A NEW CHARACTERISTIC SUBGROUP OF INFINITE GROUPS

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ABSTRACT

A NEW CHARACTERISTIC SUBGROUP
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By

Stephen F. Markstein

In this dissertation we investigate a new characteristic subgroup, $\mathcal{S}(G)$, of an arbitrary infinite group G. $\mathcal{S}(G)$ is defined by

$$\mathcal{S}(G)=\{x\in G \mid \text{whenever } H \text{ is a non-finitely generated subgroup of } G \text{ so is } \langle H,x\rangle\}.$$

$\mathcal{S}(G)$ is always a subgroup of the locally Nötherian radical, $\mathcal{N}(G)$. Let $\mathcal{S}_{\mathcal{N}}$ be the class of all groups for which $\mathcal{S}(G)=\mathcal{N}(G)$. A subgroup $H\subset G$ is nearly finitely generated in G if H is not finitely generated but every subgroup of G properly containing H is finitely generated, and $\mathcal{I}(G)=\cap\{H\subset G \mid H \text{ is nearly finitely generated in } G\}$.

In Chapter II we develop the basic properties of $\mathcal{S}(G)$. We prove, among others:

<u>Lemma</u>: $\mathcal{S}(G)\subset\mathcal{I}(G)$.

<u>Lemma</u>: All the subgroups of G are $\mathcal{S}_{\mathcal{N}}$ groups if and only if $\mathcal{N}(K)\subset\mathcal{I}(K)$ for all subgroups K of G.

<u>Corollary</u>: If $\mathcal{X}$ is a subgroup-closed class of groups, then $\mathcal{X}\subset\mathcal{S}_{\mathcal{N}}$ if and only if $\mathcal{N}(G)\subset\mathcal{I}(G)$ for all $G\in\mathcal{X}$.

<u>Lemma</u>: Normal subgroups of S_N groups are S_N groups.

In Chapter III we develop further properties of S(G) and the nearly finitely generated subgroups of G, and we prove our main results.

<u>Theorem</u>: The class of extensions of Abelian groups by polycyclic groups is contained in S_N.

<u>Definition</u>: A class X satisfies (*) if whenever G is an extension of the X group K by the nilpotent group G/K, then $N(K) \subset H$ for all the nearly finitely generated subgroups H of G.

<u>Theorem</u>: If X is a subgroup-closed class of groups, then the class of extensions of X groups by nilpotent groups is contained in S_N if and only if X satisfies (*).

<u>Theorem</u>: Finite extensions of S_N groups are S_N groups.

In Chapter IV we construct groups with remarkable properties. We show how to construct groups G for which $S(G) \neq N(G)$, and one of these examples is of a group that is an extension of a nilpotent group of class 2 by an Abelian group. We also construct groups having non-normal nearly finitely generated subgroups, and we show that a nearly finitely generated subgroup H of G need not intersect a normal subgroup K of finite index in G in a nearly finitely generated subgroup of K.

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INTRODUCTION

In this paper we introduce a new canonical subgroup, the $\mathcal{S}$-subgroup, of an arbitrary group G. The inspiration for our study stems from R. Baer's paper [2], "Group theoretical properties and functions," and J.B. Riles' paper [10], "The near Frattini subgroups of infinite groups."

The primary objective of Baer's paper is the investigation of possible relationships between group theoretical properties and functions. Suppose that $\mathcal{X}$ is any isomorphism-closed class of groups that contains the trivial group, with the convention that G is an $\mathcal{X}$-group if and only if $G\in\mathcal{X}$. Baer defines the $\mathcal{X}$-hypercenter of an arbitrary group G by

$$h_{\mathcal{X}}(G)=\{x\in G \mid H\subset G,\ H\in\mathcal{X} \text{ implies } \langle H,x\rangle\subset K\in\mathcal{X} \text{ for some } K\subset G\}.$$

It is easy to show that $h_{\mathcal{X}}(G)\diamond G$ for all such classes $\mathcal{X}$ and groups G.

Riles starts with a relative property and defines the set of non-near generators, $\mathcal{U}(G)$, of a group G. $\mathcal{U}(G)$ is given by

$$\mathcal{U}(G)=\{x\in G \mid\ |G:\langle S\rangle|=\infty \text{ implies } |G:\langle S,x\rangle|=\infty\},$$

and it is easy to see that $\mathcal{U}(G)\diamond G$.

If we alter Baer's definition to

$\bar{h}_{X}(G)=\{ x\in G | H\subset G,\ H\in X \text{ implies } \langle H,x\rangle \in X\}$,

then without additional assumptions we cannot conclude that $\bar{h}_{X}(G)$ is a subgroup of G. However if, for example, we restrict ourselves to classes X which are subgroup-closed, then this definition is equivalent to Baer's.

But it is by no means necessary that X be subgroup-closed for $\bar{h}_{X}(G)$ to be a subgroup. Indeed, the S subgroup of G, the subgroup that we shall study in this dissertation, is obtained by using for X the class of all non-finitely generated groups, which is obviously not subgroup-closed. Thus,

$$S(G) = \{ x\in G \,|\, \text{whenever H is a non-finitely generated subgroup of G so is } \langle H,x\rangle\} .$$

Our primary efforts are aimed at studying the relationship between $S(G)$ and the locally Nötherian radical, $N(G)$, of G.

Chapter I deals with notation, definitions and the statement of results from the basic group theory that will be used, for the most part, without proof or comment. In Chapter II we prove the basic facts about $S(G)$ and relate $S(G)$ to some of the well known and intensively studied subgroups of a group G.

Chapter III contains our main results. We prove that all groups G which are polycyclic extensions of Abelian groups have the property that $S(G)=N(G)$, that all finite extensions of groups G for which $\mathcal{S}(G)=\mathcal{N}(G)$ are again such groups, and we characterize the subgroup-closed classes $\mathcal{X}$ with the property that the extensions, G, of $\mathcal{X}$ groups by nilpotent groups are such that $\mathcal{S}(G)=\mathcal{N}(G)$. We also give examples of such classes $\mathcal{X}$.

In Chapter IV we present examples and constructions dealing with $\mathcal{S}(G)$ and other related subgroups of a group G. In particular, two of our examples are groups for which $\mathcal{S}(G)\neq\mathcal{N}(G)$.

CHAPTER I

BACKGROUND

In this chapter we list our notation and define the usual group theory terms as we shall use them. Interspersed with the definitions are standard theorems and facts about groups, whose proofs may be found in [6] or [11]. We delay the discussion of split extensions, wreath products and relatively free groups until Chapter IV, where they are needed.

For the remainder of this chapter let G be a group, H, K and L_ρ, $\rho \in R$, subgroups of G, x and y elements of G, S and T non-empty subsets of G.

e, 1 or 0	the identity of G
E	the identity subgroup of G
$\langle S \rangle$	the subgroup generated by S
x^y	$y^{-1}xy$, the conjugate of x by y
S^x	$\{ s^x \mid s \in S \}$, the conjugate of S by x
S^H	$\langle S^h \mid h \in H \rangle$, the normal closure of S in $\langle S,H \rangle$
$[y,x]$	$y^{-1}x^{-1}yx$, the commutator of y and x
$[S,T]$	$\langle [s,t] \mid s \in S,\ t \in T \rangle$
G'	$[G,G]$, the commutator subgroup of G

$Z(G)$	$\{z\in G \mid zx=xz \text{ for all } x\in G\}$, the center of G
ST	$\{st \mid s\in S,\ t\in T\}$
Hx	$\{hx \mid h\in H\}$, the right coset of H in G containing x
G/H, $\frac{G}{H}$	the set of right cosets of H in G
$\lvert G:H\rvert$	the cardinality of G/H, the index of H in G
$\lvert G\rvert$	the cardinality of G
Z, $\bar{Z}$	the group of integers
Z_p, $\bar{Z}_p$	the group of integers modulo p
$\emptyset$	the empty set
$K\subseteq G$	K is a subgroup of G
$K<G$	K is a proper subgroup of G
$K\simeq H$	K is isomorphic to H

S <u>normalizes</u> K if $K^x=K$ for each $x\in S$; K is a <u>normal subgroup</u> of G, written <u>$K \triangleleft G$</u>, if G normalizes K; K is <u>characteristic</u> in G, written <u>$K \Diamond G$</u>, if $K=K\alpha$ for every automorphism α of G. <u>Aut(G)</u> is the group of automorphisms of G, and <u>Lat(H,G)</u> is the lattice of subgroups of G containing H.

If $K \triangleleft G$, then G/K is a group with multiplication given by $(Kx)(Ky)=K(xy)$. G/K is a <u>factor group</u>, and every right coset of K in G is also a left coset of K in G. $H\cap K \triangleleft H$ and

$$\frac{H}{H\cap K} \simeq \frac{HK}{K}.$$

If $\lvert G:H\rvert<\infty$, then H and G are either both finitely generated or both non-finitely generated.

If $C=\{L_\rho \mid \rho\in R\neq\emptyset\}$ and if the union of every chain in C is a member of C, then Zorn's Lemma says that C has maximal members.

A class X of groups is an isomorphism-closed collection of groups containing E. If X and Y are classes of groups, then $X\circ Y$ is the class of all extensions of X groups by Y groups; that is, $G\in X\circ Y$ if there exists $H\triangleleft G$ such that $H\in X$ and $G/H\in Y$. $s(X)$ is the class of all subgroups of members of X. $L(X)$ is the class of all groups whose finitely generated subgroups are X groups. A is the class of Abelian groups; Max is the class of all groups satisfying the maximal condition on subgroups, while Max-n is the class of all groups satisfying the maximal condition on normal subgroups.

The derived series of G is defined by (i) $G^{(0)}=G$, and (ii) $G^{(n+1)}=[G^{(n)},G^{(n)}]$, for $n\geq 0$. G is solvable if $G^{(n)}=E$ for some integer n. G is polycyclic if there is a series $E\triangleleft K_1\triangleleft\dots\triangleleft K_n\triangleleft G$ with K_{i+1}/K_i cyclic for each i. G is polycyclic if and only if $G\in$ Max and G is solvable. P is the class of all polycyclic groups. $s(P)=P$.

The upper central series of G is defined by (i) $Z_0=E$, and (ii) Z_{n+1} is that subgroup of G such that $Z_{n+1}/Z_n = \mathbf{Z}(G/Z_n)$, $n\geq 0$. $Z_n \diamond G$ for each n. The lower central series of G is defined by (i) $\Gamma^1(G)=G$, and (ii) $\Gamma^{n+1}(G)=[G,\Gamma^n(G)]$, $n\geq 1$.

G is <u>nilpotent</u> if $Z_n=G$ for some integer n or equivalently, $\Gamma^m(G)=E$ for some integer m. If G is nilpotent, then the smallest value of n such that $Z_n=G$ or equivalently, the smallest value of m-1 such that $\Gamma^m(G)=E$, is the <u>class</u> of G. <u>Nilp</u> is the class of all nilpotent groups, and Nilp as well as <u>L(Nilp)</u> are subgroup-closed.

G is a <u>free group</u> if there exists a subset S of G such that every non-identity element $g\in G$ can be written exactly one way in the form $g=x_1^{n_1}\ldots x_k^{n_k}$, $x_1,\ldots,x_k\in S$, $x_i\neq x_{i+1}$, $i=1,\ldots,k-1$, n_i a non-zero integer for each i. In this event S is a set of <u>free generators</u> of G, and <u>G is free on S</u>. A <u>word</u> is a member of the free group on the set $S=\{\ldots,x_{-1},x_0,x_1,\ldots\}$. <u>$F$</u> is the class of all free groups and $s(F)=F$. Every group is a homomorphic image of a free group.

The class <u>L(Max)</u> is the class of locally Nötherian groups, groups whose finitely generated subgroups are Max-groups. If H and K are normal locally Nötherian subgroups of G, then HK is a normal locally Nötherian subgroup of G. The <u>locally Nötherian radical of G</u>, written <u>N(G)</u>, is the maximum normal locally Nötherian subgroup of G (see Baer [1] for existence.) $N(G)$ consists of all elements $x\in G$ whose normal closure, $\langle x\rangle^G$, is locally Nötherian. For every subgroup K of G, $N(G)\cap K\subset N(K)$.

The _FC-center of G, $F_1(G)$_, is the (characteristic) subgroup of G consisting of all elements of G with only finitely many distinct conjugates in G. The _FC-series of G_ is the series defined by (i) $F_0=E$, and (ii) F_{n+1} is that subgroup of G satisfying $F_{n+1}/F_n = F_1(G/F_n)$, $n\geq 0$. G is _FC-nilpotent_ if $F_n=G$ for some integer n, and G is an _FC-group_ if $F_1(G)=G$.

CHAPTER II

PRELIMINARY RESULTS

In the introduction we defined the object of our study, the $\mathcal{S}$-subgroup of a group G. We recall it here for convenience.

<u>Definition</u> 2.1: $\mathcal{S}(G) = \{ x \in G \mid \text{whenever } H \text{ is a non-finitely generated subgroup of } G\text{, so is } \langle H,x \rangle . \}$

<u>Lemma</u> 2.2: $\mathcal{S}(G)$ is a characteristic subgroup of G.

<u>Proof</u>: Suppose $x,y \in \mathcal{S}(G)$, $H \subset G$ with H not finitely generated, but $\langle H,xy^{-1} \rangle$ is finitely generated. Then $\langle H,x,y \rangle =$ $\langle \langle H,xy^{-1} \rangle ,y \rangle$ is finitely generated. However, $\langle H,x \rangle$ is not finitely generated (since $x \in \mathcal{S}(G)$), so that $\langle H,x,y \rangle = \langle \langle H,x \rangle ,y \rangle$ is not finitely generated (since $y \in \mathcal{S}(G)$). This contradiction gives $xy^{-1} \in \mathcal{S}(G)$ so that $\mathcal{S}(G)$ is a subgroup of G. That $\mathcal{S}(G)$ is characteristic follows from the fact that automorphisms take generating sets to generating sets.□

There is another characterization of $\mathcal{S}(G)$, which, although not exploited in this paper, is interesting and included for that reason.

<u>Definition</u> 2.3: $L_1(G) = \{x \in G \mid H$ is a finitely generated subgroup of G implies $Lat(H,\langle H,x\rangle) \in Max\}$,

$L_2(G) = \{x \in G \mid H$ is a finitely generated subgroup of G implies $Lat(H,\langle H,x\rangle)$ is finite $\}$.

<u>Proposition</u> 2.4: $\mathcal{S}(G) = L_1(G)$.

<u>Proof</u>: Suppose $x \in \mathcal{S}(G)$, H is a finitely generated subgroup of G but $H < H_1 < H_2 < \ldots < \langle H,x\rangle$ for some subgroups $H_1, H_2, \ldots$ of G. Let $K = \bigcup_i H_i$. Then K is the union of a properly increasing sequence of groups and cannot be finitely generated. But $\langle H,x\rangle = \langle K,x\rangle$, so that H is not finitely generated (since $x \in \mathcal{S}(G)$), which is a contradiction. Hence $x \in L_1(G)$, so $\mathcal{S}(G) \subset L_1(G)$. Conversely, suppose that $x \in L_1(G)$ and K is a subgroup of G with $\langle K,x\rangle$ finitely generated. Then $\langle K,x\rangle = \langle k_1,\ldots,k_n,x\rangle$ for some $k_1,\ldots,k_n \in K$ and

$Lat(\langle k_1,\ldots,k_n\rangle, \langle K,x\rangle) \in Max$ since $x \in L_1(G)$. In particular, then, all the members of this lattice must be finitely generated. Since $K \in Lat(\langle k_1,\ldots,k_n\rangle,\langle K,x\rangle)$, K is finitely generated so that $x \in \mathcal{S}(G)$. Thus $L_1(G) \subset \mathcal{S}(G)$ and equality follows.□

The proof of Proposition 2.4 is the only argument that we know of proving that $L_1(G)$ is a subgroup of G. Clearly $L_2(G)$ is a subset of $L_1(G)$. Whether $L_2(G)$ is a subgroup, we have been unable to decide.

Intimately connected with the study of the $\mathcal{S}$-subgroup of G are what we term the nearly finitely generated subgroups of G.

Definition 2.5: If H is a subgroup of G, then H is nearly finitely generated in G if H is not finitely generated, but every subgroup of G properly containing H is finitely generated.

Thus, H is nearly finitely generated in G if and only if H is maximal in G with respect to being non-finitely generated.

Lemma 2.6: If H is a non-finitely generated subgroup of G, then H is contained in a nearly finitely generated subgroup $\overline{H}$ of G.

Proof: Let $\mathcal{H}=\{K\subset G \mid H\subset K$, K is not finitely generated$\}$. Then $\mathcal{H}\neq\emptyset$ since $H\in\mathcal{H}$. If $\mathcal{C}$ is a chain in $\mathcal{H}$ and $C=\cup\mathcal{C}$, then C is not finitely generated since no proper union of groups is finitely generated. Hence $C\in\mathcal{H}$ and by Zorn's Lemma $\mathcal{H}$ has maximal members. It is clear that any maximal member $\overline{H}$ of $\mathcal{H}$

is nearly finitely generated in G.□

The intersection of all the nearly finitely generated subgroups of G yields another characteristic subgroup of G.

Definition 2.7: $\mathcal{I}(G)=\cap\{H \mid H$ is nearly finitely generated in G$\}$. ($\mathcal{I}(G)=G$ if $G\in$Max.)

Lemma 2.8: $\mathcal{S}(G)$ is a subgroup of $\mathcal{I}(G)$.

Proof: Suppose $x\in\mathcal{S}(G)$ and H is nearly finitely generated in G. Then H is not finitely generated and neither is $\langle H,x\rangle$ since $x\in\mathcal{S}(G)$. Hence $x\in H$ since H is maximal non-finitely generated in G, so $\mathcal{S}(G)\subset H$, whence $\mathcal{S}(G)\subset\mathcal{I}(G)$ since H is an arbitrary nearly finitely generated subgroup of G.□

Lemma 2.9: $\mathcal{S}(G)$ is a subgroup of $\mathcal{N}(G)$.

Proof: Suppose $x\in\mathcal{S}(G)$. We show that $\langle x\rangle^G$ is locally Nötherian. Since $\mathcal{S}(G)\triangleleft G$ and $x\in\mathcal{S}(G)$, $\langle x\rangle^G\subset\mathcal{S}(G)$. If $K\subset\langle x_1,\ldots,x_n\rangle\subset\langle x\rangle^G$ and K is not finitely generated, then adjoining each x_i, in turn, to K gives a non-finitely generated subgroup at each step since $x_i\in\mathcal{S}(G)$ for all i, and thus $\langle x_1,\ldots,x_n\rangle=\langle K,x_1,\ldots,x_n\rangle$ is not finitely generated, a contradiction. Hence K is finitely generated and $\langle x\rangle^G\subset\mathcal{N}(G)$ so that $\mathcal{S}(G)\subset\mathcal{N}(G)$.□

The main positive results in our paper concern classes $\mathcal{X}$ for which $G \in \mathcal{X}$ implies $\mathcal{S}(G)=\mathcal{N}(G)$. This suggests the following definition.

<u>Definition</u> 2.10: $\mathcal{S}_{\mathcal{N}}$ is the class of all groups for which $\mathcal{S}(G)=\mathcal{N}(G)$.

The following lemma is of value in proving that all subgroups of a group G are $\mathcal{S}_{\mathcal{N}}$ groups.

<u>Lemma</u> 2.11: All subgroups of a group G are $\mathcal{S}_{\mathcal{N}}$ groups if and only if for each subgroup K of G, $\mathcal{N}(K) \subset \mathcal{I}(K)$.

<u>Proof</u>:$\rightarrow$: If all subgroups of G are $\mathcal{S}_{\mathcal{N}}$ groups, then Lemma 2.8 yields $\mathcal{N}(K) \subset \mathcal{I}(K)$ for all subgroups K of G.

$\leftarrow$: Suppose that $\mathcal{N}(K) \subset \mathcal{I}(K)$ for all subgroups K of G, but that for some subgroup K and some $x \in \mathcal{N}(K)$, $x \notin \mathcal{S}(K)$. Then there exists a non-finitely generated subgroup H of K such that $\langle H,x \rangle$ is finitely generated. By Lemma2.6 there exists a nearly finitely generated subgroup $\overline{H}$ of $\langle H,x \rangle$ that contains H. Then $\langle H,x \rangle = \langle \overline{H},x \rangle$ and $x \in \mathcal{N}(\langle \overline{H},x \rangle)$ since $x \in \mathcal{N}(K)$. By hypothesis $x \in \overline{H}$ since $\overline{H}$ is nearly finitely generated in $\langle \overline{H},x \rangle$. But this says that $\langle H,x \rangle$ is not finitely generated, a contradiction.□

As an immediate corollary of this lemma, we find the groups that are their own $\mathcal{S}$-subgroups.

Corollary 2.12: $S(G)=G$ if and only if $N(G)=G$.

Proof: If $S(G)=G$, then $N(G)=G$ since $S(G)\subset N(G)$ by Lemma 2.9. Conversely, if $N(G)=G$, then $I(K)=N(K)=K$ for all subgroups K of G so that $N(G)=S(G)$ by Lemma 2.11.□

Another immediate corollary gives a criterion for proving that a subgroup-closed class is contained in S_N.

Corollary 2.13: If X is a subgroup-closed class, then X is contained in S_N if and only if $N(G)\subset I(G)$ for all $G\epsilon X$.

Of course, in applying Corollary 2.13, we need only worry about the finitely generated X groups since any non-finitely generated group trivially satisfies $N(G)\subset I(G)=G$. We may apply this corollary to the class $L(X)$ in the event that X is contained in S_N.

Corollary 2.14: If X is a subgroup-closed class of S_N groups, then $L(X)$ is a class of S_N groups.

Proof: $L(X)$ is subgroup-closed since X is, so, by Corollary 2.13, it suffices to show that $N(G)\subset H$ whenever H is a nearly finitely generated subgroup of the $L(X)$ group G. Since we need concern ourselves only with the finitely generated $L(X)$ groups, which are X groups, the result is immediate because, in this event, $N(G)=S(G)\subset H$, by hypothesis and Lemma 2.8.□

<u>Definition</u> 2.15: If $\mathcal{X}$ is a class of groups, then $\mathcal{X}$ satisfies the <u>local</u> <u>theorem</u> provided that we may conclude that a group G is an $\mathcal{X}$ group whenever all the finitely generated subgroups of G are $\mathcal{X}$ groups.

We observe that a class $\mathcal{X}$ satisfies the local theorem if and only if all the subgroups of a group G are $\mathcal{X}$ groups whenever all the finitely generated subgroups of G are $\mathcal{X}$ groups. Another corollary to Lemma 2.11 is thus:

<u>Corollary</u> 2.16: $\mathcal{S}_{\mathcal{N}}$ satisfies the local theorem.

Many of the classes of groups that are commonly studied are contained in $\mathcal{S}_{\mathcal{N}}$. We have already cited the class of locally Nötherian groups. The class of all free groups is also contained in $\mathcal{S}_{\mathcal{N}}$ since $\mathcal{N}(F)=\mathcal{S}(F)=E$ if F is free on more than 1 generator, and $\mathcal{N}(F)=\mathcal{S}(F)=F$ if F is free on 1 generator. Hence, $\mathcal{S}_{\mathcal{N}}$ is the class of all groups in the event that homomorphic images of $\mathcal{S}_{\mathcal{N}}$ groups are again $\mathcal{S}_{\mathcal{N}}$ groups, since each group is a homomorphic image of a free group. However, in Chapter IV, we construct groups for which $\mathcal{S}(G)$ is a proper subgroup of $\mathcal{N}(G)$, so the class $\mathcal{S}_{\mathcal{N}}$ is not quotient-closed. It also follows that $\mathcal{S}_{\mathcal{N}}$ is not subgroup-closed since any group can be embedded in a simple group (see [11], p. 316) and each simple group is an $\mathcal{S}_{\mathcal{N}}$ group by Corollary 2.12. $\mathcal{S}_{\mathcal{N}}$ groups are, however, closed under the taking of normal subgroups.

Lemma 2.17: If $G\in S_N$ and $K \triangleleft G$, then $K\in S_N$.

Proof: $N(K)=K\cap N(G)=K\cap S(G)\subset S(K)$ so $S(K)=N(K)$ by Lemma 2.9.□

Duguid and Mc Lain [4] have proved that all FC-nilpotent groups are locally Nötherian, (hence belong to S_N.) Thus, for an FC-nilpotent group G, $F_1(G)\subset S(G)$. This is true generally.

Proposition 2.18: For any group G, $F_1(G)\subset S(G)$.

Proof: Suppose $x\in F_1(G)$, H is a non-finitely generated subgroup of G, but $\langle H,x\rangle$ is finitely generated. Then

$$\frac{\langle H,x\rangle}{\langle x\rangle^H} = \frac{H\langle x\rangle^H}{\langle x\rangle^H} \simeq \frac{H}{\langle X\rangle^H\cap H}$$ is finitely generated so

that $\langle x\rangle^H\cap H$ is not finitely generated since H is not finitely generated. But $\langle x\rangle^H$ is a finitely generated FC-nilpotent group, hence $\langle x\rangle^H\in \mathrm{Max}$ so that $\langle x\rangle^H\cap H\in \mathrm{Max}$, a contradiction. Hence $\langle H,x\rangle$ is not finitely generated and $x\in S(G)$.□

CHAPTER III

THEOREMS

In this chapter we prove our main results, Theorems 3.10, 3.13 and 3.19.

Theorem 3.10 says that all polycyclic extensions of Abelian groups are S_N groups. Since $s(A \circ P) = A \circ P$, we need only show that if $G \in A \circ P$, then $N(G) \subset H$ for all nearly finitely generated subgroups of G (by Lemma 2.11.) Our first result in that direction is Theorem 3.8, which gives sufficient conditions for all normal Abelian subgroups to be contained in every nearly finitely generated subgroup of G. That this is not the case for an arbitrary group is shown in Theorem 4.1, in the next chapter. The example constructed in Theorem 4.2 shows that even if all normal Abelian subgroups are contained in all nearly finitely generated subgroups of G it is not necessary that the locally Nötherian radical be contained in each nearly finitely generated subgroup of G.

Theorem 3.13 gives sufficient conditions on a class X of groups that the class of $X \circ$Nilp groups is also contained in S_N. The theorem applies in the case $X = F$, the class of free groups.

Theorem 3.19 says that finite extensions of S_N groups are S_N groups.

Our first lemma has as its corollary a key theorem, which says that all normal nearly finitely generated subgroups contain the locally Nötherian radical.

<u>Lemma</u> 3.1: If $x \in N(G)$ and $H\langle x\rangle$ is a finitely generated extension of H by $\langle x\rangle$, then H is finitely generated.

<u>Proof</u>: Suppose the lemma is false and H is not finitely generated. Since $H\langle x\rangle = \langle H,x\rangle$ is finitely generated, $H \cap \langle x\rangle = E$, and there exist $h_1,\ldots,h_n \in H$ such that $\langle H,x\rangle = \langle h_1,\ldots,h_n,x\rangle$. Hence $H = \langle h_1^{\langle x\rangle}, h_2^{\langle x\rangle},\ldots,h_n^{\langle x\rangle}\rangle$. Let

$$U_1 = \{h_1,\ldots,h_n\},\ U_2 = U_1 \cup U_1^{x} \cup U_1^{x^{-1}},\ \text{and inductively}$$

define $U_{n+1} = U_n \cup U_n^{x} \cup U_n^{x^{-1}}$, $n > 1$.

Then $\langle U_i\rangle < \langle U_{i+1}\rangle$ for each i since $H = \bigcup_i \langle U_i\rangle$ and H is not finitely generated. Hence, there exists $h \in U_1$ such that the subgroup generated by the conjugates of h by all powers of x lies outside of $\langle U_i\rangle$ for each i. We complete the proof by constructing a non-finitely generated subgroup

$A \subset \langle [h,x],x\rangle \subset \langle x\rangle^G$, which contradicts the fact that $x \in N(G)$.

Because $h^{<x>} \not\subset <U_i>$ for all i, there exists a sequence $n_1 < n_2 < \ldots$ of distinct non-negative integers such that $h^{x^{n_i}} \in <U_i>$ but $h^{x^{n_i}} \notin <U_j>$ for $j<i$. Let

$$A_k = <h^{-1}h^x,\ (h^x)^{-1}h^{x^2}, \ldots,\ (h^{x^{n_k-1}})^{-1}h^{x^{n_k}}>,\ k=1,2,\ldots .$$

Then $A_k < A_{k+1}$ for all k since $h^{-1}h^{x^{n_{k+1}}} \in A_{k+1}$, but $h^{-1}h^{x^{n_{k+1}}} \notin <U_k> \supset A_k$. Let $A = \bigcup_k A_k$. Then A is not finitely generated since it is the union of a properly ascending sequence of groups. Finally,

$$(h^{x^i})^{-1}h^{x^{i+1}} = (h^{-1}h^x)^{x^i} = [h,x]^{x^i} \in <[h,x],x>,$$

so that $A_k \subset <[h,x],x>$ for all k, and thus $A \subset <[h,x],x>$, which completes the proof.□

<u>Theorem</u> 3.2: If H is nearly finitely generated in G and $H \lhd G$, then $\mathcal{N}(G) \subset H$.

<u>Proof</u>: Deny the theorem and suppose $x \in \mathcal{N}(G)$ but $x \notin H$. Then $<H,x>$ is finitely generated and $<H,x> = H<x>$ since $H \lhd G$. By Lemma 3.1, H is finitely generated, which contradicts that H is nearly finitely generated in G.□

A particular case of the following technical lemma will be of repeated use to us in proving Theorem 3.10.

<u>Lemma</u> 3.3: If Y is a subset of the group G with $H=\langle Y\rangle^G$ and $\langle Y\rangle \triangleleft H$, and if $G/H=\langle g_\rho H \mid \rho\in R\rangle$, then $G=\langle Y, g_\rho \mid \rho \in R\rangle$.

<u>Proof</u>: If $g\in G$, then $g=hw(g_{\rho_1},\ldots,g_{\rho_n})$ for some $h\in H$, w a word and $g_{\rho_1},\ldots,g_{\rho_n} \in \{g_\rho \mid \rho\in R\}$. Thus, for each $x\in Y$

$$x^g = x^{hw(g_{\rho_1},\ldots,g_{\rho_n})} = (x^h)^{w(g_{\rho_1},\ldots,g_{\rho_n})} \in \langle Y, g_\rho \mid \rho\in R\rangle$$

since $\langle Y\rangle \triangleleft H$. Therefore $H=\langle Y\rangle^G \subset \langle Y, g_\rho \mid \rho\in R\rangle$ and

$G=\langle H, g_\rho \mid \rho\in R\rangle \subset \langle Y, g_\rho \mid \rho\in R\rangle$. □

Corollary 3.4 is the form of Lemma 3.3 that we need. Its proof is omitted since it is immediate from the lemma.

<u>Corollary</u> 3.4: If $\overline{H} \triangleleft \overline{G}$, $\overline{G}/\overline{H}$ is finitely generated, $\overline{H}$ is is Abelian and not finitely generated, and if Y is a finite subset, $Y\subset\overline{H}$, such that $\langle Y\rangle^{\overline{G}}=\overline{H}$, then $\overline{G}$ is finitely generated.

Phillip Hall has proved the following theorem concerning Abelian by polycyclic groups.

<u>Theorem</u> 3.5: ([5], p. 430, theorem 3) Every finitely generated extension G of an Abelian group A by a polycyclic group $\Gamma=G/A$ satisfies Max-n.

In light of this theorem, our attention turns to the class of groups satisfying Max-n, the maximal condition on normal subgroups. Our next lemma is well known and says that in a group satisfying Max-n each normal subgroup is finitely normally generated.

Lemma 3.6: If $G \in$Max-n and $K \triangleleft G$, then $K = \langle k_1, \ldots, k_m \rangle^G$ for some $k_1, \ldots, k_m \in K$, m a positive integer.

Proof: If the lemma is not true, then we may construct a properly ascending sequence

$$\langle k_1 \rangle^G < \langle k_1, k_2 \rangle^G < \ldots < \langle k_1, k_2, \ldots, k_m \rangle^G < \ldots < K$$

of normal subgroups of G contained in K. Since $G \in$Max-n this cannot happen; the chain must break off after finitely many steps, and the lemma follows.□

Our next lemma gives us information about certain factorizations of finitely generated Max-n groups.

Lemma 3.7: If $G = AH$ is a finitely generated Max-n group with $A \triangleleft G$, and A Abelian, then H is finitely generated.

Proof: Deny the assertion and suppose that H is not finitely generated. Then $A \cap H$ is not finitely generated since $\frac{H}{A \cap H} \simeq \frac{AH}{A} = \frac{G}{A}$ is finitely generated and H is not finitely generated. Furthermore, $A \cap H \triangleleft AH = G$ since A is Abelian. We apply Corollary 3.4 with $\overline{G} = H$ and $\overline{H} = A \cap H$ and conclude that

$$\langle a_1, \ldots, a_n \rangle^H < A \cap H \text{ for all } a_1, \ldots, a_n \in A \cap H,\ n \geq 1.$$

But $\langle a_1,\ldots,a_n\rangle^H=\langle a_1,\ldots,a_n\rangle^{AH}=\langle a_1,\ldots,a_n\rangle^G$, again since A is Abelian. Thus we have that $\langle a_1,\ldots,a_n\rangle^G<A\cap H$ for all $a_1,\ldots,a_n\in A\cap H$, $n\geq 1$. This contradicts the fact that G satisfies Max-n, by Lemma 3.6. Hence, H must be finitely generated.$\square$

Theorem 3.8: If all the finitely generated subgroups of a group G satisfy Max-n, then each normal Abelian subgroup A of G is contained in each nearly finitely generated subgroup H of G.

Proof: Suppose $A\lhd G$, A Abelian, and H is nearly finitely generated in G but $A\not\subset H$. Then $K=AH$ is finitely generated and thus $K\in$Max-n by hypothesis. By Lemma 3.7, H is finitely generated, a contradiction. Hence $A\subset H$.$\square$

Our last lemma preliminary to proving Theorem 3.10 gives us further information about factorizations in a more restricted setting than Lemma 3.7.

Lemma 3.9: If $G=\mathcal{N}(G)H$ is a finitely generated Max-n group, and if $A\lhd G$, A Abelian, with $G/A\in$Max and $A\subset H$, then H is finitely generated.

Proof: If A is finitely generated, then $G\in$Max$\circ$Max so that H is finitely generated. Suppose then that A is not finitely generated. Since $G \in \text{Max-n}$ and $A \triangleleft G$,

$$A = \langle a_1, \ldots, a_n \rangle^G \text{ for some } a_1, \ldots, a_n \in A,\ n \geq 1, \text{ (by Lemma 3.6)}$$

$$= \langle a_1, \ldots, a_n \rangle^{N(G)H} = (\langle a_1, \ldots, a_n \rangle^{N(G)})^H.$$

Clearly $A \subset N(G)$, and $N(G)/A \in \text{Max}$ since $G/A \in \text{Max}$ so that

$$\frac{N(G)}{A} = \langle Ax_1, \ldots, Ax_k \rangle \text{ for some } x_1, \ldots, x_k \in N(G),\ k \geq 1.$$

Hence, $\langle a_1, \ldots, a_n \rangle^{N(G)} \subset \langle a_1, \ldots, a_n, x_1, \ldots, x_k \rangle = K$ since $\langle a_1, \ldots, a_n \rangle \subset A \subset N(G)$ and A is Abelian. Now $K \subset N(G)$ so $K \in \text{Max}$ since K is finitely generated. Thus $\langle a_1, \ldots, a_n \rangle^{N(G)}$ is finitely generated and $\langle a_1, \ldots, a_n \rangle^{N(G)} = \langle b_1, \ldots, b_r \rangle$ for some $b_1, \ldots, b_r \in A, r \geq 1$. Hence $A = \langle b_1, \ldots, b_r \rangle^H$. Now $A \triangleleft H$ and H/A is finitely generated since $G/A \in \text{Max}$. Corollary 3.4 applied to $\overline{G} = H$ and $\overline{H} = A$ yields that H is finitely generated.□

<u>Theorem</u> 3.10: $A \circ P \subset S_N$.

<u>Proof</u>: Since $s(A \circ P) = A \circ P$, it suffices by Corollary 2.13 to show that $G \in A \circ P$ implies $N(G) \subset I(G)$. Thus, suppose that $G \in A \circ P$, H is nearly finitely generated in G but $N(G) \not\subset H$. Then $K = N(G)H$ is finitely generated and $K = N(K)H$ since $N(K) \supset N(G)$. Also, H is nearly finitely generated in K. Now K is a finitely generated $A \circ P$ group so that $K \in \text{Max-n}$ by Theorem 3.5. Let $A \triangleleft K$, A Abelian, with $K/A \in \text{Max}$. Then $A \subset H$ by Theorem 3.8, and Lemma 3.9 yields that H is finitely generated. This contradicts that H is nearly finitely generated in G, so it

must be that $N(G) \subset H$, and the theorem is proved.□

Since S_N satisfies the local theorem (Corollary 2.16), it is immediate that $L(A \circ P) \subset S_N$. From this observation follows the inclusion in S_N of the subclasses of $L(A \circ P)$ described in Chapter I.

Theorem 3.11: The classes $A \circ$Nilp, $A \circ L$(Nilp) and $L(A \circ$Nilp) are contained in S_N.

Proof: This follows immediately since $A \circ \mathrm{Nilp} \subset A \circ L(\mathrm{Nilp}) \subset L(A \circ \mathrm{Nilp}) \subset L(A \circ P) \subset S_N$.□

We now investigate the circumstances under which a class X has the properties:

(i) $s(X \circ \mathrm{Nilp}) = X \circ \mathrm{Nilp}$, and (ii) $X \circ \mathrm{Nilp} \subset S_N$.

In pursuing this question it is convenient to make the following definition. Suppose X is a class of groups.

Definition 3.12: X satisfies (*) if $G \in X \circ \mathrm{Nilp}$, $K \triangleleft G$, $K \in X$ with $G/K \in \mathrm{Nilp}$ implies that $N(K) = K \cap N(G) \subset H$ for all nearly finitely generated subgroups H of G.

Returning to our question it is clear that we satisfy (i) if (iii) $s(X) = X$. Suppose that X satisfies (ii) in addition to (i). If $G \in X \circ \mathrm{Nilp}$, $K \triangleleft G$ with $K \in X$ and $G/K \in \mathrm{Nilp}$, then certainly $K\cap\mathcal{N}(G)\subset H$ whenever H is nearly finitely generated in G since $\mathcal{N}(G)=\mathcal{S}(G)\subset H$ by Lemma 2.8. Thus $\mathcal{X}$ satisfies (*). In fact the converse of this statement is true, and its proof gives us our result.

Theorem 3.13: If $s(\mathcal{X})=\mathcal{X}$, then $\mathcal{X}\circ\mathrm{Nilp}\subset\mathcal{S}_{\mathcal{N}}$ if and only if $\mathcal{X}$ has property (*).

Proof: It only remains to show that if $\mathcal{X}$ satisfies (iii) and (*), then $\mathcal{X}\circ\mathrm{Nilp}\subset\mathcal{S}_{\mathcal{N}}$. Under these assumptions we have already commented that $s(\mathcal{X}\circ\mathrm{Nilp})=\mathcal{X}\circ\mathrm{Nilp}$ so that by the remark following Corollary 2.13 we need only show that if $G\in\mathcal{X}\circ\mathrm{Nilp}$ and H is nearly finitely generated in G, then $\mathcal{N}(G)\subset H$. So let $G\in\mathcal{X}\circ\mathrm{Nilp}$, $K\triangleleft G$, with $K\in\mathcal{X}$ and $G/K\in\mathrm{Nilp}$. Then G has the invariant series

$$E\triangleleft K=Z_0\triangleleft Z_1\triangleleft\cdots\triangleleft Z_n=G,\text{ for some } n\geq 0,$$

where the Z_i are from

$$K/K=Z_0/K\triangleleft Z_1/K\triangleleft\cdots\triangleleft Z_n/K=G/K,$$

the upper central series for G/K. We show that $\mathcal{N}(G)\cap Z_i\subset H$ for $i=0,1,\ldots,n$. The proof is by induction, and the case $i=0$ is simply the assumption that $\mathcal{X}$ satisfies (*).

Suppose now that $\mathcal{N}(G)\cap Z_i\subset H$ for some i, $0\leq i\leq n-1$. We claim that $\mathcal{N}(G)\cap Z_{i+1}$ normalizes H. Let $x\in H$, $y\in\mathcal{N}(G)\cap Z_{i+1}$. Then $[x,y]\in Z_i\cap\mathcal{N}(G)\subset H$ by the inductive hypothesis. So, $[x,y]=x^{-1}x^y\in H$, which gives $x^y\in H$ since $x\in H$. Thus we have that $H \triangleleft [Z_{i+1} \cap N(G)]H = L$, and H is nearly finitely generated in L since H is nearly finitely generated in G. By Theorem 3.2, $H \supset N(L) \supset Z_{i+1} \cap N(G)$, which completes the induction. Hence, $H \supset Z_n \cap N(G) = N(G)$, and the proof is complete. □

The question of the existence of non-trivial subgroup-closed classes X satisfying property (*) is answered in the next lemma.

<u>Lemma</u> 3.14: If $s(X)=X$ and $G \in X$ implies $N(G)=E$ or $G \in A$, then X satisfies property (*).

<u>Proof</u>: Suppose $G \in X \circ \mathrm{Nilp}$, $K \triangleleft G$, $K \in X$ with $G/K \in \mathrm{Nilp}$ and suppose H is nearly finitely generated in G. Now $K \cap N(G) \subset N(K)$. If $K \cap N(G) = E$, then $K \cap N(G) \subset H$ is trivial. If $K \cap N(G) > E$, then $N(K) > E$ so that $K \in A$, by hypothesis. Hence $G \in A \circ \mathrm{Nilp} \subset S_N$ so that $K \cap N(G) \subset S(G) \subset H$. □

<u>Example</u> 3.15: The class $X=A$, of Abelian groups, satisfies (*) by Theorems 3.11 and 3.13. A more interesting case is obtained by choosing $X=F$, the class of all free groups. To see that Lemma 3.14 applies, we recall that subgroups of free groups are free, and 1-generator free groups are certainly Abelian. Thus it remains only to show that $N(F)=E$ if F is free on more than 1 generator. Suppose $K \triangleleft F$, $K>E$; K cannot be Abelian so that K is free on more than 1 generator. Let H be a 2-generator subgroup of K. Then H is free and

$|H:H'|=\infty$ so that H' is not finitely generated (see [8], p. 104.) Hence $\mathcal{N}(F)=E$ and the hypotheses of Lemma 3.14 are satisfied. Therefore, F satisfies property (*) and Theorem 3.13 yields $F\circ\mathrm{Nilp}\subset S_{\mathcal{N}}$.

Our attention now turns to proving that finite extensions of $S_{\mathcal{N}}$ groups are $S_{\mathcal{N}}$ groups. We begin with a lemma that gives an easy rule for identifying some of the non-finitely generated subgroups of a group.

<u>Lemma</u> 3.16: If $H\subset G$, H is not finitely generated, and $H\subset\overline{H}\subset HS(G)$, then $\overline{H}$ is not finitely generated.

<u>Proof</u>: Suppose to the contrary that $\overline{H}$ is finitely generated. Then there exist $s_1,\ldots,s_n\in S(G)$ such that $\overline{H}=\langle H,s_1,\ldots,s_n\rangle$. Let $H_0=H$ and inductively define $H_{i+1}=\langle H_i,s_{i+1}\rangle$, $0\le i<n$. Then $H_0=H$ is not finitely generated by hypothesis, and if we assume that H_i is not finitely generated, then $H_{i+1}=\langle H_i,s_{i+1}\rangle$ is not finitely generated since $s_{i+1}\in S(G)$ for each i. Thus $\overline{H}=H_n$ is not finitely generated, a contradiction.□

In any group G if $K\triangleleft G$, then $\mathcal{N}(K)=K\cap\mathcal{N}(G)$, so that in particular $\mathcal{N}(K)\subset\mathcal{N}(G)$. If finite extensions of $S_{\mathcal{N}}$ groups are again to be $S_{\mathcal{N}}$ groups, then we must certainly have that $S(K)\subset S(G)$ for normal subgroups K of finite index in G.

Lemma 3.17: If $K \triangleleft G$ and $|G:K| < \infty$, then $\mathcal{S}(K) \subset \mathcal{S}(G)$.

Proof: Deny the lemma and suppose that $x \in \mathcal{S}(K) - \mathcal{S}(G)$, H is a non-finitely generated subgroup of G, but $\langle H,x\rangle$ is finitely generated. Now $|H:H\cap K| = |HK:K| \leq |G:K| < \infty$ so that $H\cap K$ is not finitely generated since H is not finitely generated. Similarly $|\langle H,x\rangle : \langle H,x\rangle \cap K| = |\langle H,x\rangle K:K| \leq |G:K| < \infty$ so that $\langle H,x\rangle \cap K$ is finitely generated since $\langle H,x\rangle$ is finitely generated. Furthermore, $\langle H\cap K, x\rangle$ is not finitely generated since $x \in \mathcal{S}(K)$ and $H\cap K \subset K$ is not finitely generated, and clearly $H\cap K \subset \langle H,x\rangle \cap K$. To complete the proof, we show that $\langle H,x\rangle \cap K \subset (H\cap K)\mathcal{S}(K)$ giving $H\cap K \subset \langle H,x\rangle \cap K \subset (H\cap K)\mathcal{S}(K)$. By Lemma 3.16 we find that $\langle H,x\rangle \cap K$ is not finitely generated, which is a contradiction. Suppose then that $g \in \langle H,x\rangle \cap K = (H\langle x\rangle^H)\cap K$. Then $g = h\bar{x}$ for some $h \in H$, $\bar{x} \in \langle x\rangle^H$. But $\langle x\rangle^H \subset \mathcal{S}(K)$ since $x \in \mathcal{S}(K) \diamond K \triangleleft G$, so $\bar{x} \in K$. Now, we have that $g, \bar{x} \in K$ so that $h \in H\cap K$. Hence $g = h\bar{x} \in (H\cap K)\mathcal{S}(K)$, which completes the proof. □

Our last lemma is of the exercise variety and, although we cannot remember having seen it, we suspect that it is probably well known.

Lemma 3.18: If A, B, C, D, BD and AC are subgroups of G with $C \triangleleft BD$, and if $|B:A|$, $|D:C| < \infty$, then $|BD:AC| < \infty$.

Proof: Suppose that $Ax_1, \ldots, Ax_n$ form a complete set of right cosets of A in B, and $Cy_1, \ldots Cy_m$ form a complete set of right cosets of C in D. Let $g=bd\in BD$, $b\in B$, $d\in D$. Then $bd=(ax_i)(cy_j)$ for some $a\in A$, $c\in C$, $1\le i\le n$, $1\le j\le m$. Since $C \triangleleft BD$ $x_i c=\bar{c}x_i$ for some $\bar{c}\in C$ so that $bd=(ax_i)(cy_j)=a(x_i c)y_j=a\bar{c}x_i y_j\in (AC)x_i y_j$. Hence, $\{x_i y_j \mid 1\le i\le n,\ 1\le j\le m\}$ includes a complete set of right coset representatives of AC in BD, whence $|BD:AC|\le nm<\infty$. □

<u>Theorem</u> 3.19: Finite extensions of $\mathcal{S}_N$ groups are $\mathcal{S}_N$ groups.

<u>Proof</u>: Suppose $K \triangleleft G$, $K\in\mathcal{S}_N$, $|G:K|<\infty$, and let $x\in N(G)$. We prove that $x\in\mathcal{S}(G)$. If $x\in K$, then $x\in K\cap N(G)=N(K)=\mathcal{S}(K)\subset\mathcal{S}(G)$ by Lemma 3.17. Suppose then that $x\notin K$ and $x\notin\mathcal{S}(G)$. Then there exists a non-finitely generated subgroup H of G such that $\langle H,x\rangle=H\langle x\rangle^H$ is finitely generated. As in the proof of Lemma 3.17 $|H:H\cap K|<\infty$ and $H\cap K$ is not finitely generated, and a similar argument gives $|\langle x\rangle^H:\langle x\rangle^H\cap K|<\infty$. Furthermore, $\langle x\rangle^H\cap K \triangleleft H\langle x\rangle^H$ so that $(K\cap H)(\langle x\rangle^H\cap K)\subset H\langle x\rangle^H$. Applying Lemma 3.18 with $A=K\cap H$, $B=H$, $C=\langle x\rangle^H\cap K$ and $D=\langle x\rangle^H$ yields that $|H\langle x\rangle^H:(K\cap H)(\langle x\rangle^H\cap K)|<\infty$ so that $(K\cap H)(\langle x\rangle^H\cap K)$ is finitely generated since $H\langle x\rangle^H$ is finitely generated. But $\langle x\rangle^H\cap K\subset\langle x\rangle^G\cap K\subset N(K)=\mathcal{S}(K)$ so that by Lemma 3.16 $(K\cap H)(\langle x\rangle^H\cap K)$ is not finitely generated since $K\cap H$ is not finitely generated. This gives our contradiction so that $x\in\mathcal{S}(G)$, and the proof is complete. □

An easy extension of Theorem 3.19 is that if all the subgroups of a normal subgroup L of finite index in G are S_N groups, then all the subgroups of G are S_N groups.

Corollary 3.20: If $K \triangleleft G$, $|G:K|<\infty$ and $L \in S_N$ for all $L \subset K$, then $\bar{L} \in S_N$ for all $\bar{L} \subset G$.

Proof: Let $\bar{L} \subset G$. Then $\bar{L} \cap K \triangleleft \bar{L}$ and $|\bar{L}:\bar{L} \cap K| = |K\bar{L}:K| \leq |G:K| < \infty$. By hypothesis, then, $\bar{L} \cap K \in S_N$. Since $|\bar{L}:\bar{L} \cap K| < \infty$, Theorem 3.19 gives $\bar{L} \in S_N$. □

CHAPTER IV

EXAMPLES

In this chapter we construct groups G whose nearly finitely generated subgroups H or whose subgroup $\mathcal{S}(G)$ possess remarkable properties. The primary examples are of groups that lie outside the class $\mathcal{S}_{\mathcal{N}}$.

A general strategy for constructing such groups is to find a finitely generated group G with a non-finitely generated subgroup H such that $\langle H,\mathcal{N}(G)\rangle=G$. If G is such a group, then there exists a nearly finitely generated subgroup $\overline{H}$ of G containing H, and $\overline{H}$ clearly cannot contain $\mathcal{N}(G)$, so that we must have $\mathcal{S}(G)\neq\mathcal{N}(G)$ (since in any group we always have that $\mathcal{S}(G)$ is contained in every nearly finitely generated subgroup, by Lemma 2.8.)

All our constructions are semi-direct products (split extensions) and all but one of these are wreath products. We recall here the structure of semi-direct products and wreath products. For details and proofs concerning semi-direct products see Scott ([11], section 9.2), and for wreath products see Neumann [9].

A group G is the <u>semi-direct product</u> of its subgroups H and K if and only if $H \triangleleft G$, $G=KH$, and $K \cap H=E$. In this case, every element of G has a unique representation as kh, with $k \in K$ and $h \in H$. Multiplication in G is given by $(k_1h_1)(k_2h_2)=(k_1k_2)(h_1^{k_2}h_2)$. If T_k denotes the inner automorphism of G induced by $k \in K$ (that is, $gT_k=g^k$ for all $g \in G$), the function T defined by $kT=T_k|H$ is a homomorphism from K into Aut(H). Multiplication in G now becomes

$(k_1h_1)(k_2h_2)=(k_1k_2)((h_1(k_2T))h_2)$. In this event, G is a <u>semi-direct product with</u> (associated) <u>homomorphism T</u>.

Conversely, suppose we are given two groups H and K and a homomorphism $T:K \rightarrow \mathrm{Aut}(H)$. Let $G=K \times H$ as a set, and define multiplication in G by

$(k_1,h_1)(k_2,h_2)=(k_1k_2,(h_1(k_2T))h_2)$. Under this multiplication, G is a group. Let $K^*=\{(k,e)|k \in K\}$, $H^*=\{(e,h)|h \in H\}$. Then K^* and H^* are subgroups of G. Define U and V by $hU=(e,h)$, $kV=(k,e)$, $h \in H$ and $k \in K$. U and V are isomorphisms of H onto H^* and K onto K^*, respectively. Let T^* be the homomorphism from K^* into $\mathrm{Aut}(H^*)$ that satisfies $(hU)((kV)T^*)=(h(kT))U$. Then G is the semi-direct product of H^* by K^* with homomorphism T^*.

Because of the natural correspondence between H and H^*, K and K^*, and T and T^*, we will say that G is the semi-direct product of H by K with homomorphism T.

In our examples, we will be interested in the case where K is a subgroup of Aut(H) and T is the embedding of K into Aut(H). Multiplication then becomes

$$(\alpha_1,h_1)(\alpha_2,h_2)=(\alpha_1\alpha_2,(h_1\alpha_2)h_2),\ h_1,h_2\epsilon H,\ \alpha_1,\alpha_2\epsilon K.$$

In this instance, G is a relative holomorph of H by K, written <u>G=Hol(H,K)</u>.

We identify (α,e) with α and (e,h) with h, $\alpha\epsilon K$, $h\epsilon H$. If we compute $[h,\alpha]$, then on the one hand we get $h^{-1}h^{\alpha}$ and on the other we get

$$\begin{aligned}[h,\alpha]&=h^{-1}\alpha^{-1}h\alpha=(e,h^{-1})(\alpha^{-1},e)(e,h)(\alpha,e)\\&=(\alpha^{-1},h^{-1}\alpha^{-1})(e,h)(\alpha,e)=(\alpha^{-1},(h^{-1}\alpha^{-1})h)(\alpha,e)\\&=(e,((h^{-1}\alpha^{-1})h)\alpha)=(e,h^{-1}(h\alpha))\\&=h^{-1}(h\alpha).\end{aligned}$$

Hence, we have that the action of α on h, $h\alpha$, is given by h^{α}.

We are now ready to define the <u>standard restricted wreath product</u>, <u>$A\wr B$</u>, of the group A by the group B. Let

$$K=\{f\,|\,f:B\to A \text{ and } f(b)\neq e \text{ for only finitely many } b\epsilon B.\}$$

Then K is a group under ordinary componentwise multiplication, and each element $b\epsilon B$ induces an automorphism $\bar{b}$ of K defined by

$$f^{\bar{b}}(b')=f(b'b^{-1}) \text{ for } f\epsilon K,\ b'\epsilon B.$$

If $\bar{B}=\{\bar{b}\,|\,b\epsilon B\}$, then $\bar{B}$ becomes a group under function composition $\bar{b}_1\bar{b}_2=\overline{b_1b_2}$, and $\bar{B}$ is clearly isomorphic to B under the correspondence $b \leftrightarrow \bar{b}$. Thus, we identify B with $\bar{B}$ and consider B to be a subgroup of Aut(K). Finally, we define $A \wr B$ by

$\underline{A \wr B}=\underline{\mathrm{Hol}(K,B)}$.

It is customary to call K the <u>base group</u> of the wreath product and B the <u>top group</u>. Thus, $A \wr B=KB$, with $K \lhd A \wr B$, and $K \cap B=E$.

For each $b \in B$, there corresponds a subgroup K_b of K isomorphic to A. K_b is defined by

$K_b=\{ f \in K \mid f(x)=e \text{ if } x \neq b \}$.

Thus, K is isomorphic to the direct sum of $|B|$ copies of A. We identify A with the subgroup K_e of K, where e is the identity in B. Hence $A \wr B=\langle K_b,B\rangle$ for each $b \in B$ and $A \wr B$ is finitely generated if and only if A and B are finitely generated.

<u>Theorem</u> 4.1: $\mathcal{S}_N$ is a proper subclass of the class of all groups.

<u>Proof</u>: Let B be any finitely generated group, $B \notin \mathrm{Max}$, and let C be nearly finitely generated in B. Then $|B:C|=\infty$ because C is not finitely generated. Let $G=Z_2 \wr B$. We show that $G \notin \mathcal{S}_N$. Let K be the base group of the wreath product and let

$M = \{ f \in K \mid f$ has an even number of non-zero entries on each right coset of C in $B \}$.

M is clearly a subgroup of K, and B normalizes M since the members of B simply permute cosets of C in B. Furthermore, M is not finitely generated, for, if $\{x_1, \ldots, x_n, \ldots\}$ is a complete set of coset representatives for the right cosets of C in B, with $x_1 = e$, and if

$$M_i = \{ f \in M \mid f(x) = 0 \text{ for all } x \in Cx_j,\ j > i \},$$

then $M_1 < M_2 < \ldots < M_n < \ldots$ and $M = \bigcup_i M_i$, so that M is a union of a properly ascending chain of subgroups and thus cannot be finitely generated.

Moreover, MB is not finitely generated, for, suppose to the contrary that $MB = \langle f_1, \ldots, f_n, b_1, \ldots, b_m \rangle$ with $f_1, \ldots, f_n \in M$ and $b_1, \ldots, b_m \in B$. Now without loss of generality, we may assume that $f_i \in M_1$ for each i since $M \subset \langle M_1, B \rangle$. For each $c \in C, c \neq e$, let let $f_c \in M_1$ be given by $f_c(x) = \begin{cases} 1 \text{ if } x = c \text{ or } x = e \\ 0 \text{ otherwise} \end{cases}$. Then without loss of generality, $f_i = f_{c_i}$ for some $c_i \in C$, $i = 1, \ldots, n$, for, if $f_i(x) = \begin{cases} 1 \text{ if } x \in \{ x_{i_1}, \ldots, x_{i_{2n}} \} \\ 0 \text{ otherwise} \end{cases}$ (with the convention that $x_{i_1} = 0$ if $0 \in \{ x_{i_1}, \ldots, x_{i_{2n}} \}$), then

$$f_i = \begin{cases} f_{x_{i_1}} \cdot f_{x_{i_2}} \cdot \cdots \cdot f_{x_{i_{2n}}} & \text{if } x_{i_1} \neq 0 \\ f_{x_{i_2}} \cdot \cdots \cdot f_{x_{i_{2n}}} & \text{if } x_{i_1} = 0 \end{cases}.$$

Now if $\bar{C}=\langle c_1,\ldots,c_n\rangle$ and $c\in C-\bar{C}$, then we claim that $f_c\notin\langle f_1,\ldots,f_n,b_1,\ldots,b_m\rangle$. For, if $b\in B$, then $f_{c_i}^b$ has 2 non-zero entries on the coset of $\bar{C}$ in B containing b so that the product of conjugates of the f_{c_i} has an even number of non-zero entries on each coset of $\bar{C}$ in B. Hence, $f_c\notin\langle f_1,\ldots,f_n,b_1,\ldots,b_m\rangle$ so that MB is not finitely generated.

Now, $\langle K,MB\rangle=\langle K,B\rangle=G$ and $K\subset N(G)$ so that $G=\langle N(G),MB\rangle$ with MB not finitely generated, so that $S(G)\neq N(G)$ by the introductory remarks to this chapter. Hence, $G\notin S_N$. □

Theorem 3.11 says that $A\circ$Nilp is contained in S_N. Our next example shows that the containment does not hold for Nilp$\circ A$. In fact, we construct a group G which is an Abelian extension of a free nilpotent group of class 2, but is not an S_N group. G, of course, is an extension of an S_N group by an S_N group, and thus $S_N\circ S_N\notin S_N$. (The same conclusion could be drawn from the construction in Theorem 4.1.)

If we define the S-series of a group G by (i) $S_0=E$, and (ii) S_{i+1} is that subgroup of G such that $S_{i+1}/S_i=S(G/S_i)$, $i\geq 0$, and if we define the N-series by (i) $N_0=E$, and (ii) N_{i+1} is that subgroup of G such that $N_{i+1}/N_i=N(G/N_i)$,

$i \geq 0$, then the G in our next example has the property that $S_2 < S_3 = G$ while $N_1 < N_2 = G$. Thus, the S and N series may both reach G though not necessarily in the same number of steps.

Before proceeding with the construction, we recall some definitions and a few of the results that we need pertaining to free, free nilpotent, and free Abelian groups. Details may be found in [8].

If F is free on the set $S = \{ \ldots, x_{-1}, x_0, x_1, \ldots \}$ and w is a word (that is, $w \in F$), then w is a law in the group G if the only possible value of w in G is e, the identity. By this we mean that for every homomorphism $\alpha : F \to G$ $w\alpha = e$. (For example, $[x_i, x_j]$ is a law in every Abelian group.)

A group K is free Abelian if K is isomorphic to F/F' for some free group F. A group G is free nilpotent if G is isomorphic to $F/\Gamma^n(F)$ for some free group F and integer $n \geq 1$. Every Abelian group is a homomorphic image of a free Abelian group and every nilpotent group of class n is a homomorphic image of a free nilpotent group of class n. If G is a free nilpotent group of class 2, then $G' = Z(G)$, and if G is any free nilpotent group and $T \subset G$ is a subset containing at least 2 elements, the T freely generates a free nilpotent group if and only if T is linearly independent modulo G' (see [8], p. 121.)

<u>Theorem</u> 4.2: $\mathrm{Nilp}\circ A \not\subset S_N$, $S_N \circ S_N \not\subset S_N$.

<u>Proof</u>: Let $X=\{\ldots,x_{-1},x_0,x_1,\ldots\}$ be a countably infinite set indexed by the integers and $F=F(X)$ be the free group on X. Consider the group $G=\langle K,\alpha\rangle$ where $K=F/\Gamma^3(F)$ is the free nilpotent group of class 2 on X and α is the Neumann shift operator defined by $\alpha: x_i\Gamma^3(F) \to x_{i+1}\Gamma^3(F)$. (To simplify the notation we will omit the $\Gamma^3(F)$ in writing members of K.) Now $\alpha\in \mathrm{Aut}(K)$ and $G=\mathrm{Hol}(K,\langle\alpha\rangle)$, so that G is a split extension of K by $\langle\alpha\rangle$. Hence, $G=K\langle\alpha\rangle\in\mathrm{Nilp}\circ A$, whence $K\subset N(G)$. Note also that $\langle K,\alpha\rangle=\langle x_0,\alpha\rangle$, which says that G is finitely generated. To show that $G\notin S_N$ we need only find a non-finitely generated subgroup H of G containing α, for then $G=\langle K,\alpha\rangle = \langle N(G),H\rangle$ and $S(G)\neq N(G)$ by our comments at the beginning of this chapter. To this end consider the subgroup $H=\langle K',\alpha\rangle$.

Now $K'=Z(K)$ and K' is free Abelian, freely generated by the set $\overline{X}=\{[x_i,x_j]\mid i<j\}$, since (i) $\overline{X}$ surely generates K', and (ii) if $\sum_k n_k[x_{i_k},x_{j_k}]=0$, then every nilpotent group of class 2 on, say, the 2 generators x_{i_1} and x_{j_1} satisfies the law $[x_{i_1},x_{j_1}]^{n_1}$, which is certainly not the case. Suppose now that $\langle K',\alpha\rangle$ is finitely generated. Without loss of generality, then, $\langle K',\alpha\rangle=\langle [x_0,x_{i_1}],\ldots,[x_0,x_{i_n}],\alpha\rangle$, $x_{i_j}>0$, $j=1,\ldots,n$, so that

$$K'=K'\cap\langle K',\alpha\rangle=\langle [x_i,x_j]\mid j-i\in\{i_1,\ldots,i_n\}\rangle \ \langle K', \text{ a}$$

contradiction.

Thus far we have shown that $\mathcal{S}(G)\neq\mathcal{N}(G)$. To prove that $\mathcal{N}(G)=K$ it remains only to show the inclusion $\mathcal{N}(G)\subset K$. To obtain this, we observe that if $M>K$, then $\alpha^m\in M$ for some integer $m\neq 0$ (without loss of generality $m>0$). Hence $\langle K,\alpha^m\rangle=$

$$\langle x_0,x_1,\ldots,x_{m-1},\alpha^m\rangle \text{ is finitely generated and}$$

$|G:\langle K,\alpha^m\rangle|=m$ so that $|G:M|\leq m$ and thus M is finitely generated since G is. Since $K<M$ $M\notin\mathcal{L}(\mathrm{Max})$ so that $\mathcal{N}(G)=K$.

To show that $Z(K)\subset\mathcal{S}(G)$, suppose to the contrary that $z\in Z(K)$, $H\subset G$ is not finitely generated, but $\langle H,z\rangle$ is finitely generated. Then $H\not\subset K$ (since $K\in\mathcal{L}(\mathrm{Max})$), $\langle H,z\rangle=H\langle z\rangle^H$ and

$$\frac{\langle H,z\rangle}{\langle z\rangle^H}=\frac{H\langle z\rangle^H}{\langle z\rangle^H}\simeq\frac{H}{\langle z\rangle^H\cap H} \quad \text{is finitely generated so that}$$

$\langle z\rangle^H\cap H$ is not finitely generated since H is not finitely generated. Thus, $H=\langle\,\langle z\rangle^H\cap H,h_1,\ldots,h_n\rangle$ for some $h_1,\ldots,h_n\in H$. If $h\in H$, then $h=\alpha^m k$ for some integer m, $k\in K$, and

$$z^h=z^{\alpha^m k}=(z^{\alpha^m})^k=z^{\alpha^m} \quad \text{since } z^{\alpha^m}\in Z(K).$$

Hence, there exists an infinite subset $I\subset Z$ such that

$\langle z\rangle^H\cap H=\langle z^{\alpha^i}\mid i\in I\rangle\subset Z(K)$ and $h_i=\alpha^{m_i}k_i$ for some $m_i\in Z$, $k_i\in K$, $1\leq i\leq n$. Furthermore, $m_i\neq 0$, and without loss of generality $m_1>0$. But then,

$$\langle\,\langle z\rangle^H\cap H,h_1\rangle=\langle z^{\alpha^i},\alpha^{m_1}k_1\mid i\in I\rangle \text{ is finitely generated;}$$

for, if we choose a subset $I_{m_1} \subset I$ that contains a unique representative of every coset of $\langle m_1 \rangle$ in Z that is found in I (that is, for each $i \in I$, $i \equiv j \bmod m_1$ for some $j \in I_{m_1}$, and $j \not\equiv \bar{j} \bmod m_1$ if $j \neq \bar{j}$, $j, \bar{j} \in I_{m_1}$), then

$$\langle\langle z\rangle^H \cap H, h_1\rangle = \langle z^{\alpha^j}, \alpha^{m_1}k_1 \mid j \in I_{m_1}\rangle$$ is finitely generated

since I_{m_1} contains at most m_1 members. Hence,

$$H = \langle\langle z\rangle^H \cap H, h_1, \ldots, h_n\rangle = \langle z^{\alpha^j}, h_1, \ldots, h_n \mid j \in I_{m_1}\rangle$$ is finitely generated, a contradiction. Hence, $z \in S(G)$.

Suppose finally that $x \in K - Z(K)$. We show that $x \notin S(G)$. The construction follows the lines of the early part of this proof. Let $L = \langle x^{\alpha^m} \mid m \in Z\rangle$. Then $L \subset K$ and L is a free nilpotent group of class 2 freely generated by the x^{α^m}, $m \in Z$, since $\{x^{\alpha^m} \mid m \in Z\}$ is clearly linearly independent modulo K'. For convenience, let $y_i = x^{\alpha^i}$ for each i. Then $L = \langle \ldots, y_0, y_1, \ldots\rangle$, $L \triangleleft \langle L, \alpha\rangle$, and α simply permutes the y_i in the same manner as the x_i. As earlier in the proof, we may conclude that $\langle L', \alpha\rangle$ is not finitely generated, but

$\langle\langle L', \alpha\rangle, x\rangle = \langle L', \alpha, y_0\rangle = \langle y_0, \alpha\rangle$ is finitely generated so that $x \notin S(G)$.

The lengths of the $\mathcal{S}$ and $\mathcal{N}$ series may easily be found. $\mathcal{N}(G)=K$ and $G/K\cong\langle\alpha\rangle$ so that $\mathcal{N}(G/K)=G/K$ and $\mathcal{N}_2=G$. $\mathcal{S}(G)=Z(K)=K'$ so that $G/\mathcal{S}(G)=G/K'\in\mathcal{A}\circ\mathcal{A}$. But then, $\mathcal{S}(G/\mathcal{S}(G))=\mathcal{N}(G/\mathcal{S}(G))=\mathcal{N}(G/K')=K/K'$ (by Theorem 3.11.) Therefore, $\mathcal{S}_2=K$ and $\mathcal{S}(G/\mathcal{S}_2)=\mathcal{S}(G/K)=G/K$ since $G/K\in\mathcal{A}$. Hence $\mathcal{S}_3=G$.

In the introductory comments to Chapter III we remarked that G in this example has the property that all the normal Abelian subgroups of G are contained in $\mathcal{I}(G)$, yet $\mathcal{N}(G)\not\subset\mathcal{I}(G)$. We have already seen that $\mathcal{N}(G)\not\subset\mathcal{I}(G)$, since $\langle K',\alpha\rangle$ is contained in some nearly finitely generated subgroup H of G, and $\mathcal{N}(G)\not\subset H$ since $\langle\mathcal{N}(G),H\rangle\supset\langle\mathcal{N}(G),\alpha\rangle=G$. To see that all the normal Abelian subgroups of G are contained in $\mathcal{I}(G)$ we show that if $A\lhd G$, A Abelian, then $A\subset Z(K)=\mathcal{S}(G)\subset\mathcal{I}(G)$, by Lemma 2.8.

Suppose $x=(\alpha^r,k)\in A$, r an integer, $k\in K$, and let $y=x^{\alpha^s}$, s an integer. A straight-forward computation shows that $[x,y]=k^{-1}(k^{-1}\alpha^{s+r})(k\alpha^r)(k\alpha^s)$. The hypothesis that A is a normal Abelian subgroup of G requires that $[x,y]=e$, which is equivalent to $(k\alpha^r)(k\alpha^s)=(k\alpha^{s+r})k$. Since α is the Neumann shift operator, the only way this equality can hold if $s\neq 0$ $(x\neq y)$ is for $r=0$. Hence, we must have $x=k$ and $k(k\alpha^s)=(k\alpha^s)k$. This latter equality holds only if $k\in Z(K)=K'=\mathcal{S}(G)$. Since k is arbitrary, $A\subset Z(K)=\mathcal{S}(G)$, as required. □

Our next example shows how to construct a class of groups having nearly finitely generated subgroups that are not normal. However, we first need some preliminary information about certain subgroups of $Z_2 \wr Z$.

<u>Definition</u> 4.3: If $f \in K \subset A \wr Z = KZ$, then <u>f is on [r,s]</u>, r and s integers, if $f(z^m)=e$ for all m not between r and s inclusive.

<u>Lemma</u> 4.4: If $G=Z_2 \wr Z=KZ$ and $H>Z$, then $H \simeq Z_2 \wr Z$ and $|G:H|<\infty$, and if $L>K$, then L is finitely generated.

<u>Proof</u>: Since $H>Z$, $H \cap K>E$, and since Z may be used to shift the interval that a function is on, as per the previous definition, we choose $f \in K$ such that f is on [0,n], $n>0$, with n as small as possible. We claim that $H=\langle f,z \rangle$, for suppose that g is a function in H, g is on [k,r]. Then, by multiplying g by appropriate conjugates of f (by powers of z), we may reduce the length of the interval that g is on until we finally arrive at a function $\bar{g}$ that is on [0,n]. But $\bar{g}$ must be f since if it were not, then $\bar{g}f$ would be a non-trivial function on $[0,\bar{n}]$ for some $0<\bar{n}<n$, which contradicts our choice of f. Hence, we have that $\bar{g}$ and $g^{-1}\bar{g} \in \langle f,z \rangle$ so that $g \in \langle f,z \rangle$ and $H=\langle f,z \rangle$. Incidentally, we have also shown that f is unique in H with respect to being on [0,n].

Let $g \in K$ be the function defined by $g(z^n)=\begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$.

Then $G=\langle g,z\rangle$. The mapping $(z^k, g^{z^{n_1}} \ldots g^{z^{n_r}}) \to (z^k, f^{z^{n_1}} \ldots f^{z^{n_r}})$ is easily checked to be an isomorphism of $G=Z_2 \wr Z$ onto $H=\langle f,z\rangle$. Hence, $H \simeq Z_2 \wr Z$.

To show that $|G:H|<\infty$, let $y\in G$. Then $Hy=Hgz^r$ for some $g\in K$, r an integer, so $Hy=Hz^{-r}gz^r=H\bar{g}$, $\bar{g}\in K$, since $z\in H$. As we argued before, we may multiply $\bar{g}$ by appropriate conjugates of f, all of which are in H, until we get a function $\bar{g}'$, and $\bar{g}'$ is on $[0,n]$. Thus, $Hy=H\bar{g}'$, with $\bar{g}'$ on $[0,n]$. Since the number of functions in K on $[0,n]$ is finite we have shown that H has only a finite number of right cosets in G so that $|G:H|<\infty$.

Finally, if $L>K$, then $L=K\langle z^s\rangle$ for some positive integer s and $L \simeq (\underbrace{Z_2+\ldots+Z_2}_{s\text{-copies}}) \wr Z$ by [9, Lemma 8.1]. Thus L is finitely generated. $\square$

<u>Theorem</u> 4.5: If A is finitely generated and $G=A\wr(Z_2\wr Z)=K(Z_2\wr Z)$, then KZ is a non-normal nearly finitely generated subgroup of G.

<u>Proof</u>: G is clearly finitely generated since A and $Z_2\wr Z$ are finitely generated. Suppose $\bar{H}>H=KZ$. Then

$Z_2\wr Z \simeq \frac{G}{K} \supset \frac{\bar{H}}{K} > \frac{H}{K} \simeq Z$. Hence, $\frac{\bar{H}}{K}$ is isomorphic to a subgroup of $Z_2 \wr Z$ that properly contains the top group Z. Thus, by Lemma 4.4, $|G/K:\overline{H}/K|<\infty$ so that $|G:\overline{H}|<\infty$ and $\overline{H}$ is finitely generated since G is finitely generated.

That H is not normal in G follows from the fact that the normal closure of Z in $Z_2 \wr Z$ is larger than Z, so that the normal closure of H in G is greater than H.

If N is the base group in $Z_2 \wr Z$ and $\overline{K}=\{f\in K \mid f(x)=e \text{ if } x\notin N\}$, then certainly $K\subset\langle\overline{K},Z\rangle$ so that if $H=KZ$ is finitely generated, say $H=\langle f_1,\ldots,f_n,z\rangle$, $f_1,\ldots,f_n\in K$, then without loss of generality, $f_1,\ldots,f_n\in\overline{K}$. But then,

$\overline{K}=\langle f_1,\ldots,f_n,z\rangle\cap\overline{K}=\langle f_1,\ldots,f_n\rangle$. This says that $\overline{K}$ is finitely generated, which is impossible since N is not finitely generated. Hence, H is not finitely generated, and the proof is complete.□

Our next example shows that it is not the case that the intersection $K\cap H$ of a nearly finitely generated subgroup H of G with a normal subgroup K of finite index in G need be nearly finitely generated in K. First we need a lemma concerning the subgroups of $Z\wr Z$ that contain the top group.

Lemma 4.6: If H contains the top group in $Z\wr Z$, then H is finitely generated.

Proof: If K is the base group in $Z\wr Z$, then $H=(H\cap K)Z$ and $H\cap K$ is normal in $Z\wr Z$. $Z\wr Z$ is a finitely generated Abelian extension of an Abelian group, so $Z\wr Z$ satisfies Max-n by Theorem 3.5. Hence $H\cap K$ is finitely normally generated so that $H=(H\cap K)Z$ is finitely generated.$\square$

Example 4.7: There exists a group G containing subgroups H and K, with $K\triangleleft G$, $|G:K|<\infty$ and H nearly finitely generated in G, but $H\cap K$ is not nearly finitely generated in K.

Construction: Let $G=A\wr Z$ where A is any finitely generated group containing a nearly finitely generated subgroup B. Let us identify B with the subgroup of the base group K consisting of all functions f satisfying

$$f(z^n)=\begin{cases} e \text{ if } n\neq 0 \\ b\in B \text{ if } n=0 \end{cases}.$$ If $L=\{f\in K \mid f(z^n)\in B \text{ for all } n\}$, then

$\langle B,Z\rangle=LZ\simeq B\wr Z$, Z normalizes L and LZ is certainly not finitely generated since B is not finitely generated. Consider the subgroup $K\langle z^2\rangle$. Clearly $|G:K\langle z^2\rangle|=2$, so $K\langle z^2\rangle \triangleleft G$, and $LZ\cap(K\langle z^2\rangle)=L\langle z^2\rangle$.

$L\langle z^2\rangle$ is not nearly finitely generated in $K\langle z^2\rangle$, for consider the subgroup M of G defined by

$$M=\{f\in K \mid f(z^{2n})\in A,\ f(z^{2n+1})\in B,\ n \text{ an integer}\}.$$ Then $\langle z^2\rangle$

normalizes M, $L\langle z^2\rangle<M\langle z^2\rangle$ and $M\langle z^2\rangle$ is not finitely generated since B is not finitely generated.

Thus, it remains only to show the existence of an A such that LZ is nearly finitely generated in G. To this end, consider $A=\overline{Z}_2 \wr \overline{Z}=\overline{KZ}$. $\overline{K}$ is not finitely generated but $\overline{K}$ is nearly finitely generated in A by Lemma 4.4 Now, we have

$$G=(\overline{Z}_2 \wr \overline{Z}) \wr Z=(\overline{Z}_2 \wr \overline{Z})^Z Z=(\overline{KZ})^Z Z \simeq (\overline{K}^Z \overline{Z}^Z) Z, \quad L=\overline{K}^Z \text{ and } LZ \simeq \overline{K}^Z Z.$$

Also, $\overline{K}^Z \triangleleft G$ since $\overline{Z}^Z$, as well as Z, normalizes $\overline{K}^Z$. Now,

$$\frac{G}{\overline{K}^Z} = \frac{(\overline{KZ})^Z Z}{\overline{K}^Z} \simeq \frac{(\overline{K}^Z \overline{Z}^Z) Z}{\overline{K}^Z} \simeq \overline{Z}^Z Z = \overline{Z} \wr Z.$$ Thus, if $H > LZ \simeq \overline{K}^Z Z$, then $\frac{H}{\overline{K}^Z}$ is isomorphic to a subgroup of $\overline{Z} \wr Z$ containing the top group, Z, and so is finitely generated by Lemma 4.6. So, we have that $H=\langle \overline{K}^Z Z, g_1, \ldots, g_n \rangle = \langle \overline{K}^Z, z, g_1, \ldots, g_n \rangle$ for some $g_1, \ldots, g_n \in \overline{Z}^Z$, and without loss of generality, we may assume that $g_i(1) \neq e$, for all i, since we may conjugate the g_i by powers of z, if necessary, to effect this condition. Furthermore, $\overline{K}^Z Z=\langle \overline{K}_1^Z, z \rangle$ so that $H=\langle \overline{K}_1^Z, z, g_1, \ldots, g_n \rangle$. Now $\langle \overline{K}_1^Z, g_1 \rangle$ is finitely generated since $g_1(1) \neq e$ so that $\langle \overline{K}_1^Z, g_1 \rangle$ is isomorphic to a subgroup of $\overline{Z}_2 \wr \overline{Z}=\overline{KZ}$ containing the base group $\overline{K}$ and thus is finitely generated by Lemma 4.4. Hence, H is finitely generated, and the proof is complete. □

Our last example is a counterexample to a conjecture made by J.E. Roseblade, in discussion. The conjecture was that for polycyclic extensions of Abelian groups, G, $\mathcal{S}(G)=\mathcal{N}(G)=$ Fitt(G), where Fitt(G), the <u>Fitting subgroup</u> of G, is the maximum normal nilpotent subgroup of G (which exists for any Max-n group.) We have already seen that the first equality in the assertion is true (Theorem 3.10), but the following example shows that Fitt(G) can be a proper subgroup of $\mathcal{N}(G)$ if $G \in \mathcal{A} \circ \mathcal{P}$.

<u>Example</u> 4.8: In $G = Z_p \wr (Z_q \times Z)$, where p and q are distinct primes, Fitt(G)=K, the base group, while $\mathcal{S}(G) = \mathcal{N}(G) = KZ_q$.

<u>Proof</u>: KZ_q is clearly normal in G and locally Nötherian, but not finitely generated. If $H > KZ_q$, then $|G:H| < \infty$ so that H is finitely generated. Thus H cannot be locally Nötherian, and it follows that $\mathcal{N}(G) = KZ_q$.

We recall that in $A \wr B = KB$, if $B_1 \subset B$, then KB_1 is isomorphic to the wreath product of a direct sum of a certain number of copies of K with B_1 ([9, Lemma 8.1]), and that $A \wr B$ is nilpotent if and only if A and B are p-groups for the same prime p, B is finite and A is of finite exponent (see [3].)

Now K is a p-group and if $x \in Z_q \times Z$, then x is either of infinite order or of order q. Thus $K\langle x\rangle$ cannot be nilpotent since $p \neq q$. Hence K, which is Abelian, is the Fitting subgroup of G. □

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