28237???

This is to certify that the

thesis entitled

TOPICS IN INTERACTING CONTINUA

presented by

Mohammad Usman

has been accepted towards fulfillment
of the requirements for

M.S. degree in Mechanical Engineering

M.V. Gandhi
Major professor

Date 8/14/89

O-7639

TOPICS IN INTERACTING CONTINUA

by

Mohammad Usman

A THESIS

Submitted to
Michigan State University, East Lansing, MI
in partial fulfillment of the requirements for
the degree of

MASTER OF SCIENCE

in

The Department of Mechanical Engineering

1989

ABSTRACT

TOPICS IN INTERACTING CONTINUA

BY

MOHAMMAD USMAN

This dissertation is focused on the mathematical modeling of the mechanical response of solid-fluid mixtures within the context of the Theory of Interacting Continua. In this work, the fundamental mathematical framework for the Theory of Interacting Continua is presented, and the theory is employed to model the solid-fluid interaction in a mixture undergoing large deformations. Furthermore, the advantages of employing this theoretical approach have been demonstrated by presenting two boundary-value problems for this class of mixtures. The first boundary value problem presented herein is intrinsically interesting since it is one of the few problems involving large non-homogeneous deformations for which experimental results are available. The second problem presented in this work is more of theoretical interest which demonstrates that an infinite mixture slab of finite thickness, undergoing uniaxial extension, admits infinite solutions.

This work has addressed the issue of investigating the complete mechanical response of the solid-fluid mixtures and has served as a test-bed for evaluating the validity and predictive capability of the Theory of Interacting Continua in modeling the interaction of the elastic solids and ideal fluids.

To

My parents

whose patience and encouragement

made this work possible

ACKNOWLEDGEMENT

I express my sincerest gratitude to Prof. M. V. Gandhi for his continuous guidance and encouragement over the past three years. Prof. M. V. Gandhi has been a source of profound inspiration, and working with him has been an intellectually rewarding experience. I am thankful to Prof. M. V. Gandhi for introducing me to the research field of the Theory of Interacting Continua in which I enjoyed my research and learning growth.

I am also grateful to other members of my dissertation committee, Professor A. G. Haddow and Prof C. Y. Wang, for their comments and suggestions in the preparation of this document. Special thanks to my friend Mr. Devang J. Desai for his helpful discussions during the completion of this work and proof reading of this manuscript. Finally, I thank my wife; Safia and children; Assad and Omar for their love and cooperation during the pursuit of my master degree program.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

A	Helmholtz free energy function of the mixture per unit mass of the mixture.
A_1, A_2	Partial derivatives of Helmholtz free energy function with respect to invariants I_1 and I_2, respectively.
A_{S1}, A_{S2}	Helmholtz free energy function of the solid and fluid per unit mass of the solid and fluid, respectively.
A_e, A_m	Helmholtz free energy function of the elastic deformation and mixing per unit mass of the mixture, respectively.
b_i	Components of the interaction body force.
B_{ij}	Components of the Cauchy-Green deformation tensor.
c_1, c_2	Coefficients appearing in the dynamical part of the constitutive equations.
d_{ij}, f_{ij}	Components of the rate of the deformation tensor for the solid and fluid, respectively.
F_{ij}	Components of the deformation gradient tensor.
$f(Z)$	Stretch ratio as a function of Z coordinate.
f_i, g_i	Components of the acceleration vector for the solid and fluid, respectively.
h	Thickness of the mixture slab.
$\underset{\sim}{I}$	Identity tensor.
I_1, I_2, I_3	Invariants of the Cauchy-Green deformation tensor $\underline{B}$.

L_{ij}, M_{ij} Components of the velocity gradient tensor for the solid and fluid, respectively.

$\underset{\sim}{n}$ Unit outer normal vector.

p Scalar appearing in the constitutive equations due to the incompressibility constraint.

q Heat flux vector.

R, r Components of the reference and current radial coordinates, respectively.

S_1, S_2 Labels denoting a solid and a fluid particle, respectively.

S_{ij} Components of the total stress characterizing the state of the mixture in a saturated state ($i,j=1,2,3$ or r,θ,z).

t Time variable.

t_i Total surface traction vector.

T Absolute temperature of the mixture continua.

T_{ij} Components of the total stress tensor for the mixture.

u_i, v_i Components of the velocity vector for the solid and fluid.

w_i Components of the mean velocity vector for the mixture.

$\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ Functions defining the current configuration of the solid and fluid, respectively.

X_i, Y_i Reference position of the solid and fluid particle, respectively.

x_i, y_i Current position of a solid and fluid particle, respectively.

X,Z, x,z Components of the reference and current coordinates in the cartisian coordinate system, respectively.

ψ	Rotation of the cylindrical mixture per unit current length.
λ	Stretch ratio along the length of the cylindrical mixture.
$\lambda(Z)$	Stretch ratio in the thickness direction of the mixture slab.
λ_r, λ_θ	Radial and circumferential stretch ratios.
η	Entropy of the system.
ν_1	Volume fraction of the solid in the mixture.
ρ	Density of the mixture.
ρ_{10}, ρ_{20}	Density of the pure solid and fluid, respectively.
ρ_1, ρ_2	Mass per unit volume of the mixture for the solid and fluid, respectively.
Γ_{ij}, Λ_{ij}	Components of the vorticity tensor for the solid and fluid, respectively.
γ_i, μ_i	Coefficients appearing in the dynamical part of the constitutive equations ($i=1,2,3,4$).
Θ, θ	Components of the reference and current coordinate in the radial coordinate system.
σ_{ij}, π_{ij}	Components of the partial stress tensor for the solid and fluid, respectively ($i,j=1,2,3$ or r,θ,z).
σ_i, π_i	Components of the partial surface traction for the solid and fluid, respectively.
χ	A constant which depends on the particular combination of the solid and the fluid.

CHAPTER I

INTRODUCTION

This dissertation is focused on the mathematical modeling of the mechanical response of solid-fluid mixtures within the context of the Theory of Interacting Continua [1-4]. The study of the mechanical response of the solid-fluid mixtures has relevance to several technical problems: The problem of atherogenesis in biomechanics [5] which involves the diffusion of plasma lipoproteins through the arterial wall, processes employing cylindrical tubes for the separation of fluids, the problems of filtration and ultra-filtration [6], non-linear diffusion in soft tissues and its deformation [7] where the non-linearity in the problem arises from the permeability of elastic phase which for a number of tissues (such as articular cartilage) is strongly dependent on strain, the problem of hygro-thermo-elastic response of polymeric composite materials undergoing large deformations are but a few examples.

In the Theory of Interacting Continua the constituents of the mixture are assumed to be mixed on a molecular level and are chemically neutral. The theory postulates local balance laws for each individual constituent of the mixture, namely, balance of momentum, balance moment of momentum, balance of energy and balance of mass, and an entropy production inequality for each individual constituent of the mixture. Moreover, in this approach, thermo-dynamical variables such as the internal energy, entropy and temperature are admitted for each constituent and these ,in turn, are assumed to be related to the corresponding quantities for the whole mixture simply by algebraic relations.

The study of the mechanical response of the mixture of a non-linearly elastic solid and an ideal fluid has been of particular interest to several researchers [8-9]. The application of the theory to study problems involving large deformation, swelling and diffusion of fluid through non-linearly elastic solid has been very limited due to the lack of physically obvious ways for specifying the partial tractions, which are integral part of the theory. The purpose of this work is to investigate the mechanical response of the mixture of a non-linearly elastic solid and an ideal fluid undergoing large deformations and resolve some problems associated with the applicability of the theory. This investigation, in turn, will serve as a test-bed for evaluating the validity and predictive capability of the Theory of Interacting Continua in modeling the interaction of elastic solids and ideal fluids.

The application of the Theory of Interacting Continua to model the phenomenological behavior of the solid-fluid mixtures has been historically motivated due to limitations in the classical theories. Classical approaches which have been used to study the diffusion of fluid through solids, such as Fick's Law [10] discussed in books on diffusion in solids [11-13] and Darcy's Law discussed by Scheidegger

[14] ,for example, assume that the solid is rigid. However, this assumption is violated in solid-fluid interactions where the mixture undergoes large deformations [15-16]. Furthermore, the dependence of swelling on strain and kinematics constraints has been demonstrated by Treloar [17], and Paul and Ebra-Lima [18]. Classical theories do not adequately account for the interaction between a highly deformable solid and a fluid in a diffusion process. The limitations of the classical theories point out the need to use an appropriate theory which is capable of realistically taking into account the interaction of solid and fluid.

The Theory of Interacting Continua models the mixture as a superimposition of individual continua. Each spatial point in the mixture is assumed to be simultaneously occupied by material particles from each constituent. This essentially amounts to taking into account contributions from each constituent in a neighborhood of the point and averaging them. The theory accounts for large deformations, dependence of material properties on both constituents and interactive forces. Atkin and Craine [19] reviewed the applications of this theory through 1976. The discussion includes the work of Crochet and Naghdi [20], Mills and Steel [21], and Craine, at el. [22]. A critical review of the field makes it clearly evident that the applications of the Theory of Interacting Continua to solve boundary-value problems of physical interest have been very limited. The main difficulty in these problems arises due to the lack of physically obvious ways for specifying the partial tractions associated with the solid and fluid. At the boundary, only the total stresses can be specified. Shi, et al. [23] and Rajagopal, et al. [24] were the first to study equilibrium and steady state boundary-value problems by employing auxiliary conditions at the boundary of the solid-fluid mixtures in an effort to bypass the difficulties associated with specifying partial tractions at the boundary. The use of these auxiliary conditions rendered a whole class of boundary-value problems tractable where the boundary of the mixture could be assumed to be saturated. However, these boundary conditions were scalar in nature and derived on an ad hoc basis, which was not necessarily thermodynamically consistent.

By employing a rigorous thermodynamic criterion for closed system, Rajagopal, et al. [25] provided a systematic rationale for characterizing saturated states of homogeneously deformed and swollen cuboids. In particular, they obtained tensorial equations relating the total stresses with the stretch ratios and the volume fraction of the solid in the saturated mixture. These equations could then be used to describe additional boundary conditions assuming material elements at the boundary of the mixture continuum to be in a saturated state, and thereby bypass the difficulty associated with prescribing the partial traction conditions at the boundary.

Since 1976 considerable work by Shi et al. [23] and Rajagopal et al. [24,25] and Gandhi et al. [26] has been done in solving boundary-value problems involving large non-homogeneous deformations of solid-fluid mixtures. In this work the same approach [24-28] has been used in solving boundary-value problems involving solid-fluid interaction. Two representative boundary-value problems are presented in the context of the Theory of Interacting Continua. The first boundary-value problem presented herein is intrinsically interesting since it is one of the few problems involving large nonhomogeneous deformations for which experimental results are available. The second problem presented in this work is more of theoretical interest which demonstrates that an infinite mixture slab of finite thickness, undergoing uniaxial extension, admits infinite solutions.

The first boundary-value problem presented herein is motivated by the experimental work of Loke and Treloar [8] and theoretical work of Treloar [29] for the combined finite extension and torsion of a cylindrical mixture of a non-linearly elastic solid and an ideal fluid which is swollen to several times the volume of the original rubber cylinder. In their work, the cylinder was assumed to be saturated with the fluid. In addition, the problem was not treated within the context of Theory of Interacting Continua. The present work differs from Treloar's in two respects. First, the problem is studied within the context of Theory of Interacting Continua. Second, there is no restriction on the fluid content of the mixture, the strained state of the cylinder could range from being completely dry to fully saturated. In the problem considered here, both the solid and fluid constituents are at rest. However, the fluid can be non-homogeneously dispersed throughout the mixture domain, which gives rise to concentration gradients. The physical mechanism for the existence of such gradients is provided by the presence of an interaction body force which each constituent exerts on each other. This work is the first one of its kind to exploit Theory of Interacting Continua in order to adequately reproduce experimental results of a highly swollen mixture subjected to a complex non-homogeneous deformations.

The second problem has been motivated by the work of Rajagopal and Wineman [30]. In their work, they presented exact solutions for the problem of uniaxial extension and demonstrated that an axial variation of the stretch ratio is possible for non-linearly elastic materials. In addition, they obtained an infinite class of exact solutions for the uniaxial extension of an infinite Neo-Hookean slab of finite thickness. In this work, the boundary-value problem is studied for an infinite mixture slab of finite thickness in the context of the Theory of Interacting Continua. The applicability of the theory is demonstrated by presenting the response of a mixture slab undergoing large non-homogeneous deformations. It has been shown that infinite exact solutions are possible for the uniaxial extension of "Neo-Hookean type" mixture slab, and the possibility of axial variation of the stretch ratio is also demonstrated. This is the first such example of non-uniqueness of equilibrium deformation states presented within the context of Theory of Interacting Continua.

A brief review of the notation and basic equations relevant to a mixture of interacting continua is presented in chapter II. The constitutive equations for the mixture of a non-linearly elastic solid and an ideal fluid are discussed in chapter III. The application of the Theory of Interacting Continua is demonstrated by presenting two boundary-value problems in chapter IV.

CHAPTER II

REVIEW OF THE GENERAL THEORY OF INTERACTING CONTINUA

PRELIMINARIES: NOTATIONS AND BASIC EQUATIONS

A brief review of the notations and basic equations of the Theory of Interacting Continua is presented in this section for completeness and continuity. The historical development and a detailed exposition of the theory are succinctly presented in the comprehensive review articles by Atkin and Craine [1] and Bowen [2].

Let Ω and Ω_t denote the reference configuration and the configuration of the body at time t, respectively. For a function defined on $\Omega \times \mathbf{R}$ and $\Omega_t \times \mathbf{R}$, ∇ and grad are used to represent the partial derivative with respect to Ω and Ω_t, respectively. Also $\frac{\partial}{\partial t}$ denotes the partial derivative with respect to t. The divergence operator related to grad is denoted by div.

The solid-fluid aggregate will be considered a mixture with S_1 representing the solid and S_2 representing the fluid. At any instant of time t, it is assumed that each place in the space is occupied by particles belonging to both S_1 and S_2. Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ denote the reference positions of typical particles of S_1 and S_2. The motion of the solid and the fluid is represented by

$$\underset{\sim}{x} = \underset{\sim}{x}_1\,(\underset{\sim}{X},\ t), \quad \text{and } \underset{\sim}{y} = \underset{\sim}{x}_2\,(\underset{\sim}{Y},\ t). \tag{2.1}$$

Where the subscript ~ denotes a quantity in an orthogonal coordinate system.

These motions are assumed to be one-to-one, continuous and invertible. The various kinematical quantities associated with the solid S_1 and the fluid S_2 are

Velocity: $$\underset{\sim}{u} = \frac{D^{(1)}\underset{\sim}{x}_1}{Dt}, \qquad \underset{\sim}{v} = \frac{D^{(2)}\underset{\sim}{x}_2}{Dt}, \tag{2.2}$$

Acceleration: $$\underset{\sim}{f} = \frac{D^{(1)}\underset{\sim}{u}}{Dt}, \qquad \underset{\sim}{g} = \frac{D^{(2)}\underset{\sim}{v}}{Dt}, \tag{2.3}$$

Velocity gradient: $$\underline{L} = \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}, \qquad \underline{M} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{y}}, \text{ and} \tag{2.4}$$

Rate of deformation tensor: $$\underline{D} = \frac{1}{2}\,(\underline{L} + \underline{L}^T),\ \underline{N} = \frac{1}{2}(\underline{M} + \underline{M}^T), \tag{2.5}$$

where $D^{(1)}/Dt$ denotes differentiation with respect to t, holding $\underset{\sim}{x}$ fixed, and $D^{(2)}/Dt$ denotes a similar operation holding $\underset{\sim}{y}$ fixed and the subscript underscore (_) denotes a tensorial quantity in an orthogonal coordinate system. The deformation gradient $\underline{F}$ associated with the solid is given by

$$\underline{F} = \frac{\partial \underset{\sim}{x}_1}{\partial \underset{\sim}{X}}. \tag{2.6}$$

The total density of the mixture ρ and the mean velocity of the mixture $\underset{\sim}{w}$ are defined by

$$\rho = \rho_1 + \rho_2, \tag{2.7}$$

and

$$\rho \underset{\sim}{w} = \rho_1 \underset{\sim}{u} + \rho_2 \underset{\sim}{v}, \tag{2.8}$$

where ρ_1 and ρ_2 are the densities of the solid and the fluid in the mixed state, respectively, defined per unit volume of the mixture at time t.

The basic equations of the Theory of Interacting Continua are presented next.

(1) Conservation of mass

Assuming no interconversion of mass between the two interacting continua, the appropriate forms for the conservation of mass for the solid and the fluid are

$$\rho_1 \, |\det \underset{\sim}{F}| = \rho_{10}, \tag{2.9}$$

and

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div} (\rho_2 \, \underset{\sim}{v}) = 0, \tag{2.10}$$

where ρ_{10} is the mass density of the solid in the reference state.

(2) Conservation of linear momentum

Let $\underset{\sim}{\sigma}$ and $\underset{\sim}{\pi}$ denote the partial stress tensors associated with the solid S_1 and the fluid S_2, respectively. Then, assuming that there are no external body forces, the balance of linear momentum for the solid and fluid are given by

$$\operatorname{div} \underline{\sigma} - \underset{\sim}{b} = \rho_1 \underset{\sim}{f}, \tag{2.11}$$

and

$$\operatorname{div} \underline{\pi} + \underset{\sim}{b} = \rho_2 \underset{\sim}{g}. \tag{2.12}$$

In equations (2.11) and (2.12), $\underset{\sim}{b}$ denotes the interaction body force vector, which accounts for the mechanical interaction between the solid and the fluid.

(3) Conservation of angular momentum

This condition states that

$$\underline{\sigma} + \underline{\pi} = \underline{\sigma}^T + \underline{\pi}^T. \tag{2.13}$$

However, the partial stresses $\underline{\sigma}$ and $\underline{\pi}$ need not be symmetric.

(4) Surface tractions

Let $\underset{\sim}{\sigma}$ and $\underset{\sim}{\pi}$ denote the surface traction vectors taken by S_1 and S_2, respectively, and let $\underset{\sim}{n}$ denote the unit outer normal vector at a point on the surface of the mixture region. Then the partial surface tractions are related to the partial stress tensors by

$$\underset{\sim}{\sigma} = \underline{\sigma} \cdot \underset{\sim}{n},$$

and (2.14)

$$\underset{\sim}{\pi} = \underline{\pi} \cdot \underset{\sim}{n}.$$

(5) Thermodynamical considerations

The laws of conservation of energy and the entropy production inequality are not explicitly mentioned here for brevity. However, the relevant results are quoted. A complete discussion of these issues is presented in [31].

Let the Helmholtz free energy per unit mass of S_1 and S_2 be denoted by A_{S1} and A_{S2}, respectively. The Helmholtz free energy per unit mass of the mixture is defined by

$$\rho A = \rho_1 A_{S1} + \rho_2 A_{S2}. \tag{2.15}$$

Note that by setting

$$\underset{\sim}{b} = \text{grad}\ \phi_1 + \bar{\underset{\sim}{b}} = \text{grad}\ \phi_2 + \bar{\underset{\sim}{b}}, \tag{2.16}$$

$$\underline{\sigma} = \phi_1 \underline{I} + \bar{\underline{\sigma}}, \tag{2.17}$$

$$\underline{\pi} = \phi_2 \underline{I} + \bar{\underline{\pi}}, \tag{2.18}$$

where,

$$\phi_1 = \rho_1(A_{S1}-A),\ \phi_2 = \rho_2(A_{S2}-A),\ \text{and}$$

$$\phi_1 + \phi_2 = 0,$$

equations (2.11) - (2.13) become

$$\text{div}\ \bar{\underline{\sigma}} - \bar{\underset{\sim}{b}} = \rho_1 \underset{\sim}{f}, \tag{2.19}$$

$$\text{div}\ \bar{\underline{\pi}} + \bar{\underset{\sim}{b}} = \rho_2 \underset{\sim}{g}, \tag{2.20}$$

$$\bar{\underline{\sigma}} + \bar{\underline{\pi}} = \bar{\underline{\sigma}}^T + \bar{\underline{\pi}}^T. \tag{2.21}$$

The terms in $\underline{\sigma}$, $\underline{\pi}$ and $\underset{\sim}{b}$ which depend on ϕ_1 and ϕ_2 do not contribute to the equations of motion or the total stress.

(6) Volume additivity assumption and incompressibility constraint

Attention is restricted to a mixture of incompressible materials. It is assumed that the volume of the mixture in any deformation state and at any given time is the sum of the volumes occupied by the solid and fluid constituents at that state and time. This implies that the motion of the interacting continua at that time is such that it satisfies the following relationship [32]:

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1, \tag{2.22}$$

where ρ_{20} is the mass density of the fluid in the reference state. It may be emphasized that this assumption has a significant bearing on the form of the constitutive equations, and renders the constitutive equations to be tractable due to the elimination of the density of one of the constituents as an independent variable by virtue of equation (2.22).

CHAPTER III

CONSTITUTIVE EQUATIONS

1. CONSTITUTIVE ASSUMPTIONS

A mixture of an elastic solid and a fluid is considered. The solid is assumed to be non-linearly elastic, and the fluid is assumed to be ideal. Thus all the constitutive functions A, η, $\underset{\sim}{b}$, $\underline{\sigma}$, $\underline{\pi}$, ϕ_1, ϕ_2 and $\underset{\sim}{q}$ are required to depend on the following variables:

$$\underline{F},\ \nabla\underline{F},\ \rho_2,\ \text{grad}\ \rho_2,\ T,\ \text{grad}\ T,\ \underset{\sim}{u}\ \text{and}\ \underset{\sim}{v},$$

where A is the Helmholtz free energy for the mixture defined per unit mass of the mixture, η denotes the entropy of the system, $\underset{\sim}{q}$ represents the heat flux vector, T denotes the common absolute temperature of the solid and the fluid and rest of the variables are defined in previous part of the text.

Following Crochet and Naghdi [8] and Shi, et al. [23] , The partial stress tensors and diffusive body force vector for the solid and fluid constituents may be written as the sum of static and dynamic part as follows

$$\underline{\bar{\sigma}} = \underline{\bar{\sigma}}^{s} + \underline{\bar{\sigma}}^{d} , \qquad (3.1)$$

$$\bar{\underline{\pi}} = \bar{\underline{\pi}}^{s} + \bar{\underline{\pi}}^{d} \text{ , and} \tag{3.2}$$

$$\bar{\underline{b}} = \bar{\underline{b}}^{s} + \bar{\underline{b}}^{d} \text{ ,} \tag{3.3}$$

where superscript s denote the static part and superscript d denote the dynamical part of the constitutive equations and $\bar{\underline{\sigma}}^{s}$ $\bar{\underline{\pi}}^{s}$, $\bar{\underline{b}}^{s}$ depend upon statical variables and $\bar{\underline{\sigma}}^{d}$, $\bar{\underline{\pi}}^{d}$, $\bar{\underline{b}}^{d}$ together with A, η, $\underline{\phi}$ and the heat flux vector depend on all variables. The energy balance law and the application of the Clausius-Duhem inequality yield following constitutive relations

$$\eta = - \frac{\partial A}{\partial T} \text{ ,} \tag{3.4}$$

$$\bar{\sigma}^{s}_{ki} = \rho \frac{\partial A}{\partial F_{ij}} F_{kj} - p \frac{\rho_1}{\rho_{10}} \delta_{ki} \text{,} \tag{3.5}$$

$$\bar{\pi}^{s}_{ki} = - \rho \rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ki} - p \frac{\rho_2}{\rho_{20}} \delta_{ki} \text{, and} \tag{3.6}$$

$$\bar{b}^{s}_{k} = - \rho_2 \frac{\partial F_{ij}}{\partial x_k} \frac{\partial A}{\partial F_{ij}} + \rho_1 \frac{\partial A}{\partial \rho_2} \frac{\partial \rho_2}{\partial x_k} - \frac{p}{\rho_{10}} \frac{\partial \rho_1}{\partial x_k} \text{ ,} \tag{3.7}$$

where the Helmholtz free energy function A is assumed to depend on $\underline{F}$, ρ_2 and T. In equations (3.5)-(3.7), p is an indeterminate scalar arising from the use of volume additivity assumption/incompressibility constraint equation (2.22). The dynamical part of the partial stress tensors and diffusive body force vector satisfy the reduced entropy inequality

$$\bar{\sigma}^{d}_{(ki)}\, d_{ik} + \bar{\pi}^{d}_{(ki)}\, f_{ik} + \bar{\sigma}^{d}_{[ki]}\, (\Gamma_{ik} - \Lambda_{ik}) + \bar{b}^{d}_{k}\, (u_k - v_k)$$

$$- \frac{1}{T}\, [q_k + T(\, \rho_1\eta_1(\, u_k - w_k\,) + \rho_2\eta_2(\, v_k - w_k\,))]\, \frac{\partial T}{\partial x_k} \geq 0, \qquad (3.8)$$

where () or [] around the subscripts denote the symmetric and skew symmetric parts of the tensors, respectively.

Following the arguments based on the restrictions due to the principle of material objectivity, as presented by Crochet and Naghdi in [9] it may be concluded that the constitutive functions may depend upon the velocities of the constituents only through the relative velocity $u_i - v_i$, upon the velocity gradient only through rate of deformation tensors f_{ij} and d_{ij} and the relative vorticity tensor $\Gamma_{ij} - \Lambda_{ij}$, and upon the deformation gradient only through $B_{ij} = F_{ki}\, F_{kj}$. Furthermore, it is assumed that both the solid and fluid are initially isotropic with a center of symmetry, hence as a consequence of this assumption, the constitutive functions depend on F_{ij} through $C_{ij}\, .\, F_{jk}$, where $\underline{C} = \underline{F}^{T} . \, \underline{F}$.

It is assumed that dynamical parts of the partial stress tensors and diffusive body force vector depend linearly on the dynamical variables given by

$$\bar{\sigma}^{d}_{(ij)} = \gamma_1 d_{kk}\delta_{ij} + 2\mu_1 d_{ij} + \gamma_2 f_{kk}\delta_{ij} + 2\mu_2 f_{ij}\ , \qquad (3.9)$$

$$\bar{\pi}^{d}_{(ij)} = \gamma_3 d_{kk}\delta_{ij} + 2\mu_3 d_{ij} + \gamma_4 f_{kk}\delta_{ij} + 2\mu_4 f_{ij}\ , \qquad (3.10)$$

$$\bar{\sigma}^{d}_{[ij]} = -\bar{\pi}^{d}_{[ij]} = -c_1\, (\Gamma_{ij} - \Lambda_{ij}), \text{ and} \qquad (3.11)$$

$$\bar{b}^{d}_{k} = c_2 \, (u_k - v_k). \tag{3.12}$$

The coefficients appearing in the equations (3.9)-(3.12) are function of ρ_1, ρ_2 and T. From equation (3.8) it may be concluded that

$$\mu_1 \geq 0, \; \gamma_1 + \frac{2}{3}\mu_1 \geq 0,$$

$$\mu_4 \geq 0, \; \gamma_4 + \frac{2}{3}\mu_4 \geq 0,$$

$$(\mu_3 + \mu_2)^2 \leq 4\,\mu_1\,\mu_4, \tag{3.13}$$

$$[(\gamma_2+\gamma_3) + \frac{2}{3}(\mu_2 + \mu_3)]^2 \leq 4(\gamma_1 + \frac{2}{3}\mu_1)(\gamma_3 + \frac{2}{3}\mu_3), \text{ and}$$

$$c_1 \geq 0, \; c_2 \geq 0.$$

The constitutive equations are written in terms of the Helmholtz free energy function A per unit mass of the mixture, and the form of this function, under the assumption of isotropy, is given by

$$A = \hat{A}\,(I_1, I_2, I_3, \rho_2, T), \tag{3.14}$$

where I_1, I_2, I_3 are the principal invariants of $\underline{B} = \underline{F} \cdot \underline{F}^T$ defined through

$$I_1 = \text{tr}\,\underline{B}\,, \tag{3.15}$$

$$I_2 = \frac{1}{2}\,[(\text{tr}\,\underline{B})^2 - \text{tr}\,\underline{B}^2], \tag{3.16}$$

and

$$I_3 = \det \underline{B} = (\det \underline{F})^2. \tag{3.17}$$

Using (2.9), (2.22) and (3.17), I_3 can be expressed in terms of ρ_2 by the relation

$$I_3^{\ 1/2} = \det \underline{F} = (1 - \rho_2/\rho_{20})^{-1}. \tag{3.18}$$

Hence, The Helmholtz free energy function A is assumed to depend on I_1, I_2, ρ_2 and T, so equation (3.14) reduces to

$$A = A\,(I_1,\ I_2,\ \rho_2, T). \tag{3.19}$$

Substitution of equations (3.4)-(3.7), (3.9)-(3.12) and (3.15)-(3.18) along with the functional form of the free energy function, given by equation (3.19) , into equations (3.1)-(3.3) yields the constitutive equations as follows

$$\sigma_{ki} = \delta_{ki}\,\phi_1 + \bar{\sigma}^s_{ki} + \bar{\sigma}^d_{ki}\,,$$

$$\sigma_{ki} = \delta_{ki}\,\phi_1 - p\,\frac{\rho_1}{\rho_{10}}\,\delta_{ki} + 2\rho\left\{\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\,I_1\right)B_{ki} - \frac{\partial A}{\partial I_2}\,B_{km}\,B_{mi}\right\}$$

$$+\ \gamma_1 d_{jj}\delta_{ki} + 2\mu_1 d_{ki} + \gamma_2 f_{jj}\delta_{ki} + 2\mu_2 f_{ki} - c_1\,(\Gamma_{ki} - \Lambda_{ki}), \tag{3.20}$$

$$\pi_{ki} = -\delta_{ki}\,\phi_1 + \bar{\pi}^s_{ki} + \bar{\pi}^d_{ki}\,,$$

$$\pi_{ki} = -\delta_{ki}\,\phi_1 - p\,\frac{\rho_2}{\rho_{20}}\,\delta_{ki} - \rho\rho_2\,\frac{\partial A}{\partial \rho_2}\delta_{ki} + \gamma_3 d_{jj}\delta_{ki} + 2\mu_3 d_{ki}$$

$$+ \gamma_4 f_{jj}\delta_{ki} + 2\mu_4 f_{ki} + \frac{\rho_1\ \rho_2}{\rho_{10}\rho_{20}}\ \beta(\Gamma_{ki} - \Lambda_{ki}),\ \text{and} \tag{3.21}$$

$$b_k = \frac{\partial\phi_1}{\partial x_k} + \bar{b}_k^s + \bar{b}_k^d\ ,$$

$$b_k = \frac{\partial\phi_1}{\partial x_k} - \frac{p}{\rho_{10}}\frac{\partial\rho_1}{\partial x_k} + \rho_1\frac{\partial A}{\partial\rho_2}\frac{\partial\rho_2}{\partial x_k} - \rho_2\left\{\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\ I_1\right)\delta_{i\ell} - \frac{\partial A}{\partial I_2}\ B_{i\ell}\right\} B_{i\ell,k} + \alpha\,\frac{\rho_1}{\rho_{10}}\frac{\rho_2}{\rho_{20}}\ (u_k - v_k). \tag{3.22}$$

It is to be noted that c_1 and c_2 have been redefined, and instead in equations (3.21 and 3.22) two new constitutive parameters α and β appear which account for a contribution to the interaction body force due to relative motion between the solid and the fluid.The interaction between the solid and the fluid is evident in these equations, where the partial stress of each constituent is affected by the deformed state of both the constituents.

2. CONSTITUTIVE EQUATIONS

Steady state and equilibrium formulation of the problems where dynamical parts of the constitutive equation for σ_{ij} and π_{ij} do not contribute to the complete analysis of the solid-fluid mixtures may further simplify the constitutive equations (3.20)-(3.22). Furthermore, for isothermal condition the components of the partial stress tensors for the solid and fluid, and the interaction body force vector may be written as

$$\bar{\sigma}_{ki} = -\ p\frac{\rho_1}{\rho_{10}}\delta_{ki} + 2\rho\left\{\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}I_1\right)B_{ki} - \frac{\partial A}{\partial I_2}\ B_{km}\ B_{mi}\right\}, \tag{3.23}$$

$$\bar{\pi}_{ki} = - p \frac{\rho_2}{\rho_{20}} \delta_{ki} - \rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ki} \text{ , and} \tag{3.24}$$

$$\bar{b}_k = - \frac{p}{\rho_{10}} \frac{\partial \rho_1}{\partial x_k} + \rho_1 \frac{\partial A}{\partial \rho_2} \frac{\partial \rho_2}{\partial x_k} - \rho_2 \left\{ \left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} I_1 \right) \delta_{i\ell} - \frac{\partial A}{\partial I_2} B_{i\ell} \right\} B_{i\ell,k} + \alpha \frac{\rho_1}{\rho_{10}} \frac{\rho_2}{\rho_{20}} (u_k - v_k). \tag{3.25}$$

It is also useful to record the representation for the total stress

$$T_{ki} = \bar{\sigma}_{ki} + \bar{\pi}_{ki} = - p\delta_{ki} - \rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ki} + 2\rho \left\{ \left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} I_1 \right) B_{ki} - \frac{\partial A}{\partial I_2} B_{km} B_{mi} \right\}. \tag{3.26}$$

In the remainder of this paper, only $\bar{\sigma}$, and $\bar{\pi}$ and $\bar{b}$, will be used. Hence, for notational convenience, the superposed bars are dropped.

3. SPECIFIC FORM OF THE HELMHOLTZ FREE ENERGY FUNCTION

The application of the Theory of Interacting Continua to study diffusion and swelling phenomena of non-linearly elastic solids requires a particular form of the Helmholtz free energy function A for the solid-fluid mixture. Ideally, a broad experimental program should be setup to determine the Helmholtz free energy function for a given solid-fluid mixture. Due to the lack of experimental data for determining the specific form of A, Treloar's work [17] has been modified to suit the constitutive equations defined per unit mass of the mixture in the previous section.

The specific form of the Helmholtz free energy function is derived by assuming that the mixture is of "Neo-Hookean type," that is, A is a linear function of I_1 . The free energy function for a mixture of this type may be written as:

$$A = A_e + A_m \,, \tag{3.27}$$

Where, A_e is free energy of deformation and A_m is the free energy of mixing for the solid in the uncross-linked state both defined per unit mass of the mixture. The first term on the left hand side of the equation (3.27) represents the strain energy function for a Neo-Hookean material per unit mass of the mixture and may be given as

$$A_e = \frac{\nu_1}{\rho} \left[\frac{\hat{R}T\rho_{10}}{2M_c} (I_1 - 3) \right], \tag{3.28}$$

The second term A_m in equation (3.27) is derived from Flory-Huggins relation [17] and is given by:

$$A_m = \frac{\nu_1}{\rho} \frac{\hat{R}T}{V_1} \left[\frac{1-\nu_1}{\nu_1} \ell n(1-\nu_1) + \chi(1-\nu_1) \right], \tag{3.29}$$

where,

V_1 is the molar volume of the fluid,

χ is a constant which depends on the particular combination of the solid and the fluid,

$\hat{R}$ is the universal gas constant,

T is the absolute temperature,

M_c is the molecular weight of the polymeric solid between the cross-links.

The specific form of the free energy function given by equations (3.27)-(3.29) may be used to get an explicit form of the components of the partial stress tensors for solid constituent, for fluid constituent, and interaction body force vector, and are given as:

$$\sigma_{ki} = - p \frac{\rho_1}{\rho_{10}} \delta_{ki} + 2\rho \frac{\partial A}{\partial I_1} B_{ki} , \tag{3.30}$$

$$\pi_{ki} = - p \frac{\rho_2}{\rho_{20}} \delta_{ki} - \rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ki} , \text{ and} \tag{3.31}$$

$$b_k = - \frac{p}{\rho_{10}} \frac{\partial \rho_1}{\partial x_k} + \rho_1 \frac{\partial A}{\partial \rho_2} \frac{\partial \rho_2}{\partial x_k} - \rho_2 \frac{\partial A}{\partial I_1} B_{\ell\ell,k} + \alpha \frac{\rho_1}{\rho_{10}} \frac{\rho_2}{\rho_{20}} (u_k - v_k). \tag{3.32}$$

CHAPTER IV

APPLICATIONS OF THE THEORY OF INTERACTING CONTINUA

1. INTRODUCTION

The general Theory of Interacting Continua is useful in studying the phenomenological behavior of a mixture of multiconstituents. Mixtures for which continuum models may be proposed are fluid-fluid mixtures (e.g. bubbly liquids and suspensions), fluid-filled porous elastic solids (e.g. swollen soils), solid-solid mixtures (e.g. polymeric composites) and solid-fluid mixtures (e.g. swollen polymeric materials). The Theory of Interacting Continua may be employed to find complete system of equations governing the thermo-mechanical response of these mixtures. The fundamental work in formulating the theory was done in 60's but the process of application of the theory to solve real-life boundary-value problems has been very slow. The slow pace in solving boundary-value problems involving mixtures, in general, is due to the complexity which is the consequence of the presence of more than one constituents, and due to many unresolved issues regarding specification of the boundary conditions, and lack of specific forms of mixture characterization. The application of the theory has been successful [27,28] in solving equilibrium and steady state boundary-value problems involving solid-fluid mixtures.

In this chapter the equilibrium boundary-value problems involving a mixture of an incompressible non-linearly elastic solid and an ideal fluid are treated within the context of the Theory of Interacting Continua. In particular, attention is focused on presenting complete solutions of two boundary-value problems involving mixture of this class, and undergoing large non-homogeneous deformations.

1. COMBINED EXTENSION AND TORSION OF A SWOLLEN CYLINDER

The combined finite extension and torsion of a cylindrical mixture of an incompressible non-linearly elastic solid and an ideal fluid is considered. A 'Universal Relation' was presented by Gandhi et al. [33] for the case of small twist. In this work, the general problem for the finite deformation of a heterogeneous cylindrical mixture is formulated in the context of the Theory of Interacting Continua in order to account for the interaction between the solid and fluid constituents. This formulation permits the analysis of the individual motion of the solid and fluid constituents by incorporating the interaction between the two. However, the fluid can be non-homogeneously dispersed throughout the mixture region, which gives rise to gradients in the fluid density. The physical mechanism for the existence of such gradients is provided by the presence of an interaction body force which each constituent exerts on the other. The objective of this study is to investigate the qualitative behavior of the solid-fluid mixtures, hence only equilibrium of cylindrical mixture is considered in this work.

Consider a solid circular cylinder described by a radius R_0 and a length L_0 in the reference configuration Figure 1. The co-ordinates of a material particle in the reference configuration will be denoted by cylindrical co-ordinates (R, Θ, Z). The cylinder is assumed to be subjected to the following deformation:

$$r = r(R),$$
$$\theta = \Theta + \psi\lambda Z, \tag{4.1}$$
and
$$z = \lambda Z,$$

where (r, θ, z) denote the co-ordinates of the particle at (R, Θ, Z) in the deformed swollen configuration, λ and ψ being constants. The above deformation corresponds to a finite elongation (with an associated stretch ratio λ) along the z-co-ordinate direction, followed by a rotation of ψ per unit current length.

The Cauchy-Green tensor $\underline{B}$ which is defined as

$$\underline{B} = \underline{F} \cdot \underline{F}^T \tag{4.2}$$

takes the following form for the above deformation:

$$\underline{B} = \begin{bmatrix} \left(\frac{dr}{dR}\right)^2 & 0 & 0 \\ 0 & \left(\frac{r}{R}\right)^2 + (\psi\lambda r)^2 & \psi\lambda^2 r \\ 0 & \psi\lambda^2 r & \lambda^2 \end{bmatrix}, \tag{4.3}$$

$$= \begin{bmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 + (\psi R\lambda\lambda_\theta)^2 & \psi\lambda^2\lambda_\theta R \\ 0 & \psi\lambda^2\lambda_\theta R & \lambda^2 \end{bmatrix}, \tag{4.4}$$

where $\lambda_r = dr/dR$ and $\lambda_\theta = r/R$ denote the stretch ratios in the r and θ directions, respectively. The principal invariants of $\underline{B}$ are then given as

$$I_1 = \lambda_r^2 + \lambda_\theta^2(1 + \psi^2 R^2 \lambda^2) + \lambda^2, \tag{4.5}$$

$$I_2 = \lambda^2(\lambda_r^2 + \lambda_\theta^2) + \lambda_\theta^2\lambda_r^2(1 + \psi^2 R^2 \lambda^2), \text{ and} \tag{4.6}$$

$$I_3 = \lambda_r^2\lambda_\theta^2\lambda^2. \tag{4.7}$$

The balance of mass equation for the solid constituent (2.9) may be expressed in terms of the stretch ratios as

$$\frac{\rho_1}{\rho_{10}} = \frac{1}{\lambda_r\lambda_\theta\lambda} = \nu_1, \tag{4.8}$$

where ν_1 represents the volume fraction of the solid.

The equations of equilibrium which are appropriate for the deformation being considered are documented next. Since the assumed form of deformation implies that the stresses depend only on the radial co-ordinate r, the equations of equilibrium for the solid constituent, namely (2.11), reduce to

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - b_r = 0, \tag{4.9}$$

where σ_{rr} and $\sigma_{\theta\theta}$ denote the appropriate components of $\underset{\sim}{\sigma}$, and b_r denotes the component of the interaction body force $\underset{\sim}{b}$ in the radial direction. The equilibrium equations for the fluid constituent, namely (2.12), reduce to

$$\frac{d\pi_{rr}}{dr} + \frac{\pi_{rr} - \pi_{\theta\theta}}{r} + b_r = 0, \tag{4.10}$$

where π_{rr} and $\pi_{\theta\theta}$ denote the appropriate components of $\underline{\pi}$. Equations (3.26), (4.9) and (4.10) yield

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \tag{4.11}$$

which is the equation of equilibrium for the mixture.

For the deformation under consideration, it follows from equation (4.4) and equations (3.30) - (3.32) that the non-zero components of the partial stress tensors for the solid and fluid constituents are given by

$$\sigma_{rr} = -p\frac{\rho_1}{\rho_{10}} + 2\rho(A_1 + A_2 I_1)\lambda_r^2 - 2\rho(A_2)\lambda_r^4, \tag{4.12}$$

$$\sigma_{\theta\theta} = -p\frac{\rho_1}{\rho_{10}} + 2\rho(A_1 + A_2 I_1)\lambda_\theta^2(1 + \psi^2 R^2 \lambda^2)$$

$$-2\rho A_2\{\lambda_\theta^4(1 + \psi^2 R^2 \lambda^2)^2 + \psi^2 R^2 \lambda^4 \lambda_\theta^2\}, \tag{4.13}$$

$$\sigma_{zz} = -p\frac{\rho_1}{\rho_{10}} + 2\rho(A_1 + A_2 I_1)\lambda^2 - 2\rho A_2\{\lambda^4(1 + \psi^2 R^2 \lambda_\theta^2)\}, \tag{4.14}$$

$$\sigma_{\theta z} = 2\rho\psi R\left\{(A_1 + A_2 I_1)\lambda^2\lambda_\theta - A_2[\lambda^2\lambda_\theta((1 + \psi^2 R^2\lambda^2)\lambda_\theta^2 + \lambda^2)]\right\}, \tag{4.15}$$

and

$$\pi_{rr} = \pi_{\theta\theta} = \pi_{zz} = -p\frac{\rho_2}{\rho_{20}} - \rho\rho_2 A_{\rho_2}, \text{ respectively.} \tag{4.16}$$

The only non-zero component of the interaction body force vector is given by

$$b_r = -\frac{p}{\rho_{10}\lambda_r}\frac{d\rho_1}{dR} + \rho_1 A_{\rho_2}\frac{d\rho_2}{\lambda_r dR}$$

$$- \rho_2 A_1 \frac{d}{\lambda_r dR} \left\{ \lambda_r^2 + \lambda_\theta^2 + \psi^2 \lambda^2 R^2 \lambda_\theta^2 + \lambda^2 \right\}$$

$$- \rho_2 A_2 \frac{d}{\lambda_r dR} \left[\lambda_r^2 \lambda^2 + \lambda_\theta^2 \lambda^2 + \lambda_r^2 \lambda_\theta^2 (1 + \psi^2 R^2 \lambda^2) \right] , \qquad (4.17)$$

where $A_1 = \frac{\partial A}{\partial I_1}$, $A_2 = \frac{\partial A}{\partial I_2}$ and $A_{\rho_2} = \frac{\partial A}{\partial \rho_2}$.

It is sufficient to satisfy any two of the three equilibrium equations (4.9) - (4.11). Equations (4.12) - (4.17) are substituted in to the equilibrium equations for the solid and the mixture (4.9) and (4.11), respectively, to get the following functional forms of the equilibrium equations which are stated in terms of the co-ordinates in the reference configuration for computational convenience:

$$- \frac{dp}{dR} \frac{\rho_1}{\rho_{10}} + g_1 \, (A_1, A_2, A_{\rho_2}, \lambda_r, \lambda_\theta, R, \lambda_r', \lambda_\theta', \lambda, \psi^2 R^2) = 0, \qquad (4.18)$$

and

$$- \frac{dp}{dR} + g_2 \, (A_1, A_2, A_{\rho_2}, \lambda_r, \lambda_\theta, R, \lambda_r', \lambda_\theta', \lambda, \psi^2 R^2) = 0. \qquad (4.19)$$

In equations (4.18) and (4.19) the prime denotes differentiation with respect to the reference radial coordinate R, and the radial and tangential stretch ratio λ_r and λ_θ, respectively, are related through the compatibility condition given by

$$\frac{d\lambda_\theta}{dR} = \frac{\lambda_r - \lambda_\theta}{R}. \qquad (4.20)$$

Subsequently, the mixture is assumed to be of a "Neo-Hookean-type," that is, A is a linear function of I_1. For this case the explicit forms of the equilibrium equations for the mixture and the solid (see appendix B and C for details) are given by

$$-\frac{dp}{dR}-\frac{d}{dR}\left(\rho\rho_2\frac{\partial A}{\partial \rho_2}\right)+2\rho\, A_1\lambda_r\left[\frac{d\lambda_r}{dR}-\frac{\lambda_r}{\lambda_\theta R}\left(\lambda_r-\lambda_\theta\right)\right.$$

$$\left.-\frac{1}{\lambda_\theta R}(\lambda_\theta^2-\lambda_r^2+\psi^2\lambda^2R^2\lambda_\theta^2)\right]=0, \qquad (4.21)$$

and

$$-\nu_1\frac{dp}{dR}+2\rho\, A_1\lambda_r\left[\frac{d\lambda_r}{dR}-\frac{\lambda_r}{\lambda_\theta R}\left(\lambda_r-\lambda_\theta\right)\right.$$

$$\left.-\frac{1}{\lambda_\theta R}(\lambda_\theta^2-\lambda_r^2+\psi^2\lambda^2R^2\lambda_\theta^2)\right]-\rho_1\rho_{20}\frac{\partial A}{\partial \rho_2}\nu_1\left[\frac{1}{\lambda_r}\frac{d\lambda_r}{dR}+\frac{1}{\lambda_\theta}\frac{(\lambda_r-\lambda_\theta)}{R}\right]$$

$$+2\rho_2\, A_1\left[\lambda_r\frac{d\lambda_r}{dR}+\lambda_\theta\frac{(\lambda_r-\lambda_\theta)}{R}+\psi^2\lambda^2\lambda_\theta\lambda_r R\right]=0. \qquad (4.22)$$

Equations (4.20) - (4.22) may be solved for p, λ_r and λ_θ once the specific form of the Helmholtz free energy function for the mixture is known, and the appropriate boundary çonditions are specified. For the "Neo-Hookean type" mixture considered here the Helmholtz free energy function per unit mass of the mixture may be written from equations (3.27-3.29) as:

$$A=\frac{\nu_1}{\rho}\left[\frac{\hat{R}T\rho_{10}}{2M_c}(I_1-3)+\frac{\hat{R}T}{V_1}\left[\frac{1-\nu_1}{\nu_1}\ln(1-\nu_1)+\chi(1-\nu_1)\right]\right], \qquad (4.23)$$

Two of the appropriate boundary conditions for solving the set of ordinary differential equations (4.20) - (4.22) are given by

$$\lambda_r(0)=\lambda_\theta(0),\ \text{and} \qquad (4.24)$$

$$T_{rr}(R_o)=0 \qquad (4.25)$$

The boundary condition given by equation (4.24) arises due to the compatibility requirement between the radial and tangential stretch ratios at the axis of the cylinder. The boundary condition on the total traction vector represented by equation (4.25) is a consequence of the requirement that the outer surface of the cylinder be traction-free. Since a boundary condition for the partial traction vector is not physically obvious, following the arguments presented in [25] it is assumed that the outer surface of the cylinder is in a saturated state. This assumption results in the boundary condition represented by

$$S_{rr}(R_o) = 0, \tag{4.26}$$

where S_{rr} represents the radial stress component for a saturated state, and is given by [25]

$$S_{rr} = \rho\ (\rho_{20} - \rho_2)\ \frac{\partial A}{\partial \rho_2} + \rho_{20}A + 2\ \rho\ A_1\ \lambda_r^2\ . \tag{4.27}$$

The governing equations (4.20) - (4.22) for the combined extension and torsion of a swollen cylinder are highly non-linear and coupled, and may be solved numerically for the variables λ_r, λ_θ and p. For computational convenience, equations (4.21) and (4.22) may be combined to eliminate p, and for the Helmholtz free energy function given by (4.23) the resulting equation (see appendix D for details) is given by

$$\frac{R\lambda_\theta}{\lambda_r}\frac{d\lambda_r}{dR} = -\ \frac{(\lambda_r - \lambda_\theta)\left[K(2\chi - \frac{1}{1-\nu_1})\nu_1 - \lambda_r\lambda_\theta\right] + \psi^2\lambda^2R^2\lambda_\theta^2\lambda_r}{K\ (2\chi - \frac{1}{1-\nu_1})\nu_1 - \lambda_r^2} \tag{4.28}$$

where,

$$K = \frac{M_c}{\rho_{10}V_1}.$$

The set of ordinary differential equations given by (4.20) and (4.28) subjected to boundary conditions given by (4.24) and (4.26) were solved numerically. The following material properties [1] were used for the numerical calculations:

Density of rubber in the reference state	ρ_{10}	= .9016	gm/cc
Density of solvent in the reference state	ρ_{20}	= .862	gm/cc
Molar volume of the solvent	v_1	= 106.0	cc/mole
The molecular weight of rubber between cross links	M_c	= 8891.0	gm/mole
Rubber-solvent interaction constant	χ	= .400	

The numerical value of the universal gas constant $\hat{R}$ is given by 8.317×10^7 dyne-cm/mole - °K, and the absolute temperature T was assumed to be 303.16°K. The computational results are presented in Figures 1-6 for a value of the axial stretch ratio $\lambda = 1.938$ which was maintained in the experimental work presented in [28].

Figure 2 shows the variation of the radial and circumferential stretch ratios for two different values of the angle of twist ψ. For the case of no twist ($\psi = 0$) the cylinder is homogeneously swollen whereby the radial and tangential stretch ratios are equal and constant throughout the domain. However, when the swollen cylinder undergoes finite torsion ($\psi = 1.0$) significant gradients in the stretch ratios are evident. Furthermore, even in the case of finite torsion, the deformation in the axial domain is relatively homogeneous, and the gradients in the stretch ratios increase with the radial co-ordinate. Figure 3 shows the variation of the radial stress for two different values of the angle of twist. It is seen from Figure 3 that the non-dimensional radial stress is compressive and approaches zero at $R = R_0$ due to the boundary condition (4.26) which requires the outer surface of the deformed cylinder to be traction-free. The corresponding variation of the non-dimensional circumferential stress is shown in Figure 4. The non-dimensional radial and circumferential stresses in Figures 3 and 4 denoted by $\hat{T}_{rr}$ and $\hat{T}_{\theta\theta}$ have been non-dimensionalized with respect to $\frac{\hat{R}\ T\ \rho_{10}}{M_c}$. It is clear from Figures 2, 3, and 4 that the gradients of the radial and circumferential stretch ratios and stresses increase with increasing twist. The variation of the volume fraction of the solid along the reference radial co-ordinate is shown in Figure 5 for three different values of the angle of twist. It is evident from Figure 5 that the fluid leaves the swollen deformed cylinder as the angle of twist is increased. The non-dimensional ratio of the current volume V (in the swollen twisted state) to the volume of the original unswollen rubber cylinder V_u is presented in Figure 6. It is clear from these results that as the angle of twist increases the fluid leaves the cylindrical mixture resulting in the reduction of the current volume of the swollen cylinder. Finally, the ratio of the change in the volume $\Delta V = (V - V_0)$ to the saturated swollen untwisted volume V_0 is compared with experimental results [28] in Figure 7. The computational results based on Mixture Theory predict the same qualitative and quantitative trends as observed in experimental results, thereby illustrating the value of employing Mixture Theory in modeling the interaction of elastic solids and ideal fluids undergoing large deformations.

2. NON-UNIFORM UNIAXIAL EXTENSION OF A MIXTURE SLAB

In this section, a boundary-value problem is studied for a slab which is a mixture of a non-linearly elastic solid and an ideal fluid. Consider a mixture slab of an incompressible non-linearly elastic solid and an ideal fluid. The slab is assumed to have finite thickness h, and the other two dimensions of the slab are assumed to be infinite Figure 8. Let (X,Y,Z) denote the coordinates of a particle in the reference configuration and (x,y,z), the coordinates of the same particle in the deformed configuration. Consider the deformation

$$x = f(Z)\,X, \quad y = f(Z)\,Y \quad \text{and,} \quad z = \lambda(Z). \tag{4.29}$$

The deformation gradient $\underline{F}$ is given by

$$\underline{F} = \begin{bmatrix} f & 0 & Xf' \\ 0 & f & Yf' \\ 0 & 0 & \lambda' \end{bmatrix}. \tag{4.30}$$

The prime denotes differentiation with respect to Z. The Cauchy-Green tensor $\underline{B} = \underline{F} \cdot \underline{F}^T$ can now be represented as

$$\underline{B} = \begin{bmatrix} f^2 + X^2 f'^2 & XYf'^2 & Xf'\lambda' \\ XYf'^2 & f^2 + Y^2 f'^2 & Yf'\lambda' \\ Xf'\lambda' & Yf'\lambda' & \lambda' \end{bmatrix}. \tag{4.31}$$

The equilibrium equations are expressed in terms of the reference configuration for computational convenience. Assuming no external body forces, the equations of equilibrium for the mixture take the form

$$\frac{\partial T_{ij}}{\partial Xp} \quad F^{-1}_{pj} = 0. \tag{4.32}$$

The tensor $\underline{F}^{-1}$ that appears in these equations has the form

$$\underline{F}^{-1} = \begin{bmatrix} \frac{1}{f} & 0 & -\frac{Xf'}{f\lambda'} \\ 0 & \frac{1}{f} & -\frac{Yf'}{f\lambda'} \\ 0 & 0 & \frac{1}{\lambda'} \end{bmatrix}. \tag{4.33}$$

The equilibrium equations for a mixture of "Neo-Hookean type" reduce to

$$-\frac{\partial P}{\partial X} + 4A_1 \rho X f'^2 + 2A_1 X \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f'\lambda') = 0, \tag{4.34}$$

$$-\frac{\partial P}{\partial Y} + 4A_1 \rho Y f'^2 + 2A_1 Y \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f'\lambda') = 0, \tag{4.35}$$

and

$$-\frac{f}{\lambda'} \frac{\partial P}{\partial Z} + 4A_1 \rho f'\lambda' + 2A_1 \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho \lambda'^2)$$

$$+ \frac{f'}{\lambda'} \left[X \frac{\partial P}{\partial X} + Y \frac{\partial P}{\partial Y} \right] - \frac{f}{\lambda'} \frac{\partial}{\partial Z} \left(\rho_2 \, \rho \frac{\partial A}{\partial \rho_2} \right) = 0. \tag{4.36}$$

In equations (4.34) - (4.36),

$$A_1 = \frac{\partial A}{\partial I_1}.$$

Let

$$g(Z) = 4A_1 \rho f'^2 + 2A_1 \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f' \lambda') . \tag{4.37}$$

Then, equations (4.34) - (4.36) may be written as

$$- \frac{\partial P}{\partial X} + X\, g(Z) = 0, \tag{4.38}$$

$$- \frac{\partial P}{\partial Y} + Y\, g(Z) = 0, \tag{4.39}$$

and

$$- \frac{\partial P}{\partial Z} + \frac{4A_1 \rho f' \lambda'^2}{f} + 2A_1 \frac{\partial}{\partial Z} (\rho \lambda'^2)$$

$$+ \frac{f'}{f} [X^2 + Y^2]\, g(Z) - \frac{\partial}{\partial Z} \left(\rho_2 \rho \frac{\partial A}{\partial \rho_2} \right) = 0. \tag{4.40}$$

The scalar P in equations (4.38-4.40) is eliminated by the standard procedure of cross-differentiation to obtain

$$g'(Z) = 2 \frac{f'}{f} g(Z). \tag{4.41}$$

The equilibrium equations for the solid take the form

$$\frac{\partial \sigma_{ij}}{\partial X_p} F^{-1}_{pj} - b_i = 0. \tag{4.42}$$

The equilibrium equations for the solid reduce to

$$- \frac{\partial P}{\partial X} + 2A_1 \frac{\rho_{10}}{\rho_1} X f'^2 (2\rho + \rho_2) + 2A_1 \frac{\rho_{10}}{\rho_1} X \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f' \lambda') = 0, \tag{4.43}$$

$$- \frac{\partial P}{\partial Y} + 2A_1 \frac{\rho_{10}}{\rho_1} Yf'^2 (2\rho + \rho_2) + 2A_1 \frac{\rho_{10}}{\rho_1} Y \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f'\lambda') = 0, \qquad (4.44)$$

and

$$- \left[\frac{\rho_1}{\rho_{10}} \frac{\partial P}{\partial Z} + 2A_1 \frac{\partial}{\partial Z} (\rho\lambda') \right] \frac{f}{\lambda'} + 4\rho A_1 f'\lambda' + \frac{\rho_1}{\rho_{10}} \frac{f'}{\lambda'} \left[X \frac{\partial P}{\partial X} \right.$$

$$\left. + Y \frac{\partial P}{\partial Y} \right] - \rho_1 \frac{\partial A}{\partial \rho_2} \frac{\rho_2' f}{\lambda'} - 2A_1\rho_2 \frac{f'^3}{\lambda'} [X^2 + Y^2] + A_1 \frac{\rho_2 f}{\lambda'} \left[4ff' \right.$$

$$\left. + 2(X^2+Y^2) f'f'' + 2\lambda'\lambda'' \right] = 0. \qquad (4.45)$$

Let

$$h(Z) = 2A_1 \frac{\rho_{10}}{\rho_1} f'^2 (2\rho + \rho_2) + 2A_1 \frac{\rho_{10}}{\rho_1} \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\rho f'\lambda'). \qquad (4.46)$$

Then, equations (4.43) - (4.45) may be written as

$$- \frac{\partial P}{\partial X} + X h(Z) = 0, \qquad (4.47)$$

$$- \frac{\partial P}{\partial Y} + Y h(Z) = 0, \qquad (4.48)$$

and

$$-\left[\frac{\rho_1}{\rho_{10}} \frac{\partial P}{\partial Z} + 2A_1 \frac{\partial}{\partial Z} (\rho\lambda') \right] \frac{f}{\lambda'} + 4\rho A_1 f'\lambda'$$

$$+ \frac{\rho_1}{\rho_{10}} \frac{f'}{\lambda'} \left[X^2 + Y^2 \right] h(Z) - \rho_1 \frac{\partial A}{\partial \rho_2} \frac{\rho_2' f}{\lambda'} \qquad (4.49)$$

$$- 2A_1\rho_2 \frac{f'^3}{\lambda'} [X^2 + Y^2]$$

$$+ A_1 \frac{\rho_2 f}{\lambda'} \left[4ff' + 2(X^2+Y^2) f'f'' + 2\lambda'\lambda'' \right] = 0.$$

Again, the scalar P in equations (4.47-4.49) is eliminated using the procedure of cross-differentiation to obtain

$$h'(Z) = 2\,\frac{f'}{f}\,h(Z) + \hat{h}(Z), \tag{4.50}$$

where,

$$\hat{h}(Z) = 4A_1\rho_2\,\frac{\rho_{10}}{\rho_1}\,f'\left[f'' - \frac{f'^2}{f}\right]. \tag{4.51}$$

A simple integration of equation (4.41) yields

$$g(Z) = C_1 f^2, \tag{4.52}$$

where C_1 is a constant.

By virtue of (4.37), equation (4.52) may be written as

$$f'' + \frac{f'\lambda''}{\lambda'} + \frac{\rho'}{\rho}\,f' + \frac{2f'^2}{f} - \frac{C_1 f}{2A_1\rho} = 0. \tag{4.53}$$

Similarly, equations (4.46), (4.50), and (4.51) may be combined to yield

$$\rho_2'\,f'^2 - \frac{\rho_1'}{\rho_1}\left[\rho_2 f'^2 + \frac{C_1}{2A_1}\,f^2\right] = 0. \tag{4.54}$$

Exact solutions to equations (4.53) and (4.54) are presented next.

First, consider the case when the density of the solid remains constant. That is

$$\frac{\rho_1}{\rho_{10}} = \text{constant}. \tag{4.55}$$

Using equations (4.55) and (2.22), equation (4.54) is identically satisfied. By virtue of (2.9)

$$\frac{\rho_1}{\rho_{10}} = \frac{1}{\lambda' f^2}, \tag{4.56}$$

which reduces equation (4.53) to

$$\lambda''' - \frac{3}{2}\frac{\lambda''^2}{\lambda'} + \frac{\lambda' C_1}{2\rho A_1} = 0. \tag{4.57}$$

In equation (4.57), A_1 is a constant when the Helmholtz free energy function A for the mixture is linear in I_1 (a "Neo-Hookean-type" mixture). Then,

$$\lambda''' - \frac{3}{2}\frac{\lambda''^2}{\lambda'} = C\,\lambda', \tag{4.58}$$

where, $\quad C = \dfrac{-C_1}{2\rho A_1}.$

Next, solutions to the ordinary differential equation (4.58) will be presented for three possible cases depending on the value of the constant C being negative, zero or positive.

(i) When C > 0, it can be shown that

$$\lambda'(Z) = \frac{1}{\left[\bar{A}_1 \sin\sqrt{\frac{C}{2}}\,Z + \bar{B}_1 \cos\sqrt{\frac{C}{2}}\,Z\right]^2}$$

$$= \eta_1(\bar{A}_1, \bar{B}_1, C, \bar{Z}) , \tag{4.59}$$

and the solution to the first-order ordinary differential equation (4.59) may given in the functional form as

$$z(Z) = \lambda(Z) = \int_0^Z \eta_1(\bar{A}_1, \bar{B}_1, C, \bar{Z})\,d\bar{Z} , \tag{4.60}$$

where $\bar{A}_1$ and $\bar{B}_1$ are constants of integration, and have to be evaluated by the given boundary conditions. For a mixture layer of thickness H, fixed at the bottom, and whose deformed thickness is h, the appropriate boundary conditions may be given as

$$z(0) = 0, \tag{4.61}$$

$$z(H) = h. \tag{4.62}$$

(ii) When C < 0,

$$\lambda'(Z) = \frac{1}{\left[\bar{A}_2 e^{\sqrt{\frac{C'}{2}}\,Z} + \bar{B}_2 e^{-\sqrt{\frac{C'}{2}}\,Z}\right]^2}\ , \tag{4.63}$$

and the solution to the first-order ordinary differential equation (4.63) may given in the functional form as

$$z(Z) = \lambda(Z) = \int_0^Z \eta_2(\bar{A}_2, \bar{B}_2, C', \bar{Z})\,d\bar{Z}\ , \tag{4.64}$$

where $\bar{A}_2$ and $\bar{B}_2$ are constants of integration, and have to be evaluated by the given boundary conditions (4.61-4.62), and $C' = -C$ such that $C' > 0$.

(iii) When C = 0,

$$\lambda'(Z) = \text{constant}, \tag{4.65}$$

is a solution to equation (4.58), which corresponds to the classical solution.

Now, consider the general case where the density of the solid is a function of the space coordinates. For this case the equilibrium equations for the mixture (4.53) and the solid (4.54) reduce to

$$f^2 f' \lambda'' + (f^2 f'' + 2\, f f'^2 - K f^3)\lambda' + \left(\frac{\rho_{10} - \rho_{20}}{\rho_{20}}\right) f'' = 0, \tag{4.66}$$

and

$$f'^2 + K f^2 = 0, \tag{4.67}$$

respectively, where $K = \frac{C_1}{2\rho_{20}A_1}$. Equation (4.67) can be solved independently of equation (4.66) to obtain f(Z).

When $K < 0$, let $K = -a^2$, $a > 0$. Equation (4.67) has solutions given by

$$f_1 = \beta_1\, e^{az}, \tag{4.68}$$

$$f_2 = \beta_2\, e^{-az}, \tag{4.69}$$

where β_1 and β_2 are constants of integration. When $K > 0$, equation (4.67) has imaginary solutions, which are not physically meaningful. When $K = 0$, equation (4.67) admits the classical solution,

$$f(Z) = \text{constant}. \tag{4.70}$$

Equation (4.66) can be used to obtain the transverse stretch ratio $\lambda(Z)$ corresponding to $f_1(Z)$ and $f_2(Z)$ given by equations (4.68) and (4.69). Substituting equation (4.68) in equation (4.66) gives

$$\lambda'' + 4a\lambda' - \gamma_1\, e^{-2aZ} = 0, \tag{4.71}$$

where, $\gamma_1 = \frac{\rho_{10} - \rho_{20}}{\rho_{20}} \frac{a}{\beta_1^{\,2}}$,

which admits a one parameter family of solutions

$$\lambda(Z) = - \frac{\gamma_1}{4a^2} e^{-2aZ} + L_1 e^{-4aZ} , \qquad (4.72)$$

where L_1 is a constant of integration. Substituting equation (4.69) in equation (4.66) yields

$$\lambda'' - 4a\lambda' - \gamma_2 e^{2aZ} = 0,$$

where $\gamma_2 = \frac{\rho_{10} - \rho_{20}}{\rho_{20}} \frac{a}{\beta_2^2}$,

which admits a one parameter family of solutions

$$\lambda(Z) = - \frac{\gamma_2}{4a^2} e^{2aZ} + L_2 e^{4aZ} , \qquad (4.73)$$

where L_2 is a constant of integration. Corresponding to the classical solution for f(Z) given by (4.70), equation (4.66) admits the classical solution given by (4.65). Figure 9 shows the variation in the deformation along the thickness of the layer with respect to the reference coordinate Z for various values of the parameter a. The appropriate boundary conditions used in obtaining the results presented in Figure 9 by using equation (4.72) are;

z(1) = 2, and

$z(0) = 0.$

Figure 10 shows the corresponding variation in lateral stretch ratio f(Z) with respect to the reference coordinate Z for various values of the parameter a.

CONCLUDING REMARKS

The thermo-mechanical response of solid-fluid mixtures in problems involving non-homogeneous equilibrium swelling of solids, diffusion of fluids in solids, steady-state flow of fluids through swollen solids and wave propagation in solid-fluid mixtures may be treated within the context of the Theory of Interacting Continua. In this work, attention is focused on the mechanical response of a mixture of an incompressible, non-linearly elastic solid and an ideal fluid undergoing large deformations. The fundamental mathematical framework for the Theory of Interacting Continua is presented, and the theory is employed to model the mechanical response of solid-fluid mixtures undergoing large deformations, and in particular, two boundary-value problems have been presented for this class of mixtures. The first boundary value problem presented herein is intrinsically interesting since it is one of the few problems involving large non-homogeneous deformations for which experimental results are available. The second problem presented in this work is more of theoretical interest which demonstrates that an infinite mixture slab of finite thickness, undergoing uniaxial extension, admits infinite solutions.

The first boundary-value problem of finite extension and torsion of a swollen cylinder of an incompressible, non-linearly elastic material presented herein is formulated within the context of The Theory of Interacting Continua.

Computational results for the variation of the radial and tangential stretch ratios and the distribution of the fluid in the swollen deformed state are presented. The results demonstrate that the swollen volume of a cylinder reduces with twisting when the axial stretch ratio is held constant. Computational results for the reduction in the swollen volume predict the same qualitative and quantitative trends as observed in experimental results.

The second boundary-value problem of non-uniform uniaxial extension of a mixture slab presented herein demonstrates the possibility of an axial variation of the stretch ratio for uniaxial extension of a mixture of a non-linearly elastic solid and an ideal fluid. In addition to the classical solution, a one parameter family of solutions has also been presented.

This work investigates the complete mechanical response of the solid-fluid mixtures and has served as a test-bed for evaluating the validity and predictive capability of the Theory of Interacting Continua in modeling the interaction of elastic solids and ideal fluids, and also demonstrates the applicability of the theory to the problems involving solid-fluid mixtures. This approach may be extended to study steady-state and time-dependent problems of interest involving solid-fluid mixtures, the static and elasto-dynamic response of composite materials under hygro-thermal environment may be adequately modeled by employing the Theory of Interacting Continua, for example. Furthermore, an experimental program for the material characterization of solid-fluid mixtures will be very for pushing the frontier of understanding the complex non-linear behavior of these mixtures.

APPENDICES

Appendix A

SPATIAL DERIVATIVES

A.1 Derivatives Of ρ, ρ_1, ρ_2 and I_1 with respect to r

In this appendix some derivations are performed which will be used in appendix A3 to A5. The source of each equation is indicated respectively.

The equation for the conservation of mass may be written from the equation (2.9) as

$$\frac{\rho_1}{\rho_{10}} = \nu_1 = \frac{1}{\lambda \lambda_r \lambda_\theta} . \qquad (A1.1)$$

The equation of the volume additivity constraint from equation (2.22) may be rewritten as

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1. \qquad (A1.2)$$

Substitution of the equation (A1.1) in equation (A1.2) yields

$$\nu_1 = 1 - \frac{\rho_2}{\rho_{20}} . \qquad (A1.3)$$

The equation (2.7) may be rewritten as

$$\rho = \rho_1 + \rho_2 . \qquad (A1.4)$$

The equation (A1.4) may be rewritten with the help of equations (A1.1) and (A1.3) as

$$\rho = (\rho_{10} - \rho_{20}) \nu_1 + \rho_{20} . \qquad (A1.5)$$

Differentiating equations (A1.1),(A1.3) and (A1.5) with respect to the radial coordinate r yields following equations respectively

$$\frac{d\rho_1}{dr} = \rho_{10}\,\frac{d\nu_1}{dr} \quad , \tag{A1.6}$$

$$\frac{d\rho_2}{dr} = -\,\rho_{20}\,\frac{d\nu_1}{dr} \quad , \text{ and} \tag{A1.7}$$

$$\frac{d\rho}{dr} = (\rho_{10} - \rho_{20})\,\frac{d\nu_1}{dr} \quad . \tag{A1.8}$$

Differentiating equation (4.5) with respect to the radial coordinate r yields

$$\begin{aligned} \frac{dI_1}{dr} &= \frac{d}{dr}\left\{ \lambda_r^2 + \lambda_\theta^2\,(1 + \psi^2\,\lambda^2\,R^2) + \lambda^2 \right\} , \\ &= \frac{d}{dr}\left\{ \frac{1}{\lambda^2\,\lambda_\theta^2\,\nu_1^2} + \lambda_\theta^2 + \psi^2\,\lambda^2\,r^2 + \lambda^2 \right\} , \\ &= 2\left\{ -\frac{\lambda_r^2}{\nu_1}\frac{d\nu_1}{dr} - \frac{\lambda_r^2}{\lambda_\theta}\frac{d\lambda_\theta}{dr} + \lambda_\theta\,\frac{d\lambda_\theta}{dr} + \psi^2\,\lambda^2\,r \right\} , \\ &= 2\left\{ -\frac{\lambda_r^2}{\nu_1}\frac{d\nu_1}{dr} + \left\{ -\frac{\lambda_r^2}{\lambda_\theta} + \lambda_\theta \right\}\frac{d\lambda_\theta}{dr} + \psi^2\,\lambda^2\,r \right\} . \end{aligned} \tag{A1.9}$$

$\frac{d\lambda_\theta}{dr}$ is eliminated from equation (A1.9) by using equation (4.20) to yield

$$\frac{dI_1}{dr} = 2\left\{ -\frac{\lambda_r^2}{\nu_1}\frac{d\nu_1}{dr} + \frac{(\lambda_\theta^2 - \lambda_r^2)\,(\lambda_r - \lambda_\theta)}{\lambda_r\,r} + \psi^2\lambda^2 r \right\} . \tag{A1.10}$$

A.2 Differentiation of λ_r with respect to r

The mass balance of solid constituent may be given by the equation

$$\nu_1 = \frac{1}{\lambda\;\lambda_r\;\lambda_\theta} \quad . \tag{A2.1}$$

Differentiating the above equation with respect to the radial coordinate r yields

$$\frac{d\nu_1}{dr} = -\left\{ \lambda_r\,\frac{d\lambda_\theta}{dr} + \lambda_\theta\,\frac{d\lambda_r}{dr} \right\} 1/\,\lambda\,\lambda_\theta^2\,\lambda_r^2 \quad , \tag{A2.2}$$

or

$$\frac{d\nu_1}{dr} = -\left[\lambda_r \frac{d\lambda_\theta}{dr} + \lambda_\theta \frac{d\lambda_r}{dr} \right] \frac{\nu_1}{\lambda_r \lambda_\theta} \quad . \tag{A2.3}$$

$\frac{d\lambda_\theta}{dr}$ is eliminated from the above equation by using equation (4.20) to yield

$$\frac{d\nu_1}{dr} = -\left[\lambda_r \frac{\lambda_\theta (\lambda_r - \lambda_\theta)}{r \lambda_r} + \lambda_\theta \frac{d\lambda_r}{dr} \right] \frac{\nu_1}{\lambda_r \lambda_\theta} \quad . \tag{A2.4}$$

Or

$$\frac{d\nu_1}{dr} = -\left[\frac{(\lambda_r - \lambda_\theta)}{r} + \frac{d\lambda_r}{dr} \right] \frac{\nu_1}{\lambda_r} \quad . \tag{A2.5}$$

The above equation may be rearranged as

$$\frac{d\lambda_r}{dr} = -\left[\lambda_r \left(1 + \frac{r}{\nu_1} \frac{d\nu_1}{dr} \right) - \lambda_\theta \right] \frac{1}{r} \quad . \tag{A2.6}$$

APPENDIX B

Derivation of the equilibrium equation for the solid-fluid mixture

The equilibrium equation of the mixture may be written from equation (4.11) as

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \qquad (B.1)$$

The constitutive equations may be written for the mixture by using equations (3.30) and (3.31) in equation (3.26) as

$$T_{rr} = - p - \rho \rho_2 \frac{dA}{d\rho_2} + 2 \rho A_1 \lambda_r^2 \text{, and} \qquad (B.2)$$

$$T_{\theta\theta} = - p - \rho \rho_2 \frac{dA}{d\rho_2} + 2 \rho A_1 \lambda_\theta^2 (1 + R^2 \psi^2 \lambda^2) . \qquad (B.3)$$

For the case of "Neo-Hookean type" mixture considered, the constitutive equations may be written as

$$T_{rr} = - p - \rho \rho_2 \frac{dA}{d\rho_2} + G \nu_1 \lambda_r^2 \text{, and} \qquad (B.4)$$

$$T_{\theta\theta} = - \rho - \rho \rho_2 \frac{dA}{d\rho_2} + G \nu_1 \lambda_\theta^2 (1 + R^2 \psi^2 \lambda^2), \qquad (B.5)$$

where G is a material constant. Differentiation of equation (B.4) with respect to the radial coordinate r yields

$$\frac{dT_{rr}}{dr} = - \frac{dp}{dr} - \frac{d}{dr}(\rho \rho_2 \frac{dA}{d\rho_2}) - G \lambda_r^2 \frac{d\nu_1}{dr} - \frac{2G}{r\lambda} (\frac{\lambda_r}{\lambda_\theta} - 1) \qquad (B.6)$$

The second term in equation (B.1) may be represented in terms of the stretch ratios and the constitutive constants as

$$\frac{T_{rr} - T_{\theta\theta}}{r} = \frac{1}{r} \left[G \nu_1 (\lambda_r^2 - \lambda_\theta^2 (1 + \psi^2 \lambda^2 R^2)) \right] . \qquad (B.7)$$

Summing equations (B.6) and (B.7) yields the final equilibrium equation for the mixture of an incompressible elastic solid and an ideal fluid as follows

$$- \frac{dp}{dr} - \frac{d}{dr} \left(\rho \, \rho_2 \frac{dA}{d\rho_2} \right) - G \lambda_r^2 \frac{d\nu_1}{dr} - \frac{2G}{r\lambda} \left(\frac{\lambda_r - \lambda_\theta}{\lambda_\theta} \right)$$

$$+ \frac{G\nu_1}{r} \left(\lambda_r^2 - \lambda_\theta^2 \left(1 + \psi^2 \lambda^2 R^2 \right) \right) = 0. \qquad \text{(B.8)}$$

Appendix C

Derivation of equilibrium equation for solid constituent.

The equilibrium equation of the solid from equation (4.9) is rewritten as

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - b_r = 0 \quad , \tag{C.1}$$

Where σ_{rr} and $\sigma_{\theta\theta}$ are appropriate components of partial stress $\underset{\sim}{\sigma}$ of the solid and b_r is the radial component of diffusive body force $\underset{\sim}{b}$ and in the context of the current problem these quantities are given by the following equations, respectively

$$\sigma_{rr} = - p \frac{\rho_1}{\rho_{10}} + G \nu_1 \lambda_r^2 \ , \tag{C.2}$$

$$\sigma_{\theta\theta} = - p \frac{\rho_1}{\rho_{10}} + G \nu_1 \lambda_\theta^2 (1 + \psi^2 \lambda^2 R^2) \ , \text{ and} \tag{C.3}$$

$$b_r = - \frac{p}{\rho_{10}} \frac{d\rho_1}{dr} + \rho_1 \frac{dA}{d\rho_2} \frac{d\rho_2}{dr} - \frac{\rho_2 \nu_1 G}{2 \rho} \frac{dI_1}{dr} \ . \tag{C.4}$$

Differentiation of equation (C.2) with respect to the radial coordinate r yields

$$\frac{d\sigma_{rr}}{dr} = - \frac{\rho_1}{\rho_{10}} \frac{dp}{dr} - \frac{p}{\rho_{10}} \frac{d\rho_1}{dr} - G \lambda_r^2 \frac{d\nu_1}{dr} - \frac{2G}{r\lambda} \left(\frac{\lambda_r}{\lambda_\theta} - 1 \right)$$

(C.5)

The second term of equation (4.1) may be expressed in terms of the stretch ratios and the constitutive constants from equations (C.2) and (C.3) as follows

$$\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{G\nu_1}{r} \left(\lambda_r^2 - \lambda_\theta^2 (1 + \psi^2 \lambda^2 R^2) \right) . \tag{C.6}$$

Substitution of equations (C.4-C.6) in equation (C.1) yields the equilibrium equation of the solid constituent

$$-\nu_1 \frac{dp}{dr} - G \lambda_r^2 \frac{d\nu_1}{dr} - \frac{2G}{r\lambda} \frac{(\lambda_r - \lambda_\theta)}{\lambda_\theta}$$

$$+ \frac{G\nu_1}{r} (\lambda_r^2 - \lambda_\theta^2 (1 + \psi^2 \lambda^2 R^2)) - \rho_1 \frac{dA}{d\rho_2} \frac{d\rho_2}{dr} + \frac{\rho_2 \nu_1 G}{2\rho} \frac{dI_1}{dr} = 0 . \qquad \text{(C.7)}$$

Appendix D

Derivation of governing equation.

The governing equation of the problem of combined extension and torsion of an incompressible, non-linearly elastic swollen cylinder may be obtained from equations (B.8) and (C.7) by eliminating the quantity $\frac{dp}{dr}$ from these equations and the resulting equation may be given as

$$\nu_1 \frac{d}{dr} \left(\rho\ \rho_2 \frac{dA}{d\rho_2}\right) + \alpha(1-\nu_1) + \beta(1-\nu_1) + \gamma = 0\ , \qquad \text{(D.1)}$$

Where variables α, β, γ are defined as below:

$$\alpha = -G\ \lambda_r^2 \frac{d\nu_1}{dr} - \frac{2G}{r\lambda} \frac{(\lambda_r - \lambda_\theta)}{\lambda_\theta}\ , \qquad \text{(D.2)}$$

$$\beta = \frac{G\nu_1}{r} \left(\lambda_r^2 - \lambda_\theta^2 \left(1 + \psi^2 \lambda^2 R^2\right)\right)\ , \text{ and} \qquad \text{(D.3)}$$

$$\gamma = -\rho_1 \frac{dA}{d\rho_2} \frac{d\rho_2}{dr} + \frac{\rho_2\ \nu_1\ G}{2\rho} \frac{dI_1}{dr}\ . \qquad \text{(D.4)}$$

The explicit form of the Helmholtz free energy function A, defined per unit mass of the mixture, is used to find an explicit form of the governing equation. The Helmholtz free energy function is rewritten here from equation (4.23) as

$$A = \frac{\nu_1}{\rho} \left[\frac{\hat{R}T\rho_{10}}{2M_c} (I_1 - 3) + \frac{\hat{R}T}{V_1} \left[\frac{1-\nu_1}{\nu_1} \ln(1-\nu_1) + \chi(1-\nu_1) \right]\right]. \qquad \text{(D.5)}$$

A lengthy derivation of $\frac{dA}{d\rho_2}$ from equation (D.5) is omitted which when used in equation (D.1-D.4) along with expression for $\frac{dI_1}{dr}$ from appendix A1 yields the governing equation as

$$\frac{R\lambda_\theta}{\lambda_r}\frac{d\lambda_r}{dR} = -\frac{(\lambda_r-\lambda_\theta)\left[K(2\chi - \frac{1}{1-\nu_1})\nu_1 - \lambda_r\lambda_\theta\right] + \psi^2\lambda^2R^2\lambda_\theta^2\lambda_r}{K\ (2\chi - \frac{1}{1-\nu_1})\nu_1 - \lambda_r^2} \tag{D.6}$$

where,

$$K = \frac{M_c}{\rho_{10}V_1}.$$

FIGURES

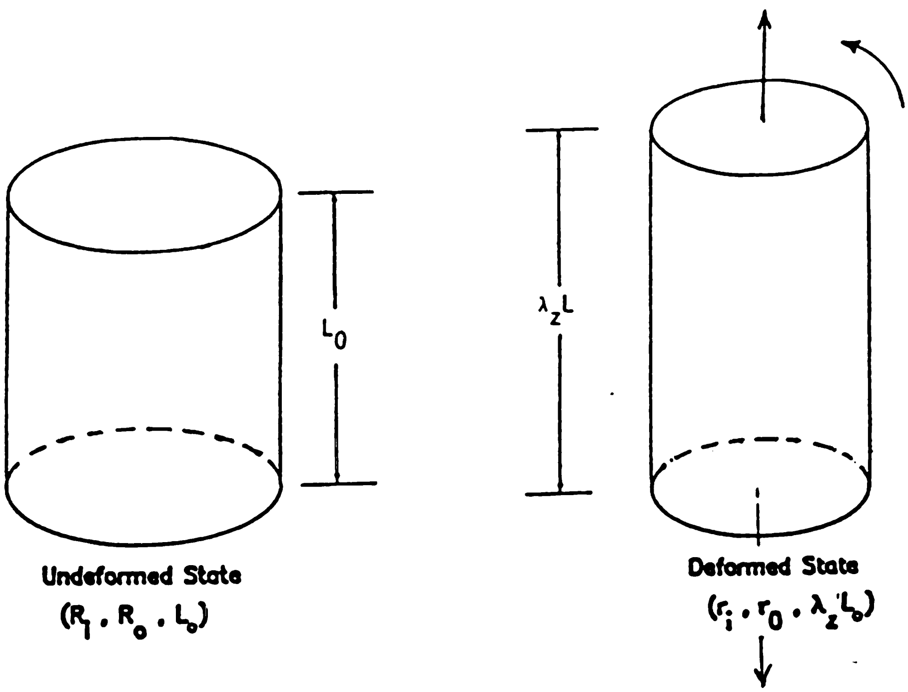

Figure 1: **Extension and Torsion of a Cylinder**

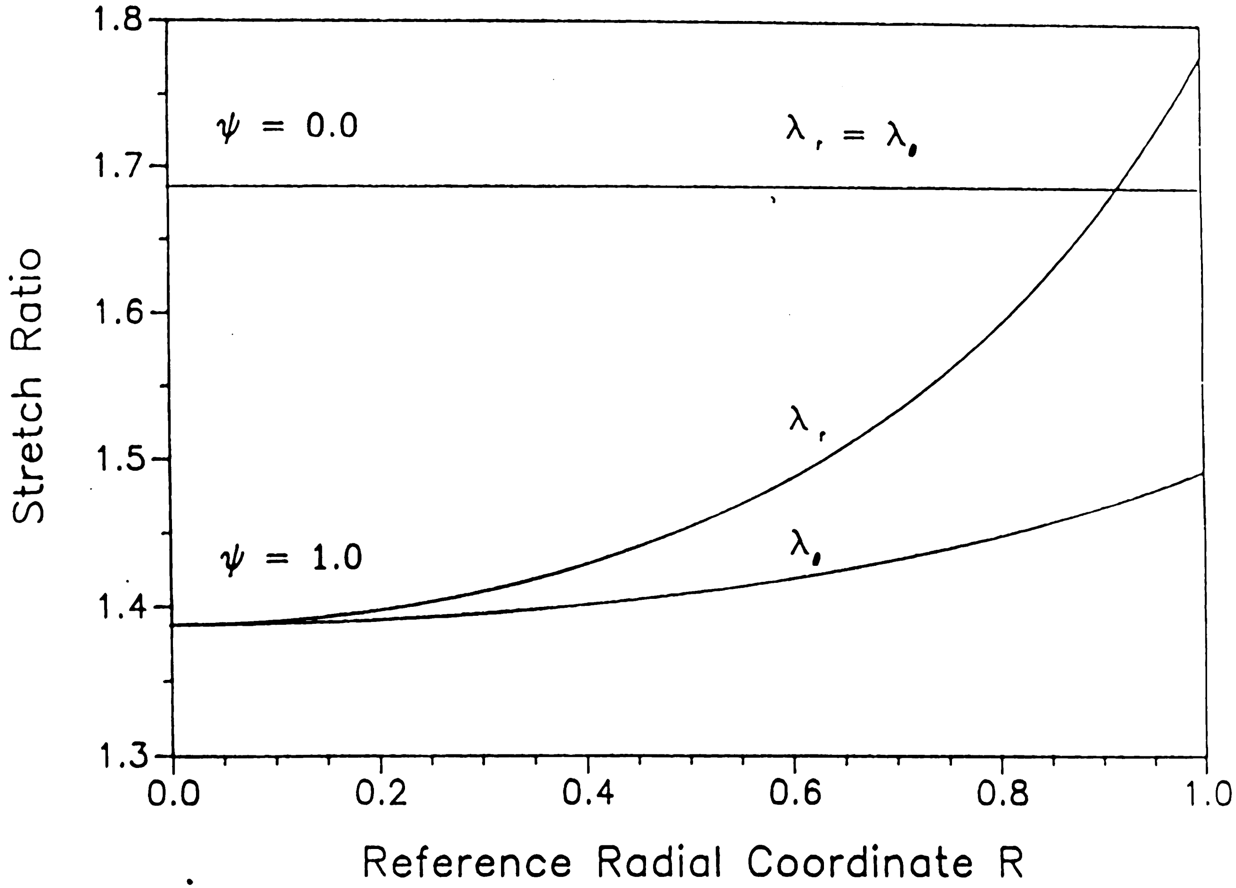

Figure 2: Variation of stretch ratios with the reference radial coordinate

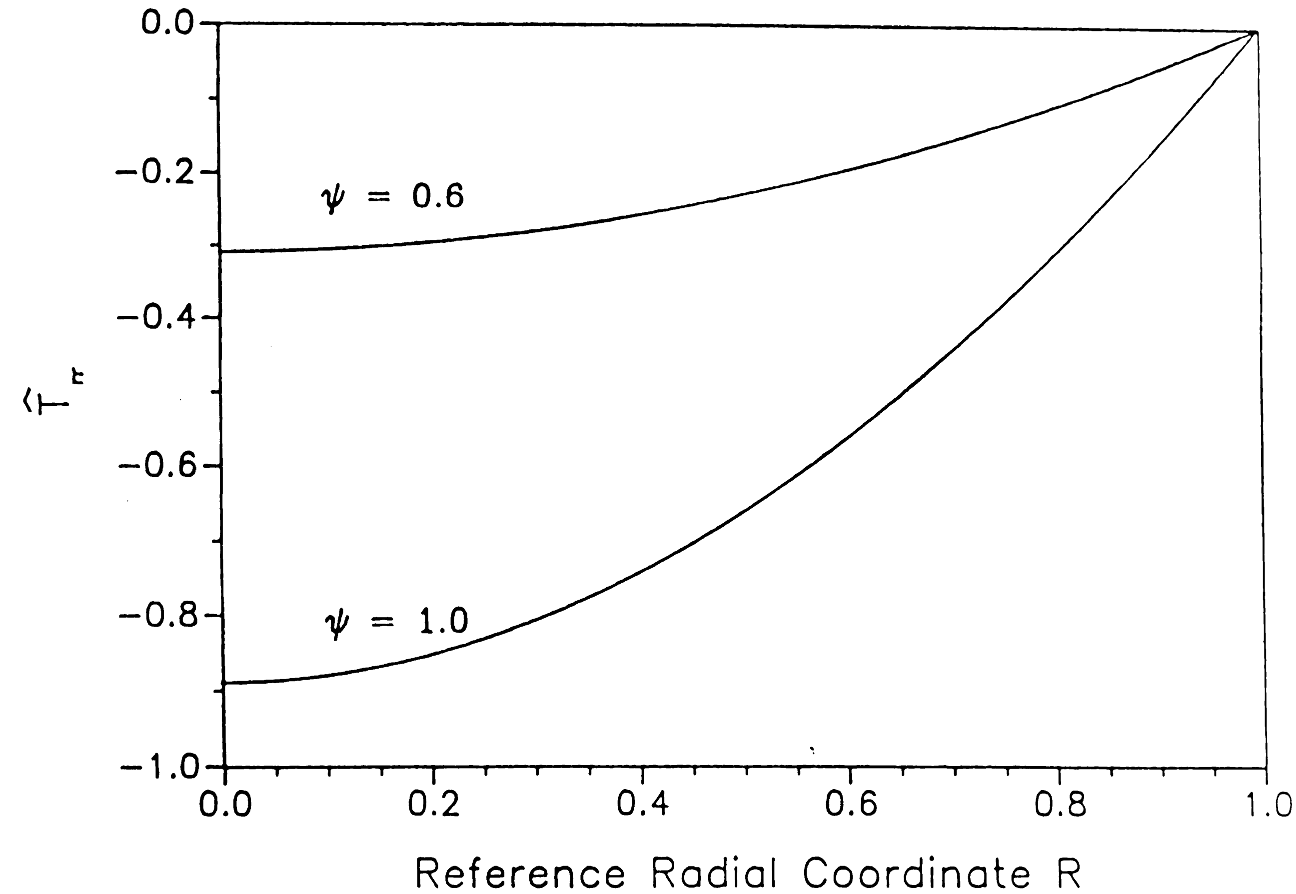

Figure 3: Variation of the radial stress with the reference radial coordinate

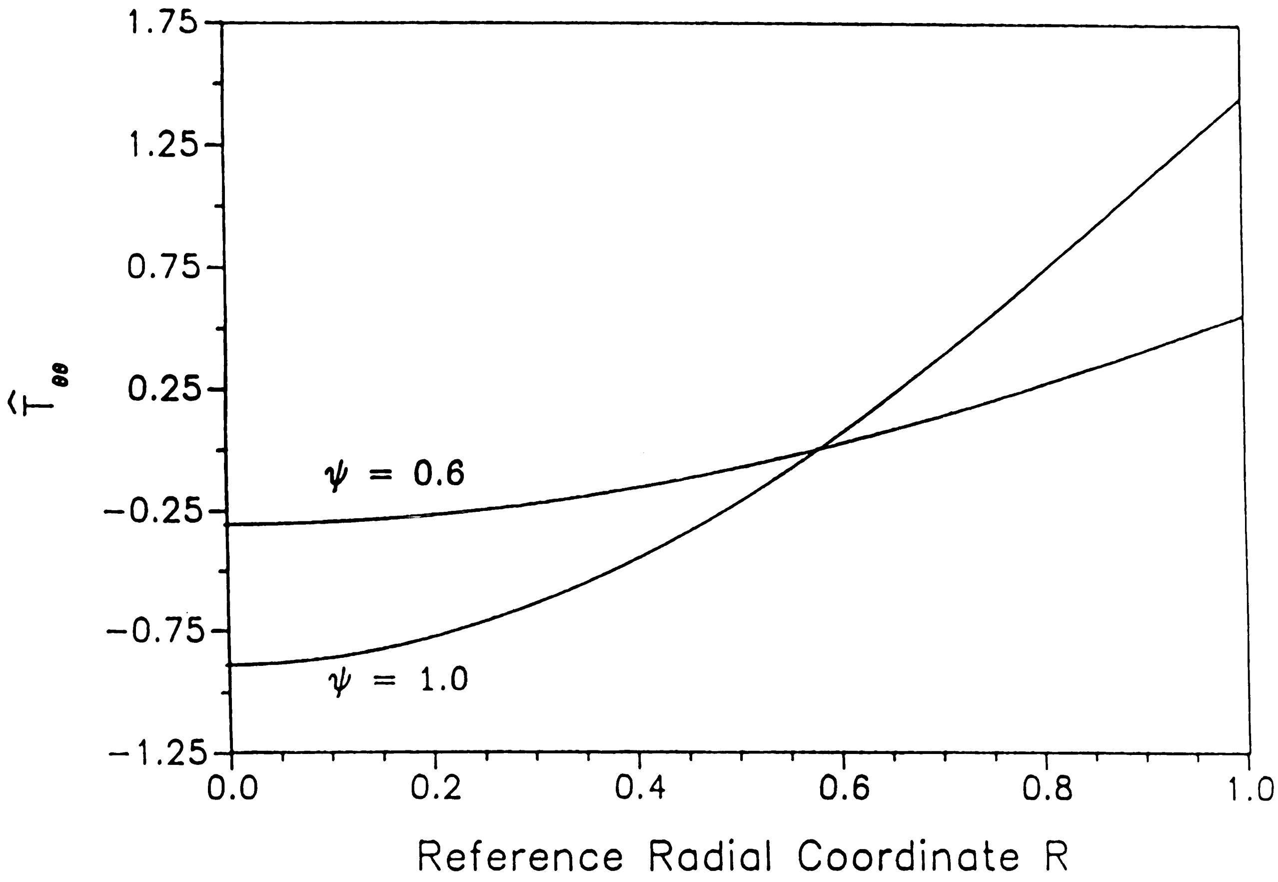

Figure 4: Variation of the circumferential stress with the reference radial coordinate

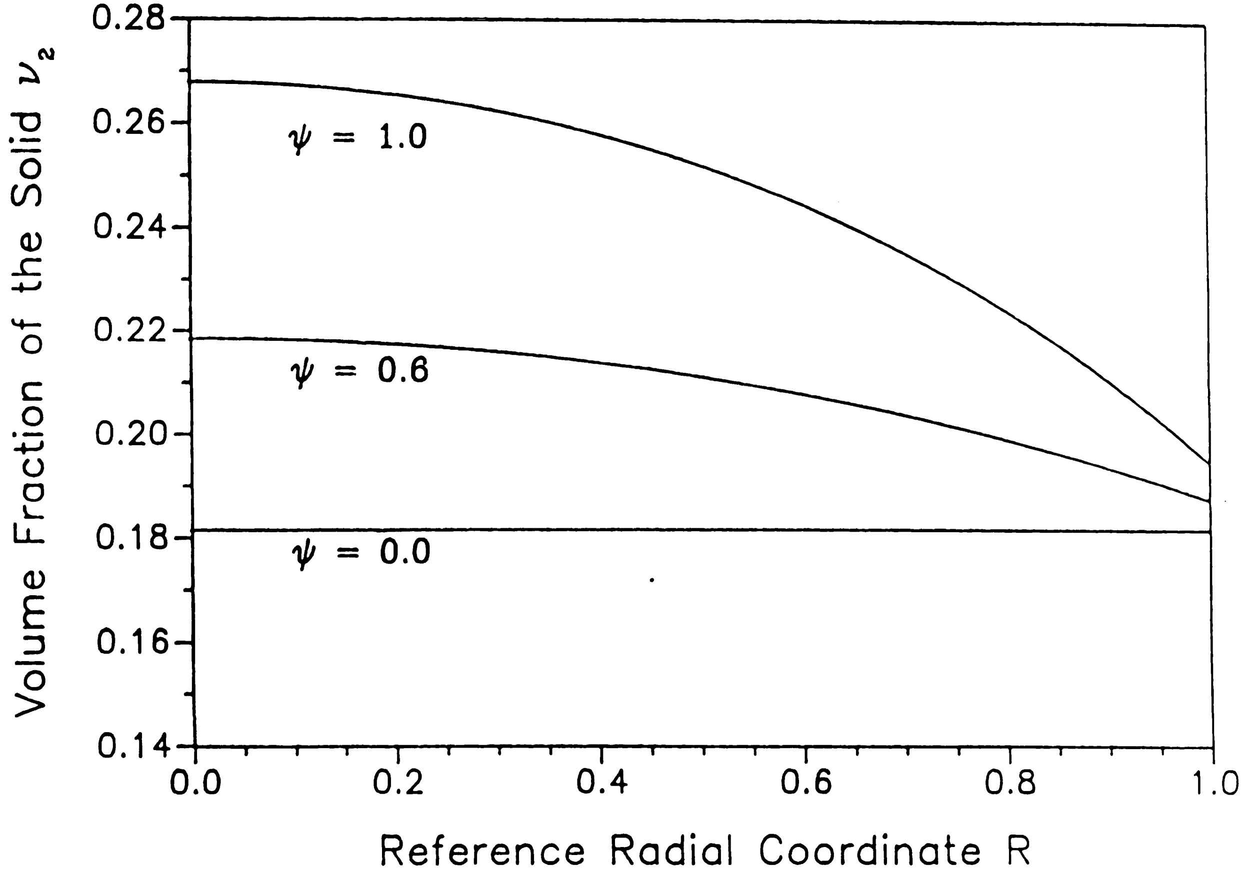

Figure 5: Variation of volume fraction of the solid with the reference radial coordinate

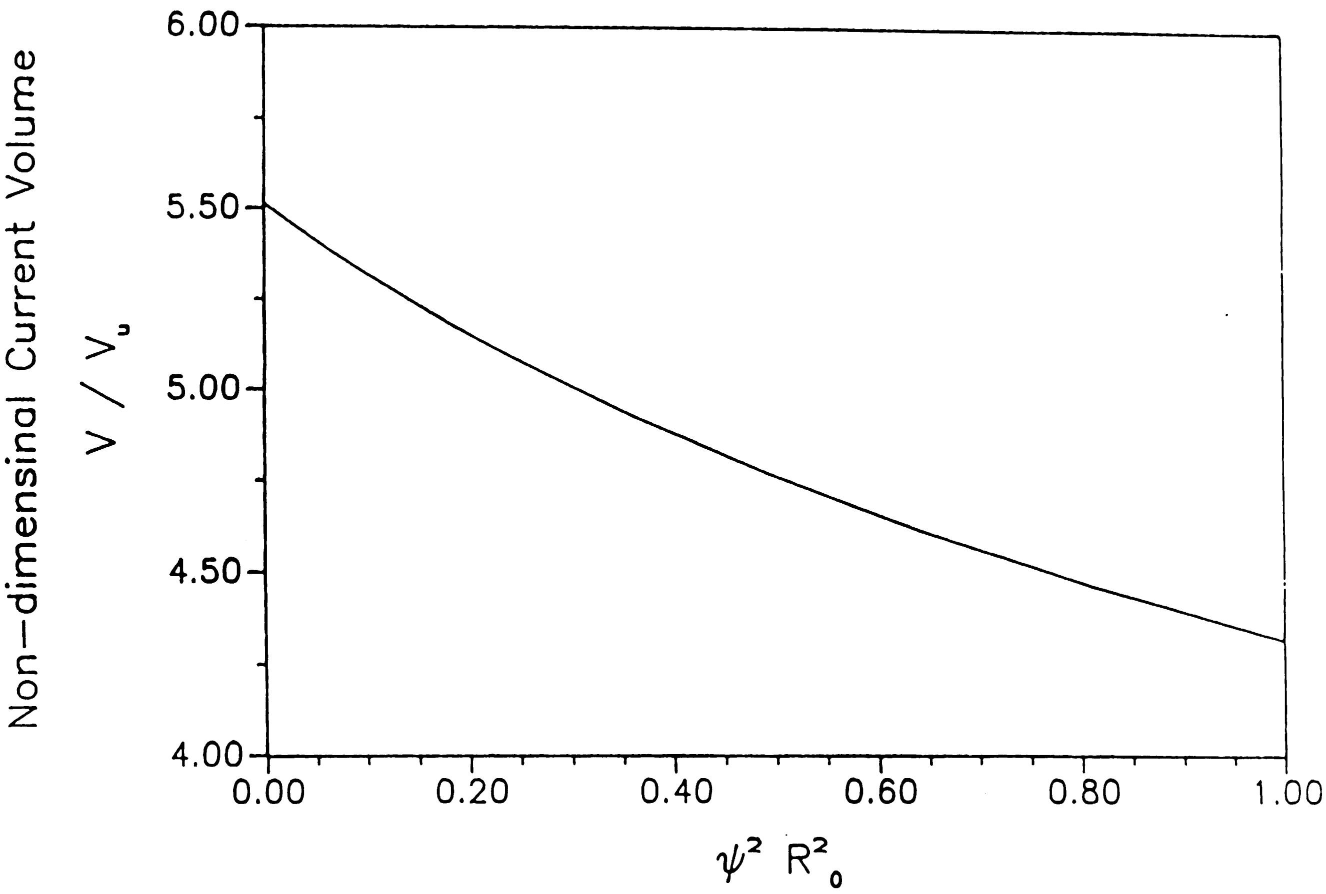

Figure 6: Variation of the current volume with twisting

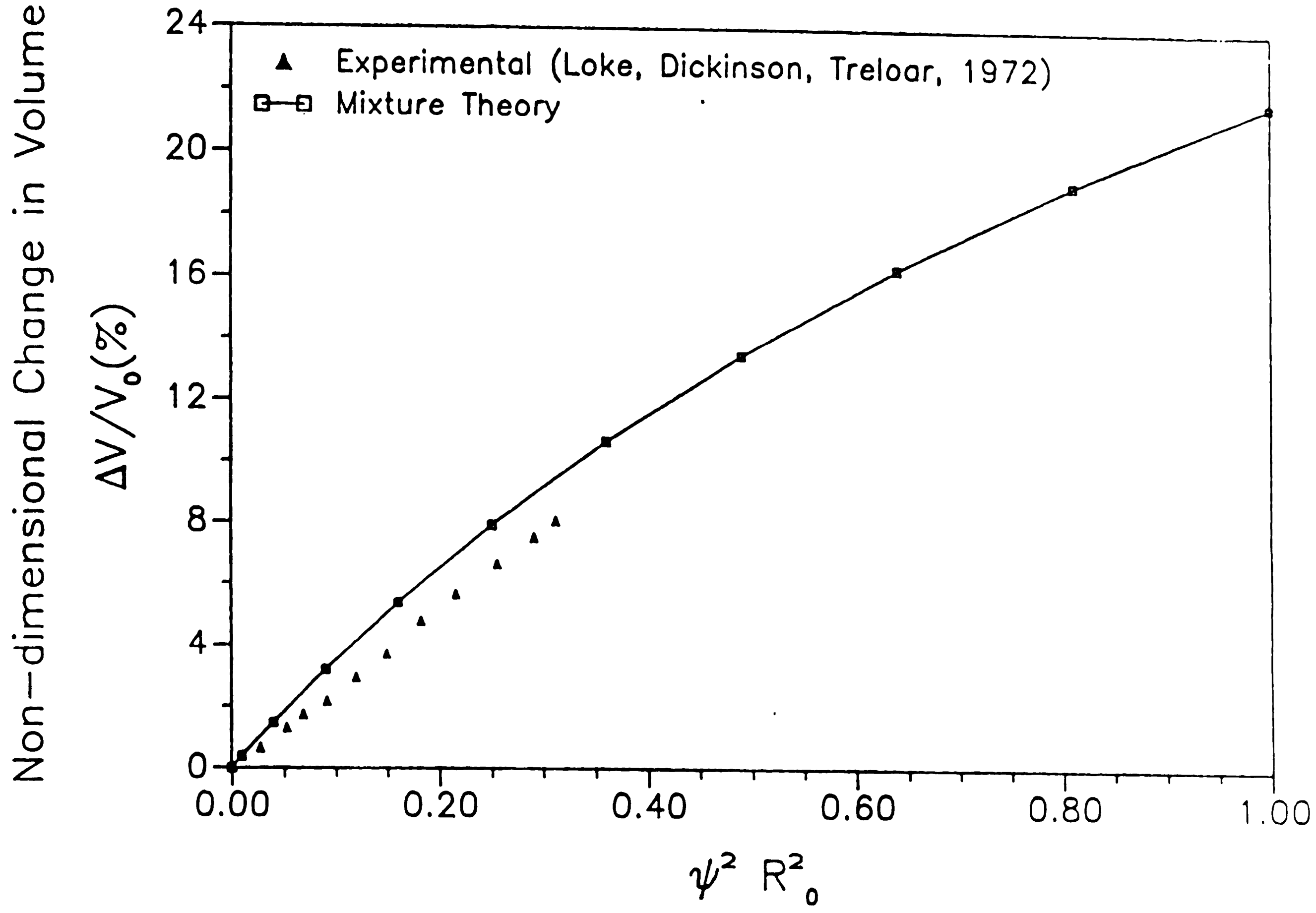

Figure 7: Change in volume with twisting

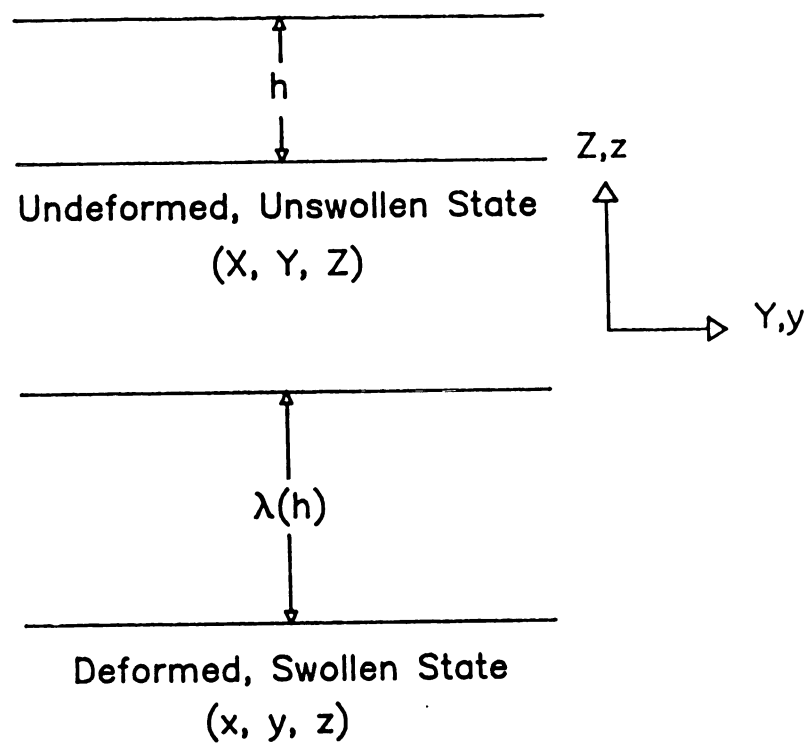

Figure 8: Uniaxial Extension of Mixture Slab

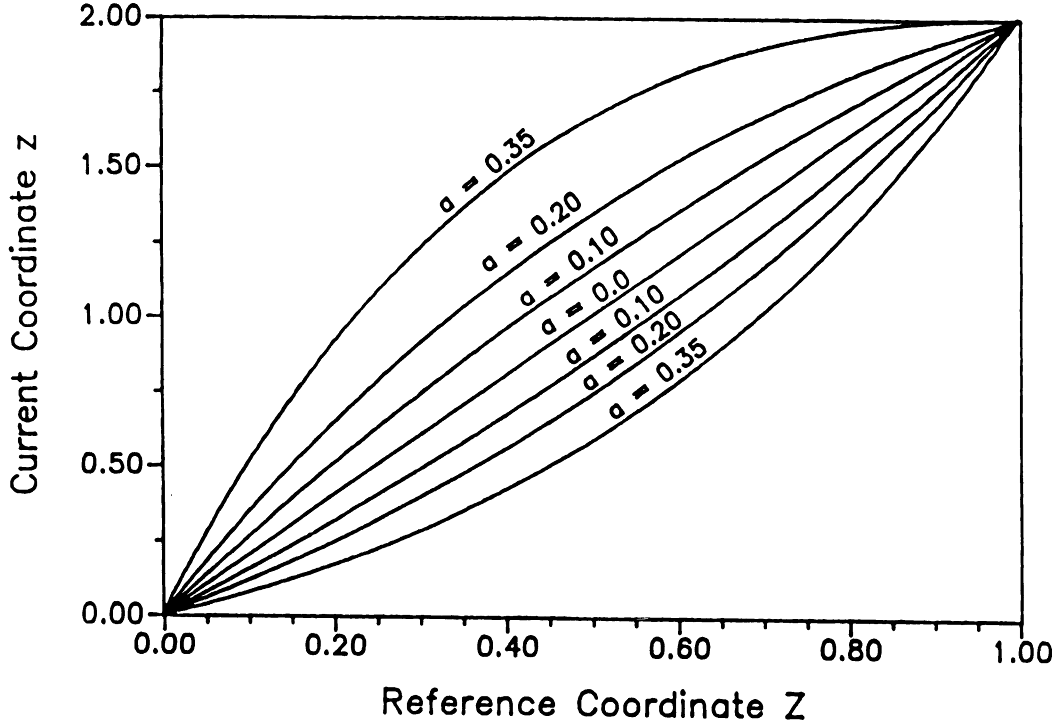

Figure 9: Variation of deformation along the thickness of the layer.

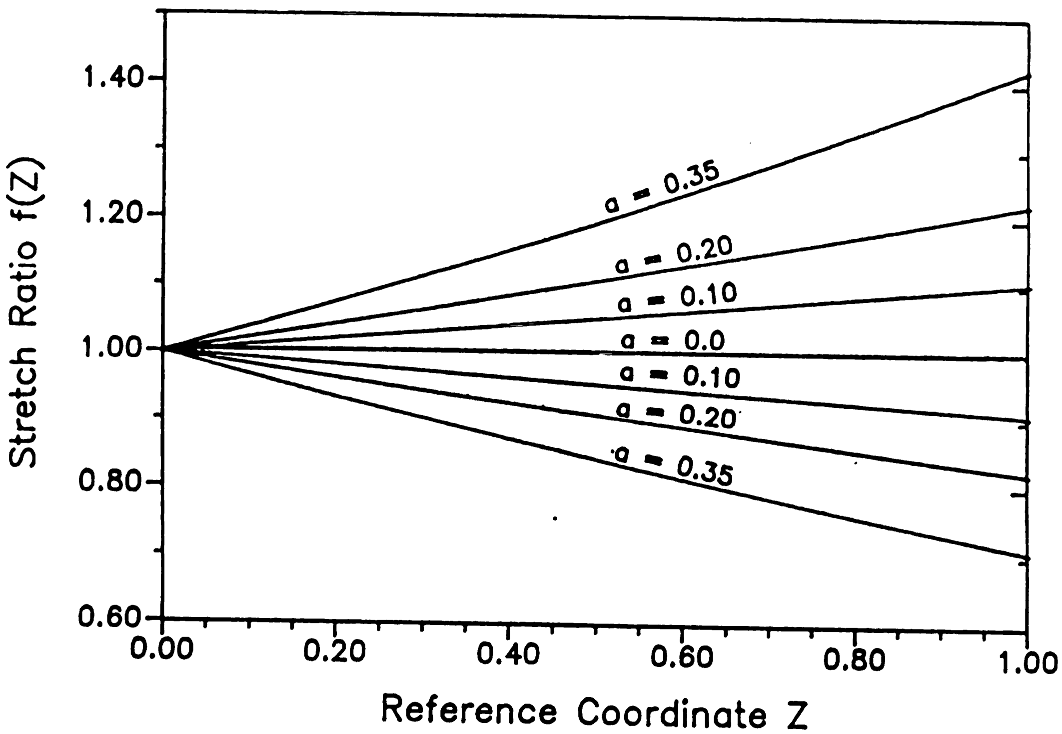

Figure 10: Variation of stretch ratio f(Z) as a function of reference coordinate Z

BIBLIOGRAPHY

BIBLIOGRAPHY

[1]. Atkin, R.J. and Craine, R.E., Continuum Theories of Mixtures: Basic Theory and Historical Development, *Q. J. Mech. Appl. Math.*, **XXIX**, (2), 209, (1976).

[2]. Bowen, R.M., *Continuum Physics*, (ed. Eringen, A.C.) **3**, Academic Press, (1975).

[3]. Bedford, A. and Drumheller, D.S., Theory of Immiscible and Structured Mixtures, *Int. J. Engineering Science*, **21**, 863 (1983).

[4]. Passman, S., Nunziato, J., and Walsh, E.K., A Theory of Multi-Phase Mixtures, *Sandia Report*, SAND 02-2261, (1983).

[5] Patel, D. and Vaishnav, R. N., *Basic Hemo-dynamics and its Role in Disease Processes*, University Park Press, Baltimore, (1980).

[6] Fillip, R. P., *Ultrafiltration, Filtration: Principles and Practices, Ptl, pp 475-518, (1977).*

[7] Misra, J. C. and Singh, S. I., A Large Deformation Analysis for Aortic Walls Under A Physiological Loading, Int. J. Engng. Sci., Vol. 21, No. 10, pp. 1193-1202, (1983).

[8]. Crochet, M.J. and Naghdi P.M., On Constitutive Equations for Flow of Fluid Through an Elastic Solid, Int. J. Engng. Sci., **4**, 383, (1966).

[9]. Green, A.E. and Steel T.R., Constitutive Equations for Interacting Continua, *Int. J. Engng. Sci.*, **4**, 483, (1966).

[10]. Fick, A., *Ann. der Phys.*, **94**, 59, (1955).

[11]. Jost, W., *Diffusion in Solids, Liquids, and Gases*, New York and London, Academic Press, (1960).

[12]. Shewman, P.G., *Diffusion in Solids*, New York, McGraw Hill, (1963).

[13]. Bird, R.B., Stewart, W.E., and Lightfoot, E.N., *Transport Phenomena,* New York and London, Wiley & Sons, Inc., (1960).

[14]. Scheidegger, A.E., *Hydro-dynamics in Porous Media , Handbuch der Physick,* III/2, Springer-Verlag, Berline, (1963).

[15]. Yasuda, H., Stammette, V., Frisch, H.L. and Peterein, *Makromolek Chem.,* **73**, 188, (1964).

[16]. Barrer, R.M., *Diffusion In and Through Solids,* New York: MacMillan, (1963).

[17]. Treloar, L.R.G., *The Physics of Rubber Elasticity.* Oxford University Press, London, (1975).

[18]. Paul, D.R. and Ebra-Lima, O.M., Pressure-induced Diffusion of an Organic Liquid Through a Highly Swollen Polymer Membrane, *J. Appl. Polymer Sci.* 14, **2201**, (1970).

[19]. Atkin, R.J. and Craine, R.E., Continuum Theories of Mixtures: Applications. *J. Inst. Maths Applies,* (1976).

[20]. Crochet, M.J. and Naghdi, P.M., Small Motions Superposed on Large Static Deformations in Porous Media, *Acta Mech.* **4**, 22, (1967).

[21]. Mills, N. and Steel, T.R., Non-linear Deformations of a Mixture of Two Isotropic Elastic Solids, *Acta Mech.* **9**, 229, (1970).

[22]. Craine, R.E., Green, A.E., and Naghdi, P.M., A Mixture of Viscous Elastic Materials with Different Constituent Temperature, *Quart. Journ. Mech. and Applied Math.,* **23**, 171, (1970).

[23]. Shi, J.J., Wineman, A.S., and Rajagopal, K.R., Application of the Theory of Interacting Continua to the Diffusion of a Fluid Through a Non-linear Elastic Medium. *Int. J. Engineering Science,* **9**, 871, (1981).

[24]. Rajagopal, K.R., Shi, J.J., and Wineman, A.S., The Diffusion of a Fluid Through a Highly Elastic Spherical Membrane, *Int. J. Eng. Science,* **21**, 1171, (1983).

[25]. Rajagopal, K.R., Wineman, A.S., and Gandhi, M.V., On Boundary Conditions for a Certain Class of Problems in Mixture Theory, *Int. J. Engineering Science,* **24** (8), 1453, (1986).

[26]. Gandhi, M.V., Rajagopal, K.R., and Wineman, A.S., Some Non-linear Diffusion Problems within the Context of the Theory of Interacting Continua, *Int. J. Engineering Science*, **25** (11/12), 1441, (1987).

[27]. Gandhi, M.V. and Usman, M. The Flexure Problem in the Context of the Theory of Interacting Continua, *SES Paper No. ESP 24-87016*, presented at the 24th Annual Technical Meeting of the Society of Engineering Science, Salt Lake City, Utah, September 1987.

[28]. Loke, K.M., Dickinson, M., and Treloar, L.R.G., Swelling of a Rubber Cylinder in Torsion: Part 2. Experimental, *Polymer*, **13**, 203, (1972).

[29]. Treloar, L.R.G., Swelling of a Rubber Cylinder in Torsion:Part 1. Theory, *Polymer*, **13**, 195, (1972).

[30]. Rajagopal, K.R. and Wineman, A.S., New Exact Solutions in Non-linear Elasticity, *Int. J. Engineering Science*, **23** (2), 217, (1985).

[31]. Green, A.E., and Naghdi, P.M., On Basic Equations for Mixtures, *Q.J. Mech. Appl. Math.*, **22** (4), 427 (1969).

[32]. Mills, N., Incompressible Mixtures of Newtonian Fluids, *Int. J. Eng. Science*. **4**, 97 (1966).

[33]. Gandhi, M.V., Rajagopal, K.R., and Wineman, A.S., A Universal Relation in Torsion for a Mixture of Solid and Fluid, *Journal of Elasticity*, **15** (2), 155 (1985).