\gamma\cap : H^p(X) \to H_{n-p}(X)$ is an isomorphism for all p. The element $\gamma \in H'_n(X)$ is called the fundamental class of X.

Suppose X is an n-PD and A is a compact, connected subset of X. A is called a <u>divisor</u> of X if

(1) $H_n(A) \approx H^n(A) = 0$

(2) the homomorphism $\varphi_A : H_q(A) \to H^{n-q}(A)$ defined by $H_q(A) \xrightarrow{i_*} H_q(X) \approx H^{n-q}(X) \xrightarrow{i^*} H^{n-q}(A)$ is an isomorphism for $q \neq 0, n$.

When we wish to emphasize the base ring L, we will write $H(X,A;L)$, and "X is an n-PD_L".

<u>Section 2. An algebraic lemma</u>

Our next step is to prove an algebraic lemma.

<u>Lemma 0.1</u>: Let

$$0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$$

and

$$0 \to B'' \xrightarrow{h} B \xrightarrow{k} B' \to 0$$

be exact and let $f : A \to B$ be an isomorphism. Suppose further that the diagrams

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "\varphi"'] & A \arrow[d, "f"] \\ B' & B \arrow[l, "k"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & A'' \\ B & B'' \arrow[l, "h"] \arrow[u, "\psi"'] \end{tikzcd}$$

are commutative,

i.e., $k \circ f \circ i = \varphi$ and $j \circ f^{-1} \circ h = \psi$.

Then φ is an isomorphism if and only if ψ is an isomorphism.

<u>Proof</u>: We will show that ψ is an isomorphism implies φ is. The converse is dual to this. To see φ is a monomorphism, let $a' \in A'$ such that $\varphi(a') = 0$. Now $f \circ i(a') \in h(B'')$ and so there exists $b'' \in B''$ such that $h(b'') =$

$f \circ i(a')$. But $\psi(b'') = j \circ i(a') = 0$ so $b'' = 0$. Thus $f \circ i(a') = 0$ and since $f \circ i$ is a monomorphism, $a' = 0$.

To see that φ is an epimorphism, let $b' \in B'$. Then there is a $b \in B$ such that $k(b) = b'$. Let $a'' \in A''$ be $j \circ f^{-1}(b)$ and $\bar{b} \in B$ be $h \circ \psi^{-1}(a'')$. Now $j \circ f^{-1}(b - \bar{b}) = a'' - j \circ f^{-1} \circ h \circ \psi^{-1}(a'') = 0$ so there is an $a' \in A'$ for which $i(a') = f^{-1}(b - \bar{b})$. Then $\varphi(a') = k \circ f \circ i(a') = k(b - \bar{b}) = k(b) - k(\bar{b}) = b' - k \circ h \circ \psi^{-1}(a'') = b'$.

<u>Corollary 0.2</u>: Let A be a compact, connected subset of X, an n-PD satisfying $H_n(A) \approx H^n(A) = 0$. Moreover assume that $H^q(X,A)$ and $H_q(X,A)$ are naturally isomorphic to $H^q(X/A)$ and $H_q(X/A)$ for $q \neq 0$. Then A is a divisor of X if and only if X/A is an n-PD with fundamental class $c_*'\gamma$.

<u>Proof</u>: In the algebraic lemma substitute the sequences

$$0 \to H_q(A) \to H_q(X) \to H_q(X/A) \to 0$$

and

$$0 \to H^{n-q}(X/A) \to H^{n-q}(X) \to H^{n-q}(A) \to 0$$

and the vertical maps φ_A, $\gamma\cap$, and $c_*'\gamma\cap$.

If A is a divisor of X, then $c_*'\gamma\cap$ is an isomorphism for $0 < q < n$ by the algebraic lemma. The commutative diagrams

$$\begin{tikzcd} H_o(X) \arrow[r, "\approx"] & H_o(X/A) \\ H^n(X) \arrow[u, "\gamma\cap", "\approx"'] & H^n(X/A) \arrow[l, "\approx"] \arrow[u, "c_*'\gamma\cap"'] \end{tikzcd} \qquad \begin{tikzcd} H_n(X) \arrow[r, "\approx"] & H_n(X/A) \\ H^o(X) \arrow[u, "\gamma\cap", "\approx"'] & H^o(X/A) \arrow[l, "\approx"] \arrow[u, "c_*'\gamma\cap"'] \end{tikzcd}$$

show that $c'_{*}\gamma\cap$ is an isomorphism for $q = 0$ and $q = n$. Thus X/A is an n-PD.

The converse is obvious.

CHAPTER I

DUALITY IN POLYHEDRAL HOMOLOGY MANIFOLDS

In this chapter we prove a special case of the main theorem of chapter 2. The proof of the general case is rather lengthy and complicated whereas the proof of the special case is essentially geometric in nature. This case should suffice for certain applications.

We recall some facts about finite polyhedra. The homology and cohomology theories used are the classical singular theories which, for simplicial complexes, are naturally equivalent to the corresponding simplicial theories.

A polyhedron X is an <u>n-dimensional homology manifold</u> over L (n-phm_L) if there is a triangulation K of X satisfying $H_q(\mathrm{St}\ v, \mathrm{LK}\ v; L) \approx H_q(B^n, S^{n-1}; L)$ for each vertex $v \in K$, where $\mathrm{St}\ v$, $\mathrm{Lk}\ v$ denote the star of v in K, link of v in K. Since this property is invariant under subdivision [1], this is equivalent to requiring that $H_q(X, X\text{-}x; L) \approx H_q(B^n, S^{n-1}; L)$ for each $x \in X$. An n-phm_L X is orientable if $H_n(X; L) \approx L$. (Hereafter we omit the coefficient domain L).

For an orientable n-phm, $H_q(X) \approx H^{n-q}(X)$ [1]. Moreover, this isomorphism is obtained as follows:

Let $\gamma \in H_n(X)$ be a generator. Then γ can be represented as the n-dimensional cycle $c = \sum \sigma_i$ where the sum runs over all principal simplexes in a triangulation of X.

There is a "cap product" [13]

$$\cap : H_r(X) \otimes H^q(X) \longrightarrow H_{r-q}(X)$$

such that $\gamma\cap : H^q(X) \to H_{n-q}(X)$ is an isomorphism for all q [10]. Similar results are known for an n-phm with non-empty boundary $\dot{X}$. That is

$$\gamma\cap : H^q(X) \to H_{n-q}(X,\dot{X})$$

$$\gamma\cap : H^q(X,\dot{X}) \to H_{n-q}(X)$$

are isomorphisms for all q, and γ can be represented by the cycle $c = \sum \sigma_i \pmod{\dot{X}}$, the summation being taken over all principal simplexes in a triangulation of X.

We say that a closed subset A of X is a _subpolyhedron_ of X if there is a triangulation $h : X \to K$ such that $h(A)$ is a subcomplex of K. It is known that there is a closed neighborhood N of A in X such that A is a strong deformation retract of N. If X is an orientable n-phm, N can be chosen to be an orientable n-phm with nonempty boundary $\dot{N}$. Namely N may be taken to be a closed simplicial neighborhood of A in a second barycentric subdivision of a triangulation of (X,A).

Recall that for compact polyhedral pairs, a relative homeomorphism induces an isomorphism of homology and cohomology [13].

Theorem 1.1: Let X be a closed n-phm_L and A a subpolyhedron of X. Then A is a divisor of X if and only if X/A is an orientable n-phm_L.

Proof: Let $* = c(A) \in X/A$, where A is a divisor of X. We first show that X/A is an n-phm. For this, it is sufficient to prove that $H_q(X/A,\ X/A - *) \approx H_q(B^n,\ S^{n-1})$. Now $H_q(X/A,\ X/A - *) \approx H_q(N/A,\ N/A - *)$ for any regular neighborhood N of A in X via the excision mapping $(N/A,\ N/A - *) \subset (X/A,\ X/A - *)$. Since A is a strong deformation retract of N, N/A is contractible. The long exact sequence of the pair $(N/A,\ N/A - *)$ quarantees that $H_q(N/A,\ N/A - *) \approx \tilde{H}_{q-1}(N/A - *)$. By virtue of the fact that $N/A - *$ and $(N/A)^{\bullet} = \dot{N}/A \cong \dot{N}$ have the same homotopy type, $H_{q-1}(\dot{N}) \approx H_q(N/A - *)$. Combining the above isomorphisms, we see that $\tilde{H}_{q-1}(\dot{N}) \approx H_q(X/A,\ X/A - *)$ so that it is sufficient to show that $\dot{N}$ has the homology of an $(n-1)$-sphere.

Note that N is also a divisor of X so that φ_N is an isomorphism for $0 < q < n$.

Consider the diagram $0 < q < n$

$$\begin{tikzcd}
H_q(X) & & H_q(N) \arrow[ll, "i_*"'] \arrow[rr, "j_*"] \arrow[ddddrr, "\varphi_N", "\approx"'] & & H_q(N,\dot{N}) \\
 & & & & \\
 & & H^{n-q}(N,\dot{N}) \arrow[uu, "\alpha\cap"', "\approx"] \arrow[ddll, "h^*\circ k^{*-1}"'] \arrow[ddrr, "j^*"] & & \\
 & & H^{n-q}(X,X-\mathring{N}) \arrow[u, "k^*", "\approx"'] \arrow[dll, "h^*"] & & \\
H^{n-q}(X) \arrow[uuuu, "\gamma\cap", "\approx"'] \arrow[rrrr, "i^*"'] & & & & H^{n-q}(N) \arrow[uuuu, "\alpha\cap"', "\approx"]
\end{tikzcd}$$

where $i : N \subset X$, $j : (N,\emptyset) \subset (N,\dot{N})$

$h : (X,\emptyset) \subset (X,X-\mathring{N})$, $k : (N,\dot{N}) \subset (X,X-\mathring{N})$

$\mathring{N}$ denotes the interior of N, and

α is the fundamental class of $(N,\dot{N})$.

Now $\gamma\cap$, $\alpha\cap$ are isomorphisms by Poincaré duality, and k^* is an isomorphism since k is an excision map.

Since γ is represented by the sum of the principal simplexes in a triangulation of X, the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] & (X,X-\mathring{N}) \\ N \arrow[u, "i"] \arrow[r] & (N,\dot{N}) \arrow[u, "k"'] \end{tikzcd}$$

shows that $h_*\gamma = j_*\alpha$. Thus the diagram

$$\begin{tikzcd} H_q(X) & & H_q(N) \arrow[ll, "i_*"'] \\ H^{n-q}(X,X-\mathring{N}) \arrow[u, "(k_*\alpha)\cap \,=\, (h_*\alpha)\cap"] \arrow[rr, "k^*"] & & H^{n-q}(N,\dot{N}) \arrow[u, "\alpha\cap"'] \end{tikzcd}$$

is commutative, due to the commutativity of the corresponding diagram at the chain and cochain levels.

Again, by a chain level argument, the following diagram is commutative.

$$\begin{tikzcd} H^{n-q}(X,X-\mathring{N}) \arrow[r, "h^*"] \arrow[d, "(h_*\gamma)\cap"'] & H^{n-q}(X) \arrow[d, "\gamma\cap"] \\ H_q(X) & H_q(X) \arrow[l, "1_*"'] \end{tikzcd}$$

Combining these results, we obtain $(\gamma\cap)\circ h^* = i_*\circ(\alpha\cap)\circ k^*$
Composing with i^* we get $i^*h^* = i^*\circ(\gamma\cap)^{-1}\circ i_*\circ(\alpha\cap)\circ k^* =$

$\varphi_N \circ (\alpha\cap) \circ k^*$. But $i^* \circ h^* = j^* \circ k^*$ so $\varphi_N \circ (\alpha\cap) \circ k^* = j^* \circ k^*$. Hence $\varphi_N \circ (\alpha\cap) = j^*$ so j^* is an isomorphism. Therefore j_* is an isomorphism for $0 < q < n$.

From the long exact sequence of the pair $(N,\dot{N})$, we see that $H_q(\dot{N}) = 0$ if $0 < q < n-1$. Moreover $0 \to H_n(N) \xrightarrow{j_*} H_n(N,\dot{N}) \xrightarrow{\partial} H_{n-1}(\dot{N}) \to 0$ is exact. Since $\dot{N}$ is an orientable n-phm, $H_{n-1}(\dot{N})$ is a free L-module, so the preceding sequence is split. Thus $L \approx H_n(N,\dot{N}) \approx H_n(N) \oplus H_{n-1}(\dot{N})$. Therefore $L \approx H_{n-1}(\dot{N}) \approx H^\circ(\dot{N})$ by Poincaré duality. This implies $\dot{N}$ is connected and so $\dot{N}$ has the homology of an (n-1)-sphere.

To show X/A is orientable, we prove that $H_n(X/A) \approx L$. The argument above demonstrates that $H_n(N) = 0$, so $H_n(A) = 0$. Hence $H_n(X) \xrightarrow{c_*} H_n(X/A)$ is a monomorphism. Because either $H_n(X/A) = 0$ or $H_n(X/A) \approx L$ [13], $H_n(X/A) \approx L$.

The proof of the reverse implication is obvious from the corollary to the algebraic lemma of the preceding chapter.

<u>Remark</u>: The proof of the above theorem also shows that $c_*\gamma$ may be taken as the fundamental class of X/A.

<u>Examples</u>: (1) Let $T = S^1 \times S^1$ and $R \subset T$ be the one point union of two circles representing the canonical generators of $\Pi_1(T)$. Then $T/R \cong S^2$. Thus R is a divisor of T. More generally, if M is a closed, orientable, polyhedral manifold, there is a closed subpolyhedron $R \subset X$ satisfying $X \cong \mathbb{R}^n \cup R$, $\mathbb{R}^n \cap R = \emptyset$ and $\dim R < n$ [6]. Since the one-point compactification of a locally compact Hausdorff

space is unique, $X/R \cong S^n$. Hence R is a divisor.

(2) If A and B are orientable polyhedral manifolds of dimension n, then $A \# B$ is obtained by removing a "nice" n-dimensional ball from each of A and B and sewing the resulting manifolds with boundary, A' and B', along the boundaries by an orientation preserving homeomorphism. It is easy to check that $A \# B/A' \cong B$. Hence A' is a divisor of $A \# B$.

(3) Let M be an orientable 3-dimensional manifold, and let S be a spine of M. Then since M/S is homeomorphic to S^3, S is a divisor of M.

CHAPTER II

DUALITY IN COHOMOLOGY MANIFOLDS

Section 1. Homology, Cohomology, and Duality for Cohomology Manifolds

In this section we develop the concepts required in the remainder of the thesis. Standard references for this material are [5] and [2].

Let X be a topological space. A presheaf (of L-modules) on X is a contravariant functor from the category of open subsets of X and inclusions to the category of L-modules and L-homomorphisms. A morphism of presheaves on X is a natural transformation of functors. A sheaf (of L-modules) on X is a pair $(\mathcal{A},\pi)$ where

(i) $\mathcal{A}$ is a topological space (not generally Hausdorff).

(ii) $\pi : \mathcal{A} \to X$ is a local homeomorphism.

(iii) For each $x \in X$, $\pi^{-1}(X) = \mathcal{A}_x$ is an L-module (and is called the stalk of $\mathcal{A}$ at x).

(iv) The module operations are continuous.

Explicitly, let $\mathcal{A} \Delta \mathcal{A}$ be the subspace of $\mathcal{A} \times \mathcal{A}$ consisting of pairs (α,β) with $\pi(\alpha) = \pi(\beta)$, and consider $L \times \mathcal{A}_x$ a subspace of $L \times \mathcal{A}$. We require that the map $\mathcal{A} \Delta \mathcal{A} \to \mathcal{A}$ taking (α,β) to $\alpha - \beta$ and that the map $L \times \mathcal{A}_x \to \mathcal{A}$ taking (ℓ,α) to $\ell\alpha$ be continuous. A morphism of sheaves $(\mathcal{A},\pi) \to (\mathcal{B},\rho)$ on X is a continuous map $f : \mathcal{A} \to \mathcal{B}$ such that $\pi = \rho \circ f$ and $f \mid \mathcal{A}_x = f_x$ is an L-homomorphism of $\mathcal{A}_x$

to $\mathcal{B}_x$.

Given a presheaf A on X, we can construct a sheaf $\mathcal{A}$ on X in a canonical way. $\mathcal{A}$ is called the sheaf generated by A. $\mathcal{A}$ is constructed as follows: For each open $U \subset X$ consider the space $U \times A(U)$, where U has the subspace topology and $A(U)$ is discrete. Form the disjoint union E of $\{U \times A(U) \mid U \subset X\}$. We define an equivalence relation R on E: if $(x,s) \in U \times A(U)$ and $(y,t) \in V \times A(V)$ then $(x,s)\ R\ (y,t)$ if and only if $x = y$ and there is a neighborhood W of x, $W \subset U \cap V$, and $s|W = t|W$ (here $s|W$ is the image of s in $A(W)$). There is an identification $q : E \to E/R = \mathcal{A}$ and a map $p : E \to X$ by $p(x,s) = x$.

$$\begin{tikzcd} E \arrow[rr, "q"] \arrow[dr, "p"'] & & \mathcal{A} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Then there is a map $\pi : \mathcal{A} \to X$ so that the above diagram commutes [7]. One easily verifies that $(\mathcal{A},\pi)$ is a sheaf on X with stalks $\mathcal{A}_x = \text{dir lim}\,\{\mathcal{A}(U) \mid U$ a neighborhood of $x\}$.

If $\mathcal{A}$ is a sheaf on X, let $\mathcal{A}(U)$ be all continuous functions $s : U \to \mathcal{A}$ for which $\pi \circ s = 1_u$. This defines a presheaf on X which generates a sheaf isomorphic to $\mathcal{A}$.

The construction of a sheaf from a presheaf or the converse is functorial. That is, a morphism of the original objects determines a morphism of the generated objects.

A _family of supports_ on X is a family φ of closed subsets of X such that

(1) A closed subset of a member of φ is in φ.

(2) φ is closed under finite unions.

φ is said to be a _paracompactifying_ family of supports if, in addition:

(3) Each element of φ is paracompact.

(4) Each element of φ has a closed neighborhood which is in φ.

The family of all compact subsets of X is denoted by C. It is paracompactifying if X is locally compact. It is customary to use no symbol for the family of closed subsets of X. This is paracompactifying if X is paracompact.

For $s \in \mathcal{A}(X)$, define $|s| = \{x \in X \mid s(x) \neq 0\}$ to be the support of s. If $\mathcal{A}$ is a sheaf on X, $\Gamma_\varphi(\mathcal{A}) = \{s \in \mathcal{A}(X) \mid |s| \in \varphi\}$. For $Y \subset X$ and φ a family of supports on X, $\varphi \cap Y$ is the family $\{K \cap Y \mid K \in \varphi\}$ and $\varphi|Y$ is the family $\{K \mid K \in \varphi$ and $K \subset Y\}$.

We say $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact if and only if $\mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x$ is exact as a sequence of L-modules.

A _graded sheaf_ $\mathcal{I}^*$ is a sequence $\{\mathcal{I}^p\}$ of sheaves, p an integer. A _differential sheaf_ is a graded sheaf together with homomorphisms $d : \mathcal{I}^p \to \mathcal{I}^{p+1}$ such that $d^2 : \mathcal{I}^p \to \mathcal{I}^{p+2}$ is zero for all p. A _resolution_ of a sheaf $\mathcal{A}$ is a differential sheaf $\mathcal{I}^*$ satisfying $\mathcal{I}^p = 0$ for $p < 0$ and a homomorphism $\epsilon : \mathcal{A} \to \mathcal{I}^\circ$ such that

$$0 \to \mathcal{A} \xrightarrow{\varepsilon} \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \to \cdots$$

is exact. Similarly one defines graded and differential presheaves.

If $\mathcal{I}^*$ is a differential sheaf, its homology sheaf $\mathcal{H}^*(\mathcal{I}^*)$ is the graded sheaf given by, as usual,

$$\mathcal{H}^p(\mathcal{I}^*) = \mathrm{Ker}\,(d : \mathcal{I}^p \to \mathcal{I}^{p+1}) \,/\, \mathrm{Im}\,(d : \mathcal{I}^{p-1} \to \mathcal{I}^p)$$

If $\mathcal{I}^*$ is generated by the differential presheaf L^* then $\mathcal{H}^p(\mathcal{I}^*)$ is generated by the presheaf $U \longmapsto H^p(L^*(U))$.

We are now ready to construct the cohomology theory we need. We first construct a canonical resolution of a sheaf $\mathcal{A}$ on X.

Let $C^\circ(U;\mathcal{A})$ be the collection of all functions (not necessarily continuous) $f : U \to \mathcal{A}$ such that $\pi \circ f = 1_U$, i.e., $C^\circ(U;\mathcal{A}) = \Pi\{\mathcal{A}_x \mid x \in U\}$. Under pointwise operations, this is an L-module and the functor $U \longmapsto C^\circ(U;\mathcal{A})$ is a presheaf on X which generates a sheaf $\mathcal{C}^\circ(X;\mathcal{A})$. Moreover $\mathcal{C}^\circ(X;\mathcal{A})(U) = C^\circ(U;\mathcal{A})$. The inclusion $\mathcal{A}(U) \subset C^\circ(U;\mathcal{A})$ provides a natural monomorphism $\varepsilon : \mathcal{A} \to \mathcal{C}^\circ(X;\mathcal{A})$. For a family of supports on X we put $C^\circ_\varphi(X;\mathcal{A}) = \Gamma_\varphi(\mathcal{C}^\circ(X;\mathcal{A}))$.

Let $\mathcal{Z}^1(X;\mathcal{A})$ be the cokernel of ε so that

$$0 \to \mathcal{A} \xrightarrow{\varepsilon} \mathcal{C}^\circ(X;\mathcal{A}) \xrightarrow{\partial} \mathcal{Z}^1(X;\mathcal{A}) \to 0$$

is exact. We definé, inductively,

$$\mathcal{C}^n(X;\mathcal{A}) = \mathcal{C}^\circ(X;\mathcal{Z}^n(X;\mathcal{A}))$$

$$\mathcal{Z}^{n+1}(X;\mathcal{A}) = \mathcal{Z}^1(X;\mathcal{Z}^n(X;\mathcal{A}))$$

so that

$$0 \to \mathcal{Z}^n(X;\mathcal{A}) \xrightarrow{\varepsilon} \mathcal{C}^n(X;\mathcal{A}) \xrightarrow{\partial} \mathcal{Z}^{n+1}(X;\mathcal{A}) \to 0$$

is exact. Let $d = \epsilon\partial$ be the composite

$$\mathcal{C}^n(X;\mathcal{A}) \xrightarrow{\partial} \mathcal{Z}^{n+1}(X;\mathcal{A}) \xrightarrow{\epsilon} \mathcal{C}^{n+1}(X;\mathcal{A})$$

Thus the sequence

$$0 \to \mathcal{A} \to \mathcal{C}^0(X;\mathcal{A}) \xrightarrow{d} \mathcal{C}^1(X;\mathcal{A}) \xrightarrow{d} \mathcal{C}^2(X;\mathcal{A}) \xrightarrow{d} \cdots$$

is exact. That is, $\mathcal{C}^*(X;\mathcal{A})$ is a resolution of $\mathcal{A}$. This is the canonical resolution of $\mathcal{A}$. Notice that $\mathcal{C}^0(X;\mathcal{A})$ is an exact functor of $\mathcal{A}$ and, therefore, so is $\mathcal{C}^*(X;\mathcal{A})$.

For φ a family of supports on X, we put

$$C^n_\varphi(X;\mathcal{A}) = \Gamma_\varphi(\mathcal{C}^n(X;\mathcal{A})) = C^0_\varphi(\mathcal{Z}^n(X;\mathcal{A}))$$

Since $C^0_\varphi(X;\mathcal{A})$ and $\mathcal{Z}^n(X;\mathcal{A})$ are exact functors of $\mathcal{A}$, so is $C^n_\varphi(X;\mathcal{A})$.

We now make the definition

$$H^n_\varphi(X;\mathcal{A}) = H^n(C^*_\varphi(X;\mathcal{A}))$$

From a short exact sequence

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

of sheaves on X we obtain a short exact sequence

$$0 \to C^*_\varphi(X;\mathcal{A}') \to C^*_\varphi(X;\mathcal{A}) \to C^*_\varphi(X;\mathcal{A}'') \to 0$$

of chain complexes and thus a long exact sequence

$$\cdots \to H^p_\varphi(X;\mathcal{A}') \to H^p_\varphi(X;\mathcal{A}) \xrightarrow{\delta} H^p_\varphi(X;\mathcal{A}'') \to H^{p+1}_\varphi(X;\mathcal{A}) \to \cdots$$

Alternatively, we can define sheaf cohomology via injective resolutions of sheaves.

A sheaf $\mathcal{J}$ on X is injective if given a monomorphism $i : \mathcal{A} \to \mathcal{B}$ of sheaves on X and $h : \mathcal{A} \to \mathcal{J}$, there exists an $\bar{h} : \mathcal{B} \to \mathcal{J}$ such that $\bar{h} \circ i = h$. That is, the functor $\mathrm{Hom}_L(\cdot,\mathcal{J})$ is exact. From the homological algebra of L-modules, we know that any L-module is a submodule of an injective L-module and that this injective module can be constructed in a canonical way [12].

For a sheaf $\mathcal{A}$ on X, let $I(\mathcal{A}_x)$ be the canonical injective L-module containing $\mathcal{A}_x$. Define $\mathcal{J}^0(X;\mathcal{A})$ to be the sheaf generated by the presheaf $\mathcal{J}^0(X;\mathcal{A})(U) = \Pi\{I(\mathcal{A}_x) \mid x \in U\}$. Then $\mathcal{J}^0(X;\mathcal{A})$ is injective and

$$\Pi\{\mathcal{A}_x \mid x \in U\} \subset \Pi\{I(\mathcal{A}_x) \mid x \in U\}$$

provides a monomorphism $\mathcal{C}^0(X;\mathcal{A}) \to \mathcal{J}^0(X;\mathcal{A})$. Composing with the canonical monomorphism $\mathcal{A} \to \mathcal{C}^0(X;\mathcal{A})$ gives a monomorphism $\mathcal{A} \to \mathcal{J}^0(X;\mathcal{A})$. Hence every sheaf is a subsheaf of an injective sheaf and thus the standard methods of homological algebra can be applied to sheaves.

Define $\mathcal{Z}^1(X;\mathcal{A}) = \mathcal{J}^0(X;\mathcal{A})/\mathcal{A}$ and

$$\mathcal{J}^n(X;\mathcal{A}) = \mathcal{J}^0(X;\mathcal{Z}^n(X;\mathcal{A}))$$

where

$$\mathcal{Z}^n(X;\mathcal{A}) = \mathcal{Z}^1(X;\mathcal{Z}^{n-1}(X;\mathcal{A})).$$

We obtain a resolution $\mathcal{J}^*(X;\mathcal{A})$ of $\mathcal{A}$ which is a covariant functor of $\mathcal{A}$. We refer to $\mathcal{J}^*(X;\mathcal{A})$ as the <u>canonical injective resolution</u> of $\mathcal{A}$.

In the usual way, if $\mathcal{L}^*$ is a resolution of $\mathcal{A}$, then there is a chain map $\mathcal{L}^* \to \mathcal{J}^*(X;\mathcal{A})$. that is, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "1"'] & \mathcal{L}^0 \arrow[r, "d"] \arrow[d] & \mathcal{L}^1 \arrow[r] \arrow[d] & \cdots \\
\mathcal{A} \arrow[r] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \cdots
\end{tikzcd}$$

Let $\mathcal{J}^* = \mathcal{C}^*(X;\mathcal{A})$ and let φ be a family of supports on X. We then have a chain map $\Gamma_\varphi(\mathcal{J}^*) \to C^*_\varphi(X;\mathcal{A})$ and hence an induced map

$$H^p(\Gamma_\varphi(\mathcal{J}^*)) \to H^p_\varphi(X;\mathcal{A}).$$

This is a natural isomorphism for all p [5]. Thus we can determine the modules $H^p_\varphi(X;\mathcal{A})$ from $\mathcal{J}^*(X;\mathcal{A})$ as well as from $\mathcal{C}^*(X;\mathcal{A})$.

To define relative cohomology, let $i : A \subset X$ and φ a family of supports on X. If $\mathcal{A}$ is a sheaf on A, there is a sheaf $i\mathcal{A}$ on X determined by $i\mathcal{A}(U) = \mathcal{A}(U \cap A)$ and a continuous map $\mathcal{A} \to i\mathcal{A}$ so that

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & i\mathcal{A} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ i\ } & X \end{array}$$

commutes.

If $(\mathcal{A},\pi)$ is a sheaf on X, $(\mathcal{A}|A,\pi')$ is the sheaf on X given by $\mathcal{A}|A = \mathcal{A} \cap \pi^{-1}(A)$, $\pi' = \pi|(\mathcal{A} \cap \pi^{-1}(A))$. For $\mathcal{A}$ on X, we have a homomorphism $i^* : \mathcal{C}^*(X;\mathcal{A}) \to i\mathcal{C}^*(A;\mathcal{A}|A)$ of sheaves on X. We introduce the notation

$$\operatorname{Ker} i^* = \mathcal{C}^*(X,A;\mathcal{A})$$

$$C^*_\varphi(X,A;\mathcal{A}) = \Gamma_\varphi(\mathcal{C}^*(X,A;\mathcal{A}))$$

$$H^*_\varphi(X,A;\mathcal{A}) = H^*(C^*_\varphi(X,A;\mathcal{A}))$$

From these definitions we obtain a short exact sequence

$$0 \to C^*_\varphi(X,A;\mathcal{A}) \to C^*_\varphi(X;\mathcal{A}) \to C^*_{\varphi\cap A}(A;\mathcal{A}|A) \to 0$$

and hence a long exact sequence

$$\cdots \to H^p_\varphi(X,A;\mathcal{A}) \to H^p_\varphi(X;\mathcal{A}) \to H^p_{\varphi\cap A}(A;\mathcal{A}|A) \to H^{p+1}_\varphi(X,A;\mathcal{A}) \to \cdots$$

For this cohomology theory, excision theorems, universal coefficient theorems, Mayer-Vietoris sequences, and many results similar to those available for "ordinary" cohomology theories are provable [5, Chapter II].

If G is an L-module we also denote the <u>constant sheaf</u> $(G\times X,\pi)$, where $\pi : G\times X \to X$ is the projection, by G. If φ and $\varphi\cap A$ are both paracompactifying,

$$H^*_\varphi(X,A;G) \approx {}_{AS}H^*_\varphi(X,A;G)$$

where the right-hand side is the Alexander-Spanier cohomology module of (X,A) with coefficients in G [5], [13]. If, in addition, X and A are homologically locally connected in the sense of singular homology (HLC),

$$H^*_\varphi(X,B;G) \approx \Delta H^*_\varphi(X,B;G)$$

where the right-hand side is the "classical" singular cohomology module of (X,B) with coefficients in G.

To define the homology theory which is "dual" to sheaf cohomology we need some additional objects.

A <u>precosheaf</u> $\mathfrak{A}$ on X is a covariant functor from the category of open subsets of X to that of L-modules. A precosheaf is a <u>cosheaf</u> if the sequence

$$\sum_{\langle\alpha,\beta\rangle} \mathfrak{A}(U_{\alpha\beta}) \overset{g}{\underset{f}{\rightrightarrows}} \sum_\alpha \mathfrak{A}(U_\alpha) \to \mathfrak{A}(U) \to 0$$

is exact for all collections $\{U_\alpha\}$ of open sets with $U = \bigcup_\alpha U_\alpha$, where $g = \sum_{\langle\alpha,\beta\rangle}(i_{U_\alpha U_\beta} - i_{U_\beta U_\alpha})$ and $f = \sum_\alpha i_{UU_\alpha}$ [i_{UV} being the canonical map $\mathfrak{A}(V) \to \mathfrak{A}(U)$ for $V \subset U$ and

$U_{\alpha\beta} = U_\alpha \cap U_\beta$]. Graded and differential cosheaves are defined in a manner analogous to that for sheaves.

Let $0 \to L \to L^0 \to L^1 \to 0$ be the canonical injective resolution of L, i.e. $L^0 = I(L)$, $L^1 = I(L)/L$ which is injective since it is divisible and L is a principal ideal domain. We define $\underline{\mathcal{H}om\,(\mathfrak{A},L)}$ to be the sheaf generated by the presheaf $U \to \mathrm{Hom}\,(\mathfrak{A}(U),L)$. Now define a differential sheaf

$$\mathcal{B}(\mathfrak{A}_*;L) = \mathcal{H}om\,(\mathfrak{A}_*,L^*)$$

for a differential cosheaf $\mathfrak{A}_*$, where L^* is the canonical injective resolution of L. As usual, the term in degree n is

$$\mathcal{B}^n(\mathfrak{A}_*;L) = \sum_{p+q=n} \mathcal{H}om\,(\mathfrak{A}_p,L^q)$$

the differential being $d' - d'' : \mathcal{B}^n \to \mathcal{B}^{n+1}$ where d' is induced by the differential $M^q \to M^{q+1}$ and $(-1)^n d''$ is induced by $\mathfrak{A}_{p+1} \to \mathfrak{A}_p$.

Let $\mathcal{L}^*$ be a "nice" differential sheaf. Then $\Gamma_C\mathcal{L}^*$ is a differential cosheaf with gradation

$$(\Gamma_C\mathcal{L}^*)_p = \Gamma_C\mathcal{L}^{-p}.$$

The differential sheaf $\mathcal{B}(\Gamma_C\mathcal{L}^*;L)$ will also be denoted by $\mathcal{B}(\mathcal{L}^*)$. Moreover, as above, we let $\mathcal{B}_n$ stand for $\mathcal{B}^{-n}$.

For a sheaf $\mathcal{a}$ on X, we define

$$\mathcal{C}_*(X;\mathcal{a}) = \mathcal{B}_*(\mathcal{J}^*(X;L)) \otimes \mathcal{a}$$

$$C_*^\varphi(X;\mathcal{a}) = \Gamma_\varphi(\mathcal{C}_*(X;\mathcal{a}))$$

$$H_*^\varphi(X;\mathcal{a}) = H_p(C_*^\varphi(X;\mathcal{a}))$$

This is called the <u>Borel-Moore homology</u> of X with support in φ.

The homology modules $H_*(X;L)$ and $H_*^c(X;L)$ correspond to the classical homology theories based on infinite and finite chains respectively. These two cases will be of primary interest to us in the remainder of this dessertation.

To get a relative homology theory, there is a chain monomorphism

$$C_*^{\varphi|A}(A;\mathcal{Q}|A) \to C_*^{\varphi}(X;\mathcal{Q})$$

for any locally closed $A \subset X$. The cokernel of this map is a chain complex whose homology we denote by $H_*^{\varphi}(X,A;\mathcal{Q})$.

A continuous map $f : X \to Y$ is <u>proper</u> (with respect to families of supports φ on X and ψ on Y) if $f^{-1}\psi \subset \varphi$. Such a map induces homomorphisms

$$f_* : H_*^{\varphi}(X;L) \to H_*^{\psi}(Y;L)$$

and

$$f^* : H^*_{\psi}(Y;L) \to H^*_{\varphi}(X;L)$$

which are natural with respect to the long exact sequences of pairs (X,A). All continuous mappings will be assumed to be proper.

By the <u>dimension</u> of X over L $(\dim_L X)$ we mean the value of the dimension function

$$\dim_L X \leq n \iff H_c^{n+1}(U;L) = 0$$
$$\text{for all open } U \subset X.$$

We say X is <u>cohomologically locally connected</u> in dimension k (X is k-clc_L) at $x \in X$ if given a neighborhood

U of x there is a neighborhood V of x with $V \subset U$ such that $H^k(U) \to H^k(V)$ is trivial (henceforth, omission of coefficients will mean the coefficients are in L). X is clc_L^n at x if it is k-clc_L for all $k \leq n$, clc_L^∞ if it is clc_L^n for all n, and clc_L if it is clc_L^∞ and for each neighborhood U of x there is a neighborhood V of x, $V \subset U$ such that $H^*(U) \to H^*(V)$ is trivial. The space X is k-clc_L, clc_L^n, clc_L^∞, or clc_L if it is so at every point. Notice that clc_L^∞ is equivalent to clc_L if X has finite cohomological dimension over L.

If $U \subset X$ is open, the restriction map $C_*(X;L) \to C_*(U;L)$ induces a homomorphism $H_*(X) \mapsto H_*(U)$ [5, p.185]. Hence the functor $U \to H_*(U)$ is a presheaf on X which generates a sheaf $\mathscr{H}_*(X;L)$ which is called the <u>sheaf of local homology groups</u>. The stalk $H_*(X;L)_x = \varinjlim H_*(U)$ (U ranging over neighborhoods of x) is called the <u>local homology group</u> of X at x.

A locally compact space X is an $(L$-$n)$-space if $\mathscr{H}_p(X;L)$ is zero for $p \neq n$ and torsion-free for $p = n$. $\mathscr{H}_n(X;L)$ is called the <u>orientation sheaf</u> of X. An $(L$-$n)$-space is an n-dimensional <u>homology manifold</u> over L (n-hm_L) if $\mathscr{H}_n(X;L)$ is locally constant with stalks isomorphic to L, and if $\dim_L X < \infty$. X is <u>L-orientable</u> if $\mathscr{H}_n(X;L)$ is the constant sheaf. We call X an n-dimensional <u>cohomology manifold</u> over L (n-cm_L) if X is an n-hm_L and clc_L. That this definition of n-cm_L is equivalent to the usual one is known [5], [2].

To obtain a "duality" between the foregoing homology and cohomology theories, we can extend the natural map

$$\Gamma_\varphi(\mathcal{C}_*(X;L)\otimes\mathcal{A})\otimes\Gamma_\psi(\mathcal{B}) \to \Gamma_{\varphi\cap\psi}(\mathcal{C}_*(X;L)\otimes\mathcal{A}\otimes\mathcal{B})$$

to a homomorphism

$$\cap : H_m^\varphi(X;\mathcal{A})\otimes H_\psi^p(X;\mathcal{B}) \to H_{m-p}^{\varphi\cap\psi}(X;\mathcal{A}\otimes\mathcal{B})$$

called the "cap product" whenever φ, ψ, and $\varphi\cap\psi$ are paracompactifying families of supports on X [5], [3].

Let $\mathcal{O} = \mathcal{H}_n(X;L)$ be the orientation sheaf of X, an n-hm_L. Then there is a unique sheaf $\mathcal{O}^{-1}$ on X such that $\mathcal{O}\otimes\mathcal{O}^{-1} \approx L$. Then

$$\cap : H_n(X;\mathcal{O}^{-1})\otimes H_\varphi^p(X;L) \to H_{n-p}^\varphi(X;\mathcal{O}^{-1}),$$

if X is paracompact and φ is paracompactifying. Moreover, in this case, there exists a $\gamma \in H_n(X;\mathcal{O}^{-1})$ for which

$$\gamma\cap : H_\varphi^p(X;L) \to H_{n-p}^\varphi(X;\mathcal{O}^{-1})$$

is an isomorphism for all p, i.e. X is an n-PD.

The "cap product" is natural in the following sense (for our purposes, the case $\mathcal{O} = L$ will be sufficient): Let $f : X \to Y$. Then for $\alpha \in H_n(X;L)$, the diagram

$$\begin{tikzcd}
H_c^p(Y;L) \arrow[r, "f_*\alpha\cap"] \arrow[d, "f^*"'] & H_{n-p}^c(Y;L) \\
H_c^p(X;L) \arrow[r, "\alpha\cap"] & H_{n-p}^c(X;L) \arrow[u, "f_*"']
\end{tikzcd}$$

is commutative for all p.

If U and V are open subsets of X with $V \subset U$, there is a homomorphism $\Gamma_c(V) \to \Gamma_c(U)$ provided by "extension by zero", i.e. for $s \in \Gamma_c(V)$, extend s to U by

$s(x) = 0$, $x \in V - |s|$. This induces a homomorphism

$$j_{UV} : H_C^*(V) \to H_C^*(U)$$

which is natural with respect to inclusions.

The Borel-Moore homology is frequently equivalent to more familiar homology theories. This is treated in [5].

Section 2. The Main Theorem.

In this section, we prove the main theorems of this thesis.

The following may be found in [5].

2.1 If A is closed in X, then we have the natural isomorphisms

(i) $H_C^*(X,A) \approx H^*_{C|X-A}(X-A) \approx H_C^*(X-A)$

(ii) $H_*^C(X,A) \approx H_*^{C\cap(X-A)}(X-A)$.

Using these we can prove a

Lemma 2.2: Let A be a closed subset of a locally compact space X. Then for $q > 0$,

(i) $H_C^q(X/A) \approx H_C^q(X,A)$

(ii) $H_q^C(X/A) \approx H_q^C(X,A)$

are naturally isomorphic.

Proof: (i) $H_C^q(X/A) \approx H_C^q(X/A,*) \approx H_C^q(X/A - *) \approx H_C^q(X-A) \approx H_C^q(X,A)$. The isomorphisms are, successively, obtained from the cohomology sequence of $(X/A,*)$, 2.1(i), relative homeomorphism, and 2.1(i) again.

(ii) Similarly, $H_q^C(X/A) \approx H_q^C(X/A,*) \approx H_q^{C\cap(X/A\,-\,*)}(X/A - *) \approx H_q^{C\cap(X-A)}(X-A) \approx H_q^C(X,A)$ from 2.1(ii).

Using 2.2(i), we get

2.3 If $i : A \subset X$ is a closed subspace and $U = X - A$, there is an exact cohomology sequence

$$\cdots \to H_C^q(U) \xrightarrow{j_{XU}} H_C^q(X) \xrightarrow{i^*} H_C^q(A) \to H_C^{q+1}(U) \to \cdots$$

__Lemma 2.4__: If W is a nondegenerate, locally compact space, and $w \in W$, then $\dim_L W = \dim_L(W-w)$.

__Proof__: In 2.3, take $A = \{w\}$ and X any open set in W.

__Corollary 2.5__: If $\dim_L X < \infty$, then $\dim_L(X/A) < \infty$.

__Proof__: By 2.4, $\dim_L(X/A) = \dim_L(X/A - *) = \dim_L(X-A) \leq \dim_L(X)$.

We list a theorem about cohomology manifolds. A proof may be found in [2].

__Theorem 2.6__: Let X be a connected $n\text{-cm}_L$. Then

(1) For every non-empty open subset U, the homomorphism
$$j_{XU} : H_C^n(U) \to H_C^n(X)$$
is surjective, hence $H_C^n(A) = 0$ for every proper closed subset A of X.

(2) X is orientable if and only if $H_C^n(X) \approx L$. If X

is orientable and U is an open subset, then U is orientable and, if U is moreover connected,

$$j_{XU} : H_C^n(U) \cong H_C^n(X).$$

We also will require a universal coefficient formula relating sheaf cohomology with compact support to Borel-Moore homology with closed support.

2.7 If U is an open subset of X, there is a sequence

$$0 \to \mathrm{Ext}\,(H_C^{p+1}(U),L) \to H_p(U) \to \mathrm{Hom}\,(H_C^p(U),L) \to 0$$

which is natural with respect to inclusions of open sets, that is, with respect to $H_p(X) \to H_p(U)$ and $H_C^*(U) \to H_C^*(X)$ [5, p. 184].

<u>Lemma 2.8</u>: Let X be an orientable n-cm_L and let U, V be open, connected subsets of X with $U \subset V$. Then the homomorphism $H_n(V) \to H_n(U)$ induced by restriction is an isomorphism.

<u>Proof</u>: From 2.6, j_{XU} and j_{XV} are isomorphisms in dimension n, and since $j_{XU} = j_{XV} \circ j_{VU}$, j_{VU} is also an isomorphism. The universal coefficient formula produces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}\,(H_C^{n+1}(V),L) & \longrightarrow & H_n(V) & \longrightarrow & \mathrm{Hom}\,(H_C^n(V),L) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Ext}\,(j_{VU},1)} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}\,(j_{VU},1)} & & \\
0 & \longrightarrow & \mathrm{Ext}\,(H_C^{n+1}(U),L) & \longrightarrow & H_n(U) & \longrightarrow & \mathrm{Hom}\,(H_C^n(U),L) & \longrightarrow & 0
\end{array}$$

Since $\dim_L X \leq n$, $\mathrm{Ext}\,(H_C^{n+1}(V),L) = \mathrm{Ext}\,(H_C^{n+1}(U),L) = 0$.

Moreover, $\mathrm{Hom}\,(j_{VU},1)$ is an isomorphism since j_{VU} is. Thus $H_n(V) \to H_n(U)$ is an isomorphism as required.

By "divisor", we now mean a divisor with respect to sheaf cohomology and Borel-Moore homology, both having constant coefficients and compact supports.

In order to establish that the orientation sheaf of Y is locally constant we need a

<u>Lemma 2.9</u>: If X is an orientable $n\text{-cm}_L$ and A is a compact, connected divisor, then for any open neighborhood U of $*$ in Y, the homomorphism $c^*\colon H_c^p(U) \to H_c^p(c^{-1}U)$ induced by $c : X \to Y$ is a monomorphism for all p and is an isomorphism for $p = n$.

<u>**Proof**</u>: Since the long exact sequence for cohomology is functorial, the algebraic lemma provides a commutative diagram, $p > 0$

$$\begin{tikzcd}
0 \arrow[r] & H_c^p(Y) \arrow[r, "c^*"] \arrow[d] & H_c^p(X) \arrow[r, "h^*"] \arrow[d, "r^*"] & H_c^p(A) \arrow[r] \arrow[d, "="] & 0 \\
\cdots \arrow[r] & H_c^p(U) \arrow[r, "c^*"] & H_c^p(c^{-1}U) \arrow[r, "k^*"] & H_c^p(A) \arrow[r] & \cdots
\end{tikzcd}$$

(since A is also a divisor of the $n\text{-cm}_L$, $c^{-1}U$) where h^*, k^*, and the vertical homomorphisms are induced by inclusions. Since $k^* \circ r^* = h^*$ is an epimorphism, so is k^* (for $p > 0$). Thus $c^* : H_c^p(U) \to H_c^p(c^{-1}U)$ is a monomorphism for $p > 1$, and, because by 2.6 $H_c^n(A) = 0$, an isomorphism for $p = n$. Due to the fact that the augmented homology module $\tilde{H}_c^0(A)$ is trivial, $c^* : H_c^1(U) \to H_c^1(c^{-1}U)$ is a monomorphism. For $p = 0$, any map of connected spaces induces an isomorphism.

Combining the preceding results we are able to show

<u>Lemma 2.10</u>: If X is an orientable $n\text{-cm}_L$ and A is a compact, connected divisor, then the orientation sheaf $\mathscr{O}_n(Y)$ of Y is locally constant. In fact, it is locally isomorphic to the constant L-sheaf.

<u>Proof</u>: This is clear for points of Y other than $*$. To prove it for $*$, let U and V be connected neighborhoods of $*$ in Y with $U \subset V$. The universal coefficient formula 2.7 provides a commutative diagram (since $\dim_L U$, $\dim_L V \leq n$)

$$\begin{tikzcd}
H_n(V) \arrow[r, "\approx"] \arrow[d, "r_*"'] & \mathrm{Hom}\,(H_c^n(V),L) \arrow[d, "\mathrm{Hom}\,(j_{VU},1)"] \\
H_n(U) \arrow[r, "\approx"] & \mathrm{Hom}\,(H_c^n(U),L)
\end{tikzcd}$$

where r_* is induced by restriction.

But in the commutative diagram

$$\begin{tikzcd}
H_c^n(U) \arrow[r, "c^*"] \arrow[d, "j_{VU}"'] & H_c^n(c^{-1}U) \arrow[d, "j_{c^{-1}V,c^{-1}U}"] \\
H_c^n(V) \arrow[r, "c^*"] & H_c^n(c^{-1}V)
\end{tikzcd}$$

the monomorphisms c^* are isomorphisms due to 2.9, and

$j_{c^{-1}U,c^{-1}V}$ is an isomorphism which causes Hom $(j_{VU},1)$ to be an isomorphism. Hence r^* is an isomorphism.

Define a presheaf F on Y by $F(U) = H_n(U')$ where U' is the component of U containing $*$, and define the homomorphism $F(U) \to F(V)$ for $V \subset U$ by restriction. Since clc^0_L is equivalent to local connectedness [15], and X is clc^0_L, connected neighborhoods of $*$ in Y are cofinal in the neighborhood system of $*$ in Y. It is then easy to see that the sheaf generated by F is isomorphic to the sheaf $\mathscr{H}_n(Y)$. The above discussion also yields that F is locally constant with stalks isomorphic to L.

In order to see that Y is clc_L, we need that this is equivalent to another condition.

2.11 [5, p. 77] Let X be a locally compact Hausdorff space, then the following two statements are equivalent.

(i) X is clc^∞_L

(ii) If U and W are open, relatively compact subspaces of X with $\overline{U} \subset W$, then Image $[j_{WU} : H^p_c(U) \to H^p_c(W)]$ is finitely generated for each p. (Here $\overline{U}$ denotes the closure of U.)

__Lemma 2.12__: If X is an orientable n-cm_L and A is a compact, connected divisor, then Y is clc_L.

__Proof__: Since $\dim_L Y$ is finite, we only need prove that

Y is clc_L^∞. In fact we only need prove that 2.11(ii) holds where U and W are neighborhoods of $*$ in Y.

To see this, consider the following commutative diagram.

$$\begin{tikzcd} H_c^p(U) \arrow[r, "c^*"] \arrow[d, "j_{WU}"'] & H_c^p(c^{-1}U) \arrow[d, "j_{c^{-1}W,c^{-1}U}"] \\ H_c^p(W) \arrow[r, "c^*"] & H_c^p(c^{-1}W) \end{tikzcd}$$

Now $\overline{U} \subset W$ implies that $\overline{c^{-1}U} \subset c^{-1}W$, but X is clc_L^∞ so that Image $[j_{c^{-1}W,c^{-1}U} : H_c^p(U) \to H_c^p(c^{-1}W)]$ is finitely generated. Thus we have that Image $[j_{c^{-1}W,c^{-1}U} \circ c^* : H_c^p(U) \to H_c^p(c^{-1}W)]$ = Image $[c^* \circ j_{WU} : H_c^p(U) \to H_c^p(c^{-1}W)]$ is also finitely generated. However, since $c^* : H_c^p(W) \to H_c^p(c^{-1}W)$ is a monomorphism by 2.9, we can conclude that Image $[j_{WU} : H_c^p(U) \to H_c^p(W)]$ is finitely generated.

Recall that a space X is <u>completely paracompact</u> if every open subset of X is paracompact. This guarantees that closed supports are paracompactifying for every open subset of X and thus Poincaré duality holds. (Actually, for our purposes it would be sufficient to assume that X and $X-A$ are paracompact).

In order to see that Y is an $(L-n)$-space we need

2.13 [5, p. 206] The homology sheaf $\mathcal{H}_p(Y;L)$ has the stalk over $y \in Y$ isomorphic to $H_p^c(Y,Y-y)$.

<u>Lemma 2.14</u>: If A is a compact, connected divisor of

X, a completely paracompact, orientable $n\text{-cm}_L$, then $\mathscr{H}_q(Y;L) = 0$ for $q \neq n$.

Proof: Since $H_p^C(Y,Y-y) = H_p^C(X,X-c^{-1}(y)) = 0$ for $y \neq *$, 2.13 allows us to consider the single case $y = *$. Let $\alpha \in H_n(X-A)$ be the fundamental class of $X-A$ abd $\gamma \in H_n(X)$ be that of X. Consider the "box" diagram

$$\begin{tikzcd}
 & H_C^{n-q}(X-A) \arrow[rr, "j_{X,X-A}"] \arrow[d, dashed, "\alpha\cap"] & & H_C^{n-q}(X) \arrow[d, "\gamma\cap"] \\
 & H_q^C(X-A) \arrow[rr, dashed, "i_*"] \arrow[ddl, dashed, "c_*"'] & & H_q^C(X) \\
H_C^{n-q}(Y-*) \arrow[uur, "c^*"] \arrow[rr, "j_{Y,Y-*}"] \arrow[d, "(c_*\alpha)\cap"'] & & H_C^{n-q}(Y) \arrow[uur, "c^*"] \arrow[d, "(c_*\gamma)\cap"'] & \\
H_q^C(Y-*) \arrow[rr, "k_*"] & & H_q^C(Y) \arrow[uur, "c_*"'] &
\end{tikzcd}$$

The inclusions $X-A \subset X$ and $Y-* \subset Y$ respectively induce the homomorphisms i_* and k_* respectively. The maps c^* and c_* are all induced by the collapsing map $c : X \to Y$. The right and left faces commute due to the functorial nature of the cap product, and the rear face commutes because of a property of Poincaré duality [5, p. 210]. The top and bottom faces commute by reason of the fact that the homomorphisms induced by a proper map on the homology and cohomology modules is functorial. We will demonstrate that the front face also commutes and that consequently k_* is an isomorphism. By chasing the diagram we get that

$$\begin{aligned}(c_*\gamma)\cap\circ j_{Y,Y-*} &= c_*\circ\gamma\cap\circ c^*\circ j_{Y,Y-*}\\ &= c_*\circ\gamma\cap\circ j_{X,X-A}\circ c^*\\ &= c_*\circ i_*\circ\alpha\cap\circ c^*\\ &= k_*\circ c_*\circ\alpha\cap\circ c^*\\ &= k_*\circ(c_*\alpha)\cap\end{aligned}$$

Moreover, the exact sequence 2.3 with $A = *$ and $X = Y$ shows that $j_{Y,Y-*}$ is an isomorphism for $q \neq 0$. But $(c_*\gamma)\cap$ is an isomorphism because A is a divisor, and $(c_*\alpha)\cap$ is an isomorphism because it is the composite, $c_*\circ\alpha\cap\circ c^*$, of isomorphisms. Thus k_* is an isomorphism for $q \neq n$. The long exact homology sequence of the pair $(Y,Y-*)$ then yields that $H_q(Y,Y-*) = 0$ for $q \neq n$. By 2.13, $\mathcal{H}_q(Y;L)_* = 0$ for $q \neq n$. We have already noted that $\mathcal{H}_q(Y;L)_y = 0$ for $q \neq n$ and $y \in Y-*$ so $\mathcal{H}_q(Y;L)$ is trivial for $q \neq n$.

<u>Corollary 2.15</u>: If A is a compact, connected divisor of X, a completely paracompact orientable $n\text{-cm}_L$, then Y is an $(L\text{-}n)$-space.

<u>Proof</u>: By 2.14, $\mathcal{H}_q(Y;L) = 0$ for $q \neq n$ and by 2.10, $\mathcal{H}_n(X;L)$ has stalks isomorphic to L so it is torsion-free.

Combining our previous results, we obtain the principal results

Theorem 2.16: If X is a completely paracompact, orientable $n\text{-cm}_L$, then A is a compact, connected divisor if and only if Y is an orientable $n\text{-cm}_L$. Moreover, in either case, the sequences

$$0 \to H_q^c(A) \to H_q^c(X) \to H_q^c(X/A) \to 0$$

and

$$0 \to H_c^q(X/A) \to H_c^q(X) \to H_c^q(A) \to 0$$

are split exact for $q \neq 0$.

Proof: By 2.5, 2.10, and 2.15, Y is a finite-dimensional $(L\text{-}n)$-space satisfying $\mathcal{H}_n(Y;L)$ is locally constant with stalks isomorphic to L. Thus Y is an $n\text{-hm}_L$. By 2.12, Y is clc_L and hence Y is an $n\text{-cm}_L$. According to 2.6(2), Y is orientable since $c^* : H_n^c(X) \to H_n^c(Y)$ is an isomorphism.

The next theorem generalizes a result due to Wilder [16, 17].

Theorem 2.17: Let X be a compact, orientable $n\text{-cm}_L$ and let $f : X \to Y$ be surjective such that for each $y \in Y$, $f^{-1}(y)$ is a connected divisor. Then Y is an orientable $n\text{-cm}_L$.

Proof: Let $\{A_i \mid i \in I\}$ be the collection of point-inverses of f which are not acyclic. Since $H_c^*(X)$ is finitely generated [15] and since A_i is a divisor, $H_c^*(X/A_i)$ is simpler than $H_c^*(X)$ in the sense that either rank $H_c^*(X/A_i) <$ rank $H_c^*(X)$ or the torsion part of $H_c^*(X/A_i)$

is a non-trivial direct summand of the torsion part of $H^*_c(X)$. Since such simplifications cannot be made infinitely many times, I must be finite. Let Z be the space obtained from X by collapsing each A_i to a point. By 2.16 and induction, Z is an orientable n-cm_L. The map $Z \to Y$ induced by f has acyclic point-inverses so by Wilder's monotone mapping theorem [16], Y is an orientable n-cm_L.

__Example__: The examples of chapter 1 are also examples here since an n-phm_L is an n-cm_L. To construct a non-polyhedral example, let C be a "sin 1/x curve" in a 4-cell so that $\bar{C} \cap S^3$ is an arc, where $S^3 = \dot{e}^4$. Let $h : S^3 \to S^2$ be the Hopf map and let $A = \text{im}\,[C \cup S^3]$ in $e^4 \cup_h S^2 = \mathbf{C}P^2$. Then A is a divisor since $\mathbf{C}P^2/A$ is homeomorphic to S^4.

__Remark__: If X is completely paracompact and f is proper, one can drop the compactness from the hypotheses of theorem 2.17. In this case, choose for $y \in Y$, neighborhoods U and V so that $\overline{f^{-1}(U)} \subset f^{-1}(V)$ are connected and relatively compact. Then by Poincaré Duality and [5, p. 77], $\text{im}\,[H^c_*(f^{-1}U) \to H^c_*(f^{-1}V)]$ is finitely generated and contains the image of $H^c_*(A)$, for any divisor A in $f^{-1}(U)$, as a direct summand. As in the proof of the theorem, only finitely many such divisors can be non-acyclic, so that U is an n-cm_L.

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