

BANACH SPACE VALUED STATIONARY
STOCHASTIC PROCESSES AND
FACTORIZATION OF NONNEGATIVE
OPERATOR VALUED FUNCTIONS ON
A BANACH SPACE

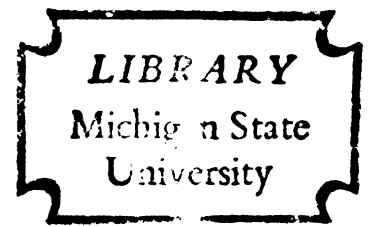
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Miamee Abolghasem

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ABSTRACT

BANACH SPACE VALUED STATIONARY STOCHASTIC PROCESSES AND FACTORIZATION OF NONNEGATIVE OPERATOR VALUED FUNCTIONS ON A BANACH SPACE

By

A.G. Miamee

In this thesis the theory of Banach space valued stationary stochastic processes and the problem of factorization of nonnegative operator valued functions are studied. The the thesis consists of eight chapters and one appendix.

Chapters I and II are introductory. In Chapter III, Banach space valued stationary stochastic processes are systematically studied. The results, such as Wold's decomposition, Cramér's decomposition, Wold-Cramér concordance theorem, etc., which are fundamental in this area are established. These include the extension to the Banach space of most of the results of R. Gangolli.

In Chapter IV the factorization problem of Banach space valued stationary stochastic processes which plays an important role in the prediction theory of Banach space valued stationary processes, is considered. Several theorems concerning this factorization are given. These involve the analysis of quasi square roots and their corresponding invariant subspaces. Continuing our study of the factorization problem, in Chapter V several necessary and sufficient conditions for factorability of these functions are given. The works of Chapters IV and V extend

to the Banach space case, most of the result of R.G. Douglas and the recent work of Yu. A. Rozanov as well as a certain result of R. Payen on factoring a nonnegative operator valued function on a Hilbert space.

Let f be a factorable nonnegative Hilbert space operator valued function, and let U be a unitary valued function. A natural question is to see if the nonnegative operator valued function UfU^* is factorable. This problem is investigated in Chapter VI. As an application of this study some results, such as a Devinatz's type necessary condition and characterization for the factorization problem are given.

In Chapter VII the important problem of finding a computable algorithm for finding the optimal factor and the linear predictor of a stochastic process is considered. An algorithm similar to the one given by N. Weiner and P. Masani for the infinite dimensional process is obtained. This involves the Fourier analysis of infinite dimensional matrix valued functions.

In Chapter VIII the problem of minimality and interpolation of infinite dimensional stationary processes is studied. Most of the results of H. Salehi for multivariate case are extended to infinite dimensional case. Also a well known result of P. Masani on minimal multivariate processes is extended to the infinite dimensional case. In the appendix the construction of quasi square roots of several operators is given.

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FACTORIZATION OF NONNEGATIVE OPERATOR VALUED
FUNCTIONS ON A BANACH SPACE**

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A THESIS

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in partial fulfillment of the requirements
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Department of Mathematics

1973

86549

TO MY WIFE EFFY

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CHAPTER .I

INTRODUCTION

The idea of Banach space valued stationary stochastic processes has been recently introduced by S.A. Chobanian in [2]. Subsequently some basic results concerning these processes were announced [24], [3]. In the Hilbert space case the basic questions of regularity, Wold's decomposition, Wold-Cramér concordance, factorability of spectral density, etc. have been studied in detail [4], [8], [10], [12], [13], [16], [18], [19]. However in the Banach space case the study of stationary stochastic processes and the related problems are in its early stages, and the results obtained in this direction are not as yet complete. In particular the important problem of factoring a nonnegative operator valued function on a Banach space has not been investigated. The problem of determining the optimal factor of a spectral density plays an important role in the prediction theory of stationary stochastic processes. This problem was tackled by Wiener and Masani [26] and later on by Masani [14] for the finite dimensional case. This problem remains open for the infinite dimensional processes.

In this thesis we first study Banach space valued stationary stochastic processes and prove some known results as well as several new results. We then consider the question of factorability for the Banach space case and establish several criteria for the

factorization problem. In particular we obtain several comparison type sufficient conditions and some analytic necessary and sufficient conditions for the factorization problem. In the second part of this thesis we provide an algorithm for finding the optimal factor and the linear predictor for the Hilbert space case. We also study the problem of minimality and interpolation of Hilbert space valued stationary stochastic processes. With this background we now summarize the content of each chapter in more detail.

In Chapter II we recall some notations and terminologies from [2] concerning Banach space valued stationary stochastic processes. We also state some facts [1], [24] regarding these processes which are needed in the later chapters.

In the first part of Chapter III we study Banach space valued stationary stochastic processes. Using a new technique (to be made clear later) we will provide proofs for Wold's decomposition, relation between regularity and factorization which were announced in [24], [3]. We also prove several new results such as a time domain and a spectral domain decomposition as well as moving average representation for these processes. In the second part of Chapter III the idea of subprocesses is introduced and most of the results of R. Gangolli [8] are extended to the Banach space case. In particular a Wold-Cramér concordance theorem for the Banach space valued stationary stochastic processes as well as some sufficient condition for the factorization problem are obtained.

In Chapter IV we consider the problem of factoring a non-negative operator valued function f on a Banach space in the form

$\Phi^* \Phi$, where Φ is a conjugate analytic operator valued function.

We give several comparison type sufficient conditions for factorization problem by extending to the Banach space case most of the results of R.G. Douglas [4]. In the Hilbert space case, $\sqrt{f}$, the positive square root of f whose existence is known is used frequently. When f is a positive operator valued function on a Banach space X the existence of a square root in the ordinary way does not make sense. Nevertheless we will prove (c.f. Lemma 4.3.1) the existence of a measurable function Q on X into some auxiliary Hilbert space which behaves almost like a square root in the sense that $f = Q^* Q$. We will call this a quasi square root. The quasi square root will play the role of square root in this work.

The results of Chapters III and IV provide only sufficient conditions for the factorization problem. In Chapter V we establish several necessary and sufficient conditions for the factorability of a nonnegative operator valued function on a Banach space. In particular our main theorem of this chapter (Theorem 5.3.8) extends to the Banach space case the recent work of Yu. A. Rozanov [19] and a certain result of R. Payen [18] on the factorization problem. The notion of quasi square root is basic in this chapter.

In Chapter VI we study the following natural question raised by M.G. Nadkarni in [16]. Given a factorable nonnegative operator valued function f on a Hilbert space, to see if the nonnegative operator valued function UfU^* is factorable, where U denotes a measurable unitary valued function. We apply these results to prove some well known facts as well as some new results regarding the factorization problem for the Hilbert space case.

In Chapter VII we consider the important problem of finding an algorithm for determining the optimal factor and the linear predictor of a Hilbert space valued stationary stochastic process. In this chapter we will adopt the notations of [16] and employ the technique of [14] in order to establish our algorithm.

In Chapter VIII we investigate the problems interpolation and minimality of a Hilbert space valued stationary stochastic process. We extend most of the results of H. Salehi [21], [22], [23] to the infinite dimensional case. Using Salehi's technique we prove infinite dimensional extensions of a result due to Masani on minimal full rank processes.

Finally in the Appendix we give the construction of a quasi square root for a particular nonnegative operator valued function on a Banach space.

CHAPTER II

PRELIMINARIES

In this chapter we introduce some basic terminologies and state some known facts which will be needed in the latter chapters.

2.1 Notation. The script letters $\mathcal{X}$ and $\mathcal{Y}$ will denote Banach spaces and the script letters $\mathcal{H}$ and $\mathcal{K}$ will stand for Hilbert spaces. If $\mathcal{X}$ is a Banach space, $\mathcal{X}^*$ will denote the Banach space of all conjugate linear functionals on $\mathcal{X}$. For any two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $B(\mathcal{X}, \mathcal{Y})$ will stand for the Banach space of all bounded linear operators on $\mathcal{X}$ into $\mathcal{Y}$.

In this work all the Banach spaces are assumed to be separable.

2.2 Definition. An operator f in $B(\mathcal{X}, \mathcal{X}^*)$ is said to be non-negative if for each $x \in \mathcal{X}$, $(fx)(x) \geq 0$. $B^+(\mathcal{X}, \mathcal{X}^*)$ will denote the class of all such operators.

2.3 Definition. Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a Hilbert space. A sequence ξ_n , $-\infty < n < \infty$ of elements of $B(\mathcal{X}, \mathcal{H})$ is called a $B(\mathcal{X}, \mathcal{H})$ -valued stationary stochastic process (SSP) if $\xi_n^* \xi_m$ depends only on $m-n$. The operators $R(m-n) = \xi_n^* \xi_m$ is called the covariance operators of the process.

The following theorem is proved in [1].

2.4 Theorem. Let $R(n)$, $-\infty < n < \infty$ be a sequence of operators on $\mathcal{X}$ into $\mathcal{X}^*$. Then $R(n)$, $-\infty < n < \infty$ is the covariance

operators of some SSP ξ_n , $-\infty < n < \infty$ if and only if it can be represented as

$$R(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} F(d\theta),$$

where F is a $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued measure and the integral is in the weak sense. In this case F is called the spectral distribution of the process ξ_n , $-\infty < n < \infty$. In case that F is a.c. with respect to (w.r.t.) the Lebesgue measure, its derivative f is called the spectral density of the process.

If $\mathcal{B}$ is a subset of some Hilbert space $\mathcal{X}$ we will denote by $\mathcal{G}\{\mathcal{B}\}$ the smallest closed subspace of $\mathcal{X}$ containing $\mathcal{B}$.

Let us give the following definition.

2.5 Definition. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP.

Then we need to define the following subspaces

$$H_\xi(\infty) = \mathcal{G}\{\xi_k x, -\infty < k < \infty, x \in \mathcal{X}\}$$

$$H_\xi(n) = \mathcal{G}\{\xi_k x, -\infty < k \leq n, x \in \mathcal{X}\}$$

$$H_\xi(-\infty) = \bigcap_n H_\xi(n).$$

When there is no danger of confusion we will omit the index ξ in the above definition.

The following definition is basic in the theory of stationary stochastic processes.

2.6 Definition. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP.

Then ξ_n , $-\infty < n < \infty$ is called

- (i) Deterministic (or singular) if $H(-\infty) = H(n)$, for all n .
- (ii) Nondeterministic if $H(-\infty) \neq H(n)$ for some n .
- (iii) Purely nondeterministic (or regular) if $H(-\infty) = 0$.

2.7 Definition. Let $\mathcal{H}$ be a separable Hilbert space and let $L_2(\mathcal{H})$ denote the Hilbert space of all $\mathcal{H}$ -valued functions on the unit circle which have a square summable norm. The $L_2(\mathcal{H})$ inner product of two functions g_1 and g_2 is given by

$$\frac{1}{2\pi} \int_0^{2\pi} (g_1(e^{i\theta}), g_2(e^{i\theta})) d\theta.$$

The subspace $L_2^{0-}(\mathcal{H})$ ($L_2^{0+}(\mathcal{H})$) consists of all functions g in $L_2(\mathcal{H})$ for which $\int_0^{2\pi} g(e^{i\theta}) e^{in\theta} d\theta = 0$ for all $n < 0$ ($n > 0$).

2.8 Definition. A weakly measurable $B(\mathcal{X}, \mathcal{H})$ -valued function

$A = A(e^{i\theta})$ is called analytic (conjugate analytic) if for each $x \in \mathcal{X}$, $A(e^{i\theta})x \in L_2^{0+}(\mathcal{H})$ ($A(e^{i\theta})x \in L_2^{0-}(\mathcal{H})$).

2.9 Definition. Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. We say that f is factorable if there exists a Hilbert space $\mathcal{H}$ and a conjugate analytic $B(\mathcal{X}, \mathcal{H})$ -valued function $A = A(e^{i\theta})$ such that

$$f(e^{i\theta}) = A^*(e^{i\theta})A(e^{i\theta}),$$

in the sense that

$$(f(e^{i\theta})x)(y) = (A(e^{i\theta})x, A(e^{i\theta})y), \text{ for all } x, y \in \mathcal{X}.$$

CHAPTER III

ANALYSIS OF BANACH SPACE VALUED STATIONARY STOCHASTIC PROCESSES

3.1 Introduction. The main aim of this chapter is to extend to the Banach space the well known results of R. Gangolli [8] on subprocesses, Wold-Cramér concordance and factorability. We also extend to the Banach space a time domain decomposition due to R. Payen [18]. In the course of our work we will have occasions to improve some of the results contained in [24], [3] as well as providing proofs for some others.

To accomplish our goal we will associate to our SSP an auxiliary Hilbert space valued stationary process. This will make it possible to utilize the available results for the Hilbert space case.

We settle preliminaries in §3.2. In §3.3 we develop some of the theory of Banach space valued stationary stochastic processes by introducing a Hilbert space valued stationary stochastic process which is relevant to our process. In this section we prove some new results as well as most of the results in [24], [3] by using our Hilbert space valued stationary process mentioned above. In §3.4 we extend most of the results of R. Gangolli [8] to Banach space valued stationary processes.

3.2 Preliminaries. All the Banach spaces and Hilbert spaces considered here will be separable.

3.2.1 Definition. Let $S \subset B(\mathcal{X}, \mathcal{X})$. By $\overline{\mathfrak{S}}(S)$ we mean the smallest closed (in strong sense) subspace of $B(\mathcal{X}, \mathcal{X})$ containing all the elements of the form SA , where $S \in S$ and $A \in B(\mathcal{X}, \mathcal{X})$ and by $\mathfrak{S}(S)$ we mean the smallest closed subspace of $\mathcal{X}$ containing all the elements of the form Sx , where $S \in S$ and $x \in \mathcal{X}$. One can prove the following theorem by an argument similar to [18], p. 335.

3.2.2 Theorem. With the notation of Definition 3.2.1, for any collection $S \subset B(\mathcal{X}, \mathcal{X})$ we have

$$\overline{\mathfrak{S}}(S) = B(\mathcal{X}, \mathfrak{S}(S)).$$

3.2.3 Definition. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP.

Then we define the following subspaces

$$H(\infty) = \mathfrak{S}\{\xi_k x, -\infty < k < \infty, x \in \mathcal{X}\}, \quad \overline{H}(\infty) = \overline{\mathfrak{S}}\{\xi_k, -\infty < k < \infty\}$$

$$H(n) = \mathfrak{S}\{\xi_k x, -\infty < k \leq n\}, \quad \overline{H}(n) = \overline{\mathfrak{S}}\{\xi_k, -\infty < k \leq n\}$$

$$H(-\infty) = \bigcap_n H(n) \quad \text{and} \quad \overline{H}(-\infty) = \bigcap_n \overline{H}(n).$$

We remark here that by Theorem 3.2.2 it is clear that

$$H(\infty) = \{Ax, A \in \overline{H}(\infty), x \in \mathcal{X}\}, \quad H(-\infty) = \{Ax, A \in \overline{H}(-\infty), x \in \mathcal{X}\}$$

and

$$H(n) = \{Ax, A \in \overline{H}(n), x \in \mathcal{X}\}.$$

3.2.4 Definition. Let A and B be in $B(\mathcal{X}, \mathcal{X})$. Then by (A, B)

we mean the unique bounded operator which is defined through

$$((A, B)x, y) = (Ax, By). \quad \text{It is clear that } (A, B) = B^*A. \quad \text{Now if for}$$

$A, B \in B(\mathcal{X}, \mathcal{X})$, $(A, B) = 0$, we say $A \perp B$.

One can prove the following theorem

3.2.5 Theorem. Let $A \in B(\mathcal{X}, \mathcal{X})$ and $\bar{M} = \bar{\mathcal{C}}(S)$, where $S \in B(\mathcal{X}, \mathcal{X})$.

Then there exists an operator in $B(\mathcal{X}, \mathcal{X})$ denoted by $(A|\bar{M})$ such that $(A|\bar{M}) \in \bar{M}$ and $\hat{A} = A - (A|\bar{M})$ is orthogonal to $\bar{M}$.

Proof. Let $(A|\bar{M})(x) = (Ax|M)$, where $M = \mathcal{C}(S)$.

3.2.6 Definition. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP.

Then we call $\zeta_n = \xi_n - (\xi_n|\bar{H}(n-1))$, $-\infty < n < \infty$ the innovation process of ξ_n , $-\infty < n < \infty$. We write $G = (\zeta_0, \zeta_0)$ and call it the predictor error operator of the SSP ξ_n , $-\infty < n < \infty$. If G is boundedly invertible then the process is called of full rank. If G is one-to-one then the process is said to be of nearly full rank.

3.2.7 Remark. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP and let G be its predictor error operator. Then it is easy to see that ξ_n , $-\infty < n < \infty$ is singular if and only if $G = 0$ and is nondeterministic if and only if $G \neq 0$.

We give the following lemma for later reference.

3.2.8 Lemma. Let ξ_n , $-\infty < n < \infty$ and η_n , $-\infty < n < \infty$ be two $B(\mathcal{X}, \mathcal{X})$ -valued SSP's with the same spectral distribution or equivalently with the same covariance structure. Then

- (i) ξ_n , $-\infty < n < \infty$ is regular iff η_n , $-\infty < n < \infty$ is regular
- (ii) ξ_n , $-\infty < n < \infty$ is singular iff η_n , $-\infty < n < \infty$ is singular
- (iii) $G_\xi = G_\eta$.

Proof. Proof depends on the fact that the operator V sending $\xi_n x$ to $\eta_n x$ can be extended to an isometry on $H_\xi(\infty)$ onto $H_\eta(\infty)$. It is clear that $H_\eta(n) = V(H_\xi(n))$ for all n . Hence $H_\eta(-\infty) = V(H_\xi(-\infty))$. Thus (i) and (ii) are obvious. For (iii) we further note that

$$v(\xi_0 | \bar{H}_\xi(-1)) = (v\xi_0 | v\bar{H}_\xi(-1)) = (\eta_0 | \bar{H}_\eta(-1)),$$

and hence

$$\begin{aligned} G_\eta &= (\eta_0 - (\eta_0 | \bar{H}_\eta(-1)), \eta_0 - (\eta_0 | \bar{H}_\eta(-1)) = \\ &= (v\eta_0 - v(\eta_0 | \bar{H}_\eta(-1)), v\eta_0 - v(\eta_0 | \bar{H}_\eta(-1)) = \\ &= (\xi_0 - (\xi_0 | \bar{H}_\xi(-1)), \xi_0 - (\xi_0 | \bar{H}_\xi(-1)) = G_\xi. \end{aligned}$$

Finally we give the following definition which we will need later.

3.2.9 Definition. Let $\xi_n, -\infty < n < \infty$ and $\eta_n, -\infty < n < \infty$ be two $B(X, X)$ -valued SSP. We say $\xi_n, -\infty < n < \infty$ is dominated by $\eta_n, -\infty < n < \infty$ if $H_\xi(n) \subset H_\eta(n)$ for all n .

3.3 Time and spectral analysis. In this section we first associate to any Banach space valued SSP $\xi_n, -\infty < n < \infty$ a Hilbert space valued SSP $u_n, -\infty < n < \infty$ (c.f. Lemma 3.3.1). We then examine the close tie which exists between $\xi_n, -\infty < n < \infty$ and $u_n, -\infty < n < \infty$. Using these new processes we can transfer the information we know for Hilbert space valued processes to get the corresponding results for Banach space case. By making use of this technique we provide a proof for Theorems 3.3.5, 3.3.7, 3.3.10, which are announced in [3]. However our moving average representation for the regular processes will have the natural form, which prevails in the one dimensional case. Using the technique mentioned above we also state and prove several new theorems which extends the known results for Hilbert space valued stationary stochastic processes.

The following lemma is essential.

3.3.1 Lemma. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP. Then there is a Hilbert space $\mathcal{Q} \subset \mathcal{X}$ and a $B(\mathcal{Q}, \mathcal{X})$ -valued SSP u_n , $-\infty < n < \infty$ such that $\xi_n = u_n \xi_0$ for all n . Moreover ξ_n and u_n have the same shift.

Proof. Define U on $\mathcal{G}\{\xi_n x, x \in \mathcal{X}, -\infty < n < +\infty\}$ by $U\xi_n x = \xi_{n+1}x$, then we have

$$(U\xi_n x, U\xi_n y) = (\xi_{n+1}x, \xi_{n+1}y) = (\xi_n x, \xi_n y).$$

So U is a unitary operator. It is now clear that $\xi_n = U^n \xi_0$. Let $\mathcal{Q} = \mathcal{G}\{\xi_0 x, x \in \mathcal{X}\}$ and let $u_n a = U^n a$, for $a \in \mathcal{Q}$. Then u_n is a $B(\mathcal{Q}, \mathcal{X})$ -valued SSP and $\xi_n = u_n \xi_0$. So u_n , $-\infty < n < \infty$ is the desired process. It is clear that u_n and ξ_n have the same shift U .

In view of the close relations between ξ_n and u_n the following definition is appropriate.

3.3.2 Definition. Let ξ_n and u_n be as above. Then u_n , $-\infty < n < \infty$ is called the associated process of ξ_n , $-\infty < n < \infty$.

The following theorem gives some relations between ξ_n and u_n .

3.3.3 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP and let u_n be its associated process. Then

$$(a) \quad R_\xi(n) = \xi_0^* R_u(n) \xi_0$$

$$(b) \quad F_\xi = \xi_0^* F_u \xi_0$$

(c) F_ξ is absolutely continuous (a.c.) iff F_u is a.c., and in this case we have $f_\xi(x, y) = f_u(\xi_0 x, \xi_0 y)$, where $f(x, y)$ denotes the density of $(F(d\theta)x)(y)$.

Proof. We observe that

$$\begin{aligned} (R_{\xi}(n)x)(y) &= (\xi_n x, \xi_0 y) = (u_n \xi_0 x, \xi_0 x) \\ &= (R_u(n)(\xi_0 x))(\xi_0 y). \end{aligned}$$

Hence $R_{\xi}(n) = \xi_0^* R_u(n) \xi_0$.

To see (b) we note that

$$(R_{\xi}(n)x)(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} (F_{\xi}(d\theta)x)(y) \quad (1)$$

and

$$\begin{aligned} ((\xi_0^* R_u(n) \xi_0)(x))(y) &= (R_u(n)(\xi_0 x))(\xi_0 y) \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} ((\xi_0^* F_u(d\theta) \xi_0)(x))(y). \end{aligned} \quad (2)$$

Now (a), (1) and (2) imply that $F_{\xi}(d\theta)$ and $\xi_0^* F_u(d\theta) \xi_0$ have the same Fourier coefficients and hence they are the same measure.

(c) Suppose F_u is a.c., then

$$\begin{aligned} \frac{d}{d\theta} (F_{\xi}(d\theta)x)(y) &= \frac{d}{d\theta} (\xi_0^* F_u(d\theta) \xi_0 x)(y) \\ &= \frac{d}{d\theta} (F_u(d\theta)(\xi_0 x)(\xi_0 y)) \end{aligned} \quad (3)$$

so $\frac{d}{d\theta} ((F_{\xi}(d\theta)x)(y))$ exists and is equal to $\frac{d}{d\theta} (F_u(d\theta)(\xi_0 x)(\xi_0 y))$.

Now to see the other way suppose F_{ξ} is a.c., then

$\frac{d}{d\theta} ((F_{\xi}(d\theta)x)(x))$ exists for all $x \in \mathcal{X}$, hence

$\frac{d}{d\theta} (F_u(d\theta)(\xi_0 x)(\xi_0 x))$ exists for each $x \in \mathcal{X}$. Therefore

$\frac{d}{d\theta} (F_u(d\theta)a)(a)$ exists for each $a \in \xi_0 \mathcal{X}$. Thus by [9], §66

$\frac{d}{d\theta} (F_u(d\theta)a)(a)$ exists for each $a \in \mathcal{A}$. Therefore F_u is a.c.

Now in this case we have

$$\begin{aligned}
\int_{\Delta} f_{\xi}(x, y) d\theta &= (F_{\xi}(\Delta)x)(y) = ((\xi_0^* F_u(\Delta) \xi_0)x)(y) \\
&= ((F_u(\Delta))(\xi_0 x))(\xi_0 y) = \int_{\Delta} f_u(\xi_0 x, \xi_0 y) d\theta.
\end{aligned}$$

Hence for all x and y in $\mathcal{X}$ we have

$$f_{\xi}(x, y) = f_u(\xi_0 x, \xi_0 y) \quad \text{for almost every } \theta.$$

The following lemma reveals a strong tie which exists between the two processes ξ_n , $-\infty < n < \infty$ and u_n , $-\infty < n < \infty$.

3.3.4 Lemma. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP with u_n , $-\infty < n < \infty$ its associated SSP, then

- (a) $H_{\xi}(n) = H_u(n)$, $H_{\xi}(-\infty) = H_u(-\infty)$.
- (b) ξ_n , $-\infty < n < \infty$ is regular iff u_n , $-\infty < n < \infty$ is regular.
- (c) ξ_n , $-\infty < n < \infty$ is singular iff u_n , $-\infty < n < \infty$ is singular.

Proof. For the proof of (a), (b) and (c) it is sufficient to show that $H_{\xi}(n) = H_u(n)$, for all n , $-\infty < n < \infty$. Consider $H_{\xi}(n) = \mathcal{G}\{\xi_k x, x \in \mathcal{X}, k \leq n\} = \mathcal{G}\{u_k \xi_0 x, x \in \mathcal{X}, k \leq n\}$. Then

$$H_{\xi}(n) = \mathcal{G}\{U^k \xi_0 x, x \in \mathcal{X}, k \leq n\} \quad (1)$$

where U is the shift operator in lemma 3.3.1. We note that

$$\mathcal{G}(\bigcup_{\alpha} A_{\alpha}) = \mathcal{G}(\bigcup_{\alpha} \mathcal{G}(A_{\alpha})), \quad (2)$$

where $\{A_{\alpha}\}$ is any collection of subsets of $\mathcal{X}$. Combining (1) and (2) we get

$$H_{\xi}(n) = \mathcal{G}\{\mathcal{G}(U^k \xi_0 x, x \in \mathcal{X}, k \leq n)\} . \quad (3)$$

Now we observe that for any subset of $\mathcal{X}$ we have $\mathcal{G}(UA) = U(\mathcal{G}(A))$, because U is unitary. Using (3) and this observation we get

$$\begin{aligned}
H_{\xi}(n) &= \mathcal{G}\{U^k(\mathcal{G}(\xi_0 \mathcal{X})), k \leq n\} = \mathcal{G}\{U^k(\mathcal{A}), k \leq n\} \\
&= \mathcal{G}\{u_k(\mathcal{A}), k \leq n\} = H_u(n) .
\end{aligned}$$

Now we give a proof for the Wold decomposition theorem.

3.3.5 Theorem. For any $B(\mathcal{X}, \mathcal{X})$ -valued SSP ξ_n , $-\infty < n < \infty$, there exists two $B(\mathcal{X}, \mathcal{X})$ -valued SSP η_n , $-\infty < n < \infty$ and ζ_n , $-\infty < n < \infty$ such that

- (i) $\xi_n = \eta_n + \zeta_n$, for all n , $-\infty < n < \infty$.
- (ii) η_n , $-\infty < n < \infty$ and ζ_n , $-\infty < n < \infty$ are dominated by ξ_n , $-\infty < n < \infty$ and have the same shift as ξ_n , $-\infty < n < \infty$.
- (iii) η_n , $-\infty < n < \infty$ is orthogonal to ζ_n , $-\infty < n < \infty$, i.e. $\eta_n \perp \zeta_n$ for all m, n .
- (iv) η_n , $-\infty < n < \infty$ is regular and ζ_n , $-\infty < n < \infty$ is singular.

Proof. Let u_n be the associated process of ξ_n , $-\infty < n < \infty$. Then by the usual Wold's decomposition for the Hilbert space valued process u_n , $-\infty < n < \infty$ [8], p. 899, we have $u_n = v_n + w_n$ where v_n , $-\infty < n < \infty$ and w_n , $-\infty < n < \infty$ are $B(\mathcal{A}, \mathcal{X})$ -valued SSP's satisfying similar conditions as (i)-(iv). Now let $\eta_n = v_n \xi_0$ and $\zeta_n = w_n \xi_0$. Then obviously $\xi_n = \eta_n + \zeta_n$. It is clear that η_n , $-\infty < n < \infty$ and ζ_n , $-\infty < n < \infty$ are SSP's with the same shift as ξ_n , $-\infty < n < \infty$. Now

$$(\eta_n x, \zeta_m y) = (v_n \xi_0 x, w_m \xi_0 y) = (v_n (\xi_0 x), w_m (\xi_0 y)) = 0$$

because $v_n \perp w_m$. This means $\eta_n \perp \zeta_m$. Now observe that $\eta_n = v_n \xi_0 = (U^n v_0) \xi_0$. Because v_n , $-\infty < n < \infty$ is dominated by u_n , $-\infty < n < \infty$ and $H_{\xi}(n) = H_u(n)$ (c.f. Lemma 3.3.4), it follows

that η_n , $-\infty < n < \infty$ is dominated by ξ_n , $-\infty < n < \infty$. Similarly one can show that ζ_n , $-\infty < n < \infty$ is dominated by ξ_n , $-\infty < n < \infty$.

It remains to verify (iv). For this we observe

$$H_{\eta}(n) = \mathcal{G}\{\eta_k x, x \in \mathcal{X}, k \leq n\} = \mathcal{G}\{v_k \xi_0 x, k \leq n, x \in \mathcal{X}\}.$$

Thus

$$H_{\eta}(n) \subset H_v(n) \quad \text{for all } n.$$

Hence $H_{\eta}(-\infty) = \bigcap_n H_{\eta}(n) \subseteq \bigcap_n H_v(n) = H_v(-\infty)$. Since v_n , $-\infty < n < \infty$ is regular therefore η_n , $-\infty < n < \infty$ is regular. Now to prove that ζ_n , $-\infty < n < \infty$ is singular, similarly we can get $H_{\zeta}(n) \subseteq H_w(n)$. But $H_w(n) = H_w(-\infty)$ for all n , since w_n , $-\infty < n < \infty$ is singular. Hence for each n , $H_{\zeta}(n) \subset H_w(-\infty)$. Now we know that for the Hilbert space case $H_w(-\infty) = H_u(-\infty)$ (c.f. [8], p. 899) so we have $H_{\zeta}(n) \subset H_u(-\infty)$. To complete the proof it suffices to show that

$$H_{\zeta}(n) = H_u(-\infty), \quad \text{for all } n. \quad (4)$$

Suppose (4) is not true, i.e. suppose $H_u(-\infty) \not\supset H_{\zeta}(n)$ for some n .

Then there exists h , $0 \neq h \in H_u(-\infty) \ominus H_{\zeta}(n)$. Hence

$$h \perp \zeta_k x \quad \text{for all } x \in \mathcal{X} \text{ and all } k \leq n. \quad (5)$$

On the other hand

$$h \perp \eta_k x \quad \text{for all } x \in \mathcal{X} \text{ and all } k \leq n, \quad (6)$$

because $\eta_k x = v_k \zeta_0 x = v_k(\zeta_0 x)$, $h \in H_u(-\infty) = H_w(-\infty)$ and the fact that $v_n \perp w_n$. Since $\xi_n = \eta_n + \zeta_n$ for all n using (5) and (6)

we get $h \perp \xi_k x$, for all $x \in \mathcal{X}$ and all $k \leq n$. So $h \perp u_k \xi_0 x$ for all $x \in \mathcal{X}$ and all $k \leq n$. Hence $h \perp u_k a$ for all $a \in \xi_0 \mathcal{X}$ and all $k \leq n$. But since $\xi_0 \mathcal{X}$ is dense in $\mathcal{A}$ we see that $h \perp u_k a$ for all $a \in \mathcal{A}$ and all $k \leq n$. Thus $h \perp H_U(n)$ which implies $h \perp H_U(-\infty)$. But $h \in H_U(-\infty)$ by the choice of h , hence $h = 0$ which is a contradiction to the choice of h . This completes the proof.

3.3.6 Remark. The Wold decomposition is unique. To see this let η_n , $-\infty < n < \infty$ and ζ_n , $-\infty < n < \infty$ be two $B(\mathcal{X}, \mathcal{X})$ -valued SSP's satisfying (i)-(iv) of the last theorem. Then we claim

$$H_\xi(-\infty) = H_\zeta(-\infty). \quad (1)$$

Granting (1), since $\xi_n x = \eta_n x + \zeta_n x$ we get

$$\begin{aligned} (\xi_n x | H_\xi(-\infty)) &= (\eta_n x | H_\xi(-\infty)) + (\zeta_n x | H_\xi(-\infty)) = (\eta_n x | H_\zeta(-\infty)) \\ &\quad + (\zeta_n x | H_\xi(-\infty)). \end{aligned}$$

Thus by (iii) and (iv) we get

$$(\zeta_n x | H_\xi(-\infty)) = \zeta_n x \quad \text{and} \quad \eta_n x = \xi_n x - (\xi_n x | H_\xi(-\infty))$$

which means the uniqueness. Now we will verify (1). By (i) we have $H_\xi(n) = \mathcal{G}\{\xi_k x, x \in \mathcal{X}, k \leq n\} = \mathcal{G}\{\eta_k x + \zeta_k x, x \in \mathcal{X}, k \leq n\}$. Thus $H_\xi(n) \subset H_\eta(n) \oplus H_\zeta(n)$. But ζ is singular and hence $H_\xi(n) \subset H_\eta(n) \oplus H_\zeta(-\infty)$. We know $H_\eta(n)$ and $H_\xi(-\infty) \subset H_\xi(n)$, hence $H_\xi(n) = H_\eta(n) \oplus H_\zeta(-\infty)$, $H_\eta(n) \perp H_\xi(-\infty)$ so we have

$$H_\xi(-\infty) = \bigcap_n H_\xi(n) = \bigcap_n (H_\eta(n) \oplus H_\zeta(-\infty)) = H_\zeta(-\infty). \quad (2)$$

The last equality in (2) follows because $\bigcap_n H_\eta(n) = \{0\}$.

In the next theorem we give a moving average representation for a regular $B(X, X)$ -valued SSP. A moving average representation was also given in [24]. We remark that our representation is exactly in the form that one obtains for the finite dimensional case. Our proof is natural and based on the associated process u_n , $-\infty < n < \infty$.

3.3.7 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(X, X)$ -valued SSP which is regular, then we have the representation

$$\xi_n = \sum_{k=0}^{\infty} S_{n-k} A_k \quad (1)$$

where S_i 's are orthogonal partial isometries and A_k 's are in $B(X, X)$, and the convergent in (1) is in the weak sense.

Proof. Let u_n , $-\infty < n < \infty$ and $\mathcal{A}$ be as in lemma 3.3.1. Then by the corresponding theorem for the Hilbert space valued process u_n , $-\infty < n < \infty$ (c.f. [18], p. 359) we have

$$u_n = \sum_{k=0}^{\infty} S_{n-k} B_k, \quad (2)$$

where S_i 's are partial isometries on X and $B_k \in B(\mathcal{A}, X)$.

Applying both sides of (2) to ξ_0 we get

$$\xi_n = u_n \xi_0 = \left(\sum_{k=0}^{\infty} S_{n-k} B_k \right) \xi_0 = \sum_{k=0}^{\infty} S_{n-k} B_k \xi_0.$$

Letting $B_k \xi_0$ be A_k , then we have

$$\xi_n = \sum_{k=0}^{\infty} S_{n-k} A_k.$$

3.3.8 Remark. The coefficients A_i 's in the moving average representation (1) is not unique, however $S_i A_j$ are unique for each i

and j . For if there is A_n 's and B_n 's satisfying in the hypothesis of the last theorem, then we have

$$\xi_n = \sum_{k=0}^{\infty} S_{n-k} A_k = \sum_{k=0}^{\infty} S_{n-k} B_k.$$

We multiply both sides of the last relation by $S_i S_i^*$ to get

$$S_i S_i^* \left(\sum_{k=0}^{\infty} S_{n-k} A_k \right) = S_i S_i^* \left(\sum_{k=0}^{\infty} S_{n-k} B_k \right).$$

Hence $S_i A_{n-i} = S_i B_{n-i}$ for all n and i . Thus $S_i A_j = S_i B_j$ for all i and j . Now we can prove the following theorem.

3.3.9 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{Y})$ -valued SSP.

Then the following are equivalent:

- (i) ξ_n , $-\infty < n < \infty$ is regular.
- (ii) $\xi_n = \sum_{k=0}^{\infty} S_{n-k} A_k$, $A_k \in B(\mathcal{X}, \mathcal{Y})$ and $\{S_n\}_{n=-\infty}^{+\infty}$ is a sequence of mutually orthogonal isometries in $B(\mathcal{Y}, \mathcal{Y})$.
- (iii) $H_{\xi}(-\infty) = \{0\}$.

Proof. (i) $\Rightarrow$ (ii) by Theorem 3.3.7. (iii) $\Rightarrow$ (i) follows by the uniqueness of Wold's decomposition. It remains to show (ii) $\Rightarrow$ (iii). Now suppose (ii) holds. Then for each x we have

$$\|\xi_0 x\|^2 = \left\| \left(\sum_{k=0}^{\infty} S_{n-k} A_k \right) x \right\|^2 = \left\| \sum_{k=0}^{\infty} S_{n-k} A_k x \right\|^2 = \sum_{k=0}^{\infty} \|S_{n-k} A_k x\|^2 < \infty. \quad (1)$$

Letting $\bar{K}(n) = \overline{\Theta(S_k)}_{k=-\infty}^n$ it follows that

$$(\xi_0 | \bar{K}(-n)) = \sum_{k=n}^{\infty} S_{n-k} A_k, \quad \|(\xi_0 | \bar{K}(-n)) x\|^2 = \sum_{k=n}^{\infty} \|S_{n-k} A_k x\|^2.$$

whence from (1) $\|(\xi_0 | \bar{K}(-n))(x)\| \rightarrow 0$, as $n \rightarrow \infty$ for each $x \in \mathcal{X}$.

But from (ii) $\bar{H}(-n) \subset \bar{K}(-n)$ and $\|(\xi_0 | \bar{H}(-n)) x\| \leq \|\xi_0 | \bar{K}(-n)\|$ and hence

$$\|(\xi_0 | \bar{H}(-n))x\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } x \in \mathcal{X}$$

i.e.

$$\|(\xi_0 x | H(-n))\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } x \in \mathcal{X}.$$

Similarly we can show for each $k \leq 0$ and each $x \in \mathcal{X}$

$$\|(\xi_k x | H(-n))\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

hence

$$\|(\xi_k x | H(-n))\| \rightarrow 0$$

i.e.

$$\begin{aligned} \|(a | H(-n))\| &\rightarrow 0 \text{ for all } a \in \mathcal{L}(\xi_k x, x \in \mathcal{X}, k \leq 0) = \\ &\text{the linear manifold generated by } \{\xi_k x, x \in \mathcal{X}, k \leq 0\}. \end{aligned} \quad (2)$$

Now given $\varepsilon > 0$ and $b \in H(0)$ then there exists $a \in \mathcal{L}(\xi_k x, x \in \mathcal{X}, k \leq 0)$ such that $\|a-b\| < \frac{\varepsilon}{2}$. Then $\|(b | H(-n))\| < \|((a-b) | H(-n))\| + \|(a | H(-n))\|$. Hence for all $n > N$ we have

$$\|(b | H(-n))\| \leq \|a-b\| + \|(a | H(-n))\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\|(b | H(-n))\| \rightarrow 0$ as $n \rightarrow \infty$ for all $b \in H(0)$. Now using this and the fact that $P_{H(-n)} \rightarrow P_{H(-\infty)}$ strongly we see that $H(-\infty) = 0$.

Now we prove the following theorem, part (b) of which was announced in [3].

3.3.10 Theorem. Let $\xi_n, -\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued stationary process with spectral distribution F . Then

(a) if ξ_n , $-\infty < n < \infty$ has a two-sided moving average representation $\xi_n = \sum_{k=-\infty}^{+\infty} \varphi_{n-k} A_k$ with $(\varphi_n, \varphi_m) = \delta_{nm} K$, $K \neq 0$, $A_k \in B(\mathcal{Y}, \mathcal{X})$ and $\varphi_n \in B(\mathcal{X}, \mathcal{X})$. Then its spectral distribution F is a.c. and we have

$$\frac{d}{d\theta} (F(d\theta)x)(y) = (\Phi(e^{i\theta})x, \Phi(e^{i\theta})x)$$

where

$$\Phi(e^{i\theta})x = \sum_{k=-\infty}^{+\infty} e^{-ik\theta} \sqrt{K} A_k x.$$

(b) ξ_n , $-\infty < n < \infty$ is regular iff F is a.c. and

$$\frac{d}{d\theta} (F(d\theta)x)(x) = \|\Phi(e^{i\theta})x\|^2$$

where $\Phi(e^{i\theta})x$ of the form $\Phi(e^{i\theta})x = \sum_{k=0}^{\infty} e^{-ik\theta} A_k x$.

Proof. (a) Consider

$$\begin{aligned} (\Phi(e^{i\theta})x, \Phi(e^{i\theta})x) &= \left(\sum_{k=-\infty}^{+\infty} e^{-ik\theta} \sqrt{K} A_k x, \sum_{k=-\infty}^{+\infty} e^{-ik\theta} \sqrt{K} A_k x \right) \\ &= \sum_{k'=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} e^{-i(k-k')\theta} (\sqrt{K} A_k x, \sqrt{K} A_{k'} x) \\ &= \sum_{n=-\infty}^{+\infty} e^{-in\theta} \sum_{k=-\infty}^{+\infty} (\sqrt{K} A_k x, \sqrt{K} A_{k-n} x). \end{aligned}$$

So the n -th Fourier coefficient of $(\Phi(e^{i\theta})x, \Phi(e^{i\theta})x)$ is

$$\sum_{k=-\infty}^{+\infty} (\sqrt{K} A_k x, \sqrt{K} A_{k-n} x).$$

On the other hand the n -th Fourier coefficient of $(F(d\theta)x)(y)$

is $(R(n)x)(x)$ which is equal to

$$\begin{aligned}
(R(n)x)(y) &= (\xi_n x, \xi_0 x) = \left(\sum_{k=-\infty}^{+\infty} \varphi_{n-k} A_k x, \sum_{k=-\infty}^{+\infty} \varphi_{-k} A_k x \right) \\
&= \sum_{k=-\infty}^{+\infty} \sum_{k'=-\infty}^{+\infty} (\varphi_{n-k} A_k x, \varphi_{-k'} A_{k'} x) \\
&= \sum_{\substack{k \text{ and } k' \\ n-k=-k'}} (\varphi_{n-k} A_k x, \varphi_{-k'} A_{k'} x) \\
&= \sum_{k=-\infty}^{+\infty} (\sqrt{K} A_k x, \sqrt{K} A_{k-n} x) .
\end{aligned}$$

Hence $(F(d\theta)x)(x)$ and $(\Phi(e^{i\theta})x, \Phi(e^{i\theta})x)$ have the same Fourier coefficients and hence the proof is complete.

(b) Necessity. Suppose $\xi_n, -\infty < n < \infty$ is regular, then by Theorem 3.3.9 it has a moving average representation. Now apply part (a) to conclude factorability.

Sufficiency. Let $\{\varphi_n\}_{n=-\infty}^{+\infty} \subset B(\mathcal{X}, \mathcal{X})$ be any sequence such that $(\varphi_n, \varphi_m) = \delta_{nm} I$ and consider the new SSP $\xi'_n = \sum_{k=-\infty}^{+\infty} \varphi_{n-k} A_k$, then by theorem 3.3.9, $\xi'_n, -\infty < n < \infty$ is regular, hence by part (a), $F_{\xi'}$ is a.c. and we have

$$\frac{d}{d\theta} (F_{\xi'}, (d\theta)x)(x) = (\Phi(e^{i\theta})x, \Phi(e^{i\theta})x)$$

hence

$$\frac{d}{d\theta} (F_{\xi'}, (d\theta)x)(x) = \frac{d}{d\theta} (F_{\xi} (d\theta)x)(x) .$$

Hence $\xi_n, -\infty < n < \infty$ and $\xi'_n, -\infty < n < \infty$ have the same spectral distributions, and so by lemma 3.2.8, $\xi_n, -\infty < n < \infty$ is regular.

As a consequence of the Wold decomposition theorem we have

3.3.11 Theorem. Let $\xi_n, -\infty < n < \infty$ be a non-deterministic $B(\mathcal{X}, \mathcal{X})$ -valued SSP with F as its spectral distribution. Then

$$F = F_{\eta} + F_{\zeta},$$

where F_{η} is a.c. and $\frac{d}{d\theta} (F_{\eta}(d\theta)x)(y) = (\phi(e^{i\theta})_x, \phi(e^{i\theta})_y)$, $\phi(e^{i\theta})$ as before

Proof. Let $\xi_n = \eta_n + \zeta_n$ be Wold's decomposition of ξ_n , $-\infty < n < \infty$. Then by part (iii) of theorem 3.3.5 we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} ((F(d\theta)x)y) &= (R_{\xi}(n)x)(y) = (\xi_n x, \xi_0 y) \\ &= (\eta_n x + \zeta_n x, \xi_0 y + \zeta_0 y) \\ &= (\eta_n x, \eta_0 y) + (\zeta_n x, \zeta_0 y) \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} [(F_{\eta}(d\theta)x)x + (F_{\zeta}(d\theta)x)(x)]. \end{aligned}$$

Thus $F(d\theta) = F_{\eta}(d\theta) + F_{\zeta}(d\theta)$. Now by theorem 3.3.10 F_{η} has the required properties.

3.3.12 Theorem. Let F be the spectral distribution of a $B(\mathcal{X}, \mathcal{X})$ -valued stationary process ξ_n , $-\infty < n < \infty$. Then ξ_n , $-\infty < n < \infty$ is regular of full rank iff F is a.c. and $\frac{d}{d\theta} (F(d\theta)x)(x) = \|\phi(e^{i\theta})x\|^2$, where $\phi(e^{i\theta})x = \sum_{k=0}^{\infty} e^{-ik\theta} A_k x$, with $A_n \in B(\mathcal{X}, \mathcal{X})$ and $A_0^* A_0$ being invertible. Furthermore if we assume that F' is bounded operator valued then ϕ is also a bounded operator valued function.

Proof. Because of theorem 3.3.10, part (b) it suffices to show that, for the function $\phi(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{-ik\theta}$, in that theorem we have $G = A_0^* A_0$. To see this, set $\xi'_n = \sum_{k=0}^{\infty} \varphi_{n-k} A_k$, where $\varphi_k \in B(\mathcal{X}, \mathcal{X})$ with $(\varphi_n, \varphi_m) = \delta_{nm} I$. Then consider the SSP ζ'_n , $-\infty < n < \infty$, defined by $\zeta'_0 x = \xi'_0 x - (\xi'_0 x | H_{\xi}, (-1)) = \varphi_0 A_0 x$.

Now for each x and y in $\mathcal{X}$ we have

$$(G_{\xi}, x)(y) = (\zeta_0' x, \zeta_0' y) = (\varphi_0 A_0 x, \varphi_0 A_0 y) = (A_0 x, A_0 y) = (A_0^* A_0 x, y).$$

Thus $G_{\xi} = A_0^* A_0$. But one can see that $\xi_n, -\infty < n < \infty$ and $\xi_n', -\infty < n < \infty$ have the same covariance structure. Hence by lemma 3.2.8 $G_{\xi} = G_{\xi'}$. Thus $G_{\xi} = A_0^* A_0$. The proof of the last statement of the theorem is clear. In fact one can show more. See remark 4.2.1.

The following is an extension of a result due to Payen [18], pp. 371-372.

3.3.13 Theorem. Every $B(\mathcal{X}, \mathcal{X})$ -valued SSP $\xi_n, -\infty < n < \infty$ is the sum of three processes

$$\xi_n = \xi_n^1 + \xi_n^2 + \xi_n^3,$$

which are mutually orthogonal, the first one being regular, the second one being deterministic with a.c. spectral distribution and the third one being deterministic with singular spectral distribution.

Proof. Let u_n and $\mathcal{Q}$ be as in lemma 3.3.1. Then let $u_n = u_n^1 + u_n^2 + u_n^3$ be the corresponding decomposition of the process $u_n, -\infty < n < \infty$, given in [18], pp. 371-372. Let $\xi_n^i = u_n^i \xi_0^i, i = 1, 2, 3$. Then $\xi_n^i, -\infty < n < \infty$ are mutually orthogonal SSP's because $u_n^i, -\infty < n < \infty$ are so. Since $u_n^1, -\infty < n < \infty$ is regular as in the proof of Wold's decomposition theorem, we can see $\xi_n^1, -\infty < n < \infty$ is regular. Now to show that $\xi_n^3, -\infty < n < \infty$ is deterministic with singular spectral distribution it suffices to show F_{ξ_3} is singular. (By Wold's decomposition theorem.) But $(F_3(d\theta)u)(a) \perp d\theta$ for all $a \in \mathcal{Q}$. Hence $((F_3(d\theta))(\xi_0 x))(\xi_0 x) \perp d\theta$ for all $x \in \mathcal{X}$ which implies that

$(F_{\xi_3}(d\theta)x)(x) = (\xi_0^* F_{\xi_3}(d\theta)\xi_0)(x) = (\xi_0^* F_{\xi_3}(d\theta)\xi_0)(x)$. Hence $(F_{\xi_3}(d\theta)x)(x) \perp d\theta$ for all $x \in \mathcal{X}$ which means F_{ξ_3} is singular. Finally by a similar argument one can show that F_{ξ_2} is a.c. using the fact that F_{ξ_2} is a.c. So we just have to show that $\xi_n^2, -\infty < n < \infty$ is deterministic. Suppose this is not the case, i.e. suppose there exists $0 \neq a \in H_{\xi_2}(0) \ominus H_{\xi_2}(-1)$. So $a \perp \xi_k^2 x$ for all $x \in \mathcal{X}$ and all $k \leq -1$. Also $a \perp \xi_k^1 x$ for all $x \in \mathcal{X}$, all k and $i = 1, 3$. (Because $a \in H_{\xi_2}(0)$ and $\xi_n^2, -\infty < n < \infty$ is orthogonal to $\xi_n^1, -\infty < n < \infty$ and $\xi_n^3, -\infty < n < \infty$). Hence $a \perp \xi_k^i x$, for all $x \in \mathcal{X}$, and all $k \leq -1$. Hence $a \perp H_{\xi}(-1)$. So we get

$$a \perp H_{\xi}(-\infty). \quad (1)$$

On the other hand $H_{\xi_2}(0) \subseteq H_{\xi_2}(0) = H_{\xi_2}(-\infty)$, because $\xi_n^2, -\infty < n < \infty$ is deterministic. So $H_{\xi_2}(0) \subseteq H_{\xi_2}(-\infty) \subset H_{\xi}(-\infty)$, by the choice of $\xi_n^2, -\infty < n < \infty$, see [18], pp. 371-372. Hence $H_{\xi_2}(0) \subset H_{\xi}(-\infty) = H_{\xi}(-\infty)$ by lemma 3.3.6, part (a). Thus $a \in H_{\xi}(-\infty)$ because $a \in H_{\xi_2}(0)$, by the choice of a . But this and (1) implies that $a = 0$, which is a contradiction.

The following corollary gives an extension of Cramér's decomposition theorem.

3.3.14 Corollary (Cramér's decomposition). Let F be the spectral distribution of a $B(\mathcal{X}, \mathcal{X})$ -valued SSP $\xi_n, -\infty < n < \infty$ then we can decompose F as

(a) $F = F_1 + F_2 + F_3$, where F_1 is a.c. and spectral distribution of a regular process, F_2 and F_3 are spectral distributions of a deterministic process with F_2 being a.c. while F_3 is singular.

(b) $F = F_a + F_s$, where F_a is a.c. and F_s is singular.

Proof. (a) Let $\xi_n = \xi_n^1 + \xi_n^2 + \xi_n^3$ be the decomposition of ξ_n , $-\infty < n < \infty$ as given in theorem 3.3.13. Then since these are mutually orthogonal processes by the standard computations one can immediately see $F = F_{\xi_1}^1 + F_{\xi_2}^2 + F_{\xi_3}^3$. We can take $F_i = F_{\xi_i}$ for $i = 1, 2, 3$.

(b) Let $F_a = F_1 + F_2$ and $F_s = F_3$, then obviously F_a is a.c. and F_s is singular.

We conclude this section with the following theorem which gives a sufficient Devinatz's type condition for the factorability of a the spectral distribution of a Banach space valued SSP.

3.3.15 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{Y})$ -valued SSP with a bounded spectral density f_{ξ} satisfying

$$\int_0^{2\pi} \log \|f_{\xi}^{-1}(e^{i\theta})\|^{-1} d\theta > -\infty. \quad (1)$$

Then f_{ξ} is factorable or equivalently ξ_n , $-\infty < n < \infty$ is regular.

Proof. We have $(\xi_0 x, \xi_0 x) = (R_0 x)(x) = \frac{1}{2\pi} \int_0^{2\pi} (f_{\xi}(e^{i\theta})x)(x) d\theta$.

Hence for each $x \in \mathcal{X}$ we have

$$\begin{aligned} \|\xi_0 x\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} (f_{\xi}(e^{i\theta})x)(x) d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} \|x\|^2 \|f_{\xi}^{-1}(e^{i\theta})\|^{-1} d\theta \\ &= \|x\|^2 \int_0^{2\pi} \frac{1}{2\pi} \|f_{\xi}^{-1}(e^{i\theta})\|^{-1} d\theta = \|x\|^2 \lambda, \end{aligned}$$

where $\lambda = \frac{1}{2\pi} \int_0^{2\pi} \|f_{\xi}^{-1}(e^{i\theta})\|^{-1} d\theta$, obviously $0 < \lambda < \infty$. By theorem

3.3.6 we have $(f_{\xi}x)(y) = f_u(\xi_0 x, \xi_0 y)$ for all $x, y \in \mathcal{X}$. So

$$\begin{aligned} |f_u(e^{i\theta})(a, b)| &= |(f_{\xi}(e^{i\theta})\xi_0^{-1}a)(\xi_0^{-1}a)| \leq \|f_{\xi}(e^{i\theta})\| \|\xi_0^{-1}a\| \\ &\leq \|f_{\xi}(e^{i\theta})\| \|\xi_0^{-1}\|^2 \|a\|^2 \leq \lambda^{-1} \|f_{\xi}(e^{i\theta})\| \|a\|^2, \end{aligned}$$

for all $a \in \mathcal{A}$. Hence $f_u(a, b)$ is a bounded bilinear form and hence there exists an operator valued function $f_u(e^{i\theta}): \mathcal{A} \rightarrow \mathcal{A}$ such that $(f_\xi(e^{i\theta})x)(y) = (\xi_0^* f_u(e^{i\theta}) \xi_0 x)(y)$. Now (1) means that

$$\int_0^{2\pi} \log \left\{ \inf_{0 \neq x \in \mathcal{X}} \frac{(f_\xi(e^{i\theta})x)(x)}{\|x\|^2} \right\} d\theta > -\infty. \quad (2)$$

Hence we get

$$\int_0^{2\pi} \log \left\{ \inf_{0 \neq x \in \mathcal{X}} \frac{(f_u(e^{i\theta})x)(x)}{\|x\|^2} \right\} d\theta = \int_0^{2\pi} \log \left\{ \inf_{0 \neq x \in \mathcal{X}} \frac{(\xi_0^* f_u(e^{i\theta}) \xi_0 x)(x)}{\|x\|^2} \right\} d\theta > -\infty.$$

Thus

$$\int_0^{2\pi} \log \left\{ \inf_{0 \neq x \in \mathcal{X}} \frac{(f_u(\xi_0 x))(\xi_0 x)}{\|\xi_0\|^{-1} \|x\|^2} \right\} d\theta > -\infty.$$

Hence

$$\int_0^{2\pi} \log \left\{ \inf_{0 \neq a \in \xi_0 \mathcal{X}} \frac{(f_u a)(a)}{\|a\|^2} \right\} d\theta > -\infty.$$

But since $\xi_0 \mathcal{X}$ is dense in $\mathcal{A}$ we get

$$\int_0^{2\pi} \log \left\{ \inf_{a \in \mathcal{A}} \frac{(f_u a)(a)}{\|a\|^2} \right\} d\theta > -\infty.$$

Hence the associated process u_n , $-\infty < n < \infty$ is regular.

By lemma 3.3.7 we see that our process ξ_n , $-\infty < n < \infty$ is regular.

3.4 Subprocesses and Spectral Conditions for Factorability of the

Spectral Density. In this section we extend to the Banach space case the results of R. Gangolli on subprocesses and their relation to the process itself and to the factorability of its spectral density. Making use of the results of §3.3 the technique employed by Gangolli can be used to establish our results.

3.4.1 Definition. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP, and let θ be a subspace of $\mathcal{X}$, then the SSP $\xi_n|_{\theta}$, $-\infty < n < \infty$ is called a subprocess of ξ_n , $-\infty < n < \infty$.

Note that in case that θ is complementary, $\xi_n|_{\theta}$ can be identified with $\xi_n P$ where P is the projection on θ . Hence in the Hilbert space case this definition coincides with Gangolli's definition. Since we will be mostly working with finite dimensional subspaces, which, are complementary we sometimes use $\xi_n P$ instead of $\xi_n|_{\theta}$. Hence $\xi_n|_{\theta} \in B(\mathcal{X}, \mathcal{X})$.

3.4.2 Lemma. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP and θ be a subspace of $\mathcal{X}$. Then

$$G_{\theta} > G,$$

i.e. $(G_{\theta}x)(x) > (Gx)(x)$ for all $x \in \theta$, here G and G_{θ} are the predictor error operator of ξ_n , $-\infty < n < \infty$ and $\xi_n|_{\theta}$, $-\infty < n < \infty$ respectively.

Proof. Since $H_{\xi|_{\theta}}(n) \subset H_{\xi}(n)$, for each $x \in \theta$ we have

$$(G_{\theta}x)(x) = (\xi_0 x - (\xi_0 x|_{H_{\xi|_{\theta}}(-1)}), \xi_0 x - (\xi_0 x|_{H_{\xi_0|_{\theta}}(-1)})). \text{ Hence} \\ (G_{\theta}x)(x) \geq (\xi_0 x - \xi_0 x|_{H_{\xi}(-1)}, \xi_0 x - \xi_0 x|_{H_{\xi}(-1)}) = (Gx)(x). \text{ Hence}$$

$$(G_{\theta}x)(x) \geq (Gx)(x) \text{ for all } x \in \theta.$$

3.4.3 Notation. Denote by $\lambda(G_{\theta}) = \inf\{(G_{\theta}x)(x), \|x\| = 1, x \in \theta\}$.

The following theorem will be useful later.

3.4.4 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP, then ξ_n , $-\infty < n < \infty$ is of full rank iff

$$\inf\{\lambda(G_{\theta})|_{\theta} \text{ a finite dimensional subspace of } \mathcal{X}\} \geq c^2 > 0. \quad (1)$$

Proof. If ξ_n , $-\infty < n < \infty$ is of full rank then $\inf_{\|x\|=1} (Gx)(x) = c^2 > 0$. Hence by lemma 3.4.2 $\lambda(G_\vartheta) \geq c^2$ for all ϑ , a subspace of $\mathcal{X}$. Now suppose that (1) holds. We must show $G \geq c^2$. Suppose not. Then there exists $x \in \mathcal{X}$ with $\|x\| = 1$ such that $(Gx)(x) = c'^2 < c^2$. But $(Gx)(x) = (\xi_0 x - (\xi_0 x|_{H_\xi(-1)}), \xi_0 x - (\xi_0 x|_{H_\xi(-1)}))$. Hence $\|\xi_0 x - (\xi_0 x|_{H_\xi(-1)})\| = c' < c$. Thus, the distance of $\xi_0 x$ from $H_\xi(-1) = c' < c$. Therefore there exists numbers a_{jk} , $j = -1, -2, \dots, -N$, $k = 1, 2, \dots, N$ and vectors $x_k \in \mathcal{X}$, $k = 1, 2, \dots, N$ such that

$$\|\xi_0 x - \sum_{j=-1}^{-N} \sum_{k=1}^N a_{jk} \xi_j x_k\| < c.$$

Letting $\vartheta = \mathcal{G}\{x, x_1, x_2, \dots, x_N\}$ we then have

$$\lambda(G_\vartheta) \leq \|\xi_0 x - (\xi_0 P x|_{H_{\xi_P}(-1)})\| \leq \|\xi_0 x - \sum_{j=-1}^{-N} \sum_{k=1}^N a_{jk} \xi_j x_k\| < c.$$

This is a contradiction to (1).

3.4.5 Remark. The finite dimensional subspaces are essentially multivariate SSP in the sense of Weiner and Masani [25] and in this case $\lambda(G_\vartheta)$ is the smallest eigenvalue of the matrix G_ϑ .

As was noted in [17] there are errors in theorem 5.3 and 5.4. Because the proof of theorem 7.3 depends on theorem 5.3 this theorem is also in doubt. Using the result of [17], p. 405, we extend correct versions of Gangolli's result to Banach space.

The next theorem gives a concordance between these two decompositions.

3.4.6 Theorem. (Concordance theorem). Suppose the $B(\mathcal{X}, \mathcal{X})$ -valued SSP ξ_n , $-\infty < n < \infty$ has full rank. Let $F = F_\eta + F_\zeta$ and

$F = F_a + F_s$ be Wold's and Cramér's decomposition of F , the spectral distribution of ξ_n , $-\infty < n < \infty$. Assume that $F'(e^{i\theta})$ is bounded and has a bounded inverse for almost every θ . Then

$$F_a = F_\eta, \quad F_s = F_\zeta.$$

Proof. Take some θ_0 such that $F'(e^{i\theta_0})$ is bounded and boundedly invertible. Then by a lemma in [1], p. 21, there exists a Hilbert space $\mathcal{Q}$ and a bounded operator $T: \mathcal{X} \rightarrow \mathcal{Q}$ such that $F'(e^{i\theta_0}) = T^* T$ and range of T dense in $\mathcal{Q}$. Now since $F'(e^{i\theta_0})$ has a bounded inverse T is onto and has a bounded inverse. Now define the process u_n , $-\infty < n < \infty$ by $u_n a = \xi_n T^{-1} a$, $a \in \mathcal{Q}$. Then u_n , $-\infty < n < \infty$ is a $B(\mathcal{Q}, \mathcal{H})$ -valued SSP and we have $\xi_n = u_n T$. One can show that ξ_n and u_n and T satisfies most of the properties we proved about ξ_n , u_n and ξ_0 in Section 3.3. In particular the results 3.3.3, 3.3.4, 3.3.5, 3.3.7, 3.3.13 and 3.3.14 hold.

Now let v_n , w_n and η_n , ζ_n be the components of Wold's decomposition of the processes u_n , $-\infty < n < \infty$ and ξ_n , $-\infty < n < \infty$ respectively, as in theorem 3.3.5. Let u_n^i , $-\infty < n < \infty$ and ξ_n^i , $-\infty < n < \infty$, $i = 1, 2, 3$ be as in theorem 3.3.13. Now as in the proof of theorem 3.3.15, we have $F'(e^{i\theta}) = T^* F'_u(e^{i\theta}) T$ and $F'_u(e^{i\theta}) = T^{*-1} F'(e^{i\theta}) T^{-1}$. Thus $F'_u(e^{i\theta})$ is bounded and has a bounded inverse a.e. We also note that $G_\xi = T^* G_u T$ and hence u_n , $-\infty < n < \infty$ is of full rank. From the results on page 405 of [17] we get

$$(F_u)_a = F_v. \quad (1)$$

But we have $\eta_n = v_n T$ and $\xi_n^i = u_n^i T$ for all n and $i = 1, 2, 3$

(c.f. theorems 3.3.5 and 3.3.13). An argument similar to the proof of theorem 3.3.3 may be used to show that

$$F_{\eta} = T^* F_v T \quad \text{and} \quad F_{\xi_i} = T^* F_{u_i} T, \quad i = 1, 2, 3. \quad (2)$$

We also have

$$(F_u)_a = F_{u_1} + F_{u_2} \quad \text{and} \quad F_a = F_{\xi_1} + F_{\xi_2}. \quad (3)$$

By (3) and (2) we get

$$F_a = T^* (F_u)_a T. \quad (4)$$

Now by (1), (4) and (2) we get $F_a = F_{\eta}$ which is the concordance.

Now as a corollary we have the following theorem.

3.4.7 Theorem. Let F and ξ_n , $-\infty < n < \infty$ be as in the previous theorem. Then the SSP ξ_n , $-\infty < n < \infty$ is regular iff the following two conditions hold

- (i) F is absolutely continuous
- (ii) there exists constant $c > 0$ such that $\Delta(G_{\theta}) \geq c$, for all finite dimensional subspaces θ .

Proof. If ξ_n , $-\infty < n < \infty$ is regular of full rank, then by theorem 3.3.10 F is a.c. and (ii) follows from theorem 3.4.4.

Now suppose (i) and (ii) hold. Then from (ii) and theorem 3.4.4 it follows that the process is of full rank and hence by the concordance theorem we get $F_{\zeta} = F_s$. But $F_s = 0$ by (i), hence $F_{\zeta} = 0$, so $F = F_{\eta}$, i.e. ξ_n , $-\infty < n < \infty$ is regular.

3.4.8 Notation. Let R be an $n \times n$ positive matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Following Gangolli for $\alpha \in [0, 1]$ we let $[\alpha_1, \alpha_2, \dots, \alpha_n] = \alpha[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}] + (1 - \alpha)[1, 0, 0, \dots]$ and

define $\Delta(\alpha, R) = \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_n^{\alpha_n}$. Note that $\Delta(0, R)$ is the smallest eigenvalue of and $\Delta(1, R)$ is the n -th root of determinant of R and for a fixed R , $\Delta(\alpha, R)$ is a continuous increasing function of α on $[0, 1]$. We also note that P^*FP is the spectral distribution of the finite dimensional subprocess $\xi_n P$. Next theorem deals with the evaluation of $\lambda(G_\vartheta)$ in terms of F .

3.4.9 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{X}, \mathcal{X})$ -valued SSP with spectral distribution F , which has a bounded derivative F' so that $F(d\theta) = F'(e^{i\theta})d\theta + F_g(d\theta)$. Suppose that ϑ is a finite dimensional subspace of dimension n_ϑ , then there exists a unique $\alpha = \alpha(\vartheta)$ in $[0, 1]$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \Delta(\alpha(P), P^*F'P) d\theta = \log \lambda(G_\vartheta).$$

Proof. Define $f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log \Delta(\alpha, P^*F'P) d\theta$ then $f(0) = \frac{1}{2\pi} \int_0^{2\pi} \log \Delta(0, P^*F'P) d\theta = \int_0^{2\pi} \log \lambda(P^*F'P) d\theta$. Hence

$$f(0) = \log \lambda(G_\vartheta).$$

On the other hand

$$\begin{aligned} f(1) &= \frac{1}{2\pi} \int_0^{2\pi} \log \Delta(1, P^*F'P) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt[n_\vartheta]{\det P^*F'P} d\theta \\ &= \frac{1}{2\pi n_\vartheta} \int_0^{2\pi} \log \det(P^*F'P) d\theta = \frac{1}{n_\vartheta} \log \det G_\vartheta = \log \lambda(G_\vartheta). \end{aligned}$$

Now $f(\alpha)$ being a continuous, increasing function on $[0, 1]$ (see p. 907, [8]) and since $f(0) \leq \log \lambda(G_\vartheta) \leq f(1)$ we see that there exists a unique $\alpha = \alpha(\vartheta)$ such that

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log \Delta(\alpha(\vartheta), P^*F'P) d\theta = \log \lambda(G_\vartheta).$$

3.4.10 Theorem. Suppose ξ_n , $-\infty < n < \infty$ is a $B(\mathcal{Y}, \mathcal{X})$ -valued SSP. Then ξ_n , $-\infty < n < \infty$ is of full rank iff each finite dimensional subprocess is of full rank and

$$\int_0^{2\pi} \log \Delta(\alpha(P)), P^* F' P d\theta \geq -c > -\infty,$$

where F' is the bounded spectral density of F , any $\alpha(P)$ is as in theorem 3.4.9. P is finite dimensional.

3.4.11 Theorem. Let ξ_n , $-\infty < n < \infty$ be a $B(\mathcal{Y}, \mathcal{X})$ -valued SSP with distribution F . For ξ_n , $-\infty < n < \infty$ to be of full rank it is necessary that for all α , $\alpha_+ \leq \alpha \leq 1$ we have

$$\int_0^{2\pi} \log \Delta(\alpha, P^* F' P) d\theta \geq -c > -\infty$$

and it is sufficient that for some α , $0 \leq \alpha \leq \alpha_-$ we have

$$\int_0^{2\pi} \log \Delta(\alpha, P^* F' P) d\theta \geq -c > -\infty,$$

where P is any finite dimensional projection and F' is the bounded derivative of F . Here α_+ and α_- are l.u.b. and g.u.b. of the set $\{\alpha(P), P \text{ finite dimensional projections}\}$.

3.4.12 Theorem. Let $f(e^{i\theta})$ be a $B^+(\mathcal{Y}, \mathcal{X}^*)$ -valued function on the unit circle. Then $(f(e^{i\theta})x)(y) = (\tilde{\Phi}(e^{i\theta})x, \tilde{\Phi}(e^{i\theta})y)$, where $\tilde{\Phi}(e^{i\theta}) = \sum_{k=0}^{\infty} \tilde{\Phi}_k e^{-ik\theta}$ with $\tilde{\Phi}_0^* \tilde{\Phi}_0$ invertible iff for each finite dimensional P , $P^* f P$ admits a factorization with $\tilde{\Phi}_{0,P}^* \tilde{\Phi}_{0,P}$ invertible. Furthermore in this case $\|(\tilde{\Phi}_{0,P}^* \tilde{\Phi}_{0,P})^{-1}\| \leq c < \infty$, where c is a constant independent of P .

Proof. $(\Rightarrow)$ Clearly if we have $(f(e^{i\theta})x)(y) = (\tilde{\Phi}(e^{i\theta})x, \tilde{\Phi}(e^{i\theta})y)$ then $((P^* f(e^{i\theta}) P)x)(y) = ((\tilde{\Phi}(e^{i\theta}) P)x, (\tilde{\Phi}(e^{i\theta}) P)y)$. Hence $P^* f P$ is factorable. Now as in the proof of theorem 3.3.12, $\tilde{\Phi}_0^* \tilde{\Phi}_0$ is the

predictor error operator G , of the corresponding SSP and hence

the invertibility of $\Phi_{0,P}^* \Phi_{0,P}$ follows from lemma 3.4.2.

($\Leftarrow$) Since $\Phi_{0,P}^* \Phi_{0,P}$ have bounded inverses then by lemma 3.4.4, the corresponding process which has density f is of full rank and

hence by the concordance theorem $F_a = F$ so $F = F_a = F_u$, i.e.

$f = f_u$, but by theorem 3.3.10 f_u , and hence f is factorable. Now

invertibility of $\Phi_0^* \Phi_0$ follows since $\Phi_0^* \Phi_0 = G$ as above.

3.4.13 Theorem. Let α_+ and α_- be as in theorem 3.4.11. Then

for f to be factorable as $f(e^{i\theta}) = \Phi^*(e^{i\theta}) \Phi(e^{i\theta})$, where

$\Phi(e^{i\theta}) = \sum_{n=0}^{\infty} e^{-in\theta} \Phi_n$, with $\Phi_0^* \Phi_0$ invertible it is necessary that

for all α , $\alpha_+ \leq \alpha \leq 1$ we have

$$\int_0^{2\pi} \log \Delta(\alpha, P^* f P) d\theta \geq -c > -\infty$$

and it is sufficient that for some α , $0 \leq \alpha \leq \alpha_-$

$$\int_0^{2\pi} \log \Delta(\alpha, P^* f P) d\theta \geq -c > -\infty,$$

where P is any finite dimensional projection and c is independent of P .

3.4.14 Remark. If we put $\alpha = 0$ in the second part of the last

theorem we get that the condition $\int_0^{2\pi} \log \Delta(P^* f P) d\theta \geq -c > -\infty$ is

sufficient for factorability of f as $f(e^{i\theta}) = \Phi^*(e^{i\theta}) \Phi(e^{i\theta})$,

where $\Phi(e^{i\theta}) = \sum_{k=0}^{\infty} \Phi_k e^{-ik\theta}$, with $\Phi_0^* \Phi_0$ invertible. This is an

improvement on theorem 3.3.15.

3.4.15 Remarks. (a) The proof of lemma 3.3.4 can be simplified

considerably. Note that $H_{\xi}(n) \subset H_u(n)$ and for $a \in \mathcal{A}$, $a = \lim_{m \rightarrow \infty} \xi_0 x_m$

for some sequence $\{x_m\} \subset \mathcal{X}$, hence $U^k a = \lim_{m \rightarrow \infty} U^k \xi_0 x_m = \lim_{m \rightarrow \infty} \xi_k x_m$

which gives $H_u(n) \subseteq H_{\xi}(n)$. Also the proof of theorem 3.3.5 can be

directly obtained by using projections in $B(\mathcal{X}, \mathcal{X})$ (see theorem 3.2.5) and standard methods.

(b) Throughout this work we shall work with the assumption that $\mathcal{X}$ is separable. In case x_n , $-\infty < n < \infty$ is a stochastic process taking values in a separable Banach space $\mathcal{X}$ then the relevant Hilbert space is $\mathcal{H} = \mathcal{G}\{x^*(x_n), x^* \in \mathcal{X}^*, n \in \mathbb{Z}\}$. It can be shown that under the condition $E\|x_0\|^2 < \infty$ (in particular where x_0 is Gaussian) $\mathcal{H}$ is separable. We note that we do not use here separability of $\mathcal{X}^*$.

3.4.16 Remark. Suppose the covariance operators R_n , $-\infty < n < \infty$ is given. Let ξ_n , $-\infty < n < \infty$ be the SSP given in theorem 2.4. In the next chapter we assume that the condition of $\mathcal{X}$ under the norm $|||x||| = (R(0)x)(x)$ is separable. In this case we can show that $H_\xi(n) = \mathcal{G}\{\xi_k x, x \in \mathcal{X}_0, k \leq n\}$. Thus for the study of prediction problem the relevant factorization problem can be studied with $\mathcal{X}$ and $\mathcal{H}$ being separable. As remarked before, this assumption is satisfied in several cases.

CHAPTER IV

FACTORIZATION OF NONNEGATIVE OPERATOR VALUED FUNCTIONS ON A BANACH SPACE

4.1 Introduction. The main purpose of this chapter is to extend most of the results of R.G. Douglas [4] on factoring nonnegative operator valued functions on a Hilbert space to nonnegative operator valued functions on a Banach space. As we mentioned before the problem of factoring nonnegative operator valued functions on a Banach space plays an important role in the study of Banach space valued stationary stochastic processes (c.f. Theorems 3.3.10 and 3.3.12).

We remark that our definition of "factorization" is exactly what Douglas called "conjugate factorization". However all our results have dual statements and hence we have the extension of Douglas' results.

When f is a positive operator valued function on a Hilbert space, $\sqrt{f}$, the square root of f whose existence is known plays a significant role. But when f is a positive operator valued function on a Banach space $\mathcal{X}$ the existence of a square root in the ordinary way does not make sense. Nevertheless we can prove (c.f. lemma 4.3.1) the existence of a measurable operator valued function A on $\mathcal{X}$ into some auxiliary Hilbert space which behaves almost like a square root in the sense that $f = A^* A$. The operator valued function A , called a quasi square root, enables

us to extend to the Banach space case a lemma of Helson [10], p. 117 and the main lemma of Douglas [4].

In §4.2 we set up necessary terminologies and state some known results. Section 4.3 includes the proof of existence of a quasi square root and two lemmas on the characterization of factorability of a positive operator on a Banach space. The results of §4.4 extend in a natural way most of the work of R.G. Douglas [4] to the Banach space case. In establishing these results we make use of our fundamental lemmas proved in §4.3 and Douglas' techniques employed in [4].

4.2 Ancillary results. In this chapter all the Banach spaces and Hilbert spaces are separable. We recall that if $f = f(e^{i\theta})$ is a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle, then we say f is factorable if there exists a Hilbert space $\mathcal{H}$ and a conjugate analytic $B(\mathcal{X}, \mathcal{H})$ -valued function $A = A(e^{i\theta})$ such that $f(e^{i\theta}) = A^*(e^{i\theta})A(e^{i\theta})$, in the sense that $(f(e^{i\theta})x)(y) = (A(e^{i\theta})x, A(e^{i\theta})y)$; $x, y \in \mathcal{X}$.

4.2.1 Remark. With the notation of the last paragraph we can show that

$$A(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{-ik\theta},$$

where A_k 's are bounded, in fact we can show that

$$\sum_{k=0}^{\infty} \|A_k x\|_{\mathcal{H}}^2 \leq C \|x\|^2, \quad x \in \mathcal{X},$$

for some finite constant C .

To prove these we proceed as follows. It is easy to see that $A(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{-in\theta}$, where A_k 's are linear operators

defined on $\mathcal{X}$. We observe that $\sum_{k=0}^{\infty} \|A_k x\|_{\mathcal{X}}^2 = \int_0^{2\pi} \|A(e^{i\theta})x\|_{\mathcal{X}}^2 d\theta = \int_0^{2\pi} (f(e^{i\theta})x)(x) d\theta < \infty$. To complete the proof it is sufficient to show that the operator $T: \mathcal{X} \rightarrow L_2(\mathcal{X})$ defined by $Tx = A(e^{i\theta})x$ is bounded. By the standard method [11], p. 85, one can show that T is closed, and being everywhere defined, it follows that T is bounded.

Let $f = f(e^{i\theta})$ be a $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Theorem 2.4 shows that there exists a stationary process ξ_n , $-\infty < n < \infty$, $\xi_n: \mathcal{X} \rightarrow \mathcal{X}$, for some Hilbert space $\mathcal{X}$, whose spectral density is $f(e^{i\theta})$. Letting $U(\xi_n x) = \xi_{n+1}x$, for all $x \in \mathcal{X}$ and $-\infty < n < \infty$, we obtain a unitary operator on

$$\mathcal{G}(\xi_n \mathcal{X}, -\infty < n < \infty).$$

Let $u_n a = U^n a$ for each $a \in \mathcal{A} = \mathcal{G}(\xi_0 \mathcal{X})$. It is clear that u_n , $-\infty < n < \infty$, is a stationary process on $\mathcal{A}$ into $\mathcal{X}$. Call F_u the spectral distribution of u_n . It is easy to see that $F_{\xi} = T^* F_u T$, where $T = \xi_0$ and F_{ξ} is the distribution of ξ_n .

4.2.2 Lemma. With the above notation, the distribution F_u is a.c. w.r.t. the Lebesgue measure, and its density f_u is a bilinear functional on $\mathcal{A} \times \mathcal{A}$. Moreover we have

$$(fx)(y) = f_u(Tx, Ty); \quad x, y \in \mathcal{X}. \quad (1)$$

Proof. Let $a = Tx$ for some $x \in \mathcal{X}$. Since $(F_{\xi}(d\theta))x(x)$ is a.c. w.r.t. the Lebesgue measure and $(F_{\xi}(d\theta)x)(x) = (F_u(d\theta)Tx)(Tx) = (F_u(d\theta)a)(a)$, it follows that $(F_u(d\theta)a)(a)$ is a.c. w.r.t. the Lebesgue measure for all $a \in T(\mathcal{X})$. By §66 [9] we have that $(F_u(d\theta)a)(a)$ is a.c. on $\mathcal{A}$. Now (1) easily follows.

4.3 Main Lemmas. In this section we will show the existence of a quasi square root for a $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. We then extend a lemma due to Helson [10], p. 117, and the main lemma of Douglas [4] to the Banach space case.

Our first lemma is on the existence of a quasi square root.

4.3.1 Lemma. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then there exists a Hilbert space $\mathcal{H}$ and a measurable $B(\mathcal{X}, \mathcal{H})$ -valued function $Q = Q(e^{i\theta})$ on the unit circle such that

$$f(e^{i\theta}) = Q^*(e^{i\theta})Q(e^{i\theta}) \quad \text{a.c.}, \quad (4.3.2)$$

in the sense that

$$(f(e^{i\theta})x)(y) = (Q(e^{i\theta})x, Q(e^{i\theta})y), \quad x, y \in \mathcal{X}.$$

Proof. By lemma 4.2.2 there exists a Hilbert space $\mathcal{A}$, an operator T in $B(\mathcal{X}, \mathcal{A})$, and a bilinear functional g on $\mathcal{A} \times \mathcal{A}$ with $\mathcal{A} = \mathcal{E}(T\mathcal{X})$ such that

$$(f(e^{i\theta})x)(y) = g(e^{i\theta})(Tx, Ty); \quad \text{for } x, y \in \mathcal{X}. \quad (1)$$

Let $\{x_i, 1 \leq i < \infty\}$ be a countable dense subset of $\mathcal{X}$. Consider $\{Tx_i\}_{i=1}^{\infty}$ and let $\{e_i\}_{i=1}^{\infty}$ be the Gram-Schmidt orthogonalization of $\{Tx_i\}_{i=1}^{\infty}$. Set

$$g_{ij}(e^{i\theta}) = g(e^{i\theta})(e_i, e_j).$$

It is clear that $g_{ij}(e^{i\theta})$ defines a nonnegative matrix (not necessarily bounded). The result of p. 112 [10] can be applied

to show the existence of a Hilbert space $\mathcal{H}$ and a sequence

$\{F_i\}_{i=1}^{\infty}$ of functions in $L_2(\mathcal{H})$ such that $g_{ij}(e^{i\theta}) = (F_i(e^{i\theta}), F_j(e^{i\theta}))_{\mathcal{H}}$. Following [10], p. 113, we obtain an operator A on the finite linear combinations of $\{e_i\}_{i=1}^{\infty}$ by

$$A\left(\sum_{i=1}^N a_i e_i\right) = \sum_{i=1}^N a_i F_i. \quad (2)$$

It is clear that

$$\left\|A\left(\sum_{i=1}^N a_i e_i\right)\right\|_{\mathcal{H}}^2 = \sum_{i,j=1}^N a_i \bar{a}_j g_{ij}(e^{i\theta}). \quad (3)$$

Let $x \in \{x_i, 1 \leq i < \infty\}$. Then $Tx = \sum_{i=1}^N a_i e_i$, for some a_i , $1 \leq i \leq N < \infty$. Then by (3) we have

$$\|ATx\|_{\mathcal{H}}^2 = \sum_{i,j=1}^N a_i \bar{a}_j g_{ij}(e^{i\theta}). \quad (4)$$

By (1), (4) and bilinearity of g we have

$$\|ATx\|_{\mathcal{H}}^2 = (f(e^{i\theta})x)(x), \quad x \in \{x_i, 1 \leq i < \infty\}. \quad (5)$$

Because f is weakly summable by (5) we have that the operator

$AT: \mathcal{X} \rightarrow L_2(\mathcal{H})$ is densely defined. For almost all θ 's we de-

fine $B(e^{i\theta})$ on $\{x_i, 1 \leq i < \infty\}$ by $B(e^{i\theta})x = (ATx)(e^{i\theta})$.

Extend $B(e^{i\theta})$ to $\mathcal{X}_0$, the finite rational linear combinations of x_i 's through linearity. From (5) we obtain

$$(B(e^{i\theta})x, B(e^{i\theta})y) = (f(e^{i\theta})x)(y), \quad x, y \in \mathcal{X}_0. \quad (6)$$

This shows that for almost all θ 's, $B(e^{i\theta})$ is well defined and bounded on $\mathcal{X}_0$. Hence a.e. $B(e^{i\theta})$ has a continuous extension to $\mathcal{X}$. Call this extension $Q(e^{i\theta})$. It follows by (6) and the continuity of Q and f that

$$Q^*(e^{i\theta})Q(e^{i\theta}) = f(e^{i\theta}).$$

Furthermore from the continuity of $Q(e^{i\theta})$, and (2) it follows that $Q(e^{i\theta})$ is weakly measurable. This completes the proof.

Lemma 4.3.1 allows us to make the following definition.

4.3.3 Definition. The function $Q = Q(e^{i\theta})$ in lemma 4.3.1 is called a quasi square root of f .

In the absense of the square root, a quasi square root has almost all the desired properties. One of its applications is demonstrated in the following lemma where we extend to the Banach space a result due to Helson [10], p. 117. First we introduce the following definition.

4.3.4 Definition. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Let Q be a quasi square root of f with values in $B(\mathcal{X}, \mathcal{X})$. $\mathcal{M}(Q)$ will denote the smallest subspace of $L_2(\mathcal{X})$ invariant with respect to U (U is multiplication by $e^{-i\theta}$) that contains the functions $\{Q(e^{i\theta})x : x \in \mathcal{X}\}$. A function $p(e^{i\theta}) = \sum_{n=0}^N e^{-in\theta} x_n$ with $x_n \in \mathcal{X}$ is said to be a conjugate analytic trigonometric polynomial and the set of all such polynomials will be denoted by $\mathcal{U}$. It is clear that $\mathcal{M}(Q)$ is the norm closure of $Q(\mathcal{U})$ in $L_2(\mathcal{X})$.

4.3.5 Lemma. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then f is factorable iff for any Q , a quasi square root of f , $\mathcal{M}(Q)$ contains no non-trivial reducing subspace of U .

Proof. Suppose $\mathcal{M}(Q) \subseteq L_2(\mathcal{X})$ contains no non-trivial reducing subspace. Then, [10], p. 61, it has the form $V(L_2^{0-}(\mathcal{W}))$, where

V is a measurable isometry operator on $\mathcal{M}$ into $\mathcal{X}$. In particular $Qx = VG_x$, $G_x \in L_2^{0-}(\mathcal{M})$. Therefore

$$(fx)(x) = (Qx, Qx)_{\mathcal{X}} = (VG_x, VG_x)_{\mathcal{X}} = (G_x, G_x)_{\mathcal{X}}. \quad (1)$$

Define the operator $\Phi(e^{i\theta})$ on $\mathcal{X}$ into $\mathcal{X}$ by

$$\Phi(e^{i\theta})x = G_x(e^{i\theta}), \quad x \in \mathcal{X} \text{ a.e.} \quad (2)$$

It is clear that $\Phi(e^{i\theta})$ is linear, and moreover by (1) and (2) we have

$$\begin{aligned} \|\Phi(e^{i\theta})x\|_{\mathcal{X}}^2 &= \|G_x(e^{i\theta})\|_{\mathcal{X}}^2 \\ &= \|V(e^{i\theta})G_x(e^{i\theta})\|_{\mathcal{M}}^2 = (f(e^{i\theta})x)(x). \end{aligned} \quad (3)$$

Hence Φ is bounded. Then by (2) and (3) and the weak summability of f it follows that Φ is a conjugate analytic $B(\mathcal{X}, \mathcal{X})$ -valued function. Hence

$$f(e^{i\theta}) = \Phi^*(e^{i\theta})\Phi(e^{i\theta}). \quad (4)$$

By (4) f is factorable.

Now assume f is factorable, say $f = \Phi^* \Phi$, where Φ is a conjugate analytic $B(\mathcal{X}, \mathcal{X})$ -valued function. Let Q be a $B(\mathcal{X}, \mathcal{M})$ -valued function which is a quasi square root of f . We can compare Φ and Q as follows. Define

$$V(\Phi p) = Qp, \quad p \in \mathcal{U}. \quad (5)$$

We have $(V(\Phi p), V(\Phi p))_{L_2(\mathcal{M})} = (Qp, Qp)_{L_2(\mathcal{M})} = \frac{1}{2\pi} \int_0^{2\pi} (f(e^{i\theta})p(e^{i\theta})(p(e^{i\theta})))d\theta = (\Phi p, \Phi p)_{L_2(\mathcal{X})}$. Hence we can extend V to an isometry on $\mathcal{M}(\Phi)$ onto $\mathcal{M}(Q)$, where $\mathcal{M}(\Phi) = \mathcal{S}(\Phi p, p \in \mathcal{U})$.

This mapping commutes with multiplication with $e^{-i\theta}$. Now $\mathcal{M}(\Phi)$ contains no non-trivial reducing subspace of the shift U , because it is a part of $L_2^{0-}(\mathcal{X})$. Hence its image $\mathcal{M}(Q)$, under V cannot contain a non-trivial reducing subspace.

Now we can extend the main lemma of Douglas as follows.

4.3.6 Lemma. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then f is factorable iff for each non-zero function $g \in \mathcal{M}(Q)$, the measure of $Z_g = \{e^{i\theta}, g(e^{i\theta}) = 0\}$ is positive.

Proof. This follows from lemma 4.3.5 and the fact that an invariant subspace of the shift U contains a non-trivial reducing subspace of U iff it contains a non-zero function g for which the measure of Z_g is positive.

4.4 Main Theorems. In this section we extend most of the results of Douglas [4] to the Banach space case. Lemma 4.3.6 is repeatedly used in the course of the proof of our theorems.

4.4.1 Theorem. Let f_1 and f_2 be weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued functions on the unit circle and Q_1 and Q_2 be $B(\mathcal{X}, \mathcal{X})$ -valued quasi square roots of f_1 and f_2 respectively such that

$$(a) \quad f_2(e^{i\theta}) \geq f_1(e^{i\theta}) \quad \text{a.e.},$$

$$(b) \quad \eta(Q_2^*) \supseteq \eta(Q_1^*) \quad \text{a.e.},$$

$$(c) \quad \varphi(e^{i\theta}) \|Q_1(e^{i\theta})x\|_{\mathcal{X}} \geq \|Q_1^*(e^{i\theta})Q_2(e^{i\theta})x\|_{\mathcal{X}^*}, \quad \text{a.e.},$$

where φ is some nonnegative scalar valued function. If f_1 is factorable then f_2 is factorable.

Proof. Let u be the set of all conjugate analytic polynomials in χ . Suppose $g \in \mathcal{M}(Q_2)$. Then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in u such that $\{Q_2 p_n\}_{n=1}^{\infty}$ converges to some g in $L_2(\chi)$. Now $f_1(e^{i\theta}) \leq f_2(e^{i\theta})$ a.e. implies that $\{Q_1 p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_2(\chi)$. Therefore there exists some $h \in \mathcal{M}(Q_1)$ such that $\{Q_1 p_n\}_{n=1}^{\infty}$ converges to h in $L_2(\chi)$. We choose a subsequence of p_n , denoting it again by p_n , such that

$$\begin{cases} Q_1(e^{i\theta})p_n(e^{i\theta}) \text{ converges a.e. to } h(e^{i\theta}) \text{ in } \chi \\ Q_2(e^{i\theta})p_n(e^{i\theta}) \text{ converges a.e. to } g(e^{i\theta}) \text{ in } \chi. \end{cases} \quad (1)$$

By (a) and (1) we have

$$\|h(e^{i\theta})\|_{\chi} \leq \|g(e^{i\theta})\|_{\chi} \text{ a.e.}$$

Hence

$$\text{the measure of } Z \setminus Z_h \text{ is zero.} \quad (2)$$

It follows from (c) that

$$\begin{aligned} \|Q_1^*(e^{i\theta})g(e^{i\theta})\|_{\chi^*} &= \lim_{n \rightarrow \infty} \|Q_1^*(e^{i\theta})Q_2(e^{i\theta})p_n(e^{i\theta})\|_{\chi^*} \\ &\leq \varphi(e^{i\theta}) \lim_{n \rightarrow \infty} \|Q(e^{i\theta})p_n(e^{i\theta})\|_{\chi} \\ &= \varphi(e^{i\theta}) \|h(e^{i\theta})\|_{\chi}. \end{aligned}$$

Hence by (b), for almost all θ 's we have the following implications.

$$\begin{aligned} h(e^{i\theta}) = 0 &\Rightarrow Q_1^*(e^{i\theta})g(e^{i\theta}) = 0 \Rightarrow Q_2^*(e^{i\theta})g(e^{i\theta}) = 0 \\ &\Rightarrow g(e^{i\theta}) \in \mathcal{M}(Q_2^*). \end{aligned}$$

But on the other hand by (1) we have $g(e^{i\theta}) \in$ closure of range of $Q_2(e^{i\theta})$. Hence $g(e^{i\theta}) = 0$ a.e., because closure of range $Q_2(e^{i\theta})$ is a subset of $\mathcal{M}^\perp(Q_2^*(e^{i\theta}))$. Therefore

$$Z_h \setminus Z_g \text{ has zero measure.} \quad (3)$$

(2) and (3) imply that Z_h and Z_g are a.e. equal. Now applying lemma 4.3.5 we conclude that f_2 is factorable.

4.4.2 Remark. In case $\mathcal{X}$ is a Hilbert space with $Q_1 = \sqrt{f_1}$, $Q_2 = \sqrt{f_2}$, (a) and (b) imply that $\mathcal{M}(f_1) = \mathcal{M}(f_2)$. Also in this case, condition (c) is the same as $\varphi(e^{i\theta})f_1(e^{i\theta}) \geq Q_2(e^{i\theta})f_1(e^{i\theta})Q_2(e^{i\theta})$ a.e. Hence our result 4.4.1 extends the main theorem of Douglas [4].

The following theorem does not seem to follow from theorem

4.4.1. However we provide a direct proof of it based on lemma 4.3.5.

4.4.3 Theorem. Let f_1 and f_2 be weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued functions on the unit circle such that

$$f_2(e^{i\theta}) \geq f_1(e^{i\theta}) \geq \varphi(e^{i\theta})f_2(e^{i\theta}) \text{ a.e.,}$$

where $\varphi(e^{i\theta})$ is a positive scalar valued function. If f_1 is factorable, then f_2 is factorable.

Proof. By lemma 4.3.5 it is sufficient to prove that for a non-zero g in $\mathcal{M}(Q_2)$, the measure of Z_g is zero. Let $Q_2 p_n \rightarrow g$ in $L_2(\mathcal{X}_2)$, then $f_1 \leq f_2$ implies that there exists $h \in L_2(\mathcal{X}_1)$ such that $Q_1 p_n \rightarrow h$ in $L_2(\mathcal{X}_1)$. Choose a subsequence of p_n , denoting it again by p_n , such that

$$\begin{cases} Q_2(e^{i\theta})P_n(e^{i\theta}) \text{ converges a.e. in } \chi_2 \text{ to } g(e^{i\theta}) \\ Q_1(e^{i\theta})P_n(e^{i\theta}) \text{ converges a.e. in } \chi_1 \text{ to } h(e^{i\theta}). \end{cases} \quad (1)$$

By (1) we have, a.e., $\|h(e^{i\theta})\|_{\chi_1} \leq \|g(e^{i\theta})\|_{\chi_2}$. Hence the measure of $Z_g \setminus Z_h = 0$. Similarly by (1) and assumption for almost all θ 's we have

$$\begin{aligned} \|g(e^{i\theta})\|_{\chi_2} &= \lim_{n \rightarrow \infty} \|Q_2(e^{i\theta})P_n(e^{i\theta})\|_{\chi_2} \\ &\leq \frac{1}{\varphi(e^{i\theta})} \lim_{n \rightarrow \infty} \|Q_1(e^{i\theta})P_n(e^{i\theta})\|_{\chi_1} \\ &= \frac{1}{\varphi(e^{i\theta})} \|h(e^{i\theta})\|_{\chi_1}. \end{aligned}$$

Hence the measure of $Z_h \setminus Z_g$ is zero. So we have shown that $Z_h = Z_g$ a.e. Now by lemma 4.3.5 and factorability of f_1 it follows that f_2 is factorable.

The following theorem is a slight extension of theorem 4.4.1.

4.4.4 Theorem. Let f_1 and f_2 be weakly summable $B^+(\chi, \chi^*)$ -valued functions on the unit circle such that

- (a) $f_2(e^{i\theta}) \geq m(e^{i\theta})f_1(e^{i\theta})$ a.e. where $m(e^{i\theta})$ is non-negative scalar valued and $\int_0^{2\pi} \log m(e^{i\theta}) d\theta > -\infty$,
- (b) $\eta(Q_2^*) \supset \eta(Q_1^*)$, a.e.,
- (c) $\varphi(e^{i\theta}) \|Q_1(e^{i\theta})x\|_{\chi} \geq \|Q_2^*(e^{i\theta})Q_1(e^{i\theta})x\|_{\chi^*}$ a.e.

where φ is a nonnegative scalar valued function.

If f_1 is factorable, then f_2 is factorable.

Proof. The proof is a combination of a standard method and theorem

4.4.1. Let $t(e^{i\theta}) = 1 \wedge m(e^{i\theta})$. Then $0 \leq t(e^{i\theta}) \leq 1$ a.e. and $\int_0^{2\pi} \log t(e^{i\theta}) d\theta > -\infty$. By Szegő's theorem there exists $k=k(e^{i\theta})$ in the Hardy class H^2 such that $t(e^{i\theta}) = |k(e^{i\theta})|^2$. Assuming $f_1(e^{i\theta}) = \phi^*(e^{i\theta})\phi(e^{i\theta})$ we have

$$t(e^{i\theta})f_1(e^{i\theta}) = (\bar{k}(e^{i\theta})\phi(e^{i\theta}))^*(\bar{k}(e^{i\theta})\phi(e^{i\theta})).$$

Applying theorem 4.4.1 to $f_1(e^{i\theta})$ and $t(e^{i\theta})f_1(e^{i\theta})$ we conclude the factorability of f_2 .

We now state the following extension of theorem 4.4.3, whose proof is omitted.

4.4.5 Theorem. Let f_1 and f_2 be weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued functions on the unit circle such that

$$f_2(e^{i\theta}) \geq m(e^{i\theta})f_1(e^{i\theta}) \geq \varphi(e^{i\theta})f_2(e^{i\theta}) \quad \text{a.e.},$$

where m and φ are positive scalar valued functions, with $\int_0^{2\pi} \log m(e^{i\theta}) d\theta > -\infty$. If f_1 is factorable, then f_2 is factorable.

Now we shall give some Devinatz' type theorems. First we introduce the following definition.

4.4.6 Definition. Let f be a $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle, we say that

(a) f has a "conjugate analytic null function" if

(i) $\eta(f(e^{i\theta}))$ is complementary a.e. and

(ii) $\|P(e^{i\theta})x\|_{\mathcal{X}}^2 = (\phi(e^{i\theta})x, \phi(e^{i\theta})x)_{\mathcal{X}}$; $x \in \mathcal{X}$,

where P is the projection into the complement of $\eta(f(e^{i\theta}))$

along $\eta(f(e^{i\theta}))$, and ϕ is a conjugate analytic $B(X, X)$ -valued function.

(b) f has a "quasi conjugate analytic null function" if condition (a) (i) holds and (a) (ii) replaced by

$$\begin{aligned} (\phi(e^{i\theta})x, \phi(e^{i\theta})x)_X &\leq \|p(e^{i\theta})x\|_X^2 \\ &\leq \varphi(e^{i\theta})(\phi(e^{i\theta})x, \phi(e^{i\theta})x)_X, \quad x \in X, \end{aligned}$$

where φ is a positive scalar valued function.

We remark that in the Hilbert space case the terminologies "conjugate analytic null", "quasi conjugate analytic null" and "conjugate analytic range" are all equivalent.

The following result is a generalization of theorem 2 of [4] to the Banach space case.

4.4.7 Theorem. Let f be a weakly summable $B^+(\chi, \chi^*)$ -valued function on the unit circle such that

(a) f has a quasi conjugate analytic null function,

(b) $(f(e^{i\theta})x)(x) \geq n(e^{i\theta})\|p(e^{i\theta})x\|_X^2$,

where $n(e^{i\theta})$ is a positive valued function with $\int_0^{2\pi} \log n(e^{i\theta}) d\theta > -\infty$.

Then f is factorable.

Proof. Let $t(e^{i\theta}) = 1 \wedge n(e^{i\theta})$. Then as in the proof of theorem

4.4.4, $t(e^{i\theta}) = |p(e^{i\theta})|^2$, with $p(e^{i\theta}) \in H^2$. Since $f = Q^*Q$,

$\eta(f) = \eta(Q)$, and $(f(x))(y) = 0$ if either x or y is in $\eta(Q)$.

We then have

$$n(e^{i\theta})\|p(e^{i\theta})x\|_X^2 \leq (f(e^{i\theta})x)(x) \leq |f(e^{i\theta})|^2\|p(e^{i\theta})x\|_X^2. \quad (1)$$

By (1) and our assumptions we have

$$(\bar{\varphi}(e^{i\theta})x, \bar{\varphi}(e^{i\theta})x)_{\mathcal{X}} \leq (f(e^{i\theta})x)(x) \leq \|f(e^{i\theta})\| \varphi(e^{i\theta}) (\bar{\varphi}(e^{i\theta})x, \bar{\varphi}(e^{i\theta})x)_{\mathcal{X}}$$

where φ is as in definition 4.4.6(b). Hence we have

$$\begin{aligned} (p(e^{i\theta})\bar{\varphi}(e^{i\theta}))^* (p(e^{i\theta})\bar{\varphi}(e^{i\theta})) &\leq f(e^{i\theta}) \\ &\leq \psi(e^{i\theta}) (\bar{p}(e^{i\theta})\bar{\varphi}(e^{i\theta}))^* (\bar{p}(e^{i\theta})\bar{\varphi}(e^{i\theta})), \end{aligned} \quad (2)$$

where $\psi = |f|\varphi/t$. Hence by theorem 4.4.3, f is factorable.

Using this result we can prove the following theorem which generalizes Devinatz' theorem [10], p. 119 to the Banach space case.

4.4.8 Theorem. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle such that $f^{-1}(e^{i\theta})$ exists a.e. and is bounded. If $\int_0^{2\pi} \log[\|f^{-1}(e^{i\theta})\|^{-1}] d\theta > -\infty$, then f is factorable.

Proof. Let us denote $\|f^{-1}(e^{i\theta})\|^{-1}$ by $n(e^{i\theta})$, so we have

$$n(e^{i\theta})\|x\|^2 \leq (f(e^{i\theta})x)(x) \quad \text{and} \quad \int_0^{2\pi} \log n(e^{i\theta}) d\theta > -\infty. \quad (1)$$

Denoting the positive quantity $\frac{1}{2\pi} \int_0^{2\pi} n(e^{i\theta}) d\theta$ by N^2 from (1) we obtain

$$N^2|x|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} (f(e^{i\theta})x)(x) d\theta. \quad (2)$$

By [1] there exists a Hilbert space $\mathcal{H}$ and an operator T in $B(\mathcal{X}, \mathcal{H})$ such that

$$(T^*Tx)(x) = \frac{1}{2\pi} \int_0^{2\pi} (f(e^{i\theta})x)(x) d\theta, \quad x \in \mathcal{X}. \quad (3)$$

By (2), (3) and boundedness of T we have

$$N\|x\| \leq \|Tx\|_{\mathcal{H}} \leq M\|x\| \quad \text{for all } x \in \mathcal{X}, \quad (4)$$

where $0 < N \leq M < \infty$. We note that $\mathcal{M}(f(e^{i\theta})) = \{0\}$, so that the projection operator occurring in the last theorem is identity. Then (1) and (4) guarantee the validity of the hypothesis of theorem 4.4.7. Hence f is factorable.

4.4.9 Remark. As we have seen above the condition

$\int_0^{2\pi} \log[\|f^{-1}(e^{i\theta})\|^{-1}] d\theta > -\infty$ implies the existence of a Hilbert space $\mathcal{H}$ and a bounded linear operator T on $\mathcal{H}$ onto $\mathcal{H}$ which is one to one. This means that the topology of $\mathcal{H}$ can be obtained through an inner product. Hence one could also obtain our theorem 4.4.9 by appealing directly to the Hilbert space case.

It is useful to know under what condition the finite sum, limit and series of factorable $B^+(\mathcal{H}, \mathcal{H}^*)$ -valued functions is factorable. Having our main lemma 4.3.6 available we can prove the following theorems.

4.4.10 Theorem. Let f_1 and f_2 be weakly summable $B^+(\mathcal{H}, \mathcal{H}^*)$ -valued functions on the unit circle. If f_1 and f_2 are factorable, then $f = f_1 + f_2$ is factorable.

Proof. Let Q_1, Q_2 and Q be quasi square roots of f_1, f_2 and f respectively, and $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}$ be the corresponding Hilbert spaces. Let $g \in \mathcal{M}(Q)$, then there exists a sequence $p_n \in \mathcal{U}$ such that $\lim_{n \rightarrow \infty} Q p_n = g$ in $L_2(\mathcal{H})$. Since $f \geq f_j$, $j = 1, 2$, giving similar argument as in the proof of theorem 4.4.3 we can show the existence of a subsequence of p_n , denoting it again by p_n , such that

$$\begin{cases} \lim_{n \rightarrow \infty} Q p_n = g \text{ in } L_2(\mathcal{H}); \lim_{n \rightarrow \infty} Q_j p_n = g_j \text{ in } L_2(\mathcal{H}_j), j = 1, 2, \\ \lim_{n \rightarrow \infty} Q p_n = g \text{ in } \mathcal{H} \text{ a.e.}; \lim_{n \rightarrow \infty} Q_j p_n = g_j \text{ in } \mathcal{H}_j \text{ a.e.}, j = 1, 2. \end{cases} \quad (1)$$

From (1) it follows that

$$\|g_1(e^{i\theta})\|_{\mathcal{X}_1}^2 + \|g_2(e^{i\theta})\|_{\mathcal{X}_2}^2 = \|g(e^{i\theta})\|_{\mathcal{X}}^2 \quad \text{a.e.} \quad (2)$$

Since f_j ($j = 1, 2$) is factorable, by lemma 4.3.6 either g_j ($j = 1, 2$) is a zero function or the measure of Zg_j is zero. In any case from (2) it follows that either g is a zero function or the measure of Zg is zero. Hence by lemma 4.3.6 the proof is complete.

4.4.11 Theorem. Let $\{f_j\}_{j=1}^{\infty}$ be an increasing sequence of factorable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued functions on the unit circle and φ be a non-negative scalar valued function such that

- (a) $\lim_{j \rightarrow \infty} f_j(e^{i\theta}) = f(e^{i\theta}) \quad \text{a.e.},$
- (b) $\lim_{n \rightarrow \infty} \int_0^{2\pi} (f_j(e^{i\theta})x)(x) d\theta < \infty, \text{ for all } x \in \mathcal{X},$
- (c) $\|f(e^{i\theta})\| \leq \varphi(e^{i\theta}) \quad \text{a.e.}$

Then f is factorable.

Proof. Let $\{Q_j\}_{j=1}^{\infty}$ and Q denote quasi square roots of $\{f_j\}_{j=1}^{\infty}$ and f respectively. By (b) and (c) f is weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued. If f is not factorable then there exists a sequence $p_n \in \mathcal{U}$ and a function $g \in L_2(\mathcal{X})$ such that $\lim_{n \rightarrow \infty} Q p_n = g$ in $L_2(\mathcal{X})$ with g non-zero and the measure of Zg is positive (c.f. lemma 4.3.6). As in the proof of the last theorem, there exists a subsequence of p_n , say p_n , so that $\lim_{n \rightarrow \infty} Q_j p_n = g_j$, for each j , in $L_2(\mathcal{X}_j)$ norm, and $\lim_{n \rightarrow \infty} Q_j p_n = g_j$ a.e. in $\mathcal{X}_j$. (These limits are uniform w.r.t. j because f dominates all f_j 's.) Since $\lim_{j \rightarrow \infty} f_j(e^{i\theta}) = f(e^{i\theta})$ in the

strong sense we have $\lim_{j \rightarrow \infty} (f_j(e^{i\theta})_x)(x) = (f(e^{i\theta})_x)(x)$, $x \in X$,
and hence for almost all θ we have

$$\lim_{j \rightarrow \infty} \|g_j(e^{i\theta})\|_{X_j} = \|g(e^{i\theta})\|_X.$$

Thus the measure of Z_g is pointwise positive for some j which implies by lemma 4.3.6, $g_j = 0$ a.e. and hence $g = 0$ a.e. This contradiction completes the proof.

4.4.12 Theorem. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of factorable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued functions on the unit circle and φ be a nonnegative scalar valued function such that

- (a) $\sum_{j=1}^{\infty} f_j(e^{i\theta}) = f(e^{i\theta})$
- (b) $\sum_{j=1}^{\infty} \int_0^{2\pi} (f_j(e^{i\theta})_x)(x) d\theta < \infty$, $x \in X$,
- (c) $\|f(e^{i\theta})\| < \varphi(e^{i\theta})$ a.e.

Then f is factorable.

Proof. Apply theorem 4.4.11 to the increasing sequence of partial

sums $\left\{ \sum_{j=1}^N f_j \right\}_{N=1}^{\infty}$, which are factorable by theorem 4.4.10.

CHAPTER V

NECESSARY AND SUFFICIENT CONDITIONS FOR FACTORABILITY OF NONNEGATIVE OPERATOR VALUED FUNCTIONS ON A BANACH SPACE

5.1 Introduction. In this chapter we continue to study the important problem of factoring a nonnegative operator valued function on a Banach space. In Chapter IV we were able to extend to the Banach space the work of R.G. Douglas [4] on factoring nonnegative operator valued functions. However these results provided only sufficient condition for the factorization problem. Our purpose here is to establish some necessary and sufficient conditions for factorability of nonnegative operator valued functions on a Banach space. This extends to the Banach space the recent work of Yu. A. Rozanov [19] and a certain result of R. Payen [18] on necessary and sufficient conditions for the factorization problem. It also reveals the close connection which exists between these characterizations.

In §5.2 we set up necessary terminologies and state some known results. In §5.3 we prove our main theorem on characterizing factorable operator valued functions on a Banach space. In establishing our main theorem we make use of quasi square roots and technique employed in [19].

5.2 Preliminaries. In this chapter all Banach spaces and Hilbert spaces will be separable.

Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\chi, \chi^*)$ valued function on the unit circle. Then by lemma 4.3.1 a quasi square root of f , $Q = Q(e^{i\theta})$, with values in $B(\chi, \chi)$ exists. Let $\xi_n = e^{in\theta} Q(e^{i\theta})$. Then ξ_n , $-\infty < n < \infty$, is a $B(\chi, L_2(\chi))$ -valued SSP whose spectral density is f . From here on ξ_n , $-\infty < n < \infty$, represents this process.

In §5.3 we need a lemma due to Rozanov. Because of its importance and for ease of reference we state this lemma here. First we introduce some notations (c.f. [19]). Let B be a linear manifold in $L_2(\chi)$ and $S = \{g_n(e^{i\theta})\}_{n=1}^\infty$ be a complete orthonormal system of functions in B . We denote by $B_S(e^{i\theta})$ the linear manifold in the Hilbert space χ generated by all values $g_1(e^{i\theta}), g_2(e^{i\theta}), \dots$. Obviously the closure $\overline{B}(e^{i\theta}) = \overline{B_S}(e^{i\theta})$ does not depend on S in the sense that $\overline{B_{S_1}}(e^{i\theta}) = \overline{B_{S_2}}(e^{i\theta})$ a.e. θ , if S_1 and S_2 are any two complete orthonormal systems in B .

In case $B = Q(e^{i\theta})\chi$ it easily follows that

$$\overline{B}(e^{i\theta}) = \overline{Q(e^{i\theta})\chi} \text{ a.e.,}$$

where $\overline{Q(e^{i\theta})\chi}$ denotes the closure of the range of the operator $Q(e^{i\theta})$.

5.2.1 Lemma (Rozanov). Let B be a linear manifold in $L_2(\chi)$. Then the subspace $\mathcal{E}(e^{in\theta} B, -\infty < n < \infty)$ generated by $e^{in\theta} B$, $-\infty < n < \infty$, consists of all functions $g \in L_2(\chi)$ such that

$$g(e^{i\theta}) \in \overline{B}(e^{i\theta}) \text{ a.e.}$$

5.3 Main results. In this section we prove our main results. χ is a separable Banach space and χ is a separable Hilbert space.

f is a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle.

Q will denote a quasi square root of f with values in $B(\mathcal{X}, \mathcal{X})$.

$\xi_n = e^{in\theta} Q(e^{i\theta})$, $-\infty < n < \infty$, is a $B(\mathcal{X}, L_2(\mathcal{X}))$ -valued SSP with the

spectral density f . Let $H = \mathcal{G}(\xi_n x, x \in \mathcal{X}, -\infty < n < \infty)$ and

$H(n) = \mathcal{G}(\xi_n x, x \in \mathcal{X}, -\infty \leq n)$.

We shall be interested in the structure of the subspaces

$B = H(T) \ominus H(S)$, where T, S are some sets of integers and for

any set T

$$H(T) = \mathcal{G}(\xi_n x, x \in \mathcal{X}, n \in T).$$

One can say that B is the innovation of $H(T)$ in comparison with $H(S)$.

L_T will denote the linear space of all $\mathcal{X}^*$ valued integrable functions φ {for each $x \in \mathcal{X}$, $\varphi(e^{i\theta})x$ is summable} with Fourier decomposition of the form

$$\varphi(e^{i\theta}) \sim \sum_{n \in T} a_n e^{ni\theta}, a_n \in \mathcal{X}^* \quad (5.3.1)$$

i.e.

$$\varphi(e^{i\theta})x \sim \sum_{n \in T} a_n x e^{ni\theta}, x \in \mathcal{X}$$

such that

$$\varphi(e^{i\theta}) \in Q^*(e^{i\theta})\mathcal{X} \quad (5.3.2)$$

and

$$\int_0^{2\pi} \|Q^{*-1}(e^{i\theta})\varphi(e^{i\theta})\|^2 d\theta < \infty, \quad (5.3.3)$$

where $Q^{*-1}(e^{i\theta})$ is the inverse operator from $Q^*(e^{i\theta})\mathcal{X}$ onto

$\mathcal{N}_{Q^*(e^{i\theta})}^\perp$ = orthogonal complement of the null subspace of $Q^*(e^{i\theta})$.

5.3.4 Lemma. Let S be the complement of T in the set of all integers and $B_T = H(T) \ominus H(S)$. Then

$$B_T = Q^{*-1} L_T. \quad (5.3.5)$$

Proof. Let $\varphi(e^{i\theta}) \in B_T$. Define $\psi(e^{i\theta}) = Q^*(e^{i\theta})\varphi(e^{i\theta})$. From the relations $\int_0^{2\pi} (Q^*(e^{i\theta})\varphi(e^{i\theta}))(x) d\theta = \int_0^{2\pi} (\varphi(e^{i\theta}), Q(e^{i\theta})x) d\theta$
 $\leq (\int_0^{2\pi} \|\varphi(e^{i\theta})\|^2 d\theta)^{\frac{1}{2}} (\int_0^{2\pi} \|Q(e^{i\theta})x\|^2 d\theta)^{\frac{1}{2}} < \infty$, it follows that $\psi(e^{i\theta})(x)$ is summable, $x \in \mathcal{X}$. Also we have that $0 = \int_0^{2\pi} e^{-is\theta} (\varphi(e^{i\theta}), Q(e^{i\theta})x) d\theta = \int_0^{2\pi} e^{-is\theta} (Q^*(e^{i\theta}), \varphi(e^{i\theta}))(x) d\theta$, $s \in S$. Since $\varphi \in H$, by lemma 5.2.1 and the paragraph preceding it $\varphi(e^{i\theta}) \in Q(e^{i\theta})\mathcal{X}$ a.e. Hence $\varphi(e^{i\theta}) = Q^{*-1}(e^{i\theta})\psi(e^{i\theta})$ and $\psi(e^{i\theta}) \in L_T$. Thus $B_T \subseteq Q^{*-1} L_T$. Now let $\psi(e^{i\theta})$ be in L_T . Set $\varphi(e^{i\theta}) = Q^{*-1}(e^{i\theta})\psi(e^{i\theta})$. We will show that $\varphi(e^{i\theta}) \in B_T$. We note that for each $x \in \mathcal{X}$, $(\varphi(e^{i\theta}), Q(e^{i\theta})x) = \psi(e^{i\theta})(x)$. Hence $\int_0^{2\pi} e^{-is\theta} (\varphi(e^{i\theta}), Q(e^{i\theta})x) d\theta = 0$, $s \in S$, $x \in \mathcal{X}$. It suffices to show that $\varphi \in H = \overline{\mathfrak{S}(e^{in\theta} Q(e^{i\theta})\mathcal{X}, -\infty < n < \infty)}$. But $\varphi(e^{i\theta}) \in \text{range of } Q^{*-1}(e^{i\theta}) = \overline{Q(e^{i\theta})\mathcal{X}}$. Hence by lemma 5.2.1 and the paragraph preceding it we have that $\varphi(e^{i\theta}) \in B_T$. This completes the proof of $Q^{*-1} L_T \subseteq B_T$.

5.3.6 Corollary. The relation $\mathfrak{S}(e^{in\theta} B_T, -\infty < n < \infty) = H$ holds if and only if

$$\overline{Q(e^{i\theta})\mathcal{X}} = \overline{Q^{*-1} L_T(e^{i\theta})} \quad \text{a.e.} \quad (5.3.7)$$

Proof. Since $H = \mathfrak{S}(e^{in\theta} Q(e^{i\theta})\mathcal{X}, -\infty < n < \infty)$, by lemma 5.2.1 $H = \mathfrak{S}(e^{in\theta} B_T, -\infty < n < \infty)$ if and only if $\overline{B_T}(e^{i\theta}) = \overline{Q(e^{i\theta})\mathcal{X}}$. But by the last lemma $\overline{B_T}(e^{i\theta}) = \overline{Q^{*-1} L_T(e^{i\theta})}$.

We note that this result in particular is useful for regularity if we take T to be the set of all nonnegative integers. In this case the space L_T is included in the Hardy class of functions

$$\varphi(e^{i\theta}) \sim \sum_0^{\infty} a_n e^{in\theta}.$$

More precisely they are so that for each $x \in \mathcal{X}$, the scalar valued function $\varphi(e^{i\theta})(x)$ is in the classical Hardy class H^1 . We also note that in this case the relation

$$\mathcal{G}(e^{in\theta}_{B_T}, -\infty < n < \infty) = H$$

of corollary 5.3.6 is equivalent to the regularity of the process $\xi_n = e^{in\theta} Q(e^{i\theta}), -\infty < n < \infty$.

We are now ready to prove our main result.

5.3.8 Theorem. Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then the following statements are equivalent:

1. f is factorable
2. there exists a conjugate analytic $B(\mathcal{X}, \mathcal{X})$ -valued function ψ such that

$$(i) \quad \psi^*(e^{i\theta})\mathcal{X} \subseteq Q^*(e^{i\theta})\mathcal{X}$$

$$(ii) \quad \overline{Q^{*-1}(e^{i\theta})\psi^*(e^{i\theta})\mathcal{X}} = \overline{Q(e^{i\theta})\mathcal{X}} \quad \text{a.e.}$$

$$(iii) \quad \int_0^{2\pi} \|Q^{*-1}(e^{i\theta})\psi^*(e^{i\theta})k\|^2 d\theta < \infty, \quad k \in \mathcal{X}.$$

3. The process $e^{in\theta} Q(e^{i\theta}), -\infty < n < \infty$, is regular.

4. There exists a sequence $\{\varphi_n(e^{i\theta})\}_{n=1}^{\infty}$ of measurable $\mathcal{X}$ -valued functions such that

(a) a.e. θ , $\{\varphi_n(e^{i\theta})\}_{n=1}^{\infty}$ is an orthonormal basis for $\overline{Q(e^{i\theta})\mathcal{X}}$,

(b) For each n , $g_n(e^{i\theta})x = Q^*(e^{i\theta})\varphi_n(e^{i\theta})x$ is in the usual Hardy class H^2 for all $x \in \mathcal{X}$.

Proof. (1) $\Rightarrow$ (2). Let $f = \bar{\Phi}^* \Phi$, where Φ is conjugate analytic.

Then

$$\|Q(e^{i\theta})x\|^2 = \|\bar{\Phi}(e^{i\theta})x\|^2, \quad x \in \mathcal{X}. \quad (5.3.9)$$

Define

$$V(e^{i\theta})Q(e^{i\theta})x = \bar{\Phi}(e^{i\theta})x, \quad x \in \mathcal{X}. \quad (5.3.10)$$

By (5.3.9) - (5.3.10), $V(e^{i\theta})$ can be extended to an isometry on $\overline{Q(e^{i\theta})\mathcal{X}}$ onto $\overline{\bar{\Phi}(e^{i\theta})\mathcal{X}}$. Then the operator valued function $\Psi = VQ$ satisfies the conditions (i) - (iii) of (2).

(2) $\Rightarrow$ (3). Let Ψ be a conjugate analytic $B(\mathcal{X}, \mathcal{X})$ -valued function satisfying (i) - (iii). By corollary 5.3.6 and the paragraph following this corollary, it suffices to show that

$$\overline{Q(e^{i\theta})\mathcal{X}} = \overline{Q^{*-1}L_T(e^{i\theta})} \quad \text{a.e.}$$

Clearly the right hand side is a subset of the left hand side.

The other inclusion follows from (ii) - (iii) and the fact that for each $k \in \mathcal{X}$, $\Psi^* k \in L_T$.

(3) $\Rightarrow$ (4). Since $e^{in\theta}Q(e^{i\theta})$, $-\infty < n < \infty$ is a regular process, $H(0) = \mathfrak{G}(e^{in\theta}Q(e^{i\theta})\mathcal{X}, n \leq 0)$ does not contain a non-trivial

doubly invariant subspace of $\mathcal{H}$. Hence by [10], p. 61 it is of the form $V L_2^0(\mathcal{M})$, where V is a measurable isometry operator on some Hilbert space $\mathcal{M}$ into $\mathcal{H}$. Let $\{\varphi_n(e^{i\theta})\}_{n=1}^\infty$ be an orthonormal basis for $H(0) \ominus H(-1)$. An argument similar to one used in [18], p. 380 and [10], p. 61 may be used to show that for almost all θ , $\{\varphi_n(e^{i\theta})\}_{n=1}^\infty$ forms an orthonormal basis for $Q(e^{i\theta})\mathcal{H}$. Let $x \in \mathcal{H}$. To show $g_n(e^{i\theta})(x) = Q^*(e^{i\theta})\varphi_n(e^{i\theta})(x)$ is in the Hardy class H^2 , we observe that for each n ,

$$\begin{aligned} \varphi_n \perp e^{ik\theta} Q(e^{i\theta})x \text{ for } x \in \mathcal{H} \text{ and } k \leq -1. \text{ Hence} \\ \int_0^{2\pi} e^{-ik\theta} g_n(e^{i\theta})(x) d\theta = \int_0^{2\pi} e^{-ik\theta} Q^*(e^{i\theta})\varphi_n(e^{i\theta})(x) d\theta = \\ \int_0^{2\pi} e^{-ik\theta} (\varphi_n(e^{i\theta}), Q(e^{i\theta})x) d\theta = 0. \text{ Thus } g_n x \in H^2. \end{aligned}$$

(4) $\Rightarrow$ (1). Let (a) - (b) hold. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. We define the operator valued function Φ by

$$\Phi(e^{i\theta})x = \sum_n \overline{g_n(e^{i\theta})(x)} e_n, \quad x \in \mathcal{H}. \quad (5.3.11)$$

We note that for all $x, y \in \mathcal{H}$

$$\begin{aligned} (fx)(y) &= (Qx, Qy) = \sum_n (Qx, \varphi_n) \overline{(Qy, \varphi_n)} \\ &= \sum_n \overline{(Q^* \varphi_n x)} (Q^* \varphi_n y) = \sum_n \overline{(g_n x)} (g_n y) \\ &= (\sum_n \overline{(g_n x)} e_n, \sum_n \overline{(g_n y)} e_n) = (\Phi x, \Phi y), \end{aligned} \quad (5.3.12)$$

where the second equality follows by (a). Hence $\Phi(e^{i\theta}) \in B(\mathcal{H}, \mathcal{H})$.

Because of weak summability of f it follows that for each

$x \in \mathcal{H}$, $\Phi x \in L_2(\mathcal{H})$. Using (5.3.11) and (5.3.12) it is not hard

to show that the conjugate analytic $B(\mathcal{H}, \mathcal{H})$ -valued sequence

$$\Phi_N(e^{i\theta})x = \sum_{n=1}^N g_n(e^{i\theta})(x) e_n \text{ converges to } \Phi(e^{i\theta})x \text{ in } L_2(\mathcal{H}).$$

Therefore $\phi \in L_2^{0-}(\mathcal{H})$. Since by (5.3.12) $f = \phi^* \phi$ the result follows.

We remark that if one assumes $\int_0^{2\pi} \|f(e^{i\theta})\| d\theta < \infty$, then part (b) of (4) in the above theorem can be replaced by (b'):

For each n , $g_n(e^{i\theta}) = Q^*(e^{i\theta}) \varphi_n(e^{i\theta})$ satisfies

$$\int_0^{2\pi} \|g_n(e^{i\theta})\|^2 d\theta < \infty.$$

CHAPTER VI

FACTORIZATION OF UfU^*

6.1 Introduction. Let $f = f(e^{i\theta})$ be a $B^+(\mathcal{X}, \mathcal{X})$ -valued function on the unit circle. Suppose $U(e^{i\theta})$ is a $B(\mathcal{X}, \mathcal{X})$ unitary valued function. Then UfU^* is also nonnegative operator valued function. Suppose f is factorable, then the natural question raised by M.G. Nadkarni in [16] is to investigate the factorability of UfU^* . We study this problem here in this chapter. Using the results of Yu. A. Rozanov on factorization problem [19] (c.f. theorem 5.3.8) we give several necessary and sufficient conditions for the factorability of UfU^* . As a natural application of our theorems we obtain a result similar to the one given in [18], p. 381 on the factorization of a nonnegative operator valued function, involving the eigenvalues of f . We also obtain a Devinatz's type necessary condition for the factorability of nonnegative operator valued functions. In connection with these results we consider the problem of factoring a 2×2 nonnegative matrix valued function of rank one which has been discussed earlier by N. Wiener and P. Masani [27].

6.2 Factorability of UfU^* . Let $\mathcal{X}$ and $\mathcal{Y}$ be two separable Hilbert spaces. A $B^+(\mathcal{X}, \mathcal{X})$ -valued function $f = f(e^{i\theta})$ is called uniformly summable if $\int_0^{2\pi} \|f(e^{i\theta})\|^2 d\theta < \infty$, and weakly summable if $\int_0^{2\pi} (f(e^{i\theta})x, x) d\theta < \infty$, for all $x \in \mathcal{X}$. A weakly summable $B^+(\mathcal{X}, \mathcal{X})$ -

valued function is called factorable if $f(e^{i\theta}) = \phi^*(e^{i\theta})\phi(e^{i\theta})$ where $\phi(e^{i\theta})$ is a conjugate analytic $B(\mathcal{X}, \mathcal{X})$ -valued function (c.f. definition 2.9).

To stay in the framework of the standard results on Hilbert space valued stationary stochastic processes in the rest of this thesis we take the following definition of factorability which in the Hilbert space case is equivalent to our earlier notion of factorability given in definition 2.9.

We say that a weakly summable $B^+(\mathcal{X}, \mathcal{X})$ -valued function $f = f(e^{i\theta})$ is factorable if there exists an analytic $B(\mathcal{X}, \mathcal{X})$ -valued function $\phi = \phi(e^{i\theta})$ such that

$$f(e^{i\theta}) = \phi(e^{i\theta})\phi^*(e^{i\theta}).$$

The following theorem is a natural extension of Devinatz's theorem [10], p. 119.

6.2.1 Theorem. Let $f = f(e^{i\theta})$ be a uniformly summable $B^+(\mathcal{X}, \mathcal{X})$ -valued function which is a.e. invertible and

$$\int_0^{2\pi} \log \|f^{-1}(e^{i\theta})\| d\theta < \infty. \quad (1)$$

Then the $B^+(\mathcal{X}, \mathcal{X})$ -valued function UfU^* is factorable.

Proof. We observe that for each $x \in \mathcal{X}$ we have

$$((UfU^*)^{-1}x, x) = (Uf^{-1}U^*x, x) = (f^{-1}U^*x, U^*x) \leq \|f^{-1}\| \|x\|^2.$$

So we have $\|(UfU^*)^{-1}\| \leq \|f^{-1}\|$. Hence by (1) we see that

$\int_0^{2\pi} \log \|(UfU^*)^{-1}\| d\theta < \infty$. But by Devinatz's theorem we see that UfU^* is factorable.

The following theorem is an immediate consequence of theorem 6.2.1.

6.2.2 Corollary. Let $f = f(e^{i\theta})$ be a weakly summable non-negative finite dimensional matrix valued function which is of full rank. Then f is factorable if and only if UfU^* is factorable.

Recently Yu. A. Rozanov [19] gave a necessary and sufficient condition for factorability of a weakly summable $B^+(\chi, \chi)$ -valued function. We extended his results to the Banach space case in Chapter V.. However since we are going to use his result in this chapter, we will state his theorem in the context of the notations of this chapter. Before doing so we recall the following necessary notation.

6.2.3 Notation. As before $L_2(\chi)$ is the Hilbert space of all measurable χ valued functions $k(e^{i\theta})$ such that

$$\int_0^{2\pi} \|k(e^{i\theta})\|^2 d\theta < \infty, \text{ with the inner product of any two elements } k_1 = k_1(e^{i\theta}) \text{ and } k_2 = k_2(e^{i\theta}) \in L_2(\chi) \text{ defined by } (k_1, k_2) = \frac{1}{2\pi} \int_0^{2\pi} (k_1(e^{i\theta}), k_2(e^{i\theta})) d\theta.$$

6.2.4 Theorem (Rozanov). Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\chi, \chi)$ -valued function. Then f is factorable if and only if there exists an analytic operator valued function Ψ such that

$$(a) \quad \Psi(e^{i\theta})\chi \subseteq f^{\frac{1}{2}}(e^{i\theta})\chi \text{ a.e.}$$

$$(b) \quad \overline{f^{-\frac{1}{2}}(e^{i\theta})\Psi(e^{i\theta})\chi} = \overline{f^{\frac{1}{2}}(e^{i\theta})\chi} \text{ a.e.}$$

$$(c) \quad \int_0^{2\pi} \|f^{-\frac{1}{2}}(e^{i\theta})\Psi(e^{i\theta})x\|^2 d\theta < \infty,$$

where $f^{-\frac{1}{2}}(e^{i\theta})$ is the inverse operator from $f^{\frac{1}{2}}(e^{i\theta})\chi$ into $f^{\frac{1}{2}}(e^{i\theta})\chi$.

6.2.6 Theorem. Let f be a nonnegative finite dimensional matrix valued function such that $M(e^{i\theta})/m(e^{i\theta})$ is in L_1 , where $M(e^{i\theta})$ and $m(e^{i\theta})$ denote the largest and smallest non zero eigenvalue of $f(e^{i\theta})$. Suppose $f^{\frac{1}{2}}(e^{i\theta})\chi$ is an invariant subspace of $U(e^{i\theta})$. Then f is factorable if and only if UfU^* is factorable.

Proof. Suppose f is factorable as $f(e^{i\theta}) = \Phi(e^{i\theta})\Phi^*(e^{i\theta})$, with $\Phi(e^{i\theta})$ being analytic valued function. We will show that conditions (a) - (d) hold for UfU^* with $\Psi(e^{i\theta})$ being $\Phi(e^{i\theta})$. By [5], p. 413 we have

$$f^{\frac{1}{2}}(e^{i\theta})\chi = \Phi(e^{i\theta})\chi. \quad (2)$$

But by hypothesis $Uf^{\frac{1}{2}}\chi = f^{\frac{1}{2}}\chi$. Hence by (2) we have

$$\Phi\chi = f^{\frac{1}{2}}\chi = Uf^{\frac{1}{2}}\chi = Uf^{\frac{1}{2}}(U^*\chi) = (Uf^{\frac{1}{2}}U^*)\chi,$$

so we have $\Phi\chi = (Uf^{\frac{1}{2}}U^*)\chi$ for a.e. θ which is (a). Now by definition of $f^{-\frac{1}{2}}$ we have $f^{-\frac{1}{2}}f^{\frac{1}{2}}\chi = f^{\frac{1}{2}}\chi$. So we have

$$Uf^{-\frac{1}{2}}f^{\frac{1}{2}}\chi = Uf^{\frac{1}{2}}\chi. \quad (3)$$

Now $f^{\frac{1}{2}}\chi$ is invariant under U . Hence $f^{\frac{1}{2}}\chi$ is also invariant under U^* . Thus from (3) we get

$$Uf^{\frac{1}{2}}\chi = Uf^{-\frac{1}{2}}(f^{\frac{1}{2}}\chi) = Uf^{-\frac{1}{2}}(U^*f^{\frac{1}{2}}\chi) = Uf^{-\frac{1}{2}}U^*f^{\frac{1}{2}}\chi.$$

So by (2) we get $Uf^{\frac{1}{2}}\chi = Uf^{-\frac{1}{2}}U^*f^{\frac{1}{2}}\chi = Uf^{-\frac{1}{2}}U^*\Phi\chi$. Hence we get

$\overline{Uf^{\frac{1}{2}}U^*\chi} = \overline{Uf^{-\frac{1}{2}}U^*\Phi\chi}$ which is (b). To see (c) we have $f^{\frac{1}{2}}f^{\frac{1}{2}} = \Phi\Phi^*$.

By [5], p. 413 there exists an operator valued function $C(e^{i\theta})$ with Banach norm one such that $\Phi(e^{i\theta}) = f^{\frac{1}{2}}(e^{i\theta})C(e^{i\theta})$. So we have

$$\begin{aligned} \int_0^{2\pi} \|Uf^{-\frac{1}{2}}U^*\Phi\|^2 d\theta &\leq \int_0^{2\pi} \|f^{-\frac{1}{2}}U^*\Phi\|^2 d\theta \leq \int_0^{2\pi} \|f^{-\frac{1}{2}}U^*f^{-\frac{1}{2}}\|^2 d\theta \\ &\leq \int_0^{2\pi} \|f^{-\frac{1}{2}}\|^2 \|f^{\frac{1}{2}}\|^2 d\theta \leq \int_0^{2\pi} (M(e^{i\theta})/m(e^{i\theta})) d\theta < \infty. \end{aligned}$$

Now suppose UfU^* is factorable. Note that UfU^* and f have the same eigenvalues. We also note that $Uf^{\frac{1}{2}}U^*\chi$ is invariant under U^* , because $U^*(Uf^{\frac{1}{2}}U^*\chi) = U^*U(f^{\frac{1}{2}}U^*\chi) = f^{\frac{1}{2}}U^*\chi = f^{\frac{1}{2}}\chi = U(f^{\frac{1}{2}}\chi) = (Uf^{\frac{1}{2}})(\chi) = (Uf^{\frac{1}{2}})(U^*\chi) = Uf^{\frac{1}{2}}U^*\chi$. So repeating the argument we gave above, with UfU^* and U^* instead of f and U respectively we can show that f is factorable.

We now state the following infinite dimensional extension of the last theorem whose proof is exactly as in the last theorem and hence is omitted.

6.2.7 Theorem. Let U be an unitary $B(\mathcal{H}, \mathcal{M})$ -valued function. Let $f(e^{i\theta})$ be a uniformly summable $B^+(\mathcal{H}, \mathcal{H})$ -valued function such that $R(e^{i\theta}) = f(e^{i\theta})\mathcal{H}$ is a reducing subspace (closed) of $U(e^{i\theta})$. Suppose that $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ on $R(e^{i\theta})$, with $M(e^{i\theta})/m(e^{i\theta})$ being in L_1 . Then f is factorable if and only if UfU^* is factorable.

As a consequence to theorem 6.2.7 we obtain the following corollary.

6.2.8 Corollary. Let f be a uniformly summable $B^+(\mathcal{H}, \mathcal{H})$ -valued function satisfying $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$, with $M(e^{i\theta})/m(e^{i\theta}) \in L_1$. Let U be a unitary $B(\mathcal{H}, \mathcal{M})$ -valued function. Then f is factorable if and only if UfU^* is factorable.

Proof. From $M(e^{i\theta})/m(e^{i\theta}) \in L_1$, and

$$m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I,$$

it follows that $\overline{f(e^{i\theta})}\chi = \chi$ a.e. We now apply theorem 6.2.7 to conclude the proof of this corollary.

The next two theorems provide some necessary and sufficient conditions for the factorability of UfU^* .

6.2.9 Theorem. Let $f = f(e^{i\theta})$ be a $B^+(\mathcal{X}, \mathcal{X})$ -valued function such that $\int_0^{2\pi} \|f(e^{i\theta})\| d\theta < \infty$. Then UfU^* is factorable if and only if there exists a partial isometry valued function $V(e^{i\theta})$, with initial range in $\mathcal{M}$ and terminal range $\overline{U(e^{i\theta})f^{\frac{1}{2}}(e^{i\theta})U^*(e^{i\theta})\mathcal{M}}$, such that $U(e^{i\theta})f^{\frac{1}{2}}(e^{i\theta})U^*(e^{i\theta})V(e^{i\theta})$ is an analytic $B(\mathcal{M}, \mathcal{X})$ -valued function.

Proof. Sufficiency. Let $\Phi = Uf^{\frac{1}{2}}U^*V$. Then Φ is an analytic operator valued function and $\Phi\Phi^* = Uf^{\frac{1}{2}}U^*VV^*Uf^{\frac{1}{2}}U^* = UfU^*$.

Necessity. Suppose UfU^* is factorable of the form $UfU^* = \Phi\Phi^*$, where Φ is analytic operator valued function operator valued function. We then get

$$(Uf^{\frac{1}{2}}U^*)(Uf^{\frac{1}{2}}U^*) = \Phi\Phi^*. \quad (1)$$

So for each $x \in \mathcal{X}$ we have

$$\|(Uf^{\frac{1}{2}}U^*)^*x\|^2 = \|\Phi^*x\|^2. \quad (2)$$

Then we can define $W(e^{i\theta})$ on $(Uf^{\frac{1}{2}}U^*)^*\mathcal{X}$ into $\Phi^*\mathcal{X}$ by $W(e^{i\theta})((Uf^{\frac{1}{2}}U^*)^*x) = \Phi^*x$ and by (2) we can extend it to an isometry W on $\overline{(Uf^{\frac{1}{2}}U^*)^*\mathcal{X}}$ onto $\overline{\Phi^*\mathcal{X}}$. We then have

$$WUf^{\frac{1}{2}U^*} = \Phi^* . \quad (3)$$

Taking adjoint on both sides of (3) we get $\Phi = Uf^{\frac{1}{2}U^*}W^*$ and letting W^* to be V we get $Uf^{\frac{1}{2}U^*}V = \Phi$. Thus $Uf^{\frac{1}{2}U^*}V$ is analytic operator valued because Φ is so.

The following theorem gives a relation between the factors of f and UfU^* .

6.2.10 Theorem. Let f be a $B^+(\mathcal{X}, \mathcal{X})$ -valued function such that $\int_0^{2\pi} \|f(e^{i\theta})\| d\theta < \infty$. Suppose that Φ is an analytic factor of f . Then UfU^* is factorable if and only if there exists a partial isometry valued function $V(e^{i\theta})$ with terminal range being $\overline{U(e^{i\theta})\Phi(e^{i\theta})U^*(e^{i\theta})\mathcal{X}}$ such that $U\Phi U^*V$ is an analytic operator valued function. In this case $U\Phi U^*V$ is the analytic factor of UfU^* .

6.3 Application. In this section we establish some new results and prove some well known facts using the materials of §6.2.

The following is a special case of a result due to Weiner and Masani [25].

6.3.1 Corollary. Let f be a nonnegative finite dimensional matrix valued function such that $M/m \in L_1$, where $M(e^{i\theta})$ and $m(e^{i\theta})$ are the largest and smallest eigenvalues of $f(e^{i\theta})$ respectively. Then f is factorable if and only if $\log \det f \in L_1$.

Proof. Let $U(e^{i\theta})$ be the unitary operator valued function which is measurable and diagonalizes f . Suppose we have

$$UfU^* = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} ,$$

where $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$ are the eigenvalues of f . We know by corollary 6.2.8 that f is factorable iff UfU^* is factorable. But obviously UfU^* is factorable iff $\log \lambda_i \in L_1$ for all $i = 1, 2, \dots, n$. Hence f is factorable if and only if $\log \lambda_1 \in L_1$. Thus f is factorable if and only if $\log \det f \in L_1$.

A well known sufficient condition for factorability of a weakly summable $B^+(\chi, \chi)$ -valued function f is

$$\int_0^{2\pi} \log \|f^{-1}(e^{i\theta})\|^{-1} d\theta > -\infty.$$

The following theorem shows that under some extra conditions the above condition is also necessary. First we prove the following lemma.

6.3.2 Lemma. Let f be a uniformly summable $B^+(\chi, \chi)$ -valued function which is factorable. Suppose $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ with $M/m \in L_1$. Let $\lambda(e^{i\theta})$ be an eigenvalue of finite multiplicity for $f(e^{i\theta})$ a.e. Then

$$\int_0^{2\pi} \log \lambda(e^{i\theta}) d\theta > -\infty.$$

Proof. Let $v(e^{i\theta})$ be a measurable unit eigenvalue of $f(e^{i\theta})$ corresponded to $\lambda(e^{i\theta})$. (For the existence of such an eigenfunction one may give a proof similar to the one in the proof of part (b) of theorem 6.3.5.) Let $U(e^{i\theta})$ be a measurable unitary operator valued function such that $UfU^* = \phi\phi^*$, where ϕ is analytic. Hence $(UfU^*, a) = (\phi\phi^*, a) = (\phi^* a, \phi^* a)$. Thus we have

$$\begin{aligned}\lambda(e^{i\theta}) &= (f(e^{i\theta})v(e^{i\theta}), v(e^{i\theta})) = (f(e^{i\theta})U^*(e^{i\theta})e_1, U^*(e^{i\theta})e_1) \\ &= (U(e^{i\theta})f(e^{i\theta})U^*(e^{i\theta})e_1, e_1) = (\phi^*e_1, \phi^*e_1).\end{aligned}$$

Since $\phi^*e_1 = \sum_{n=1}^{\infty} (\phi^*e_1, e_n)e_n$ we get

$$\lambda(e^{i\theta}) = \sum_{k=1}^{\infty} |(\phi^*e_1, e_n)|^2. \quad (1)$$

But since $\phi^*(e^{i\theta}) = \sum_{n=0}^{\infty} \phi_n^* e^{-in\theta}$ we see that

$$(\phi^*(e^{i\theta})e_1, e_m) = \sum_{n=0}^{\infty} (\phi_n^*e_1, e_m)e^{-in\theta}$$

so $\int_0^{2\pi} \log |(\phi^*(e^{i\theta})e_1, e_m)|^2 d\theta > -\infty$ for some m . Hence by (1)

$$\int_0^{2\pi} \log \lambda(e^{i\theta}) d\theta > -\infty.$$

6.3.3 Theorem. Let f be a $B^+(\mathcal{X}, \mathcal{X})$ -valued uniformly summable function such that $M/m \in L_1$, where $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$. Suppose $f(e^{i\theta})$ has at least one eigenfunction of multiplicity one. Then f being factorable implies that

$$\int_0^{2\pi} \log \|f^{-1}(e^{i\theta})\|^{-1} d\theta > -\infty.$$

Proof. Since $M(e^{i\theta}) \geq \lambda(e^{i\theta})$, using lemma 6.3.2 we get

$$-\infty < \int_0^{2\pi} \log \lambda(e^{i\theta}) d\theta \leq \int_0^{2\pi} \log M(e^{i\theta}) d\theta. \quad (1)$$

We also know that

$$\begin{aligned}0 &\leq \int_0^{2\pi} (\log M(e^{i\theta}) - \log m(e^{i\theta})) d\theta = \int_0^{2\pi} \log(M(e^{i\theta})/m(e^{i\theta})) d\theta \\ &= \int_0^{2\pi} (M(e^{i\theta})/m(e^{i\theta})) d\theta < \infty.\end{aligned} \quad (2)$$

From (1) and (2) we see that $\log m(e^{i\theta})$ is summable, because we have $\log m(e^{i\theta}) = \log M(e^{i\theta}) - (\log M(e^{i\theta}) - \log m(e^{i\theta}))$.

Hence we get $\int_0^{2\pi} \log m(e^{i\theta}) d\theta > -\infty$, which means

$$\int_0^{2\pi} \log \|f^{-1}(e^{i\theta})\|^{-1} d\theta > -\infty.$$

The next theorem is an interesting consequence of corollary

6.2.8. We shall need the following lemma first.

6.3.4 Lemma. Let f be a measurable $B^+(\mathcal{X}, \mathcal{X})$ -valued function whose spectrum consists only of the eigenvalues, each eigenvalue having finite multiplicity. Let $\iota_1(e^{i\theta}) \geq \iota_2(e^{i\theta}) \geq \iota_3(e^{i\theta}) \geq \dots$ denote the eigenvalues of $f(e^{i\theta})$ listed according to their multiplicity. Then $\iota_1, \iota_2, \iota_3, \dots$ are measurable and there exists a measurable unitarily valued function $U(e^{i\theta})$ such that

$$U(e^{i\theta}) f(e^{i\theta}) U^*(e^{i\theta}) = \sum_{j=1}^{\infty} \iota_j(e^{i\theta}) Q_j(e^{i\theta}),$$

where Q_j 's are constant one dimensional projectors.

Proof. Here is an outline of the proof.

By a similar argument as in [6], p. 653, one can show that $\sum_{j=1}^q \iota_j(e^{i\theta})$ is measurable for each $q \geq 1$. Hence each $\iota_j(e^{i\theta})$ is measurable. Following the proof of [7], p. 391, we can show the existence of a complete orthonormal sequence

$\{u_j\}_{j=1}^{\infty}$ of eigenvectors of f which are measurable. Let

$\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence of vectors in $\mathcal{X}$.

Define the unitary operator valued function U by $U(e^{i\theta})u_j = e_j$.

We then see that U has the desired properties.

6.3.5 Theorem. Let f be a uniformly summable $B^+(\mathcal{X}, \mathcal{X})$ -valued function such that $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$, with M/m being summable. Then

(a) Let the spectrum of f consist of only eigenvalues, each one of which being of finite multiplicity. Then f is factorable if and only if for each j , $1 \leq j < \infty$, we have

$$\int_0^{2\pi} \log \iota_j(e^{i\theta}) d\theta > -\infty,$$

where $\iota_1(e^{i\theta}) \geq \iota_2(e^{i\theta}) \geq \iota_3(e^{i\theta}) \geq \dots$ are the eigenvalues of $f(e^{i\theta})$ listed according to their multiplicity.

(b) If

$$f(e^{i\theta}) = \sum_{j=1}^{\infty} p_j(e^{i\theta}) P_j(e^{i\theta}),$$

P_j 's are measurable one-dimensional projection valued functions such that $P_j(e^{i\theta})$ are mutually orthogonal. Then f is factorable if and only if $\int_0^{2\pi} \log p_j(e^{i\theta}) d\theta > -\infty$, for all $j = 1, 2, 3, \dots$

Proof. (a) Let $U(e^{i\theta})$ be the unitary operator valued function obtained in lemma 6.3.4. Then we have

$$U(e^{i\theta}) f(e^{i\theta}) U^*(e^{i\theta}) = \sum_{j=1}^{\infty} \iota_j(e^{i\theta}) Q_j,$$

where Q_j is the orthonormal projection on e_j . Now by corollary 6.2.8 we know that UfU^* is factorable if and only if f is factorable.

Now suppose that for each $j = 1, 2, 3, \dots$ we have

$$\int_0^{2\pi} \log \iota_j(e^{i\theta}) d\theta > -\infty.$$

Then there exists scalar valued conjugate analytic functions

φ_j such that $\iota_j(e^{i\theta}) = |\varphi_j(e^{i\theta})|^2$, $j = 1, 2, 3, \dots$. Let $x \in \mathcal{X}$.

Then we observe that

$$\begin{aligned}
\left\| \sum_{j=1}^N \varphi_j(e^{i\theta}) Q_j(e^{i\theta}) x \right\|^2 &= \left(\sum_{j=1}^N \varphi_j(e^{i\theta}) Q_j x, \sum_{j=1}^N \varphi_j(e^{i\theta}) Q_j x \right) \\
&= \left(\sum_{j=1}^N \ell_j(e^{i\theta}) Q_j x, x \right) + \left(\sum_{j=1}^{\infty} \ell_j(e^{i\theta}) Q_j x, x \right).
\end{aligned}$$

Hence we have

$$\left\| \sum_{j=1}^N \varphi_j(e^{i\theta}) Q_j x \right\|^2 \leq (U(e^{i\theta}) f(e^{i\theta}) U^*(e^{i\theta}) x, x). \quad (1)$$

Similarly we get

$$\left\| \sum_{j=N}^M \varphi_j(e^{i\theta}) Q_j x \right\|^2 = \left(\sum_{j=N}^M \ell_j(e^{i\theta}) Q_j x, x \right). \quad (2)$$

By (1) and (2) we get $\sum_{j=1}^{\infty} \overline{\varphi_j}(e^{i\theta}) Q_j x$ converges and defines a bounded operator, say Φ , through $\Phi x = \sum_{j=1}^{\infty} \overline{\varphi_j} Q_j x$. We also have

$$\left\| \sum_{j=1}^N \varphi_j(e^{i\theta}) Q_j x \right\|^2 = \left(\sum_{j=1}^N \ell_j(e^{i\theta}) Q_j x, x \right). \quad (3)$$

Now letting $N \rightarrow \infty$ in (3), we get

$$(U f U^* x, x) = \left\| \sum_{j=1}^{\infty} \varphi_j(e^{i\theta}) Q_j x \right\|^2 = \left(\sum_{j=1}^{\infty} \varphi_j(e^{i\theta}) Q_j x, \sum_{j=1}^{\infty} \varphi_j(e^{i\theta}) Q_j x \right)$$

which means that $U f U^* = \Phi \Phi^*$. Now clearly for each $m = 1, 2, 3, \dots$

we know that $\Phi_m x = \sum_{j=1}^m \overline{\varphi_j}(e^{i\theta}) Q_j x$, has only nonnegative Fourier coefficients. We also observe that

$$\begin{aligned}
\int_0^{2\pi} \left\| \sum_{j=N}^M \overline{\varphi_j}(e^{i\theta}) Q_j x \right\|^2 d\theta &= \int_0^{2\pi} \left(\sum_{j=N}^M \ell_j(e^{i\theta}) Q_j x, x \right) d\theta \\
&= \int_0^{2\pi} \left(\sum_{j=1}^M \ell_j(e^{i\theta}) Q_j x, x \right) d\theta \\
&\quad - \int_0^{2\pi} \left(\sum_{j=1}^N \ell_j(e^{i\theta}) Q_j x, x \right) d\theta.
\end{aligned}$$

Hence by the dominated convergence theorem we see that the function $\sum_{j=1}^M \bar{\phi}_j(e^{i\theta}) Q_j x$ is Cauchy in $L_2(\mathcal{M})$. Hence we see that

$\phi(e^{i\theta})x$ has only nonnegative Fourier coefficients. Now for each

$n \leq 0$ we let $\phi_n x$ be the n -th Fourier coefficient of ϕx , then

$\phi x = \sum_{n=0}^{\infty} e^{in\theta} \phi_n x$. But we have $UfU^* = (\phi^* x, \phi^* x)$, which together

with the weak integrability of UfU^* and closed graph theorem

we see that ϕ_n 's are bounded (Remark 4.2.1). Hence UfU^* is factor-

able, which means that f is factorable. Then by lemma 6.3.2

$$\int_0^{2\pi} \log(\iota_j(e^{i\theta})) d\theta > -\infty, \text{ for all } j = 1, 2, 3, \dots$$

(b) For each fixed $j = 1, 2, 3, \dots$ let $u_j(e^{i\theta})$ be a unit vector in $P_j(e^{i\theta})$. Let $\{e_m\}_{m=1}^{\infty}$ be an orthonormal base for $\mathcal{M}$. Then

for all $j \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, $P_j(e^{i\theta})e_m = (e_m, u_j(e^{i\theta})u_j(e^{i\theta}))$ is measur-

able. We can divide the unit circle as the disjoint union of

countable sequences of $\{E_{mj}\}_{m=1}^{\infty}$ such that $P_j(e^{i\theta})e_m$ is different from zero on E_{mj} and zero on E_{nj} for all $n > m$. Then obviously

we have

$$u_j(e^{i\theta}) = \frac{P_j(e^{i\theta})e_m}{\|P_j(e^{i\theta})e_m\|} \text{ if } \theta \in E_{mj}.$$

Now since each of $P_j(e^{i\theta})e_m$ is measurable, we see that $u_j(e^{i\theta})$

are measurable. But $\iota_j(e^{i\theta}) = (f(e^{i\theta})u_j(e^{i\theta}), u_j(e^{i\theta}))$, so

$\iota_j(e^{i\theta})$ is measurable for each $j = 1, 2, 3, \dots$. Now one can

define the unitary operator valued function U through

$$U(e^{i\theta})u_j(e^{i\theta}) = e_j.$$

Now U is measurable and the rest of the proof is exactly similar to the proof of (a).

Based on theorem 6.2.9 we give a proof of a result due to Wiener and Masani. However we point out here that for the proof of sufficiency we make use of an argument contained in [27].

First we introduce some notation and state a lemma from [27].

We denote by L_{δ}^{0+} the boundary values of the functions in Hardy class H_{δ} and by Q_{δ}^{0+} the class of all functions f such that $f = h_1/h_2$ a.e. with h_1 and $h_2 \in L_{\delta}^{0+}$.

6.3.6 Lemma. Every function in Q_{δ}^{0+} on the unit circle, $0 \leq \delta < \infty$, such that $|f| = 1$ a.e., is in Q_{∞}^{0+} and admits a factorization $f = \psi_1 \bar{\psi}_2$, where $\psi_1, \psi_2 \in L_{\infty}^{0+}$ and $|\psi_1| = |\psi_2| = 1$.

6.3.7 Theorem. Let $f = [f_{ij}]_{i,j=1}^2$ be a weakly summable 2×2 (non zero) nonnegative matrix valued function such that $\det f = 0$ a.e. Then the following two conditions are equivalent.

(a) $f(e^{i\theta}) = \phi(e^{i\theta}) \phi^*(e^{i\theta})$, where $0 \neq \phi \in L_2^{0+}$.

(b) For $i = 1$ or 2 , $\log f_{ii} \in L_1$ and for $i \neq j$, $\frac{f_{ji}}{f_{ii}} \in Q_{\delta}^{0+}$.

Proof. Assuming $f_{11} \neq 0$ a.e. we have

$$f(e^{i\theta}) = \begin{bmatrix} f_{11}(e^{i\theta}) & f_{12}(e^{i\theta}) \\ f_{21}(e^{i\theta}) & f_{22}(e^{i\theta}) \end{bmatrix} = f_{11} \begin{bmatrix} 1 & \bar{\psi} \\ \psi & |\psi|^2 \end{bmatrix},$$

where $\psi = \frac{f_{21}}{f_{11}}$. Clearly the eigenvalues of f are zero and $f_{11} + f_{22} = f_{11}(1 + |\psi|^2)$. Let $a(e^{i\theta})$ be a scalar valued function, then the vector $(-\bar{\psi}a, a)$ and $(\bar{a}, \psi\bar{a})$ are eigenvalues of f corresponded to zero and $f_{11}(1 + |\psi|^2)$ respectively. If we let $a = \frac{1}{\sqrt{1 + |\psi|^2}}$ then the unit vectors $(-\bar{\psi}a, a)$ and $(\bar{a}, \psi\bar{a})$ are eigenvectors of f corresponded to zero and $f_{11}(1 + |\psi|^2)$ respectively. Hence the unitary valued function

$$U(e^{i\theta}) = \begin{bmatrix} a & -a \overline{\psi(e^{i\theta})} \\ a \psi(e^{i\theta}) & a \end{bmatrix}$$

sends the vectors $(1,0)$ and $(0,1)$ to $(a, a\psi)$ and $(-a\bar{\psi}, a)$ respectively. In other words we have

$$\begin{bmatrix} f_{11} & f_{21} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} a & -a\bar{\psi} \\ a\psi & a \end{bmatrix} \cdot \begin{bmatrix} f_{11} + f_{22} & \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & a\bar{\psi} \\ -a\psi & a \end{bmatrix}.$$

Let us denote by

$$f' = \begin{bmatrix} f_{11} + f_{22} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then by theorem 6.2.9, $f = Uf'U^*$ is factorable iff there exists a partial isometry valued function V with terminal range $\overline{Uf' \frac{1}{2} U^*}$ such that $Uf' \frac{1}{2} U^* V$ is analytic. Now since

$$\begin{aligned} Uf' \frac{1}{2} U^* &= \begin{bmatrix} a & -\bar{\psi}a \\ \psi a & a \end{bmatrix} \cdot \begin{bmatrix} \sqrt{f_{11}+f_{22}} & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & \bar{\psi}a \\ -\psi a & a \end{bmatrix} \\ &= a^2 \sqrt{f_{11}+f_{22}} \begin{bmatrix} 1 & \bar{\psi} \\ \psi & |\psi|^2 \end{bmatrix}. \end{aligned} \quad (1)$$

Hence $Uf' \frac{1}{2} U^*$ is the subspace generated by the vector $(1, \psi)$.

So the operator $V(e^{i\theta})$ has to send some vector, say $(s(e^{i\theta}), t(e^{i\theta}))$ to $(\frac{1}{\sqrt{1+|\psi|^2}}, \frac{\psi}{\sqrt{1+|\psi|^2}})$ and $(-t(e^{i\theta}), s(e^{i\theta}))$ to $(0,0)$, where $s(e^{i\theta})$ and $t(e^{i\theta})$ are some scalar valued

functions with the property $|s(e^{i\theta})|^2 + |t(e^{i\theta})|^2 = 1$. So v must be of the form

$$v(e^{i\theta}) = \frac{1}{\sqrt{1 + |\psi|^2}} \begin{bmatrix} \bar{s} & \bar{t} \\ \bar{\psi}\bar{s} & \bar{\psi}\bar{t} \end{bmatrix}, \quad (2)$$

where $|s|^2 + |t|^2 = 1$. If we define $v(e^{i\theta})$ by (2) then from (1) we get

$$Uf^{\frac{1}{2}}U^* = \sqrt{f_{11}} \begin{bmatrix} \bar{s} & \bar{t} \\ \bar{s}\bar{\psi} & \bar{t}\bar{\psi} \end{bmatrix} \quad (3)$$

where $|s|^2 + |t|^2 = 1$. Now suppose (a) holds, then we have $f = \Phi\Phi^*$, $0 \neq \Phi = [\varphi_{ij}]_{i,j=1}^2 \in L_2^{0+}$. So we get $f_{11} = |\varphi_{11}|^2 + |\varphi_{12}|^2$, $f_{22} = |\varphi_{21}|^2 + |\varphi_{22}|^2$. Thus

$$\log f_{11} > 2 \log |\varphi_{11}|, \quad 2 \log |\varphi_{12}|$$

$$\log f_{22} > 2 \log |\varphi_{21}|, \quad 2 \log |\varphi_{22}|.$$

Since $\Phi \neq 0$ then $\varphi_{ij} \neq 0$ for some i and j . Hence

$\log f_{11} \in L_1$ or $\log f_{22} \in L_1$. Without loss of generality we assume $\log f_{11} \in L_1$. Now by (3) and theorem 6.2.9 there exists functions $s(e^{i\theta})$ and $t(e^{i\theta})$ such that

$$Uf^{\frac{1}{2}}U^*v = \sqrt{f_{11}} \begin{bmatrix} \bar{s} & \bar{t} \\ \bar{s}\bar{\psi} & \bar{t}\bar{\psi} \end{bmatrix} = \begin{bmatrix} \bar{s}\sqrt{f_{11}} & \bar{t}\sqrt{f_{11}} \\ \bar{s}\bar{\psi}\sqrt{f_{11}} & \bar{t}\bar{\psi}\sqrt{f_{11}} \end{bmatrix} \in L_2^{0+}.$$

Now since $\bar{\psi} = \frac{\bar{s}\sqrt{f_{11}}}{\bar{s}\sqrt{f_{11}}}$ and $\bar{s}\sqrt{f_{11}}\bar{\psi}, \bar{s}\sqrt{f_{11}} \in L_2^{0+}$, we see that $\bar{\psi} \in Q_2^{0+}$, hence $f_{21}/f_{11} \in Q_2^{0+}$.

We now show that (b) implies (a). Suppose (b) holds.

For definiteness we assume that $i = 1$, so that we have

$\log f_{11} \in L_1$ and $f_{21}/f_{11} \in Q_\delta^{0+}$. If $f_{22} = 0$ then obviously f is factorable. Assuming that $f_{22} > 0$ on a set of positive

measure. Then as shown in [27], p. 306, the condition

$f_{21}/f_{11} \in Q_\delta^{0+}$ implies that $\log f_{22} \in L_1$. Let $f_{11} = \varphi_1 \bar{\varphi}_1$ and $f_{22} = \varphi_2 \bar{\varphi}_2$ be the analytic factorization of f_{11} and f_{22} .

Apply lemma 6.3.5 to the function $\frac{f_{22}}{f_{11}} \frac{\varphi_1}{\varphi_2}$ to get

$\frac{f_{22}}{f_{11}} \cdot \frac{\varphi_1}{\varphi_2} = \frac{\psi_1}{\psi_2}$, where $|\psi_1| = 1$ and $\psi_i \in L_\infty^{0+}$ for $i = 1, 2$.

Let $s(e^{i\theta}) = t(e^{i\theta}) = \frac{\sqrt{2}}{2} \psi_2 \frac{\varphi_1}{\sqrt{f_{11}}}$. Then we have

$$Uf^{\frac{1}{2}}U^*V = \sqrt{f_{11}} \begin{bmatrix} \bar{s} & \bar{t} \\ \bar{s}\psi & \bar{t}\psi \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} \psi_2 \varphi_1 & \psi_2 \varphi_1 \\ \psi_1 \varphi_2 & \psi_1 \varphi_2 \end{bmatrix}.$$

So $Uf^{\frac{1}{2}}U^*V \in L_2^{0+}$, because $\psi_i \in L_\infty^{0+}$ and $\varphi_i \in L_2^{0+}$ for $i = 1, 2$.

CHAPTER VII

ALGORITHMS FOR DETERMINING THE OPTIMAL FACTOR AND THE LINEAR PREDICTOR

7.1 Introduction. The theory of multivariate stationary stochastic processes as developed by Wiener and Masani [25], [26], [14], essentially consists of two parts. Part one deals with the analysis of time and spectral domain. This part has been studied by several authors and has been extended to the infinite dimensional case (c.f. [4], [8], [12], [16], [18], [19]). Part two is concerned with the important problem of determining the generating function, namely given a nonnegative Hermitian $q \times q$, ($1 \leq q < \infty$) matrix valued function on the unit circle, such that $f(e^{i\theta})$ is weakly summable and $\log \det f$ summable, to find a $q \times q$ matrix valued function Φ such that

$$f(e^{i\theta}) = \Phi(e^{i\theta}) \Phi^*(e^{i\theta}),$$

where Φ is an analytic optimal factor.

An iterative procedure which yields an infinite series for Φ in terms of f has been given there by Wiener and Masani [26] under the following assumptions

$$c_1 I \leq f(e^{i\theta}) \leq c_2 I, \tag{1}$$

where $0 < c_1 \leq c_2 < \infty$.

In [14] Masani was able to improve the result he and Wiener give in [26] by assuming in lieu of condition (1), that

- (i) f is a weakly summable hermitian matrix valued function.
- (ii) f^{-1} exists a.e. and f^{-1} is weakly summable.
- (iii) if $\nu(e^{i\theta})$ and $\mu(e^{i\theta})$ denote the smallest and largest eigenvalue of $f(e^{i\theta})$, then μ/ν is summable.

The problem of determining the optimal factor was also the subject of discussion by Salehi [21], where some improvements were made in the field.

The problem of determining the optimal factor for the infinite dimensional case has not been discussed in the literature. In this chapter we wish to obtain an algorithm for determining the optimal factor and the linear predictor for the infinite dimensional case. As seen from Wiener and Masani's work, it looks as though one has to assume that the spectral density is bounded away from zero. On the other hand a trace class operator on an infinite dimensional Hilbert space is not bounded away from zero. Hence processes with finite trace will not satisfy the stipulation and purpose of this chapter. This suggests the adoption of notations and terminologies provided by M.G. Nadkarni. In doing so we can extend the algorithm given by Masani [14] for multivariate process to the infinite dimensional case. Section 7.2 is devoted to preliminary results. In §7.3 the relation between the two sided predictor error matrices of a process and its subprocesses is studied. Using this relation we show that under some boundedness condition our process is minimal fullrank. We then show the crucial fact that for the optimal factor Φ , Φ^{-1} is in L_2^{-0+} . In

§7.4, under some extra conditions, we obtain an algorithm for finding the linear predictor.

We would like to mention that our method of attacking the problem of determination of the generating factor and the linear predictor is in the spirit of the work of N. Wiener and P. Masani [26] and Masani [14].

7.2 Preliminaries. In this section we shall set down the notations and preliminaries which will be needed in the next sections. Although some of these notations have been introduced in Chapter II, since we will sometimes deal with unbounded operators, this re-introduction is necessary. Most of the notations and results of the first half of this section are given in the work of M.G. Nadkarni [16]. In the second half we prove some results on the Fourier analysis of infinite dimensional matrix valued functions which will be needed later.

7.2.1 Definition. Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\bar{\mathcal{H}}$ the collection of all $\xi = [\xi^n]$, $n \in \mathbb{Z}_+$ of elements in $\mathcal{H}$. Clearly $\bar{\mathcal{H}}$ is a linear space and we give $\bar{\mathcal{H}}$ the product topology, i.e. $\xi_n \rightarrow \xi$ if for each $k \in \mathbb{Z}_+$, $\xi_n^k \rightarrow \xi^k$ in $\mathcal{H}$. Let $\xi, \eta \in \bar{\mathcal{H}}$. We denote by (ξ, η) the Gramian of ξ and η to be the matrix whose (i, j) -th entry is (ξ^i, η^j) . Clearly (ξ, ξ) is nonnegative and $(\xi, \xi) = 0$ if and only if $\xi = 0$. The Gramian has the following properties:

- (i) $\xi_n \rightarrow \xi, \eta_n \rightarrow \eta$ implies that $(\xi_n, \eta_n) \rightarrow (\xi, \eta)$, elementwise.
- (ii) $\xi_n \rightarrow 0$ if and only if $(\xi_n, \xi_n) \rightarrow 0$.
- (iii) If A and B are infinite dimensional matrices such that $A\xi$ and $B\eta$ are defined, then $(A\xi, B\eta) = A(\xi, \eta)B^*$, where B^* is

the adjoint matrix of B .

A closed subset $\bar{H}$ of $\bar{X}$ is called a subspace if $\bar{H}$ is closed under addition and $A\xi \in \bar{H}$ for any matrix A and any $\xi \in \bar{H}$ for which $A\xi$ is defined. We say $\xi \perp \eta$ if $(\xi, \eta) = 0$. A vector ξ is called normal if $(\xi, \xi) = I$, where I is the identity matrix. A sequence $\{\xi_n\}_{n=-\infty}^{+\infty}$ is called orthonormal if $(\xi_n, \xi_m) = \delta_{nm} I$.

For any $\mathcal{B} \subset \mathcal{X}$, $\bar{\mathcal{B}} \subset \bar{X}$, we write

- (i) $\mathcal{G}(\mathcal{B})$ = subspace of $\mathcal{X}$ spanned by $\mathcal{B}$
- (ii) $\bar{\mathcal{G}}(\bar{\mathcal{B}})$ = subspace of $\bar{X}$ spanned by $\bar{\mathcal{B}}$
- (iii) $\mathcal{G}(\bar{\mathcal{B}})$ = subspace of $\mathcal{X}$ spanned by coordinates of vectors in $\bar{\mathcal{B}}$
- (iv) $\bar{\mathcal{G}}(\mathcal{B})$ = the set of all vectors in $\bar{X}$ with coordinates in $\mathcal{B}$.

It is easy to see that for a subspace $\mathcal{B}$ of $\mathcal{X}$, $\bar{\mathcal{G}}(\mathcal{B})$ is a subspace of $\bar{X}$ and if $\bar{\mathcal{B}}$ is a subspace of $\bar{X}$ then we have

$\bar{\mathcal{B}} = \bar{\mathcal{G}}(\mathcal{G}(\bar{\mathcal{B}}))$. Hence for any subspace $\bar{\mathcal{B}}$ of $\bar{X}$ we have $\bar{\mathcal{B}} = \bar{\mathcal{G}}(\mathcal{B})$, for some $\mathcal{B}$. Let $\xi \in \bar{X}$. We write $(\xi|\bar{\mathcal{B}})$ to denote the vector whose i -th coordinator is given by $(\xi|\bar{\mathcal{B}})^i = (\xi^i|\mathcal{B})$.

7.2.2 Definition. A sequence ξ_n , $-\infty < n < \infty$ of elements of $\bar{X}$ is called an infinite dimensional stationary stochastic process

(SSP) if the Gramian (ξ_m, ξ_n) depends only on $m-n$. It is easy to see that there exists a unitary operator U on $\mathcal{X}$ such that

$U^n \xi_0^i = \xi_n^i$. Let $\bar{U}$ be its inflation operator defined on $\bar{X}$.

Hence we have $\xi_n = \bar{U}^n \xi_0$. Let ξ_n , $-\infty < n < \infty$ be a SSP. We write

$$\bar{H}(n) = \bar{\mathcal{G}}(\xi_k, k \leq n), \quad \bar{H}(\infty) = \bar{\mathcal{G}}(\xi_k, k < \infty)$$

$$\bar{H}(-\infty) = \bigcap_n \bar{H}(n) \quad \text{and} \quad \bar{K}(n) = \bar{\mathcal{G}}(\xi_k, k \neq n).$$

7.2.3 Definition. Let $\eta_n = \xi_n - (\xi_n | \bar{H}(n-1))$. One can see that $\eta_n = \bar{U}^n \eta_0$. We call η_n , $-\infty < n < \infty$ the innovation process of ξ_n , $-\infty < n < \infty$. We write $G = (\eta_0, \eta_0)$ and call G the predictor error matrix of ξ_n , $-\infty < n < \infty$. We say ξ_n , $-\infty < n < \infty$ is of full rank if its predictor error matrix is of full rank, i.e. if $G \geq \lambda I$ for some positive number λ .

7.2.4 Definition. Let $\zeta_n = \xi_n - (\xi_n | \bar{K}(n))$, the processes ζ_n , $-\infty < n < \infty$ is called the two sided innovation process of ξ_n . We write $\Sigma = (\zeta_0, \zeta_0)$ and we call the process ξ_n , $-\infty < n < \infty$ to be minimal if $\Sigma \neq 0$, and minimal full rank if $\Sigma \geq \lambda I$, for some positive number λ .

7.2.5 Definition. Let μ be an infinite dimensional nonnegative matrix valued measure on the Borel subsets of the unit circle [16], and let $g = [g_k]$, $k \in \mathbb{Z}_+$ be a row vector valued function such that $g_k = 0$ for all except finitely many k , and for these k , g_k is a trigonometric polynomial. Let $L_2'(\mu)$ be the set of all such g 's with norm given by

$$\|g\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m, \ell=1}^{\infty} g_m(e^{i\theta}) \mu_{m\ell}(d\theta) \overline{g_\ell(e^{i\theta})} d\theta.$$

Two elements of $L_2'(\mu)$ are identified if their difference has zero norm. The inner product of two elements g and h is given by

$$(g, h) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m, \ell=1}^{\infty} g_m(e^{i\theta}) \mu_{m\ell}(d\theta) \overline{h_\ell(e^{i\theta})} d\theta.$$

7.2.6 Definition. Let $L_2(\mu)$ denote the completion of $L_2'(\mu)$.

In case of f being a spectral density we denote by $L_2(f)$ the space $L_2(\mu)$, where $\mu_{m\ell}(A) = \int_A f_{m\ell}(e^{i\theta}) d\theta$. Hence the inner product in this case is given by

$$\begin{aligned} (g, h)_f &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m, \ell=1}^{\infty} g_m(e^{i\theta}) f_{m\ell}(e^{i\theta}) \overline{h_\ell(e^{i\theta})} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) f(e^{i\theta}) h^*(e^{i\theta}) d\theta. \end{aligned}$$

7.2.7 Definition. Let f be the spectral density of a SSP

ξ_n , $-\infty < n < \infty$. There is a natural isomorphism between $L_2(f)$ and $H_\xi(\infty)$ which can be obtained by linearity from the mapping $S: \xi_n^k \rightarrow \psi_n^k$, where $\psi_n^k = [e^{-in\theta} \delta_{k\ell}]$. We now state the following lemma (c.f. [16], p. 152).

7.2.8 Lemma. Let $f(e^{i\theta})$ be an infinite dimensional positive matrix valued function which satisfies

$$0 < m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I \quad \text{a.e.}$$

Then $L_2(f)$ consists of all ℓ^2 valued functions $g = [g_1, g_2, \dots]$ with measurable entries such that $\|g \sqrt{f}\|^2 = g(e^{i\theta}) f(e^{i\theta}) g^*(e^{i\theta}) = \lim_{N \rightarrow \infty} \sum_{m, \ell=1}^N g_m(e^{i\theta}) f_{m\ell}(e^{i\theta}) g_\ell(e^{i\theta})$ exists a.e. and the resulting function is summable.

Now we give the following two definitions.

7.2.9 Definition. Let $f(e^{i\theta})$ be a positive infinite-dimensional matrix valued function which satisfies

$$0 < m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I \quad \text{a.e.}$$

We denote by $\bar{L}_2(f)$ the set of all infinite dimensional matrix valued functions, each row of which being in $L_2(f)$. We then give

$\bar{L}_2(f)$ the row-wise convergent topology, i.e. if $\Phi_n, \Phi \in \bar{L}_2(f)$ then $\Phi_n \rightarrow \Phi$ in $\bar{L}_2(f)$ sense if and only if $\Phi_n^i \rightarrow \Phi^i$ for all $i \in \mathbb{Z}_+$, where Φ^i denotes the i -th row of Φ . When $f = I$, then we write L_2 and $\bar{L}_2$ instead of $L_2(f)$ and $\bar{L}_2(f)$ respectively.

7.2.10 Definition. Let $\bar{S}$ be the inflation of S , defined on $\bar{H}_S(\infty)$ into $\bar{L}_2(f)$ by the relation $(\bar{S}(\xi))^i = S(\xi^i)$, where $\xi \in \bar{H}_S(\infty)$ and $i \in \mathbb{Z}_+$.

The following lemma gives some properties of $\bar{S}$.

7.2.11 Lemma. With the above notation we have

- (i) $(\xi, \eta) = (\bar{S}\xi, \bar{S}\eta)_f$, where for each Φ and Ψ in $\bar{L}_2(f)$ we let $(\Phi, \Psi)_f = [(\Phi^i, \Psi^j)]_{i,j=1}^\infty$
- (ii) $\bar{S}$ is one-to-one
- (iii) $\bar{S}(\xi + \eta) = \bar{S}(\xi) + \bar{S}(\eta)$
- (iv) $\bar{S}$ is a continuous transformation
- (v) $\bar{S}(A\xi) = A(\bar{S}\xi)$, whenever $A\xi$ is defined.

Proof. (i) - (iv) is obvious. To see (v) consider

$$\begin{aligned} \bar{S}(A\xi) &= \bar{S}([A\xi^i]_{i=1}^\infty) = [S((A\xi)^i)]_{i=1}^\infty \\ &= [S(\sum_{j=1}^\infty a_{ij}\xi^j)]_{i=1}^\infty = [\sum_{j=1}^\infty a_{ij}S(\xi^j)]_{i=1}^\infty = A(S\xi). \end{aligned}$$

Now we digress to discuss some Fourier analysis of infinite dimensional matrix valued functions. For a matrix valued function $\Phi = [\varphi_{m\ell}]_{m,\ell=1}^{+\infty}$ whose elements are summable, we define its n -th Fourier coefficient $A_n = (a_{m\ell}^n)$ by

$$a_{m\ell}^n = \frac{1}{2\pi} \int_0^{2\pi} \varphi_{m\ell}(e^{i\theta}) e^{-in\theta} d\theta.$$

We first prove the Parseval identity.

7.2.12 Lemma (Parseval identity). Let $\Phi = [\varphi_1, \varphi_2, \varphi_3, \dots]$ and $\Psi = [\psi_1, \psi_2, \psi_3, \dots]$ belong to L_2 . Let A_k^m and B_k^m be the k -th Fourier coefficients of φ_m and ψ_m respectively. Then

$$(a) \quad (\Phi, \Psi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(e^{i\theta}) \Psi^*(e^{i\theta}) d\theta = \sum_{k=-\infty}^{+\infty} \sum_{m=1}^{\infty} A_k^m \overline{B_k^m}$$

$$(b) \quad \|\Phi\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|\Phi(e^{i\theta})\|_2^2 d\theta = \sum_{k=-\infty}^{+\infty} \sum_{m=1}^{\infty} |A_k^m|^2.$$

Proof. Let $e_k^m = [e^{ik\theta} \delta_{m\ell}]$. Then e_k^m , $-\infty < k < \infty$, $1 \leq m < \infty$,

becomes a complete orthonormal system. We also observe that

$(\Psi, e_k^m) = A_k^m$ and $(\Phi, e_k^m) = B_k^m$. Now standard Hilbert space arguments can be used to complete the proof of the theorem.

7.2.13 Remark. The space L_2 consists of all weakly measurable ℓ_2 -valued functions $\Psi = [\psi_1, \psi_2, \dots]$, for which $\|\Psi\|_2^2 = \sum_{i=1}^{\infty} |\psi_i|^2$ is integrable.

We now prove the Riesz-Fisher theorem for infinite dimensional case.

7.2.14 Lemma. Let $\{A_n\}_{n=-\infty}^{+\infty}$ be a sequence of infinite dimensional matrices. Then $\{A_n\}_{n=-\infty}^{+\infty}$ is the Fourier coefficients of a function Ψ in $\overline{L}_2$ if and only if $\sum_{\ell=1}^{\infty} \sum_{n=-\infty}^{+\infty} |a_{m\ell}^n|^2 < \infty$ for all $m \in \mathbb{Z}_+$. Furthermore in this case we have $\Psi = \sum_{n=-\infty}^{+\infty} A_n e^{in\theta}$.

Proof. If $\Psi \in \overline{L}_2$, then for each $m \in \mathbb{Z}_+$, we have

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \sum_{\ell=1}^{\infty} |a_{m\ell}^n|^2 &= \sum_{\ell=1}^{\infty} \sum_{n=-\infty}^{+\infty} |a_{m\ell}^n|^2 = \sum_{\ell=1}^{\infty} \frac{1}{2\pi} \int |\psi_{m\ell}(e^{i\theta})|^2 d\theta \\ &= \frac{1}{2\pi} \int \sum_{\ell=1}^{\infty} |\psi_{m\ell}(e^{i\theta})|^2 d\theta < \infty. \end{aligned}$$

The last inequality follows because each row of Ψ belongs to L_2 .

For the other way assume that $\sum_{n=-\infty}^{+\infty} \sum_{\ell=1}^{\infty} |a_{m\ell}^n|^2 < \infty$, for each $m \in \mathbb{Z}_+$.

Then for each $m, l \in \mathbb{Z}_+$, $\sum_{n=-\infty}^{+\infty} |a_{ml}^n|^2 < \infty$. Hence for each $m, l \in \mathbb{Z}_+$ there exists a square integrable function ψ_{ml} whose n -th Fourier coefficient is a_{ml}^n . Let $\Psi = [\psi_{ml}]_{m,l=1}^{\infty}$. Then Ψ has A_n as its n -th Fourier coefficient, and Ψ is in $\bar{L}_2$, because

$$\int_0^{2\pi} \sum_{l=1}^{\infty} |\psi_{ml}(e^{i\theta})|^2 d\theta = \sum_{l=1}^{\infty} \int_0^{2\pi} |\psi_{ml}(e^{i\theta})|^2 d\theta = \sum_{l=1}^{\infty} \sum_{n=-\infty}^{+\infty} |a_{ml}^n|^2 < \infty.$$

Now standard arguments show that $\Psi = \sum_{n=-\infty}^{+\infty} A_n e^{in\theta}$.

An important consequence of the last lemma is that if A_n is the n -th Fourier coefficient of a function in $\bar{L}_2$ and $\{B_n\}_{n=-\infty}^{+\infty}$ is composed from A_n 's and zeros, then $\{B_n\}_{n=-\infty}^{+\infty}$ is also the Fourier coefficients of a function in $\bar{L}_2$. This allows us to give the following definition.

7.2.15 Definition. (a) If $\Psi \in \bar{L}_2$ and has Fourier coefficients A_n , then Ψ_+ , Ψ_{0+} , Ψ_- and Ψ_{0-} will denote the function in $\bar{L}_2$ whose n -th Fourier coefficient is A_n for all $n > 0$; $n \geq 0$, $n < 0$ and $n \leq 0$ and zero for the remaining n 's respectively. Ψ_0 will be the constant function $\Psi_0 = A_0$.

(b) $\bar{L}_2^+$, $\bar{L}_2^{0+}$, $\bar{L}_2^-$, $\bar{L}_2^{0-}$ will denote the subset of all functions in $\bar{L}_2$ whose n -th Fourier coefficient vanishes for all $n \leq 0$, $n < 0$, $n \geq 0$, $n > 0$ respectively. Note that Ψ_{0+} , Ψ_+ , Ψ_- , Ψ_{0-} belongs to $\bar{L}_2^{0+}$, $\bar{L}_2^+$, $\bar{L}_2^-$, $\bar{L}_2^{0-}$ whenever $\Psi \in \bar{L}_2$.

The proof of the following lemma is obvious and hence is omitted.

7.2.16 Lemma. (a) The sets $\bar{L}_2^{0+}$, $\bar{L}_2^+$, $\bar{L}_2^-$, $\bar{L}_2^{0-}$ are closed subspaces of $\bar{L}_2$ with $\bar{L}_2^+ \perp \bar{L}_2^-$.

(b) Let $\Psi \in \bar{L}_2$ and let Ψ^i denote the i -th row of Ψ . Then

$$(i) \quad \Psi = \Psi_+ + \Psi_0 + \Psi_- = \Psi_{0-} + \Psi_+ = \Psi_- + \Psi_{0+}$$

$$(ii) \quad \|\Psi^i\|^2 = \|\Psi_+^i\|^2 + \|\Psi_0^i\|^2 + \|\Psi_-^i\|^2 = \|\Psi_{0+}^i\|^2 + \|\Psi_+^i\|^2, \quad i \in \mathbb{Z}_+$$

$$(iii) \quad \|\Psi_+^i\|, \|\Psi_{0+}^i\|, \|\Psi_0^i\|, \|\Psi_-^i\| \leq \|\Psi\|, \quad i \in \mathbb{Z}_+.$$

$$(iv) \quad (\Psi_+)^* = (\Psi_-)^*, \quad (\Psi_-)^* = (\Psi_+)^*.$$

7.2.17 Remark. Similar definitions and properties can be given for L_2 instead of $\bar{L}_2$.

We now prove a convolution rule for the functions in L_2 .

7.2.18 Lemma (Convolution rule). Let A_n and B_n be the n -th Fourier coefficients of Ψ and $\Phi \in \bar{L}_2$ respectively. Then the n -th Fourier coefficient of $\Psi\Phi^*$ is $\sum_{k=-\infty}^{\infty} A_k B_{n-k}^*$.

Proof. The n -th Fourier coefficients of the (j, l) -th element of $\Psi\Phi^*$ is given by $\frac{1}{2\pi} \int_0^{2\pi} (\Psi(e^{i\theta})\Phi^*(e^{i\theta}))_{j,l} e^{-in\theta} d\theta$, which is

$$\frac{1}{2\pi} \int_0^{2\pi} (\Psi(e^{i\theta})\Phi^*(e^{i\theta})) e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=1}^{\infty} (\psi_{jm}(e^{i\theta}) \bar{\varphi}_{lm}(e^{i\theta})) e^{-in\theta} d\theta.$$

Consider

$$\begin{aligned} \int_0^{2\pi} \sum_{m=1}^{\infty} |\psi_{jm}(e^{i\theta}) \bar{\varphi}_{lm}(e^{i\theta})| e^{-in\theta} d\theta &= \int_0^{2\pi} \sum_{m=1}^{\infty} |\psi_{jm}(e^{i\theta})| |\varphi_{lm}(e^{i\theta})| d\theta \\ &\leq \int_0^{2\pi} \left(\sum_{m=1}^{\infty} |\psi_{jm}(e^{i\theta})|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} |\varphi_{lm}(e^{i\theta})|^2 \right)^{\frac{1}{2}} d\theta \\ &\leq \left(\int_0^{2\pi} \sum_{m=1}^{\infty} |\psi_{jm}(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \sum_{m=1}^{\infty} |\varphi_{lm}(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}. \end{aligned}$$

Now since Ψ and $\Phi \in \bar{L}_2$ we get $\int_0^{2\pi} \sum_{m=1}^{\infty} |\psi_{jm} \bar{\varphi}_{lm}| d\theta < \infty$. Hence we can change the order of integration and summation,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} (\Psi(e^{i\theta}) \Phi^*(e^{i\theta}))_{j\ell} e^{-in\theta} d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=1}^{\infty} \Psi_{jm}(e^{i\theta}) \overline{\Phi_{\ell m}(e^{i\theta})} e^{-in\theta} d\theta \\
&= \sum_{m=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \Psi_{jm}(e^{i\theta}) \overline{\Phi_{\ell m}(e^{i\theta})} e^{-in\theta} d\theta = \sum_{m=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{jm}^k \overline{b_{\ell m}^{n-k}}.
\end{aligned}$$

The last equality is by the usual convolution rule. If we change the order in the last term we get

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} (\Psi(e^{i\theta}) \Phi^*(e^{i\theta}))_{j\ell} e^{-in\theta} d\theta \\
= \sum_{k=-\infty}^{+\infty} \sum_{n=1}^{\infty} a_{jm}^k \overline{b_{\ell m}^{n-k}} = \sum_{k=-\infty}^{+\infty} (A_k B_{n-k}^*)_{j\ell} = \left(\sum_{k=-\infty}^{+\infty} A_k B_{n-k}^* \right)_{j\ell}.
\end{aligned}$$

We also need the following definition.

7.2.19 Definition. Let Φ be in $\overline{L}_2^{0+}$. Then Φ is called the optimal factor of f if the following three conditions hold

- (i) $f(e^{i\theta}) = \Phi(e^{i\theta}) \Phi^*(e^{i\theta})$ a.e.
- (ii) $\Phi_0 \geq 0$
- (iii) For any $\Psi \in \overline{L}_2^{0+}$, $f(e^{i\theta}) = \Psi(e^{i\theta}) \Psi^*(e^{i\theta}) = \Phi(e^{i\theta}) \Phi^*(e^{i\theta})$,
we have $\Phi_0 \geq \sqrt{\Psi_0 \Psi_0^*}$.

The following uniqueness theorem can be proved exactly as in the finite dimensional case (c.f. [15]), and hence we omit its proof.

7.2.20 Theorem (Uniqueness theorem). Let Φ, Ψ be bounded linear operator valued functions such that Φ^{-1}, Ψ^{-1} exist and are bounded. Let $\Phi, \Phi^{-1}, \Psi, \Psi^{-1}$ be in $\overline{L}_2^{0+}$. If

$$\Phi(e^{i\theta}) \Phi^*(e^{i\theta}) = \Psi(e^{i\theta}) \Psi^*(e^{i\theta}),$$

then there exists a unitary operator U_0 such that

$$\Phi(e^{i\theta}) = \Psi(e^{i\theta})U_0.$$

Furthermore $U_0 = I$, if either Φ_0 and Ψ_0 are equal, or they are positive definite.

7.3 Further Analysis of Time and Spectral Domain. In this section we develop some results which will be needed in the next sections.

Let ξ_n , $-\infty < n < \infty$, be a SSP. We denote by $\xi_{L,n}$, $-\infty < n < \infty$ the L-dimensional subprocess of it, i.e. for each $n \in \mathbb{Z}$, we have $(\xi_{L,n})^i = \xi_n^i$ if $0 < i \leq L$ and zero if $i > L$.

The following theorem gives some relation between Σ and Σ_L , the predictor error matrix of ξ_n , $-\infty < n < \infty$ and $\xi_{L,n}$, $-\infty < n < \infty$ respectively.

7.3.1 Theorem. Let ξ_n , $-\infty < n < \infty$ be a SSP such that (ξ_0, ξ_0) is a bounded operator. Let Σ and Σ_L be the two sided predictor error matrices of ξ_n , $-\infty < n < \infty$ and $\xi_{L,n}$, $-\infty < n < \infty$ respectively. Then $\Sigma \geq \lambda^2 I$ if and only if $\Sigma_L \geq \lambda^2 I_L$ for all $L > 0$.

Proof. If $\Sigma > \lambda^2 I$ then clearly $\Sigma_L > \lambda^2 I_L$ for all $L > 0$. To prove the other way, let us assume $\Sigma_L > \lambda^2 I_L$ for all $L > 0$, and suppose $\Sigma < \lambda^2 I$, i.e. suppose there exists a sequence c_n , $-\infty < n < \infty$ with $\sum_{i=1}^{\infty} |c_i|^2 = 1$ such that

$$\sum_{i,j=1}^{\infty} c_i \Sigma_{ij} \overline{c_j} = \lambda'^2 < \lambda^2. \quad (1)$$

Let $\epsilon = \lambda - \lambda'$ and take $N_1 > 0$ such that

$$\sum_{j=n}^{\infty} |c_j|^2 < \frac{\lambda^2 - (\lambda' + \epsilon/2)^2}{\lambda^2}, \text{ for all } n > N_1. \quad (2)$$

We have

$$\left\| \sum_{i=1}^{\infty} c_i \xi_0^i \right\|^2 = \left(\sum_{i=1}^{\infty} c_i \xi_0^i, \sum_{i=1}^{\infty} c_i \xi_0^i \right) = \sum_{k,j=1}^{\infty} c_i (\xi_0^i, \xi_0^j) \overline{c_j} < \infty,$$

because (ξ_0, ξ_0) is bounded. Hence there exists $N_2 > 0$ such that

$$\left\| \sum_{i=n}^{\infty} c_i \xi_0^i \right\| < \epsilon/4, \text{ for all } n > N_2. \quad (3)$$

If ζ_0 is the two sided linear predictor of ξ_n , $-\infty < n < \infty$ then (1) means

$$\begin{aligned} \lambda^2 > \lambda'^2 &= \sum_{i,j=1}^{\infty} c_i \sum_{i,j} \overline{c_j} = \sum_{i,j} c_i (\zeta_0, \zeta_0)_{ij} \overline{c_j} \\ &= \left\| \sum_{i=1}^{\infty} c_i \zeta_0^i \right\|^2 = \left\| \sum_{i=1}^{\infty} c_i \xi_0^i - \left(\sum_{i=1}^{\infty} c_i \xi_0^i \middle| K(0) \right) \right\|^2. \end{aligned}$$

Hence we get $\|Q(\sum_{i=1}^{\infty} c_i \xi_0^i)\| = \lambda' < \lambda$, where Q is the projection on $K(0)^\perp \cap H(\infty)$. Let Q_L be the projection on $H(\infty) \cap K_L(0)^\perp$.

Then since $K_L(0) \uparrow K(0)$ we obtain $Q_L \downarrow Q$. So there exists $N_3 > 0$ such that

$$\|Q_n(\sum_{i=1}^{\infty} c_i \xi_0^i)\| < \lambda' + \epsilon/4, \text{ for all } n > N_3. \quad (4)$$

Let $N = \max(N_1, N_2, N_3)$ then by (4) we obtain that

$$\|P_N(\sum_{i=1}^{\infty} c_i \xi_0^i)\| < \lambda' + \epsilon/4. \text{ Hence we get}$$

$$\|Q_N(\sum_{i=N+1}^{\infty} c_i \xi_0^i) + Q_N(\sum_{i=1}^N c_i \xi_0^i)\| < \lambda' + \epsilon/4.$$

So we get

$$\begin{aligned} &\|P_N(\sum_{i=1}^N c_i \xi_0^i)\| - \|Q_N(\sum_{i=N+1}^{\infty} c_i \xi_0^i)\| \\ &\leq \|Q_N(\sum_{i=N+1}^{\infty} c_i \xi_0^i) + Q_N(\sum_{i=1}^N c_i \xi_0^i)\| \leq \lambda' + \epsilon/4. \end{aligned}$$

Hence

$$\|Q_N(\sum_{i=1}^N c_i \xi_0^i)\| \leq \lambda' + \epsilon/4 + \|Q_N(\sum_{i=N+1}^{\infty} c_i \xi_0^i)\| < \lambda' + \epsilon/2.$$

Thus $\|Q_N(\sum_{i=1}^N c_i \xi_0^i)\|^2 < (\lambda' + \epsilon/2)^2$ and hence by (2) we get

$$\begin{aligned} \|Q_N(\sum_{i=1}^N c_i \xi_0^i)\|^2 &< (\lambda' + \epsilon/2)^2 + [(\lambda^2 - (\lambda' + \epsilon/2)^2 - \lambda^2 \sum_{i=N+1}^{\infty} |c_i|^2)] \\ &< \lambda^2 - \lambda^2 \sum_{i=N+1}^{\infty} |c_i|^2 = \lambda^2 (\sum_{i=1}^{\infty} |c_i|^2 - \sum_{i=N+1}^{\infty} |c_i|^2). \end{aligned}$$

Hence $\|Q_N(\sum_{i=1}^N c_i \xi_0^i)\|^2 < \lambda^2 \sum_{i=1}^N |c_i|^2$ and thus we get

$$\|\sum_{i=1}^N c_i \xi_0^i - (\sum_{i=1}^N c_i \xi_0^i |K_N(0))\|^2 \leq \lambda^2 \sum_{i=1}^N |c_i|^2,$$

or

$$\|\sum_{i=1}^N c_i (\xi_0^i - (\xi_0^i |K_N(0)))\|^2 < \lambda^2 \sum_{i=1}^N |c_i|^2.$$

So we get $\|\sum_{i=1}^N c_i \xi_{L,0}^i\|^2 < \lambda^2 \sum_{i=1}^N |c_i|^2$, which implies

$$\sum_{i,j=1}^N c_i \sum_{i,j} \overline{c_j} < \lambda^2 \sum_{i=1}^N |c_i|^2.$$

Hence we get $\Sigma_N < \lambda^2 I_N$, which is a contradiction.

A similar theorem for the one sided predictor errors,

G and G_L was proved by Gangolli [8].

We will need the following theorem due to Masani [14].

7.3.2 Theorem. Let ξ_n , $-\infty < n < \infty$, be a finite dimensional SSP with spectral distribution F and two sided predictor error matrix Σ . Then ξ_n , $-\infty < n < \infty$ is minimal full rank if and only if F'^{-1} exists a.e. and $F'^{-1} \in L_1$. In this case we have

$$\left[\frac{1}{2\pi} \int_0^{2\pi} F^{-1}(e^{i\theta}) d\theta \right]^{-1} = \Sigma.$$

To progress further in this section, we suppose our stationary stochastic process satisfies the following condition.

7.3.3 Assumption. Let ξ_n , $-\infty < n < \infty$ be a SSP with a spectral density f such that $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ a.e. with $1/m(e^{i\theta})$ and $M(e^{i\theta})$ being summable.

Let f_L be the spectral density of the L -dimensional subprocess of ξ_n , $-\infty < n < \infty$. Then $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ and hence for all $L > 0$, $(1/m(e^{i\theta}))I \leq f_L^{-1}(e^{i\theta}) \leq (1/m(e^{i\theta}))I$. Hence by theorem 7.3.2, the subprocess $\xi_{L,n}$, $-\infty < n < \infty$ is minimal full rank for each $L > 0$. Now applying theorem 7.3.2 to these subprocesses we get

$$\Sigma_L = \left[\frac{1}{2\pi} \int_0^{2\pi} f_L^{-1}(e^{i\theta}) d\theta \right]^{-1}, \text{ for all } L > 0.$$

Thus by theorem 7.3.1, $\Sigma > \left[\frac{1}{2\pi} \int_0^{2\pi} (1/m(e^{i\theta})) d\theta \right]^{-1} I_1$. We also have $G > \Sigma > \left[\frac{1}{2\pi} \int_0^{2\pi} (1/m(e^{i\theta})) d\theta \right]^{-1} I$.

Summarizing we get the following lemma.

7.3.4 Lemma. If ξ_n , $-\infty < n < \infty$ is a SSP with density f satisfying the assumption 7.3.3, then $G > \Sigma > \lambda I$ for some $\lambda > 0$. Now since G^{-1} exists we let $\theta_k = \sqrt{G^{-1}} \eta_k$, and we call it the one sided normalized innovation process of the processes ξ_n , $-\infty < n < \infty$.

Now using the last lemma in conjunction with several results in [12], we prove the following theorem.

7.3.5 Theorem. Let ξ_n , $-\infty < n < \infty$ be a SSP with density f satisfying assumption 7.3.3. Then ξ_n , $-\infty < n < \infty$ is purely

nondeterministic and f is factorable as

$$f(e^{i\theta}) = \Phi(e^{i\theta}) \Phi^*(e^{i\theta}),$$

where $\Phi(e^{i\theta}) = \sum_{k=0}^{\infty} C_k e^{ik\theta}$ and $\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} |C_{ij}^k|^2 < \infty$, for each $i \in \mathbb{Z}_+$. Furthermore Φ is the optimal factor with $C_0 = \sqrt{G}$ and $C_k = (\xi_0, \theta_{-k})$, for each $k \in \mathbb{Z}_+$.

Proof. Let $d(e^{i\theta}) = (1 \wedge m(e^{i\theta}))I$, then $f(e^{i\theta}) \geq d(e^{i\theta})$ a.e.

and we have

$$\begin{aligned} \int_0^{2\pi} |\log(1 \wedge m(e^{i\theta}))| d\theta &\leq \int_0^{2\pi} \log(1/(1 \wedge m(e^{i\theta}))) d\theta \\ &\leq \log \int_0^{2\pi} (1/(1 \wedge m(e^{i\theta}))) d\theta \leq \log \left[\int_E (1/m(e^{i\theta})) d\theta + \int_{E^c} d\theta \right], \end{aligned}$$

where $E = \{\theta, 0 \leq \theta \leq 2\pi, m(e^{i\theta}) < 1\}$. Hence $\int_0^{2\pi} |\log(1 \wedge m(e^{i\theta}))| d\theta \leq \log \left[\int_0^{2\pi} (1/m(e^{i\theta})) d\theta + 1 \right] < \infty$. Now by [17], p. 165, we see that f is factorable and hence by [17], p. 163, the SSP $\xi_n, -\infty < n < \infty$ is purely nondeterministic. Thus by [17], pp. 155-156, $\xi_n, -\infty < n < \infty$, has a one sided moving average, $\xi_n = \sum_{k=0}^{\infty} C_k \theta_{n-k}$, such that for each $i \in \mathbb{Z}_+$, $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |C_{ij}^k|^2 < \infty$ and $\theta_n, -\infty < n < \infty$ is the one sided normalized innovation process of $\xi_n, -\infty < n < \infty$. Now take $\Phi(e^{i\theta}) = \sum_{k=0}^{\infty} C_k e^{ik\theta}$. It is clear that Φ is an analytic factor of f , and we have

$$(\xi_0, \theta_0) = \left(\sum_{k=0}^{\infty} C_k \theta_{-k}, \theta_0 \right) = C_0(\theta_0, \theta_0) = C_0.$$

Also we have

$$(\xi_0, \theta_{-k}) = \left(\sum_{k=0}^{\infty} C_k \theta_{-k}, \theta_{-k} \right) = (C_k \theta_{-k}, \theta_{-k}) = C_k(\theta_{-k}, \theta_{-k}) = C_k.$$

Finally, in order to show Φ is an optimal factor, assume that

Ψ is an analytic factor, then we have $f = \Phi \Phi^* = \Psi \Psi^*$. Let $\bar{H}(-N, -1) = \mathcal{G}(\xi_k, -N \leq k \leq -1)$, $(\xi_0 | \bar{H}(-N, -1)) = \sum_{k=1}^N A_k \xi_{-k}$. Then

$G = \lim_{N \rightarrow \infty} G_N$, where

$$\begin{aligned} G_N &= (\xi_0 - \sum_{k=1}^N A_k \xi_{-k}, \xi_0 - \sum_{k=1}^N A_k \xi_{-k}) \\ &= \frac{1}{2\pi} \int_0^{2\pi} (I - \sum_{k=1}^N A_k e^{ik\xi})^* f(e^{i\xi}) (I - \sum_{k=1}^N A_k e^{ik\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} (I - \sum_{k=1}^N A_k e^{ik\theta})^* \Psi^* \Psi (I - \sum_{k=1}^N A_k e^{ik\theta}) d\theta. \end{aligned}$$

Hence

$$G_N > \sum_{k=0}^N (E_k^N)^* (E_k^N) > (E_1^N)^* (E_1^N),$$

where E_k^N is the k -th Fourier coefficient of $\Psi(I - \sum_{k=1}^{\infty} A_k e^{ik\theta})$.

It is easy to see that $E_1^N = \Psi_0$ for all N . Thus

$$G_N \geq \Psi_0 \Psi_0^*.$$

Now let $N \rightarrow \infty$ to get

$$\Phi_0^* \Phi_0 = G \geq \Psi_0^* \Psi_0.$$

The proof of optimality given above is adapted from the proof of the similar result due to Masani, for finite dimensional case.

We will need the following lemma.

7.3.6 Lemma. Let ξ_n , $-\infty < n < \infty$ be a SSP with spectral density f satisfying assumption 7.3.3. Let θ_n , $-\infty < n < \infty$ and Φ be its normalized innovation and generating function respectively. Then

(a) $e^{-ni\theta} \Phi^{-1}(e^{i\theta})$ belongs to $\bar{L}_2(f)$ and corresponds to θ_n in $\bar{H}(\infty)$.

(b) For Ψ in $\bar{L}_2(f)$, $\Psi\Phi$ belongs to $\bar{L}_2$.

(c) For any Ψ in $\bar{L}_2(f)$, if we let A_k be the k -th Fourier coefficient of $\Psi\Phi$, then $(\sum_{k=-n}^n A_k e^{ik\theta})\Phi^{-1} \rightarrow \Psi$ in $\bar{L}_2(f)$ sense.

Proof. (a) Since θ_n belongs to $H(\infty)$, there exists a corresponding element of $\bar{L}_2(f)$, say Ψ . Now consider

$$(e^{in\theta}\Phi)_{\ell,m} = (\sum_{j=0}^{\infty} (\xi_0, \theta_j) e^{i(n+j)\theta})_{\ell,m} = \sum_{j=n}^{\infty} (\xi_0, \theta_{n-j})_{\ell,m} e^{ij\theta}. \quad (1)$$

On the other hand, we have

$$\begin{aligned} \int_0^{2\pi} e^{-ik\theta} (f(e^{i\theta})\Psi^*(e^{i\theta}))_{\ell,m} d\theta &= \int_0^{2\pi} e^{-ik\theta} (\sum_{j=1}^{\infty} f_{\ell j}(e^{i\theta}) \Psi_{jm}(e^{i\theta})) d\theta \\ &= (e^{-ik\theta} I, \Psi)_{\ell,m}. \end{aligned} \quad (2)$$

The last equality is by definition, since for arbitrary Ψ, Φ

in $\bar{L}_2(f)$ we have $(\Phi, \Psi)_{\ell,m} = (\Phi^{\ell}, \Psi^m) = \int \sum_{i,j=1}^{\infty} \varphi_{\ell i} f_{ij} \bar{\psi}_{jm} d\theta$. Now by (2) we get

$$\int_0^{2\pi} e^{-ik\theta} (f(e^{i\theta})\Psi^*(e^{i\theta}))_{\ell,m} d\theta = (e^{-ik\theta} I, \Psi)_{\ell,m} = (\xi_0, \theta_{n-k})_{\ell,m}. \quad (3)$$

Now (1) and (3) imply that for each $\ell, m \in \mathbb{Z}_+$, $(e^{ni\theta}\Phi(e^{i\theta}))_{\ell,m}$

and $(f(e^{i\theta})\Psi^*(e^{i\theta}))_{\ell,m}$ have the same Fourier coefficients.

Hence for each $\ell, m \in \mathbb{Z}_+$ we have $(e^{ni\theta}\Phi(e^{i\theta}))_{\ell,m} = (f(e^{i\theta})\Psi^*(e^{i\theta}))_{\ell,m}$ which means

$$e^{ni\theta}\Phi(e^{i\theta}) = f(e^{i\theta})\Psi^*(e^{i\theta}) = \Phi(e^{i\theta})\Phi^*(e^{i\theta})\Psi^*(e^{i\theta}).$$

Now since $\Phi(e^{i\theta})$ is invertible we get $\Psi^*(e^{i\theta}) = e^{in\theta}\Phi^{*-1}(e^{i\theta})$.

Thus

$$\Psi(e^{i\theta}) = e^{-ni\theta}\Phi^{-1}(e^{i\theta}) \quad \text{a.e.}$$

so $e^{-ni\theta} \Phi^{-1}(e^{i\theta})$ corresponds to θ_n .

(b) Suppose $\Psi \in \bar{L}_2(f)$. It suffices to show that $(\Psi\Phi)^k \in L_2$, for all $k \in \mathbb{Z}_+$. We observe that $(\Psi\Phi)^k = [\psi_{k,1}, \psi_{k,2}, \dots]\Phi$.

Now we observe

$$\begin{aligned} \int_0^{2\pi} \|(\Psi\Phi)^k\|^2 d\theta &= \int_0^{2\pi} \|[\psi_{k,1}, \psi_{k,2}, \dots]\Phi\|^2 d\theta \\ &= \int_0^{2\pi} ([\psi_{k,1}, \psi_{k,2}, \dots]\Phi, [\psi_{k,1}, \psi_{k,2}, \dots]\Phi) d\theta \\ &= \int_0^{2\pi} ([\psi_{k,1}, \psi_{k,2}, \dots]f, [\psi_{k,1}, \psi_{k,2}, \dots]) d\theta. \end{aligned}$$

Hence $\int_0^{2\pi} \|(\Psi\Phi)^k\|^2 d\theta = \int_0^{2\pi} \sum_{i,j=1}^{\infty} \psi_{k,j} f_{i,j} \bar{\psi}_{k,j} d\theta < \infty$. The last inequality follows from $\Psi^k \in L_2(f)$.

(c) Using the Parseval identity of lemma 7.2.12 twice, we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \|(\sum_{k=-n}^n A_k e^{ik\theta})^{\mathcal{L}}\|^2 d\theta &= \sum_{k=-n}^n \sum_{j=1}^{\infty} |a_{\mathcal{L}j}^k|^2 \leq \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} |a_{\mathcal{L}j}^k|^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|(\Psi\Phi)^{\mathcal{L}}\|^2 d\theta = \|(\Psi\Phi)^{\mathcal{L}}\| < \infty. \end{aligned}$$

The last inequality follows by (b). So $\sum_{k=-n}^n A_k e^{ik\theta}$ is in $\bar{L}_2$.

Hence $\sum_{k=-n}^n (A_k e^{ik\theta})_{\Phi}^{-1}$ belongs to $\bar{L}_2(f)$ for each $n \geq 0$.

(This follows by similar argument provided below.) Now to show

$\sum_{k=-n}^n (A_k e^{ik\theta})_{\Phi}^{-1} \Psi$ converges to zero in $\bar{L}_2(f)$, it is sufficient to show that $((\sum_{k=-n}^n A_k e^{ik\theta})_{\Phi}^{-1} - \Psi)^{\mathcal{L}} \rightarrow 0$ for each $\mathcal{L} \in \mathbb{Z}_+$. Consider

$$\begin{aligned} \|(\sum_{k=-n}^n A_k e^{ik\theta})_{\Phi}^{-1} - \Psi)^{\mathcal{L}}\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} \|((\sum_{k=-n}^n A_k e^{ik\theta})_{\Phi}^{-1} - \Psi)^{\mathcal{L}}\|^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|(((\sum_{k=-n}^n A_k e^{ik\theta})_{\Phi}^{-1} - \Psi)\sqrt{f})^{\mathcal{L}}\|^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|((\sum_{k=-n}^n A_k e^{ik\theta} - \Psi\Phi)_{\Phi}^{-1} \sqrt{f})^{\mathcal{L}}\|^2 d\theta = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} \left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} - \psi_{\Phi} \right) \mathcal{L}_{\Phi}^{-1} \sqrt{f} \right\|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\left(\sum_{k=-n}^n A_k e^{ik\theta} - \psi_{\Phi} \right) \mathcal{L}_{\Phi}^{-1} \sqrt{f} \sqrt{f} \Phi^{-1*}, \left(\sum_{k=-n}^n A_k e^{ik\theta} - \psi_{\Phi} \right) \mathcal{L}_{\Phi} \right) d\theta.
\end{aligned}$$

Hence

$$\left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} \right)_{\Phi}^{-1} - \psi \right\|_{\mathcal{L}}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} - \psi_{\Phi} \right) \mathcal{L}_{\Phi} \right\|^2 d\theta.$$

Hence

$$\left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} \right)_{\Phi}^{-1} - \psi \right\|_{\mathcal{L}}^2 = \left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} - \psi_{\Phi} \right) \mathcal{L}_{\Phi} \right\|^2 d\theta. \quad (4)$$

Now since A_k 's are the Fourier coefficients of ψ_{Φ} , by the Parseval identity the right hand side of (4) converges to zero and hence $\left\| \left(\sum_{k=-n}^n A_k e^{ik\theta} \right)_{\Phi}^{-1} - \psi \right\|_{\mathcal{L}}^2$ converges to zero.

Now we prove the following corollary which is important.

7.3.7 Corollary. Let $\Gamma_v(e^{i\theta}) = [e^{-iv\theta}_{\Phi}(e^{i\theta})]_{0+} \Phi^{-1}(e^{i\theta})$, $v > 0$. Then $\Gamma_v(e^{i\theta})$ belongs to $\bar{L}_2(f)$ and corresponds to the linear predictor $\hat{\xi}_v = (\xi_v | \bar{H}(0))$ under the isomorphism $\bar{S}$ given in definition 7.2.10.

Proof. Consider

$$\begin{aligned}
\|(\Gamma_v)_{\mathcal{L}}\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} \|(\Gamma_v(e^{i\theta})) \mathcal{L}_{\Phi} \sqrt{f}(e^{i\theta})\|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|(\Gamma_v(e^{i\theta}) \sqrt{f}(e^{i\theta})) \mathcal{L}_{\Phi}\|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|([e^{-iv\theta}_{\Phi}(e^{i\theta})]_{0+} \Phi^{-1} \sqrt{f}(e^{i\theta})) \mathcal{L}_{\Phi}\|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|([e^{-iv\theta}_{\Phi}(e^{i\theta})]_{0+}) \mathcal{L}_{\Phi}^{-1}(e^{i\theta}) \sqrt{f}(e^{i\theta})\|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|([e^{-iv\theta}_{\Phi}(e^{i\theta})]_{0+}) \mathcal{L}_{\Phi}\|^2 d\theta.
\end{aligned}$$

The last equality follows as in the proof of part (c) of the last lemma. Hence $\|(\Gamma_v(e^{i\theta}))^\ell\|^2 = \|([e^{-iv\theta}\tilde{\Phi}(e^{i\theta})]_{0+})^\ell\|^2 \leq \|\tilde{\Phi}(e^{i\theta})^\ell\|^2$. Thus $\|(\Gamma_v)^\ell\|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} (e_\ell \tilde{\Phi}^*, e_\ell) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (e_\ell f, e_\ell) d\theta < \infty$, because f is weakly summable. Hence $\Gamma_v(e^{i\theta}) = [e^{-iv\theta}\tilde{\Phi}(e^{i\theta})]_{0+} \tilde{\Phi}^{-1}(e^{i\theta})$ belongs to $\bar{L}_2$. Now let C_k be the k -th Fourier coefficient of the generating function $\tilde{\Phi}$; $\Gamma_v(e^{i\theta})\tilde{\Phi}(e^{i\theta}) = [e^{-iv\theta}\tilde{\Phi}(e^{i\theta})]_{0+} = \sum_{k=0}^{\infty} C_{v+k} e^{ik\theta}$ belongs to $\bar{L}_2$ and by lemma 7.3.6 part (c), we see that

$$\left(\sum_{k=0}^N C_{v+k} e^{ik\theta} \right) \tilde{\Phi}^{-1}(e^{i\theta}) \rightarrow \Gamma_v(e^{i\theta}) \quad \text{in } \bar{L}_2(f).$$

Now let γ_v be the random function in $\bar{H}(\infty)$ corresponded to Γ_v in $\bar{L}_2(f)$. Then by part (a) of lemma 7.3.6, we have

$$\sum_{k=0}^N C_{v+k} \theta_{-k} \rightarrow \gamma_v, \quad \text{in } \bar{H} \text{ sense.}$$

On the other hand, $\sum_{k=0}^N C_{v+k} \theta_{-k}$ converges to $\hat{\xi}_v$, because $\xi_v = \sum_{k=0}^{\infty} C_{v+k} \theta_k$. Hence $\gamma_v = \hat{\xi}_v$, i.e. $\hat{\xi}_v$ corresponds to Γ_v , under the isomorphism $\bar{S}$.

Let ξ_n , $-\infty < n < \infty$ be a minimal full rank process with two sided innovation process ζ_n , $-\infty < n < \infty$. We write $\alpha_n = \zeta_n^{-1}$ and call α_n , $-\infty < n < \infty$ the normalized two sided innovation process of ξ_n , $-\infty < n < \infty$.

Now we prove the following lemma which is crucial in getting our algorithm.

7.3.8 Lemma. Let ξ_n , $-\infty < n < \infty$ be a SSP with density f satisfying assumption 7.3.3 and let α_n , $-\infty < n < \infty$ be its two sided normalized innovation process. Then

(a) The sequences α_n , $-\infty < n < \infty$ and ξ_n , $-\infty < n < \infty$ are bi-orthogonal.

- (b) $\alpha_n = \sum_{k=0}^{\infty} D_k^* \theta_{n+k}$, where θ_n , $-\infty < n < \infty$ is the one sided normalized innovation process of ξ_n , $-\infty < n < \infty$, and D_k is the k -th Fourier coefficient of the inverse Φ^{-1} of the generating function $\Phi(e^{i\theta}) = \sum_{k=0}^{\infty} c_k e^{ik\theta}$.
- (c) $\Phi^{-1} \in \bar{L}_2^{0+}$.

Proof. (a) We have

$$\begin{aligned} (\alpha_0, \xi_0) &= (\Sigma^{-1} \zeta_0, \xi_0) = \Sigma^{-1}(\zeta_0, \xi_0) \\ &= \Sigma^{-1}(\xi_0 - (\xi_0 | \bar{K}(0)), \xi_0) \\ &= \Sigma^{-1}(\xi_0 - (\xi_0 | \bar{K}(0)), \xi_0 - (\xi_0 | \bar{K}(0))). \end{aligned}$$

So we get

$$(\alpha_0, \xi_0) = \Sigma^{-1}(\zeta_0, \zeta_0) = \Sigma^{-1} \Sigma = I. \quad (1)$$

For $n \neq 0$ we have $\xi_0 \in \bar{K}(n) \perp \alpha_k$. Hence

$$(\alpha_k, \xi_0) = 0. \quad (2)$$

Using (1), (2) and the stationarity of our process, we get (a).

(b) and (c). Since $\alpha_k \perp \bar{K}(0) \supset \bar{H}(n)$ for $n < 0$, we get

$$\alpha_0 \perp \bigcap_n \bar{H}(n) = \bar{H}(-\infty). \quad (3)$$

Now since $\alpha_0 \in \bar{H}(\infty)$ and $\bar{H}(\infty) = \bar{H}(-\infty) \oplus \bar{\Theta}(\theta_k, -\infty < k < \infty)$, we get $\alpha_0 \in \bar{\Theta}(\theta_k, -\infty < k < \infty)$. In fact $\alpha_0 \in \bar{\Theta}(\theta_k, k \geq 0)$, because for each $k < 0$, $\theta_k \subset \bar{H}(-1) \subset \bar{K}(0) \perp \alpha_0$. Hence

$$\alpha_0 = \sum_{k=0}^{\infty} A_k \theta_k, \text{ with } \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} |a_{kj}^k|^2 < \infty, \quad (4)$$

for each $i \in \mathbb{Z}_+$. Now by Wold's decomposition we have

$\xi_n = \sum_{k=0}^{\infty} C_k \theta_{n-k} + v_n$, where $v_n \in \overline{H}(-\infty)$. Hence by (3) and (4) we obtain

$$(\alpha_0, \xi_n) = \left(\sum_{j=0}^{\infty} A_j \theta_j, \sum_{k=0}^{\infty} C_k \theta_{n-k} \right) = \sum_{j=0}^n A_j C_{n-j}^*.$$

Hence by part (a) we have

$$\sum_{j=0}^n A_j C_{n-j}^* = \delta_{n0} I. \quad (5)$$

Then by the convolution rule we get

$$\left(\sum_{k=0}^{\infty} A_k e^{-ik\theta} \right) \Phi^*(e^{i\theta}) = I \text{ a.e.} \quad (6)$$

(6), together with the invertibility of Φ^* , implies that

$$\Phi^{*-1} = \sum_{k=0}^{\infty} A_k e^{-ik\theta} \text{ a.e.} \quad (7)$$

Taking adjoint, we get

$$\Phi^{-1} = \left(\sum_{k=0}^{\infty} A_k e^{-ik\theta} \right)^* \text{ a.e.} \quad (8)$$

So the (ℓ, m) -th entry of Φ^{-1} is $(\Phi^{-1})_{\ell, m} = \sum_{k=0}^{\infty} \overline{a_{m, \ell}^k} e^{ik\theta}$. Hence the k -th Fourier coefficients of $(\Phi^{-1})_{\ell, m}$ is $\overline{a_{m, \ell}^k}$, for each m, ℓ , $1 \leq m, \ell < \infty$. Hence if we let $D_k = A_k^*$ then we have

$$\Phi^{-1} = \sum_{k=0}^{\infty} D_k e^{ik\theta}$$

in the following sense

$$(\Phi^{-1}(e^{i\theta}))_{\ell, m} = \sum_{k=0}^{\infty} d_{\ell, m}^k e^{-ik\theta}. \quad (9)$$

Now from $\|\Phi^{-1}(e^{i\theta})\|^2 = \|f^{-1}(e^{i\theta})\|^2 \leq 1/m(e^{i\theta})$ we see that $\Phi^{-1} \in \overline{L}_2$, which means $\sum_{m=1}^{\infty} \sum_{k=0}^{\infty} |d_{\ell, m}^k|^2 < \infty$, for all ℓ , $1 \leq \ell < \infty$. This and (9) implies that

$$\Phi^{-1}(e^{i\theta}) = \sum_{k=0}^{\infty} D_k e^{ik\theta}, \text{ in } \bar{L}_2(f).$$

This completes the proof of (b) and (c).

7.4 Determination of the generating function and the linear predictor.

In this section we shall express the generating function of a SSP, ξ_n , $-\infty < n < \infty$ satisfying some boundedness conditions (to be made precise later) in terms of the spectral density f by an iterative procedure as in the finite dimensional case [14]. We shall then derive a computable expression for the linear predictor error matrix. We mention here that because of infinite dimensionality our convergents here would be in a weaker sense than the convergents of the corresponding results in [14].

Here we suppose that our process ξ_n , $-\infty < n < \infty$ has a spectral density f satisfying the following assumption.

7.4.1 Assumption. Let $f(e^{i\theta})$, the spectral density of our process satisfy

$$m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I,$$

with $M(e^{i\theta})$, $1/m(e^{i\theta})$ and $M(e^{i\theta})/m(e^{i\theta})$ being summable.

We need the following lemma.

7.4.2 Lemma. Let ξ_n , $-\infty < n < \infty$ be a SSP whose spectral density f , satisfies the assumption 7.4.1. Then there exists a nonnegative real valued function $f_1(e^{i\theta})$ and a nonnegative infinite dimensional matrix valued function $f_2(e^{i\theta})$ such that

$$(a) \quad f(e^{i\theta}) = f_1(e^{i\theta})f_2(e^{i\theta})$$

$$(b) \quad f_2(e^{i\theta}) = I + N(e^{i\theta}), \text{ where } \|N(e^{i\theta})\|_B < 1 \text{ a.e.}$$

$$(c) \quad f_2(e^{i\theta}) \text{ satisfies assumption 7.3.3}$$

(d) $f_1(e^{i\theta})$ and $1/f_1(e^{i\theta})$ being summable.

Proof. (a) Let $f_1(e^{i\theta}) = \frac{1}{2}(M(e^{i\theta}) + m(e^{i\theta}))$ and $f_2(e^{i\theta}) = f(e^{i\theta})/f_1(e^{i\theta})$ which are defined a.e. and satisfies (a).

(b) Let $N(e^{i\theta}) = f_2(e^{i\theta}) - I = (f_1(e^{i\theta})/f_2(e^{i\theta})) - I$. Since $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$, for all θ 's we get

$$\|N(e^{i\theta})\|_B = \left| \frac{2}{m(e^{i\theta}) + M(e^{i\theta})} f(e^{i\theta}) - I \right| \leq \frac{M(e^{i\theta}) - m(e^{i\theta})}{M(e^{i\theta}) + m(e^{i\theta})}.$$

Now since $m(e^{i\theta}) > 0$ a.e. we get $\|N(e^{i\theta})\|_B < 1$ a.e.

(c) Now from $f_2(e^{i\theta}) = I + N(e^{i\theta}) = f(e^{i\theta})/f_1(e^{i\theta})$, we get

$$\frac{2m(e^{i\theta})}{M(e^{i\theta}) + m(e^{i\theta})} I \leq f_2(e^{i\theta}) \leq \frac{2M(e^{i\theta})}{M(e^{i\theta}) + m(e^{i\theta})} I.$$

Hence $(m(e^{i\theta})/M(e^{i\theta}))I \leq f_2(e^{i\theta}) \leq 2(M(e^{i\theta})/M(e^{i\theta})) = 2I$, which completes the proof of (c).

(d) $f_1(e^{i\theta})$ and $1/f_1(e^{i\theta})$ are summable, because we have $0 \leq f_1(e^{i\theta}) \leq M(e^{i\theta})$ and $0 \leq 1/f_1(e^{i\theta}) \leq 2/m(e^{i\theta})$.

The following theorem gives the relation between the generating functions of the spectral densities f , f_1 and f_2 .

7.4.3 Theorem. Let f be the spectral density of a SSP which satisfies the assumption 7.4.1. Let $f_1(e^{i\theta})$, $f_2(e^{i\theta})$ and $N(e^{i\theta})$ be as in theorem 7.4.2. Let Φ , φ_1 , Φ_2 be the generating functions and G , g_1 , G_2 be the predictor error matrices of the spectral densities f , f_1 , f_2 respectively. Then

(a) Φ^{-1} , $(1/\varphi_1)I$ and Φ_2^{-1} are in $\overline{L}_2^{0+}$.

(b) $\Phi = \varphi_1 \Phi_2$.

(c) $G = g_1 G_2$.

Proof. (a) is clear from theorem 7.4.2 and lemma 7.3.8. For

(b) consider

$$\Phi \Phi^* = f = f_1 f_2 = \varphi_1 \bar{\varphi}_1 \Phi_2 \Phi_2^* = (\varphi_1 \Phi_2) (\varphi_1 \Phi_2)^* = \Psi \Psi^*,$$

where $\Psi = \varphi_1 \Phi_2$. By (a) and convolution rule we see that Ψ and Ψ^{-1} belong to $\bar{L}_2^{0+}$. Since the 0-th Fourier coefficients of Φ and Ψ are positive matrices, we can apply the uniqueness theorem 7.2.21 to conclude $\Phi = \Psi$. Hence $\Phi = \varphi_1 \Phi_2$.

Now (c) follows because

$$\begin{aligned} G &= \Phi_0 \Phi_0^* = (\varphi_1 \Phi_2)_0 (\varphi_1 \Phi_2)_0^* = (\sqrt{g_1} \sqrt{G_2}) (\sqrt{g_1} \sqrt{G_2})^* \\ &= \sqrt{g_1} \sqrt{G_2} \sqrt{G_2} \sqrt{g_1} = g_1 G_2. \end{aligned}$$

Since f_1 is a real valued spectral density one can find its generating function by the usual method. So in order to find Φ we just have to get an algorithm to find the optimal factor of f_2 . Hence in view of the last theorem, we can assume that our spectral f satisfies the following condition.

7.4.4 Assumption. Let f be a spectral density of a SSP such that $f(e^{i\theta}) = I + N(e^{i\theta})$, where $N(e^{i\theta})$ is a Hermitian valued function with the following two properties.

- (i) $\|N(e^{i\theta})\|_B < 1$ a.e.
- (ii) $m(e^{i\theta})I \leq I + N(e^{i\theta}) \leq MI$, where M is a positive constant and $1/m(e^{i\theta})$ is summable.

From now on we will be working under the set up of assumption 7.4.4.

7.4.5 Definition. For any $\Psi \in L_2$ define $\theta(\Psi) = (\Psi N)_+$, this makes sense because $\|N(e^{i\theta})\|_B \leq 1$. Now for each $\Psi \in \bar{L}_2$ we define $\bar{\theta}$ by $(\bar{\theta}\Psi)^i = \theta(\Psi^i)$, for all $i \in \mathbb{Z}_+$.

We omit the easy proof of the following lemma.

7.4.6 Lemma. (a) ϑ is a bounded operator on L_2 into L_2 with the Banach norm less than or equal to one.

(b) $\bar{\vartheta}(I) = N_+$, $\bar{\vartheta}^2(I) = (N_+N)_+, \dots$

Now we prove the following lemma.

7.4.7 Lemma. Let $\bar{\vartheta}$ and G be the generating function and the predictor error matrix of the spectral density $f(e^{i\theta}) = I + N(e^{i\theta})$ satisfying condition 7.4.4. Then

$$(\bar{J} + \bar{\vartheta})(\sqrt{G} \bar{\vartheta}^{-1}) = I,$$

where $\bar{J}$ is the identity operator on $\bar{L}_2$.

Proof. Let $\Psi = \sqrt{G} \bar{\vartheta}^{-1}$. Then by theorem 7.3.8 part (c), Ψ belongs to $\bar{L}_2^{0+}$ and $\Psi_0 = I$. Hence $\Psi = I + \Psi_+$. Next, since $I + N(e^{i\theta}) = \bar{\vartheta}(e^{i\theta}) \bar{\vartheta}^*(e^{i\theta})$ we get

$$\Psi + \Psi N = \sqrt{G} \bar{\vartheta}^{-1}(I + N) = \sqrt{G} \bar{\vartheta}^* \in \bar{L}_2^{0-}.$$

Hence $\Psi_+ + (\Psi N)_+ = (\Psi + \Psi N)_+ = 0$. Thus $\Psi - I + (\Psi N)_+ = 0$. Hence

$$(\bar{J} + \bar{\vartheta})(\Psi) = I.$$

We next state the following theorem.

7.4.8 Theorem. Let ϑ and $\bar{\vartheta}$ be as in the definition 7.4.5.

Then

(a) ϑ is a strict contraction on L_2^{0+} , i.e. $0 \neq \Psi \in L_2^{0+}$ implies that $\|\vartheta\Psi\| < \|\Psi\|$.

(b) $\bar{J} + \bar{\vartheta}$ is one-to-one on $\bar{L}_2$ into itself.

(c) $(\vartheta\Psi, X) = (\Psi, \vartheta X)$, for all $\Psi, X \in L_2^+$.

Proof. (a) By assumption 7.4.4 there exists an $\epsilon > 0$ and a set C_ϵ with positive measure such that

$$\|N(e^{i\theta})\|_B < \sqrt{1-\epsilon}, \text{ for all } \theta \in C_\epsilon.$$

Let $0 \neq \psi \in L_2^{0+}$. Since $\|\phi\psi\| = \|(\psi N)_+\| < \|\psi N\|$ we have

$$\begin{aligned} \|\phi\psi\|^2 &\leq \frac{1}{2\pi} \int_0^{2\pi} \|\psi(e^{i\theta})N(e^{i\theta})\|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \|\psi(e^{i\theta})\|^2 \|N(e^{i\theta})\|_B^2 d\theta \\ &\leq (1-\epsilon) \frac{1}{2\pi} \int_{C_\epsilon} \|\psi(e^{i\theta})\|^2 d\theta + \frac{1}{2\pi} \int_{C/C_\epsilon} \|\psi\|^2 d\theta. \end{aligned}$$

Hence we get $\|\phi\psi\|^2 \leq \|\psi\|^2 - \frac{\epsilon}{2\pi} \int_{C_\epsilon} \|\psi\|^2 d\theta$. Now since $0 \neq \psi \in L_2^{0+}$ one can see that $\int_{C_\epsilon} \|\psi(e^{i\theta})\|^2 d\theta > 0$, which means $\|\phi\psi\| < \|\psi\|$.

(b) Let $\psi \in \bar{L}_2$ and suppose that $(\bar{\phi} + \bar{\phi})(\psi) = 0$. Then $(\bar{\phi} + \bar{\phi})(\psi^i) = \psi^i + \bar{\phi}(\psi^i) = 0$ for all $i \in Z_+$. Hence $\psi^i = -\bar{\phi}(\psi^i)$. So $\psi^i \in L_2^{0+}$ and consequently $\|\psi^i\| = \|\bar{\phi}(\psi^i)\|$. Hence by part (a) we get $\psi^i = 0$ for all $i \in Z_+$, which means $\psi = 0$. This completes the proof of (b).

(c) For ψ and X in L_2^+ we have

$$(\phi\psi, X) = ((\psi N)_+, X) = (\psi N, X_+) .$$

(The last equality follows from Parseval identity.) Hence

$$\begin{aligned} (\phi\psi, X) &= (\psi N, X_+) = (\psi N, X) = (\psi, XN) \\ &= (\psi_+, XN) = (\psi, (XN)_+) = (\psi, \phi X) . \end{aligned}$$

We now show that for the operator $(\bar{\phi} + \bar{\phi})^{-1}$, the usual geometric series converges; the convergence, as one expects, is strongly and in $\bar{L}_2(f)$ sense.

7.4.9 Theorem. Let ϕ and $\bar{\phi}$ be as in definition 7.4.5. Then

(a) $\phi^n \rightarrow 0$ strongly in L_2 , as $n \rightarrow \infty$, i.e. for each $\psi \in \bar{L}_2$,

$$\lim_{n \rightarrow \infty} \|\phi^n \psi\| = 0.$$

(b) $\bar{\theta}^n \rightarrow 0$ strongly in $\bar{L}_2$, i.e. for each $\psi \in \bar{L}_2$, $i \in \mathbb{Z}_+$,

$$\lim_{n \rightarrow \infty} \|\bar{\theta}^n \psi^i\| = 0.$$

(c) If ψ is in the range of $\bar{\mathcal{J}} + \bar{\theta}$ then

$$(\bar{\mathcal{J}} + \bar{\theta})^{-1}(\psi) = \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \bar{\theta}^k(\psi), \text{ in } \bar{L}_2.$$

Proof. Let $\psi \in L_2$. Then using theorem 7.4.8 (a) and an argument similar to the one used by Masani in theorem 4.8 of [14], one can show that $\|\bar{\theta}^n \psi\| \rightarrow 0$, as $n \rightarrow \infty$. This completes (a). Now (b) and (c) immediately follow from (a).

We know that the range of $\bar{\mathcal{J}} + \bar{\theta}$ is a subset of L_2^+ , containing I . Let us write

$$\psi = (\bar{\mathcal{J}} + \bar{\theta})^{-1}(I) = I - N_+ + (N_+ N)_+ + \dots \in \bar{L}_2.$$

The function ψ is thus available from the spectral density by an iterative method. We shall now show that the generating function Φ of our SSP and its predictor error matrix G are easily obtainable from ψ .

7.4.10 Theorem. Let f , the spectral density of our SSP, satisfy the assumption 7.4.4. Then (a) $\psi = \sqrt{G} \Phi^{-1}$, (b) $\psi f \psi^* = G$ a.e.

Proof. (a) Since by theorem 7.4.9 (c) and lemma 7.4.7 we have

$$(\bar{\mathcal{J}} + \bar{\theta})(\psi) = I = (\bar{\mathcal{J}} + \bar{\theta})(\sqrt{G} \Phi^{-1}).$$

On the other hand, by theorem 7.4.8 (b), $\bar{\mathcal{J}} + \bar{\theta}$ is one-to-one.

Hence $\psi = \sqrt{G} \Phi^{-1}$ which gives (a).

(b) By (a) we get $\sqrt{G} = \psi \Phi$ and hence we get

$$G = \sqrt{G} \sqrt{G} = (\psi \Phi)(\psi \Phi)^* = \psi \Phi \Phi^* \psi^* = \psi f \psi^*.$$

Since Φ and $\Phi^{-1} \in \overline{L}_2^{0+}$ it follows that Ψ and $\Psi^{-1} \in \overline{L}_2^{0+}$.

Let $\Psi(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{ik\theta}$, $\Psi^{-1}(e^{i\theta}) = \sum_{k=0}^{\infty} B_k e^{ik\theta}$.

From the series $\Psi = I - N_+ + (N_+ N)_+ - \dots$ we see that

$A_0 = I$, and for each $n > 0$

$$A_n = -\Gamma'_n + \sum_{n=1}^{\infty} \Gamma'_n \Gamma'_{m-n} - \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \Gamma'_p \Gamma'_{n-p} \Gamma'_{m-n} + \dots \quad (1)$$

where Γ'_k is the k -th Fourier coefficient of N . Thus the coefficient A_k is determinable. The coefficient B_k can be found from the relations

$$A_0 B_0 = I = B_0 A_0$$

$$A_0 B_1 + A_1 B_0 = 0 = B_0 A_1 + B_1 A_0 \quad (2)$$

$$\dots = \dots$$

Since $A_0 = I$ matrix inversion will not be encountered in finding B_k . Now for C_k and D_k , the k -th Fourier coefficients of Φ and Φ^{-1} respectively, we have $A_k = \sqrt{G} D_k$, $B_k = C_k \sqrt{G}^{-1}$. But G can be evaluated from $G = \Psi \Psi^*$, so we can get $\Phi = \sum_{k=0}^{\infty} C_k e^{ik\theta}$.

The last thing we are going to do in this section is, given the SSP ξ_n , $-\infty < n < \infty$, with spectral density f , to find a scheme for computing $\hat{\xi}_v$, the linear predictor of $\log v$.

Let ξ_n , $-\infty < n < \infty$ be a SSP with spectral density f .

Let M be a constant and $m(e^{i\theta})$ be a scalar valued function with $1/m(e^{i\theta})$ being summable. Suppose $m(e^{i\theta})I \leq f(e^{i\theta}) \leq MI$.

It is clear that under this condition, f satisfies assumption 7.4.1, and hence one is able to find C_k and D_k , the k -th Fourier coefficients of Φ and Φ^{-1} respectively. So one can compute

E_{vk} , the k -th Fourier coefficients of $[e^{-iv\theta} \Phi(e^{i\theta})]_{0+}^{-1}$, in fact $E_{vk} = \sum_{n=0}^{\infty} C_{v+n} D_{k-n}$. Also in this case, we have $[e^{-iv\theta} \Phi(e^{i\theta})]_{0+}^{-1} \in \bar{L}_2^{0+}$. Hence $\sum_{k=0}^n E_{vk} e^{ik\theta} \rightarrow [e^{-iv\theta} \Phi(e^{i\theta})]_{0+}^{-1}$, as $n \rightarrow \infty$, in $\bar{L}_2$ sense. From $f(e^{i\theta}) \leq MI$ it follows that, as $n \rightarrow \infty$,

$$\sum_{k=0}^n E_{vk} e^{ik\theta} \rightarrow [e^{-iv\theta} \Phi(e^{i\theta})]_{0+}^{-1}(e^{i\theta}), \text{ in } \bar{L}_2(f).$$

Applying the isomorphism $\bar{S}$ we see that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n E_{vk} \xi_{-k} = \hat{\xi}_v, \text{ in } \bar{H}(\infty) \text{ sense.}$$

So we have a scheme for finding $\hat{\xi}_v$.

Meanwhile we proved the following theorem.

7.4.11 Theorem. Let ξ_n , $-\infty < n < \infty$ be a SSP with spectral density f satisfying $m(e^{i\theta})I \leq f(e^{i\theta}) \leq MI$, where M is a constant and $m(e^{i\theta})$ is a scalar valued function with a summable reciprocal. Then with the above notation we have

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n E_{vk} e^{ik\theta} = [e^{-iv\theta} \Phi(e^{i\theta})]_{0+}^{-1}(e^{i\theta}), \text{ in } \bar{L}_2(f).$$

Hence we get $\lim_{n \rightarrow \infty} \sum_{k=0}^n E_{vk} \xi_{-k} = \hat{\xi}_v$, in $\bar{H}(\infty)$.

CHAPTER VIII

MINIMALITY AND INTERPOLATION OF INFINITE DIMENSIONAL STATIONARY STOCHASTIC PROCESSES

8.1 Introduction. In this chapter we investigate the problems of interpolation and minimality for infinite dimensional stationary stochastic processes. We will continue with the notation of the last chapter. We will first extend to the infinite dimensional case, most of the results of H. Salehi [21], [22], [23]. Using his technique in [23] we prove two infinite dimensional extensions for a result due to P. Masani [14] on the minimal full rank stationary stochastic processes (c.f. theorem 7.3.2).

8.2 Minimality and interpolation. We assume that our stationary stochastic process satisfies the following assumption.

8.2.1 Assumption. Our SSP has a spectral density f , with $0 < m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ a.e.

8.2.2 Definition. Let M and N be weakly measurable $1 \times \infty$ matrix valued functions. We say $\langle M, N \rangle$ is Hellinger integrable with respect to f if

$$\langle M, N \rangle_f = \frac{1}{2\pi} \int_0^{2\pi} M(e^{i\theta}) f^{-1}(e^{i\theta}) N^*(e^{i\theta}) d\theta < \infty.$$

We denote by $H_2(f)$ the class of all $1 \times \infty$ matrix valued functions M , for which $\langle M, M \rangle$ is Hellinger integrable with respect to f .

The following lemma gives some properties of $H_2(f)$ functions.

8.2.3 Lemma. (a) $M \in H_2(f)$ if and only if $Mf^{-\frac{1}{2}} \in L_2$.

(b) M and N in $H_2(f)$ implies that $\langle M, N \rangle$ is Hellinger integrable with respect to f .

(c) M and N in $H_2(f)$ implies that $M + N \in H_2(f)$.

Proof. Since $\langle M, N \rangle_f = (M, N)_f$, the proof follows from the corresponding properties of $L_2(f)$.

The following lemma is needed later to establish the isomorphism between $H_2(f)$ and the space $L_2(f)$ introduced in §7.2.

8.2.4 Lemma. Let $M(e^{i\theta}) = \phi(e^{i\theta})f(e^{i\theta})$ and $N(e^{i\theta}) = \psi(e^{i\theta})f(e^{i\theta})$, where ϕ and $\psi \in L_2(f)$. Then $\langle M, N \rangle$ is Hellinger integrable with respect to f and

$$\langle M, N \rangle_f = (\phi, \psi)_f.$$

The proof is clear because

$$\begin{aligned} \langle M, N \rangle_f &= \frac{1}{2\pi} \int_0^{2\pi} M(e^{i\theta}) f^{-1}(e^{i\theta}) N^*(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\phi(e^{i\theta}) f(e^{i\theta})) f^{-1}(e^{i\theta}) (\psi(e^{i\theta}) f(e^{i\theta}))^* d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) f(e^{i\theta}) \psi^*(e^{i\theta}) d\theta = (\phi, \psi)_f. \end{aligned}$$

Let T be the linear transformation defined on $L_2(f)$ into $H_2(f)$ by $T\phi = \phi f$.

Some important properties of T are stated in the next theorem.

8.2.5 Theorem. (a) T is a linear operator on $L_2(f)$ into $H_2(f)$, i.e. for any $a, b \in \mathbb{C}$ and any $\Phi, \Psi \in L_2(f)$, we have $T(a\Phi + b\Psi) = aT\Phi + bT\Psi$.

(b) T is an isomorphism, in fact $\langle T\Phi, T\Psi \rangle_f = \langle \Phi, \Psi \rangle_f$ for all Φ and $\Psi \in L_2(f)$.

(c) T is onto. In fact, if $M \in H_2(f)$ then $T(Mf^{-1}) = M$.

Proof. (a) is obvious. (b) follows from lemma 8.2.4. To see (c) we just have to show that Mf^{-1} is in $L_2(f)$, which is the case because

$$\begin{aligned} (Mf^{-1}, Mf^{-1}) &= \frac{1}{2\pi} \int_0^{2\pi} (M(e^{i\theta})f^{-1}(e^{i\theta}))f(e^{i\theta})(M(e^{i\theta})f^{-1}(e^{i\theta}))^* d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} M(e^{i\theta})f^{-1}(e^{i\theta})M^*(e^{i\theta})d\theta < \infty. \end{aligned}$$

Now since $L_2(f)$ is a Hilbert space, the following corollary whose proof is omitted, is an immediate consequence of the last theorem.

8.2.6 Corollary. $H_2(f)$ is a Hilbert space over complex numbers.

8.2.7 Definition. We denote by $\bar{H}_2(f)$ the space of all $\infty \times \infty$ matrix valued functions, each row of which being in $H_2(f)$. For M and N in $\bar{H}_2(f)$, we define $\langle M, N \rangle_f$ by

$$(\langle M, N \rangle_f)_{i,j} = \langle M^i, N^j \rangle_f.$$

Let the transformation $\bar{T}$ on $\bar{L}_2(f)$ into $\bar{H}_2(f)$ be the inflation of T . The results of theorem 8.2.5 and the usual technique can be used to show that $\bar{T}$ is a one-to-one transformation on $\bar{L}_2(f)$ onto $\bar{H}_2(f)$ which is an isometry, in fact for all Φ and $\Psi \in \bar{L}_2(f)$,

$$\langle \bar{T}\bar{\Phi}, \bar{T}\Psi \rangle_f = (\bar{\Phi}, \Psi)_f.$$

Furthermore for any $M \in \bar{H}_2(f)$, $Mf^{-1} \in \bar{L}_2(f)$ and $\bar{T}(Mf^{-1}) = M$.

Now let us give the following definition.

8.2.8 Definition. Let ξ_n , $-\infty < n < \infty$ be a SSP. Let J be a subset of integers Z . We write $\bar{m}_J = \mathcal{G}(\xi_j, j \in J)$, $\bar{n}_J = \bar{m}_J^\perp \cap \bar{H}(\infty)$, where $J' = Z \setminus J$. We say that

- (a) J is interpolable with respect to ξ_n , $-\infty < n < \infty$ if $\bar{n}_J = \{0\}$.
- (b) ξ_n , $-\infty < n < \infty$ is interpolable if each bounded subset, J , of integers is interpolable with respect to ξ_n , $-\infty < n < \infty$.
- (c) The process ξ_n , $-\infty < n < \infty$ is minimal if for each integer j , $J = \{j\}$ is not interpolable with respect to ξ_n , $-\infty < n < \infty$.

8.2.9 Definition. (a) For each element $\xi \in \bar{n}_J$ we write

$$P_\xi(e^{i\theta}) = \sum_{j \in J} (\xi, \xi_j) e^{-ij\theta}.$$

(b) We define the operator Q on $\bar{n}_J$ into $\bar{H}_2(f)$ to be

$$Q\xi = P_\xi.$$

Part (a) of the next theorem shows that $Q\xi \in \bar{H}_2(f)$, for each $\xi \in \bar{n}_J$.

8.2.10 Theorem. (a) Let $\xi \in \bar{n}_J$ and $\Psi \in \bar{L}_2(f)$ such that $\bar{S}\Psi = \xi$. Then $P_\xi = \Psi f$.

(b) Q is an isometry on $\bar{n}_J$ into $H_2(f)$, in fact

$$(\xi, \eta)_f = \langle Q\xi, Q\eta \rangle_f,$$

where ξ and η are any two elements of $\bar{H}(\infty)$.

Proof. (a) Let $\Psi \in \bar{L}_2(f)$ and $\xi = \bar{S}\Psi$. Then

$$(\xi, \xi_k) = (\psi, e^{-i\theta k})_f = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta}) f(e^{i\theta}) \psi^*(e^{i\theta}) d\theta.$$

On the other hand we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} P_\xi(e^{i\theta}) d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \left(\sum_{j \in J} (\xi, \xi_j) e^{-ij\theta} \right) e^{ik\theta} \right\} d\theta \\ &= \sum_{j \in J} \frac{1}{2\pi} \int_0^{2\pi} (\xi, \xi_j) e^{i(k-j)\theta} d\theta = (\xi, \xi_j). \end{aligned}$$

Hence ψf and P_ξ have the same Fourier coefficients, which implies

$$P_\xi = \psi f.$$

(b) Let ξ and $\eta \in \bar{n}_J$ and $\phi, \psi \in \bar{L}_2(f)$ such that $\bar{S}\psi = \xi$ and $\bar{S}\phi = \eta$. Then we have

$$\begin{aligned} \langle Q\xi, Q\eta \rangle_f &= \langle P_\xi, P_\eta \rangle_f = \langle T\psi, T\phi \rangle_f \\ &= (\psi, \phi)_f = (\xi, \eta). \end{aligned}$$

The following theorem gives a characterization for interpolability of an infinite dimensional SSP.

8.2.11 Theorem. Let ξ_n , $-\infty < n < \infty$ be a SSP. Then it is interpolable if and only if for any trigonometric polynomial P with matrix coefficients, either P is zero in $H_2(f)$ or $P \notin \bar{H}_2(f)$.

Proof. Sufficiency. Suppose $\bar{n}_J \neq \{0\}$ for some bounded subset J of Z . Then there exists $0 \neq \xi \in \bar{n}_J$. Hence we get $0 \neq (\xi, \xi) = \langle P_\xi, P_\xi \rangle_f$. Hence P_ξ is a non zero trigonometric polynomial in $\bar{H}_2(f)$.

Necessity. Suppose there exists a non zero trigonometric polynomial in $\bar{H}_2(f)$. Then $\phi = Pf^{-1} \in \bar{L}_2(f)$. Hence there exists

$\xi, 0 \neq \xi \in \bar{H}(\infty)$ such that $\bar{S}\xi = \xi$. We have

$$\begin{aligned} (\xi, \xi_k) &= \frac{1}{2\pi} \int_0^{2\pi} \bar{\xi}(e^{i\theta}) f(e^{i\theta}) e^{ik\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} P(e^{i\theta}) f^{-1}(e^{i\theta}) f(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} P(e^{i\theta}) d\theta = \sum_{j \in J} \frac{1}{2\pi} \int_0^{2\pi} A_{-j} e^{i(k-j)\theta} d\theta. \end{aligned}$$

Hence

$$(\xi, \xi_k) = \begin{cases} A_{-k} & k \in J \\ 0 & k \notin J \end{cases},$$

where $P(e^{i\theta}) = \sum_{j \in J} A_{-j} e^{-ij\theta}$. So we see that $(\xi, \xi_k) = 0$ if $j \in J$ and hence $\xi \in \bar{m}_J^\perp$. But obviously $\xi \in \bar{H}(\infty)$. Therefore $\xi \in \bar{n}_J$ and furthermore

$$P_\xi = \sum_{j \in J} (\xi, \xi_j) e^{-ij\theta} = \sum_{j \in J} A_{-j} e^{-ij\theta} = P.$$

Hence $P = P_\xi$. Now since $0 \neq (\xi, \xi) = (T\xi, T\xi)_f = \langle P_\xi, P_\xi \rangle_f = \langle P, P \rangle_f$. Therefore $\bar{n}_J \neq \{0\}$. Hence J is not interpolable with respect to $\xi_n, -\infty < n < \infty$. Thus $\xi_n, -\infty < n < \infty$ is not interpolable.

In the next two theorems we give generalizations of theorem 7.3.2.

8.2.12 Theorem. Let $\xi_n, -\infty < n < \infty$ be a SSP whose density, f satisfies $m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ with $M(e^{i\theta})$ and $1/m(e^{i\theta})$ being summable. Then the process $\xi_n, -\infty < n < \infty$ is minimal full rank and we have

$$\Sigma = \left[\frac{1}{2\pi} \int_0^{2\pi} f^{-1}(e^{i\theta}) d\theta \right]^{-1}.$$

Proof. By lemma 7.3.4, $\Sigma > \lambda I$ for some positive number $\lambda > 0$.

Hence the process is minimal full rank. By theorem 8.2.10 (b)

we have

$$\Sigma = (\zeta_0, \zeta_0) = \langle Q\zeta_0, Q\zeta_0 \rangle_f .$$

But $Q\zeta_0 = P\zeta_0 = (\zeta_0, \zeta_0) = \Sigma$. So we get

$$\Sigma = \langle \Sigma, \Sigma \rangle_f = \frac{1}{2\pi} \int_0^{2\pi} \Sigma f^{-1}(e^{i\theta}) \Sigma d\theta .$$

Now since $1/m(e^{i\theta})$ is summable one can see that

$$\Sigma = \Sigma \left[\frac{1}{2\pi} \int_0^{2\pi} f^{-1}(e^{i\theta}) d\theta \right] \Sigma .$$

Now because $\Sigma \geq \lambda I$ we get

$$\Sigma = \left[\frac{1}{2\pi} \int_0^{2\pi} f^{-1}(e^{i\theta}) d\theta \right]^{-1} .$$

Theorem 8.2.11 and lemma 7.3.4 give sufficient conditions for minimal full rank processes. The next theorem provides a necessary and sufficient condition for a process to be minimal full rank. The next theorem also gives a natural extension of theorem 7.3.2.

8.2.13 Theorem. Let $\xi_n, -\infty < n < \infty$ be a SSP with spectral density f satisfying $0 < m(e^{i\theta})I \leq f(e^{i\theta}) \leq M(e^{i\theta})I$ a.e., where $M(e^{i\theta})$ is a summable scalar valued function. Let f_L be the top left $L \times L$ submatrix of f . Then

(a) the process $\xi_n, -\infty < n < \infty$ is minimal full rank if and only if there exists a constant μ such that

$$\int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta \leq \mu I_L ,$$

uniformly for all L , $1 \leq L < \infty$.

(b) We have

$$\Sigma = \inf_{1 \leq L < \infty} \left[\frac{1}{2\pi} \int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta \right]^{-1}.$$

Proof. (a) The process ξ_n , $-\infty < n < \infty$ is minimal full rank if and only if $\Sigma \geq \lambda I$ for some $\lambda > 0$. By lemma 7.3.1 we see that

$\Sigma \geq \lambda I$ if and only if $\Sigma_L \geq \lambda I_L$ uniformly in L , $1 \leq L < \infty$.

Hence the process ξ_n , $-\infty < n < \infty$ is minimal if and only if

$\Sigma_L \geq \lambda I_L$ uniformly in L , $1 \leq L < \infty$. But by lemma 7.3.2 we know that $\Sigma_L^{-1} = \frac{1}{2\pi} \int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta$. So ξ_n , $-\infty < n < \infty$ is minimal full rank if and only if $\int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta < \mu I_L$ uniformly in L , $1 \leq L < \infty$.

(b) We know that $\Sigma_L \rightarrow \Sigma$ strongly, as $n \rightarrow \infty$. Now taking limit on both sides of

$$\Sigma_L = \left[\frac{1}{2\pi} \int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta \right]^{-1}$$

we get

$$\Sigma = \inf_{1 \leq L < \infty} \left[\frac{1}{2\pi} \int_0^{2\pi} (f_L(e^{i\theta}))^{-1} d\theta \right]^{-1}.$$

APPENDIX

APPENDIX

Here we consider an example of a weakly summable $B(\chi, \chi^*)$ valued function $f = f(e^{i\theta})$ on the unit circle. Using the technique of lemma 4.3.1 we construct a quasi square root of f . We adhere to the notations of lemma 4.3.1.

Let $\chi = \ell^1$, $\chi^* = \ell^2$. Hence $\chi^* = \ell^\infty$. Take the partition $E_n = [\frac{(2^n - 2)\pi}{2^{n-1}}, \frac{(2^n - 1)\pi}{2^{n-1}})$, $1 \leq n < \infty$ of $[0, 2\pi)$, for each $x = (x_1, x_2, \dots) \in \ell^1$ we define

$$f(e^{i\theta})x = (x_1, x_2, \dots, x_n, 0, 0, \dots) \quad \text{if } \theta \in E_n.$$

Since $(f(e^{i\theta})x)(x) = \sum_{j=1}^n |x_j|^2$ for $\theta \in E_n$ we see that f is a nonnegative operator valued function. It is clear that $f = f(e^{i\theta})$ is measurable and $|f(e^{i\theta})|_B \leq 1$ for each θ . Now we give the construction of a quasi square root of f as outlined in lemma 4.3.1. We first obtain the operator T . Since

$$\begin{aligned} (Tx, Tx) &= (T^*Tx)(x) = \frac{1}{2\pi} \int_0^{2\pi} (f(e^{i\theta})x)(x) d\theta \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{E_n} |x_j|^2 d\theta = \frac{1}{2\pi} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{2\pi}{2^n} |x_j|^2 \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^n} |x_j|^2 = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{1}{2^n} |x_j|^2 \\ &= \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} |x_j|^2 \end{aligned}$$

one can see that $T: \ell^1 \rightarrow \ell^2$ can be taken as

$$Tx = (x_1, x_2/\sqrt{2}, x_3/\sqrt{2^2}, \dots).$$

Hence the function g in the proof of lemma 4.3.1 is given by

$$g(e^{i\theta})(a, b) = \sum_{j=1}^n 2^{j-1} a_j \bar{b}_j, \quad \theta \in E_n;$$

$$a = (a_1, a_2, a_3, \dots) \quad \text{and} \quad b = (b_1, b_2, b_3, \dots) \quad \text{in} \quad \ell^2.$$

Consider the countable set $\{x_i\}_{i=1}^\infty$ in the proof of lemma 4.3.1 to be given by $x_i = (\delta_{ji})_{j=1}^\infty$. Then $Tx_i = (\delta_{ji}/\sqrt{2^{j-1}})_{j=1}^\infty$. Now the Gram-Schmidt orthogonalization of $\{Tx_i\}_{i=1}^\infty$ becomes $\{e_i\}_{i=1}^\infty$, where $e_i = (\delta_{ji})_{j=1}^\infty$. Hence the matrix valued function $[g_{ij}]_{i,j=1}^\infty$ is given by

$$g_{i,j}(e^{i\theta}) = g(e^{i\theta})(e_i, e_j) = \begin{cases} 2^{i-1} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Now it is easy to see that the functions $F_j \in L_2(X)$ of the proof of lemma 4.3.1, can be taken to be the constant functions

$$F_j(e^{i\theta}) = (\sqrt{2^{j-1}} \delta_{ji})_{i=1}^\infty. \quad \text{The operator } A \text{ is given by}$$

$$A(e^{i\theta})(a_1, a_2, a_3, \dots) = (a_1, a_2\sqrt{2}, a_3\sqrt{2^2}, \dots, a_n\sqrt{2^{n-1}}, 0, 0, \dots) \quad \theta \in E_n.$$

Now for each $\theta \in E_n$, we have

$$\begin{aligned} B(e^{i\theta})x &= (ATx)(e^{i\theta}) = A(x_1, x_2/\sqrt{2}, x_3/\sqrt{2^2}, \dots) \\ &= (x_1, x_2, \dots, x_n, 0, 0, \dots). \end{aligned}$$

Hence we have

$$Q(e^{i\theta})x = (x_1, x_2, \dots, x_n, 0, 0, \dots), \quad \theta \in E_n.$$

Obviously $Q(e^{i\theta})$ is measurable and

$$(f(e^{i\theta})x)(y) = (Q(e^{i\theta})x, Q(e^{i\theta})y), \quad x, y \in \mathcal{X}.$$

Clearly $Q(e^{i\theta})$ is bounded. So we have obtained an explicit form for this quasi square root.

In this example since f is countably valued function, following [1] one could factor each value of f separately to determine Q , whose measurability is automatic. However in general when f is not countably valued this procedure may not yield a measurable quasi square root.

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