FOSTER DISTRIBUTED-LUMPED NETWORK SYNTHESIS

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ABSTRACT

FOSTER DISTRIBUTED-LUMPED
NETWORK SYNTHESIS

by G. T. Daryanani

Distributed network theory is approached from a basic and unifying standpoint. Sufficient conditions are developed for the realizability of frequency domain, non-rational, immittance functions. The networks consist of distributed and lumped elements and have Foster-type topologies. As a starting point for a comprehensive theory for distributed network synthesis, they are recommended by their mathematical tractability.

The approach used is to classify functions by their singularities. RC and RL networks are the only ones considered. All singularities lie on the negative real axis of the complex s-plane. The first class considered consists of functions that have a discontinuity across a line on the axis and are holomorphic elsewhere. This class includes branches of multivalued functions which have branch points as their singularities. The theory depends on the properties of an integral with a Cauchy-type kernel evaluated along the line of discontinuity. The Russian mathematician, Muskhelishvili, discusses such integrals in his work on singular integral equations.

In the second class the functions may have a countable (finite or infinite) number of poles. The Mittag-Leffler theorem gives representations for functions with infinite numbers of poles which yield Foster-type infinite-lumped networks.

An open question, the answer to which is possibly in the negative, is--can p. r. immittance functions which are RC, RL realizable have any other singularities?

FOSTER DISTRIBUTED-LUMPED

NETWORK SYNTHESIS

By

G. T. Daryanani

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Carol

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CHAPTER I

INTRODUCTION

With the development of integrated circuit technology, distributed networks have taken an important place in the synthesis of network functions. A problem the network designer faces is to translate a given set of graphical or tabular data into a mathematical function for which a network synthesis procedure exists. There is much information available on the approximation of rational network functions. These lead to networks with lumped elements, which are a sub-class of distributed elements. Distributed networks can be used to realize functions which are not rational. Infinite numbers of lumped elements would be needed to realize non-rational functions. Distributed networks are preferred for their ease of fabrication.

The networks considered are linear, time-invariant and passive. They may consist of distributed elements and a countable number of lumped elements.

1.1 Literature Survey

References 1-3 and 13 deal with the realizability of uniformly distributed RC line ($\overline{\text{URC}}$) networks by means of certain transformations of the frequency variable to a form suitable for lumped synthesis procedures. Wyndrum[1] uses positive real transformations to obtain the realizability conditions for driving point synthesis of $\overline{\text{URC}}$ networks with identical RC products. The network sections have a cascade or a series-parallel structure. O'Shea's[2] transformations (not p. r.) are more general in that they yield realizations consisting of an arbitrary interconnection of $\overline{\text{URC}}$ networks with constant RC products. Rao, Schaffer and Newcomb[3] treat the realizability of arbitrary n-port connections of $\overline{\text{URC}}$ networks with rationally related $\sqrt{rcl}$ products.

Heizer[4] shows how a class of immittances in rational form can be realized using a single tapered distributed network.

Networks consisting of both distributed lines and lumped elements are treated in references 5-7. Rao and Newcomb[6] apply the works of Youla[5] and Koga[10] to the synthesis of arbitrarily interconnected networks consisting of $\overline{\text{URC}}$ lines with rationally related $\sqrt{rcl}$ products, lumped resistors, capacitors, and ideal transformers. Protonotarius and Wing[7] suggest a description for a non-uniform RC line with lumped elements along the line. The networks are characterized by some analytic properties of the network functions (ABCD parameters). Conditions are given for a function to be realizable by a non-uniform RC line with an RC impedance termination.

The theory of functions of two complex variables has been utilized for the synthesis of variable parameter networks in references 8-10. Ansell[8] applies the two-variable theory to the synthesis of lossless transmission lines with commensurate delays (i.e., all the line delays in the network are whole multiples of some unit delay). The lumped elements are allowed to have a frequency-dependent behavior.

1.2 Objectives of the Thesis

In this thesis distributed network theory is approached from a basic and unifying standpoint. The frequency-domain, non-rational immittance functions considered are classified by their singularities. Synthesis procedures are developed for functions with various types of singularities.

Syntheses are effected by distributed networks with Foster-type topologies. Such topologies have not previously been investigated, yet their study is well-motivated on both theoretical and applied bases. Physical components with such models occur in nature, albeit frequently in forms equivalent to lumped models. For example, a lumped resistor R may be written as $R = \int r_x dx$ which has a Foster-distributed representation. As a starting point for a comprehensive theory of distributed network synthesis, they are recommended by their mathematical tractability.

RC and RL networks are the only ones considered. All singularities lie on the negative real axis of the complex s-plane for such network functions. The first class considered consists of functions that have a discontinuity across a line (or a union of line segments) on the axis and are holomorphic elsewhere. This class includes branches of multivalued functions which have branch points as their singularities (Ref. 12 p. 59-64). The theory depends on the properties of an integral with a Cauchy-type kernel evaluated along the line of discontinuity. The Russian mathematician, Muskhelishvili, discusses this integral in his monograph on singular integral equations.[11]

In the second class considered the functions may have both a line of discontinuity and a countably infinite number of poles. The Mittag-Leffler theorem (Appendix B) gives representations for functions with infinite numbers of poles which yield Foster-type infinite-lumped networks.

1.3 Summary of Chapters

Properties of the Cauchy integral and related definitions are discussed in Chapter 2. In Chapter 3 synthesis procedures are developed for functions with a line discontinuity. Various network interpretations of the Cauchy integral are given. Extensions to admittances, to LC distributed networks, and to positive real transformations are also covered. Chapter 4 deals with functions

which have a countably infinite number of poles together with line discontinuities. Sufficient conditions are developed for the synthesis of such functions. In the conclusions some possible extensions and practical aspects of this thesis are considered.

CHAPTER II

THE CAUCHY INTEGRAL

In this chapter immittance functions with a line discontinuity are considered.

2.1 Motivation for Cauchy Representation

An integral representation with a Cauchy-type kernel forms the basis of the development. Such a representation is motivated by considering the following state description of the driving point impedance of a linear, passive, time-invariant network: [22]

$$\frac{d\Psi(s,t)}{dt} = s\Psi(s,t) + i(t) \tag{2.1}$$

$$v(t) = \int_{Br.} \frac{1}{2\pi j} \cdot Z(s)\Psi(s,t)\,ds \tag{2.2}$$

where,

i(t) is the input current,

Br. denotes the Bromwich path of integration

v(t) is the voltage developed at the input port,

Z(s) is the driving point impedance,

$\Psi(s,t)$ the state "vector" is the solution to equation 1 and $s \varepsilon C$ the class of complex numbers $(s = \sigma + j\omega)$

For RC and RL networks Z(s) has singularities only on the negative real axis. The driving point impedance then has the alternate state description (see Appendix A for proof).

$$\frac{d\Psi(-\sigma,t)}{dt} = -\sigma\Psi(-\sigma,t) + i(t) \qquad (2.3)$$

$$v(t) = \int_0^{\infty} f(\sigma)\ \Psi(-\sigma,t)d\sigma \qquad (2.4)$$

These equations are the state space description of a distributed network, one form of which is shown in fig. 2. The voltage at the input port (fig. 2) is

$$v(t) = \int_0^{\infty} v(\sigma)d\sigma = \int_0^{\infty} f(\sigma)\Psi(-\sigma,t)d\sigma$$

The current i(t) is the sum of the currents i_r and i_c through any section.

$$i = i_r + i_c$$

$$= \sigma\frac{v(\sigma)d\sigma}{f(\sigma)d\sigma} + \frac{1}{f(\sigma)d\sigma}\cdot\frac{d(v(\sigma)d\sigma)}{dt}$$

$$i(t) = \frac{\Psi(-\sigma,t)\ f(\sigma)d\sigma.\sigma}{f(\sigma)d\sigma} + \frac{f(\sigma)d\sigma\ d\Psi(-\sigma,t)}{f(\sigma)d\sigma\ dt}$$

$$\frac{d\Psi(-\sigma,t)}{dt} = -\sigma\Psi(-\sigma,t) + i(t)$$

So the network of figure 1 is seen to have the state description given by equations 2.3 and 2.4.

The impedance of this network is an integral with a Cauchy-type kernel

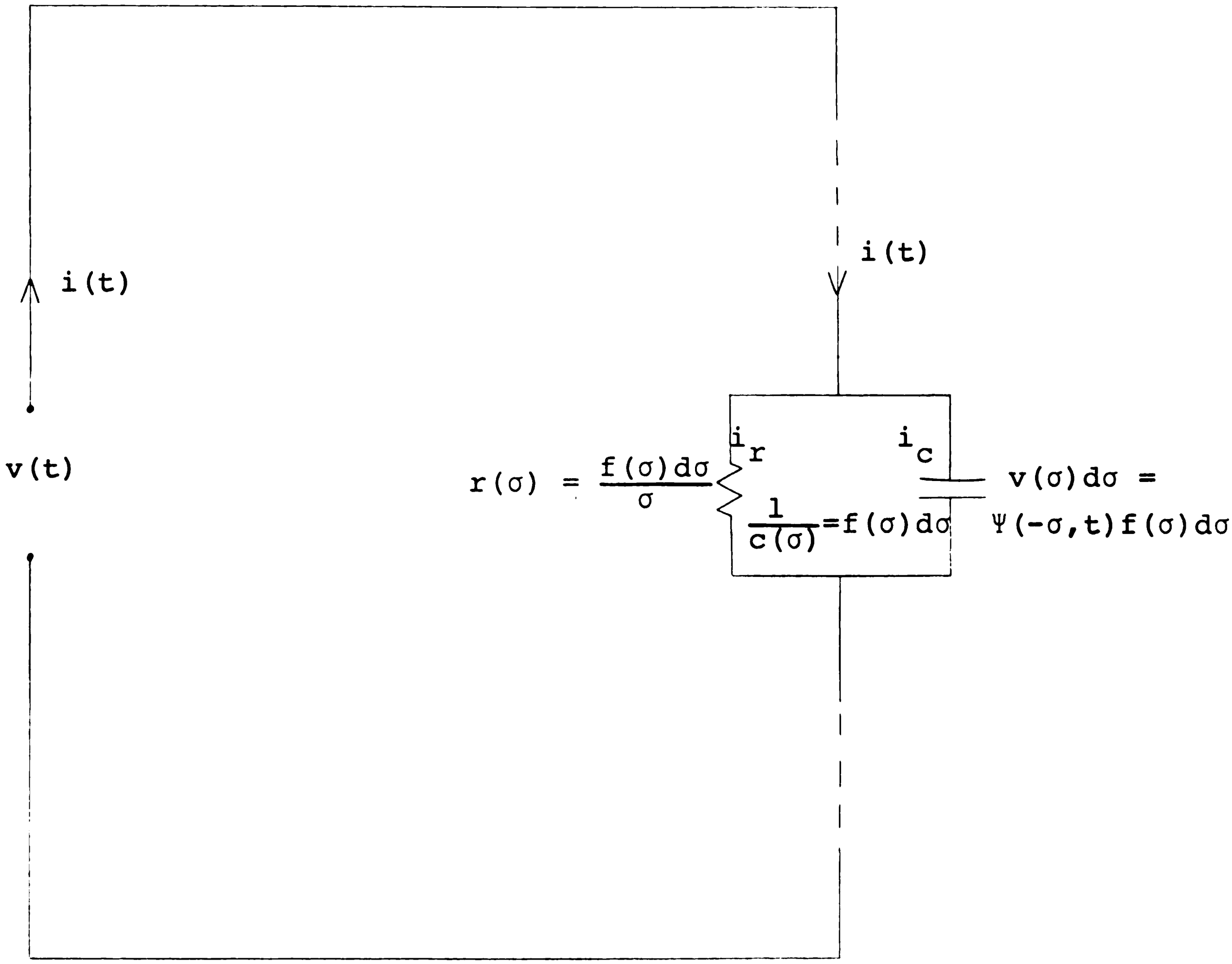


Figure 1. Network Satisfying State Description Equations 2.3 and 2.4

$$Z(s) = \int_{0}^{\infty} \frac{f(\sigma)d\sigma}{s+\sigma} \tag{2.5}$$

The above development indicates that immittance functions with line discontinuities on the negative real axis could have a Cauchy-type integral representation which could lead to a network interpretation of the function. The conditions under which such functions do have a Cauchy representation are given in Theorem 2.2. Synthesis procedures for the integral are discussed in Chapter 3. Some definitions[11] required for the development are given in the next section.

2.2 Definitions

Def. 2.1 A line.--The union of a finite number of non-intersecting arcs L_j is called a smooth line L. A smooth arc possesses a continuous tangent at each point and is open or closed. The ends of the arcs L_j will be denoted by c_j (see fig. 2). Infinity may be an end point. The orientation chosen for the arc is from c_j to c_{j+1}. For RC, RL realisable network functions the arcs will lie on the negative real axis. The upper half plane is called the L^+ region and the lower half plane the L^- region.

Def. 2.2 Cauchy Integral.--Let f(t) be a function of the point t on a line L, bounded everywhere on L, with the possible exception of a finite number of points c_j where

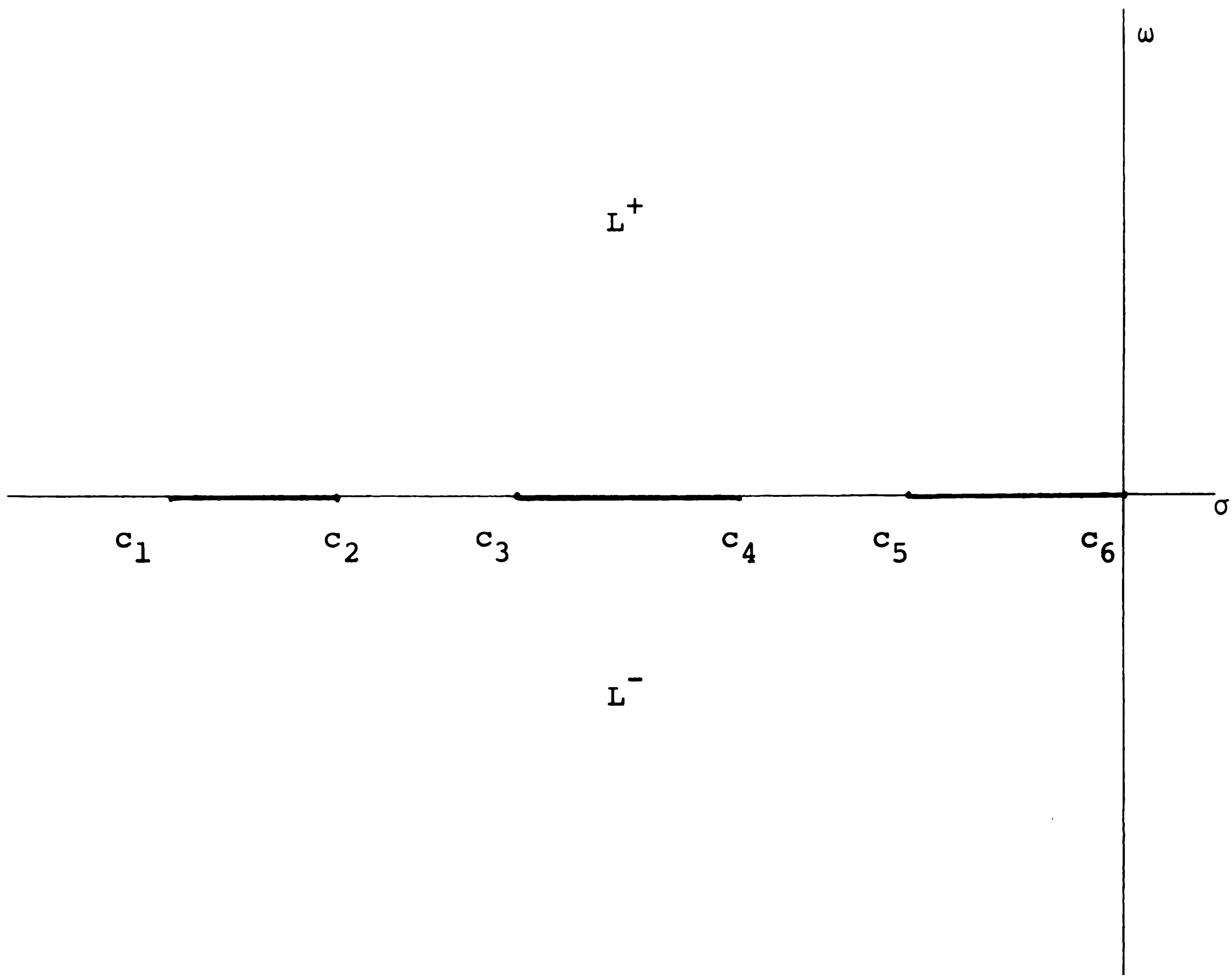


Figure 2. The Line of Discontinuity

$$|f(t)| \leq \frac{K}{|t-c|^{\alpha}}$$

c stands for any of the points c_j, K and α are positive constants and $\alpha<1$. Then

$$F(s) = \frac{1}{2\pi j} \int_L \frac{f(t)}{t-s}\, dt \qquad (2.6)$$

where s is any point in the plane not on the line, is called the Cauchy Integral.

Def. 2.3 Density Function.--In equation 2.6 $f(t)/2\pi j$ is called the density function. (This definition differs from the one in Ref. 11 where f(t) is referred to as the density function.)

Def. 2.4 Sectionally Holomorphic Function.--A function F(s) is sectionally holomorphic with line of continuity L, if F(s) is holomorphic in the plane not including L and if F(s) is continuous (in the sense defined below) on L from the left and right with the possible exception of the ends near which the following inequality holds

$$|F(s)| < \frac{\text{const.}}{|s-c|^{\alpha}} \qquad 0\leq\alpha<1 \qquad (2.7)$$

F(s) is said to be continuous at t on L from the left if F(s) tends to a definite limit $F^{+}(t)$ as s approaches t along any path which remains in L^{+}. A similar definition holds for $F^{-}(t)$.

Def. 2.5 Hölder Condition.--A function f(t) is said to satisfy the Hölder (or H) condition on an arc if for any two points t_1, t_2 on the arc

$$|f(t_2) - f(t_1)| \leq A|t_2 - t_1|^{\mu} \tag{2.8}$$

where A and μ are positive constants, A is called the Hölder constant and μ the Hölder index.

Def. 2.6 Class H.--f(t) belongs to the class H on L, if it satisfies the H (μ) condition on each of the closed arcs L_j of L for some $\mu>0$.

Def. 2.7 Class H*.--If f(t) satisfies the H (μ) condition on every closed part of L not containing the ends and if near any end c it is of the form

$$f(t) = \frac{f^*(t)}{(t-c)^{\alpha}} \quad 0 \leq \alpha < 1 \tag{2.9}$$

where $f^*(t)$ belongs to the class H, then f(t) is said to belong to the class H^* on L.

It may be noted that a function satisfies the H condition for all $0 < \nu \leq \mu$, if it does so for μ. Also, if $f_1(t)$ and $f_2(t)$ satisfy the H(μ) and H (ν) conditions respectively then $f_1(t) + f_2(t)$, $f_1(t) \cdot f_2(t)$ and $\frac{1}{f_1(t)}$ ($f_1(t) \neq 0$) satisfy the H condition. (The latter two hold on compact domains.)

2.3 Properties of the Cauchy Integral

The next two theorems give the conditions under which an immittance function has a Cauchy Integral representation.

Theorem 2.1.--If a function $F(s)$ has the following properties:

(1) $F(s) \to 0$ as $s \to \infty$

(2) $F(s)$ is sectionally holomorphic with line of discontinuity L

(3) $F(s)$ satisfies the boundary condition

$F^{-}(t) - F^{+}(t) = 2\pi j f(-t)$

where $f(-t)$ is a function of the point t on L; then $F(s)$ is uniquely determined in the entire complex plane, except where $s \varepsilon L$.

Proof: Suppose there is another function $G(s)$ that satisfies these three properties with the same line of discontinuity and density function. Then $G^{-}(t) - G^{+}(t) = 2\pi j f(-t)$.

Consider $W(s) = F(s) - G(s)$. $F(s)$ and $G(s)$ are sectionally holomorphic so $W(s)$ also is. $F(s)$ and $G(s)$ have the same density function so $W^{+}(t) - W^{-}(t) =$ $F^{+}(t) - F^{-}(t) - G^{+}(t) - G^{-}(t) = 0$ or $W^{+}(t) = W^{-}(t)$.

Define

$$\overline{W}(s) = \begin{cases} W(s) & \text{for } s \notin L \\ W^{+}(t) & \text{for } t \varepsilon L \end{cases}$$

It follows easily from Morera's theorem (Ref. 12 p. 188) that $\overline{W}(s)$ is holomorphic in the neighborhood of any point t on L. Hence $\overline{W}(s)$ is holomorphic in the entire complex plane. By Liouville's theorem[12] $\overline{W}(s)$ must be a constant. But $\overline{W}(s) \to 0$ as $s \to \infty$. Thus $\overline{W}(s) \equiv 0$ in the entire plane. Or, $W(s) \equiv 0$ for $s \notin L$ which implies that $F(s) \equiv G(s)$ for $s \notin L$.

<u>Theorem 2.2</u>.--If a function F(s) satisfies the three conditions of theorem 2.1 and in addition if

(4) f(-t) belongs to the class H^* on L then the Cauchy Integral, with density function -f(-t), is the unique representation for F(s)

$$\text{i.e.} \quad F(s) = \int_L \frac{-f(-t)}{t-s}\, dt \qquad (2.10)$$

It suffices to show that equation 2.8 with $f(-t) \varepsilon H^*$ satisfies the three conditions of theorem 2.1. Uniqueness follows from theorem 2.1. Condition 4 states that f(-t) is Hölder continuous on L and that it satisfies equation 2.9 at the end points.

Consider a function F(s) which satisfies the Cauchy Integral conditions. By Theorem 2 it has the Cauchy Integral representation. Consider a network which realizes this integral. The immittance function of this network satisfies condition 1, 2, 3 and 4. By Theorem 1, this immittance function is identical to F(s).

<u>Def. 2.8 Cauchy Integral Conditions</u>.--The following four conditions will henceforth be referred to as the Cauchy Integral conditions for F(s):

(1) $F(s) \to 0$ as $s \to \infty$

(2) F(s) is sectionally holomorphic with line of discontinuity L

(3) F(s) satisfies the boundary condition

$F^{-}(t) - F^{+}(t) = 2\pi j f(-t)$ for t on L

(4) f(t) belongs to the class H^{*}

The line L is in general the union of disjoint line segments in $[0,\infty]$. For convenience of notation L may be taken to be the whole interval $[0,\infty]$ if f(-t) is assumed to be zero where Z(s) does not have a discontinuity. It should be remembered however that Z(s) and f(-t) must still satisfy the conditions of equations 2.7 and 2.9 at the end points of the original line. Then,

$$Z(s) = \int_{-\infty}^{0} \frac{-f(-t)dt}{t-s}$$

$$= \int_{0}^{\infty} \frac{f(x)}{s+x} \qquad (2.11)$$

Equation 2.9 is the form used in the synthesis procedures. f(-t) may be calculated using the boundary condition $Z^{-}(t) - Z^{+}(t) = 2\pi j f(-t)$. An alternate method which is often more convenient is to use the Cauchy Inversion Integral[11,14]

$$f(-t) = \int_{L} \frac{F(s)}{t-s}\, ds \qquad (2.12)$$

CHAPTER III

FOSTER-DISTRIBUTED NETWORKS

In Chapter II some properties of the Cauchy Integral were discussed. The conditions for a function with a line discontinuity to have a Cauchy Integral representation were given. The purpose of this chapter is to find sufficient conditions for the Cauchy integral to have a network interpretation. The realizations use distributed networks having a Foster-type configuration with resistances, inductances, and elastances.

The discontinuity of an impedance function can be described by the density function f(-x) defined by the condition

$$Z^{-}(x) - Z^{+}(x) = 2\pi j f(-x) \text{ for } x \text{ on } L \tag{3.1}$$

It can be shown that for a linear, passive, time-invariant, real network the density function is real-valued (see Appendix B). This suggests that the discontinuity of Z(s) can be classified according to the sign of the density function.

Sufficient conditions for the realization of Z(s) when f(-x) is non-negative are developed in section 1. Various network interpretations are suggested. The non-negative and the real density function cases are discussed in sections 2 and 3 respectively. An example illustrates each case.

The steps for the synthesis procedure in the general case are summarized in section 4. In section 5 it is conjectured that the Cauchy Integral is a basic description for all RC networks. The facts supporting this conjecture are discussed.

Extensions to admittance functions and to LC networks are given in section 6.

3.1 Non-Negative Density Functions

Consider an impedance function Z(s) whose only singularity is a line of discontinuity on the negative real axis. If Z(s) satisfies the Cauchy Integral conditions (cf. Def. 2.8) then Z(s) may be represented as in equation 2.11

$$Z(s) = \int_0^\infty \frac{f(x)dx}{s+x} \qquad f(x) \geq 0 \qquad (3.2)$$

The density function f(-x) is given to be non-negative for $x \varepsilon [-\infty, 0]$. This is the same as saying f(x) is non-negative for $x \varepsilon [0, \infty]$. Three RC network interpretations for Z(s) are given here.

Figure 3(a) is a distributed RC Foster-type realization of equation 3.2. The distributed elements are a resistance $r_x = \frac{f(x)dx}{x}$ and an elastance $\frac{1}{c_x} = f(x)dx$. Both are non-negative elements.

The Riemann-sum approximation of this integral is

$$\sum_{n=1}^{N} \frac{f(x_i)}{s+x_i} \Delta x_i \tag{3.3}$$

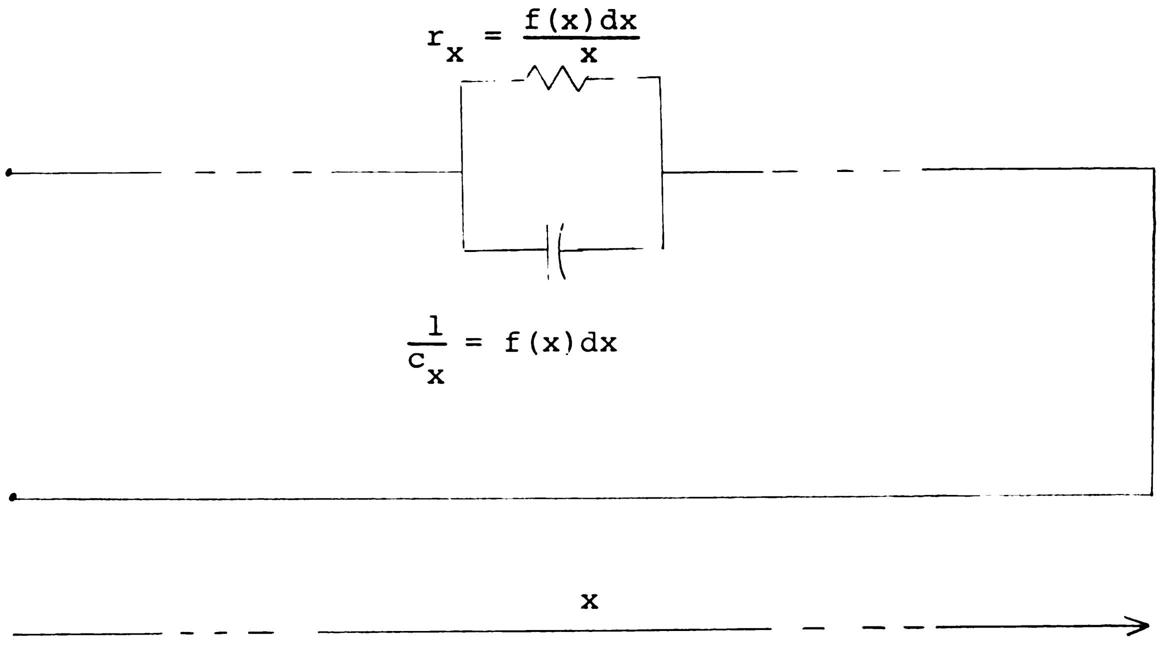
This approximation has the finite-lumped network representation of figure 3(b). The lumped elements are $r_{x_i} = \frac{f(x_i)}{x_i}$ and $\frac{1}{c_{x_i}} = f(x_i)\Delta x_i$. The approximation becomes exact as $\Delta x \to 0$.

A transformer with a real, varying turns ratio could be used to realize the integral of equation 3.2. Figure 4 illustrates this. The turns ratio corresponding to each section is $f(x)dx$ which varies with x.
The v-i characteristics of this transformer network are given by

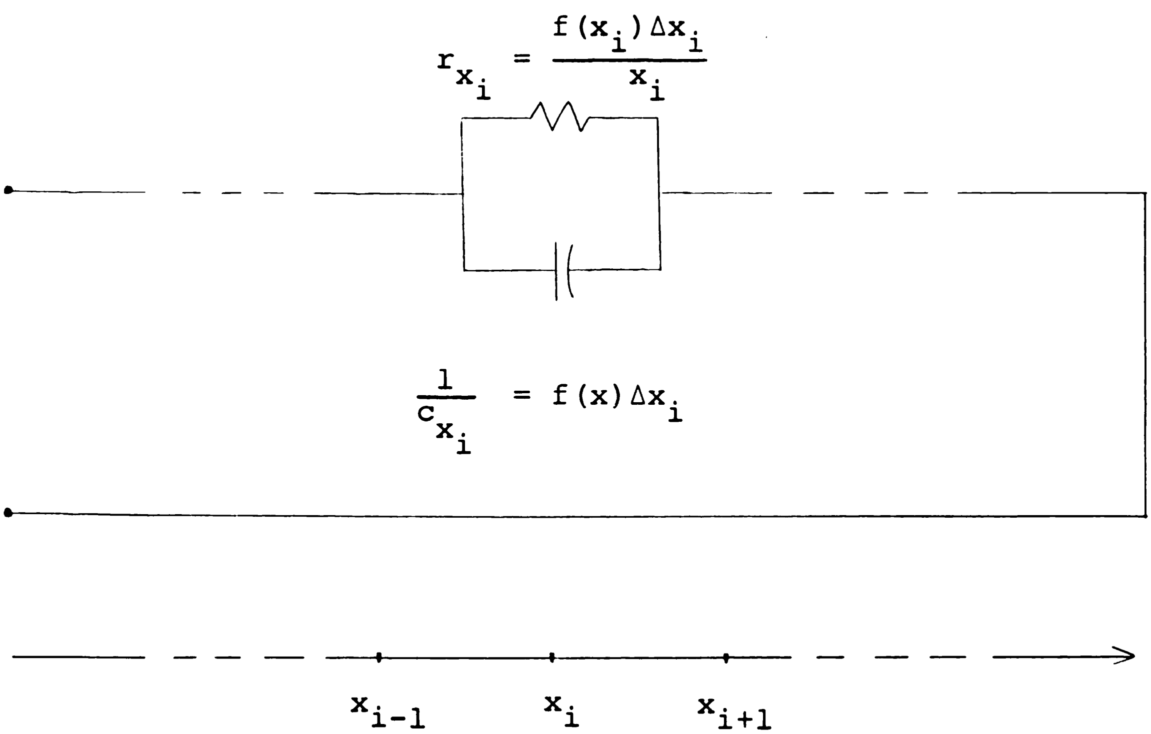
$$v(t) = \int_0^\infty v_x(t)\ f(x)dx$$

$$i_x(t) = -i(t)\ f(x)dx$$

Here $v(t)$ and $i(t)$ are the primary voltage and current; $v_x(t)$, $i_x(t)$ are the secondary voltage and current. (The Riemann sum approximation of eq. 3.3 may be realized by taking the turns ratio to be $f(x_i)\Delta x_i$.) The secondary sections consist of a lumped capacitance of 1 farad in parallel with a lumped resistance of $\frac{1}{x}$ ohms. Such a transformer is a new element and is introduced in this work. The network sections on the secondary side do not



3(a)



3(b)

Figure 3. Networks for Non-Negative Density Function
(a) Distributed Network
(b) Approximate Finite-Lumped Network

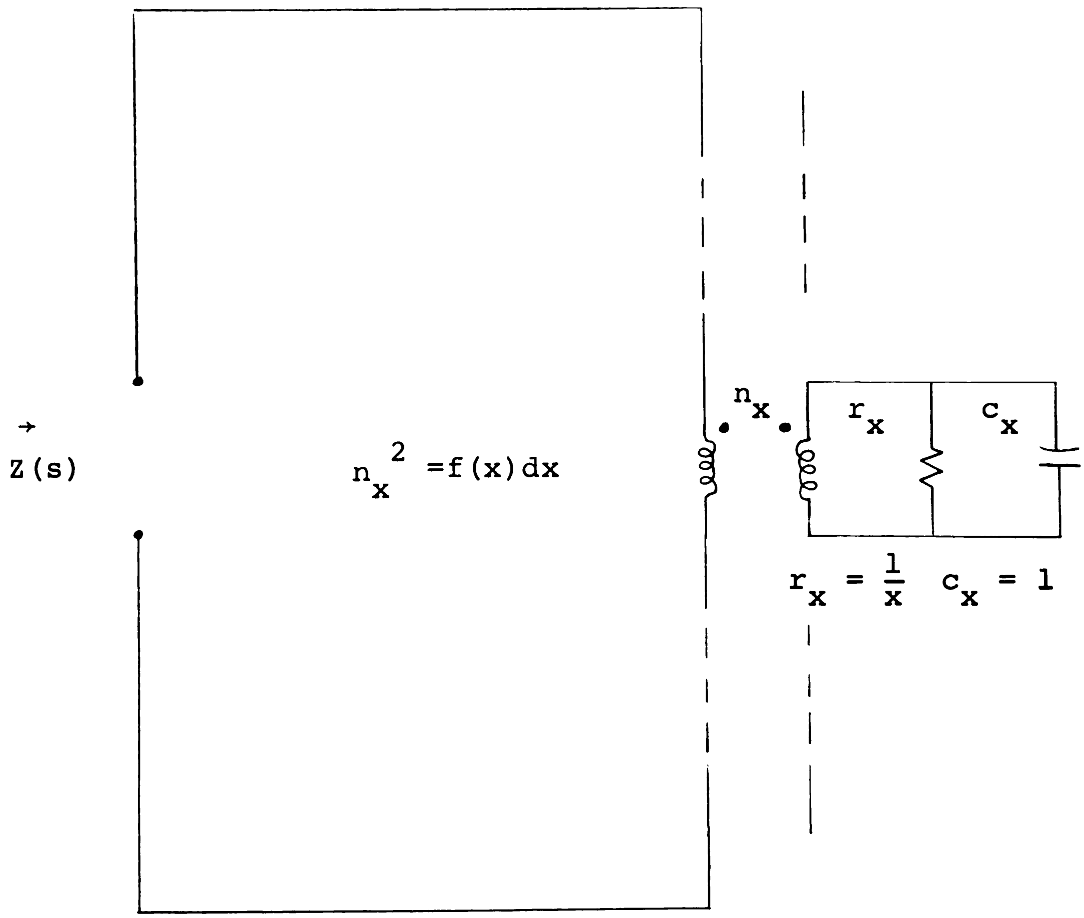


Figure 4. Network for Non-Negative Density Function Using a Transformer

depend on the density function. They only depend on the interval over which there is a discontinuity. The density function of Z(s) determines the turns ratio. This network representation is conceptually simpler and could be of analytic value in extensions of this work.

Henceforth the form of fig. 3(a) will be used to represent network functions.

Note that if Z(s) has a finite limit k as $s \to \infty$, the limit is represented as a lumped resistor k in series with the distributed network.

Example 1.--Consider the multivalued function $Z(s) = \frac{1}{s^{\alpha}}$ $(0<\alpha<1)$. Z(s) has branch points at the origin and infinity. The principal branch of Z(s) determined by a branch cut along the negative real axis is positive real. It is this branch that will be realized. This branch is holomorphic in the whole plane excluding the negative real axis which is the line of discontinuity.

Check on Cauchy Integral conditions:

(i) $\frac{1}{s^{\alpha}} \to 0$ as $s \to \infty$

(ii) $\frac{1}{s^{\alpha}}$ has the negative real axis as the line of discontinuity. The limiting value $Z^{+}(x)$ is $\frac{1}{-x^{\alpha}} e^{-j\pi/\alpha}$ and $Z^{-}(x)$ is $\frac{1}{-x^{\alpha}} e^{+j\pi/\alpha}$.

At the origin, which is an end point

$$|Z(s)| = |\frac{1}{s^\alpha}| \qquad 0<\alpha<1.$$

The end point condition of eq. 2.7 is also satisfied at $c = \infty$.

Hence $Z(s)$ is sectionally holomorphic.

(iii) $$Z^-(x) - Z^+(x) = \frac{e^{+j\pi/\alpha}}{-x^\alpha} - \frac{e^{-j\pi/\alpha}}{-x^\alpha}$$

$$= \frac{1}{-x^\alpha} 2j \sin\frac{\pi}{\alpha}$$

So, $$f(-x) = \frac{1}{-x^\alpha \pi}\sin\frac{\pi}{\alpha} \qquad x\varepsilon[-\infty,0]$$

$$f(x) = \frac{1}{\pi x^\alpha}\sin\frac{\pi}{\alpha} \qquad x\varepsilon[0,\infty]$$

The density function could also be obtained from the Cauchy Inversion Integral (eq. 2.12).

(iv) That the Hölder condition is satisfied by $f(x)$ on every closed part of L not containing the ends is shown as follows.

$$|x_1^\alpha - x_2^\alpha| \leq |x_1 - x_2|^\alpha$$

which says that x^α satisfies the H condition. This implies that $\frac{1}{x^\alpha}$ and hence $\frac{1}{x^\alpha} \cdot (\frac{1}{\pi}\sin\frac{\pi}{\alpha})$ satisfies the condition of equation 2.9.

The Cauchy Integral conditions are satisfied. So, the p.r. branch of $\frac{1}{s^\alpha}$ has the representation of equation 2. The realization is as in fig. 3 with $f(x) = \frac{1}{\pi x^\alpha}\sin\frac{\pi}{\alpha}$; $x\varepsilon\ [0,\infty]$

3.2 Non-Positive Density Functions

If an impedance function Z(s) satisfies all the Cauchy Integral conditions and if its density function is non-positive, it is not p.r. and hence does not have a network realization. This is seen by considering the real part of such a function for $\sigma \geq 0$.

$$\text{Re } Z(s) = \text{Re} \int_0^\infty \frac{f(x)dx}{s+x}$$

$$= \int_0^\infty \frac{f(x)\ (\sigma+x)dx}{(\sigma+x)^2 + \omega^2}$$

which is negative for $\sigma \geq 0$.

3.2(a) Case 1. A function Z(s) with a non-positive density function could be realized if $\frac{Z(s)}{s}$ (but not Z(s)) satisfied the Cauchy conditions. The function $Z_1(s) = \frac{Z(s)}{s}$ may be written as

$$Z_1(s) = \frac{Z(s)}{s} = \int_0^\infty \frac{f_1(x)}{s + x}\ dx$$

where $f_1(x)$ is the density function of $Z_1(s)$. Note that $f_1(x) = \frac{f(x)}{-x}$ is non-negative.

$$Z(s) = \int_0^\infty \frac{s\ f_1(x)}{s + x}\ dx. \tag{3.4}$$

This integral form of Z(s) has an RL distributed network realization as shown in figure 5. The inductance $\frac{f_1(x)dx}{x}$ and the resistance $f_1(x)$ dx are non-negative. The Riemann sum approximate networks with and without the transformer are obtained as before.

The functions considered in this case need not be regular at infinity.

Example 2. Consider the p.r. branch of s^{α} $(0<\alpha<1)$. The density function of $Z(s) = s^{\alpha}$ is

$$f(-x) = \frac{1}{2\pi j}(x^{\alpha}e^{j\pi\alpha} - x^{\alpha}e^{-j\pi\alpha})$$

$$= \frac{x^{\alpha}}{\pi}\sin\pi\alpha \qquad x \varepsilon [-\infty,0]$$

so $f(x) = \frac{-x^{\alpha}}{\pi}\sin\pi\alpha \quad x\varepsilon\ [0,\infty]$

$Z_1(s) = \frac{Z(s)}{s} = \frac{1}{s^{\beta}}$ $(\beta = 1-\alpha)$. It was shown in example 1 that $\frac{1}{s^{\beta}}$ satisfied the Cauchy Integral conditions.

$f_1(x) = \frac{1}{\pi x^{\beta}}\sin\frac{\pi}{\beta}$ is non-negative.

$Z(s)$ thus has the network realization of fig. 5 with

$f_1(x) = \frac{1}{\pi x^{\beta}}\sin\frac{\pi}{\beta}$, $x\varepsilon\ [0,\infty]$.

3.2(b) Case 2. Next, sufficient conditions are obtained for a function which is regular at infinity. Suppose the density function $f(x)$ is non-positive and $Z(s)$ has the limit $Z(\infty)$ (a constant) as $s \to \infty$. Let $Z(s) = Z_1(s) + Z(\infty)$ and suppose

(1) $Z_1(s)$ satisfies the Cauchy Integral conditions.

(2) $Z(\infty) = \int_0^{\infty}\frac{f_1(x)}{-x}dx + R_1$ where $0\leq R_1<\infty$ and

$\frac{f_1(x)}{x}$ is integrable.

Then $Z(s)$ may be written as

$$Z(s) = Z_1(s) + Z(\infty) = \int_0^{\infty}\frac{f_1(x)}{s+x}dx + \int_0^{\infty}\frac{f_1(x)}{-x}dx + R_1 \tag{3.5}$$

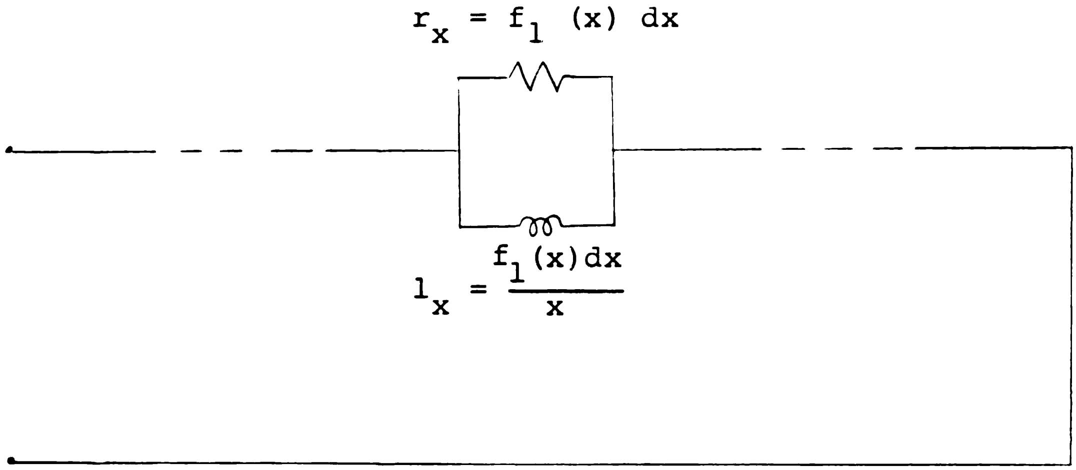


Figure 5. Network for Non-Positive Density Function (Eq. 3.4)

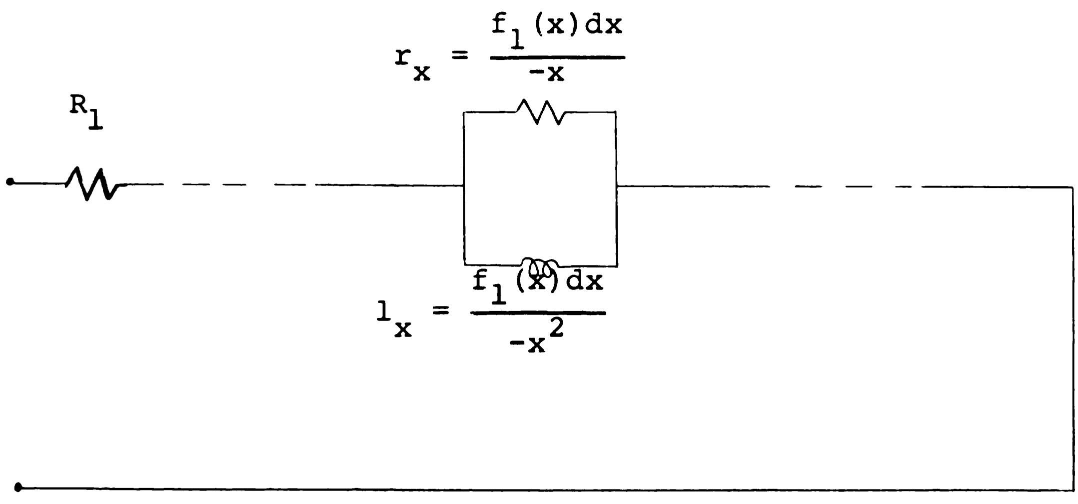


Figure 6. Network for Non-Positive Density Function (Eq. 3.6)

since $Z_1(s)$ has a Cauchy integral representation. The density function of $Z_1(s)$ is the same as that of $Z(s)$ - so $f_1(x)$ is non-positive.

Since $\frac{f_1(x)}{x}$ is integrable the two integrals in eq. 3.5 may be combined

$$Z(s) = \int_0^\infty \left(\frac{f_1(x)}{s+x} + \frac{f_1(x)}{-x}\right) dx + R_1$$

$$= \int_0^\infty \frac{s \cdot f_1(x)/-x}{s+x} dx + R_1 \qquad (3.6)$$

$\frac{f_1(x)}{-x}$ is non-negative. Equation 3.6 is easily seen to have the RL distributed network realization of figure 6.

In the above $Z_1(s)$ satisfies the Cauchy conditions and has a non-positive density function. So $Z_1(s)$ is not p.r. The addition of a large enough positive constant $(Z(\infty))$ makes the sum $(Z(s))$ p.r.

Example 3. An example of such a function is $\log \frac{2(s+1)}{(s+2)}$

$$Z(s) = \log 2 \left(\frac{s+1}{s+2}\right) = \log 2 + \log \left(\frac{s+1}{s+2}\right)$$

$$= \int_1^2 \frac{1}{x} dx + \int_1^2 \frac{-1}{s+x} dx$$

$$= \int_1^2 \frac{s}{s+x} dx$$

Here $Z_1(s) = \log \left(\frac{s+1}{s+2}\right)$; $Z(\infty) = \log 2$; $R_1 = 0$ and $f_1(x) = -1$.

The realization is as in fig. 6 with $f_1(x) = -1$; $x \varepsilon [1,2]$. A more general form of example 3 is discussed in detail later (Example 4).

3.3 Real Density Functions--the General Case

In the most general case the density function is real (see Appendix B). The realization in this case consists of a distributed RC in series with a distributed RL network. Sufficient conditions for this case are now developed.

Theorem 3.1.--Let $Z(s)$ have the limit $Z(\infty)$ (a constant) as $s \to \infty$. Let $Z(s) = Z_1(s) + Z(\infty)$. Let $f_1(x) = f_1^+(x) + f_1^-(x)$ where $f_1^+(x)$ and $f_1^-(x)$ are the non-negative and non-positive components of $f(x)$, respectively. Let $Z(s) = \int_0^\infty \frac{f_1^-(x)}{x} dx + R_1$. $Z(s)$ has a distributed network realization if,

(1) $Z_1(s)$ satisfies the Cauchy Integral conditions

(2) $\frac{f_1^+(x)}{x}$ and $\frac{f_1^-(x)}{x}$ are integrable

(3) $0 \leq R_1 < \infty$

Proof: $Z_1(s)$ has a Cauchy integral representation.

$$Z_1(s) = \int_0^\infty \frac{f_1(x)}{s+x}\, dx$$

$$= \int_0^\infty \left(\frac{f_1^+(x)}{s+x} + \frac{f_1^-(x)}{s+x} \right) dx$$

This integral may be split into two integrals since $\frac{f_1^+(x)}{x}$ and $\frac{f_1^-(x)}{x}$ are integrable

$$Z_1(s) = \int_0^\infty \frac{f_1^+(x)}{s+x}\, dx + \int_0^\infty \frac{f_1^-(x)}{s+x}\, dx.$$

$$Z(s) = Z_1(s) + Z(\infty)$$

$$= \int_0^\infty \frac{f_1^+(x)}{s+x}\, dx + \int_0^\infty \frac{f_1^-(x)}{s+x}\, dx + \int_0^\infty \frac{f_1^-(x)}{-x}\, dx + R_1$$

Again by condition 2 the last two integrals may be combined.

$$Z(s) = \int_0^\infty \frac{f_1^+(x)}{s+x}\, dx + \int_0^\infty \frac{s.f_1^-(x)/-x}{s+x}\, dx + R_1 \quad (3.7)$$

The first integral is realized by an RC Foster-type distributed network; the second by an RL network and the third by a lumped resistor. This is depicted in figure 7.

The Riemann approximation of equation 3.7, which leads to a lumped network, is

$$Z(s) = \sum_{i=1}^{N} \frac{f_1^+(x_i)\Delta x_i}{s+x_i} + \sum_{j=1}^{N} \left(\frac{f_1^-(x_j)\Delta x_j}{s+x_j} + \frac{f_1^-(x_j)\Delta x_j}{-x_j} \right) + R_1 \quad (3.8)$$

The residues at the poles $-x_i$ and $-x_j$ are $f_1^+(x_i)\Delta x_i$ and $f_1^-(x_j)\Delta x_j$, respectively. In the distributed case of equation 3.7 $f_1^+(x)dx$ and $f_1^-(x)dx$ are seen to play a role similar to residues in the lumped case. They could be

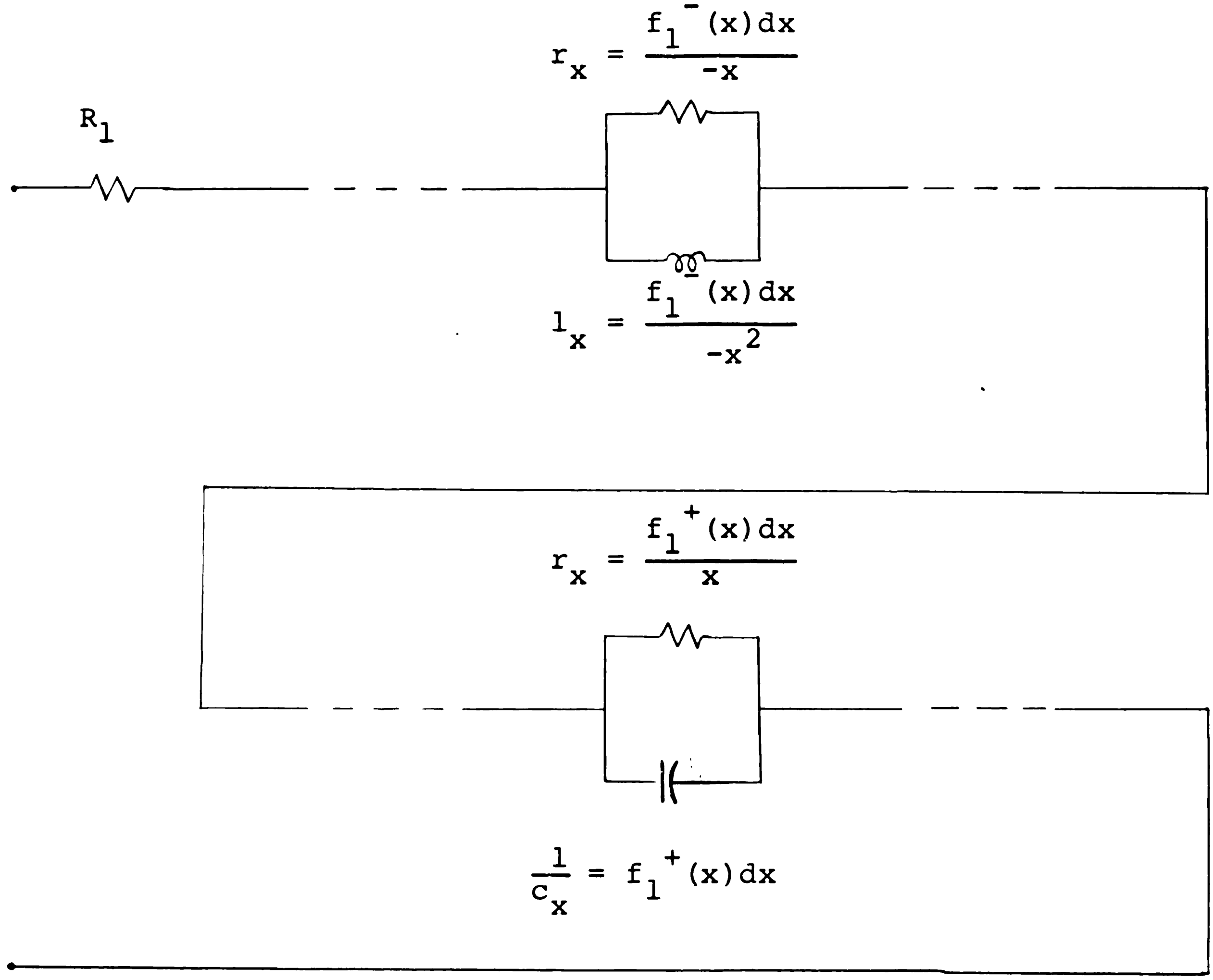


Figure 7. Network for Real Density Function--The General Case (Eq. 3.7)

called "differential residues" defined on the line of discontinuity.

Example 4.

$$Z(s) = \log \left(K \frac{\prod_{i=1}^{n} s+a_i}{\prod_{i=1}^{n} s+b_i} \right)$$

$a_i < a_{i+1}$, $b_i < b_{i+1}$ a_i and b_i are non-negative.

Z(s) may be written as

$$Z(s) = \sum_{i=1}^{n} \left(\log \frac{s+a'_i}{s+b'_i} + \log \frac{s+a''_i}{s+b''_i} + \log \frac{b_i''}{a_i''} \right) + \log K_1 \qquad (3.9)$$

It is assumed that $\log K_1 > 0$.

$$a_i = a_i' \text{ and } b_i = b_i' \text{ if } a_i \geq b_i$$

$$a_i = a_i'' \text{ and } b_i = b_i'' \text{ if } a_i < b_i$$

The density function is evaluated for a typical term in equation 3.9

$$z(s) = \log \left(\frac{s+a}{s+b}\right)$$

$$z^+(\sigma) = \lim_{\omega \to 0} \left(\log \frac{\sigma+j\omega+a}{\sigma+j\omega+b}\right)$$

$$z^-(\sigma) = \lim_{\omega \to 0} \left(\log \frac{\sigma-j\omega+a}{\sigma-j\omega+b}\right)$$

$$z^-(\sigma) - z^+(\sigma) = \lim_{\omega \to 0} \left(\log \frac{[(\sigma+a)(\sigma+b)-j\omega][(\sigma+a)(\sigma+b)-j\omega]}{[(\sigma+b)^2+\omega^2]\ [(\sigma+a)^2+\omega^2]} \right)$$

$$= \lim_{\omega \to 0} \left(\log \frac{[(\sigma+a)(\sigma+b)-j\omega][(\sigma+a)(\sigma+b)-j\omega]}{(\sigma+a)^2\ (\sigma+b)^2} \right)$$

$$z^{-}(\sigma) - z^{+}(\sigma) = \begin{cases} 0 & \text{when } \sigma\varepsilon[0,1) \\ -2\pi i & \text{when } \sigma\varepsilon[1,2] \\ 0 & \text{when } \sigma\varepsilon(2,\infty] \end{cases}$$

From this it is seen that the density function for z(s) is -1 when $\sigma\varepsilon[a,b]$ and is zero elsewhere.

Returning to equation 3.9, the density function for Z(s) is seen to be

$$f(x) = \begin{cases} +1 & x\varepsilon[a_i', b_i'] \\ -1 & x\varepsilon[a_i'', b_i''] \\ 0 & \text{elsewhere.} \end{cases}$$

It can be shown that the Cauchy Integral conditions are satisfied. Z(s) may thus be written as

$$Z(s) = \sum_{i=1}^{N} \left(\int_{a_i'}^{b_1'} \frac{1}{s+x}\,dx + \int_{a_i''}^{b_1''} \frac{-1}{s+x}\,dx + \int_{a_i''}^{b_1''} \frac{1}{x}\,dx \right) + \log K_1$$

$$Z(s) = \sum_{i=1}^{N} \left(\int_{a_i'}^{b_1'} \frac{1}{s+x}\,dx + \int_{a_i''}^{b_1''} \frac{s/x}{s+x}\,dx \right) + \log K_1$$

This is of the form of equation 3.7 and is realized using an RL, RC distributed network in series with the resistor $\log K_1$. (Figure 7.) $f_1^{+}(x)$ is +1 for $x\varepsilon\ [a_i', b_i']$ and $f_1^{-}(x)$ is -1 for $x\varepsilon\ [a_i'', b_i'']$.

3.4 Outline of Synthesis Procedure

In this section it is assumed that the only singularities of the impedance function Z(s) are line discontinuities. If Z(s) does have poles they are removed as a rational function and the latter is realized by conventional lumped synthesis techniques (or possibly, as in Chapter 4).

In the preceding sections of this chapter synthesis procedures were developed for a driving point impedance

$$Z(s) = Z_1(s) + Z_2(s) \qquad (3.10)$$

where $Z_2(s)$ had the form of equation 3.4 (section 2a) and its density function was non-positive. $Z_2(s)$ was not required to be regular at infinity. $Z_1(s)$ in the general case had a real density function and was expressible as in equation 3.7. $Z_1(s)$ was required to be regular at infinity (section 3). Impedance functions with non-negative density functions (section 1) and non-positive density functions (section 2b) are special cases of the function $Z_1(s)$. In the following, the sequence of steps required for the synthesis of $Z_1(s)$ and $Z_2(s)$ are outlined.

Given a function Z(s) which is not regular at infinity the first step is to decompose it in the form of equation 3.10. If $Z(s) \to ks^{\alpha}$ $(0<\alpha<1)$ as $s \to \infty$, then $Z_2(s)$ is identified as ks^{α} and is subtracted from Z(s) to give $Z_1(s)$. But in general this is a difficult problem and it has not been developed here. This decomposition is obviously not needed if Z(s) is regular at infinity.

The steps for realizing $Z_1(s)$ are:

(A) Find $Z_1(\infty)$. This must be a non-negative finite quantity.

$$Z_1(s) = Z_1'(s) + Z_1(\infty)$$

$Z_1'(s)$ tends to zero as $s \to \infty$.

(B) Find L (the line of discontinuity). If the multi-valued form of function $Z_1'(s)$ has branch points as its singularities, these must lie on the negative real axis. The p.r. branch of $Z_1'(s)$ is determined. This has a line of discontinuity joining the branch points.

(C) Find the density function $f_1'(x)$. The density function may be evaluated from the boundary condition,

$$\frac{1}{2\pi j} \left(Z_1'^{-}(x) - Z_1'^{+}(x)\right) \quad \text{for } x \text{ on } L.$$

$Z_1'^{-}(x)$ and $Z_1'^{+}(x)$ will always be defined if L is a branch cut. Alternately, $f_1'(x)$ may be calculated from the Cauchy Inversion integral

$$f_1'(-x) = \int_L \frac{Z_1'(s)}{x-s} \, dx$$

(D) Check $f_1'(x)$ for the H^* condition. This involves checking $f(x)$ for Hölder continuity on closed arcs of L not containing the ends, and checking the end point condition $f_1'(t) = \frac{f_1'^{*}(t)}{(t-c)^{\alpha}}$ $(0<\alpha<1)$

(E) Check the end point condition for $Z_1'(s)$.

$$|Z_1'(s)| < \frac{\text{const.}}{|s-c|^\alpha} \qquad (0 \leq \alpha < 1).$$

Conditions A to E constitute the Cauchy Integral conditions.

(F) Find $f_1'^{+}(x)$ and $f_1'^{-}(x)$. This may involve finding the roots of the function $f_1'(x)$. If $f_1'(x)$ is of constant sign this step is not needed.

(G) Check the integrability of $\frac{f_1'^{+}(x)}{x}$ and $\frac{f_1'^{-}(x)}{x}$ on L. This step is omitted if $f_1'(x)$ is of constant sign.

(H) Check that $Z_1(\infty) \geq \int_0^\infty \frac{f_1'^{-}(x)}{-x}\, dx$

Evaluate $R_1 = Z_1(\infty) - \int_0^\infty \frac{f_1'^{-}(x)dx}{-x}$

Next consider the conditions on $Z_2(s)$. The function $\frac{Z_2(s)}{s}$ must satisfy conditions A to E as above. Also the density function must be non-negative on L.

3.5 Positive-Real Transformations

Consider the function

$$Z(s) = F_1(F_2(s))$$

where $F_2(s)$ is a lumped RC realizable rational function and where $F_1(z)$ has a Cauchy Integral representation. The line of discontinuity L for $F_1(z)$ lies on the negative real axis and the density function is non-negative. $F_1(z)$ is then realizable using an RC distributed network. There is no loss in generality in assuming L = [-a, -b].

Z(s) may be written as

$$Z(s) = \int_a^b \frac{f_1(x)}{F_2(s) + x}\, dx \qquad (3.11)$$

From this equation it is easily seen that Z(s) has the RC distributed realization of figure 8.

It will now be shown that any RC network with such a Foster configuration (fig. 8) can also be represented in the form of the basic Cauchy Integral

$$Z(s) = \int_{L'} \frac{g(x)}{s+x}\, dx$$

This fact suggests the conjecture that "the Cauchy Integral is a basic description of all RC networks."

Theorem 3.2.--If $F_1(z)$ has a Cauchy Integral representation with a non-negative integrable density function and line of discontinuity L = [-a, -b], and if $F_2(s)$ is an RC realizable rational admittance function, then Z(s) = $F_1(F_2(s)$ has the Cauchy Integral representation.

$$Z(s) = \int_{L'} \frac{g(x)}{s+x}\, dx$$

within an additive constant. The line of discontinuity is on the negative real axis of the s-plane and g(x) is non-negative.

The following Lemma is required.

Lemma 3.2. If $\frac{q(s)}{p(s)}$ is a lumped RC realizable admittance function and if the roots of

$$\frac{q(s)}{p(s)} + x = 0 \qquad (x \text{ non-negative})$$

are $-\alpha_i(x)$, then

$$\frac{d\alpha_i(x)}{dx} > 0$$

Proof (of Lemma 3.2):

$\frac{q(s)}{p(s)}$ is an RC realizable admittance. Its poles and zeros alternate on the negative real axis. The lowest critical frequency is a zero and the largest critical frequency is a pole.

$\frac{q(s)}{p(s)} + x$ $(x \geq 0)$ is RC realizable. Its roots lie on the negative real axis. They are also the roots of

$$\frac{q(\sigma)}{p(\sigma)} = - x \qquad (s = \sigma + j\omega)$$

The plot of $\frac{q(\sigma)}{p(\sigma)}$ versus σ is shown in figure 9.

It is evident from figure 9 that as x increases $\alpha_i(x)$ increase in magnitude

i.e., $\frac{d\alpha_i(x)}{dx} > 0$.

Theorem 3.2 is now proved.

Proof (of Theorem 3.2):

Let $F_2(s) = \frac{q(s)}{p(s)}$ where $\frac{q(s)}{p(s)}$ is an RC realizable

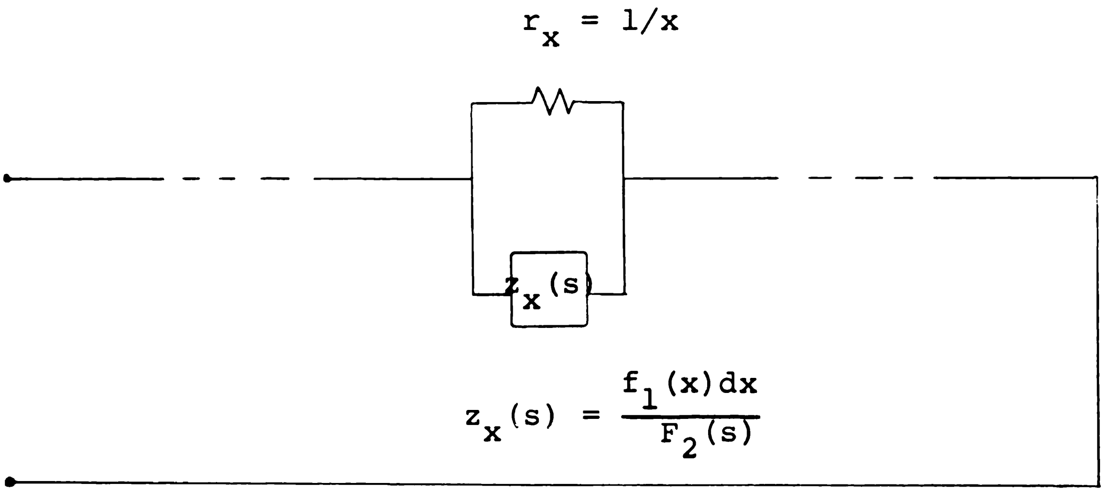


Figure 8. Distributed Network for Equation 3.11

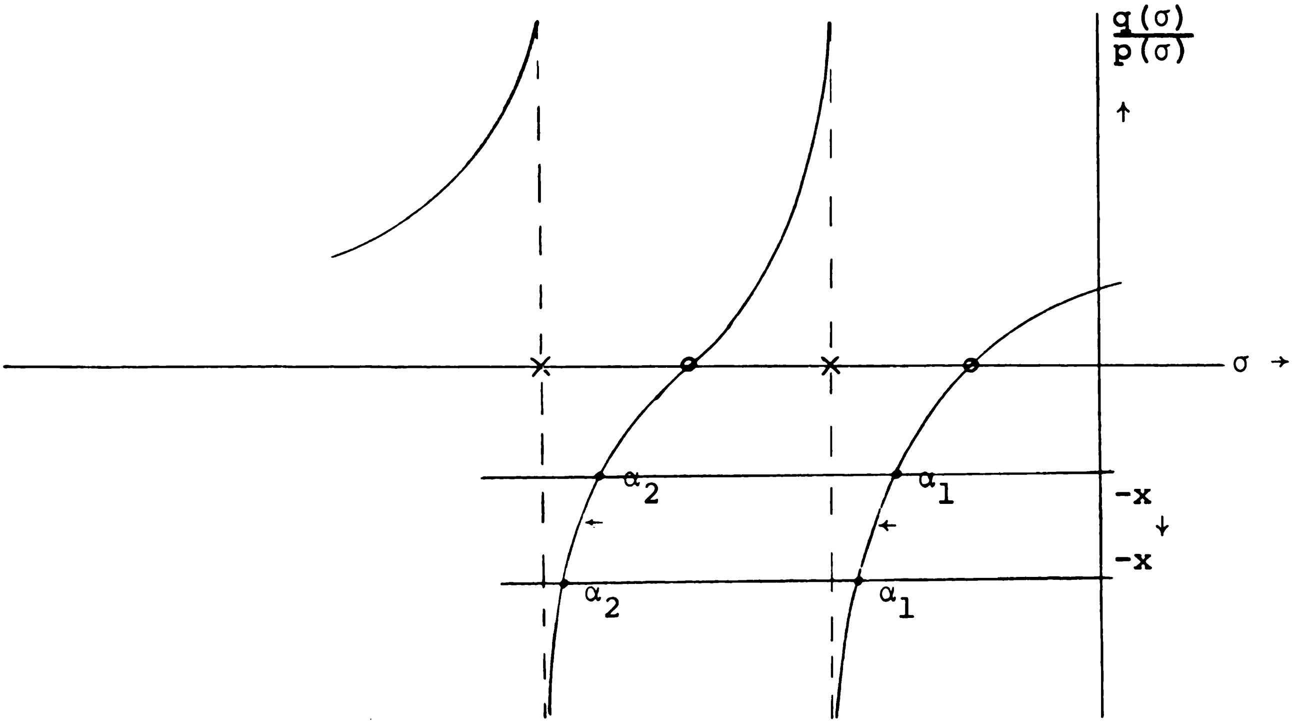


Figure 9. Roots of $\frac{q(s)}{p(s)} + x = 0$

rational impedance function. Since $F_1(z)$ has a Cauchy Integral representation

$$Z(s) = F_1(F_2(s)) = \int_a^b \frac{f_1(x)}{\frac{q(s)}{p(s)}+x}\, dx \tag{3.12}$$

Consider $\dfrac{1}{\frac{q(s)}{p(s)} + x}$ $(x \geq 0)$. This function is also RC realizable. It may be written as

$$\frac{1}{\frac{q(s)}{p(s)} + x} = r_\circ(x) + \sum_{i=1}^{n} \frac{r_i(x)}{s+\alpha_i(x)}$$

where $-\alpha_i(x)$ are the roots of $\frac{q(s)}{p(s)} + x = 0$.

The residues r_i are all non-negative. The poles $-\alpha_i(x)$ are non-positive.

Then,

$$Z(s) = \int_a^b \left(\sum_{i=1}^{n} \frac{r_i(x)}{s+\alpha_i(x)} + r_\circ(x) \right) f_1(x)\, dx$$

$$= \sum_{i=1}^{n} \int_a^b \frac{r_i(x)\; f_1(x)}{s+\alpha_i(x)} + \int_a^b r_\circ(x)\; f_1(x)\; dx. \tag{3.13}$$

The decomposition is valid since $f_1(x)$ is integrable.

Let $\alpha_i(x) = y_i$; $dy_i = \dfrac{d\alpha_i(x)}{dx}\, dx$

Let $\dfrac{d\alpha_i(x)}{dx} = c_i(x)$; $dy_i = c_i(x)\, dx$

By the above lemma $c_i(x)$ is positive, hence

$$dx = \frac{dy_i}{c_i(x)}$$

Substituting in equation 3.13,

$$Z(s) = \sum_{i=1}^{n} \int_{\alpha_i(a)}^{\alpha_i(b)} \left(\frac{r_i(x)\ f_1(x)}{s + y_i} \cdot \frac{dy_i}{c_i(x)} \right) + \int_a^b r_o(x) f_1(x) dx$$

$$\text{Let, } \frac{r_i(x)\ f_1(x)}{c_i(x)} = g_i(y_i)$$

Since $r_i(x)$, $f_1(x)$, $c_i(x)$ are all non-negative

$$g_i(y_i) \geq 0$$

Then,

$$Z(s) = \sum_{i=1}^{n} \int_{\alpha_i(a)}^{\alpha_i(b)} \frac{g_1(x)}{s + x} dx + \int_a^b r_o(x) f_1(x) dx$$

$$Z(s) = \int_{L'} \frac{g_1(x)}{s + x} dx + \text{constant.}$$

Here $g_i(x)$ is non-negative and each section of the line discontinuity $[-\alpha_i(a), -\alpha_i(b)]$ is on the negative real axis.

In the above theorem suppose $F_1(z)$ and $F_2(z)$ both have Cauchy Integral representations and are RC distributed realizable. Then the conjecture stated above would require that $F_1(F_2(s))$, which is RC realizable, also have the basic Cauchy Integral representation. This may very well be true as is indicated by the following.

$$Z(s) = \int_a^b \frac{f_1(x)}{\left(\int_c^d \frac{f_2(y)}{s + y} dy\right) + x} dx \qquad (3.14)$$

Using a Riemann-sum approximation for $F_2(s)$,

$$Z(s) \cong \int_a^b \frac{f_1(x)}{\sum_{i=1}^{N}\left(\frac{f_2(y_i)}{s + y_i} \; dy_i\right) + x} \, dx$$

$F_2(s)$ is now in a form which is lumped RC realizable. Hence by the above theorem,

$$Z(s) \cong \sum_{i=1}^{N} \int_{c_i}^{d_i} \frac{g_i(x)}{s + x} \, dx$$

As N tends to infinity c_i approaches d_i, thus

$$Z(s) \cong \sum_{i=1}^{N} \frac{g_i(x_i)(d_i - c_i)}{s + x_i}$$

It seems reasonable that in the limit this approximation would have a Cauchy integral representation with line discontinuity [-c, -d].

$$Z(s) \cong \int_c^d \frac{k_i(x)}{s + x} \, dx$$

Another fact supporting the conjecture that the Cauchy Integral is a basic description of all RC networks is that a function with a simple pole at p (with residue r) also has such a representation. The density function is $r_\circ \delta(x-p)$, where $\delta(x-p)$ is a delta function at p.

$$\frac{r}{s-p} = \int_0^\infty \frac{r_\circ \delta(x-p)}{s + x} \, dx$$

Example 5

Consider $Z(s) = \sqrt{\dfrac{s(s+2)}{(s+1)(s+3)}}$

Here $F_1(z)$ is distributed realizable and $F_2(s)$ is lumped realizable, let $z = \dfrac{s(s+2)}{(s+1)(s+3)}$ be the p.r. transformation. Its reciprocal is lumped RC realizable.

$$Z(s) = \int_0^\infty \frac{1}{\pi\sqrt{x}\ (z+x)}\, dx$$

The network is shown in figure 10.

Example 6

$$Z(s) = \log \left(\frac{\sqrt{s} + a}{\sqrt{s} + b}\right) \qquad a>b$$

Here $F_1(s)$ and $F_2(s)$ are distributed realizable.

let $z = \sqrt{s}$

$$Z(s) = \int_a^b \frac{1}{z + x}\, dx = \int_a^b \frac{1}{\sqrt{s} + x}\, dx$$

The network is shown in figure 11. The impedance $z_i = \dfrac{\Delta x_i}{\sqrt{s}}$ is realized as in example 1. x varies from a to b.

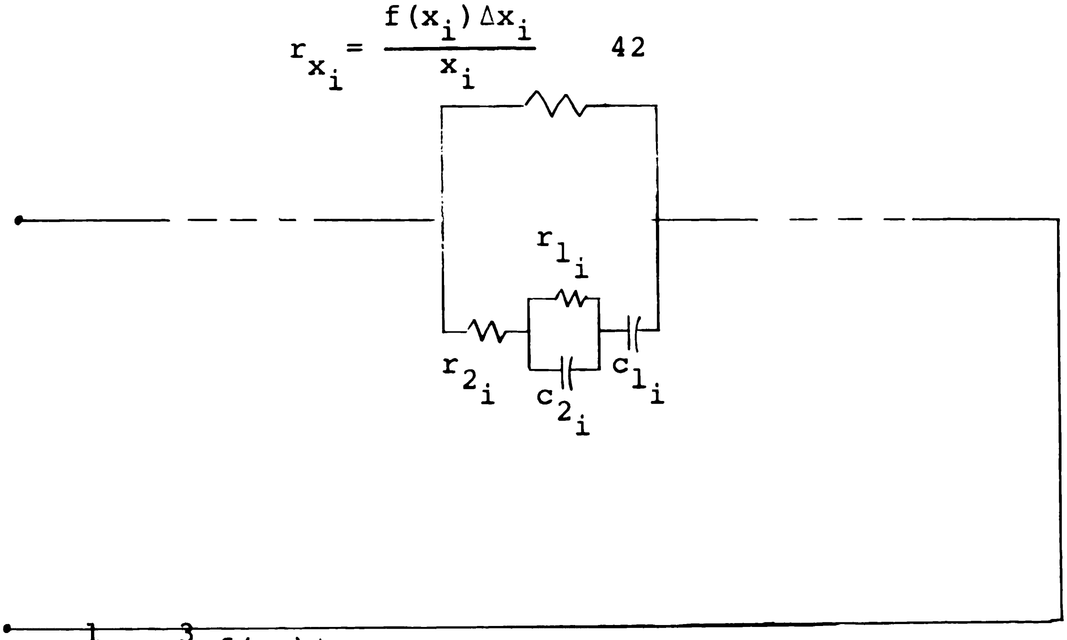
Example 7

$$Z(s) = \frac{(\sqrt{s} + 1)(\sqrt{s} + 3)}{\sqrt{s}\ (\sqrt{s} + 2)}$$

Here $F_1(s)$ is lumped realizable and $F_2(Z)$ is distributed realizable.

let $\sqrt{s} = z$

then, $Z(s) = \dfrac{(z + 1)(z + 3)}{z\ (z + 2)}$



$$\frac{1}{c_{1_i}} = \frac{3}{2}\, f(x_i)\Delta x_i$$

$$\frac{1}{c_{2_i}} = 2\, f(x_i)\Delta x_i$$

$$r_{1_i} = \frac{f(x_i)\Delta x_i}{4}$$

$$r_{2_i} = f(x_i)\Delta x_i$$

Figure 10. Network for Example 5

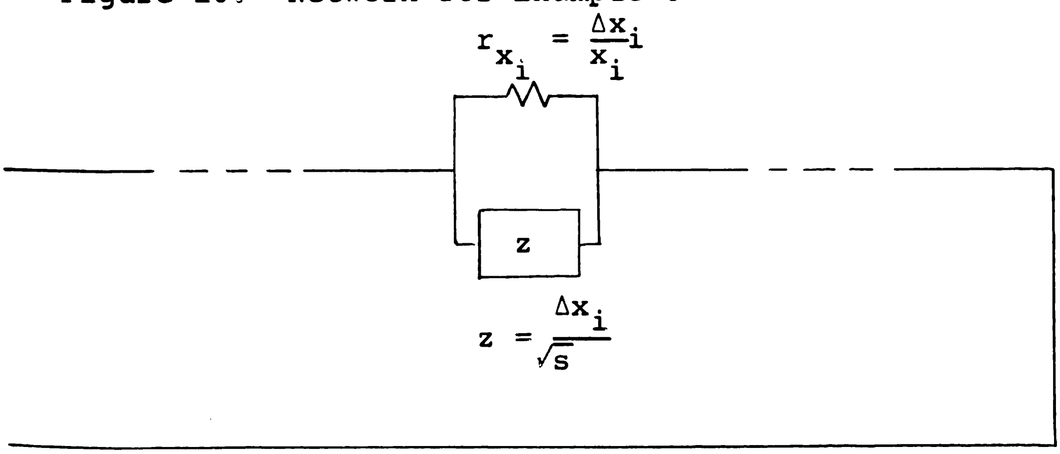


Figure 11. Network for Example 6

The network is given in figure 12.

3.6 Extensions

3.6(a) Admittances. The discussion so far has been restricted to impedance functions, which yield type I Foster networks. The sufficient conditions for realizing admittance functions are entirely analogous to the ones developed in sections 1-3. The Cauchy Integral is realized by type II Foster networks. For instance, in the case of a non-negative density function the admittance

$$Y(s) = \int_0^\infty \frac{f(x)}{s+x} \, dx$$

is realized by the network of figure 13.

It was mentioned in the last section that the decomposition of an impedance function which is not regular at infinity (equation 3.10) can be difficult. A possible way of avoiding such a decomposition would be to realize the reciprocal of the impedance function as an admittance function. The admittance function will be regular at infinity and the decomposition is not needed.

3.5(b) LC Distributed Networks. One way of obtaining an LC distributed realization is by considering the transformation

$$Z_1(s) = \frac{1}{\sqrt{s}} \cdot Z(\sqrt{s})$$

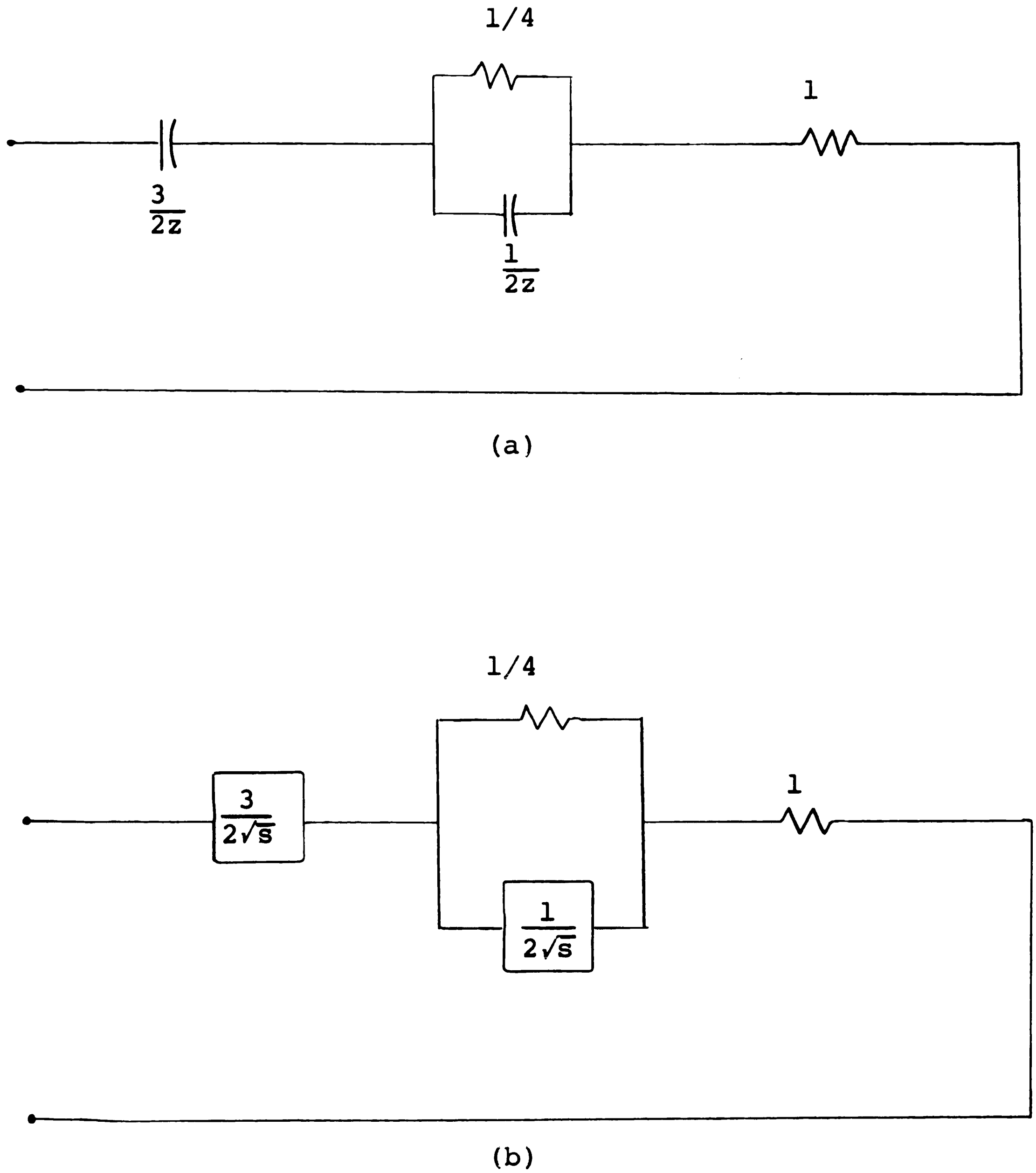


Figure 12. Network for Example 7
(a) in the z Plane, Z(z)
(b) in the s Plane, Z(s)

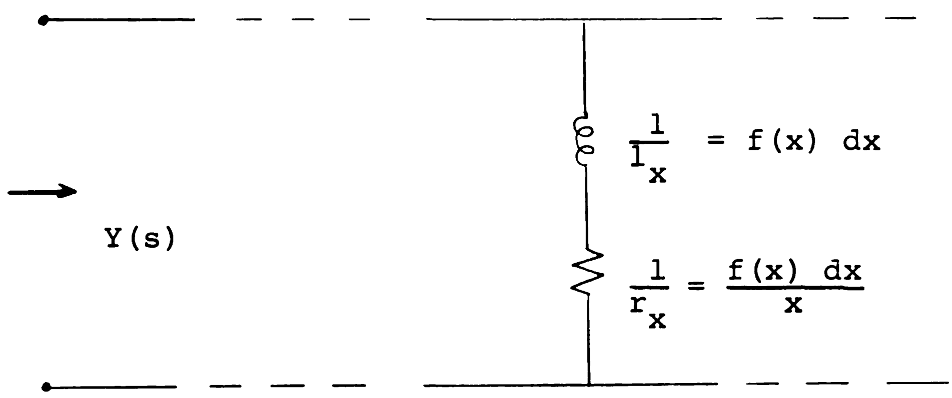


Figure 13. Network for an Admittance with a Non-Negative Density Function

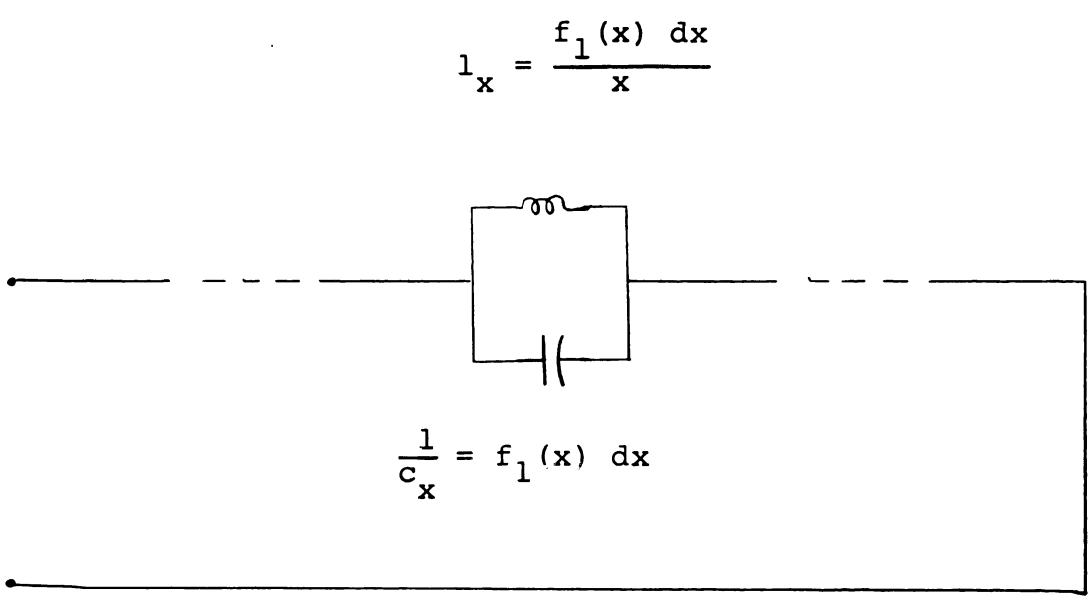


Figure 14. LC Distributed Network for Eq. 3.15

If $Z_1(s)$ satisfies the Cauchy Integral conditions and $f_1(x)$ is non-negative, then

$$Z_1(s) = \int_0^\infty \frac{f_1(x)\, dx}{s + x}$$

$$Z(s) = \int_0^\infty \frac{s\, f_1(x)\, dx}{s^2 + x} \qquad (3.15)$$

This representation of Z(s) has singularities along conjugate lines on the imaginary axis. The LC distributed realization for Z(s) is shown in figure 14. An example of such a function is $Z(s) = \overset{+}{\sqrt{s^2+1}}$ ($\sqrt{s^2+1}$ has branch points at +j and =j). Here $Z_1(s) = \overset{+}{\sqrt{1+1/s}}$ and $f_1(x)$ is $\overset{+}{\sqrt{1-1/x}}$, $x \varepsilon\ [1,\infty]$.

CHAPTER IV

FOSTER LUMPED-INFINITE NETWORKS

In Chapter II the functions considered were characterized by line discontinuities. The next class considered consists of functions which have a countable number of simple poles on the negative real axis. In general, the driving point impedance Z(s) may be the sum of two impedances $Z_1(s)$ and $Z_2(s)$, where $Z_1(s)$ is realizable as a distributed network (as in the last chapter) and $Z_2(s)$ has a countable number of poles.

If $Z_2(s)$ has a finite number of poles, classical finite-lumped network synthesis techniques may be used to realize it. This chapter considers the realization of functions which have countably infinite numbers of poles. Given a function which has a sequence of poles p_n with corresponding residues r_n, the problem is to find a representation for this function which leads to a network interpretation. A simple choice might have been the sum of the principal parts i.e., $F(z) = \sum_{n=1}^{\infty} \frac{r_n}{z - p_n}$. But this function does not always converge. (Consider for instance $\sum_{n=1}^{\infty} \frac{1}{z-n}$. This function does not converge anywhere in the z plane.) The answer to this problem is found in a corollary of Mittag-Leffler's theorem (see Appendix C). This theorem gives representations which lead to Foster-type networks consisting of infinite numbers of lumped elements.

The countably infinite lumped realization of some irrational functions have been treated in references 17-19. Halijak[17] uses a Newton and Halley rational form of $\sqrt{s}$ which leads to an RC infinite lattice structure. Stiegletz[18] approximates $\frac{1}{\sqrt{s}}$ as a rational function whose poles and zeros alternate on the negative real axis yielding a Foster-type RC network. Dutta-Roy[19] shows that a continued fraction expansion of s^{α}, $-1 \leq \alpha \leq 1$, gives an infinite RC ladder structure which is the approximant to an RC distributed line. The function s^{α} occurs in the design of constant-argument immittances which have many applications. The image admittance of a constant K filter with a cut-off frequency ω_o is $\sqrt{1 + \frac{s^2}{\omega_o^2}}$. This can be realized using an infinite structure.[15] s^{α} and $\sqrt{1 + s^2}$ belong to the class of functions that were treated in Chapter 3.

In section 1 sufficient conditions are developed for an RC infinite-network realization. Sections 2 and 3 consider the LC and RL cases respectively. Two examples are given in section 4.

4.1 RC Infinite-Lumped Networks

Theorem 4.1.--A driving point impedance function $Z(s)$ having a sequence of distinct, simple poles $\{p_n\}$ ($\lim_{n\to\infty} p_n = \infty$) and a corresponding sequence of residues $\{r_n\}$ can be realized by a network consiting of a countably infinite number of lumped resistors and capacitors if,

1. all the poles lie on the negative real axis
2. the residues are all real and non-negative
3. $\sum_{n=1}^{\infty} \left|\frac{r_n}{p_n}\right| < \infty$

Proof: Condition (3) is used to show that $Z(s)$ has a form similar to the one for k=0 in Mittag-Leffler's theorem. For $|s| \leq R$, $\frac{s}{p_n} \to 0$ uniformly as $n \to \infty$. So for $|s| \leq R$ and $n > N$, where N is some integer,

$$\left|\frac{r_n}{s-p_n}\right| = \left|\frac{r_n}{p_n(1-s/p_n)}\right| \leq 2\left|\frac{r_n}{p_n}\right|$$

$$\sum_{n=1}^{\infty} \left|\frac{r_n}{s-p_n}\right| \leq \sum_{n=1}^{\infty} \left|\frac{r_n}{p_n}\right| < \infty$$

Hence, $$Z(s) = \sum_{n=1}^{\infty} \frac{r_n}{s-p_n} \qquad (4.1)$$

converges uniformly and absolutely on compact sets not containing any of the poles. $\sum_{n=1}^{\infty} \frac{r_n}{s-p_n}$ has simple poles at p_n, $n = 1,2,\ldots\infty$, with the corresponding residues r_n.

The network of figure 15 is a Foster lumped-infinite RC realization for Z(s) represented as in eq. 4.1.

Since $|\frac{r_n}{p_n}| \to 0$ as $n \to \infty$, the impedance of sections $Z_n(s) = \frac{r_n}{s-p_n}$ decreases uniformly to zero as $n \to \infty$. So a finite network approximation may be attained by omitting the sections $Z_n(s)$ beyond some N.

A similar set of conditions holds for an admittance function Y(s). The RL network representation for an admittance

$$Y(s) = \sum_{n=1}^{\infty} \frac{r_n}{s-p_n} \tag{4.2}$$

is shown in figure 16.

4.2 LC Infinite Lumped Networks

Theorem 4.2.--A driving point impedance function Z(s) having two sequences of distinct, simple poles $\{+jp_n\}$ and $\{-jp_n\}$ $(\lim_{n\to\infty} p_n = \infty)$, the corresponding sequences of residues for both sequences being r_n, can be realized by a network consisting of a countably infinite number of lumped inductors and capacitors if,

1. all the poles are on the imaginary axis i.e., p_n is real
2. the residues are all real and non-negative
3. $\sum_{n=1}^{\infty} |\frac{r_n}{p_n^2}| < \infty$

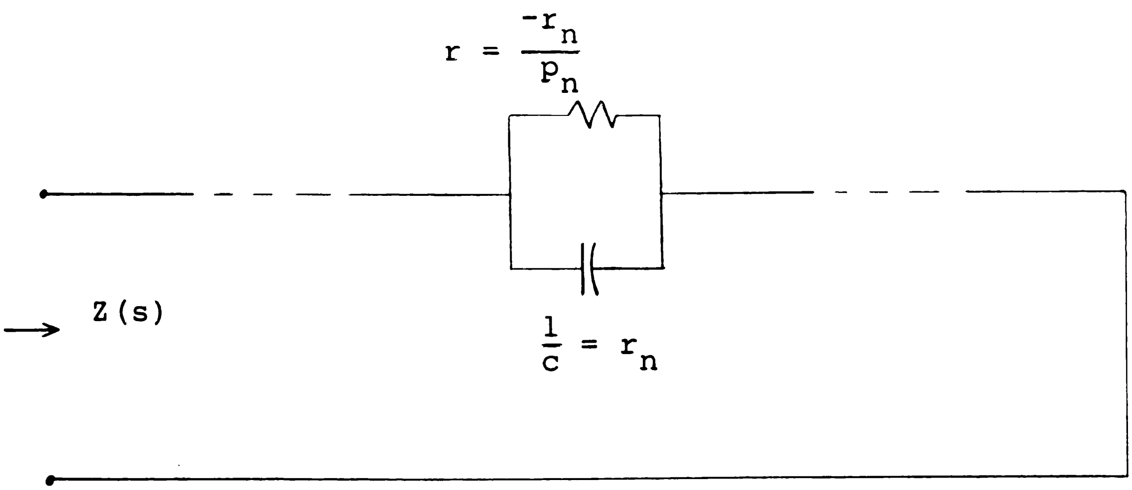


Figure 15. RC Infinite-Lumped Network for Eq. 4.1

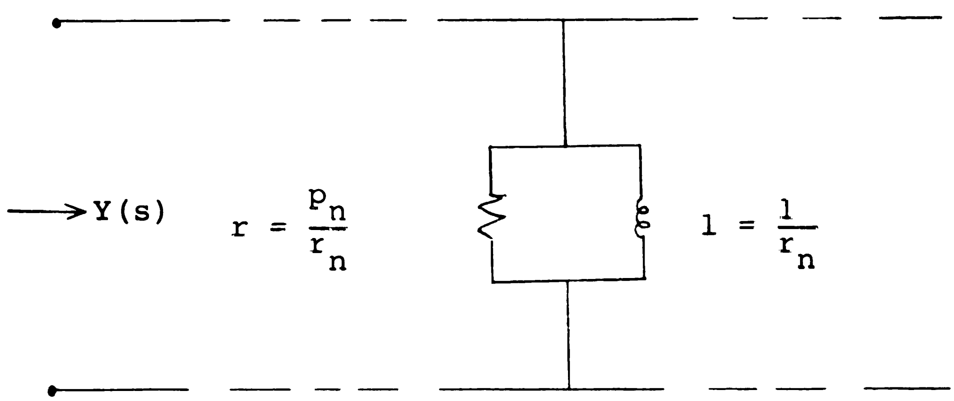


Figure 16. RL Infinite-Lumped Network for Eq. 4.2

Proof: Z(s) will be shown to have a form similar to that of k=1 in the Mittag-Leffler theorem

$$\frac{r_n}{s-jp_n} + \frac{r_n}{jp_n} = \frac{r_n s}{p_n(s-jp_n)}$$

For $|s| \leq R$, $\frac{s}{jp_n} \to 0$ uniformly as $n \to \infty$. So, for $|s| \leq R$ and $n > N$, where N is some integer,

$$\left|\frac{r_n}{s-jp_n} + \frac{r_n}{jp_n}\right| = \left|\frac{r_n}{p_n^{\,2}(1-\frac{s}{jp_n})}\right| \leq \left|2\,\frac{r_n \cdot R}{p_n^{\,2}}\right|$$

$$\sum_{n=1}^{\infty} \left|\frac{r_n}{s-jp_n} + \frac{r_n}{jp_n}\right| \leq 2R \sum_{n=1}^{\infty} \left|\frac{r_n}{p_n^{\,2}}\right| < \infty$$

Hence, $Z_1(s) = \sum_{n=1}^{\infty} \left(\frac{r_n}{s-jp_n} + \frac{r_n}{jp_n}\right)$ will converge absolutely and uniformly on compact sets not containing any of the poles.

Similarly $Z_2(s) = \sum_{n=1}^{\infty} \left(\frac{r_n}{s+jp_n} - \frac{r_n}{jp_n}\right)$ will also converge.

So $Z(s) = Z_1(s) + Z_2(s)$

$$= \sum_{n=1}^{\infty} \left(\frac{r_n}{s-jp_n} + \frac{r_n}{jp_n} + \frac{r_n}{s+jp_n} - \frac{r_n}{jp_n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2r_n \cdot s}{s^2+p_n^{\,2}} \qquad (4.3)$$

converges.

This representation of $Z(s)$ (eq 4.3) has the required poles at $\pm jp_n$, $n = 1,2,\ldots,\infty$; and the corresponding residues r_n.

Figure 17 is an LC lumped infinite realization for $Z(s)$. The $\frac{1}{r_0}$ capacitor is present if $Z(s)$ has a pole at the origin with residue r_0.

Admittances are treated as in the last section.

4.3 RL Infinite Lumped Networks

Theorem 4.3.--A driving point impedance $Z(s)$ having a sequence of distinct, simple poles p_n $(\lim_{n\to\infty} p_n = \infty)$ and a corresponding sequence of residues r_n, can be realized by a network consisting of a countably infinite number of lumped resistors and inductors if,

1. all the poles lie on the negative real axis
2. the residues are all real and non-positive
3. $\sum_{n=1}^{\infty} \left|\frac{r_n}{p_n^2}\right| < \infty$

Proof: Proceeding as in Theorem 4.2 it is seen that $Z(s) =$

$$- \sum_{n=1}^{\infty} \left(\frac{r_n}{s-p_n} + \frac{r_n}{p_n}\right) \quad \text{converges.}$$

$$= -\sum_{n=1}^{\infty} \frac{r_n + (r_n/p_n)\cdot(s-p_n)}{s-p_n}$$

$$= -\sum_{n=1}^{\infty} \frac{r_n/p_n \cdot s}{s - p_n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{-p_n/r_n + p_n^2/r_n \cdot 1/s} \qquad (4.4)$$

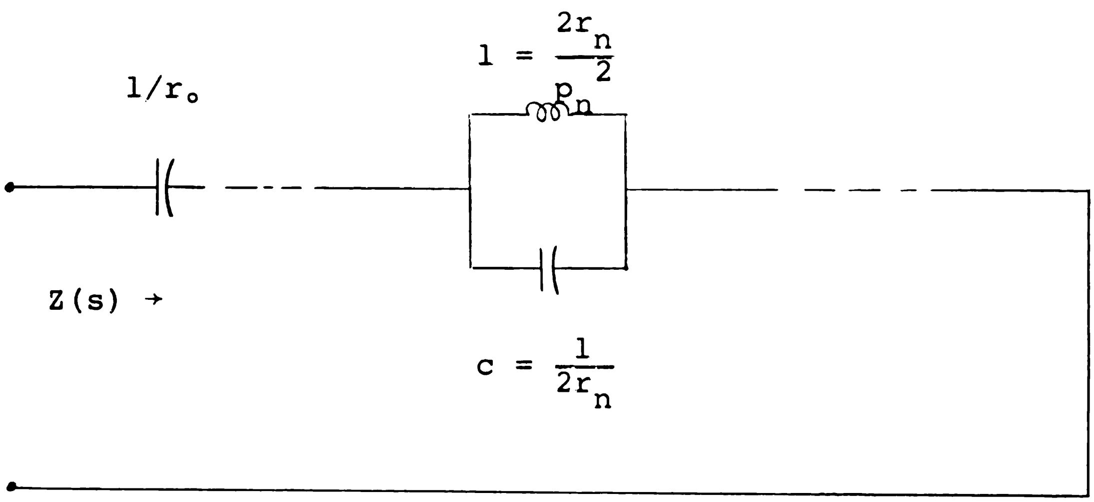


Figure 17. LC Infinite-Lumped Network for Eq. 4.3

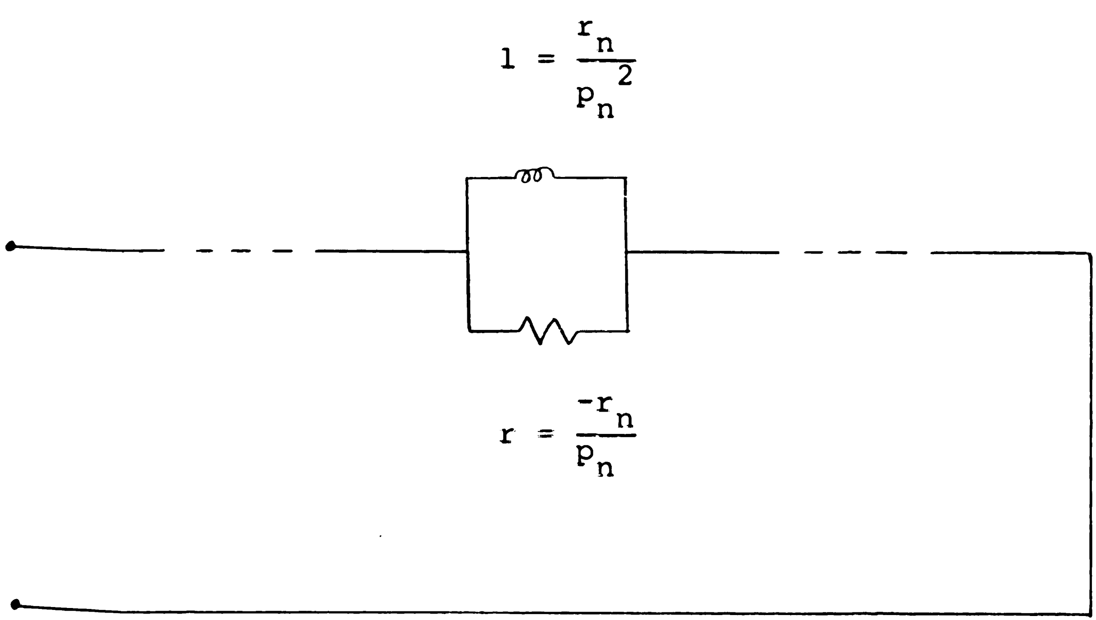


Figure 18. RL Infinite Lumped Network for Eq. 4.4

This representation of Z(s) has poles at p_n, n=1,2,...∞ and the corresponding residues r_n.

Figure 18 is an infinite RL realization for Z(s).

4.4 Examples

The following two examples yield LC infinite realizations.

Example 8: Z(s) = coth s

cot s has simple poles at s = $\pm jk\pi$ with residue 1 at each pole.

$$\sum_{k=1}^{\infty} |k\pi|^{-2} < \infty$$

cot s has a pole at the origin so by Mittag-Leffler's theorem

$$\cot s = \frac{1}{s} + \sum_{k=-\infty}^{+\infty} \left(\frac{1}{s-k\pi} + \frac{1}{k\pi}\right)$$

Hence coth s = -j cot (-js)

$$= \frac{1}{s} + \sum_{k=1}^{\infty} \left(\frac{1}{s+jk\pi} + \frac{1}{s-jk\pi}\right)$$

$$= \frac{1}{s} + \sum_{k=1}^{\infty} \frac{2s}{s^2+k^2\pi^2}$$

The realization is of the form of figure 17 with

$$r_o = 1 \; ; \; r_k = 1 \; ; \; p_k = k\pi$$

CHAPTER V

CONCLUSIONS

Foster-type network representations were found for a class of driving point immittances. The distributed networks of Chapter III have a finite lumped network representation, attained by considering the Riemann-sum approximation of the Cauchy Integral. The lumped-infinite networks of Chapter IV could also be approximated by using a finite number of sections, since the impedance of the sections converged to zero as $n \to \infty$.

Fabrication procedures for such networks have not been proposed in this work. It would be desirable to find a transformation leading to an equivalent Cauer RC infinite network, since this could be implemented by a single non-uniform RC distributed line. In the case of lumped networks whenever a Foster form exists a Cauer form does also exist. Both are canonical forms. This indicates that such a transformation should exist for lumped networks. The Riemann-sum approximation of the Cauchy Integral leads to a lumped network and the transformation postulated could be applied to it. In order to know how closely the transformed Cauer network represented the original integral, it would be necessary to estimate the approximations incurred in forming the Riemann sum and in applying the transformation. The former approximation could involve forming a sequence of Riemann-sums which converged to the integral. The convergence of the corresponding sequence of Cauer networks would need investigation.

The driving point immittance functions of passive, linear, time-invariant networks are characterized by their singularities. Functions with a countable number of poles and a line discontinuity were treated. An open question, the answer to which is possibly in the negative, is--can positive real driving point impedances which are RC, RL realizable have any other singularities? This question is illustrated in figure 19 as question 1. The shaded regions indicate the class of functions for which realizations have been developed. All p.r. functions with a finite number of poles and line discontinuities. Necessary and sufficient conditions would be needed for all such functions to be realizable.

Extensions of this work to RLC realizable functions will involve conjugate lines of discontinuity and countable numbers of conjugate poles in the negative half plane.

POSITIVE REAL IMMITTANCE FUNCTIONS
RC AND RL REALIZABLE

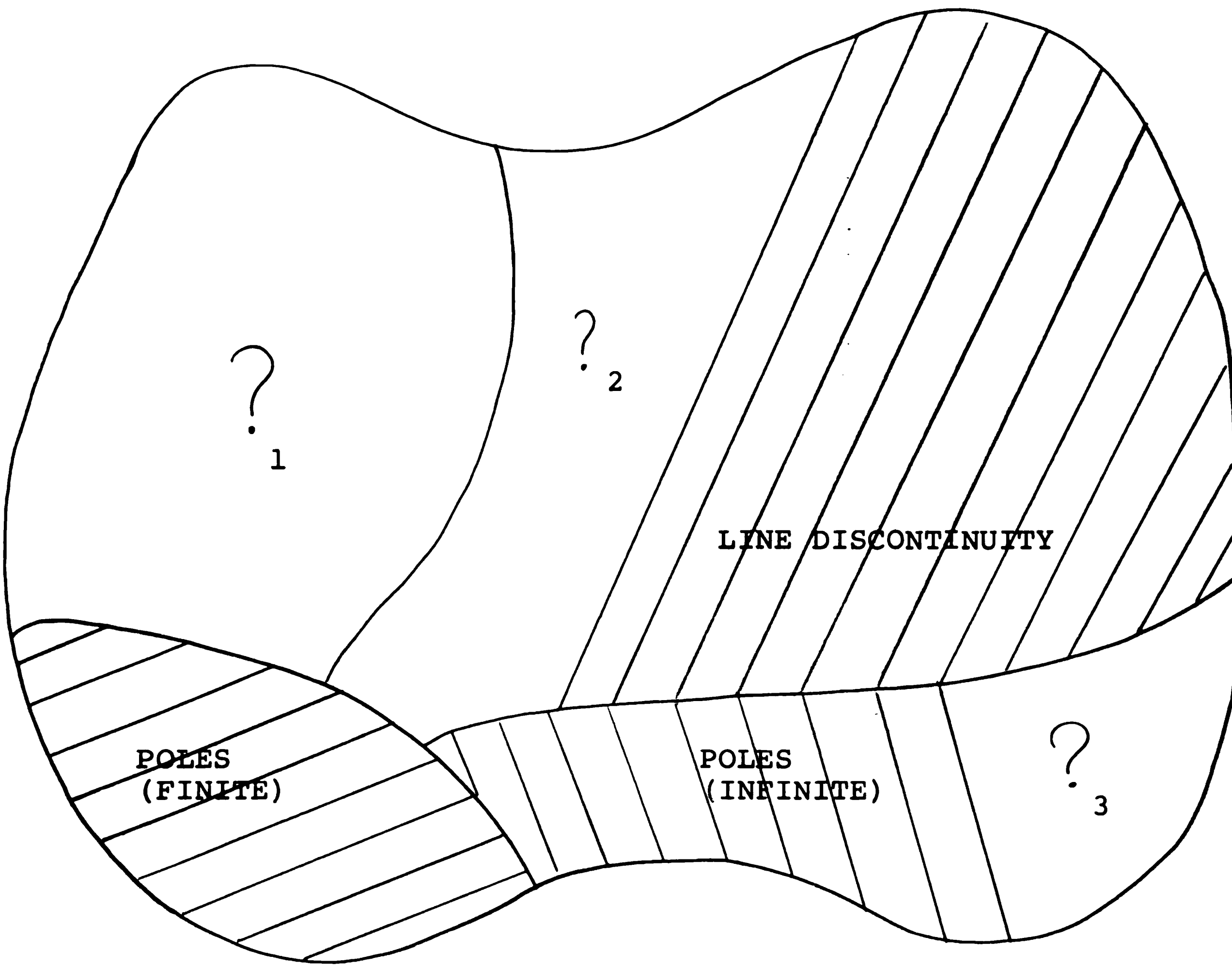


Figure 19. RC and RL Realizable Immittance Functions

APPENDICES

APPENDIX A

State Description for RC, RL Realizable Driving Point Impedances

A state description for the driving point impedance of a linear, time-invariant, passive network is given by equations 2.1 and 2.2. For RC and RL networks the singularities of Z(s) are on the negative real axis. In the following it is shown that equations 2.3 and 2.4 are a state description for RC, RL realizable impedance functions.

The contour drawn in figure 20 excludes the negative real axis. The path Br is broken up into three parts B_1, B_2 and B_3.

$$\frac{1}{2\pi j}\int_{Br} Z(s)\Psi(s,t)ds = \frac{1}{2\pi j}\left(\int_{B_1} Z(s)\Psi(s,t)ds+\int_{B_2} Z(s)\Psi(s,t)ds+\int_{B_3} Z(s)\Psi(s,t)ds\right) \quad \text{(A1.1)}$$

By Cauchy's Theorem (Ref. 12, p. 163)

$$\int_{B_1} Z(s)\Psi(s,t)ds = -\int_{C_1} Z(s)\Psi(s,t)ds-\int_{C_3+} Z(s)\Psi(s,t)ds-\int_{C_4+} Z(s)\Psi(s,t)ds \quad \text{(A1.2)}$$

The solution to equation 2.1 is

$$\Psi(s,t) = \int_{-\infty}^{t} e^{s(t-T)}\, i(T)dT$$

Assuming $i(T) = 0$ for $t < -T$ and $i(T) \leq M$ for all $t \geq T \geq -T$

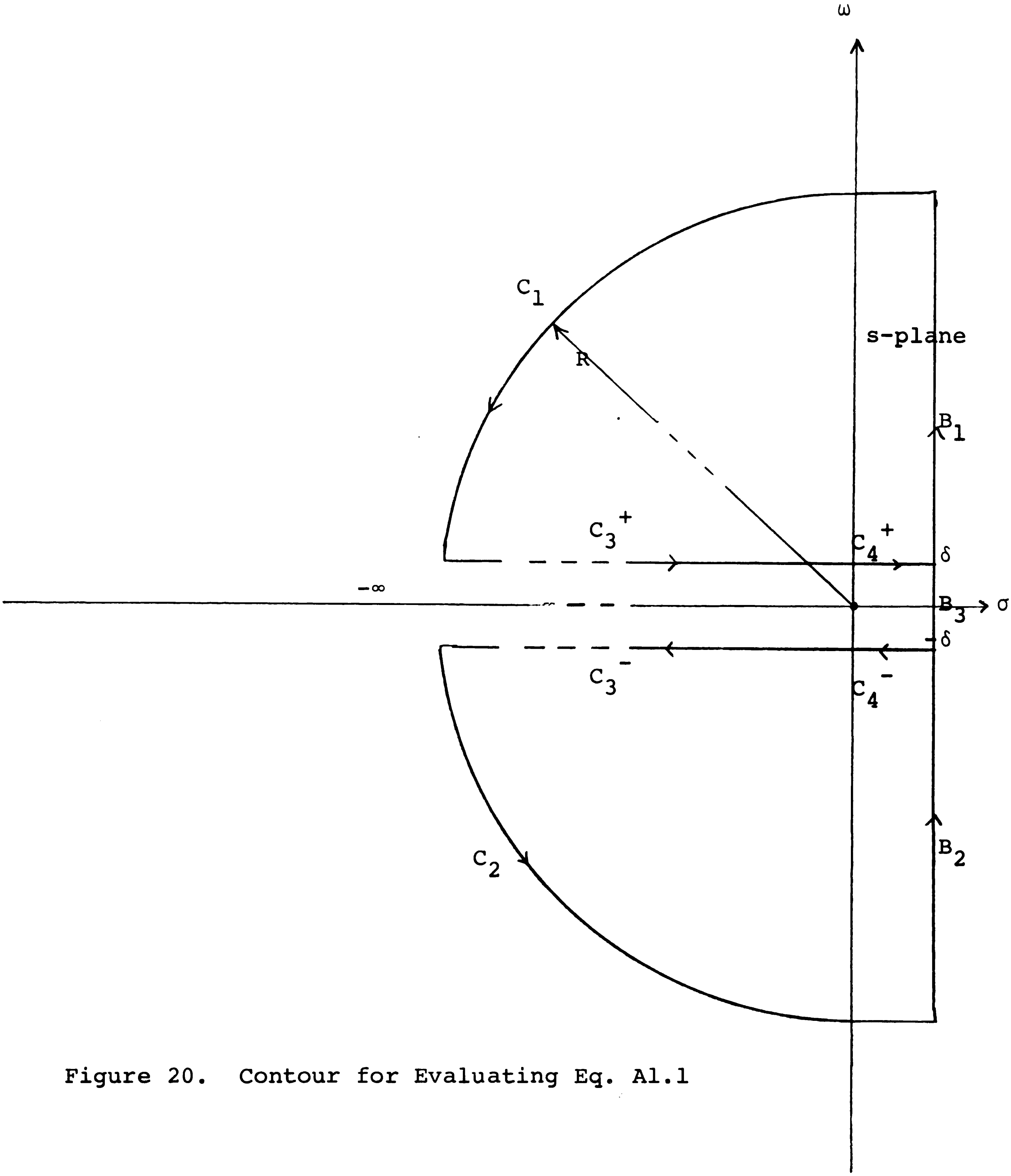


Figure 20. Contour for Evaluating Eq. A1.1

$$|\Psi(s,t)| \leq |Me^{st} \int_{-T}^{t} e^{-s\tau} d\tau|$$

$$= |Me^{st} [\frac{e^{-s\tau}}{-s}]_{-T}^{t}|$$

$$= |\frac{M}{s}(e^{s(T+t)} - 1)| \qquad \text{(A1.3)}$$

Estimating the integral along C_1 in equation A1.2 where

$$s = Re^{j\theta} \text{ or } s = \sigma + j\omega$$

$$ds = jRe^{j\theta}d\theta \text{ and } |s| \to \infty$$

by equation A1.3

$$|\int_{C_1} Z(s)\Psi(s,t)ds| \leq \int_{C_1} |Z(s)\frac{M}{s}(e^{s(T+t)}-1)ds|$$

$$\leq \int_{C_1} |\frac{Z(s)}{R} \; e^{s(T+t)} R \; d\theta|$$

$$\leq M \int_{C_1} |Z(s)e^{\sigma(T+t)} \; d\theta|$$

σ varies from 0 to $-\infty$. It is required that $Z(s) \to 0$ as $s \to \infty$ (cf. Theorem 2.1). Thus this integral tends to zero. Equation A1.2 reduces to

$$\int_{B_1} Z(s)\Psi(s,t)ds = - \int_{C_3^+} Z(s)\Psi(s,t)ds - \int_{C_4^+} Z(s)\Psi(s,t)ds \qquad \text{(A1.4)}$$

Similarly considering the integral along B_2

$$\int_{B_2} Z(s)\Psi(s,t)ds = - \int_{C_3^-} Z(s)\Psi(s,t)ds - \int_{C_4^-} Z(s)\Psi(s,t)ds. \qquad \text{(A1.5)}$$

Consider the integral along B_3

$$\left|\int_{B_3} Z(s)\Psi(s,t)ds\right| < \int_{-j\delta}^{+j\delta} |Z(s)\Psi(s,t)ds| \tag{A1.6}$$

Z(s) and Ψ(s,t) are analytic in the right half plane. Hence they are bounded along the interval of integration. As $\delta \to 0$ the integral along B_3 tends to zero.

Another fact that is obtained from the analyticity of Z(s) and Ψ(s,t) in the right half plane is that

$$\int_{C_4^+} Z(s)\Psi(s,t)ds = - \int_{C_4^-} Z(s)\Psi(s,t)ds \tag{A1.7}$$

Equations A1.4, A1.5, A1.6 and A1.7 are substituted in equation A1.1 to give

$$\frac{1}{2\pi j} \int_{Br} Z(s)\Psi(s,t)ds = - \int_{C_3^+} Z(s)\Psi(s,t)ds - \int_{C_3^-} Z(s)\Psi(s,t)ds \tag{A1.8}$$

On the line C_3^+, σ varies from $-\infty$ to 0. Ψ(s,t) is analytic in the entire s-plane, so $\Psi(s,t) = \Psi(\sigma,t)$. Let $Z(s) = Z^+(\sigma)$ on C_3^+.

On the line C_3^-, σ varies from $-\infty$ to 0. As before $\Psi(s,t) = \Psi(\sigma,t)$ and let $Z(s) = Z^-(\sigma)$ here. Then by equation 2.2 and A1.8,

$$v(t) = - \int_{C_3^+} \frac{1}{2\pi j} Z^+(\sigma)\Psi(\alpha,t)d\alpha - \int_{C_3^-} \frac{1}{2\pi j} Z^-(\sigma)\psi(\sigma,t)d\sigma$$

Define $f(-\sigma) = \frac{1}{2\pi j} \left(Z^{-}(\sigma) - Z^{+}(\sigma)\right)$

as in page 14.

Then,

$$v(t) = \int_{-\infty}^{0} f(-\sigma)\ \Psi\ (\sigma,t)d\sigma$$

$$= \int_{0}^{\infty} f(\sigma)\ \Psi\ (-\sigma,t)d\sigma \qquad \text{(A1.9)}$$

Equation 2.1 reduces to

$$\frac{d\ (-\sigma,t)}{dt} = -\sigma\ \Psi\ (-\sigma,t) + i(t) \qquad \text{(A1.1)}$$

Equations A1.9 and A1.10 are the reduced state description of the driving point impedance Z(s).

APPENDIX B

Proof of Real Valuedness of Density Function

Consider a one-port linear, time-invariant, passive, real network. Let the input $i(t) = I_o e^{p_o t}$ be applied at time t_o (p_o is a point in the complex frequence p plane). The output can be written as

$$v(t) = Z(p_o) I_o e^{p_o t} \qquad t \varepsilon \quad [t_o, \infty) \tag{A2.1}$$

Where Z(p) is the driving point impedance function. The assumption made in writing the solution as in equation A2.1 is that no transients occur as the excitation is applied at time t_o and after. In the case of distributed networks the appropriate distribution of initial voltages and currents need to be established to assure that this particular solution alone is excited. This can always be brought about as long as p_o is not a singular point.

Consider a real input

$$i(t) = I_o e^{p_o t} + I_o e^{\overline{p}_o t}$$

the output is real since the network is real. That is,

$$Z(p_o)\ I_o e^{p_o t} + Z(\overline{p}_o)\ I_o e^{\overline{p}_o t} \tag{A2.2}$$

is real. Hence the expression in equation A2.2 is equal to its own conjugate

$$\overline{Z(P_o)}\ I_o e^{\overline{p}_o t} + \overline{Z(\overline{p}_o)}\ I_o e^{p_o t} \tag{A2.3}$$

If it assumed that $I_o \neq 0$ then equation A2.2 and A2.3 are equal only if

$$\overline{Z(p)} = Z(\overline{p}) \tag{A2.4}$$

The above discussion may be found in references 15 and 16.

Consider the boundary condition defining the density function

$$Z^{-}(x) - Z^{+}(x) = 2\pi j\ f(-x)$$

By equation A2.4 $Z(s) = \overline{Z(\bar{s})}$ for $|\omega| > 0$. $Z^{-}(x) = \overline{Z^{+}(x)}$ and $Z^{-}(x) - Z^{+}(x)$ is pure imaginary. Hence $f(-x)$ is real valued.

APPENDIX C

A Corollary of Mittag-Leffler's Theorem

Given a sequence of distince complex numbers z_n having no limit point in the finite plane (arranged so that $|z_n| < |z_{n+1}|$), if there exists an integer $k \geqq 0$ such that

(1) $\sum_{n=1}^{\infty} |z_n|^{-k} = \infty$ and

(2) $\sum_{n=1}^{\infty} |z_n|^{-k-1} < \infty$

then there exists a meromorphic function f(z) having principal parts $\frac{1}{z-z_n}$ at $z = z_n$ for $n = 0,1,2\ldots\infty$.

The function f(z) may be taken to be

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-z_n} + \frac{1}{z_n} + \frac{z}{z_n^{\,2}} + \ldots \frac{z^{k-1}}{z_n^{\,k}} \right) \text{ if } k>0$$

(The $\frac{1}{z}$ term is present if there is a pole at the origin.)

If $k = 0$, $f(z) = \sum_{n=1}^{\infty} \frac{1}{z-z_n}$.

The series for f(z) converges absolutely and uniformly on compact sets not containing any of the poles.

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