SYMBOLIC, ALGEBRAIC, AND NUMERIC SOLUTIONS TO HEAT CONDUCTION PROBLEMS USING GREEN'S FUNCTIONS

By

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ABSTRACT

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Symbolic calculations that involve tedious, error-prone evaluation have plagued the scientist and engineer for many centuries. A new tool called computer algebra can be used to evaluate complex mathematical operations, such as differentiation and integration, which can be repetitious when applied to partial differential equations. Symbolic results from computer algebra systems can offer insight to problems which numerical results lack.

Symbolic manipulation of expressions and operations are used extensively in this thesis. A new technique, using computer algebra, for the symbolic solution of heat diffusion-type problems is examined. The new technique involves the Green's function approach and uses Laplace transforms and separation of variables for determining appropriate Green's functions. Data bases of Green's functions and integrals are used to speed up the calculation time of solutions.

A systematic and orderly procedure for developing Green's functions which are computationally efficient for small dimensionless times is examined. The small time Green's functions are used in a partitioning scheme which accelerates the evaluation time of the symbolic solutions.

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Two computer programs are presented that symbolically calculates temperature distributions for a limited number of heat transfer problems. The one dimensional program called CANSS generated solutions for various types of boundary conditions, initial conditions, and volume energy heat sources. The two dimensional program called CANSS2D generates temperature distributions for boundary conditions of the zeroth, first and second kinds. The temperature distributions of examples presented in this thesis match solutions found in the literature and are partitioned in time to increase evaluation efficiency.

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Last, to my wife, Mary Ellen, the author dedicates this dissertation for her unflagging confidence and optimism in me. Through her, this dissertation was made possible.

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GREEK SY

α
β
ϕ
ϵ
λ

LIST OF SYMBOLS

a	Parameter in Equation (3.2.37)
b	Thin Film Thickness
c	Specific Heat
d	Parameter in Equation (3.2.37)
e	Exponential Function
f	Forcing Function
g	Volume Energy Heat Source
h	Heat Transfer Coefficient
i	Index
j	Index
k	Thermal Conductivity
m	Index
n	Index
p	Laplace Transform Parameter
q	Heat Flux
r	Spatial Coordinate
s and s'	Number of Boundaries
t	Time
u	Instantaneous Source Solution
v	Volume Element
w	Laplace Solution
x,y,z	Length Variables
A	Function for Equation (3.2.29)
B	Function for Equation (3.2.29)
C	Constants
D	Constants
F	Initial Condition
G	Green's Function
H	Heavyside Function
I	Index for Numbering System
J	Index for Numbering System
K	Kernel Function
L	Length of Finite Slab
M	Matrix in Equation (3.2.20)
N	Matrix in Equation (3.2.20)
Q	Heat Source Strength
R	Region of Space
S	Surface Boundary
T	Temperature
W	Transformation Variable
X,Y,Z	Numbering System Parameters

GREEK SYMBOLS

α	Thermal Diffusivity
β	Eigenvalue
ϕ	Start-up Time
ϵ	Small Number
λ	Dummy Variable

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γ
θ
τ
ξ
δ
∇
∞
∂
ℓ
Γ
π

Superscr

$*$
$+$

Subscript

L

$\surd$	Square Root Sign
ρ	Density
γ	Constant
θ	Dummy Variable
τ	Dummy Time Variable
ξ	Constant
δ	Delta Function
∇	Laplacian Operator
∞	Infinity
∂	Partial Derivative Sign
ℓ	Large Time Index
Γ	Gamma Function
π	Pi (3.1415....)

Superscripts

*	Dimensionless
+	Dimensionless

Subscripts

L	Normalized With Respect to L

CHAPTER 1

INTRODUCTION

The first introduction of a child to mathematics is frequently symbolic. The child learns that symbols have meaning - such as, two crossed lines mean the numbers are added and two horizontal lines means what follows is the sum total. Calculus brings the onset of the numerical approach and the acceptance of symbols becomes lost in the quest for approximate solutions.

Computers, long used to reduce the repetition and increase the accuracy of numerical calculation, can now do the same for symbolic analysis. Computer algebra systems are used to define new mathematical concepts and increase the speed of repetitive symbolic analysis previously performed by hand. Symbolic analysis can give insight to the structure of the physics of problems that purely numerical solutions miss.

Computer algebra programs work best on algorithmic representations of systems that involve complex mathematical operations, such as integration or differentiation, but are straightforward. Symbolic calculations of this type are said to be "fierce". At another extreme, computer algebra programs also work well on systems that represent a broad class of problems and can be expressed using an algebraic structure. The calculations for this case are not "fierce", but tedious and repetitive.

Symbolic manipulation, or computer algebra, was brought into the public domain by a program called MACSYMA [The MATHLAB Group, 1983] in the early 70's. Since that time, many computer algebra systems have been brought to the market. The early versions of computer algebra programs were expensive, memory intensive, and dependent on specific computer hardware. Today, the computer algebra software is more user friendly, less machine dependent, and less expensive.

The introduction of symbolic software compatible with micro-computers, personal computers, and, more recently, hand held programmable calculators encourages the use of computer algebra and will ultimately lead to educational programs in symbolic manipulation. Computer algebra can be used as an educational tool in many branches of science including calculus, physics, chemistry, and engineering. It is now possible for most universities to offer computer algebra to students as a learning aid.

This thesis presents a study of the application of computer algebra techniques to a field of engineering and can be described as computer aided symbolic engineering (CASE). Symbolic computer programs use mathematical concepts to describe, analyze, and evaluate the mathematical operations that formerly required pencil-and-paper analysis. Symbolic methods improve accuracy by evaluating solution in closed form. The accuracy of the solutions obtained using computer algebra techniques is dependent on the accuracy of the parameters or variables input to the problem thus reducing or eliminating human error caused by evaluating repetitive algebraic processes. The pattern revealed by a group of variables or parameters will cover an infinite number of cases. Computer algebra can be used in the areas of fluid dynamics, to solve large full symbolic matrices, finite element methods, to generate accurate trial functions, and machine dynamics to name a few. The CASE field will continue to grow as more uses are discovered for computer algebra.

Hayes and Michie [1984] state an expert system applies a structured set of rules to a data base to evaluate input and calculate output, and is a rudimentary form of artificial intelligence (AI). The expert system discussed in this thesis employes a rule based procedure (inference engine) along with known facts and assertions (knowledge base) and is used to treat problems in heat diffusion. Figure 1.1 represents the structure of a expert system.

An expert system called computer algebraic, numeric, and symbolic system (CANSS) applies symbolic manipulation to problems in mathematical physics. The CANSS program developed in this thesis has the capacity to solve heat diffusion problems not found in the literature. The CANSS program represents a new technique in CASE for obtaining temperature distributions for linear, multi-dimensional, transient heat diffusion problems by the application of symbolic manipulation computer software. As a tool for heat transfer engineers in the 80's and 90's, symbolic analysis can be compared to the introduction of finite element techniques to transient heat transfer in the early 70's.

The CANSS program uses a Green's function approach and a symbolic manipulation program called SMP [1983] to generate analytical temperature distributions for problems that involve various linear boundary conditions, initial conditions and volume energy heat sources. The technique can calculate temperature distributions for some nonlinear heat diffusion problems by using a transformation found in Ozisik [1980, pg. 440] using the Kirchhoff transformation. Some nonlinear diffusion problems can be treated as linear for special combinations of the physical properties.

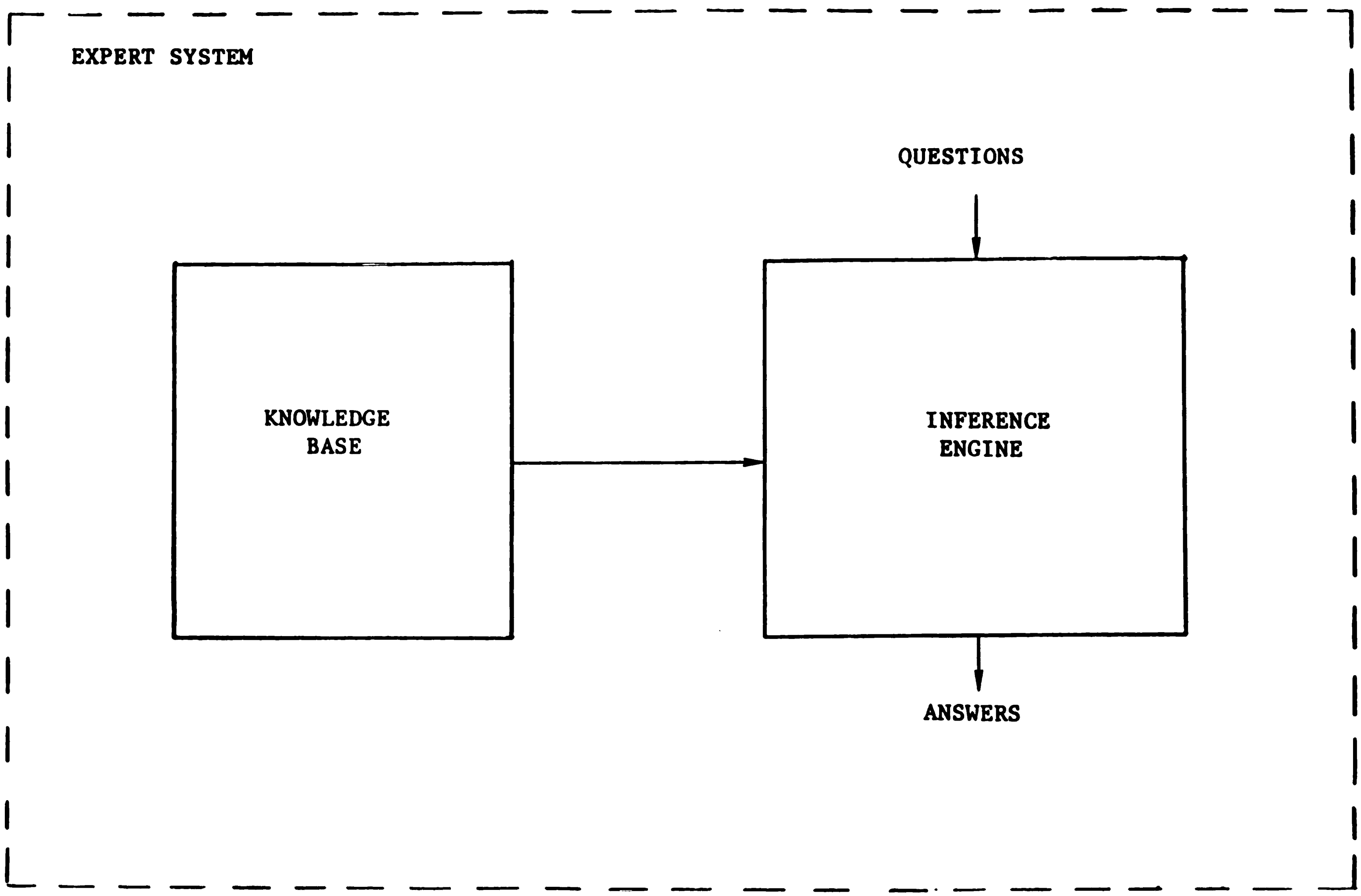

Figure 1.1 Structure of an Expert System

Transient temperature distributions for finite, infinite, and semi-infinite single or multi-dimensional bodies are typically calculated using finite difference or finite element methods. These methods generate information about the temperature for a finite set of points or nodes and times. Care must be taken with the distribution size of the points or mesh and the time step to insure stability of the solution. Solutions obtained by numerical techniques must be re-calculated for the entire domain of space and time when one or more input parameters change. Simple substitution of new parameters into the symbolic form of the solution result in new solutions without extensive re-calculation.

Computer algebra methods offer accurate symbolic results that can act as test cases for the calibration of purely numerical techniques such as finite element or finite difference. Specifically, symbolic solutions to diffusion problems can complement the numerical solutions that are normally used in heat transfer analysis. Symbolic solutions to basic geometries can be included as trial functions in the finite element and finite difference methods. Symbolic temperature distributions can be also used as a starting point for numerical procedures or for procedures where a poor initial guess will lead to solutions that do not converge or oscillate.

1.1 Previous Work

The Green's function approach to the solution of partial differential equations of heat diffusion is well documented. Morse and Feshbach [1953], and more recently, Ozisik [1980] and Beck [1984a] provide a structure for determining temperature distributions using the Green's function approach.

Walters [1949] uses the Green's function approach to solve transient heat conduction and vibration problems analytically and, more recently, Hassanein and Kulcinski [1984] use the Green's function approach to examine the rapid heating of fusion reactor walls. Hassanein and Kulcinski reported comparisons of the Green's function approach to the finite difference method. They report the Green's function approach requires more analytical calculations than the finite element method but the time step may be larger and the calculations are more straightforward. The analytical calculations in their study could be calculated by a computer algebra system to decrease the complexity of the problem.

The Green's function approach has been used to generate influence, or kernel, functions for the unsteady surface element method developed by Keltner and Beck [1981]. The unsteady surface element method splits up the boundary and uses the influence functions to obtain solutions for both linear and nonlinear boundary conditions. Cole [1986] used the Green's function approach to determine some influence functions for conjugate heat transfer problems in the unsteady surface element method.

The Green's function is geometry specific - a different Green's function is needed for each geometry and each set of boundary conditions. Many references and texts such as Butkovski [1982], Stakgold [1979] and Beck [1984b] include lists of the Green's functions. Beck and Litkouhi [1985] proposed a numbering system, which is used extensively in this thesis, to generate a data base for the Green's functions.

Numerically inefficient solutions to heat and mass diffusion problems have been the bane of the heat transfer engineer for many years. Miller and Gordon [1937] report the solutions obtained using the traditional methods of Fourier series, while mathematically acceptable, are inefficient for small regions of time because of the slow rate of convergence. Aizen, et. al., [1971] show additional methods of increasing the speed of convergence for certain problems in heat diffusion.

The Green's function approach is well suited for symbolic calculation because it is algorithmic and uses operations not typically found in numerical analysis. The capability for integration and differentiation of the computer algebra systems make them the unique vehicle for calculating temperature distributions using the Green's function method. Integration and differentiation are the key to the Green's function approach. The algorithmic structure of the approach is used with a computer algebra system to generate new symbolic or numeric solutions.

In a recent paper by Haji-Sheikh and Lakshminarayanan [1986], symbolic analysis is applied to the solution of diffusion type problems through the use of the Galerkin method. The temperature solutions reported in this work are efficient for large dimensionless times and for complex geometrical shapes.

The integration routines in SMP are based on an algorithm first proposed by Risch [1969]. Risch bases his algorithm on the text by Ritt [1949] which describes the integration of expressions in finite terms. The Risch algorithm states that the integral must be represented as an elementary function and the integral solution must also be expressible in elementary functions. An elementary function is a function composed of polynomials, exponentials, and logarithms using only rational and algebraic operations. The Risch algorithm is still in its infancy, but the work of Cherry [1986] and Knowles [1986] continue to expand the functions to which the algorithm is applied. Cherry extended the types of elementary functions to include some logarithmic integrals and Knowles extended Cherry's work to include some exponential integrals and a broader class of logarithmic integrals. Ng [1977] describes procedures based on the Risch algorithm as pattern recognition strategies.

Roach and Steinberg [1984] use symbolic manipulation in the area of computational fluid dynamics. They report the ability of speeding up code development time and the prospect of virtually error-free testing of constitutive equations and difference forms.

Rand [1984] describes the application of a computer algebra program for solving ordinary differential equations, finding eigensolutions to eigenvalue problems, and solving some examples of boundary value problems. Rand reports the application of computer algebra for finding approximate solutions to differential equations that contain small parameter by using perturbation methods. Mathematicians and engineers have been interested in finding closed form expressions for summations. Moenck [1977] describes a method using symbolic manipulation to express the sum of a rational function as a rational function part and a transcendental part involving derivitives of the gamma function.

Char, et. al., [1986] describes the application of computer algebra to undergraduate mathematics curriculum. They have found it feasible to offer courses in computer algebra to large groups of students. The initial findings suggest the introduction of a computer algebra system to undergraduates has met with limited success. They suggest more powerful facilities for integration, a smooth interface to numerical procedures, and a more user friendly interface.

1.2 Thesis Objectives

The first objective of this thesis is to use the computer algebra system SMP [1983] to develop two computer algebra programs that calculate symbolic temperature distributions, one for one-dimensional bodies and the second program for two-dimensional bodies. A unified method of solution based on Green's functions is developed. The symbolic solutions generated by the programs are to be computationally efficient for the whole range of dimensionless times.

The second objective is the generation of symbolic temperature distributions for some heat diffusion problems using the programs mentioned above. Experience needs to be obtained with the capabilities and limitations of such computer programs. The solutions to basic heat transfer problems can be used as kernel functions in numerical techniques such as finite element, finite difference, and boundary element methods. The programs in this thesis are based on the Green's function approach to heat transfer which implies a linear problem model. Litkouhi [1982] shows the distributions developed by the linear Green's function approach can be used in a numerical surface element method to obtain distributions to more physically complex or nonlinear problems. Some temperature distributions that are not available are also generated.

The third objective of this thesis is to investigate a new class of applications of computer algebra systems to mathematical problems in engineering. The programs and examples in this thesis are chosen from heat transfer, but the computer algebra environment is not restricted to the diffusion equation.

The fourth objective of this thesis is to study the need for a structure in the field of interest. Structure in a field of interest means the availability of a formalism for the calculation of solutions and a data base of pertinent information relating to the field. The structure is important because cases with different parameters can be treated with the same formalism. For example, the formalism for solution of a conduction heat transfer problem could be the application of Fourier series analysis and the data base would include solutions to fundamental ordinary differential equations and some basic integrals.

The fifth objective is an examination of the knowledge bases necessary for efficient computation of solutions. Hayes-Roth, et. al., [1983] state that the knowledge base aid the efficiency of an expert system by eliminating "blind alleys", eliminating repetitive calculations, and applying specific information about the problem.

A sixth objective is to use the new concepts of time partitioning by Beck and Keltner [1984] to obtain symbolic solutions to heat transfer problems that converge rapidly for the whole range of dimensionless times. Efficient symbolic solutions can give valuable insight to the physics of the solutions. An investigator can determine how the solution to a specific system would react to a change of input parameters without re-calculating the solution over the entire domain of time and space.

The final objective is to survey areas associated with mathematical physics in engineering for the mathematical structure or formalism that can effectively use the elements of computer algebra.

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1.3 Synopsis of the Thesis

This thesis is divided into six chapters. Chapter 1 is the introduction and gives the motivation for studying heat diffusion using a Green's function approach and computer algebra. Previous work and the scope of the thesis are examined in Chapter 1.

Chapter 2 introduces the mathematical development and structure of the Green' function approach to the solution of heat diffusion problems that appear in this thesis. The partial differential equations and the associated boundary and initial conditions are examined. A procedure for obtaining the Green's functions and a procedure for generating the products of one dimensional Green's functions to obtain the Green's functions for two and three dimensional cases in rectangular coordinates are presented.

Chapter 3 describes a method for obtaining approximate expressions for Green's functions in one dimension using the technique of Laplace transforms. The expressions generated by this method converge quickly for small times. Examples of Green's functions are given which examine the effect of various boundary conditions for semi-infinite bodies. Three example problems are examined for finite, one dimensional bodies using various boundary conditions.

Chapter 4 describes a computer algebra program, called CANSS, that calculates symbolic temperature distributions for semi-infinite and finite bodies in one dimension. Various types of boundary conditions, initial conditions, and volume energy heat sources may be applied to the bodies using the CANSS algorithm. Three example problems are examined along with some important integrals that relate to one dimensional problems. A flowchart/example is presented.

Chapter 5 describes a symbolic computer algorithm, called CANSS2D, that generates symbolic temperature distributions for two dimensional plates. Two example problems are examined along with some special integrals that occur during the calculation. A discussion of time partitioning in two and three dimensions concludes the chapter.

Chapter 6 presents the conclusions and summary. This chapter also offer suggestions for the extension of this thesis.

CHAPTER 2

GREEN'S FUNCTION FORMULATION

2.1 Introduction

This chapter traces the development of the Green's function approach to the solution of linear partial differential equations for heat conduction. This approach has become an accepted solution technique due to the recent work by Greenberg [1971], Ozisik [1980] and Beck [1984b], and the previous work of Morse and Feshbach [1953]. These authors show that the Green's function approach to the solution to heat conduction problems offers both simplicity and structure and gives the engineer and scientist an alternate approach for solving diffusion type problems. The unifying structure implicit in the Green's function approach provides an ideal testing vehicle for demonstrating the use of symbolic computation.

The Green's function approach is simple because of the straightforward manner in which the boundary condition, initial condition and heat generation terms of the solution are generated. The approach has structure due to the use of a single Green's function for the complete solution.

Classical heat conduction theory states that the heat transfer rate, q, is proportional to the temperature gradient in the medium, or,

$$q \sim \frac{\partial T}{\partial r_i} \qquad (2.1.1)$$

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where r_i is a coordinate in the direction of the heat flow in the medium. Inserting a constant of proportionality called the thermal conductivity, k, the constitutive equation that describes the rate of heat flow by conduction in the r_i direction as,

$$q = -k \frac{\partial T}{\partial r_i}. \tag{2.1.2}$$

One consequence of the above equation is that a thermal disturbance on or in the medium is propagated everywhere instantaneously. The constitutive equation is very accurate even though instantaneous diffusion is unrealistic. Maxwell [1867] and, more recently, Vick and Ozisik [1984] modified the constitutive equation by including a relaxation term that includes some start-up time, ϕ, for the initiation of heat flow. The modified constitutive equation,

$$q + \phi \frac{\partial q}{\partial t} = -k \frac{\partial T}{\partial r_i} \tag{2.1.3}$$

was first suggested by Maxwell to account for finite diffusion velocity and shows that the spontaneous release of a finite pulse of energy causes a thermal wave front to move through the body at a finite velocity. The wave of thermal energy dissipates exponentially as it moves through the medium, see Figure 2.1. Other researchers (Vernotte [1958] and Chester [1963]) have used the modified constitutive equation to account for thermal waves in their experiments with helium at temperatures close to absolute zero.

This thesis will not concern itself with the property of the thermal wave that moves through the medium. When the relaxation time ϕ is small, equation (2.1.3) reduces to equation (2.1.2) except for extremely small real times or temperatures near absolute zero.

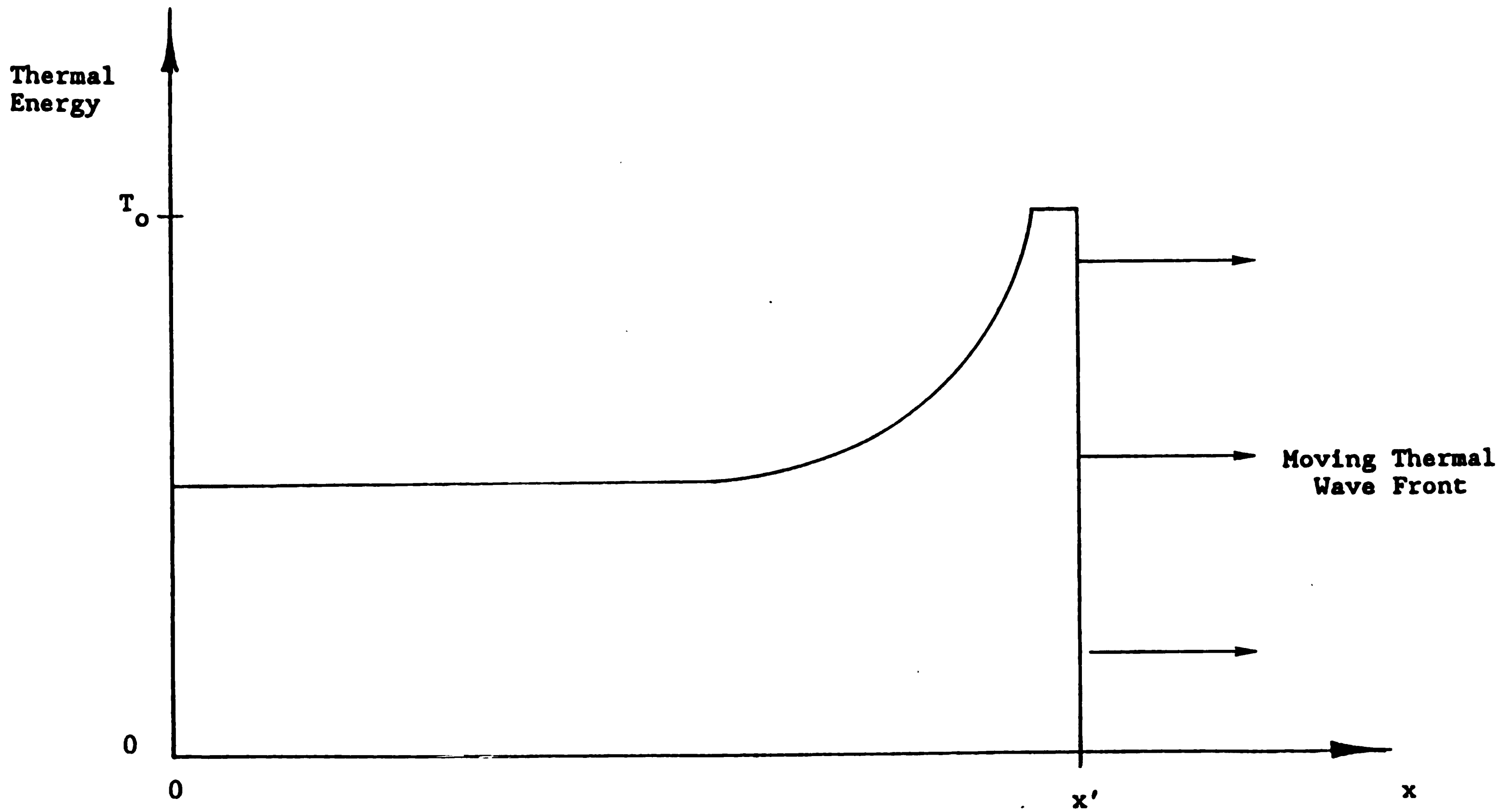

Figure 2.1   Thermal Energy Wave With Exponential Decay.

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$$\nabla^2 = \frac{\partial^2}{\partial \underline{r}^2} ,$$

The solutions to heat conduction problems using the Green's function approach is presented in Section 2.2. The partial differential equations for multi-dimensional heat conduction are developed along with the boundary and initial conditions. In Section 2.3, the concepts of the time-reversed auxiliary Green's function are presented along with a technique to determine the Green's function and multiplying Green's functions to obtain multi-dimensional Green's functions. The structure and formalism of the approach are discussed in Section 2.4. Section 2.5 summarizes and concludes this chapter.

2.2 Mathematical Derivation of Heat Diffusion in a Body

2.2.1 Heat Conduction Equation and Boundary Conditions

The purpose of this section is to discuss the partial differential equation and its boundary and initial conditions used to generate temperature distributions in infinite, semi-infinite and finite bodies. The partial differential equation that describes the transient, multi-dimensional, linear heat conduction in cartesian coordinates is,

$$\nabla^2 \mathbf{T} + \frac{g(\underline{r},t)}{k} - \beta \frac{\partial T}{\partial r_j} + \gamma T = \frac{1}{\alpha} \frac{\partial T}{\partial t}. \qquad (2.2.1)$$

The symbol ∇^2 is the Laplacian operator defined as,

$$\nabla^2 = \frac{\partial^2}{\partial \underline{r}^2}, \qquad (2.2.2)$$

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the symbol $g(\underline{r},t)$ is the internal heat generated in the solid body and $\underline{r}$ is the rectangular coordinate (i.e. x, y, or z). The boundaries are assumed to be parallel to the rectangular coordinates. The symbols α and k are the thermal diffusivity and conductivity, respectively, of the solid body. The diffusion equation is nonlinear when the thermal diffusivity or $\rho\ C_p$ are functions of the temperature. Ozisik [1980, pg. 440] shows that for certain boundary conditions and restrictions, the nonlinear equation can be transformed into a linear equation. For example, if the thermal conductivity can be expressed as,

$$k = 1 + \beta_1 T \tag{2.2.3}$$

and $\rho\ C_p$ is expressed as,

$$\rho\ C_p = 1 + \beta_2 T, \tag{2.2.4}$$

the diffusion equation is nonlinear. If the constants β_1 and β_2 are equal, the diffusion equation can be transformed into a linear equation by the Kirchhoff transformation and the thermal diffusivity becoming a constant.

The term $\beta \frac{\partial T}{\partial r_j}$ represents energy carried by a convective flow in the r_j direction and the term γT could represent generation that is proportional to the local temperature. The terms β and γ are constant. In heat transfer analysis, the γT term may also represent the effect of side losses for a fin.

The nonhomogeneous boundary conditions that are applied to the surfaces are written as,

$$k_i \frac{\partial T}{\partial n_i} + (\rho c b)_i \frac{\partial T}{\partial t} + h_i T = f_i(r_i,t), \tag{2.2.5}$$

where the integer i represents the surface to which the boundary condition is applied and n_i is the outward pointing normal to the i-th surface. The symbol h_i is the convection coefficient and $f_i(r_i,t)$ is the nonhomogeneous forcing function associated with the i-th surface. The k_i symbol will represent the thermal diffusivity of the body, except when a temperature condition is imposed on the surface, k_i is set equal to zero.

An additional term, $(\rho cb)_i \frac{\partial T}{\partial t}$, has been added to the traditional boundary equation that takes into account a thin film occurring at the i-th surface. It could represent a thin film of gold on a glass or silicon substrate or (see Carslaw and Jaeger [1959, pg. 128]) a slab in contact with a well stirred fluid. A laminar sublayer will form across a boundary of a slab placed in a flow of fluid. If the flow velocity of the sublayer can be considered constant, the sublayer can be considered a thin film.

A thin film term acts to hold or store some energy. It is assumed that the thickness of the thin film is small enough so that the temperature at the surface of the film is the same as the temperature at the surface of the body. The term $(\rho cb)_i$ represents the storage capacity of the thin film at the i-th surface.

The initial condition necessary to complete the description for the solution of equation (2.2.1) is,

$$T(\underline{r},0) = F(\underline{r}). \qquad (2.2.6)$$

The heat conduction equation (2.2.1) in the rectangular coordinate system with convection in a direction parallel to the rectangular coordinate directions can be reduced to a more desirable form by defining a new variable (see Ozisik [1980, pg. 75]), $W(\underline{r},t)$, where

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$$T(\underline{r},t) = W(\underline{r},t)\ \exp[-\gamma t]\ \exp[\frac{\beta_1}{2\alpha}\ (x - \frac{\beta_1}{2}\ t)]\ \exp[\frac{\beta_2}{2\alpha}\ (y - \frac{\beta_2}{2}\ t)]\ \cdot$$
$$\exp[\frac{\beta_3}{2\alpha}\ (z - \frac{\beta_3}{2}\ t)] \qquad (2.2.7)$$

Substituting equation (2.2.7) into equation (2.2.1) yields,

$$\nabla^2 W + \frac{g'(\underline{r},t)}{k} = \frac{1}{\alpha}\frac{\partial W}{\partial t} \qquad (2.2.8)$$

where,

$$g'(\underline{r},t) = g(\underline{r},t)\ \exp[-\gamma t]\ \exp[-\frac{\beta_1}{2\alpha}\ (x - \frac{\beta_1}{2}\ t)]\ \exp[-\frac{\beta_2}{2\alpha}\ (y - \frac{\beta_2}{2}\ t)]\ \cdot$$
$$\exp[-\frac{\beta_3}{2\alpha}\ (z - \frac{\beta_3}{2}\ t)] \qquad (2.2.9)$$

represents the heat generation in the solid body.

Equation (2.2.8) is easier to solve than equation (2.2.1), and it represents a broad class of conduction problems that include convective diffusion, heat generation and generation proportional to the local temperature. The transformation given by equation (2.2.7) must be applied to the boundary and initial condition as well as the heat conduction equation. Applying the transformation to the general boundary condition, equation (2.2.5), yields,

$$k_i\frac{\partial W}{\partial n_i} + (\rho cb)_i\frac{\partial W}{\partial t} + h_i' W = f_i'(r_i,t), \qquad (2.2.9)$$

where,

$$h_i' = h_i - k_i\frac{\beta_i}{2\alpha} - (\rho cb)_i\left(\gamma - \frac{\beta_i^2}{4\alpha}\right) \qquad (2.2.10)$$

the new convection coefficient, and,

$$f_i' = f_i(r_i,t)\ \exp[-\gamma t]\ \exp[-\frac{\beta_1}{2\alpha}(r_1 - \frac{\beta_1}{2}t)]\ \exp[-\frac{\beta_2}{2\alpha}(r_2 - \frac{\beta_2}{2}t)]\ \cdot$$
$$\exp[-\frac{\beta_3}{2\alpha}(r_3 - \frac{\beta_3}{2}t)] \qquad (2.2.11)$$

represents the new forcing function.

The boundary conditions for a diffusion problem has been stated previously as,

$$k_i \frac{\partial T}{\partial n_i} + (\rho cb)_i \frac{\partial T}{\partial t} + h_i T = f_i(r_i,t). \qquad (2.2.12)$$

Five distinct classes of boundary conditions can be obtained from this equation.

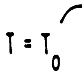
The first kind is when the temperature is prescribed condition at a boundary, see Figure 2.2a,

$$T = f_i'(r_i,t). \qquad (2.2.13)$$

Equation (2.2.13) can be obtained from equation (2.2.12) by letting k_i = 0, $(\rho cb)_i = 0$, and $h_i = 1$. The term $f_i(r_i,t)$ is the prescribed temperature history at the boundary. If the temperature at the boundary is zero, equation (2.2.13) on the boundary becomes the homogeneous boundary condition,

$$T = 0. \qquad (2.2.14)$$

The temperature boundary condition is also called a Dirichlet condition.


T = T0

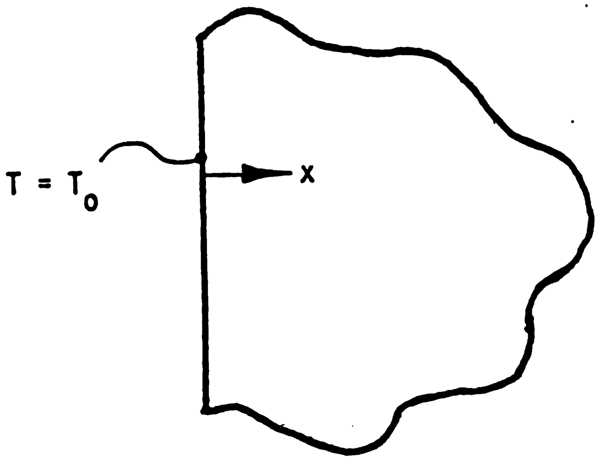


Figure 2.2a

Semi-infinite Slab
With a Temperature Boundary
Condition (X10).

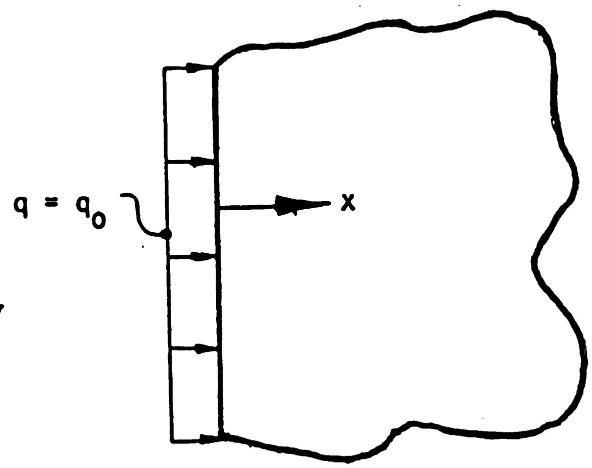


Figure 2.2b

Semi-infinite Slab
With a Heat Flux Boundary
Condition (X20).

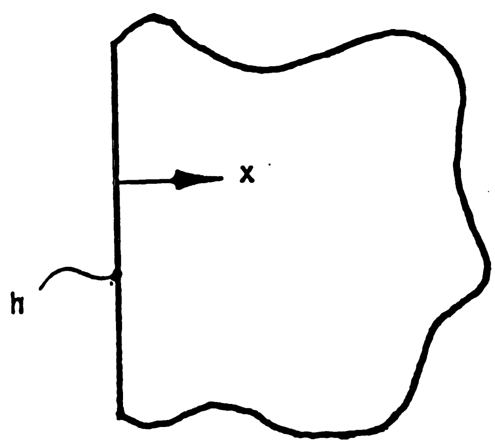


Figure 2.2c

Semi-infinite Slab
With a Convective Boundary
Condition (X30).

The second kind of boundary condition (also called a Neumann condition) is one in which the heat flow is prescribed, see Figure 2.2b,

$$k_i \left.\frac{\partial T}{\partial n_i}\right|_{r_i} = f_i(r_i, t), \tag{2.2.15}$$

where $f_i(r_i, t)$ is a prescribed heat flux. This equation is obtained from equation (2.2.12) by letting h_i and $(\rho cb)_i$ equal zero.

If the heat flux at the boundary is zero, the boundary condition becomes homogeneous and is referred to as being insulated,

$$\left.\frac{\partial T}{\partial n_i}\right|_{r_i} = 0. \tag{2.2.16}$$

A Robin condition or convective boundary condition occurs when there is a linear combination of the temperature and the normal derivative of the temperature -- a boundary condition of the third kind; see Figure 2.2c. The value of the thin film coefficient in equation (2.2.12) is set to zero and the boundary condition equation becomes,

$$k_i \left.\frac{\partial T}{\partial n_i}\right|_{r_i} + h_i\, T = f_i(r_i, t) \tag{2.2.17}$$

or,

$$L \left.\frac{\partial T}{\partial n_i}\right|_{r_i} + Bi\, T = f''(r_i, t) \cdot L, \tag{2.2.18}$$

where $f''(r_i, t)$ was previously defined, $Bi = \frac{h_i L}{k}$ is the Biot number for the solid body at the i-th surface and L may be the thickness of the body. T

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body. The Biot number is the convective coefficient divided by the thermal conductivity. If $f''(r_i,t)$ is zero, the boundary conditions is homogeneous.

A boundary condition of the fourth kind, or Carslaw condition, exists when a "thin" film with no convective heat loss is prescribed at the surface of the thin film's outer boundary; see Figure 2.3a. The thin film is assumed to have high conductivity therefore no temperature gradient exists through the film. The value of h_i is set to zero and the boundary condition equation (2.2.12) becomes,

$$k_i \frac{\partial T}{\partial n_i}\bigg|_{r_i} + (\rho cb)_i \frac{\partial T}{\partial t} = f_i(r_i,t) \tag{2.2.19}$$

or,

$$\frac{\partial T}{\partial n_i}\bigg|_{r_i} + \frac{(\rho cb)_i}{k_i} \frac{\partial T}{\partial t} = \frac{f_i(r_i,t)}{k_i}, \tag{2.2.20}$$

where k_i is the thermal conductivity of the solid body and $(\rho cb)_i$ is the thin film coefficient. This equation is homogeneous if $f_i(r_i,t)$ is set equal to zero.

The boundary condition of the fifth kind, or the Jaeger condition, includes all of the terms in the boundary condition equation described by equation (2.2.12), see Figure 2.3b. The term k_i is set to the thermal conductivity of the solid body, the term h_i is the convection coefficient and $(\rho cb)_i$ is the thin film coefficient. It is homogeneous if $f_i(r_i,t)$ is set equal to zero.

Another type of boundary condition that is considered will be called a boundary condition of the zeroth kind. This boundary condition

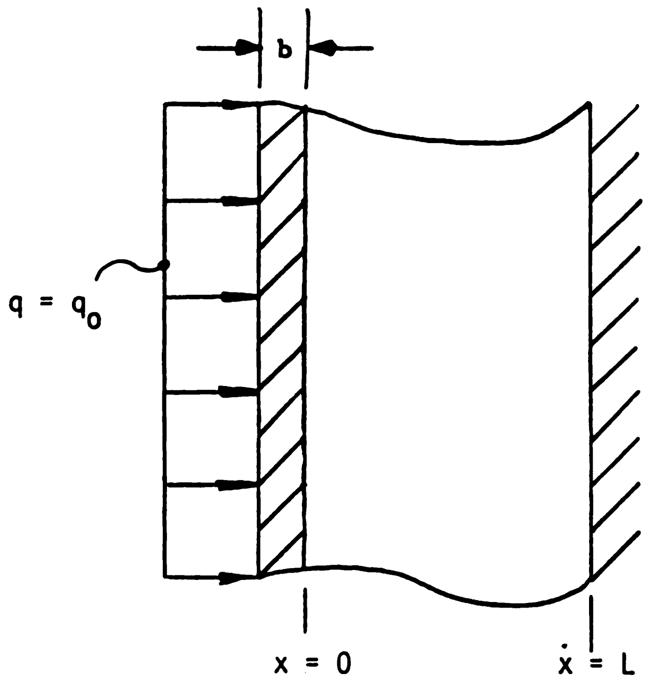

Figure 2.3a Finite Slab With a Non-convective Thin Film and a Heat Flux Condition (X42).

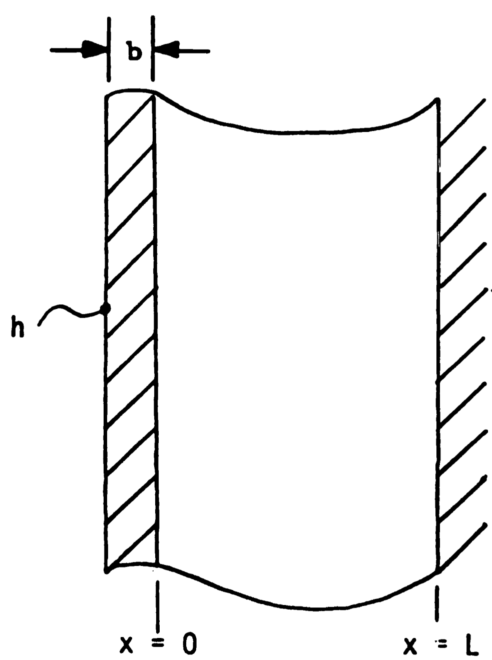

Figure 2.3b Finite Slab With a Convective Thin Film and a Heat Flux Condition (X52).

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has no physical boundary, such as in the case of an infinite or semi-infinite body; see Figure 2.4. The zeroth condition has also been referred to as the natural condition.

The initial condition for the heat conduction equation,

$$T(\underline{r},0) = F(\underline{r}). \tag{2.2.21}$$

This concludes the description of the transient linear heat diffusion equation that will be used in this thesis.

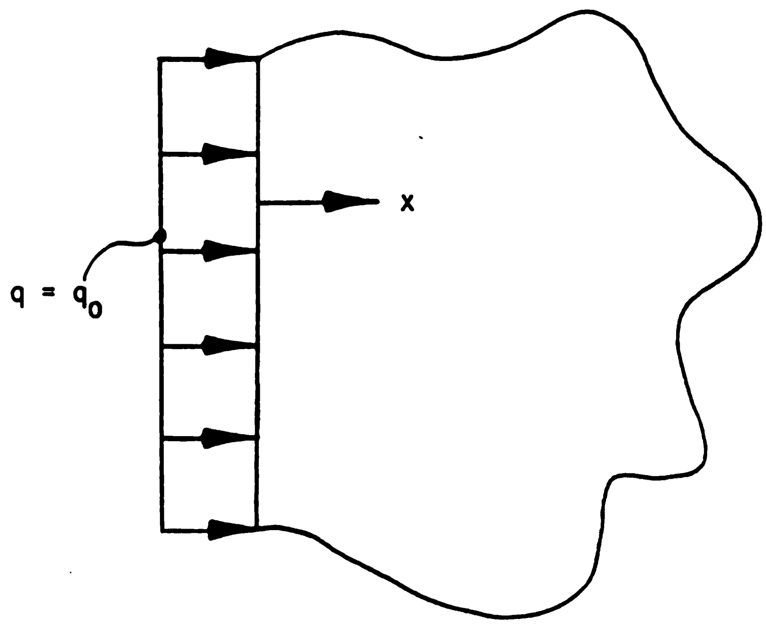
2.2.2 The Auxiliary Green's Function Equation

The purpose of this section is to generate the auxiliary Green's function equation. This equation, when combined with the heat diffusion equation, gives a formalism to the solution of heat diffusion for temperature distribution.

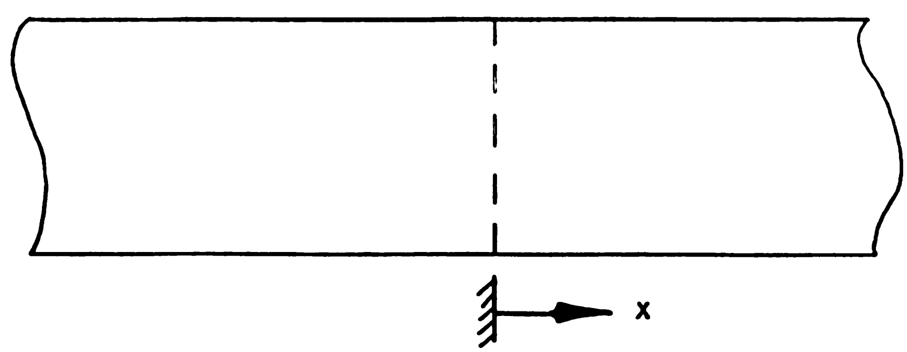
The development of the auxiliary Green's function equation follows the work of Morse and Feshbach [1959], Ozisik [1980], and Beck [1984]. The auxiliary Green's function equation for heat diffusion is obtained from the heat conduction equation model with an instantaneous heat pulse source of unit strength. The boundary conditions for this auxiliary problem are homogeneous and the solution to this auxiliary problem is called the Green's function, G. The auxiliary Green's function equation is,

$$\alpha \nabla^2 G(\underline{r},t|\underline{r}',\tau) + \delta(\underline{r} - \underline{r}')\,\delta(t - \tau) = \frac{\partial G}{\partial t} \tag{2.2.22}$$

in the region R and $t > \tau$,



(a) Semi-infinite Body (X20)



(b) Infinite Body (X00)

Figure 2.4 A Semi-infinite Body and an Infinite Body.

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subject to the boundary conditions,

$$k_i \frac{\partial G}{\partial n_i} + (\rho cb)_i \frac{\partial G}{\partial t} + h_i G = 0 \qquad (2.2.23)$$

on the i-th surface,

and subject to the causality relationship,

$$G(\underline{r},t|\underline{r}',\tau) = 0 \qquad (2.2.24)$$

for $t < \tau$.

The physical interpretation of the Green's function is the response of the system to a unit impulse of heat that occurs at some time τ and at some position $\underline{r}'$. The solution to the auxiliary problem is given the symbol $G(\underline{r},t|\underline{r}',\tau)$ where $\underline{r}$ and t are the point and time of interest respectively, $\underline{r}'$ and τ are the position and time of the impulse.

Equation (2.2.24) is called the causality relationship because it expresses the relationship between the impulse, which occurs at time τ, and the effect of the impulse, which can occur only after time τ. This means that for times $t < \tau$ or, mathematically, for times $-t > -\tau$, there is no effect and the Green's function is zero.

The heat diffusion equation and the auxiliary Green's function equation are parabolic in time due to the appearance of the first derivative with respect to time. This means, for example, that a solution to the heat conduction equation, $T(r,t)$, is not the same as the solution $T(r,-t)$. The diffusion equation and the auxiliary Green's function equation are asymmetric in time; they can distinguish between the past and future.

The Green's function also satisfies a reciprocity condition,

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$$G(\underline{r},t|\underline{r}',\tau) = G(\underline{r}',-\tau|\underline{r},-t). \qquad (2.2.25)$$

The function $G(\underline{r}',-\tau|\underline{r},-t)$ is the effect of a source at location $\underline{r}$ and time $-t$ at a point $\underline{r}'$ and at time $-\tau$. The causality condition is still true and the reciprocal function $G(\underline{r}',-\tau|\underline{r},-t)$ satisfies the time reversed auxiliary Green' function equation,

$$\alpha \nabla_0^2 G + \delta(\underline{r} - \underline{r}')\, \delta(t - \tau) = - \frac{\partial G}{\partial \tau}, \qquad (2.2.26)$$

where ∇_0^2 is the Laplacian operator for the $\underline{r}'$ coordinates. This auxiliary Green's function now describes the development of the effect of a source placed at position $\underline{r}$ and at time $-t$.

2.3 The Green's Function Approach

2.3.1 Mathematical Derivation of the Green's Function Approach

The purpose of this section is to combine the heat diffusion and the auxiliary Green's function to obtain a formalism for the solution of heat diffusion problems for the temperature distribution. A general expression for the solution to the heat conduction equation [Ozisik, 1980] can be generated by combining the heat conduction equation and the time-reversed auxiliary Green's function equation. Multiplying the heat conduction equation (equation (2.2.8) with r replaced by r' and t replaced by τ) by the Green's function, multiplying the time-reversed auxiliary Green's function equation (equation (2.2.26)) by the temperature and subtracting gives,

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$$(G \nabla_0^2 T - T \nabla_0^2 G) + \frac{g(\underline{r}',\tau)}{k} G - \frac{1}{\alpha} \delta(\underline{r} - \underline{r}')\delta(t - \tau) T = \frac{1}{\alpha} \frac{\partial(G\ T)}{\partial \tau} . \tag{2.3.1}$$

This equation is integrated over the total region, R, with respect to $\underline{r}'$ and τ. The term τ goes from 0 to $t+\epsilon$, where ϵ is arbitrarily small value of time which will be made to approach zero. This yields,

$$\int_{\tau=0}^{t+\epsilon} \int_R \alpha (G \nabla_0^2 T - T \nabla_0^2 G) \, dv' \, d\tau + \int_{\tau=0}^{t+\epsilon} \int_R \frac{\alpha}{k} G \, g(\underline{r}',\tau) \, dv' \, d\tau$$

$$- T(\underline{r},t) = \int_R \Big[G\ T \Big]_{\tau=0}^{t+\epsilon} dv' . \tag{2.3.2}$$

where dv' is an volume element in the region R. Rearranging the above equation for the temperature distribution gives,

$$T(\underline{r},t) = - \int_R \Big[G\ T \Big]_{\tau=0}^{t+\epsilon} dv' + \int_{\tau=0}^{t+\epsilon} \int_R \frac{\alpha}{k} G \, g(\underline{r}',\tau) \, dv' \, d\tau$$

$$+ \int_{\tau=0}^{t+\epsilon} \int_R \alpha (G \nabla_0^2 T - T \nabla_0^2 G) \, dv' \, d\tau . \tag{2.3.3}$$

The only term on the left hand side is the temperature distribution of the body at location $\underline{r}$ and at time t. This equation is now examined term by term.

The first term on the right hand side of equation (2.3.3) is,

$$- \int_R \Big[G\ T \Big]_{\tau=0}^{t+\epsilon} dv' . \tag{2.3.4}$$

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Evaluating $-\left[G\ T \right]_{\tau=0}^{t+\epsilon} = -\ (0 - G\ T|_{\tau=0}) = +\ G\ T|_{\tau=0}$. The Green's function evaluated at $\tau=t+\epsilon$ is zero because the effect cannot begin before the impulse. When $\tau = 0$, the temperature distribution, $T(\underline{r},0)$, is the initial temperature distribution, $F(\underline{r})$. This term becomes,

$$I_1 = \int_R G(\underline{r},t|\underline{r}',0)\ F(\underline{r}')\ dv', \tag{2.3.5}$$

and is the effect of the initial distribution on the temperature distribution and is designated I_1.

The second term on the right hand side of equation (2.3.3) is,

$$I_3 = \int_{\tau=0}^{t+\epsilon} \int_R \frac{\alpha}{k}\, G(\underline{r},t|\underline{r}',\tau)\ g(\underline{r}',\tau)\ dv'\ d\tau. \tag{2.3.6}$$

This term is the effect of a distributed heat source, $g(\underline{r},t)$, on the temperature distribution and is designated I_3.

The third term on the right hand side of equation (2.3.3) is,

$$\int_{\tau=0}^{t+\epsilon} \int_R \alpha\ (\ G\ \nabla_0^2 T\ -\ T\ \nabla_0^2 G\ )\ dv'\ d\tau. \tag{2.3.7}$$

Green's theorem can be used to change the volume integral to a surface integral so that,

$$\int_{\tau=0}^{t+\epsilon} \int_R \alpha\ (\ G\ \nabla_0^2 T\ -\ T\ \nabla_0^2 G\ )\ dv'\ d\tau$$

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$$- \int_{\tau=0}^{t+\epsilon} \sum_{i=1}^{s} \int_{S_i} \alpha \left(G \frac{\partial T}{\partial n_i}\bigg|_{r'=r_i} - T \frac{\partial G}{\partial n_i}\bigg|_{r'=r_i} \right) dS_i \, d\tau, \qquad (2.3.8)$$

where the term $\frac{\partial}{\partial n_i}$ is differentiation along an outward drawn normal to the boundary surface S_i, see Figure 2.5, where $i=1,2,3,\ldots,s$, and s is the number of boundaries.

The integrand of this integral can be expressed in terms of the boundary conditions of the heat conduction equation and the auxiliary Green's function equation. Multiplying the boundary condition equation (2.2.12) by the Green's function, multiplying equation (2.2.23) by the temperature and subtracting yields,

$$\left(G \frac{\partial T}{\partial n_i}\bigg|_{r'=r_i} - T \frac{\partial G}{\partial n_i}\bigg|_{r'=r_i}\right) = \frac{f_i(r_i', t)}{k_i} G + \frac{(\rho \ c \ b)_i}{k_i} \left(T \frac{\partial G}{\partial \tau} + G \frac{\partial T}{\partial \tau}\right),$$

or, (2.3.9)

$$\left(G \frac{\partial T}{\partial n_i}\bigg|_{r'=r_i} - T \frac{\partial G}{\partial n_i}\bigg|_{r'=r_i}\right) = \frac{f_i(r_i', t)}{k_i} G + \frac{(\rho \ c \ b)_i}{k_i} \frac{\partial (G \ T)}{\partial \tau} .$$

(2.3.10)

Integrating over the surface S_i and the time, τ, for constant thermal diffusivity and boundary conditions of the second through fifth kinds, yields,

$$\int_{\tau=0}^{t+\epsilon} \sum_{i=1}^{s} \int_{S_i} \alpha \left(G \frac{\partial T}{\partial n_i}\bigg|_{r'=r_i} - T \frac{\partial G}{\partial n_i}\bigg|_{r'=r_i}\right) dS_i \, d\tau$$

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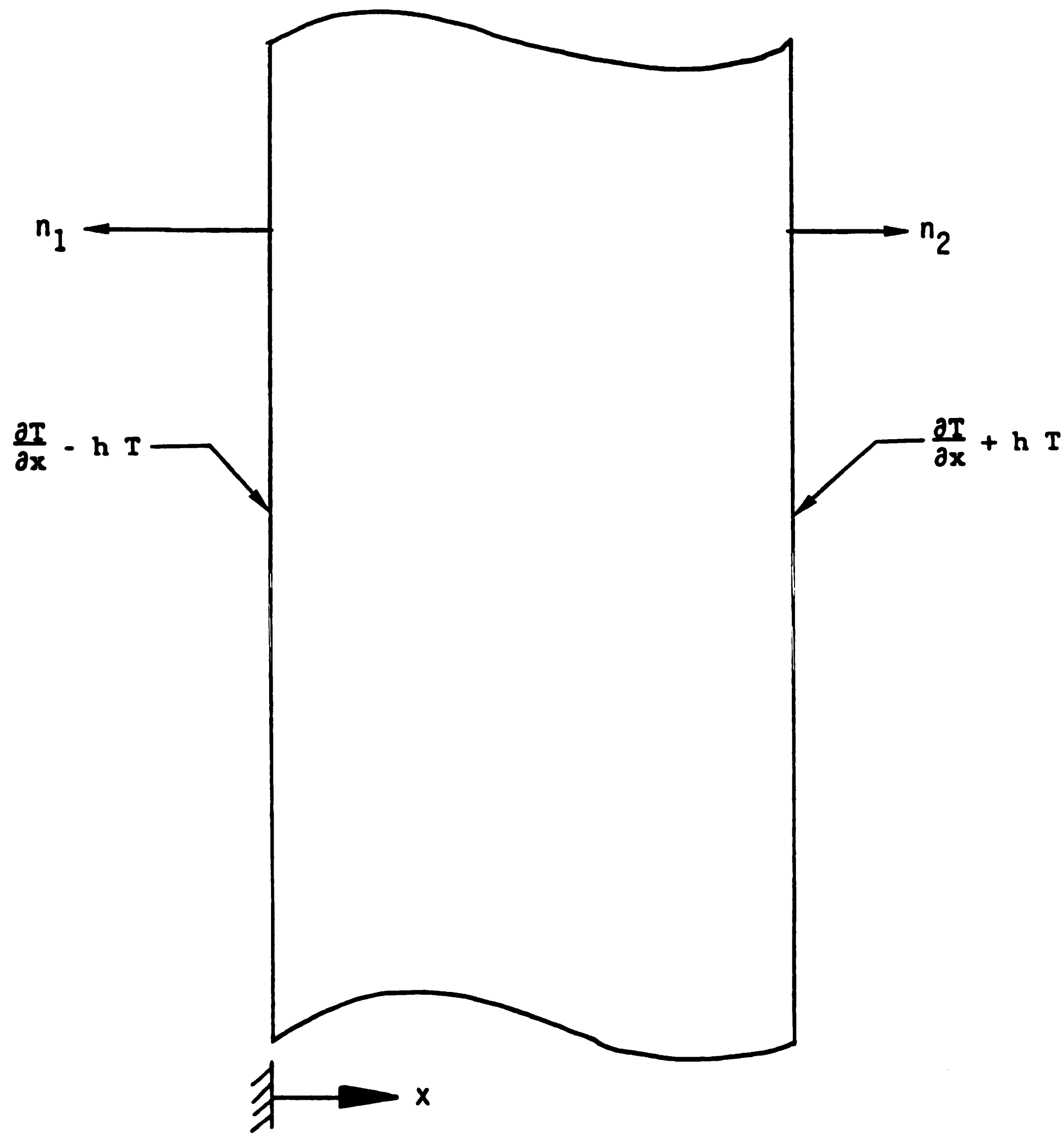

Figure 2.5 A Description of Normals at the Surface of a Finite Body.

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$$- \alpha \int_{\tau=0}^{t+\epsilon} \sum_{i=1}^{s} \int_{S_i} \frac{f_i(r_i',\tau)}{k_i} G(\underline{r},t|\underline{r}',\tau)\, dS_i\, d\tau$$

$$+ \alpha \sum_{i=1}^{s} \int_{S_i} \frac{(\rho\ c\ b)_i}{k_i} G(\underline{r},t|\underline{r}',0)\, F(\underline{r}')\, dS_i. \qquad (2.3.11)$$

The first term on the right hand side of equation (2.3.11) is the effect of the boundary conditions on the temperature distribution and will be designated as I_4. The second term on the right hand side of equation (2.3.11) is the effect of the thermal storage capacity of the thin film on the temperature distribution of the solid body and is designated I_2.

For a boundary condition of the first kind, since G at the boundary is zero, equation (2.3.8) becomes,

$$I_4 = - \alpha \int_{\tau=0}^{t+\epsilon} \sum_{j=1}^{s'} \int_{S_j} f_j(r_j',\tau) \left.\frac{\partial G}{\partial n_j}\right|_{r'=r_j} dS_j\, d\tau. \qquad (2.3.12)$$

where s' is the number of boundary conditions of the first kinds.

Drawing together the four terms yields an expression for the temperature distribution as,

$$T(\underline{r},t) = I_1 + I_2 + I_3 + I_4$$

$$= \int_R G(\underline{r},t|\underline{r}',0)\, F(\underline{r}')\, dv'$$

$$+ \alpha \sum_{i=1}^{s} \int_{S_i} \frac{(\rho\ c\ b)_i}{k_i} G(\underline{r},t|\underline{r}',0)\, F(\underline{r}')\, dS_i$$

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$$+ \int_{\tau=0}^{t+\epsilon} \int_{R} \frac{\alpha}{k} G(\underline{r},t|\underline{r}',\tau)\; g(\underline{r}',\tau)\; dv'\; d\tau$$

$$+ \alpha \int_{\tau=0}^{t+\epsilon} \sum_{i=1}^{s} \int_{S_i} \frac{f_i(r_i',\tau)}{k_i} G(\underline{r},t|\underline{r}',\tau)\; dS_i\; d\tau$$

(for boundary conditions of the second through fifth kinds)

$$- \alpha \int_{\tau=0}^{t+\epsilon} \sum_{j=1}^{s'} \int_{S_j} f_j(r_j',\tau) \left.\frac{\partial G}{\partial n_j}\right|_{r'=r_j} dS_j\; d\tau. \qquad (2.3.13)$$

(for boundary condition of the first kind only)

The four terms on the right hand side of equation (2.3.13) describe a formalism to be used to solve for the temperature distribution in a body for boundary conditions of the second through fifth kind. The first two terms represent the effect on the temperature distribution caused by a nonzero initial condition. The third term represents the effect caused by a volume energy source and the last two terms represents the effect caused by nonhomogeneous boundary conditions.

If a boundary condition of the zeroth kind occurs at a surface $i=j$, the last two terms in equation (2.3.13) are omitted for that surface.

The temperature distribution in the one dimensional cartesian coordinate x for a one dimensional slab is,

$$T(x,t) = \int_{x'=0}^{L} G(x,t|x',0)\; F(x')\; dx'$$

$$+ \alpha \sum_{i=1}^{2} \frac{(\rho\; c\; b)_i}{k_i} \left[G(x,t|x',0)\; F(x')\right]_{x'=x_i}$$

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$$+ \int_{\tau=0}^{t} \int_{x'=0}^{L} \frac{\alpha}{k} G(x,t|x',\tau)\ g(x',\tau)\ dx'\ d\tau$$

$$+ \alpha \int_{\tau=0}^{t} \sum_{i=1}^{s} \frac{f_i(x_i',\tau)}{k_i} G(x,t|x',\tau)_{x'=x_i} \ d\tau \ ,$$

(for boundary conditions of the second through fifth kinds)

$$- \alpha \int_{\tau=0}^{t} \sum_{j=1}^{s'} f_j(x_j',\tau) \left.\frac{\partial G}{\partial n_j}\right|_{x'=x_j} d\tau, \qquad (2.3.14)$$

(for boundary condition of the first kind only)

where L is the length of the slab in the x direction.

The temperature distribution in the two dimensional cartesian coordinates x and y for a two dimensional plate is,

$$T(x,y,t) = \int_{x'} \int_{y'} G(x,y,t|x',y',0)\ F(x',y')\ dx'\ dy' \ +$$

$$\alpha \sum_{i=1}^{4} \frac{(\rho\ c\ b)_i}{k_i} \Big[G(x,y,t|x',y',0)\ F(x',y')\Big]_{\substack{x'=x_i \\ y'=y_i}} \ +$$

$$\int_{\tau=0}^{t} \int_{x'} \int_{y'} \frac{\alpha}{k} G(x,y,t|x',y',\tau)\ g(x',y',\tau)\ dA\ d\tau \ +$$

$$\alpha \int_{\tau=0}^{t} \sum_{i=1}^{s} \frac{f_i(x_i',y_i',\tau)}{k_i} G(x,y,t|x',y',\tau)_{\substack{x'=x_i \\ y'=y_i}} d\tau \ ,$$

(for boundary conditions of the second through fifth kinds)

$$- \alpha \int_{\tau=0}^{t} \sum_{j=1}^{s'} f_j(x_j',y_j',\tau) \left.\frac{\partial G}{\partial n_j}\right|_{x'=x_j,\,y'=y_j} d\tau, \qquad (2.3.15)$$

(for boundary condition of the first kind only)

2.3.2 Determination of the Green's Function

The objective of this section is to discuss some methods to determine the correct Green's function based on the boundary conditions. There are many procedures for the determination of the Green's function. One procedure, which will be discussed in the next chapter, is by the method of Laplace transforms. A second method, which produces results similar to the Laplace transform method, is the method of images. Both of these techniques produce Green's functions that are computationally efficient at small dimensionless time. The third method, which produces Green's functions that are computationally efficient at large dimensionless times, is obtained by using the traditional method of separation of variables.

The method of images requires the temperature in a finite or semi-infinite body caused by a supply of heat at certain points, and the removal of heat at other points. A supply of heat at a point is called a source while the removal of heat at a point is called a sink. In a one dimensional infinite body, the temperature distribution due to an instantaneous heat pulse is,

$$T(x,x'|t,\tau) = Q\,[4\,\pi\,\alpha\,(t-\tau)]^{-1/2}\,\exp[-(x-x')^2/(4\,\alpha\,(t-\tau))], \tag{2.3.16}$$

where Q is the strength of the source, x' and τ are the location and starting time of the source, and x and t are the location and time of interest. Multiplying this temperature distribution by ρc and integrating over the entire infinite body yields,

$$\int_{-\infty}^{\infty} \rho c T \, dx' = Q\rho c \int_{-\infty}^{\infty} [4 \pi \alpha (t-\tau)]^{-1/2} \exp[-(x-x')^2/(4 \alpha (t-\tau))] \, dx'$$

$$= Q\rho c. \qquad (2.3.17)$$

which shows the total amount of heat liberated in the infinite solid is $Q\rho c$.

When the infinite body is bounded by planes, the bounding planes can be considered to act as heat mirrors which reflect the sources and sinks. The Green's function of a bounded body is simply the distribution of the original source plus the effects of these reflections, see Figure 2.6. The method in Chapter 3 will give a technique for the placement of the sources, sinks and correction terms for the positions shown in Figure 2.6.

In his book on partial differential equations, Sommerfeld [1967] describes the method of images for heat diffusion in a slab as a room with parallel mirrors. This simulates a finite body of length L. A light placed in the room will be reflected by both mirrors not once, but in infinite repetition. The reflections of the light source are the heat poles with a period 2nL where L is the distance between the mirrors and n is an integer index. The summation of the poles yield the Green's function for small dimensionless time. The reflections can be considered to form background correction factors to the effect of the source.

The method of images is restricted to systems that have boundaries composed of straight lines. Other shapes may be considered but the method may be applied only approximately.

A second approach that is frequently used to solve heat conduction problems is by using the separation of variables. This technique gives Green's functions that are efficient for large times. The homogeneous

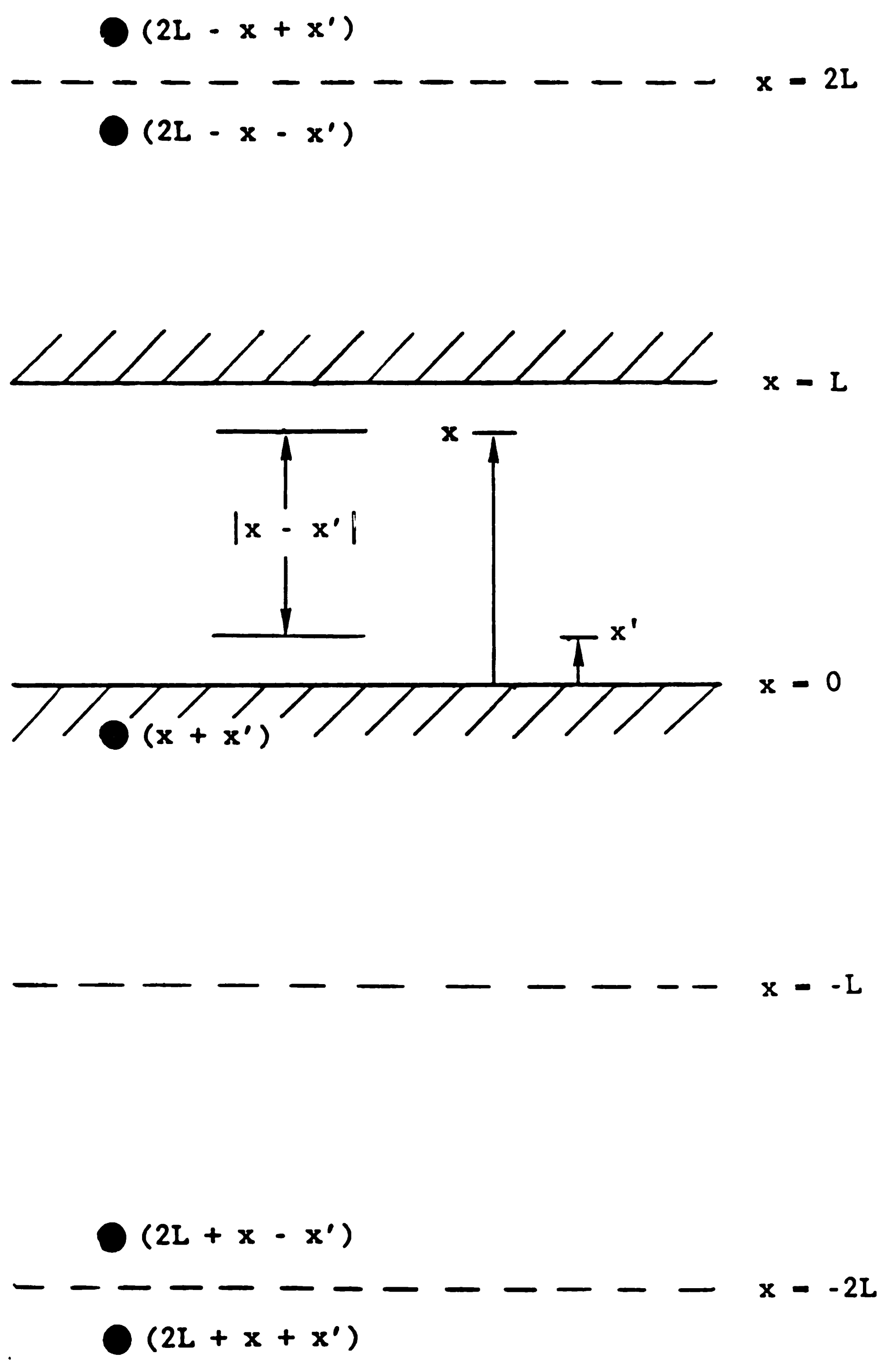

Figure 2.6 Reflections of Sources and Sinks in a Finite Body.

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heat conduction equation is solved using the separation of variables technique, with homogeneous boundary conditions and an initial condition, $F(\underline{r})$, and the solution is expressed in the form,

$$T(\underline{r},t) = \int_R K(\underline{r},\underline{r}',t)\, F(\underline{r}')\, dv'. \tag{2.3.18}$$

Since the Green's function solution for this case is,

$$T(\underline{r},t) = \int_R G(\underline{r},\underline{r}'|t,0)\, F(\underline{r}')\, dv', \tag{2.3.19}$$

and it follows that,

$$G(\underline{r},\underline{r}'|t,0) = K(\underline{r},\underline{r}',t). \tag{2.3.20}$$

Replacing t with $t-\tau$ in $K(\underline{r},\underline{r}',t)$ gives the general Green's function for solution to nonhomogeneous problems, $G(\underline{r},\underline{r}'|t,\tau)$.

It is important to note that to solve the nonhomogeneous problem for temperature, the homogeneous problem for the Green's function is all that is needed to be considered.

2.3.3 Products of Green's Functions

The purpose of this section is to show the combinations of Green's functions that are mathematically possible for rectangular coordinates.

A unique feature of the Green's function approach to the solution of transient heat diffusion problems is the capability of multiplying one dimensional Green's functions to obtain Green's functions for two and th

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and three dimensions in cartesian coordinates for boundary conditions of the zeroth through third kinds. The development of this idea follows the techniques of Ozisik [1980] and Beck and Yen[1984b].

In cartesian coordinates it is desired to prove,

$$G(x,y,z,t|x',y',z',\tau) = G_1(x,t|x',\tau) \cdot G_2(y,t|y',\tau) \cdot G_3(z,t|z',\tau). \qquad (2.3.21)$$

To prove this, the right hand side of equation (2.3.21) is substituted into the auxiliary Green's function equation and the boundary conditions and then compared to the equation of the individual coordinate equations added together.

Substituting the right hand side of equation (2.3.21) into the auxiliary Green's function equation (equation (2.2.26)) yields,

$$\alpha \left[G_2 G_3 \frac{\partial^2 G_1}{\partial x^2} + G_1 G_3 \frac{\partial^2 G_2}{\partial y^2} + G_1 G_2 \frac{\partial^2 G_3}{\partial z^2} \right] - \left[G_2 G_3 \frac{\partial G_1}{\partial \tau} + G_1 G_3 \frac{\partial G_2}{\partial \tau} + G_1 G_2 \frac{\partial G_3}{\partial \tau} \right] = \delta(x-x')\, \delta(y-y')\, \delta(z-z')\, \delta(t-\tau). \qquad (2.3.22)$$

The one dimensional equations that describe G_1, G_2, and G_3 are,

$$\alpha \left[\frac{\partial^2 G_1}{\partial x^2} - \frac{1}{\alpha} \frac{\partial G_1}{\partial \tau} \right] = \delta(x-x')\, \delta(t-\tau) \qquad (2.3.23)$$

$$\alpha \left[\frac{\partial^2 G_2}{\partial y^2} - \frac{1}{\alpha} \frac{\partial G_2}{\partial \tau} \right] = \delta(y-y')\, \delta(t-\tau) \qquad (2.3.24)$$

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$$\alpha \left[\frac{\partial^2 G_3}{\partial z^2} - \frac{1}{\alpha} \frac{\partial G_3}{\partial \tau} \right] = \delta(z-z')\ \delta(t-\tau) \tag{2.3.25}$$

Multiplying these three equations by G_2G_3, G_1G_3, and G_1G_2 respectively, and adding gives,

$$\alpha \left[G_2G_3 \frac{\partial^2 G_1}{\partial x^2} + G_1G_3 \frac{\partial^2 G_2}{\partial y^2} + G_1G_2 \frac{\partial^2 G_3}{\partial z^2} \right] - \left[G_2G_3 \frac{\partial G_1}{\partial \tau} + G_1G_3 \frac{\partial G_2}{\partial \tau} + G_1G_2 \frac{\partial G_3}{\partial \tau} \right] =$$

$$G_2G_3\ \delta(x-x')\ \delta(t-\tau) + G_1G_3\ \delta(y-y')\ \delta(t-\tau) + G_1G_2\ \delta(z-z')\ \delta(t-\tau). \tag{2.3.26}$$

The terms on the left hand side of equations (2.3.22) and (2.3.26) match therefore it must be shown that the right hand terms of these equations are equal. This means, for example, that,

$$G_2G_3\ \delta(x-x')\ \delta(t-\tau) = \frac{1}{3}\ \delta(x-x')\ \delta(y-y')\ \delta(z-z')\ \delta(t-\tau) \tag{2.3.27}$$

Integrating equation (2.3.24) with respect to τ from minus to plus infinity ,

$$\int_{-\infty}^{\infty} \alpha \left[\frac{\partial^2 G_2}{\partial y^2} - \frac{1}{\alpha} \frac{\partial G_2}{\partial \tau} \right] d\tau = \delta(y-y') \int_{-\infty}^{\infty} \delta(t-\tau)\ d\tau \quad , \tag{2.3.28}$$

which yields,

$$G_2 = \delta(y-y')\ H(t-\tau). \tag{2.3.29}$$

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When $t \neq \tau$, the left and right hand side of equation (2.3.27) are zero. When $t = \tau$, G_2 and G_3 act like unit step functions, $H(t-\tau)$, multiplied by functions independent on time. Therefore,

$$G_2 G_3 = \delta(y-y')\ \delta(z-z')\ H^2(t-\tau). \tag{2.3.30}$$

Substituting equation (2.3.30) in equation (2.3.27), integrating over time yields,

$$\int_{-\infty}^{\infty} H^2(t-\tau)\ \delta(t-\tau)\ dt = \int_{-\infty}^{\infty} \frac{1}{3}\ \delta(t-\tau)\ dt$$

$$= \frac{H^3}{3} \Big|_{t=\tau} = \frac{1}{3}. \tag{2.3.31}$$

The same procedure is used for the remaining terms on the right hand side of equation (2.3.27) and results in,

$$3\left\{ \frac{1}{3}\ \delta(x-x')\ \delta(y-y')\ \delta(z-z')\ \delta(t-\tau) \right\} =$$

$$\delta(x-x')\ \delta(y-y')\ \delta(z-z')\ \delta(t-\tau), \tag{2.3.32}$$

which completes the proof for the differential equation.

The boundary conditions must also be satisfied for the product relationship to hold. The general boundary condition for boundary conditions of the first, second and third kind is,

$$k_i \frac{\partial G}{\partial n_i} + h_i\ G = 0. \tag{2.3.33}$$

Substituting equation (2.3.21) in this expression gives,

$$G_2G_3\ k_i \frac{\partial G_1}{\partial n_i} + G_2G_3\ \ h_i\ G_1 = 0, \tag{2.3.34}$$

or, dividing through by G_2G_3 gives,

$$k_i \frac{\partial G_1}{\partial n_i} + h_i\ G_1 = 0, \tag{2.3.35}$$

which is the same expression as equation (2.3.33). This proves that the product relationship will hold for boundary conditions of the first, second and third kind.

The general equation for boundary conditions of the fourth and fifth kind is,

$$k_i \frac{\partial G}{\partial n_i} + (\rho cb)_i \frac{\partial G}{\partial t} + h_i\ G = 0. \tag{2.3.36}$$

Substituting equation (2.3.21) in this equation gives,

$$G_2G_3\ \frac{\partial G_1}{\partial x} \pm G_1G_2G_3h_i$$

$$\pm\ (\rho cb)_i \left[G_2G_3\ \frac{\partial G_1}{\partial t} + G_1G_3\ \frac{\partial G_2}{\partial t} + G_1G_2\ \frac{\partial G_3}{\partial t}\right] = 0 \tag{2.3.37}$$

which cannot be reduced to the form of equation (2.3.36). Consequently, the product relationship of the one dimensional Green's functions is not valid for two or three dimensional diffusion problems when there are boundary conditions of the fourth or fifth kinds.

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2.4 Formalism of the Green's Function Approach

The description of the temperature distribution based on the Green's function approach, equation (2.3.13), provides a logical and mathematical structure to the solution of heat diffusion problems. This formalism is used to begin a structured data base of solutions based solely on the geometry and the boundary condition types of the specified problems.

The dictionary defines formalism as a rigorous adherence to recognized forms. A mathematical formalism uses some basic "building blocks" or functions in a structure to obtain analytical solutions. In the Green's function approach, the basic "building blocks" are contained in the data bases. Both data bases in the Green's function approach, the Green's function data base and the integral data base, are finite and analytical. This will restrict the capabilities of the method to a finite number of cases, but the solutions will be analytical. The data bases do no need to be analytical in general. For example, in the integral data base, an integral that does not have a closed form could be expressed in numerical form, but the solution based on the numerical form will not be analytical.

The Green's function approach to heat diffusion problems relies on the availability of Green's functions. A single Green's function is used in each term on the right hand side of equation (2.3.13). Independent of the initial condition and volume energy source, the Green's function is determined from the boundary conditions. This suggests that a look-up table or data base of Green's functions is important to the formalism. Furthermore, a complete table of one dimensional Green's functions for the cartesian coordinate system is available.

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Beck and Litkouhi [1984] have suggested a numbering system for single and multi-dimensional Greens functions based on the types of boundary conditions that occur on the surfaces of the body. Figure 2.7 shows the distinct number of cases for the Green's function for the one dimensional rectangular coordinate system. For example, a one dimensional slab with a boundary condition of the first kind at the left surface and insulated on the right surface is designated as X12. The numbering scheme allows the Green's functions to be manually and computer catalogued and the effort necessary for locating the functions will be the effort necessary to establish the catalogue number for the specified problem. Figure 2.7 shows the distinct number of cases for the Green's function for the rectangular coordinate system.

The numbering system proposed is important since the temperature distributions obtained by the Green's function approach can be catalogued and stored and need not be re-evaluated. The numbering system uses the types of boundary conditions and determines a Green's function that represents a plane, line, or point source that occurs at some point x' in the slab (or on the boundary), and at some time τ. Due to the linearity of the problems, this function is then multiplied by the forcing function and integrated over the boundary, area, and/or time of interest.

Since all of the Green's functions for the six linear boundary conditions have been catalogued, it is easy to calculate new solutions by implementing the Green's function formalism. The convergence of the new solutions is greatly improved over the solutions based solely on the method of separation of variables or the method of Laplace transforms.

For small times, the Laplace transform method is most efficient. For dimensionless times greater than 0.05, more terms are necessary in

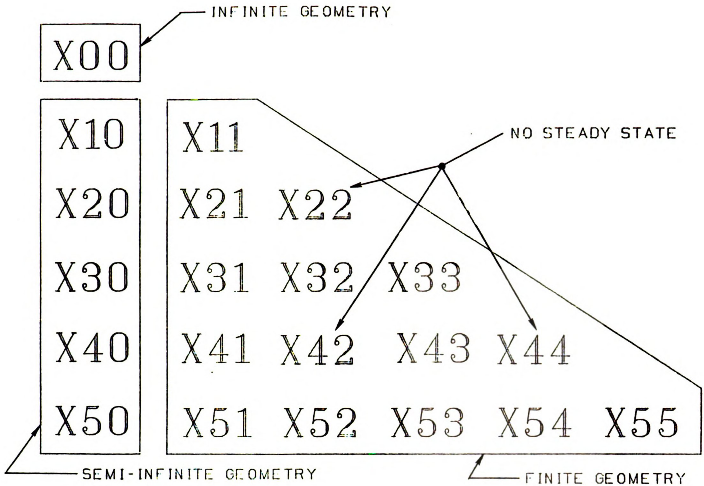

Figure 2.7 Distinct Cases of Green's Function For One Dimensional Problems

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the Laplace transform series for convergence. At this point, by switching to the separation of variables function, convergence is retained and the number of terms evaluated in the series remains small or at least tractable.

The ability to use the product of Green's functions for multidimensional temperature distributions is advantageous since the same solution kernel may be programmed for one, two, or three dimensional problems for boundary conditions of the zeroth, first, second, and third kinds. The Green's function approach does not allow boundary conditions of the fourth and fifth kinds for problems in two and three dimensions.

The great usefulness of the Green's function procedure is startling due to the vast number of cases that are involved. The CANSS program will treat heat conduction problems that deal with nonhomogeneous boundary conditions of the zeroth through fifth kind, time and space variation in the boundary conditions and initial condition, constant volume energy sources, terms associated with fins, and terms associated with flow.

The symbolic formalism of the Green's function approach permits the use of either the small time Green's function or the large time function in the solution to the stated problem. Time partitioning of the solution, as suggested by Beck and Keltner [1985] may be used effectively to reduce the computational load.

Time partitioning for the solution of linear, transient diffusion type equations uses a linear combination of two separate, but equivalent expressions. One expression is obtained using a Laplace transform technique and converges quickly when time is small; the other expression can be obtained using the method of separation of variables and will converge rapidly when the dimensionless time is large. Therefore, two expressions are available for use, each having a region of dimensionless

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time for which it is best suited. The Green's function expressions can be used to generate a solution that converges for any dimensionless time by combining both the small and large dimensionless time expressions .

An example of the combined Green's function for a one dimensional slab insulated on both boundaries is,

$$G(x_L, x_L', t^*) =$$

$$\frac{1}{L} [4 \pi t^*]^{-1/2} \sum_{m=-\infty}^{\infty} (\exp[-(2m+x_L-x_L')^2/4t^*] + \exp[-(2m+x_L+x_L')^2/4t^*]) \tag{2.4.1}$$

for t^* small, and for large t^* is,

$$G(x_L, x_L', t^*) = \text{Constant}$$

$$+ \frac{1}{L} \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp[-n^2 \pi^2 t^*] \cos[n\pi x_L] \cos[n\pi x_L'] \tag{2.4.2}$$

where x_L and x_L' are normalized with respect to L and $t^* = \frac{\alpha(t-\tau)}{L^2}$. Note that the exponential term converges rapidly to zero for small dimensionless times in the small time expression and rapidly to zero for large dimensionless times in the large time expression.

2.5 Summary

A formalism for solving simple, multi-dimensional, linear, transient heat diffusion problems is presented in this chapter. This formalism consists of a multi-dimensional equation and a group of kernal functions called Green's functions. The formalism allows the application of six distinct types of boundary condition, and the facility for handling nonhomogeneous initial conditions and volume energy sources.

The unique structure of the Green's function formalism and the availability of the Green's functions themselves are ideal candidates for symbolic solution. A simple transformation allows the formalism to handle terms that include convective diffusion and temperature generation proportional to the local temperature.

Two methods are discussed to determine the Green's functions. A third method will be described in the following chapter. These methods provide a data-base form for cataloging the Green's functions. Multiplying one dimensional Green's functions for certain combinations of boundary conditions is shown.

Chapter 3

SMALL TIME GREEN'S FUNCTIONS OBTAINED USING LAPLACE TRANSFORMS

3.1 Introduction

Carslaw and Jaeger [1959] and Morse and Feshbach [1953] demonstrate the use of the small time Green's functions, and Ozisik [1980] summarizes the importance of these functions in their classical heat transfer and physics texts. The small time functions permits efficient investigation of transient thermal activity at very small times. For small times, the heat conducting bodies are thermally semi-infinite; that is, temperature changes are contained only in the region of the body near the heated surface. The small time Green's functions are particularly important in problems where the forcing functions resemble instantaneous sources, such as in the areas of robotics, electronics, and measuring energy deposition by pulsed lasers.

A limited group of small time Green's functions can be found in the previously cited references, but no care was taken to organize these functions. This chapter extends the work of Beck[1984], Ozisik[1980], and Morse and Feshbach[1954] by (a) organizing and systematizing the small time solutions for boundary conditions of the first, second, and third kind, and (b) generating additional small time functions for fourth and fifth types of boundary conditions.

The boundary condition of the fourth kind involves a surface film of finite heat capacity and the fifth kind involves a surface film but

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but also has convective heat transfer at the outside surface of the film.

The method for obtaining small time Green's functions incorporates a symbolic manipulation program called SMP [1983] for the manipulation and evaluation of large algebraic expressions. The ability of SMP to perform complicated algebraic manipulations and the necessity of dealing with advanced mathematical constructs such as integration, differentiation, factoring, and expansions, to name a few, is the prime motivation for using SMP. The procedures written in the SMP language have a generality not found in the traditional scientific numerical programs (i.e. BASIC, FORTRAN, PASCAL) due to the ability of the program to manipulate symbolic expressions, as well as the capability for numerical evaluation.

Table 3.1 and Figure 3.1 show a comparison of the Green's function for a semi-infinite body versus a finite body at small times. The location of the source is at coordinate x', the location of the point of interest is the coordinate x, and the thickness of the finite body is L. The symbol t is the time of interest and the symbol τ is the time when the source begins. Each body has a heat flux condition at the left boundary and the finite bodies are insulated on the right side. The diffusivity of the medium is constant and equal to one.

Figure 3.1 shows the semi-infinite body solution is a good approximation to the finite body solution when the time is small and the thickness of the body is large. It is not a good approximation when the time becomes large or the thickness of the body is small. As the thickness of the finite body increases, the Green's function for the finite bodies converge to the solution of the semi-infinite body, as expected. The Green's function for the larger width bodies can be approximated at small times by the semi-infinite Green's function. The dependence of

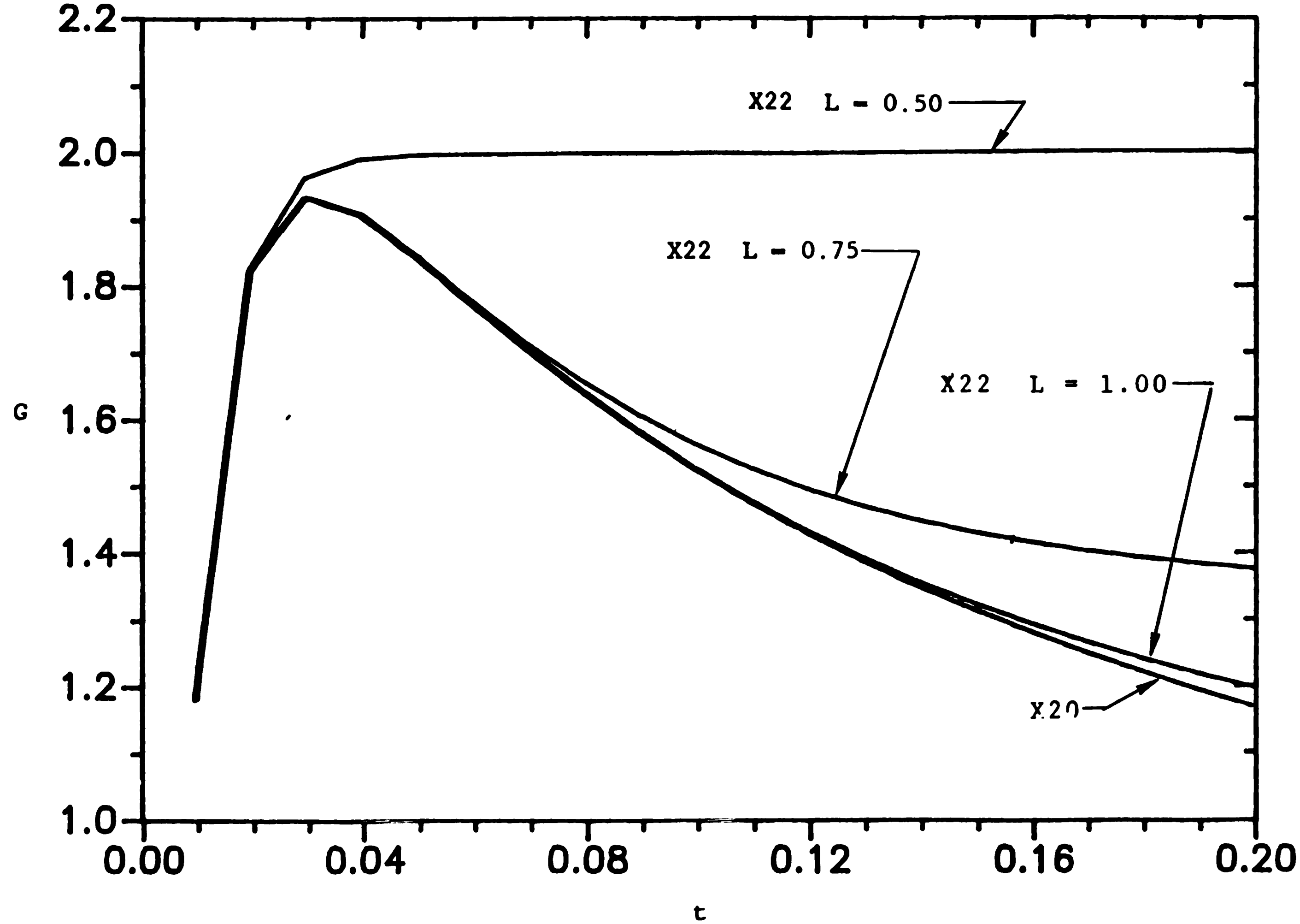

Figure 3.1 Small Time Green's Functions for Various Lengths of a Finite Body Versus Time.

Table 3.1 Small Time for Finite and Infinite Body
Source at x' = 0 , Point of Interest at x = 0.25

time	X20 L = ∞	X22 L = 0.5	X22 L = 0.75	X22 L = 1.0
0.01	1.18261	1.18261	1.18261	1.18261
0.02	1.82649	1.83002	1.82649	1.82649
0.03	1.93495	1.96495	1.93496	1.93495
0.04	1.90875	1.99278	1.90891	1.90875
0.05	1.84596	1.99851	1.84698	1.84596
0.06	1.77522	1.99969	1.77865	1.77523
0.07	1.70583	1.99994	1.71391	1.70587
0.08	1.64080	1.99999	1.65606	1.64094
0.09	1.58090	2.00000	1.60579	1.58129
0.10	1.52604	2.00000	1.56278	1.52689
0.11	1.47584	2.00000	1.52626	1.47747
0.12	1.42983	2.00000	1.49542	1.43264
0.13	1.38757	2.00000	1.46943	1.39199
0.14	1.34862	2.00000	1.44758	1.35516
0.15	1.31262	2.00000	1.42922	1.32178
0.16	1.27924	2.00000	1.41380	1.29154
0.17	1.24820	2.00000	1.40086	1.26414
0.18	1.21924	2.00000	1.38999	1.23932
0.19	1.19216	2.00000	1.38088	1.21683
0.20	1.16676	2.00000	1.37323	1.19645

Table 3.2 Exponential as a Function of Time

time	LT expansion $e^{-1/4t}$	LT expansion $e^{-4/4t}$	SOV expansion $e^{-\pi^2 t}$	SOV expansion $e^{-4\pi^2 t}$
0.001	0.00000	0.00000	0.99018	0.96129
0.005	0.00000	0.00000	0.95185	0.82087
0.010	0.00000	0.00000	0.90602	0.67383
0.025	0.00005	0.00000	0.78134	0.37271
0.050	0.00674	0.00000	0.61050	0.13891
0.100	0.08209	0.00005	0.37271	0.01930
0.250	0.36788	0.01832	0.08480	0.00005
0.500	0.60653	0.13534	0.00719	0.00000
1.000	0.77880	0.36788	0.00005	0.00000
5.000	0.95123	0.81873	0.00000	0.00000

the thickness scale may be removed from consideration of the solution by defining a non-dimensional time parameter, t^*, where,

$$t^* = \frac{\alpha(t-\tau)}{L^2} . \tag{3.1.1}$$

Small and large time solutions can be combined to obtain solutions that are computationally efficient for the total time region. The solutions are split into time partitions for which the resulting solutions may be evaluated. At early times, the transient solutions are more efficiently represented by a summation of exponential functions derived from the Laplace transform (LT) or the method of images. The exponential terms generated by the Laplace transform technique are functions of $-C_m/t^*$ where C_m is a function of m^2 and increases in value as the index m increases, and t^* is the dimensionless time. As the dimensionless time becomes small, the exponential terms rapidly approaches zero. For example, the first two typical exponential terms of the summation derived by the Laplace transform technique for a one dimensional slab, insulated on both boundaries, are shown in Table 3.2 under the LT heading, namely $e^{-(1/4t^*)}$ and $e^{-(4/4t^*)}$. Notice that for $t^* < 0.05$, the term $e^{-(1/4t^*)}$ is less than 0.0068 and for $t^* < 0.01$ there is no contribution out to the fifth decimal place. This means that as the dimensionless time gets small, only the first few terms need be kept for an accurate approximation. The additional terms may be dropped without diminishing the accuracy of the computation. When m=2 (m^2=4), one additional term, $e^{-(4/4t^*)}$, is also shown in Table 3.2. Five decimal place accuracy is obtained for the sum of these two terms by retaining only the first term when the dimensionless time is less than 0.1.

The solutions derived using separation of variables (SOV) are more efficient at large times. Like the solutions found in the Laplace transform technique, the solutions for the SOV method are functions of a summation of exponential terms, but the exponential functions are typically functions of $-C_n t^*$ where t^* is the dimensionless time. The value C_n is a function of n^2 and it increases in value as the value of the index n increases. As dimensionless time increases, these exponential terms decrease rapidly towards zero. The first two typical exponential terms, $e^{-\pi^2 t^*}$ and $e^{-4\pi^2 t^*}$, of the summation found by using the SOV method for a one dimensional slab with insulated boundary conditions, are shown in Table 3.2 under the SOV heading. When $t^* > 0.500$, the contribution of the first summation term, $e^{-\pi^2 t^*}$, is less than 0.0072. For t^* greater than 0.250, five decimal place accuracy is obtained when only the first term in the summation is retained.

Accuracy can be increased for both the LT and the SOV method at any dimensionless time by including additional terms to their respective series. Inclusion of the additional terms not only increases the accuracy of the solution, but unfortunately also increases the amount of computation time necessary to arrive at the solution due to the time of evaluation of the additional terms.

Table 3.2 shows for any dimensionless time, the second term in the LT or SOV series is always numerically equal to or less than the first term in the series. This means the first term will dominate the solution if the appropriate series is chosen based on the time. Retaining an infinite number of terms in either the SOV or LT methods will yield an exact solution but the choice of series that results in the most efficient method is dependent of the time.

The analysis methods used in heat transfer are typically the finite element (FE) method or the finite difference (FD) method and are numerical in nature. These numerical analysis methods generate information at the locations designated by the resolution of both the spatial grid size and the time step size. A smoothing function is used to develop information at locations that differ from the spatial grid points or at times that differ from the time step size. Very fine temporal and spatial resolution are needed for solution convergence at early times for FE and FD methods; therefore, at early times these solution techniques are not efficient in the use of computer time.

The procedure for the small time Green's functions is useful in the continuing research to develop the unsteady surface element (USE) method [Keltner and Beck,1981], which is a method for solving linear transient heat conduction problems. Solutions to certain basic transient heat conduction problems, called influence or kernel functions, are used as building blocks to solve problems of complex geometry and problems that deal with nonlinear boundary conditions in the USE method. The small and large time Green's functions, along with a combination of the small and large time Green's functions, can be used to generate the influence functions in the USE method.

Section 3.2 of this chapter gives a mathematical statement of the general linear transient heat conduction problem and the development of the small time Green's funct terms of a one dimensional slab.
The objective of Section 3.3 is to examine some examples of Green's functions for semi-infinite bodies. Section 3.4 gives examples of how the small time Green's functions are developed for finite one dimensional bodies. Section 3.5 is a summary of the chapter.

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3.2 Mathematical Development of the Small Time Green's Function

The objective of this section is to develop a procedure to generate small time Green's functions for one dimensional bodies with boundary conditions of the zeroth through fifth kinds. The technique uses the partial differential equation of heat diffusion, the associated boundary conditions and Laplace transforms to generate an expression for the Green's function that is accurate and efficient at small times. As the dimensionless time approaches zero, the approximate solution goes to the exact solution.

The partial differential equation for linear transient heat conduction developed in Chapter 2 is, after the appropriate transformations,

$$\nabla^2 T + \frac{g(\underline{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in region R.} \tag{3.2.1}$$

The associated boundary conditions are,

$$k_i \frac{\partial T}{\partial n_i} + (\rho\ c\ b)\frac{\partial T}{\partial t} + h_i T = f_i(r_i,t), \tag{3.2.2}$$

for $i=1,2,\ldots,s$, where the symbol s is the number of boundaries, and the initial condition is,

$$T(\underline{r},0) = F(\underline{r}). \tag{3.2.3}$$

The thermal conductivity, k, and the thermal diffusivity, α, are constant with position, time, and temperature, $(\underline{r},t,T)$; ∇^2 is the Laplacian operator and n is an outward pointing normal.

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For a one dimensional slab without heat generation, i.e. $g(\underline{r},t) = 0$, equation (3.2.1) reduces to,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad 0 < x < L \qquad (3.2.4)$$

The one dimensional transient heat conduction equation (equation 3.2.4) is satisfied by,

$$T = [4\,\pi\,\alpha(t-\tau)]^{-1/2}\, e^{-(x-x')^2/(4\,\alpha(t-\tau))} \qquad (3.2.5)$$

which tends to zero when $\tau \to t$ at all points except x', where it goes to infinity. This solution is called the **temperature due to an instantaneous plane source through x' and at time τ and of strength unity per unit area**. It is the fundamental transient solution in a planar heat-conducting body and is actually a Green's function. Since the describing equation and boundary and initial conditions are linear, this solution must be included in any solution to a planar region.

A solution, $T(x,t)$, to the transient heat conduction equation (equation 3.2.4) is required that goes to infinity at $x = x'$ when $\tau \to t$, but is zero for every other value of x in $0 < x < L$ when $\tau \to t$ and will satisfy the boundary conditions. The method of obtaining this solution is similar to that given in Carslaw and Jaeger[1959, pp. 359-360].

Let the solution for the instantaneous source at $\tau = 0$ and at x' be equal to,

$$u = [4\,\pi\,\alpha\,t]^{-1/2}\, e^{-(x-x')^2/(4\,\alpha\,t)}. \qquad (3.2.6)$$

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The complete solution for the temperature, T, that satisfies the differential equation and the boundary conditions is,

$$T = u + w \tag{3.2.7}$$

Since T and u are described by the transient heat conduction equation, w is also; hence w is given by the solution of a subsidiary equation,

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha}\frac{\partial w}{\partial t} \qquad 0 < x < L\ ,\ t > 0. \tag{3.2.8}$$

The Laplace transformation of the subsidiary equation for w is,

$$\frac{d^2 \bar{w}}{dx^2} - q^2 \bar{w} = 0 \qquad 0 < x < L \tag{3.2.9}$$

$$\text{for} \qquad q^2 = p/\alpha \tag{3.2.10}$$

$$\text{and} \qquad \bar{w} = \int_{t=0}^{\infty} e^{-pt}\, w(x,t)\, dt \tag{3.2.11}$$

where p is the Laplace transform parameter.

A solution for $\bar{w}$ that is convenient for satisfying boundary conditions as $x \to 0$ and $x \to L$ is of the form,

$$\bar{w} = D_1 \sinh(q\ x) + D_2 \cosh(q\ x) \tag{3.2.12}$$

where D_1 and D_2 are constants determined from the boundary conditions and sinh and cosh are the hyperbolic sine and cosine functions, which are defined as,

$$\sinh(qL) = \frac{1}{2}\left[e^{qL} - e^{-qL} \right] \tag{3.2.13}$$

and

$$\cosh(qL) = \frac{1}{2}\left[e^{qL} + e^{-qL} \right]. \tag{3.2.14}$$

The Laplace transform solution for $\bar{T}$ is,

$$\bar{T} = \bar{u} + \bar{w} \tag{3.2.15}$$

$$= \frac{L}{\alpha}\left(\frac{e^{-q|x-x'|}}{2\,q\,L} + D_1 \sinh(q\,x) + D_2 \cosh(q\,x)\right] \tag{3.2.16}$$

subject to the boundary conditions,

$$\frac{d\bar{T}}{dn_i} + \xi_i \bar{T} = 0 \qquad \text{when } x = 0 \text{ or } x = L \tag{3.2.17}$$

where $\xi_i = 0$ for the Neumann condition (second kind)

$= h_i/k$ for the Robin condition (third kind)

$= (\rho\ c\ b)_i q^2/(\rho\ c)$ for the Carslaw condition (fourth kind)

$= h_i/k + (\rho\ c\ b)_i q^2/(\rho\ c)$ for the Jaeger condition (fifth kind).

Setting k_i equal to zero will result in a boundary condition of the first kind (Dirichlet) and causes ξ_i to go to infinity.

Substituting the solution $\bar{T}$ into these boundary conditions to find the constants D_1 and D_2 for all the possible cases is a tedious and error-prone process when done by hand. The process is made more efficient when symbolic manipulation is used. In SMP, the differentiation operator D[$expr,{$x,$n,$pt}] forms the partial derivative of expression $expr successively $n times with respect to coordinate $x, evaluating the final result symbolically at the point $pt.

Differentiation of $\bar{T}$ with respect to x (for a one-dimensional slab) is executed by using the differentiation operator D[$\bar{T}$,{x,1,pt}] where the location (pt) is set to zero or the thickness L. Applying the two boundary conditions to the solution $\bar{T}$ results in two expressions and two unknowns, D_1 and D_2, as functions of sinh(qL) and cosh(qL). Expressing the hyperbolic functions in terms of negative exponentials and expanding the result in a series by the binomial theorem yields the solution $\bar{T}$ as a summation of negative exponential terms.

The solution $\bar{T}$ is inverted term by term either by the Laplace transform inversion theorem or, more simply, by a table of Laplace transforms. Some important Laplace transforms that occur when using this method can be found in Appendix A. Typically, a table of Laplace transforms is all that is necessary for the inversion of $\bar{T}$ when only a few of the terms in the series are retained.

As an example of the method, consider the X42 case, a one dimensional slab with no heat generation, a nonconvective thin film boundary condition at $x = 0$ and insulation at $x = L$. The transient heat conduction equation for this case is described in equation (3.2.4). The left boundary is described by equation (3.2.2) with h_1 and $f_1(t)$ set to zero. The right boundary condition is described by equation (3.2.2) with h_2, $(\rho cb)_2$, and $f_2(t)$ set to zero. Substituting the solution, equation (3.2.16) for $\bar{T}$ into the left boundary condition yields,

$$(1 - C_1 qL)\,\frac{e^{-qx}}{2qL} + D_1 - C_1 qL\, D_2 = 0 \qquad (3.2.18)$$

Substituting the solution into the right boundary condition yields,

$$D_1 \cosh(qL) \; + \; D_2 \sinh(qL) \; - \; \frac{e^{-q(L-x')}}{2qL} \; = \; 0 \tag{3.2.19}$$

Using matrix notation gives,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{2qL} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \tag{3.2.20}$$

where $M_{11} = 1$, $M_{12} = -C_1qL$, $M_{21} = \cosh(qL)$, $M_{22} = \sinh(qL)$, $N_1 = -(1-C_1qL)e^{-qx'}$ and $N_2 = e^{-q(L-x')}$. The determinant of the $\mathbf{M}$ matrix is equal to,

$$\text{Determinant} = \sinh(qL) + C_1qL \cosh(qL). \tag{3.2.21}$$

Expanding the hyperbolic functions using equations (3.2.13) and (3.2.14) gives,

$$\text{Determinant} = \frac{(e^{qL} - e^{-qL})}{2} + C_1qL \; \frac{(e^{qL} + e^{-qL})}{2}, \tag{3.2.22}$$

or,

$$\text{Determinant} = \frac{1}{2}(1 + C_1qL)\, e^{qL} - \frac{1}{2}(1 - C_1qL)\, e^{-qL}, \tag{3.2.23}$$

or,

$$\text{Determinant} = \frac{1}{2}(1+C_1qL)\, e^{qL} \left[1 - \frac{(1-C_1qL)}{(1+C_1qL)}\, e^{-2qL} \right]. \tag{3.2.24}$$

When 2qL is large, which is appropriate when time is small, the second term in the brackets in equation (3.2.24) goes to zero and an approximation to the determinant becomes,

$$\text{Determinant} = \frac{1}{2}\,(1+C_1qL)\;e^{qL} \qquad (3.2.25)$$

Cramer's Rule is used with the approximation to the determinant (equation (3.2.25)) to solve the matrix equation (equation (3.2.20)) for the constants D_1 and D_2.

$$D_1 = \frac{1}{2qL}\left[\frac{(1-C_1qL)}{(1+C_1qL)}\;e^{-q(2L+x')} - \frac{(1-C_1qL)}{(1+C_1qL)}\;e^{-qx'} + \frac{2C_1qL}{(1+C_1qL)}\;e^{-q(2L-x')}\right] \qquad (3.2.26)$$

$$D_2 = \frac{1}{2qL}\left[\frac{(1-C_1qL)}{(1+C_1qL)}\;e^{-q(2L+x')} + \frac{(1-C_1qL)}{(1+C_1qL)}\;e^{-qx'} + \frac{2C_1qL}{(1+C_1qL)}\;e^{-q(2L-x')}\right] \qquad (3.2.27)$$

Substituting D_1 and D_2 into the solution for $\bar{T}$ and expanding the hyperbolic functions yield the Laplace transform Green's function, $\tilde{G}$, for this case, which is,

$$\tilde{G}_{X42} = \frac{1}{2qL}\left[e^{-q(x-x')} + \frac{(1-C_1qL)}{(1+C_1qL)}\;e^{-q(x+x')} + e^{-q(2L-x-x')} + \frac{(1-C_1qL)}{(1+C_1qL)}\left(e^{-q(2L+x-x')} + e^{-q(2L-x+x')}\right)\right], \qquad (3.2.28)$$

and is valid for e^{-2qL} being small. The term qL is large when time is small.

When equation (3.2.28) is expanded with partial fractions, the coefficients of each term can be matched with a transform in a Laplace table of transforms found in Appendix A.

The exponentials in the complete series (with no approximation to the determinant) for boundary conditions of the first and second kind have simple coefficients for the transforms that lead to solutions that are valid for all times, but for large times, the solutions converge slowly.

For the more complicated boundary condition of the third, fourth or fifth kind, and for larger times with the first and second boundary condition, the coefficients of the successive exponentials in the complete series for the transforms become more complicated functions of q^2L^2; hence only the first few terms of the series are readily used and the solutions are valid for relatively small times. The coefficients for the five types of boundary conditions are calculated by combining the similar terms of the expansion. A general form is obtained below for the small time Green's functions.

Only a few terms in the small time expressions need be generated because the small time solution can be combined with the large time (Fourier) solution to get a solution that is accurate and efficient for any time. Beck and Keltner [1985] demonstrate this idea in a paper on the time partitioning of transient heat conduction solutions. Time partitioning of the Green's function allows the solution to transient heat conduction problems to be more efficient since the number of terms in the solution are tractable. The integration of a Green's function with respect to time may have poor convergence properties when the Green's function has not been properly partitioned.

Following the notation of Beck and Litkouhi [1985], the equation below can be used for all cases for small times (large qL), XIJ, where I represents the boundary condition on the left side (1,2,3,4, or 5) and J represents the boundary condition on the right side (1,2,3,4, or 5) of the slab,

$$\tilde{G}_{XIJ}(x,x',s) = \frac{L}{\alpha}\left\{ \frac{1}{2qL} e^{-q|x-x'|} \right.$$
$$+ A(a_I)\, e^{-q(x+x')} \quad + A(b_J)\, e^{-q(2L-x-x')}$$
$$\left. + B(a_I,b_J)\left[e^{-q(2L+x-x')} + e^{-q(2L-x+x')}\right] \right\} \tag{3.2.29}$$

A maximum of five terms is retained for the small time solution. The five terms in equation (3.2.29) represents the original source and four sources or sinks closest to the original source term as shown in Figure 3.2. The coefficients A(•) and B(•,•) are given below.

$$A(c) = \frac{1}{2qL} - \frac{c}{qL(qL + c)} \tag{3.2.30}$$
$$= -\frac{1}{2qL} + \frac{1}{qL+c} \tag{3.2.31}$$
$$= \frac{qL-c}{2qL(qL + c)} \tag{3.2.32}$$
$$B(c,d) = \frac{1}{2qL} - \frac{c + d}{(qL+c)(qL+d)} \tag{3.2.33}$$
$$= \frac{(qL-c)\ (qL-d)}{2qL(qL+c)(qL+d)} \tag{3.2.34}$$

If $c \neq d$, B(c,d) can be written as,

$$B(c,d) = \frac{1}{2qL} + \frac{c+d}{c-d}\left(\frac{1}{qL+c} - \frac{1}{qL+d} \right) \tag{3.2.35}$$

If $c = d$, B(c,c) can be written as,

$$B(c,c) = \frac{1}{2qL} - \frac{2c}{(qL+c)^2} \tag{3.2.36}$$

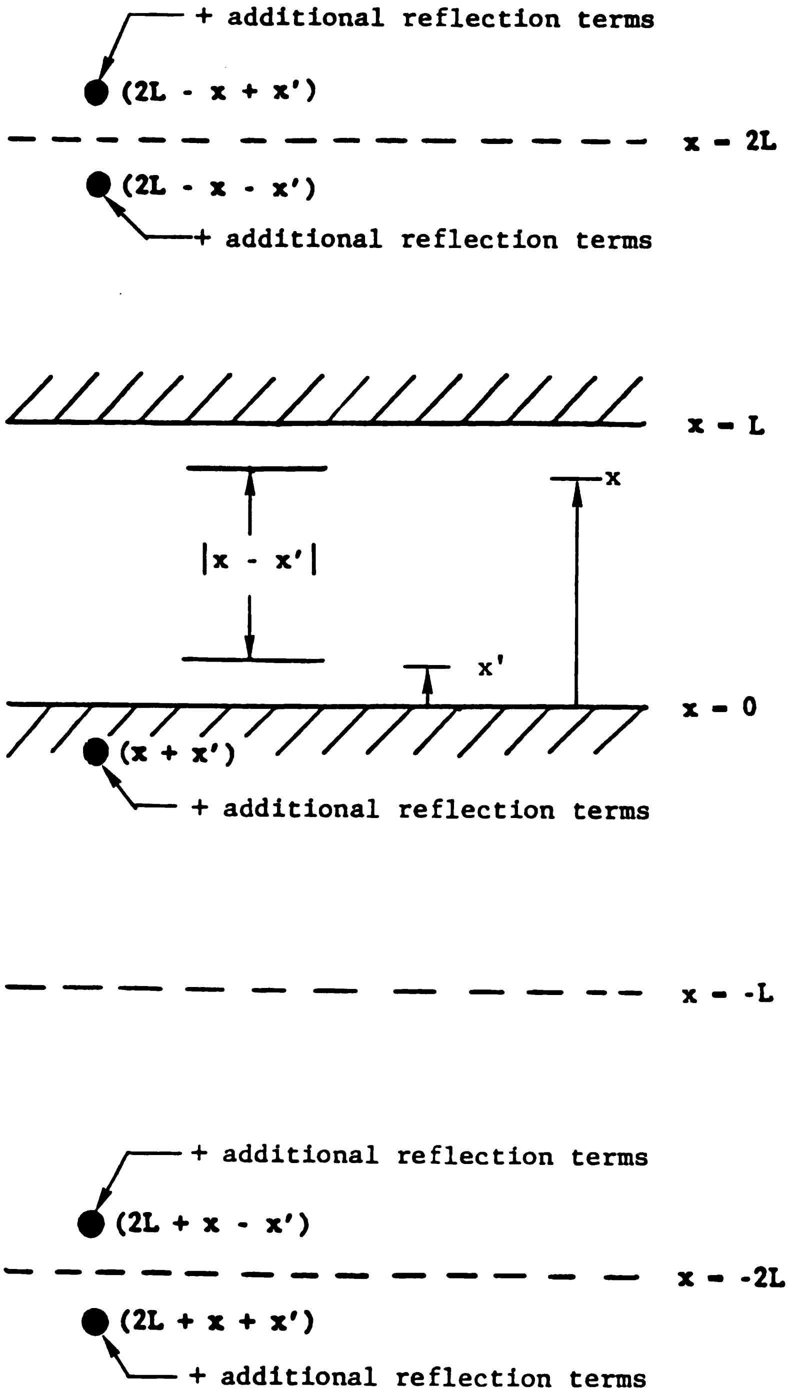

Figure 3.2 Reflections of Sources and Sinks in a Finite Body and the Locations of Additional Reflection Terms.

The a_I and b_J values used in equation (3.3.29) are given by,

$$a_1 = \infty \qquad b_1 = \infty \tag{3.2.37}$$

$$a_2 = 0 \qquad b_2 = 0 \tag{3.2.38}$$

$$a_3 = B_1 \qquad b_3 = B_2 \tag{3.2.39}$$

$$a_4 = C_1 q^2 L^2 \qquad b_4 = C_2 q^2 L^2 \tag{3.2.40}$$

$$a_5 = B_1 + C_1 q^2 L^2 \qquad b_5 = B_2 + C_2 q^2 L^2 \tag{3.2.41}$$

where,

$$B_1 = \frac{h_1 L}{k} \qquad B_2 = \frac{h_2 L}{k} \tag{3.2.42}$$

$$C_1 = \frac{(\rho c b)_1}{(\rho c L)} \qquad C_2 = \frac{(\rho c b)_2}{(\rho c L)} \tag{3.2.43}$$

Equations (3.2.37) through (3.2.41) give the simplified version of the coefficients A(•) and B(•,•) to be used in equation (3.2.29). Table 3.3 gives a summary of the coefficients of $e^{-q(x+x')}$ for boundary conditions of the first through fifth kinds and indicates the two types of coefficients that need to be transformed.

The coefficient $\frac{1}{2qL}$ and the coefficient $\frac{1}{(qL + \beta)}$ coupled with the exponential term, each have a simple transform found in the table of Laplace transforms in Appendix A. The symbol β is a constant that depends on the type of condition at the boundary. The Laplace Inversion Theorem could be applied if the transform does not appear in the table of Laplace transforms, but for small times, this is not necessary. Table 3.4 is similar to Table 3.2 except it contains the coefficients for $e^{-q(2L-x-x')}$.

Table 3.5 contains the coefficients for the two exponential terms $e^{-q(2L+x-x')}$ and $e^{-q(2L-x+x')}$. This table depends on the previous two tables with the exception of nine coefficients. The nine coefficients that appear in Table 3.5 and do not appear in Tables 3.3 and 3.4

Table 3.3 Coefficients of $e^{-q(x+x')}$

$$I = 0 \qquad A = 0$$

$$I = 1 \qquad A(a_1) = -\frac{1}{2qL}$$

$$I = 2 \qquad A(a_2) = \frac{1}{2qL}$$

$$I = 3 \qquad A(a_3) = -\frac{1}{2qL} + \frac{1}{qL+B_1}$$

$$I = 4 \qquad A(a_4) = \frac{1}{2qL} - \frac{1}{qL+1/C_1}$$

$$I = 5 \qquad A(a_5) = -\frac{1}{2qL} + \frac{1}{C_1(S_1-S_2)}\left[\frac{1}{qL-S_1} - \frac{1}{qL-S_2}\right]$$

$$\text{for } C_1 < \frac{1}{4B_1}$$

where,
$$S_1 = \frac{1}{2C_1}\left[-1 + (1 - 4B_1C_1)^{1/2}\right]$$

$$S_2 = \frac{1}{2C_1}\left[-1 - (1 - 4B_1C_1)^{1/2}\right]$$

Table 3.4 Coefficients of $e^{-q(2L-x-x')}$

$$J = 0 \qquad A = 0$$

$$J = 1 \qquad A(b_1) = -\frac{1}{2qL}$$

$$J = 2 \qquad A(b_2) = \frac{1}{2qL}$$

$$J = 3 \qquad A(b_3) = -\frac{1}{2qL} + \frac{1}{qL+B_2}$$

$$J = 4 \qquad A(b_4) = \frac{1}{2qL} - \frac{1}{qL+1/C_2}$$

$$J = 5 \qquad A(b_5) = -\frac{1}{2qL} + \frac{1}{C_2(S_3-S_4)}\left[\frac{1}{qL-S_3} - \frac{1}{qL-S_4}\right]$$

$$\text{for } C_2 < \frac{1}{4B_2}$$

where,
$$S_3 = \frac{1}{2C_2}\left[-1 + (1 - 4B_2C_2)^{1/2}\right]$$

$$S_4 = \frac{1}{2C_2}\left[-1 - (1 - 4B_2C_2)^{1/2}\right]$$

Table 3.5 Coefficients of $e^{-q(2L+x-x')}$ and $e^{-q(2L-x+x')}$

	J = 0	J = 1	J = 2	J = 3	J = 4	J = 5
I = 0	0	0	0	0	0	0
I = 1	0	$-A(a_1)$	$A(a_1)$	$-A(b_3)$	$-A(b_4)$	$-A(b_5)$
I = 2	0	$-A(a_2)$	$A(a_2)$	$A(b_3)$	$A(b_4)$	$A(b_5)$
I = 3	0	$-A(a_3)$	$A(a_3)$	$B(a_3,b_3)$	$B(a_3,b_4)$	$B(a_3,b_5)$
I = 4	0	$-A(a_4)$	$A(a_4)$	$B(a_4,b_3)$	$B(a_4,b_4)$	$B(a_4,b_5)$
I = 5	0	$-A(a_5)$	$A(a_5)$	$B(a_5,b_3)$	$B(a_5,b_4)$	$B(a_5,b_5)$

where
$$B(a_3,b_3) = \frac{1}{2qL} + \frac{B_1 + B_2}{B_1 - B_2}\left[\frac{1}{qL + B_1} - \frac{1}{qL + B_2}\right], \quad B_1 \neq B_2$$
$$= \frac{1}{2qL} - \frac{2B_1}{(qL + B_1)^2}, \quad B_1 = B_2$$
$$B(a_4,b_3) = -\frac{1}{2qL} - \frac{1 + C_1B_2}{1 - C_1B_2}\left[\frac{1}{qL + 1/C_1} - \frac{1}{qL + B_2}\right], \quad C_1B_1 \neq 1$$
$$= -\frac{1}{2qL} + \frac{2/C_1}{(qL + 1/C_1)^2}, \quad B_2C_1 = 1$$
$$B(a_5,b_3) = \frac{1}{2qL}\left[1 + (B_1 + B_2)\left(\frac{-1}{S_1S_2} + \frac{1}{(2-S_1)(2-S_2)}\right)\right]$$
$$+ \frac{1}{(qL+S_1)}\left[\frac{C_1S_1^2 + (B_1 + B_2)}{S_1(S_2 - S_1)(2 - S_1)}\right]$$
$$+ \frac{1}{(qL+S_2)}\left[\frac{C_1S_2^2 + (B_1 + B_2)}{S_2(S_1 - S_2)(2 - S_2)}\right]$$

and where S_1 and S_2 are defined in Table 3.3 and $C_1 < 1/4B_1$
$$B(a_4,b_4) = \frac{1}{2qL} + \frac{C_1 + C_2}{C_1 - C_2}\left[\frac{1}{qL + 1/C_1} - \frac{1}{qL + 1/C_2}\right], \quad C_1 \neq C_2$$
$$= \frac{1}{2qL} - \frac{2/C_1}{(qL + 1/C_1)^2}, \quad C_1 = C_2$$
$$B(a_5,b_4) = \frac{1}{2qL}\left[1 + B_1\left(\frac{-1}{S_1S_2} + \frac{1}{(2-S_1)(2-S_2)}\right)\right]$$
$$+ \frac{1}{(qL+S_1)}\left[\frac{C_1S_1^2 + B_1}{S_1(S_2 - S_1)(2 - S_1)}\right]$$

Table 3.5 (cont.)

$$+ \frac{1}{(qL+S_2)} \left[\frac{C_1 S_2^2 + B_1}{S_2(S_1 - S_2)(2 - S_2)} \right]$$

$$B(a_5,b_5) = \frac{1}{2qL} - \frac{(C_1 + C_2)S_1^2 + (B_1 + B_2)}{(S_2-S_1)(S_3-S_1)(S_4-S_1)} \left[\frac{1}{qL+S_1} \right]$$

$$- \frac{(C_1 + C_2)S_2^2 + (B_1 + B_2)}{(S_1-S_2)(S_3-S_2)(S_4-S_2)} \left[\frac{1}{qL+S_2} \right]$$

$$- \frac{(C_1 + C_2)S_3^2 + (B_1 + B_2)}{(S_1-S_3)(S_2-S_3)(S_4-S_3)} \left[\frac{1}{qL+S_3} \right]$$

$$- \frac{(C_1 + C_2)S_4^2 + (B_1 + B_2)}{(S_1-S_4)(S_2-S_4)(S_3-S_4)} \left[\frac{1}{qL+S_4} \right]$$

where S_3 and S_4 are defined in Table 3.4 and,

$B_1 > 0$, $B_2 > 0$, $C_1 > 0$, $C_2 > 0$, $C_1 < 1/4B_1$, $C_2 < 1/4B_2$

$$B(a_5,b_5) = \frac{1}{2qL} - \frac{2(C_1S_1^2 + B_1 - 2\,C_1S_1)}{(S_2 - S_1)^2} \left[\frac{1}{(qL + S_1)^2} \right]$$

$$+ \frac{C_1S_1^2}{(S_2 - S_1)} \left[\frac{1}{qL+S_1} \right]$$

$$- \frac{2(C_1S_2^2 + B_1 - 2\,C_1S_2)}{(S_2 - S_1)^2} \left[\frac{1}{(qL + S_2)^2} \right]$$

$$+ \frac{C_1S_2^2}{(S_1 - S_2)} \left[\frac{1}{qL+S_2} \right]$$

for $B_1 = B_2$, $C_1 = C_2$, and $B_1C_1 \neq 1/4$

$$B(a_3,b_4) = B(a_4,b_3)\Big|_{B_2 \to B_1,\ C_1 \to C_2}$$

$$B(a_3,b_5) = B(a_5,b_3)\Big|_{B_2 \to B_1,\ C_1 \to C_2,\ S_1 \to S_3,\ S_2 \to S_4}$$

$$B(a_4,b_5) = B(a_5,b_4)\Big|_{B_2 \to B_1,\ C_1 \to C_2,\ S_1 \to S_3,\ S_2 \to S_4}$$

are expanded and attached to the end of Table 3.5. Each of the nine coefficients can be expressed as a combination of the functions $\frac{1}{2qL}$, $\frac{1}{(qL + \beta)}$, or $\frac{1}{(qL + \beta)^2}$, where, as before, β depends on the condition at the boundary. Two of these coefficients have been previously discussed and the third coefficient can also be found in the table of Laplace transforms in Appendix A.

Equation (3.2.29), Tables 3.3, 3.4 and 3.5, and a brief table of Laplace transforms are all that is necessary to determine an approximation to the small time Green's function for boundary conditions of the first through fifth kinds. Table 3.6 gives the inverse Laplace transforms for the terms that are included in Table 3.3.

The four locations nearest the original source, along with the source's location, correspond to the locations of the exponential terms that will lead to an approximate expression of the small time Green's function. This procedure lends to itself a simple physical interpretation. Each term in the approximate series corresponds to the solution of a related problem for an infinite slab, see Figure 3.2, and thus the solution for the finite region can be interpreted as the effects of adding sources and sinks to an infinite body. Since the coefficients of equation (3.2.29) can always be written in terms of $\frac{1}{2qL}$ + additional terms, the approximate solution to problems with boundary conditions of the first through fifth kinds will include the solution for a slab that is insulated at both boundaries. The choice of placing a source or sink at a particular location depends on whether the $\frac{1}{2qL}$ term is positive or negative. A positive value for this term gives a source at the location. Any additional terms associated with the location represent the effects of a boundary condition that is not insulated. Figure 3.2 shows

Table 3.6 Inverse Laplace Transforms of $A(\cdot)\ e^{-qx}$

I	Inverse Laplace Transform
0	0
1	$-\frac{\alpha}{L^2}\ EX(x,t)$
2	$\frac{\alpha}{L^2}\ EX(x,t)$
3	$\frac{\alpha}{L^2}\left[EX(x,t) - B_i\ ER(x,t,B_i) \right]$
4	$-\frac{\alpha}{L^2}\left[EX(x,t) - C_i^{-1} ER(x,t,C_i^{-1}) \right]$
5	$-\frac{\alpha}{L^2}\left\{ EX(x,t) - \frac{1}{C_i(S_1 - S_2)}\left[ER(x,t,S_2) - ER(x,t,S_1) \right] \right\}$

$$EX(x,t) = [4\pi t^*]^{-1/2}\ e^{-\frac{(x^+)^2}{4t^*}},\quad x^* = \frac{x}{L},\quad t^* = \frac{\alpha t}{L^2}$$

$$ER(x,t,u) = e^{-(x^*)^2/(4t^*)}\ \mathrm{rerf}\left[\frac{x^*}{(4t^*)^{1/2}} + u\ (t^*)^{1/2} \right]$$

where the function $\mathrm{rerf}(z) = e^{z^2}\ \mathrm{erfc}(z)$, t^* = dimensionless time, and x^* is the normalized coordinate.

the locations of the sources and sinks for a slab along with the locations of any additional terms that might occur for a boundary that is not insulated.

The procedure may be extended to include more reflections of the source by retaining the second term in equation (3.2.19) for the determinant of matrix equation (3.2.20). A third term, C(c,d), appears in the coefficient list of equation (3.2.29) and multiplies the reflection locations $e^{-q(2L+x+x')}$ and $e^{-q(4L-x-x')}$. The coefficient C(c,d) is,

$$C(c,d) = \frac{(qL-c)^2\,(qL-d)}{2qL(qL+c)^2\,(qL+d)} \quad \text{or} \quad \frac{(qL-d)^2\,(qL-c)}{2qL(qL+d)^2\,(qL+c)} \,. \tag{3.2.44}$$

The third term, C(c,d), complicates the evaluation of the solution by including additional terms and functions that are not easily transformed. Recently, a paper dealing with the generalization and application of Laplace transformation formulas in diffusion problems [Shibata and Kugo, 1983] has eased the calculation for some of the inverse transforms of the coefficient C(c,d), but for small dimensionless times, the term C(c,d) is not necessary.

3.3 Green's Functions for Some Semi-infinite Cases in One Dimension

The objective of this section is to show the effects of boundary conditions on some semi-infinite geometries. The Green's functions for semi-infinite geometries are developed from the source solution and, for boundary conditions of the first, second and third kinds, can be found in Carslaw and Jaeger [1959].

The basic building block Green's function for infinite and semi-infinite bodies (called the source solution by Carslaw and Jaeger [1959, pg. 50]) is,

$$G(x,x'|t,\tau) = [4\ \pi\ \alpha(t-\tau)]^{-1/2} \exp[-(x-x')^2/(4\ \alpha(t-\tau))]. \quad (3.3.1)$$

This function represents a unit impulse occurring at time τ and at position x'. In an infinite or semi-infinite medium the impulse has an effect on the medium a long time after the impulse is generated. This effect is shown in the figures below.

Figure 3.3 is a plot of the Green's function of a semi-infinite body normalized with respect to the position of the point of interest for boundary conditions of the zeroth, first, and second kind. The impulse occurs at $x'=0$ and $\tau=0$. One curve is the function $1/\sqrt{(4t^*)}$, which is the leading coefficient of the Green's function for an infinite body, where the symbol t^* is the dimensionless time. The Green's function for an infinite medium, represented by $n = 0$, is, as expected, about one half the value the leading coefficient curve except at small dimensionless times because the impulse can move in two directions. When $n = 1$, which means that there is a boundary condition of the first kind occurring at the surface of a semi-infinite body, the Green's function is zero for all times. This means the effect of an impulse at the surface of a body with a boundary condition of the first kind at that surface is zero.

When $n = 2$, which means a boundary condition of the second kind occurs at the surface of a semi-infinite body, the curve is twice the value of the infinite curve after a dimensionless time of five. The boundary condition of the second kind reflects the impulse which is the

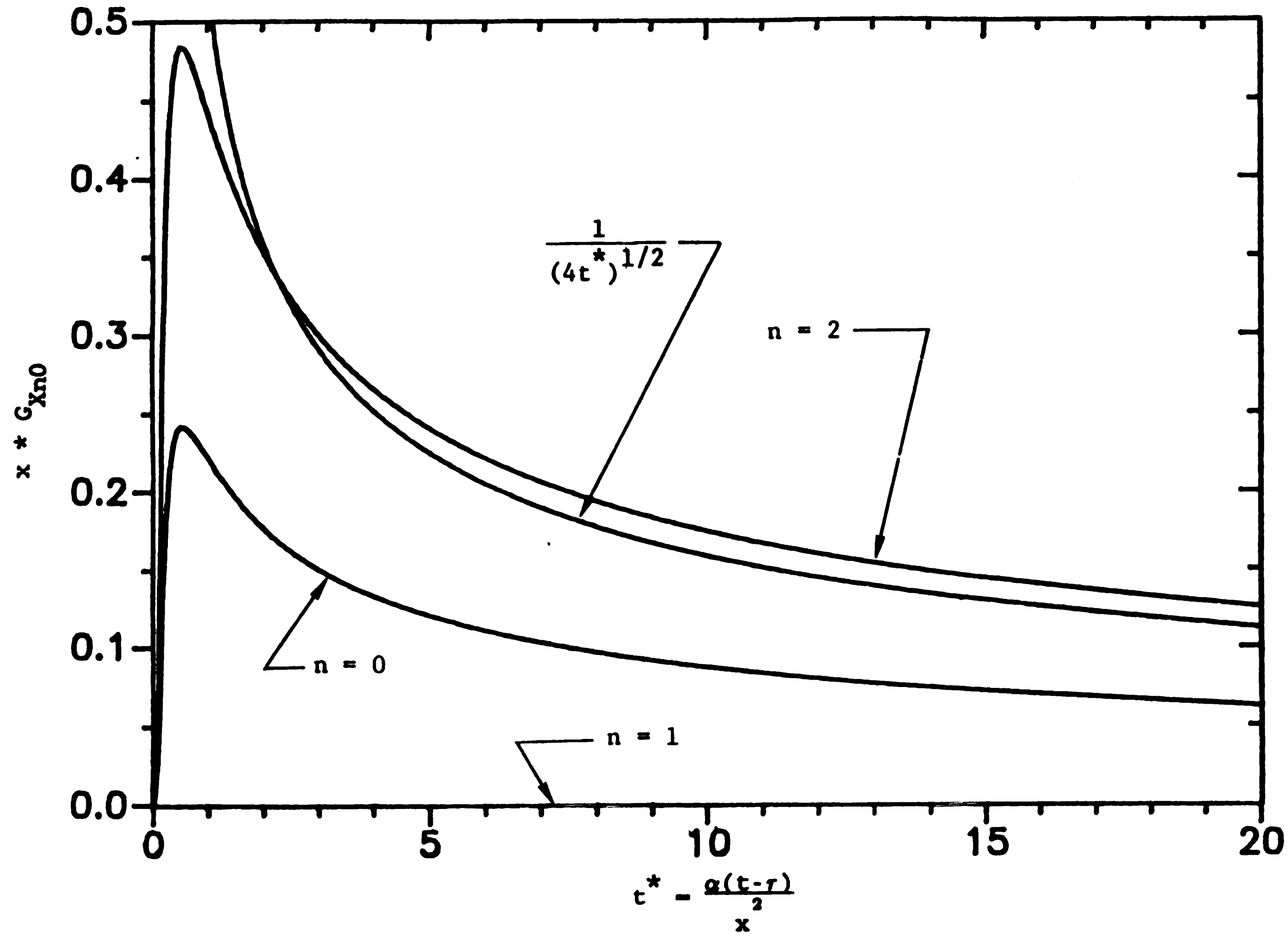

Figure 3.3 Green's Function Versus Time for a Semi-infinite Body With Boundary Conditions of the Zeroth, First, and Second Kind.

cause for the doubling. All of the curves, except when $n = 1$, converge very slowly as the time increases.

Figure 3.4 is a plot of the Green's function for a one dimensional, semi-infinite body with a convective boundary condition (X30) at the surface, normalized with respect to x'. The three curves show the effect of increasing the Biot number on the Green's function. The $1/\mathrm{Sqrt}(4\ t^*)$ curve is shown for reference. If the Biot number is very small, the Green's function approaches the Green's function for an insulated case (X20) as expected. As the Biot number increases, the Green's function decreases until it becomes zero, which represents a boundary condition of the first kind.

Figure 3.5 is a plot of the Green's function for a semi-infinite body in one dimension with a nonconvective thin film at the surface (X40) and normalized with respect to x'. The three curves represent the effect of increasing the Carslaw number, C_1, which is the thermal storage capacity of the thin film divided by the thermal storage capacity of the solid. When the thermal capacity of the thin film approaches zero, the Green's function approaches the Green's function for an insulated body (X20).

Figure 3.6 are the Green's functions, normalized with respect to the source location, x', for various positions of interest and dimensionless times. Curves for an infinite medium (Figure 3.6a, X00) and a semi-infinite medium with boundary conditions of the first (Figure 3.6b, X10) and second kind (Figure 3.6c, X20) are shown. An important feature of these plots is the shape of the Green's function at very small times. Notice for $t^* < 0.05$, the shape of the curves are independent of the type of boundary condition that occurs at the surface when the location of the source and the point of interest coincide.

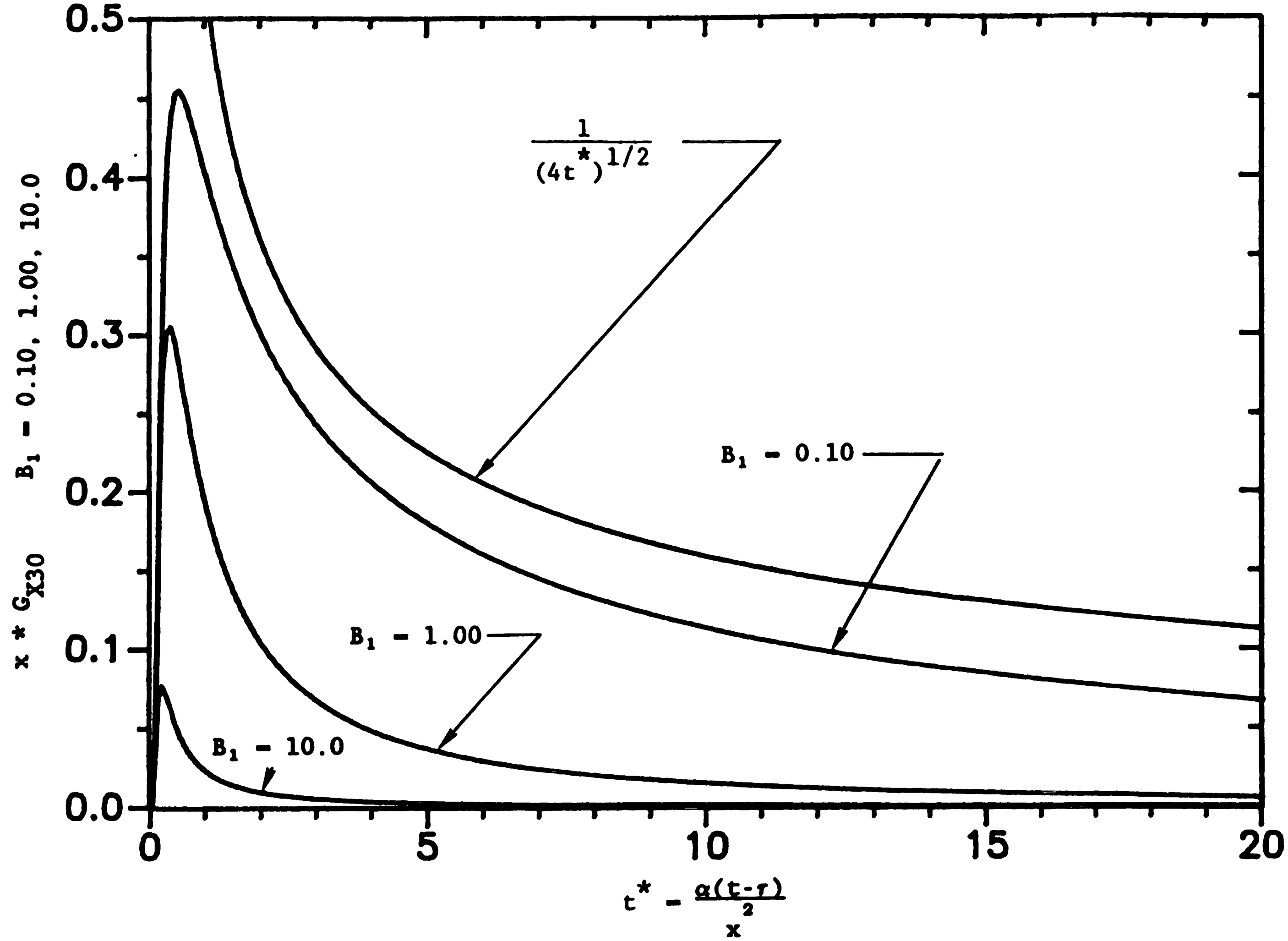

Figure 3.4 Green's Function Versus Time for a Semi-infinite Body With a Boundary Condition of the Third Kind and Various Values of the Parameters.

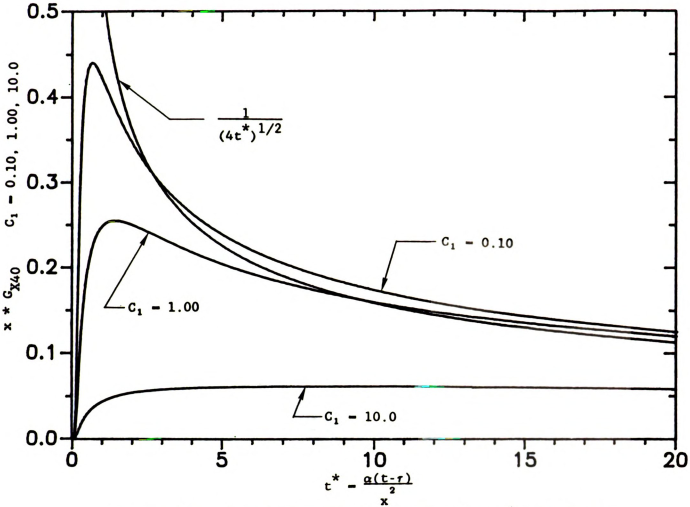

Figure 3.5 Green's Function Versus Time for a Semi-infinite Body With a Boundary Condition of the Fourth Kind and Various Values of the Parameters.

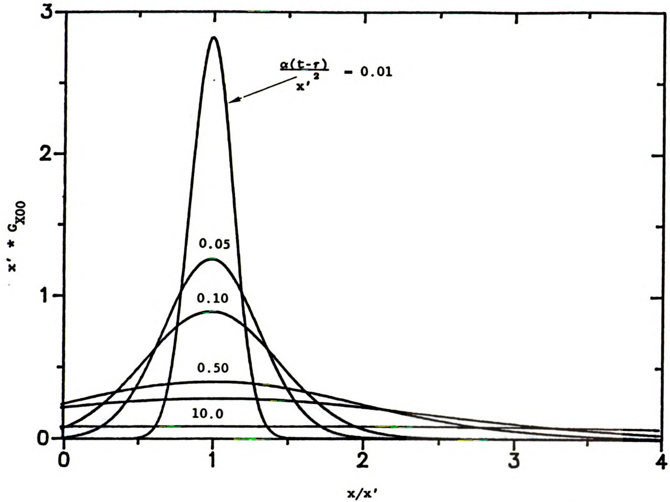

Figure 3.6a Green's Function Versus Dimensionless Position for an Infinite Body.

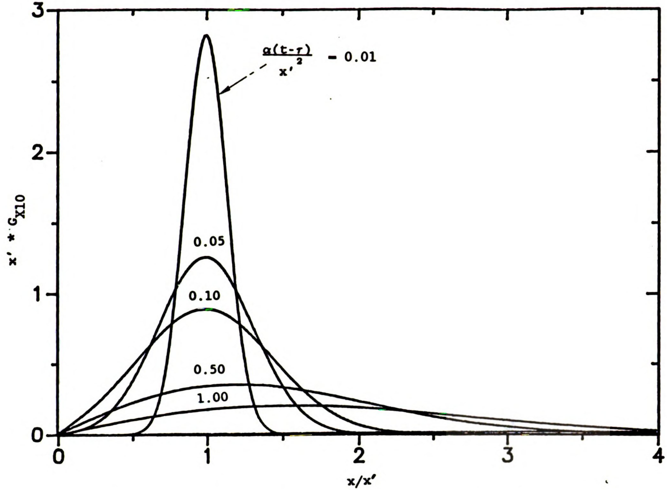

Figure 3.6b Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the First Kind.

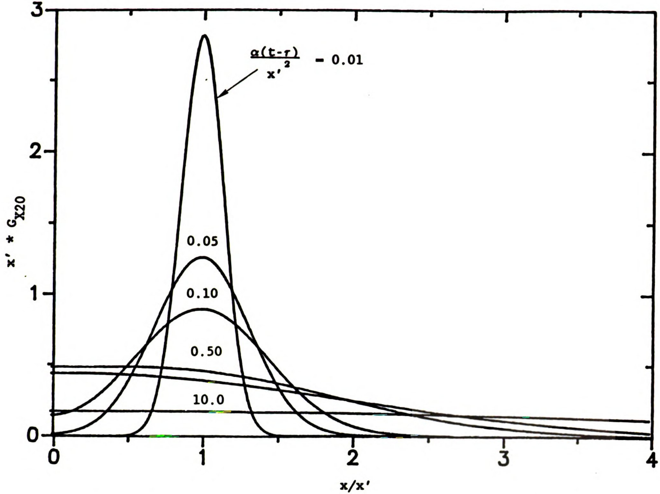

Figure 3.6c Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Second Kind.

Another important feature of these curves is the difference between the curves when the time is not small and the point of interest is at the surface ($x = 0$). For boundary conditions of the first kind (X10), the Green's function is zero for all times. The Green's function for the zeroth boundary (X00) is one half the Green's function for a boundary condition of the second kind (X20). The result is expected and it has been shown previously.

Figures 3.7a, 3.7b, and 3.7c show the Green's functions for a semi-infinite body, normalized with respect to x', with a boundary condition of the third kind on the surface (X30). The three plots represent the Biot number increasing by factors of ten from 0.1 to 10. When the Biot number is small, the X30 Green's function approach the Green's function for a boundary of the second kind (X20), as expected. When the Biot number is large, the X30 Green's function approaches the Green's function for a boundary condition of the first kind (X10).

Similar conclusions can be observed from Figures 3.8 and 3.9 which are the Green's functions for a semi-infinite body with boundary conditions of the fourth (X40) and fifth (X50) kind occurring at the surface with various parameter values.

3.4 **Small Time Green's Functions for Finite Bodies**

The objective of this section is to use the general results of Section 3.2 to generate approximate Green's functions that are accurate and efficient at small times.

Two example problems will be discussed in this section. Both problems involve a one dimensional slab with constant thermal conductivity, k, constant thermal diffusivity, α, and no heat generation. The

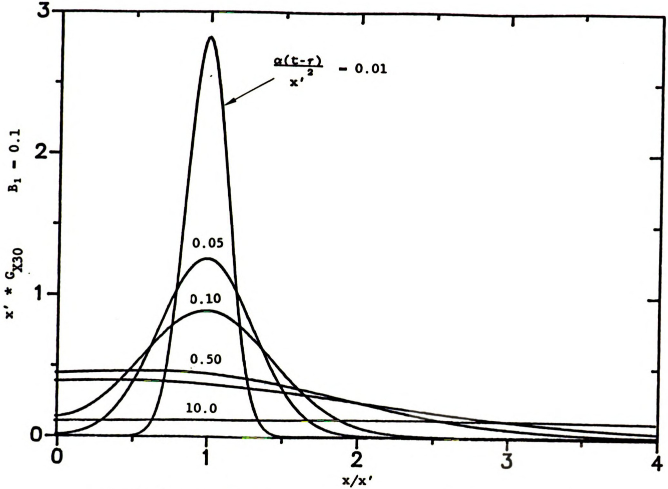

Figure 3.7a Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Third Kind, B1 = 0.1.

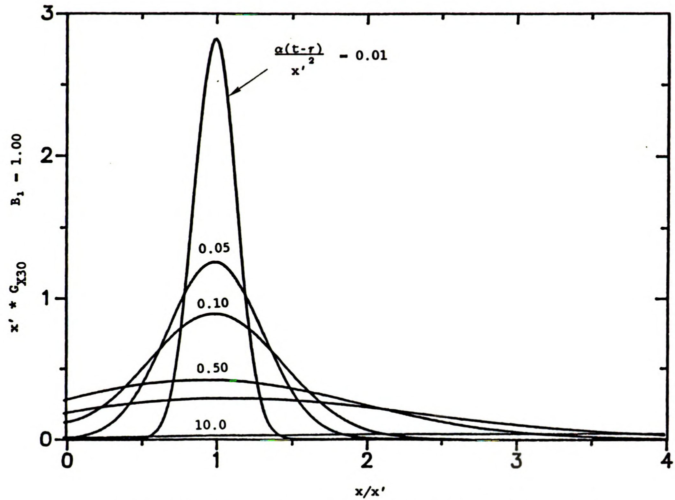

Figure 3.7b Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Third Kind, B1 - 1.0.

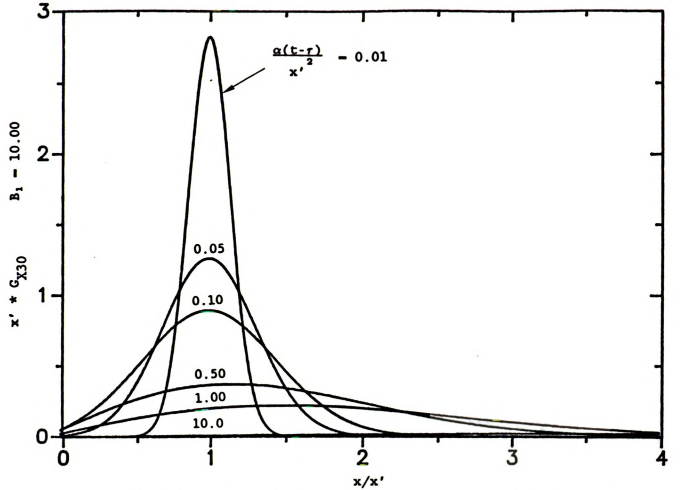

Figure 3.7c Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Third Kind, B1 = 10.0.

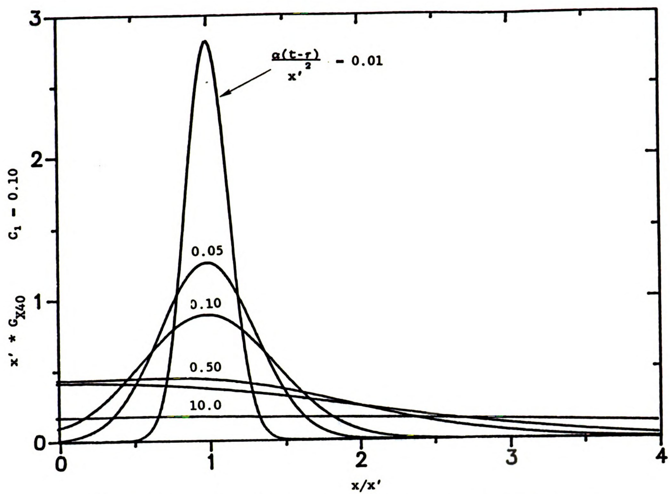

Figure 3.8a Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fourth Kind, C1 - 0.1.

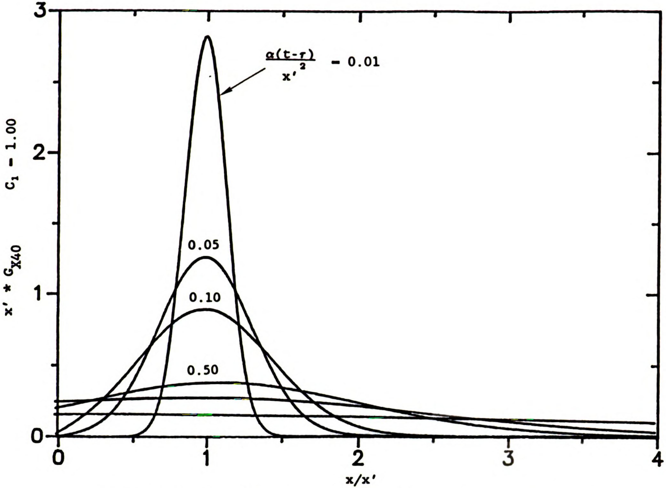

Figure 3.8b Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fourth Kind, Cl - 1.0.

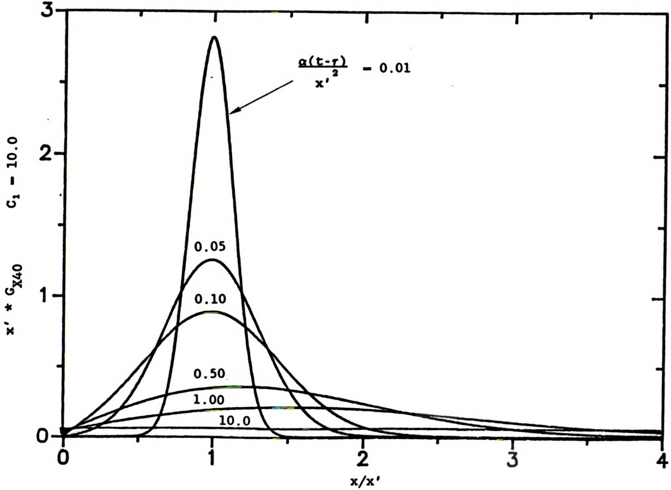

Figure 3.8c Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fourth Kind, C1 - 10.0.

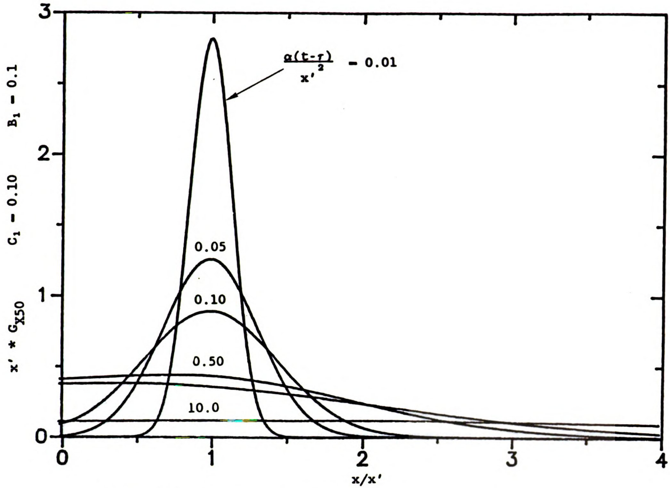

Figure 3.9a Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fifth Kind, C1 - 0.1, B1 - 0.1.

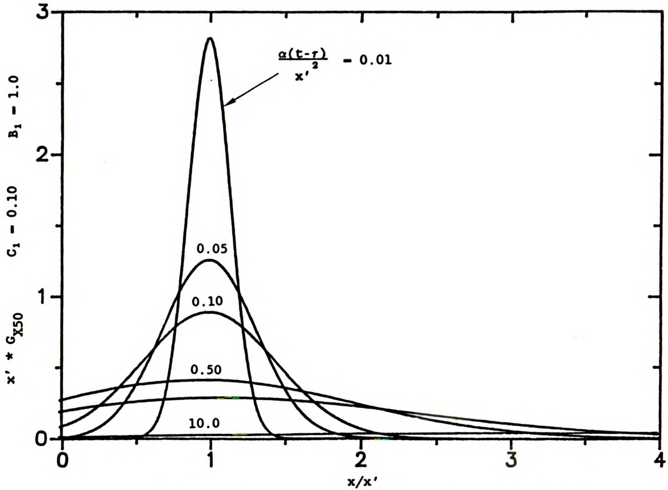

Figure 3.9b Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fifth Kind, Cl - 0.1, Bl - 1.0.

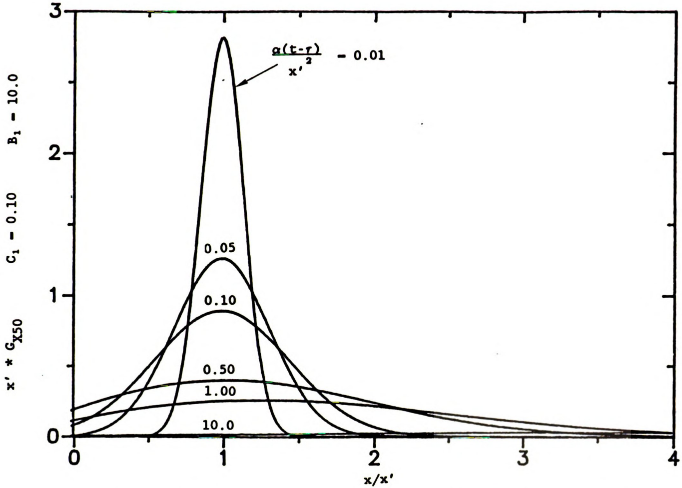

Figure 3.9c Green's Function Versus Dimensionless Position for an Finite Body With a Boundary Condition of the Fifth Kind, C1 = 0.1, B1 = 10.0.

transient heat conduction equation that describes this case is given in equation (3.2.4). If $\tau \neq 0$ in the following expressions, substitute for the time t the value $(t - \tau)$. The first example has insulated boundary conditions on both sides (Figure 3.10a) and the second example has a nonconvective thin film (Figure 3.10b) at $x = 0$ and is insulated at $x = L$. The first example is called a X22 case since it has boundary conditions of the second kind on either side, while the second example is called a X42 case since it has a boundary condition of the fourth kind on the left boundary and is insulated on the right boundary.

3.4.1 <u>Slab Insulated On Both Boundaries (X22)</u>

For a slab insulated on both boundaries (X22), the coefficients from Tables 3.3, 3.4 and 3.5 are,

$$A[0] = \frac{1}{2qL} \quad \text{and} \quad B[0,0] = \frac{1}{2qL}$$

since $a_1 = 0$ and $b_2 = 0$. Adding the five terms of the Laplace transform solution for small times for the Green's function, equation (3.2.29) gives an approximate solution of the form,

$$\tilde{G}_{X22} = \frac{L}{\alpha}\left\{ \frac{1}{2qL}\left(e^{-q|x-x'|} + e^{-q(x+x')} + e^{-q(2L-x-x')} + e^{-q(2L+x-x')} + e^{-q(2L-x+x')} \right)\right\} \tag{3.4.1}$$

Using a table of Laplace transforms (Appendix A) or Table 3.6 for the inversion of these types of Laplace transform gives an approximation to the small time Green's function for the X22 case as,

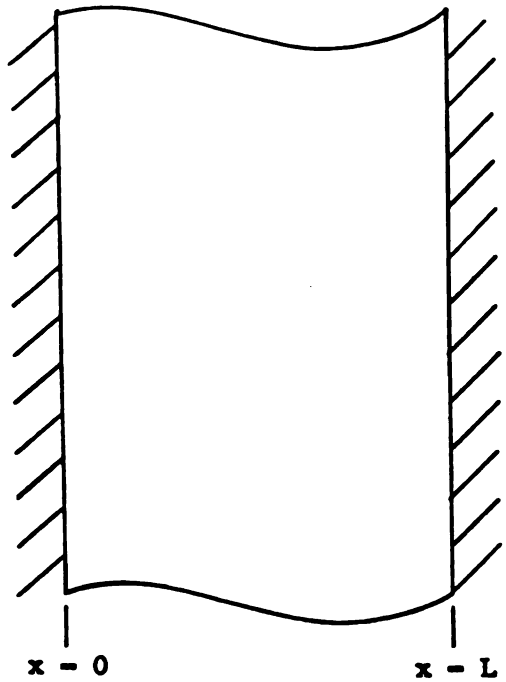

Figure 3.10a Finite Body Insulated on Two Sides.

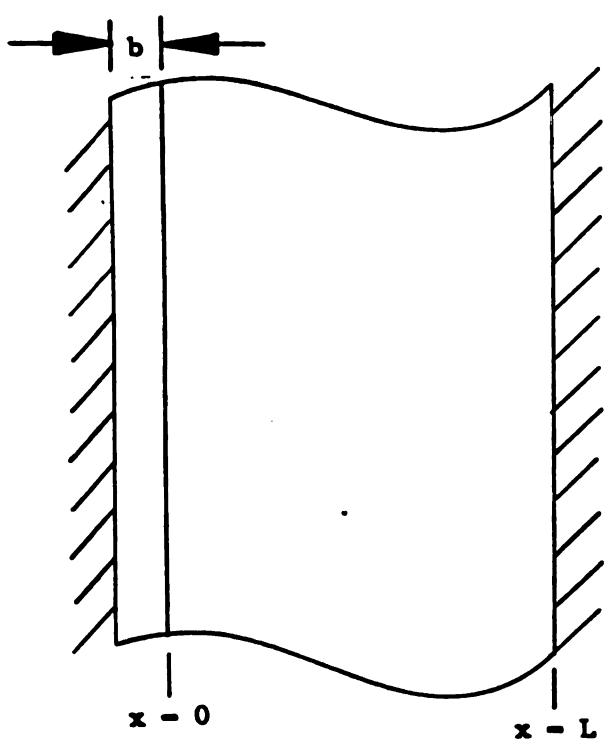

Figure 3.10b Finite Body Insulated on One Side and With a Non-convective Thin Film on the Opposite Side.

$$G_{X22}(x,x'|t,0) = \frac{1}{L}\,[4\,\pi\,(\alpha\,t/L^2)]^{-1/2}\left(e^{-(x_L-x_L')^2/(4\,\alpha\,t/L^2)} + e^{-(x_L+x_L')^2/(4\,\alpha\,t/L^2)} + e^{-(2-x_L-x_L')^2/(4\,\alpha\,t/L^2)} + e^{-(2+x_L-x_L')^2/(4\,\alpha\,t/L^2)} + e^{-(2-x_L+x_L')^2/(4\,\alpha\,t/L^2)} \right) \quad (3.4.2)$$

where $x_L = \frac{x}{L}$.

If the dimensionless time $(\alpha\,t/L^2)$ is less than 0.1, this function is accurate to five decimal places. The Fourier or long time Green's function to the X22 case from Beck [1986] is,

$$G_{X22}(x,x'|t,0) = \frac{1}{L}\left(1 + 2\sum_{m=1}^{\infty} e^{-m^2\pi^2\alpha t/L^2}\cos(m\,\pi\,x/L)\,\cos(m\,\pi\,x'/L) \right) \quad (3.4.3)$$

A plot of the number of terms necessary for convergence to within 0.00001 versus dimensionless time $(\alpha\,t/L^2)$ is shown in Figure 3.11 for the X22 case when the point of interest and the location of the source term are both at the left boundary. Notice that as the dimensionless time gets small, the number of terms in the small time solution goes down, while the number of terms in the long time solution goes up. By keeping the dimensionless time less than 0.025, the number of terms in the LT method is reduced to three. When the Green's functions are transformed into real time and used in the formalism to determine temperature distributions, the integration over τ is possible. Additional terms leads to integration over τ that are in closed form.

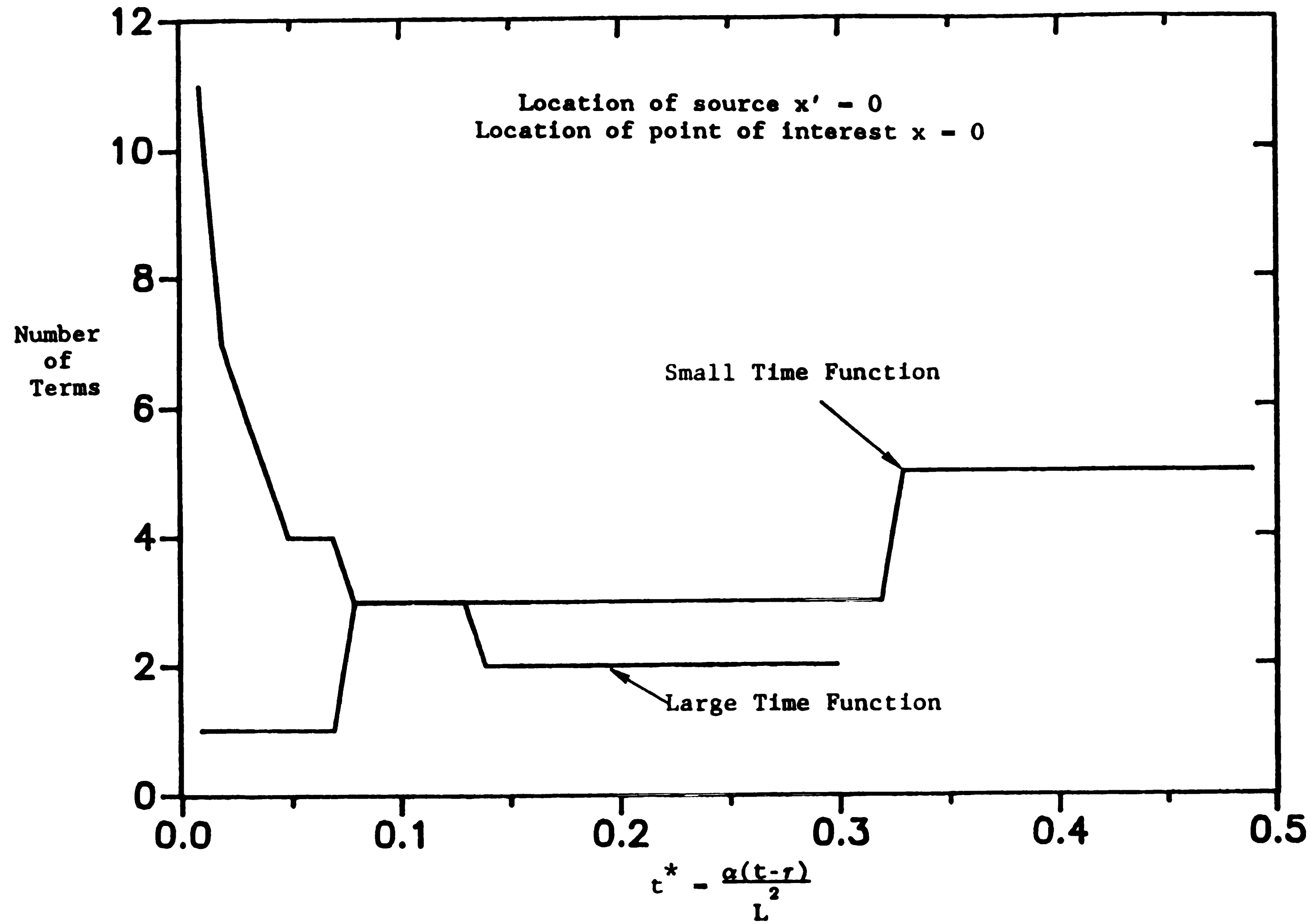

Figure 3.11 Number of Terms for Convergence of the Green's Function Versus Dimensionless Time for an Insulated Body (X22). Accuracy = 0.00001.

3.4.2 Slab With a Carslaw Left Boundary Condition and Right Boundary Insulated (X42)

For the X42 case, the coefficients from Tables 3.3, 3.4, and 3.5 are,

$$A[a_4] = \frac{1}{2qL} - \frac{1}{(qL+1/C_1)}$$

$$B[a_4,b_2] = \frac{1}{2qL} - \frac{1}{(qL+1/C_1)}$$

Adding the five terms to get an approximate Laplace transform solution gives,

$$\tilde{G}_{X42} = \tilde{G}_{X22} - \frac{1}{(qL+1/C_1)}\left[e^{-q(x+x')} + e^{-q(2L+x-x')} + e^{-q(2L-x+x')} \right] \tag{3.4.4}$$

where $\bar{G}_{X22}$ is the Laplace transform of the Green's function of the X22 case. The additional three terms represent the reflections of the images of the sources located at (x+x'), (2L + x - x'), and (2L - x + x'). Using a table of Laplace transforms (Appendix A) for the inversion of these types of Laplace functions gives an approximation to the small time Green's function for the X42 case as,

$$\begin{aligned} G_{X42}(x,x'|t,0) = G_{X22} &- \frac{2}{L}\,\mathrm{EX}[x+x',t] + \frac{1}{LC_1}\,\mathrm{ER}[x+x',t,C_1^{-1}] \\ &- \frac{2}{L}\,\mathrm{EX}[2L+x-x',t] + \frac{1}{LC_1}\,\mathrm{ER}[2L+x-x',t,C_1^{-1}] - \frac{2}{L}\,\mathrm{EX}[2L-x+x',t] \\ &+ \frac{1}{LC_1}\,\mathrm{ER}[2L-x+x',t,C_1^{-1}] \end{aligned} \tag{3.4.5}$$

where, $EX[x^+,t^+] = [4\pi t^+]^{-1/2} e^{x^+/4t^+}$,

$ER[x^+,t^+,P] = e^{x^+P + t^+P^2} \operatorname{erfc}\left(x^+/\sqrt{4t^+} + P\sqrt{t^+} \right)$

and $x^+ = x/L$, $t^+ = \alpha t/L^2$.

This solution is accurate to five decimal places for dimensionless time ($\alpha t/L^2$) less than 0.1.

The Fourier solution to the X42 case from Beck [1986] is,

$$G_{X42} = (1/L)\left(\frac{1}{N_0} + \sum_{m=1}^{\infty}\left[\frac{e^{-\beta_m^2 \alpha t/L^2}}{N_m} X_m(x)\, X_m(x')\right]\right) \quad (3.4.6)$$

where, $X_m(x) = \cos(\beta_m x/L) - (\beta_m C_1)\sin(\beta_m x/L)$ (3.4.7)

$$N_0 = 1 + C_1 \quad (3.4.8)$$

$$N_m = ((\beta_m C_1)^2 + C_1 + 1)/2 \quad (3.4.9)$$

$$\beta_m \cot(\beta_m) + 1/C_1 = 0 \quad (3.4.10)$$

Equation (3.4.5) converges rapidly when ($\alpha t/L^2$) > 0.025. Small and large time Green's functions for the Carslaw number equal to 0.10 are shown in Table 3.7. The small time solution is composed of the small time solution for the X22 case plus six additional terms. It is important to notice that as the dimensionless time gets small, the number of terms in the small time solution goes down, while the number of terms in the long time solution goes up.

Table 3.7 X42 Case C1 = 0.10
Small and Large Time for Various Number of Terms

Dimensionless time	Approximate small time Green's function	Number of terms	Large time Green's function	Number of terms
0.001	8.92062	1	5.64245	4
	7.23578	4	6.87825	8
	7.23578	8	7.16274	12
0.005	3.98942	1	4.86146	4
	5.23157	4	5.22249	8
	5.23157	8	5.23150	12
0.010	2.82095	1	4.18044	4
	4.27584	4	4.27565	8
	4.27584	8	4.27583	12
0.050	1.26157	1	2.32323	4
	2.32326	4	2.32326	8
	2.32326	8	2.32326	12
0.100	0.89206	1	1.70582	4
	1.70582	4	1.70582	8
	1.70582	8	1.70582	12
0.500	0.39894	1	0.93720	4
	0.84412	4	0.93720	8
	0.91489	8	0.93720	12

3.5 Summary

A method for generating and organizing the small time Green's functions is given for boundary conditions of the first through third kind, and extended to two additional boundary conditions called the fourth (Carslaw) and fifth (Jaeger) kinds. The small time Green's functions developed here can be formed into fundamental building blocks, called influence functions, since they are the response of the system to a unit point source. The small time Green's function can be used in conjunction with the large time Green's function to generate a Green's function that is efficient at both small and large times. The availability of symbolic software makes the procedure for finding the small time Green's function both attractive and efficient.

The following equation and Table 3.8 are a summary of the small time Green's function for boundary conditions of the zeroth through fifth kind for dimensionless time less than 0.025.

$$
\begin{aligned}
G_{XIJ}(x,t|x',\tau) = {} & [4\pi\alpha(t-\tau)]^{-1/2} \cdot \\
& \left[e^{-\frac{(x-x')^2}{4\alpha(t-\tau)}} + M\, e^{-\frac{(x+x')^2}{4\alpha(t-\tau)}} + N\, e^{-\frac{(2L-x-x')^2}{4\alpha(t-\tau)}} \right] \\
& + \frac{1}{L} \Bigg[- M\, \theta_1\, ER(x-x',t-\tau,\theta_1) - N\, \theta_2\, ER(2L-x-x',t-\tau,\theta_2) \\
& + \frac{E_1}{(1-4B_1C_1)^{1/2}} \left\{ S_2\, ER(x+x',t-\tau,S_2) - S_1\, ER(x+x',t-\tau,S_1) \right\} \\
& + \frac{E_2}{(1-4B_2C_2)^{1/2}} \left\{ S_4\, ER(2L-x-x',t-\tau,S_4) - S_3\, ER(2L-x-x',t-\tau,S_3) \right\} \Bigg],
\end{aligned}
\tag{3.5.1}
$$

where $ER(\cdot,\cdot,\cdot)$ and S_i are defined in Table 3.6.

Table 3.8 Small time Green's functions for $\alpha(t-\tau)/L^2 \leq 0.025$

Case No.	M	N	θ_1	θ_2	E_1	E_2
X00*	0	0	0	0	0	0
X10*	-1	0	0	0	0	0
X11	-1	-1	0	0	0	0
X12	-1	1	0	0	0	0
X13	-1	1	0	B_2	0	0
X14	-1	-1	0	C_2^{-1}	0	0
X15	-1	-1	0	0	0	1
X20*	1	0	0	0	0	0
X21	1	-1	0	0	0	0
X22	1	1	0	0	0	0
X23	1	1	0	B_2	0	0
X24	1	-1	0	C_2^{-1}	0	0
X25	1	-1	0	0	0	1
X30*	1	0	B_1	0	0	0
X31	1	-1	B_1	0	0	0
X32	1	1	B_1	0	0	0
X33	1	1	B_1	B_2	0	0
X34	1	-1	B_1	C_2^{-1}	0	0
X35	1	-1	B_1	0	0	1
X40*	-1	0	C_1^{-1}	0	0	0
X41	-1	-1	C_1^{-1}	0	0	0
X42	-1	1	C_1^{-1}	0	0	0
X43	-1	1	C_1^{-1}	B_2	0	0
X44	-1	1	C_1^{-1}	C_2^{-1}	0	0
X45	-1	-1	C_1^{-1}	0	0	1
X50*	-1	0	0	0	1	0
X51	-1	-1	0	0	1	0
X52	-1	1	0	0	1	0
X53	-1	1	0	B_2	1	0
X54	-1	-1	0	C_2^{-1}	1	0
X55	-1	-1	0	0	1	1

* Valid for any time

CHAPTER 4

TRANSIENT ONE DIMENSIONAL CANSS PROGRAM

4.1 Introduction

A program that solves linear, transient heat diffusion problems in the cartesian coordinate system for a variety of boundary conditions, initial conditions, and heat generation terms in one dimension is presented in this chapter. The program includes infinite, semi-infinite and finite slab geometries. The boundaries of the slabs are restricted to be parallel to the cartesian coordinate axes. The program will calculate temperature distributions of nonlinear problems for certain combinations of physical properties as shown in Chapter 2.

The one dimensional computer algebraic, numeric, and symbolic solution program, (CANSS), is compiled in a computer algebraic software system called the symbolic manipulation program (SMP). SMP [1983] is one of a group of computer software programs designed to analyze simple or complex mathematical problems interactively or in batch mode. A partial list of other programs that manipulate expressions and symbols is MACSYMA [The MATHLAB Group, 1985], MAPLE [Char, et. al., 1985], REDUCE [Hearn, 1985] and mu-MATH [mu-MATH, 1985].

The one dimensional CANSS program is comprised of a main procedure, which directs the overall analysis of the indicated problem, and a group of specialized sub-procedures or libraries. The sub-procedures contain functions for developing the small and large time Green's functions and include provisions for treating the boundary and initial conditions and the volume energy source terms. Moreover, integration of special functions is accomplished using the sub-procedures. The temperature distributions developed by the CANSS algorithm cover the entire range of dimensionless time.

The CANSS program runs on a DEC MicroVax under the VMS operating system. This device is a virtual memory machine with approximately eight megabytes of main memory. One element of a one dimensional symbolic calculation of a temperature distribution uses about 30 to 40 CPU seconds.

Section 4.2 of this chapter explains in detail the input and operation to the CANSS algorithm. Section 4.3 shows results of three one dimensional problems solved using the CANSS program. Section 4.4 describes some integrals that occur during the solution of one dimensional problems. Section 4.5 describes time partitioning in the one dimensional case. Section 4.6 is a flowchart/example of the logic of the CANSS program. Section 4.7 summarizes the chapter.

4.2 One Dimensional CANSS Program

The one dimensional CANSS program is designed to generate symbolic temperature distributions that are rapidly convergent for small and large dimensionless times in a infinite, semi-infinite, and finite slab. The one dimensional slab may have non-homogeneous boundary conditions of the zeroth through fourth kind occurring on the surfaces that are functions of time but not functions of position. The slab may have an initial condition that is a function of the spatial coordinate and a volume energy heat source that is a constant in time and space.

The forcing function applied to the boundaries are of the form,

$$f_i(t) = C_i \, t^{n/2} \, , \qquad (4.2.1)$$

where n = -1, 0, 1, 2, ..., i=1 is for the left hand boundary and i=2 is for the right hand boundary. Equation (4.2.1) allows the forcing functions on the boundary of a one dimensional slab to be functions of time, constant or zero depending on the value of the constants C_i and n.

The initial condition is a function of the spatial coordinate only and is restricted to polynomials in the CANSS program. It is noted that the CANSS program can solve some problems that have initial condition that are transcendental and of the form,

$$F(x) = T_0 \sin(\pi \, x/L) \quad \text{or} \quad T_0 \cos(\pi \, x/L), \qquad (4.2.2)$$

where T_0 is a constant temperature and the coordinate x is normalized with respect to the length of the slab, L. In the case of a semi-infinite or infinite slab, the coordinate x is normalized by a unit measure of length.

The volume energy heat source is constant in both position and time. The heat source must be constant because many of the solutions to the integrals necessary to solve non-constant heat sources that are a function of time or position are not available.

Tzeng and Beck [1985] describe a numbering system data base to solutions of heat diffusion. The numbering system is used to catalog solutions and in a numerical program to evaluate new solutions. A short su

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summary of the numbering system parameters used in this thesis is excerpted and shown in Table 4.1.

The first set of parameters in the numbering system give the boundary condition types at the surfaces of the body. The second set of parameters gives the time variation of the boundary condition. The initial condition is described along with additional parameters for heat generation, fin type terms, etc. The examples discussed in the next section show how the numbering system is applied to heat conduction problems.

Applying this system to the CANSS program yields distinct cases for four semi-infinite slabs (X10 ... X40) and ten finite slabs (X11 - X44). The four distinct semi-infinite cases are allowed five types of time variation on the boundary (B0 ... B4), four types of initial conditions (T0,T1,T2,T6), and two types of volume energy heat sources (G0,G1). The ten distinct finite cases are allowed five types of boundary condition on each side (B0 ... B4), four types of initial conditions (T0,T1,T2,T6), and two types of volume energy heat sources (G0,G1). The CANSS program will give closed form expressions for over two hundred (204) distinct problems in heat diffusion. These cases do not include the superposition of solutions due to the linearity of the diffusion problem.

The fact that many types of convolution integrals are not available analytically is the limiting factor for all non-homogeneous terms in the analytical Green's function approach for obtaining temperature distributions. The ability of the CANSS integration algorithm of recognizing and symbolically solving the integrals is crucial to the solution of the problem. Many integration procedures are included in the internal SMP integrator, but due to the complex convolution nature of the Green's function approach, the internal SMP integrator may not calculate

Table 4.1 Key for Heat Conduction Data Base (Excerpted from Tzeng and Beck [1985])

Boundary Conditions (Rectangular Coordinates)

X0	**Infinite Boundary Condition**
X1	$T = f(t)$ **(Dirichlet Boundary Condition)**
X2	$\frac{dT}{dx} = f(t)$ **(Neumann Boundary Condition)**
X3	$k \frac{dT}{dx} + hT = f(t)$ **(Robin Boundary Condition)**
X4	$k \frac{dT}{dx} + (\rho cb) \frac{dT}{dt} = f(t)$ **(Carslaw Boundary Condition)**
X5	$k \frac{dT}{dx} + hT + (\rho cb) \frac{dT}{dt} = f(t)$ **(Jaeger Boundary Condition)**

Time Variation of the Boundary Condition

B0	$f(t) = 0$
B1	$f(t) = T_0$ (Constant)
B2	$f(t) = T_0 t$ (Linear)
B3	$f(t) = T_0 t^n$ (Integer Polynomial)
B4	$f(t) = T_0 t^{n/2}$ (Rational Polynomial)

Initial Condition

T0	$f(x) = 0$
T1	$f(x) = T_0$ (Constant)
T2	$f(x) = T_0 x$ (Linear)
T3	$f(x) = T_0 x^n$ (Integer Polynomial)
T4	$f(x) = T_0 x^{n/2}$ (Rational Polynomial)
T5	Step Change in $f(x)$
T6	$f(x) = T_0 \sin(C\ x)$ or $T_0 \cos(C\ x)$ (Transcendental)

Heat Generation Source Term

G0	$g(x,t) = 0$
G1	$g(x,t) = G_0$

the indicated integrals. In such cases, the integrals can be numerically evaluated.

Special integration functions have been added to the SMP program by means of a library of integration which resides in an external file called onedim.int and is called by the main CANSS program to assist the internal SMP integrator. The internal SMP integrator is invoked when an integral is not evaluated by the external CANSS integrator. The integral is returned to the user if it is not evaluated by either the CANSS external integrator or the SMP internal integrator. The procedure is halted until a solution to the integral is found. If the user can evaluate the unknown integral, the solution is placed into the external CANSS integration procedure and the problem is restarted.

The one dimensional CANSS program begins with the loading of external procedures that query the user for information regarding the right and left boundary condition. These procedures set up the geometry for the description and calculation of the Green's function. Also, the procedures that query the user for the initial condition and the volume energy heat source are loaded.

Two assumptions are made at this point of the algorithm. The partition time, t_1, which is defined as the dimensionless time when the convergence characteristics of the small time Green's function require more than three reflection terms, is set to 0.025. The value of the partition time may be increased up to 0.10 if boundary conditions of the third and fourth kinds are not applied to the boundaries. The second assumption is that the forcing functions for all the boundary, initial, and volume energy term begin at time $\tau = 0$.

The first step in generating a temperature distribution for a slab is to determine the left and right boundary conditions thus describing the correct Green's function. The one dimensional CANSS algorithm allows the user to choose from five type of boundary conditions; boundary conditions of the zeroth through the fourth kind. The boundary condition of the fifth kind is not included in the one dimensional CANSS algorithm because specific numerical conditions must be met and the solution will not be symbolic.

The next step in the algorithm is the input of the initial condition. The initial condition condition in the CANSS algorithm, by definition, is only a function of position. For some problems, the initial condition may be composed of functions of sines and cosines where the argument is a function of position. The heat generation term is input next and must be a constant.

The program has the information necessary to develop a temperature distribution for the body. The conditions of the problem are displayed to the user and a library of small and large time Green's function is loaded. The small and large time Green's function for the stated problem are completely described by the boundary conditions and displayed for the user. The Green's functions are infinite summations, but, the summation sign is not available for display in the CANSS environment at this time.

The summation for the small time Green's function goes from minus infinity to plus infinity. Only the most dominant terms of the series ($n = 0, \pm 1$) are retained because of the way the time is partitioned. The summation of the large time Green's function begins at one and goes to plus infinity. Again, only the most dominant terms of the summation series ($m = 1,2,3,\ldots,N$) will be retained due to the way in which the time is partitioned. The number N is always less than ten with proper time partitioning.

An additional equation called an eigencondition must be solved for each term in the large time temperature distribution to obtain the eigenvalue. This means an additional function must be solved for each term in the large time solution. Mikhailov, et. al., [1983] considers the safe and efficient calculation of the eigenconditions for the eigenvalues to be tricky, and in many cases, avoided and ignored. A Newton-Raphson iterative procedure may be used to estimate the eigenvalues from the eigencondition but, for boundary conditions of the third, fourth, and fifth kinds, the iteration process adds additional calculations and decreases the efficiency of obtaining the temperature distributions. An additional problem with the Newton-Raphson iteration process is the choice of the initial guess. The eigenconditions are functions of transcendental functions and the appropriate initial guess is critical for obtaining the eigenvalue. Beck [1986] lists some procedures for estimating or approximating a good guess for the eigenvalues in heat transfer problems.

The number of terms in the small time Green's function is kept small because the integration of certain convolution integrals in time is difficult or not possible for some terms in the series. The large time Green's function allows integration on convolution integrals for all terms in the series but will typically need an additional calculation of the eigencondition for the eigenvalues.

Once the Green's functions for small and large time have been determined, the CANSS routine begins to direct the integration of the four terms in equation (2.3.14) that will lead to the temperature distribution. The integrands of the integrals are checked against the external CANSS integrator and a temperature distribution is generated. If the external CANSS integrator can not recognize or match the integrand, the internal SMP integrator is tried. If both integrators fail, the procedure halts because the integration cannot be performed in the CANSS environment and the integral is returned to the user for evaluation. A list of special integrals included in the CANSS program, but not readily available, is provided in Appendix B.

The user must decide at this point if the integration is possible, and if so, extend the CANSS external integration library to include the new integral. The procedure is then restarted from the beginning. When a solution is presented, it is a function of the position and time and has convergent expressions for small and large times. The temperature distribution is not exact when only a few of the terms in the appropriate series are retained but it is accurate to within four decimal places of the exact distribution. This should be sufficient for most applications.

The one dimensional CANSS program treats the infinite and semi-infinite geometries as special cases. The infinite and semi-infinite problems have only one form of the solution and are not time partitioned. There are no convergence problems and the solutions to the infinite and semi-infinite geometry problems are exact. The small time solutions for the finite cases reduce to similar expressions as for the semi-infinite body for small times.

4.3 One Dimensional CANSS Examples

Three example problems solved by the CANSS program are presented in this section. The first example is a semi-infinite slab with a constant heat flux, q_0, at the surface x=0. There is no internal heat generation in the body and the initial temperature condition is zero. The Green's function for this problem is assigned the number X20 because of the boundary condition of the second kind on the left hand side of the body and a natural or zeroth boundary condition on the right hand side; see Figure 4.1a. Beck and Litkouhi [1985] and Tzeng and Beck [1985] define a numbering system that assigns the number X20B1T0 to the solution of the example.

The second example problem is a finite slab with a constant heat flux on the left surface, q_0, and a constant temperature, T_0, on the right boundary; see Figure 4.1b. The initial condition is a transcendental function of the space variable x and no heat is generated in the slab. The Green's function is assigned the number X21 and the solution is assigned the number X21B10T6.

The third example problem is a finite slab with heat flux a function of the square root of time on the left surface, $q_0 \left(\frac{t}{t_0}\right)^{1/2}$, and a temperature condition equal to zero on the right hand side; see Figure 4.1c. The initial temperature is zero and there is no volume energy heat source in the slab. The Green's function for this case is X21. The solution is assigned the number X21B30T0.

4.3.1 Semi-infinite Example Problem (X20B1T0)

The mathematical description of the semi-infinite problem is,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad , \text{ for } x > 0 \text{ and } t > 0, \tag{4.3.1}$$

where,

$$-\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 \quad , \text{ for } t > 0, \tag{4.3.2}$$

and,

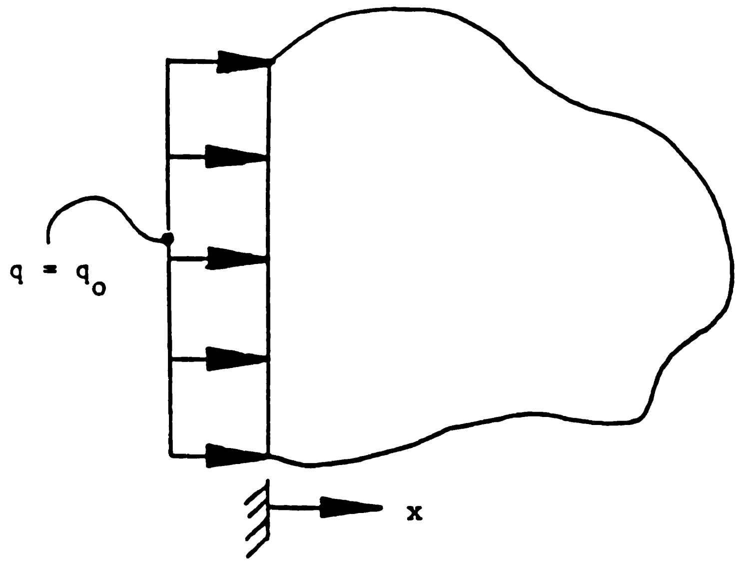

Figure 4.1a

Semi-infinite Body With Constant Heat Flux on the Surface (X20B1T0).

Initial Condition

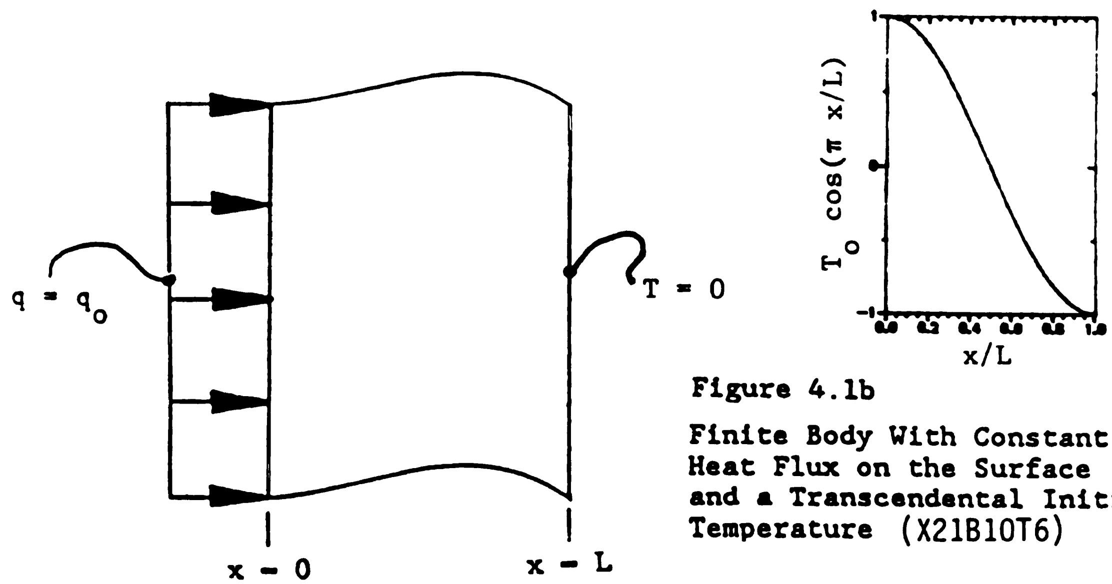

Figure 4.1b

Finite Body With Constant Heat Flux on the Surface and a Transcendental Initial Temperature (X21B10T6)

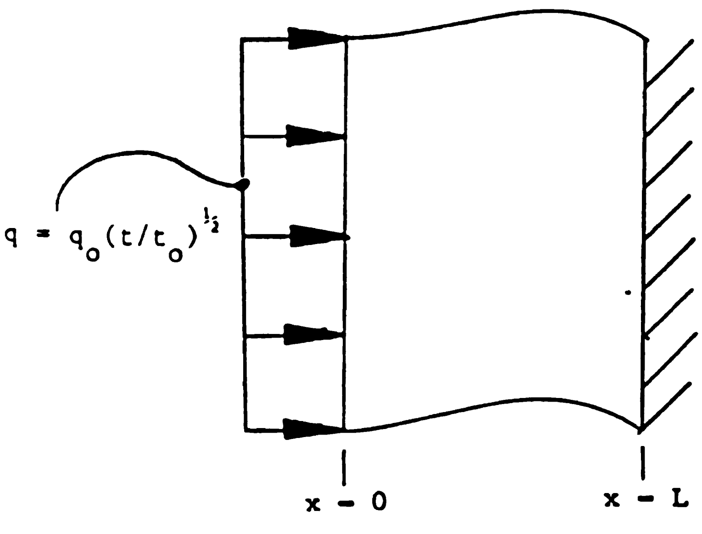

Figure 4.1c

Finite Body With a Heat Flux Condition as a Function of Time (X22B30T0).

$$T(x,0) = 0 \quad , \text{ for } x > 0. \tag{4.3.3}$$

Carslaw and Jaeger [1959, pg. 75, #6] show the exact solution to be,

$$v(x,t) = \frac{2\ q_0\ (\alpha t)^{1/2}}{k}\ \text{ierfc}\left(\frac{x}{2\ \sqrt{(\alpha t)}} \right). \tag{4.3.4}$$

The CANSS program returns the temperature distribution,

$$T(x,t) = \frac{2\ q\ L}{k}\ (t^*)^{1/2}\ \text{Gamma}[1]\ \text{IErfc}\left(\frac{x_L}{2\ \sqrt{t^*}}\ ,\ 1 \right). \tag{4.3.5}$$

In the CANSS program, t^* is the dimensionless time, $\frac{\alpha t}{L^2}$, and the symbol x_L is the dimensionless position, $\frac{x}{L}$.

The symbolic solution from the CANSS routine matches the solution found in Carslaw and Jaeger exactly since Gamma[1] = $\Gamma(1) = 1$ and IErfc[y,1] = $i^1\text{erfc}(y) = \text{ierfc}(y)$. A plot of the dimensionless temperature distribution in the semi-infinite slab is shown in Figure 4.2. This figure shows the temperature decreases at a specific time when the point of interest is moved into the body. The effect of the heat flux is felt instantaneously at all points in the semi-infinite slab, though when the point of interest is far from the source at the boundary, the effect is insignificant.

The repeated integrals error function are important in heat diffusion problems because they appear often. The integral of the error function, $i^n\text{erfc}$, is

$$i^n\text{erfc}(y) = \int_y^\infty i^{n-1}\text{erfc}(\xi)\ d\xi, \tag{4.3.6}$$

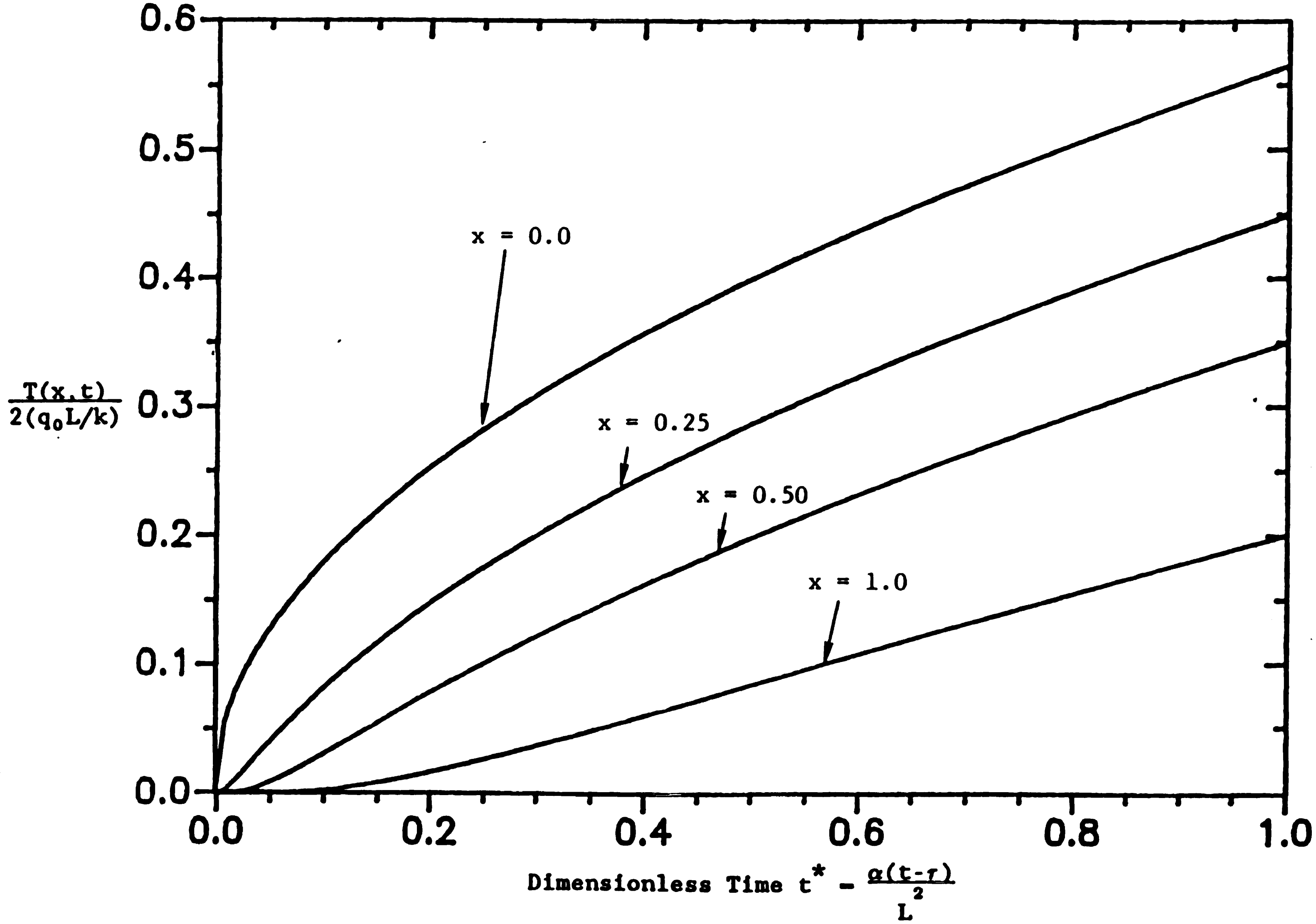

Figure 4.2 Dimensionless Temperature Versus Dimensionless Time for a Semi-infinite Slab With Constant Heat Flux.

for n = 1,2,3..., and $i^0\mathrm{erfc}(y) = \mathrm{erfc}(y)$, where erfc(•) is the complementary error function. Integrating equation (4.3.6) by parts and letting n=1 yields,

$$i^1\mathrm{erfc}(y) = \mathrm{ierfc}(y) = \frac{e^{-y^2}}{\sqrt{\pi}} - y\,\mathrm{erfc}(y). \qquad (4.3.7)$$

A general recurrence relationship for the integral error functions [Carslaw and Jaeger, 1959] is,

$$2\,n\,i^n\mathrm{erfc}(y) = i^{n-2}\mathrm{erfc}(y) - 2\,y\,i^{n-1}\mathrm{erfc}(y). \qquad (4.3.8)$$

A plot of the integral error function, $i^n\mathrm{erfc}(y)$, as a function of n and y is shown in Figure 4.3. The integral error functions approach zero quickly as the argument y increases and the order of the index n increases. Solutions to heat transfer problems which contain integral error functions tend to converge quickly.

4.3.2 Finite Slab with a Transcendental Initial Condition (X21B10T6)

The second example problem is a finite slab of length, L, without heat generation. The slab has a constant heat flux, q_0, at the boundary x = 0 and a constant temperature of zero at x = L. The initial condition of the slab is a periodic function of space. The mathematical description of this case is,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}\quad, \text{ for } 0 < x < L \text{ and } t > 0, \qquad (4.3.9)$$

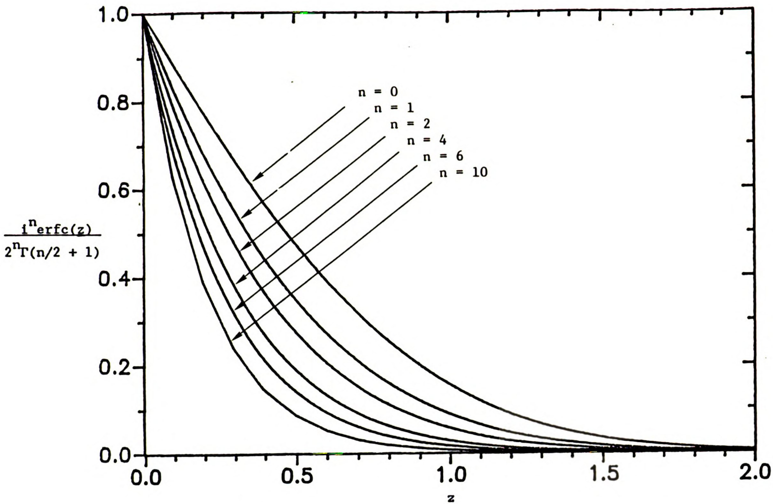

Figure 4.3 Integral Error Function as a Function of n.

where,

$$- \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 \quad , \text{ for } t > 0, \tag{4.3.10}$$

$$T(L,t) = 0 \quad , \text{ for } t > 0, \tag{4.3.11}$$

and,

$$T(x,0) = T_0 \cos(\pi\, x/L) \quad , \text{ for } 0 < x < L. \tag{4.3.12}$$

The temperature distribution from the CANSS routine for this case is composed of two parts. The first portion, denoted X21B10T0, is the effect of the non-homogeneous boundary condition on the temperature distribution. It is composed of a small time function, which is accurate when the dimensionless time is less than the partition time, t_1, and a large time function, which is accurate when the dimensionless time of interest is greater than the partition time. The small time temperature distribution is,

$$T(x,t) = 2 \left(\frac{q_0 L}{k} \right) (t^*)^{1/2}\, \Gamma(1) \sum_{n=-\infty}^{\infty} \mathrm{IErfc}\left(\frac{|0.5\ (2\ n + x_L)|}{(t^*)^{1/2}} \ , \ 1 \right), \tag{4.3.13}$$

while the large time temperature distribution is,

$$T(x,t) = 2 \left(\frac{q_0 L}{k} \right) (t_1^*)^{1/2}\, \Gamma(1) \sum_{n=-\infty}^{\infty} \mathrm{IErfc}\left(\frac{|0.5\ (2\ n + x_L)|}{(t_1^*)^{1/2}} \ , \ 1 \right)$$
$$+ \ 2 \left(\frac{q_0 L}{k} \right) \sum_{m=1}^{\infty} \mathrm{Cos}[(m - 0.5)\pi\, x_L] \ \cdot$$

.

$$\left[\frac{e^{-(m - 0.5)^2 \pi^2 t_1^*} - e^{-(m - 0.5)^2 \pi^2 t^*}}{\pi^2 (m - 0.5)^2} \right], \qquad (4.3.14)$$

where x_L is the position of interest normalized with respect to the length of the slab. The index n in equations (4.3.13) and (4.3.14) need not extend from minus infinity to plus infinity. Only three terms are necessary ($n = -1, 0, 1$) for the distribution to converge when the partition time is chosen correctly. The index m in equation (4.3.14) may be kept small ($m \leq 10$) for efficient calculation of the temperature distribution at large times.

The second term in the solution to the temperature distribution (X21T6) is caused by the non-zero initial condition. Time partitioning is not always needed for the initial condition. The second term, generated by the initial condition, is,

$$2\,T_0 \sum_{m=1}^{\infty} e^{-(m - 0.5)^2 \pi^2 t^*} \, \mathrm{Cos}[(m - 0.5)\pi\, x] \;\bullet$$
$$\frac{\mathrm{Cos}[m\,\pi]}{\pi} \left(\frac{1}{1 + 2\,m} - \frac{1}{3 - 2\,m} \right) \qquad (4.3.15)$$

The temperature distribution of the slab is determined by adding either equation (4.3.13) or (4.3.14) to (4.3.15) depending on whether the dimensionless time is greater than or less than the partition time. The temperature distribution of the second example problem has not been previously determined. The temperature distributions of the slab for equations (4.3.13) and (4.3.14) are plotted in Figure 4.4a for the appropriate dimensionless time. The partition time is set to 0.10 and splits the region into two areas for problems with boundary conditions of the first and second kind.

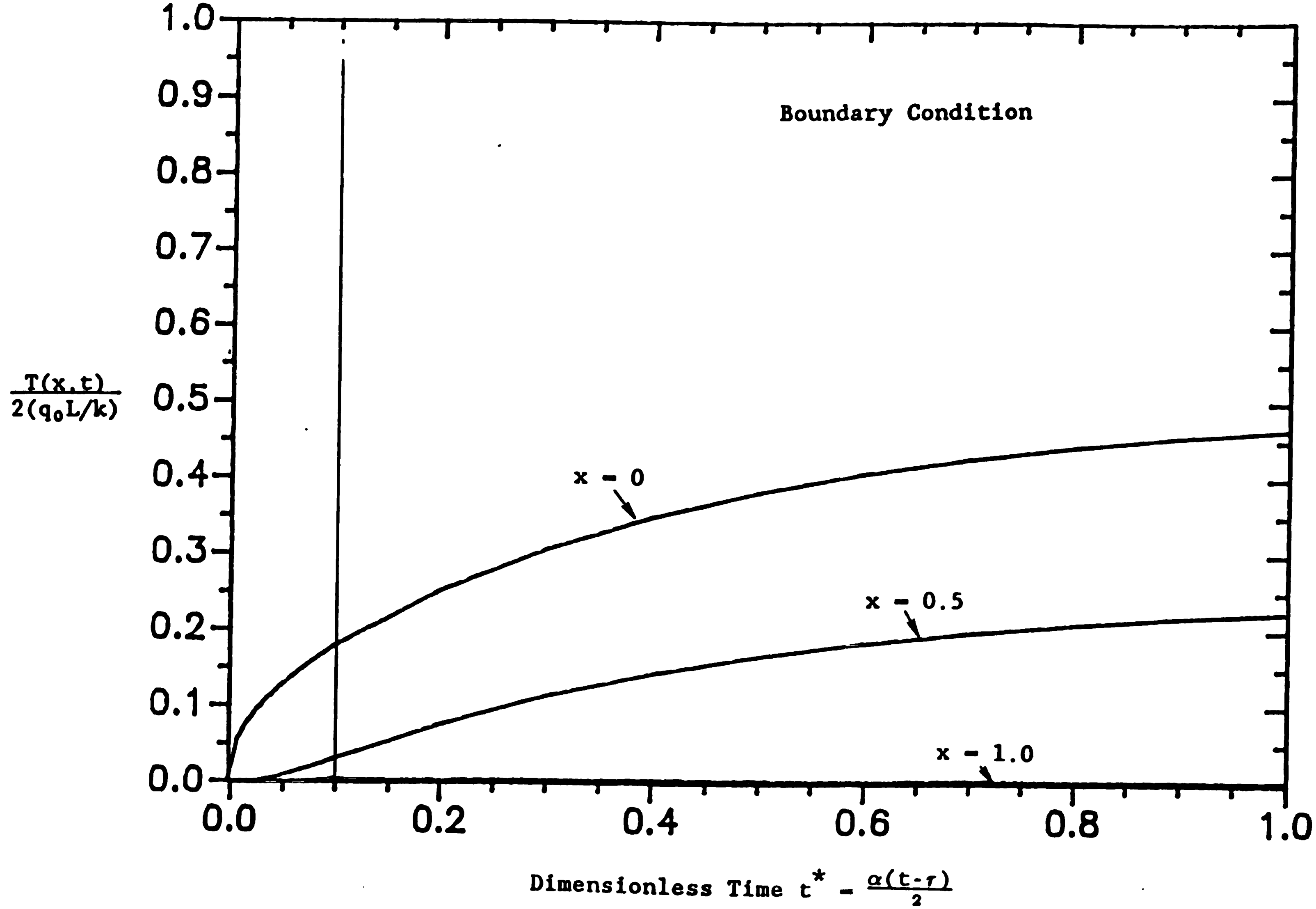

Figure 4.4a Dimensionless Temperature Versus Dimensionless Time for a Finite Body in Example Two for the Boundary Condition.

Equation (4.3.15) is plotted in Figure 4.4b for the constant $T_0 = \frac{q_0 L}{k}$ so that both curves can be plotted on the same scale. The temperature distributions for the boundary source (Figure 4.4a) look similar to the distributions for the semi-infinite case in Figure 4.2 but the temperature distributions in Figure 4.4a converge to a steady state temperature due to the imposed boundary conditions.

Figure 4.4c shows the temperature distribution in the slab when the boundary condition solution is added to the initial condition solution and $T_0 = \frac{q_0 L}{k}$. The nature of the initial condition solution's effect is quickly damped as the time increases.

The temperature distribution in Figure 4.4c when the location of the point of interest is at $x_L = 1$ shows a small error. The exact solution at the point $x_L = 1$ should be zero but the graph shows a value of 0.0036. The error occurs when the small time function is required to calculate the approximate solution without regarding the effect of the boundary on the right hand side. Since the point of interest is the boundary on the right hand side the small time function picks up a small amount of error because additional terms in the small time series become significant. The approximation at this boundary can be readily improved by reducing the value for the partition time or adding more terms to the small time series.

4.3.3 Finite Slab With a Boundary Condition a Function of the Square Root of Time (X21B30T0)

The third example, denoted X21B30T0, is a finite slab of length, L, with no heat generation, zero as the initial condition, heat flux a function of the square root of time on the left boundary and temperature

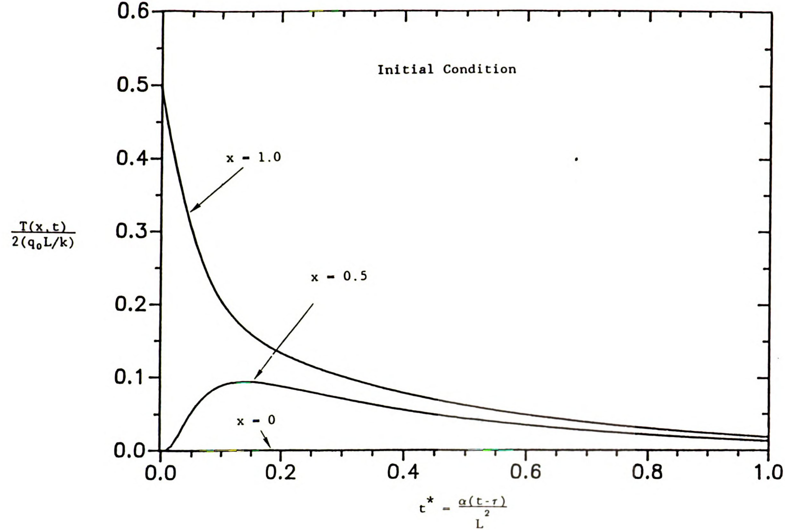

Figure 4.4b Dimensionless Temperature Versus Dimensionless Time for a Finite Body in Example Two for the Initial Condition.

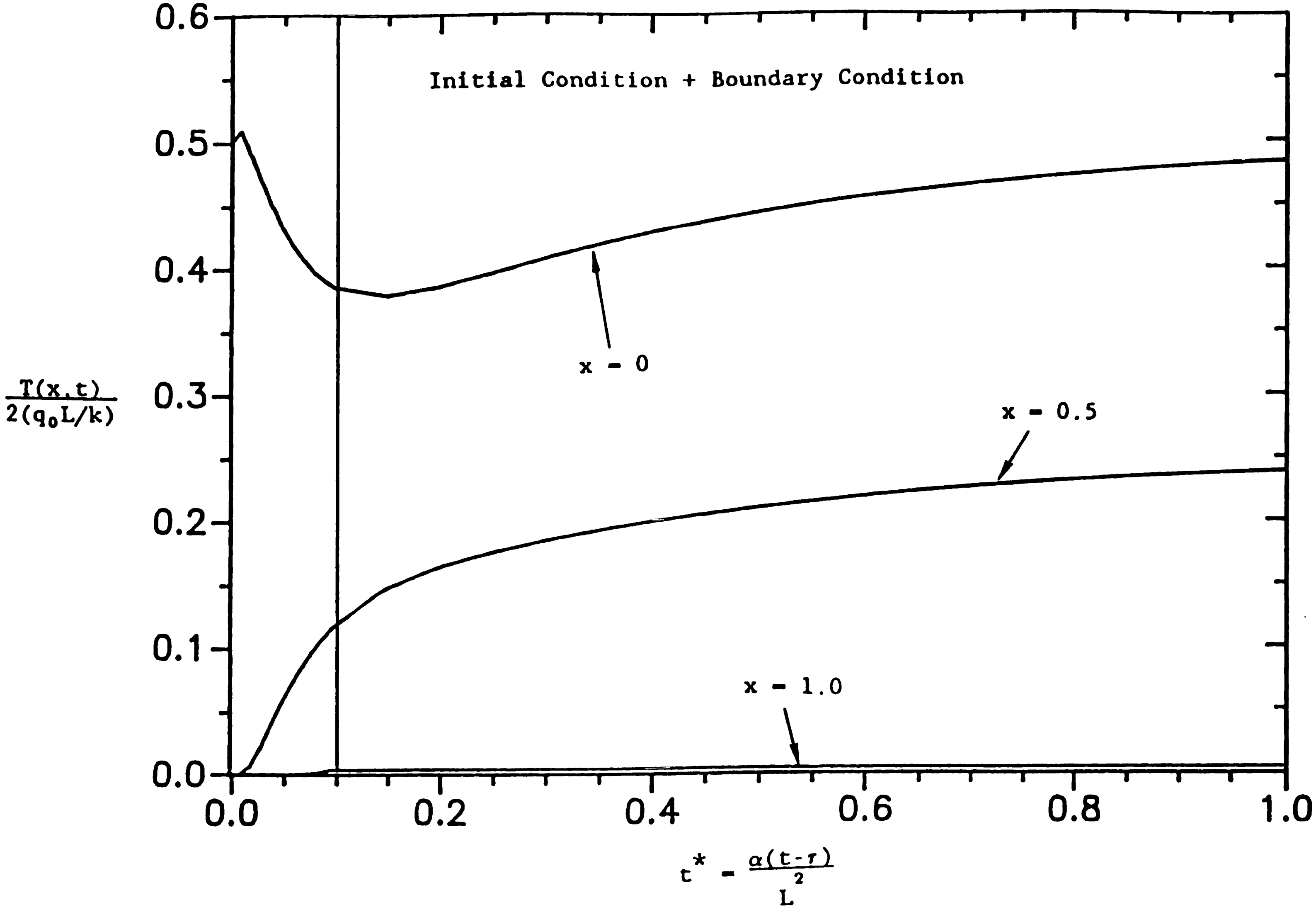

Figure 4.4c Dimensionless Temperature Versus Dimensionless Time for a Finite Body in Example Two for the Initial and Boundary Condition Added Together.

set to zero on the right hand boundary; see Figure 4.1c. The mathematical description of this case is,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad , \text{ for } 0 < x < L \text{ and } t > 0, \tag{4.3.16}$$

where,

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 (t/t_0)^{1/2}, \text{ for } t > 0, \tag{4.3.17}$$

$$T = 0 \quad , \text{ for } t > 0, \tag{4.3.18}$$

and,

$$T(x,0) = 0 \quad , \text{ for } 0 < x < L. \tag{4.3.19}$$

The constant t_0 is an arbitrary unit of time.

The partitioned temperature distribution cannot be calculated by the CANSS program in closed form. The integral expression for the temperature at small dimensionless times that remains is,

$$T(x,t) = 4 \left[\frac{q_0 \ L}{k \ \sqrt{t_0^*}} \right] \Gamma(1.5) \ t^* \sum_{n=-\infty}^{\infty} \mathrm{IErfc} \left[\frac{|0.5 \ (2 \ n + x_L)|}{(t^*)^{1/2}} \ , \ 2 \right] \tag{4.3.20}$$

and for large dimensionless times is,

$$T(x,t) = \left[\frac{q_0 \ L}{k \ \sqrt{t_0^*}} \right] \left\{ \frac{1}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \int_{\lambda=0}^{t_1^*} (t^* - \lambda)^{1/2} \frac{e^{\frac{-(2n+x_L)^2}{4 \ \lambda}}}{\sqrt{\lambda}} \ d\lambda \right.$$

$$+ \left[2 \sum_{m=1}^{\infty} \frac{\cos(\pi(m-.5)x_L)\, e^{-(\pi^2 (m-.5)^2 t_1^*)}}{\pi^3 \, (m-.5)^3} \cdot \right.$$

$$\left. \left\{ (\pi(m-.5))\ (t^* - t_1^*)^{1/2} - \mathrm{Daw}[\pi(m-.5)(t^* - t_1^*)^{1/2} \right\} \right] \Bigg\}, \tag{4.3.21}$$

where Daw(•) is the Dawson integral and is discussed in the next section, t_1^* is the dimensionless partition time, and t^* is the dimensionless time of interest.

The integral in the large time expression for the temperature distribution has not been evaluated in closed form. Many approaches may be used to evaluate this integral approximately. The most significant approach for evaluating the integral in a symbolic context is to define the integral as a new function. This new function would be evaluated for all values of the parameters and would become a well known function. The exponential integral function, $E_n(z)$, is an example of defining an integral that cannot be expressed in closed form and is considered a well known function.

Another approach for the evaluation of the temperature distribution at small times is to approximate the unknown integral using numerical integration. The solution will be dependent on the specific parameters associated with the problem. Since the solution can be partitioned in time, the integral in the large time expression can be evaluated using a small number of terms in the infinite series.

A third approach for the solution of the integral in the large time expression is to approximate the forcing function over the range of integration for small times. Figure 4.5 plots the forcing function $f(\tau) = (\tau/t_0)^{1/2}$ from the third example problem for a time of interest

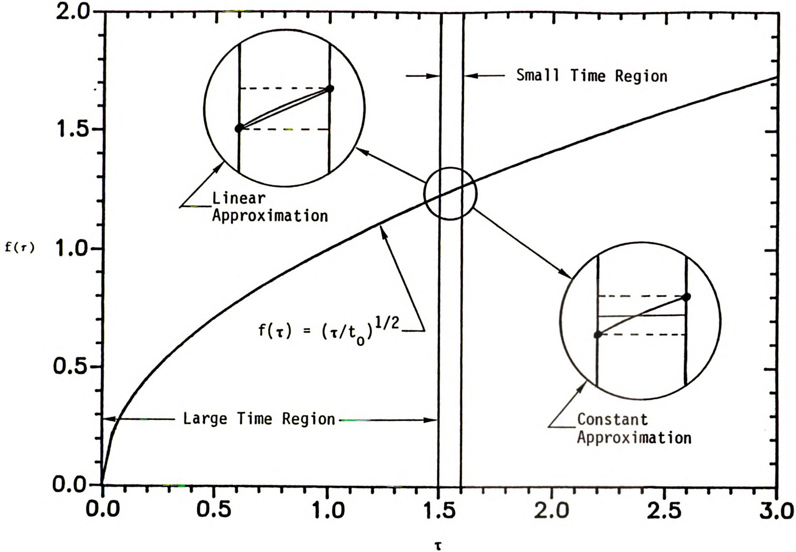

Figure 4.5 Two Types of Approximations for Small Time Integrals

greater than the partition time. The small time portion of the forcing function could be approximated by a constant value of the function averaged over the range of integration or more accurately by a linear function. A closed form for the solution of the small time integral can be found for a constant or linear forcing function.

The small time portion of the solution may be approximated using only the first three terms ($n = -1, 0, 1$) of the series when the dimensionless time is small. The $n = 0$ term will dominate the solution at small times. The index m, for large times, will also remain small because the first few terms of the large time series will dominate the solution.

The large time expression for the temperature distribution represents a new solution that has not appeared in the literature previously. The second term in the large time solution for temperature distribution will represent the total solution when the partition time is set to zero. This solution is exact but needs many terms in the series to converge.

4.4 Some Integrals Used in One Dimensional Problems

The objective of this section is to discuss two types of integral functions that occur during the calculation of the temperature distribution of a one dimensional slab. These integrals are the Dawson integral and an integral which occurs calculating large time solutions.

4.4.1 The Dawson Integral

The Dawson integral, $F(x)$, discussed and analysed by Dawson [1898], was first analysed in heat diffusion problems by Gordon and Miller [1931] and is of the form,

$$F(x) = e^{-x^2} \int_{u=0}^{x} e^{u^2}\, du. \tag{4.4.1}$$

The Dawson integral can be approximated for all ranges of its argument as a function of the confluent hypergeometric function. Lebedev [1965, pg. 20] shows the Dawson integral satisfies the linear differential equation,

$$F'(x) + 2\, x\, F(x) = 1, \tag{4.4.2}$$

with the initial condition, $F(0) = 0$. When a series expansion of $F(x)$ is substituted in equation (4.4.2) and the coefficient of similar powers of the argument x are collected, the expansion yields,

$$F(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, 2^m\, x^{2m+1}}{1\cdot 3\cdots(2\, m + 1)}. \qquad -\infty < x < \infty \tag{4.4.3}$$

Equation (4.4.3) can be expressed as,

$$F(x) = x \sum_{m=0}^{\infty} \frac{(-1)^m\, 2^m\, x^{2m}}{1\cdot 3\cdots(2\, m + 1)}, \tag{4.4.4}$$

which is a form of the confluent hypergeometric function, $\Phi(\cdot,\cdot,\cdot)$,

$$F(x) = x\, \Phi(1,\tfrac{3}{2},-x^2). \tag{4.4.5}$$

The properties of the confluent hypergeometric function can be found in Abramowitz and Stegun [1964, Chapter 13].

The Dawson integral provides the solution to one dimensional diffusion equations for finite bodies when the non-homogeneous forcing function on a boundary is a function of the square root of time. A plot of the Dawson integral versus its argument is shown in Figure 4.6. A disadvantage of this integral is its slow convergence as the argument gets large but since a difference in time is needed, the difference converges quickly. The temperature distributions obtained for finite bodies with boundary conditions a function of time to the one half power or, more generally, to the $\frac{n}{2}$ power, where n is an odd index, have not been previously discovered and represent a new type of solution. The argument to the Dawson integral is a function of the eigenvalue of the diffusion problem that increases in value as more terms are added to the series. This helps the solution convergence.

When the forcing function is $t^{1/2}$, $t^{3/2}$,..., $t^{(2n-1)/2}$, the Dawson integral is involved and it is divided by the eigenvalue to an odd power other than one. The general equation to be solved is,

$$I_n(t,t_1) = \int_{\tau=0}^{t-t_1} \tau^{(2n-1)/2} \exp(-\beta^2\alpha(t-\tau)/L^2)\, d\tau \,.$$

$$\text{for } n = 0, 1, 2, \ldots \qquad (4.4.6)$$

When n = 0, the solution of the integral is,

$$I_0(t,t_1) = \left[\frac{L^2}{\alpha\beta_m^2}\right]\left[\frac{L^2}{\alpha\beta_m^2}\right]^{-1/2} e^{-\beta_m^2 t_1^*}\ \mathrm{Daw}(\beta_m(t^*-t_1^*)^{1/2}),$$

$$(4.4.7)$$

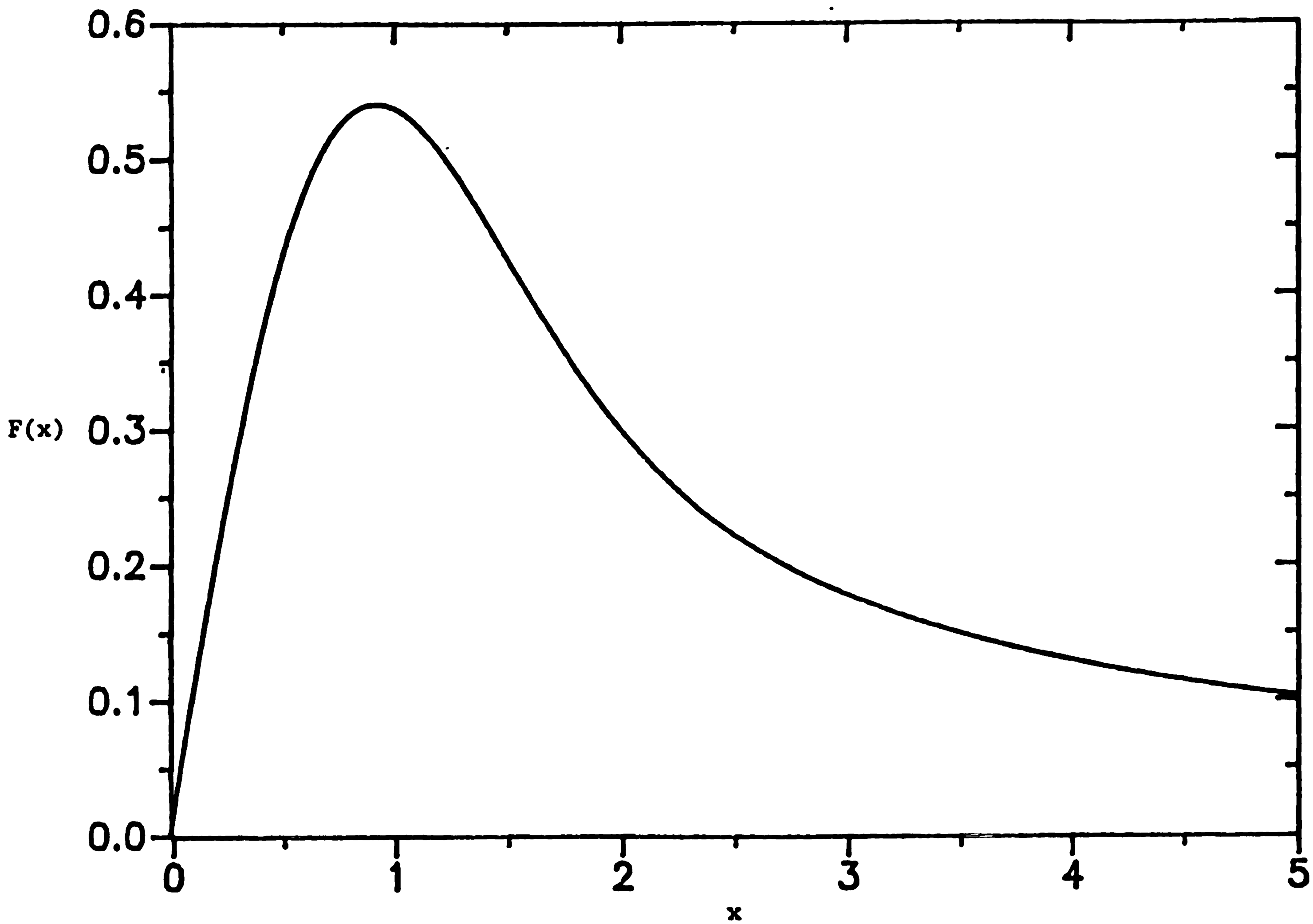

Figure 4.6 Dawson Integral.

where $t^* = \frac{\alpha t}{L^2}$ and $t_1^* = \frac{\alpha t_1}{L^2}$.

The general solution to equation (4.4.6) when n = 1, 2, 3, ..., is,

$$I_n(t,t_1) = \left(\frac{L^2}{\alpha \beta_m^2}\right) \left[\left(\frac{L^2}{\alpha \beta_m^2}\right)^{\frac{(2n-1)}{2}} e^{-\beta_m^2 t_1^*} \left(\beta_m^2 (t^* - t_1^*)\right)^{\frac{(2n-1)}{2}} - (2n-1)\, I_{n-1}(t,t_1) \right] \tag{4.4.8}$$

4.4.2 An Exponential Integral in One Dimensional Problems

An exponential integral that occurs in solving the one dimensional heat diffusion problem using a large time Green's function approach when the forcing function is a function of time raised to an integer power is presented in this section. The general form of this integral is,

$$Z_n(t,t_1) = \int_{\tau=0}^{t-t_1} \tau^n \exp(-\beta_m^2 \alpha (t-\tau)/L^2)\, d\tau , \tag{4.4.9}$$

where n = 0, 1, 2, ..., β_m is the eigenvalue associated with the eigencondition, and t_1 is the boundary partition time between small and large times.

As an example, consider the case of n = 1,

$$Z_1(t,t_1) = \int_{\tau=0}^{t-t_1} \tau \exp(-\beta^2 \alpha (t-\tau)/L^2)\, d\tau , \tag{4.4.10}$$

may be written as,

$$Z_1(t,t_1) = e^{-\beta_m^2 t^*} \int_{\tau=0}^{t-t_1} \tau \, e^{\beta_m^2 \frac{\alpha\tau}{L^2}} \, d\tau \; , \tag{4.4.11}$$

where $t^* = \frac{\alpha\, t}{L^2}$. Integrating by parts, letting,

$$u = \tau \qquad \text{and} \qquad dv = e^{\beta_m^2 \frac{\alpha\tau}{L^2}} \, d\tau$$

$$du = d\tau \qquad \text{and} \qquad v = \left(\frac{L^2}{\alpha\, \beta_m^2} \right) e^{\beta_m^2 \frac{\alpha\tau}{L^2}} \, d\tau$$

yields,

$$Z_1(t,t_1) = \left(\frac{L^2}{\alpha\, \beta_m^2} \right) e^{-\beta_m^2 t^*} \left[\tau \, e^{\beta_m^2 \frac{\alpha\tau}{L^2}} \Bigg|_{\tau=0}^{t-t_1} - \int_{\tau=0}^{t-t_1} e^{\beta_m^2 \frac{\alpha\tau}{L^2}} \, d\tau \right]. \tag{4.4.12}$$

Performing the indicated operations gives,

$$Z_1(t,t_1) = \left(\frac{L^2}{\alpha\, \beta_m^2} \right)^2 \left[e^{-\beta_m^2 t^*} + \left(\beta_m^2 \, (t^* - t_1^*) - 1 \right) e^{-\beta_m^2 t_1^*} \right], \tag{4.4.13}$$

where t^* and t_1^* have been described in the previous section. The number of terms necessary for this expression to converge to six decimal place accuracy is three.

A general expression for the solution of integrals of the type in equation (4.4.9) is the recursion relation,

$$Z_n(t,t_1) = \left(\frac{L^2}{\alpha\,\beta_m^2}\right)\left[\left(\frac{L^2}{\alpha\,\beta_m^2}\right)^n e^{-\beta_m^2 t_1^*}\left[\beta_m^2\,(t^* - t_1^*)\right]^n - n\,Z_{n-1}(t,t_1)\right] \qquad (4.4.14)$$

where,

$$Z_0(t,t_1) = \left(\frac{L^2}{\alpha\,\beta_m^2}\right)\left[e^{-\beta_m^2 t_1^*} - e^{-\beta_m t^*}\right] \qquad (4.4.15)$$

While the general expression for the integral in equation (4.4.9) does not represent a new type of solution, it does allow the expression to be presented in a compact form and is easy to program.

4.5 Time Region Partitioning for One Dimensional Problems

Solutions to linear, transient heat conduction problems in the cartesian coordinate system must be split into solutions that are convergent for the dimensionless times that are specified. The lack of a convergent solution causes an unnecessary use of computation resources. A "tuned" solution, one which is optimized for speedy convergence characteristics, is much preferred over a solution that converges slowly, i.e., required hundreds or even thousands of terms.

The solutions to heat diffusion problems for temperature distributions in cartesian coordinate systems can be tuned readily using the Green's function approach. The solutions using the Green's function approach will be tuned for small and large times because the

Green's function for heat diffusion problems have already been placed in the convergent form for small and large times.

The Green's function solutions are generated for one dimensional cartesian coordinate systems, but it has been shown in Chapter 2 that the one dimensional Green's functions may be multiplied together to obtain multi-dimensional Green's functions for certain boundary conditions. This creates some minor adjustments in the procedure to partition the dimensionless time for the spatially multi-dimensional solutions which will be discussed in the next chapter.

In a one dimensional problem of heat diffusion, the dimensionless time where both the small and large time Green's function yield acceptable convergence rates will be called the partition time, t_1^*, where the star (*) denotes dimensionless time. There exists a Green's function solution that is tuned for convergence for each time partitioned region. In the one dimensional case, the two time regions are separated by the partitioning time, t_1^*, such that in the small time region,

$$t^* - t_1^* \le \tau^* \le t^*, \qquad (4.5.1)$$

and in the large time region,

$$0 \le \tau^* < t^* - t_1^*, \qquad (4.5.2)$$

where t^* is the dimensionless time of interest, see Figure 4.7. The integration of the Green's function for times that fall in the region of small time is,

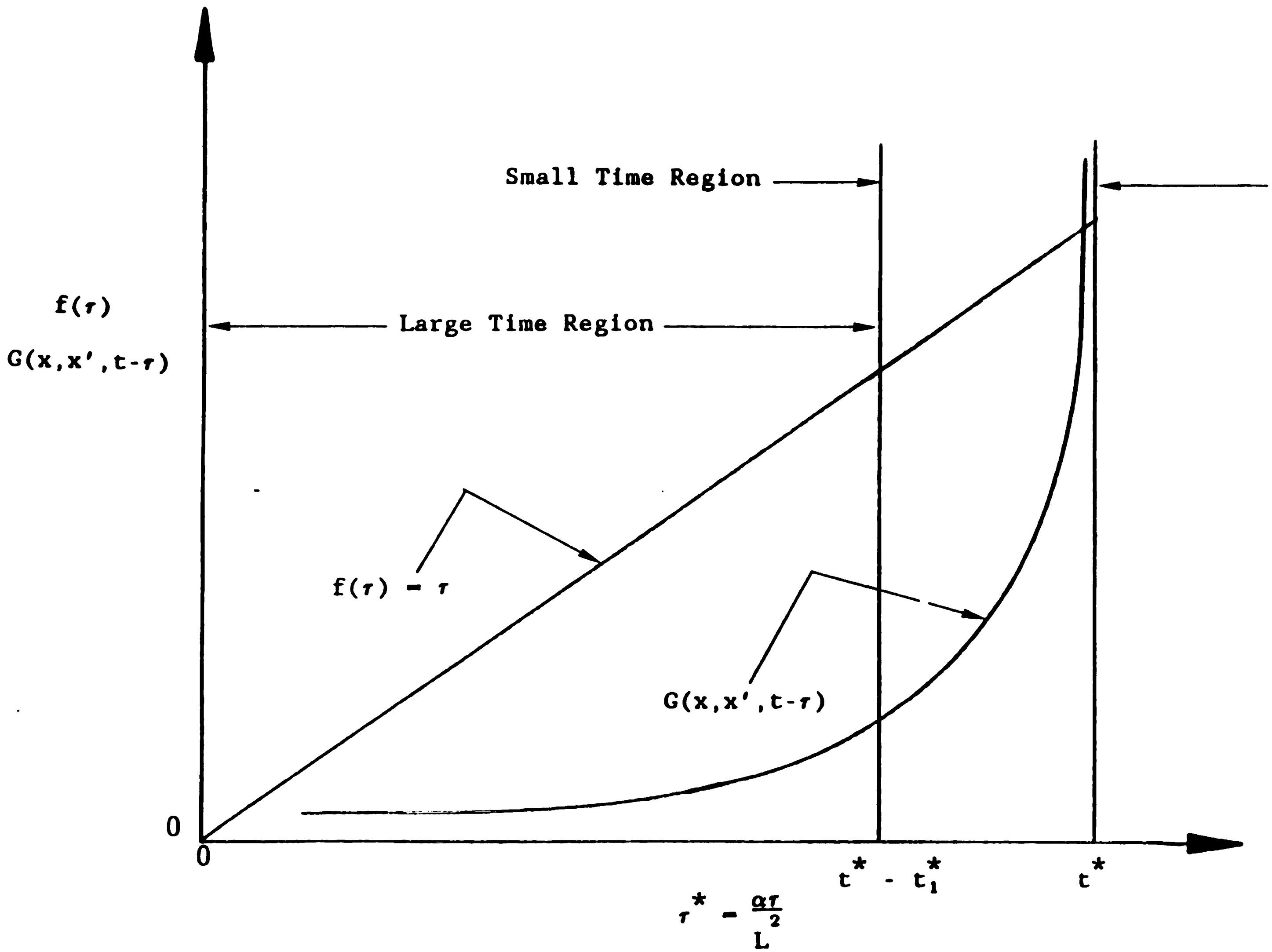

Figure 4.7 Convolution of the Green's Function and a Function of Time.

$$I_s = \int_{\tau^*=t^*-t_1^*}^{t^*} \text{GFS}\ d\tau \qquad \text{for} \quad t^* - t_1^* \le \tau^* \le t^*, \qquad (4.5.3)$$

and the integration of the Green's function for times that fall in the large time region is,

$$I_\ell = \int_{\tau^*=t^*-t_1^*}^{t_1^*} \text{GFS}\ d\tau^* + \int_{\tau^*=0}^{t^*-t_1^*} \text{GFL}\ d\tau^* \qquad \text{for } 0 \le \tau^* < t^* - t_1^*, \qquad (4.5.4)$$

where GFS and GFL are the small and large time Green's functions respectively that have been integrated over the surfaces or volume of the body and include forcing functions that are functions of time. Of particular interest is that the small time Green's function is quickly convergent for dimensionless times at or close to zero but slowly convergent for large times.

Due to the choice of t_1^*, the integral I_s is quickly convergent. The large time Green's function converges very slowly when the time, t^*, approaches zero, but since the time integration of the second term on the right hand side of I_ℓ does not involve zero or a time close to zero by the definition of t_1^*, the integral converges quickly. It is noted that for constant forcing functions, and if the time of interest is greater than the partition time, t_1^*, the first integral on the right hand side of I_ℓ becomes a constant and needs to be calculated only once.

4.6 One Dimensional CANSS Flowchart/Example

This section contains the flowchart/example for the one dimensional CANSS program. The example problem is a finite body, insulated on the right hand side, zero as the initial temperature, and zero as the volume energy heat source. The left hand boundary has a linear heat flux of the form $q = q_0 t$. The symbol <CR> means carriage return.

Section 1 of Appendix C shows the input and output from running this example.

One Dimensional CANSS Flowchart

Flowchart	Enter	Explanation
Start Program	<"canss.prg"<CR>	Begin program while in SMP
Load Subroutines		The subroutines exp.int, grab.int and the external CANSS integration routines are loaded.
Display Environment	<CR>	A banner is displayed that defines the CANSS environment
Load B.C.,I.C.,& HG terms	<CR>	Load the left and right boundary input conditions. The initial condition and heat generation input routines are loaded and note special values.
Input LB Condition	2<CR> Qo<CR> 1<CR>	Input the left boundary condition(0-4), constants, and the index on the time variable
Input RB Condition	2<CR> 0<CR> 0<CR>	Input the right boundary condition(0-4), constants, and the index on the time variable
Input Initial Condition	0<CR> 0<CR>	Input the initial condition. Type constants and polynomial power of coordinate x.

Input Source	0<CR>	Input the volume energy source term as a constant.
Display Status	0<CR>	Display the description of the problem.
Load GF Library		Load the Green's function library.
Generate GF	<CR>	Calculate the Green's function based on boundary conditions for small and large times.
Calculate small boundary solution	<CR>	Calculate and display the small time boundary solution.
Calculate large boundary solution	<CR>	Calculate and display the large time boundary solution.
Calculate initial condition solution	<CR>	Calculate and display the initial condition solution.
Calculate volume source solution	<CR>	Calculate and display the volume energy source solution.
STOP		End calculation and display calculation time in CPU sec.

4.7 **Summary**

A one dimensional program for the symbolic solution of transient heat diffusion problems is presented in this chapter. Three example problems have been examined and checked against known solutions when possible. Some integrals that occur during the calculation of the temperature distribution have been examined. The importance of time

partitioning for one dimensional problems is discussed. A flowchart of the one dimensional CANSS algorithm is presented.

CHAPTER 5

TRANSIENT TWO DIMENSIONAL CANSS2D PROGRAM

5.1 Introduction

The computer algebraic, numeric, and symbolic system called CANSS2D calculates temperature distributions for transient, two dimensional heat diffusion problems using a Green's function approach. The program CANSS2D is written in the language of SMP [1983] and generates a symbolic solution for the temperature distribution in a homogeneous plate having boundary conditions of the zeroth, first or second kind on any surface. A distinctive feature of the two dimensional CANSS2D program is the ability of the program to allow nonzero, but constant, boundary conditions to cover only part of a surface. The remaining portion of the boundary condition at the surface is set to zero. Special forms of nonlinear problems can be addressed for certain combinations of the physical parameters as was shown in Chapter 4.

The Green's function approach to the solution of the temperature distribution for linear, transient heat diffusion problems in two dimensions leads to integrals that have not previously been discovered or integrals that are not well known. These integrals are discussed later in this chapter or appear in Appendix B. Again, for certain combinations of physical parameters, nonlinear problems may be solved for temperature distribution.

Many of the integrals to be evaluated for two dimensional problems are found in Appendix B of this thesis. Typically, the integrations of a one dimensional system provide integrals that are not well known but have been studied extensively. The two dimensional systems generate integrals that are not well known and have not been studied as extensively.

Most temperature distributions for two dimensional heat diffusion problems that appear in references and textbooks are left as a function of an integral on time. Ozisik [1980], Carslaw and Jaeger [1959] and other references generally avoid generating explicit, closed form two dimensional temperature distributions and leave the distributions in terms of integrals. This is due to the complexity of the evaluation of the integrals in closed form. The two dimensional CANSS2D program returns functions that are recognizable and can be evaluated. The CANSS2D two dimensional program solves the integrals and presents the temperature distribution in terms of partitioned dimensionless time so that the temperature distribution for each time region converge quickly.

The Laplace transform technique, developed in Section 3.2, produces Green's functions that are accurate and efficient at small dimensionless times. Green's functions expressed in terms of Fourier expansion and developed using the separation of variables technique for finite bodies (see Churchill and Brown [1978]) can involve infinite series that converge slowly at small times. In many cases, the temperature distributions obtained using the Green's functions developed by the Laplace transform technique involve integrals that are unfamiliar, have not been evaluated, or not been tabulated.

The Laplace transform approach is used in this thesis to determine unknown integrals caused by the convolution of time. Arpachi

[1966] describes the theory of convolution integrals in his text of conduction heat transfer. Doetch [1961] relates the convolution integral to an effect (forcing function) multiplied by a weighting function (Green's function). The individual integrands in the integral are transformed into Laplace transform space, multiplied together, then inverse Laplace transformed. Typically, all that is necessary for the inverse Laplace transform is a short table in Appendix A.

A description of the CANSS2D algorithm for obtaining temperature distribution in a plate and the flowchart for the CANSS2D algorithm are presented in Section 5.2 of this chapter. Two example problems solved using the CANSS2D program are discussed in Section 5.3. Some integrals that arise from the calculation of temperature distribution in two dimensional problems are reviewed in Section 5.4. Two and three dimensional time partitioning is discussed in Section 5.5. Section 5.6 summarizes this chapter.

5.2 **Transient Two Dimensional CANSS2D Program**

The objective of this section is to present a program that will calculate the temperature distribution in a two dimensional body. A flowchart/example of the CANSS2D program appears at the end of this section.

5.2.1 CANSS2D Program

CANSS2D is a computer program designed to generate two dimensional symbolic temperature distributions in a homogeneous plate using symbolic manipulation. The temperature distributions symbolically calculated by the algorithm are partitioned with respect to time. The numerical evaluation of the temperature distribution expressions are quickly convergent in each time region. No heat generation is allowed in the plate and the initial temperature of the plate is constant at zero. Only constant boundary conditions of the zeroth, first, or second kind may occur at the surfaces of the plate. The describing partial differential equation is given in equation (3.2.4) and the boundary conditions are given in equations (2.2.11) and (2.2.13).

The non-homogeneous portion of the boundary condition may extend to any percentage of the surface length but the boundary cannot have a mixed condition. This means, for example, a boundary condition of the first kind cannot coexist with a boundary condition of the second kind on the same surface. The products of Green's functions that include boundary conditions of the fourth and fifth kinds are not allowed in the Green's function method for two dimensional bodies as was shown in Chapter 2.

The temperature distributions are generated by the Green's function approach described in Chapter 2, equation (2.3.15), where both the initial condition and volume energy heat source are zero. The equation for the temperature distribution is,

$$T(x,y,t) = \alpha \int_{\tau=0}^{t} \int_{S_i} \sum_{i=1}^{s} \frac{f_i(x_i,y_i,\tau)}{k_i} G(x,y,t|x',y',\tau)_{\substack{x'=x_i \\ y'=y_i}} dS_i \, d\tau.$$

(for boundary conditions of the second kind)

$$- \alpha \int_{\tau=0}^{t} \int_{S_j} \sum_{j=1}^{s'} f_j(x_j',y_j',\tau) \left.\frac{\partial G}{\partial n_j}\right|_{\substack{x'=x_j \\ y'=y_j}} dS_j \, d\tau, \qquad (5.2.1)$$

(for boundary conditions of the first kind)

where s is the number of Robin conditions and s' is the number of Dirichlet conditions on the boundaries of the plate. The boundary condition of the zeroth kind is automatically included in equation (5.2.1).

The CANSS2D program for obtaining temperature distributions of spatially two dimension plate problems begins by loading five external files. The names of procedures that accompany the SMP program begin with capital letters and are underlined. The names of procedures with a lower case letters and underlined were written by this author.

The first two external files loaded help simplify expressions so that the integration routines do not waste time expanding or searching through expressions. The Repexplg external file of SMP performs replacements on terms that involve exponentials and logarithms. The grab.int file removes constants from expressions so that the integration routines do not need to parse through full expressions.

The next external file loaded by the CANSS2D routine is called exp.int. It contains the procedures used to integrate the forcing functions over space. The fourth file loaded in the CANSS2D algorithm is the temporal integration procedure called z2d.int. This file contains procedures to calculate the integration with respect to the time variable. When an integral is not recognised by the external integration routines over space or time, the internal SMP integrator is invoked.

Some constants are loaded along with some mathematical rules that define properties of the transcendental functions. A banner is displayed defining the environment for which the program is run.

The first step in calculating the temperature distribution in the plate is to describe the boundary conditions. The CANSS2D algorithm can use boundary conditions of the zeroth, first and second kinds that have a forcing function which is either zero or a constant. The input data routine also asks for the length of the plate in the x and y direction and calculates the aspect ratio ($r^+ = \frac{L_y}{L_x}$) for the plate. The aspect ratio is important in describing the partition times for each of the three time regions and will be discussed in a following section. The input routine also queries the user for the forcing function placed on each boundary based on the type of boundary condition that occurs on the surface. The CANSS2D routine has all the information necessary to describe the temperature distribution for the stated problem.

The CANSS2D program begins the calculation of temperature distribution by deciding if the two dimensional problem is actually a special case of a one dimensional problem. This may occur when a boundary condition is semi-infinite. A special routine is loaded to handle problems with boundary conditions of the zeroth kinds. This problem may not be strictly two dimensional depending on the boundary conditions on the other surfaces.

If the problem is not a special case, the Green's functions data base is loaded and the appropriate Green's function for the three time regions are calculated and displayed. The next step is to integrate each of the Green's function with respect to the spatial variables and display the results for the three time regions. The pattern that is used in the integration is 1) check the integral against the external CANSS2D integration library and if a match is not found, 2) let the SMP internal integrator operate on the integral and if a solution is not obtained, 3) return the integral to the user for further evaluation.

The CANSS2D program will generate closed form solutions for cases associated with the limitations placed on the boundary and initial conditions, and the volume energy heat source.

The last calculation step is to generate the temperature distribution for the appropriate time region by integrating with respect to time. The temperature distribution is displayed and the program stops and displays the calculation time in CPU seconds. A flowchart of the CANSS2D is shown in the next section and includes the input to the example problem of a partially heated plate examined in the next section.

5.2.2 Two Dimensional CANSS2D Flowchart/Example

This section contains a flowchart/example of a two dimensional plate partially heated on one side. The input and output from the CANSS2D program is shown in Section 2 of Appendix C.

Flowchart	Enter	Explanation
Start Program	<"canss2d.prg"<CR>	
Load Subroutines		The subroutines for the Green's function, grab.int, exp.int and the external integration routines are loaded.
Display Banner	<CR>	A banner is displayed that defines the environment for 2-D problems.
Display Defaults	<CR>	Give the default values and define some constants.

Enter Boundary Condition Numbers	2<CR> 2<CR> 1<CR> 1<CR>	Enter the types of boundary condition for the bottom, left, top, and right sides.
Display GF Number		Display the Green's function number for this problem.
Input Lengths	1<CR> 2<CR>	Input the length of the plate in the x and y direction.
Input Bottom Forcing Function	y<CR>	Input the forcing function on the bottom of the plate. For zero, type y
Input Left Forcing Function	n<CR> Qo<CR> 0<CR> .5<CR>	Input the forcing function on the left side of the plate. For zero, type y.
Input Top Forcing Function	y<CR>	Input the forcing function on the top of the plate. For zero, type y.
Input Right Forcing Function	y<CR>	Input the forcing function on the right side of the plate. For zero, type y.
Load 2-D Routine		Check for BC of zeroth kind. Load appropriate subroutine.
Display Small-Small GF		Display the two dimensional Green's function for the small-small time region.
Display Small-Large GF		Display the two dimensional Green's function for the small-large time region.
Display Large-Large GF		Display the two dimensional Green's function for the large-large time region.
Calculate	<CR>	Calculate the spatial integration for

Spatial Integration	<CR> <CR>	the three time regions.
Calculate Temporal Integration	<CR> <CR> <CR>	Calculate the temporal integration for the three time regions.
STOP		Stop program and display the calculation time in CPU seconds.

5.3 **Two Dimensional CANSS2D Examples**

Two example problems of the two dimensional CANSS2D algorithm are presented in this section. The first example problem is a plate with an aspect ratio of 1/2. A constant heat flux and temperature occur on two of the four boundaries (X21B10Y21B01T0); see Figure 5.1a. The second example problem is a a plate insulated on the bottom and with zero temperature on the top and right hand side. The left boundary has a constant heat flux over half the boundary and insulated otherwise (X22B(1,0)0Y22T0), see Figure 5.1b.

5.3.1 A Two Dimensional Plate with Heating

The first example problem solved using the CANSS2D program is a plate with a constant heat flux, q_0, on the left surface, a constant temperature, T_0, on the upper surface, insulated on the bottom and having the temperature on the right hand side zero; see Figure 5.1a. The aspect ratio of the plate is 1/2 or the length of the plate in the y direction is L and the length of the plate in the x direction is 2L. The terms x and y, which represent distances in the x and y direction,

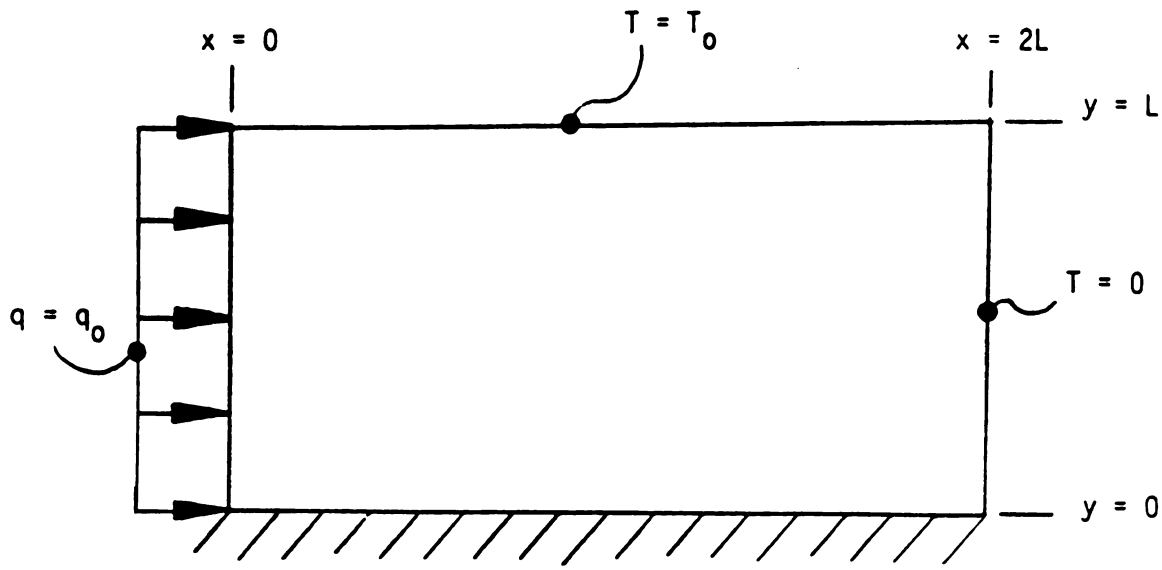

Figure 5.1a Thin Plate With Heat Flux and Temperature Conditions

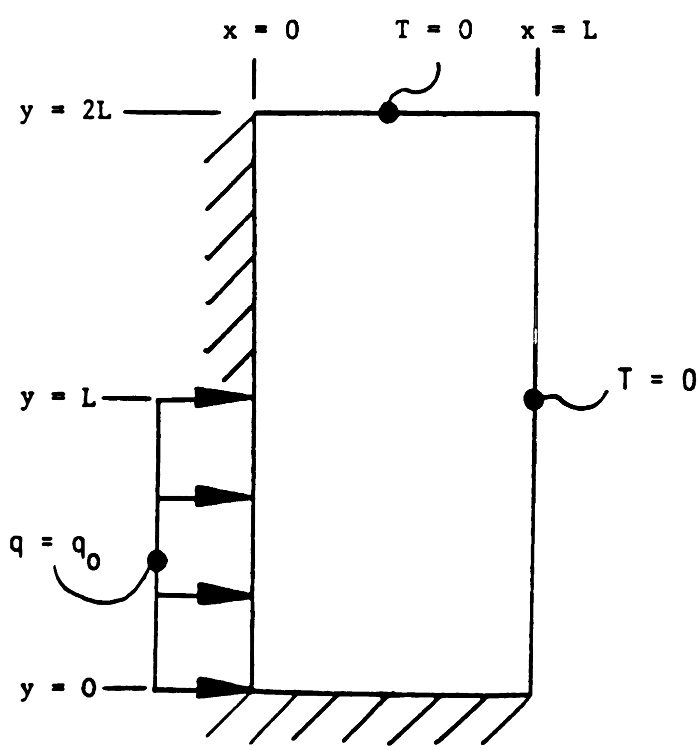

Figure 5.1b Partially Heated Thin Plate

are made dimensionless in the CANSS2D algorithm by dividing by the respective lengths of the plate in the x and y direction. The Green's function for this problem (X21Y21) is split into three convergent expressions for three regions of dimensionless time.

The third expression of the temperature distribution, the part applicable when the dimensionless times for the x and y direction are considered large, is examined first because an analytical solution is available. Beck [1984a, pg. 1242] has obtained an analytical solution for the temperature distribution to be,

$$T(x,y,t) = 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[1 - e^{-\alpha t[\cdot]} \right] \cos(\frac{\beta_m x}{2\,L}) \cos(\frac{\beta_n y}{L}) \cdot$$

$$\frac{(-1)^n}{(\beta_m^2 + 4\,\beta_n^2)} \left\{ 4\,T_0 \frac{\beta_n}{\beta_m} (-1)^m - \frac{2\,q_0\,L}{\beta_n\,k} \right\}, \qquad (5.3.1)$$

where the term β_m is an eigenvalue and m is the summation index for the Green's function in the x direction, the term β_n is the eigenvalue and n is the summation index for the Green's function in the y direction. The term α is the thermal diffusivity and k is the thermal conductivity of the plate. The term $[\cdot]$, when the length of the plate in the x direction is 2L and the length in the y direction is L, is given by,

$$[\cdot] = \left(\frac{\beta_m}{2L} \right)^2 + \left(\frac{\beta_n}{L} \right)^2 \qquad (5.3.2)$$

The CANSS2D algorithm splits the dimensionless time t^* into three regions. In the third region, t_2^* is defined as the dimensionless partition time between the second and third time region and t^* is the dimensionless time of interest. A description of the partition times for two and three dimensional heat diffusion problems is given later in this chapter.

The thermal conductivity and the thermal diffusivity default to one in the CANSS2D program and give the change in temperature distribution in the third time region as,

$$\Delta T(x,y,t) = 8 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ 2\, T_0 \frac{\beta_n}{\beta_m} (-1)^m - \frac{q_0 L}{\beta_n} \right\} \cos\left(\frac{\beta_m x}{2\,L}\right) \cos\left(\frac{\beta_n y}{L}\right) \cdot$$

$$\frac{(-1)^n}{(\beta_m^2 + 4\,\beta_n^2)} \left[e^{-t_2^*(\beta_m^2 + 4\,\beta_n^2)} - e^{-t^*(\beta_m^2 + 4\,\beta_n^2)} \right] \qquad (5.3.3)$$

If the dimensionless partition time t_2^* is set to zero, the temperature distribution from Beck and from the CANSS2D program match exactly.

In the second time region, the change in temperature distribution that is efficient for times greater than the dimensionless partition time t_1^* and less than the dimensionless partition time t_2^* is given as,

$$\Delta T(x,y,t) = \frac{q_0 L}{k} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n+1} \cos\left(\frac{\beta_n y}{L}\right)}{\beta_n^2}$$

$$\left\{ \left[\operatorname{erfc}\left(2\beta_n (t_1^*)^{1/2} - \frac{\beta_{mo}}{(t_1^*)^{1/2}}\right) - \operatorname{erfc}\left(2\beta_n (t^*)^{1/2} - \frac{\beta_{mo}}{(t^*)^{1/2}}\right) \right] e^{-4\,\beta_n \beta_{mo}} \right.$$

$$\left. + \left[\operatorname{erfc}\left(2\beta_n (t_1^*)^{1/2} + \frac{\beta_{mo}}{(t_1^*)^{1/2}}\right) - \operatorname{erfc}\left(2\beta_n (t^*)^{1/2} + \frac{\beta_{mo}}{(t^*)^{1/2}}\right) \right] e^{4\,\beta_n \beta_{mo}} \right\}$$

$$+ \quad \frac{T_0}{2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n+1} \cos(\frac{\beta_n y}{L})}{\beta_n} \Big[\operatorname{coshe}(t^*, \beta_n, \beta_{m+})$$

$$+ \operatorname{coshe}(t^*, \beta_n, \beta_{m-}) - \operatorname{coshe}(t_1^*, \beta_n, \beta_{m-}) - \operatorname{coshe}(t_1^*, \beta_n, \beta_{m+}) \Big], \tag{5.3.4}$$

where the term $\beta_n = \pi\ (n - 0.5)$ is the eigenvalue in the y direction. The terms $\beta_{mo} = 0.5\ (2\ m + x_L)$, $\beta_{m+} = 0.5\ (2\ m + x_L + 1)$, $\beta_{m-} = 0.5\ (2\ m + x_L - 1)$, and,

$$\operatorname{coshe}(t, \beta_n, \beta_{m\pm}) = \frac{1}{\beta_n^2} \Bigg\{ \frac{e^{-2\beta_n \beta_{m\pm}}}{2} \operatorname{erfc}\left[\beta_n (t^*)^{1/2} - \frac{\beta_{m\pm}}{(t^*)^{1/2}} \right]$$

$$- \frac{e^{2\beta_n \beta_{x\pm}}}{2} \operatorname{erfc}\left[\beta_n (t^*)^{1/2} + \frac{\beta_{m\pm}}{(t^*)^{1/2}} \right] - e^{-\beta_n^2 t} \operatorname{erfc}\left[\frac{\beta_{m\pm}}{(t^*)^{1/2}} \right] \Bigg\} \tag{5.3.5}$$

The term coshe(•,•,•) will be examined more closely in the next section.

The term q_0 is the constant heat flux, L is the length of the plate in the y direction and k is the thermal diffusivity. The most dominant terms in the summations are m = 1 and n = 0. The term β_n is the eigenvalue for the large time Green's function in the y direction. The term $\beta_{m\pm}$ can be thought of as similar to an eigenvalue for the small time Green's function in the x direction. The term $\beta_{m\pm}$ is not an eigenvalue.

When the dimensionless time in both the x and y direction is small the temperature in the first time region is,

$$T(x,y,t) = 2 \frac{q_0 L}{k} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} \cdot$$

$$\left\{ \frac{(t^*)^{1/2}}{\sqrt{\pi}} \; e^{-\beta_{mo}^2/t^*} \left[\mathrm{erf}\left(\frac{\beta_{n+}}{(t^*)^{1/2}} \right) - \mathrm{erf}\left(\frac{\beta_{n-}}{(t^*)^{1/2}} \right) \right] \right.$$

$$+ \frac{\beta_{mo}}{\pi} \left[\mathrm{expi}\left(1, \frac{\beta_{mo}^2 + 4\,\beta_{n+}^2}{t^*} \right) - \mathrm{expi}\left(1, \frac{\beta_{mo}^2 + 4\,\beta_{n-}^2}{t^*} \right) \right]$$

$$\left. + \beta_{mo} \left[H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{\beta_{mo}}{2\,\beta_{n-}} \right) - H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{\beta_{mo}}{2\,\beta_{n+}} \right) \right] \right\}$$

$$+ \frac{T_0}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} \left[H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{2\,\beta_{no}}{\beta_{m+}} \right) - H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{2\,\beta_{no}}{\beta_{m-}} \right) \right], \tag{5.3.6}$$

where the function $H(\cdot,\cdot)$ is discussed by Litkouhi [1982] and in Appendix B. The symbols $\beta_{no} = 0.5\ (2\ n - y_L + 1)$, $\beta_{n+} = 0.5\ (2\ n + y_L + 1)$, $\beta_{n-} = 0.5\ (2\ n + y_L - 1)$, and the term $\mathrm{expi}(\cdot,\cdot)$ is the exponential integral. The exponential integral is defined in Abramowitz and Stegun [1964] as,

$$\mathrm{expi}[n,x] = E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n}\, dt. \tag{5.3.7}$$

Other symbols in the temperature distribution equation, when the dimensionless times for the x and y direction are considered small, have been defined previously. The convergent equations for the change in temperature distribution in the first and second time region have not been previously investigated.

5.3.2 A Partially Heated Plate

The second example of the CANSS2D program, see Figure 5.1b, is for a plate with zero temperature on the top and right hand side, insulated on the bottom, a heat flux, q_0, applied to the lower half of the left hand side and insulated on the top half of the left hand side. The aspect ratio for this case is two.

The temperature distribution in the plate for the first time region generated by the CANSS2D program (when the dimensionless time in the x and y directions is small) is,

$$T(x,y,t) = \frac{q_0 L}{2\ k} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n}$$

$$\left\{ \frac{(t^*)^{1/2}}{\sqrt{\pi}} e^{-\beta_{mo}^2/t} \left[\operatorname{erf}\left(\frac{2\beta_{n+}}{(t^*)^{1/2}} \right) - \operatorname{erf}\left(\frac{2\beta_{n-}}{(t^*)^{1/2}} \right) \right] \right.$$

$$+ \frac{\beta_{mo}}{\pi} \left[\operatorname{expi}\left(1, \frac{\beta_{mo}^2 + 4\ \beta_{n+}^2}{t^*} \right) - \operatorname{expi}\left(1, \frac{\beta_{mo}^2 + 4\ \beta_{n-}^2}{t^*} \right) \right]$$

$$\left. + \beta_{mo} \left[H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{\beta_{mo}}{2\ \beta_{n-}} \right) - H\left(\frac{\beta_{mo}}{(t^*)^{1/2}}, \frac{\beta_{mo}}{2\ \beta_{n+}} \right) \right] \right\} . \qquad (5.3.8)$$

The terms $\beta_{n+} = 0.5\ (2\ n + y_L + \frac{1}{2})$ and $\beta_{n-} = 0.5\ (2\ n + y_L - \frac{1}{2})$ for this problem and β_{mo} has been defined in the previous example.

The change in temperature distribution for the second time region generated by the CANSS2D program, where one dimensionless time is large and one is small, is,

$$\Delta T(x,y,t) = \frac{1}{2} \frac{q_0 L}{k} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cos(\frac{\beta_m x}{L})}{\beta_m^2} \left[\operatorname{coshe}(t^*, \beta_m, 2\ \beta_{n-}) \right.$$

$$+ \text{coshe}(t^*, \beta_m, 2\,\beta_{n+}) \;-\; \text{coshe}(t_1^*, \beta_m, 2\,\beta_{n+}) \;-\; \text{coshe}(t_1^*, \beta_m, \beta_{n-}) \Big], \tag{5.3.9}$$

where the term β_m is the eigenvalue for the x direction, the terms $\beta_{n\pm}$ were described in the previous time region and $\text{coshe}(t^*, \beta_m, \beta_{n\pm})$ will be described in the next section.

The change in temperature distribution for the third time region generated by the CANSS2D program, when both dimensionless times are large, is,

$$\Delta T(x,y,t) = 16\,\frac{q_0 L}{k} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\frac{\beta_m x}{L})\,\cos(\frac{\beta_n y}{2\,L})\quad \sin(\frac{\beta_n}{4})}{\beta_n^2\,(4\,\beta_m^2 + \beta_n^2)}\,\cdot$$

$$\Big[\, e^{-t_2^*(\beta_m^2 + \beta_n^2/4)} - e^{-t^*(\beta_m^2 + \beta_n^2/4)} \Big]. \tag{5.3.10}$$

The change in temperature distribution when the time in the x and y direction are considered large is shown in Table 5.1 for the position $x = \frac{L}{2}$ and a partition time $t_2 = 0.8$.

5.4 Two Dimensional Integrals

Three integrals that often appear in calculating the temperature distribution in two dimensional diffusion problems are presented in this section. The first integral examined occurs in the calculation of temperature distribution in the first time region. The next integral occurs in the calculation of the temperature distribution in second time region. The two integrals are difficult to solve and evaluate because

Table 5.1 Dimensionless Change in Temperature From Equation (5.3.10) When the Partition Time Is Set to 0.1 and the x Coordinate Is Set to L/2.

Position y	time = 1.0	time = 1.5	time = 2.0
0.0	3.48250E-4	6.68336E-4	7.36753E-4
0.1	3.47147E-4	6.66239E-4	7.34445E-4
0.2	3.43846E-4	6.59963E-4	7.27537E-4
0.3	3.38373E-4	6.49553E-4	7.16078E-4
0.4	3.30769E-4	6.35082E-4	7.00148E-4
0.5	3.21092E-4	6.16653E-4	6.79859E-4
0.6	3.09412E-4	5.94393E-4	6.55349E-4
0.7	2.95815E-4	5.68458E-4	6.26788E-4
0.8	2.80399E-4	5.39023E-4	5.94368E-4
0.9	2.63273E-4	5.06288E-4	5.58306E-4
1.0	2.44554E-4	4.70470E-4	5.18841E-4
1.1	2.24369E-4	4.31803E-4	4.76228E-4
1.2	2.02851E-4	3.90537E-4	4.30743E-4
1.3	1.80138E-4	3.46933E-4	3.82672E-4
1.4	1.56373E-4	3.01262E-4	3.32315E-4
1.5	1.31703E-4	2.53807E-4	2.79982E-4
1.6	1.06275E-4	2.04855E-4	2.25991E-4
1.7	8.02387E-5	1.54700E-4	1.70667E-4
1.8	5.37464E-5	1.03638E-4	1.14338E-4
1.9	2.69494E-5	5.19707E-5	5.73369E-5
2.0	0.00000	0.00000	0.00000

of the types of functions involved and the appearance of the convolution of time. Other special integrals which may occur in two dimensional problems are given in Appendix B.

5.4.1 An Integral in Time Region One

An integral that often occurs when the dimensionless time in the x and y direction is small and the coordinates x_L and y_L are normalized with respect to the lengths L_x and L_y is,

$$I_1 = \int_{\tau=0}^{t} \frac{e^{-x_L^2/(4\alpha(t-\tau)/L_x^2)}}{\sqrt{(t-\tau)}} \operatorname{erfc}\left(\frac{y_L}{\sqrt{4\alpha(t-\tau)/L_y^2}} \right) d\tau \qquad (5.4.1)$$

Dimensionless groups used to eliminate the convolution on time are defined to be,

$$w = \frac{x_L}{\sqrt{4\alpha(t-\tau)/L_x^2}} \quad , \quad X = \frac{x_L}{\sqrt{4\alpha t/L_x^2}} \quad , \quad Y = \frac{y_L\, r^+}{\sqrt{4\alpha t/L_x^2}}$$

where (r^+) is the aspect ratio. The integral I_1 becomes,

$$I_1 = 2\sqrt{t}\;\; X \int_{w=X}^{\infty} \frac{e^{-w^2}}{w^2} \operatorname{erfc}\left(\frac{Y}{X} w \right) dw \;, \qquad (5.4.2)$$

where t is the time of interest. Litkouhi [1982, pg. 123] has shown that this integral can be written as,

$$I_1 = 2\sqrt{t}\left\{\sqrt{\pi}\,\mathrm{ierfc}(X) - e^{-X^2}\,\mathrm{erf}(Y) - \frac{Y}{\sqrt{\pi}}\,E_1(X^2 + Y^2)\right.$$

$$\left. + \sqrt{\pi}\,X\,H\!\left(Y, \frac{X}{Y}\right)\right\}, \qquad (5.4.3)$$

where $H(\cdot,\cdot)$ is an integral discussed by Litkouhi and obtained from a text by Rosser [1948]. When $X = 0$, integral I_1 becomes,

$$I_1 = 2\sqrt{t}\left\{\mathrm{erfc}(Y) - \frac{Y}{\sqrt{\pi}}\,E_1(Y^2)\right\}. \qquad (5.4.4)$$

A plot of the integral when $X = 0$ is shown in Figure 5.2. This figure shows the immediate effect on the temperature distribution of the integral at the surface $Y = 0$ as time increases. As the point of interest is moved deeper in the body (i.e. $Y = 0.25$), the effect on the distribution is smaller and rises slower.

When $Y = 0$, integral I_1 becomes,

$$I_1 = 2\sqrt{t}\left\{\sqrt{\pi}\,\mathrm{ierfc}(X)\right\}. \qquad (5.4.5)$$

A plot of this function is shown in Figure 5.3. This figure shows the effect of the integral on the temperature distribution as the time increases. Due to the nature of the ierfc function, a small increase in the argument to the function leads to a large decrease in the value of the function.

5.4.2 Two Integrals in Time Region Two

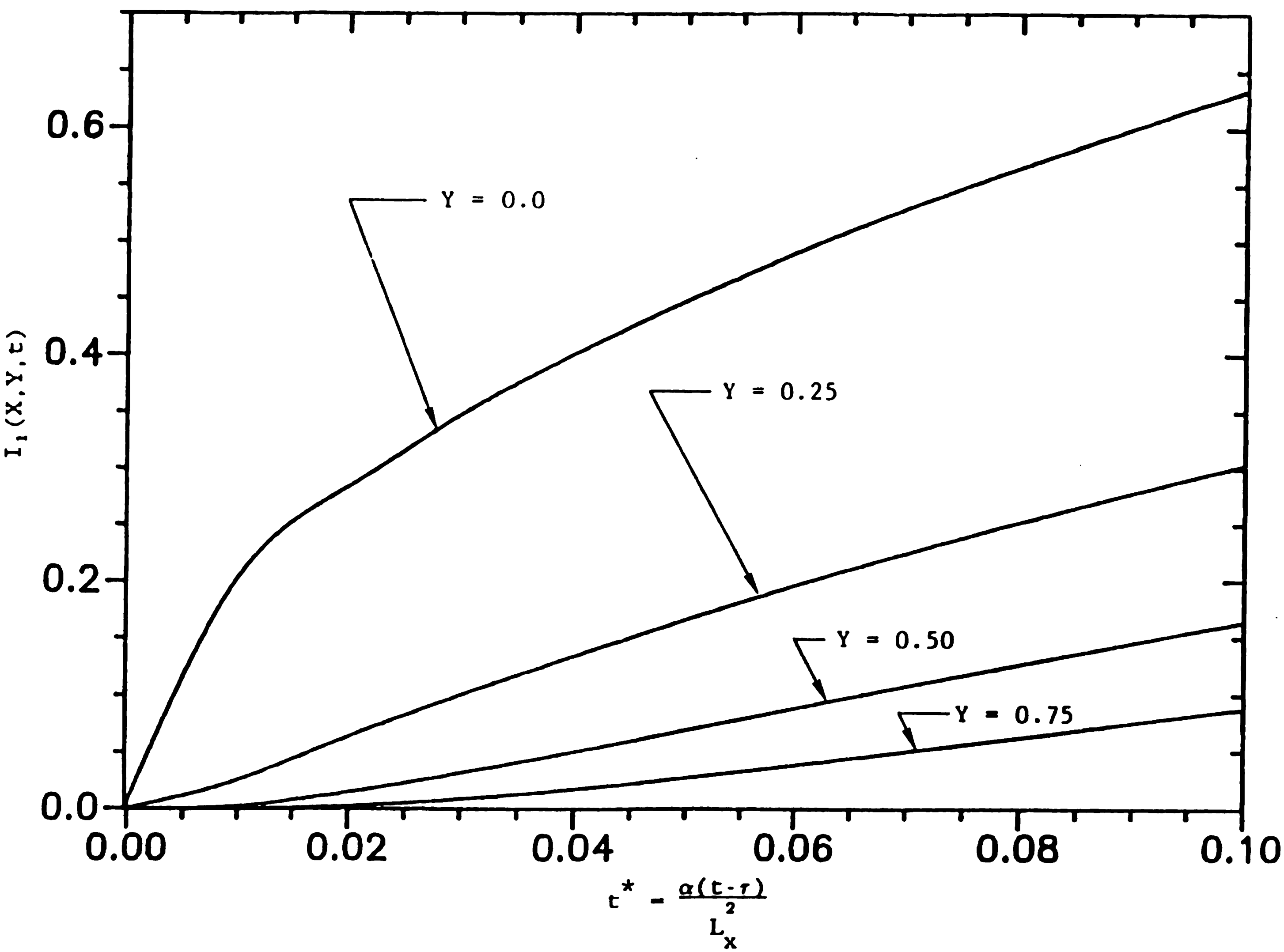

Figure 5.2 Integral in Equation (5.4.4) Versus Dimensionless Time When X = 0.

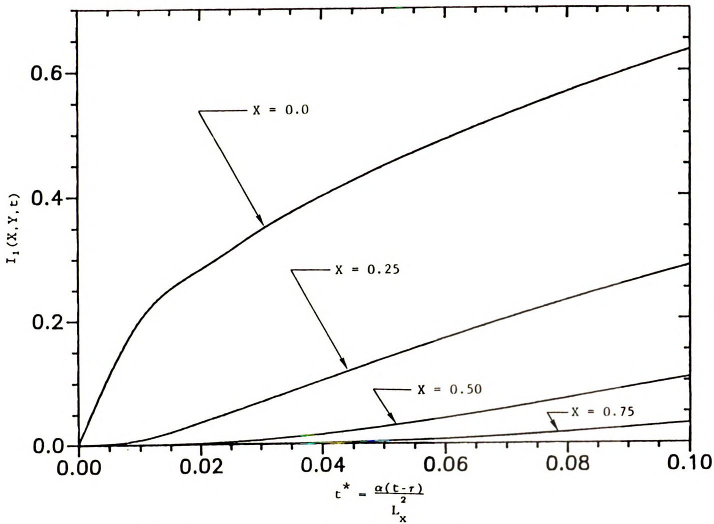

Figure 5.3 Integral in Equation (5.4.5) Versus Dimensionless Time When Y = 0.

Two integrals that often occur in two dimensional problems when the dimensionless time in one direction is small and in the other direction large are,

$$I_2 = \int_{\theta=t_1^*}^{t^*} e^{-C_1^2\theta} \operatorname{erfc}\left(\frac{C_2}{\sqrt{\theta}} \right) d\theta \ , \tag{5.4.6}$$

and,

$$I_3 = \int_{\theta=t_1^*}^{t^*} \theta^{-1/2} \ e^{-(C_1^2\ \theta \ + \ C_2^2/\theta)} \ d\theta, \tag{5.4.7}$$

where $\theta = \frac{\alpha\ (t-\tau)}{L^2}$ is the convoluted time variable and C_1 and C_2 are constants.

The solution to the integral I_2 is found in Cho [1971] to be,

$$I_2 = \frac{1}{2\ C_1^2} \Bigg[e^{-2\ C_1\ C_2} \operatorname{erfc}(C_1\sqrt{\theta} - \frac{C_2}{\sqrt{\theta}}) - e^{2\ C_1\ C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}})$$

$$- 2\ e^{-C_1^2} \operatorname{erfc}(\frac{C_2}{\sqrt{\theta}}) \Bigg]_{\theta=t_1^*}^{t^*} \ . \tag{5.4.8}$$

By defining a new function $\operatorname{coshe}(t,C_1,C_2)$ to be,

$$\operatorname{coshe}(t,C_1,C_2) = \frac{1}{2\ C_1^2} \Bigg[e^{-2\ C_1\ C_2} \operatorname{erfc}(C_1\sqrt{t} - \frac{C_2}{\sqrt{t}})$$

$$- e^{2\ C_1\ C_2} \operatorname{erfc}(C_1\sqrt{t} + \frac{C_2}{\sqrt{t}}) \ - 2\ e^{-C_1^2\ t} \operatorname{erfc}(\frac{C_2}{\sqrt{t}}) \Bigg], \tag{5.4.9}$$

the integral I_2 may be written in compact form as,

$$I_2 = \text{coshe}(t^*, C_1, C_2) - \text{coshe}(t_1^*, C_1, C_2). \qquad (5.4.10)$$

Two plots of the coshe function versus dimensionless time for various values of the constants C_1 and C_2 are shown in Figures 5.4 and 5.5. Figure 5.4 plots the coshe function when the value of the parameter $C_1 = \frac{\pi}{2}$. This value corresponds to the first eigenvalue of a Green's function for a X21 case. The difference between any two values of dimensionless time is positive. The difference between two values of the coshe function decreases as the value of the parameter, C_2, increases.

Figure 5.5 plots the coshe function when the value of the first parameter $C_1 = \pi$. This value corresponds to the first eigenvalue of a Green's function for a X22 case. This figure shows the coshe function decreases rapidly as the parameter C_1 is increased. Additional eigenvalues are not necessary for computations involving the coshe function.

A summary of the coshe function is:

1) $t \to 0$	$\text{coshe}(0, C_1, C_2) \to 0$
2) $t \to \infty$	$\text{coshe}(\infty, C_1, C_2) \to 0$
3) $C_1 \neq 0$	
4) $C_1 \to \infty$	$\text{coshe}(t, \infty, C_2) \to 0$
5) $C_2 \to 0$	$\text{coshe}(t, C_1, 0) \to - \dfrac{e^{-C_1^2 t}}{C_1^2}$
6) $C_2 \to \infty$	$\text{coshe}(t, C_1, \infty) \to 0.$

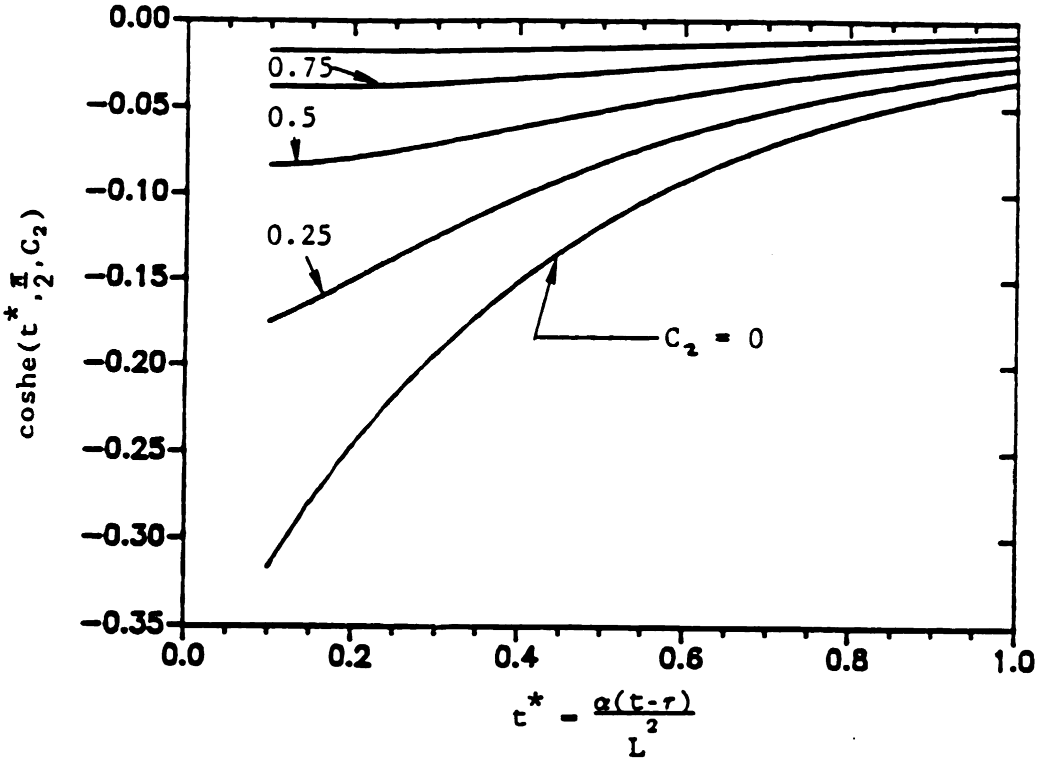

Figure 5. 4 The coshe Function Versus Dimensionless Time When $C_1 = \frac{\pi}{2}$ and Various Values of C_2.

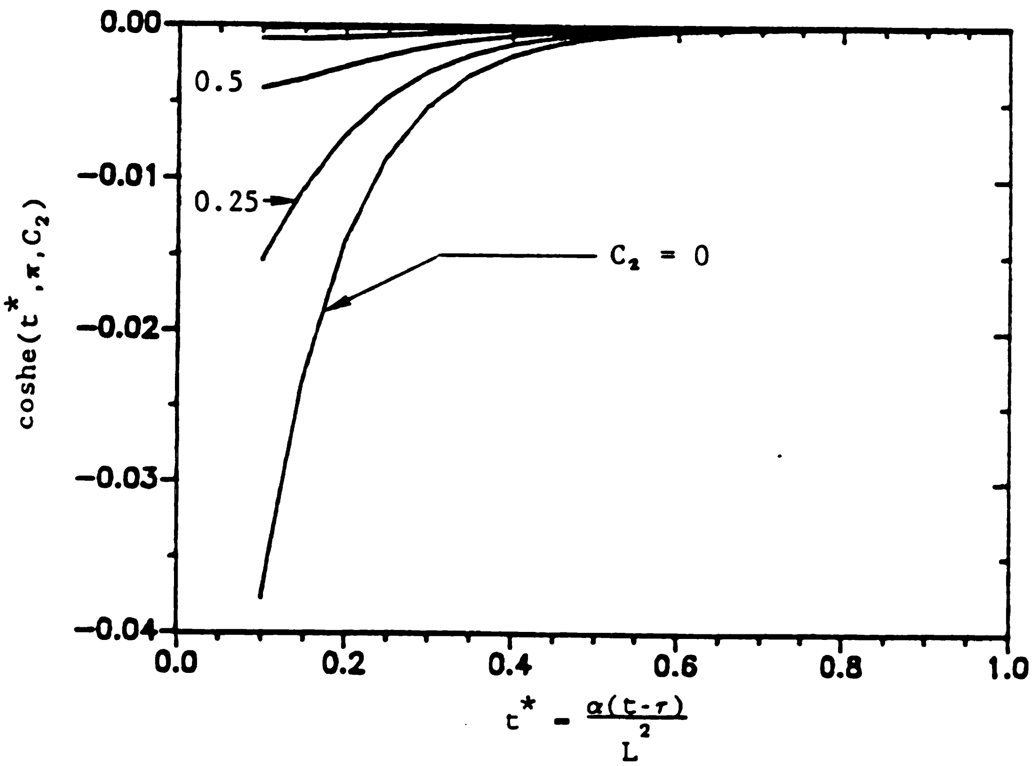

Figure 5. 5 The coshe Function Versus Dimensionless Time When $C_1 = \pi$ and Various Values of C_2.

The solution of the next integral, I_3, which appears regularly in the second time region can be found in Abramowitz and Stegun [1964, pg. 304] to be,

$$I_3 = \frac{\sqrt{\pi}}{2\ C_1} \left[e^{2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) + e^{-2C_1C_2} \operatorname{erfc}(2C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) \right]_{\theta=t^*}^{t_1^*} \tag{5.4.11}$$

The solutions to both integrals, I_2 and I_3, include a term that may cause numerical instability of the total solution. The term,

$$e^{2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}), \tag{5.4.12}$$

may not be easy to evaluate due to the positive argument in the exponential function. The complementary error function can be approximated for large values of the argument (> 1.7) and is,

$$\operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) = \frac{1}{\sqrt{\pi}}\ e^{-(C_1^2\theta + C_2^2/\theta)}\ e^{-2C_1C_2}\ \cdot$$
$$\left[\frac{1}{(C_1\sqrt{\theta} + C_2/\sqrt{\theta})} - \frac{1}{2(C_1\sqrt{\theta} + C_2/\sqrt{\theta})^3} + \frac{3}{4(C_1\sqrt{\theta} + C_2/\sqrt{\theta})^5} - \cdots \right], \tag{5.4.13}$$

No approximation is necessary when the argument to the complimentary error function is small (< 1.7).

5.5 Time Partitioning in Two and Three Dimensions

The time partitioning scheme for two or three dimensional heat diffusion problems is more complicated than for a one dimensional heat diffusion problem because the dimensionless time variables for the additional coordinates are not equivalent unless the lengths of the sides of the plate are equal.

The CANSS2D program for a two dimensional plate bases the dimensionless time on the x-coordinate length of the plate. This choice is arbitrary and need not be followed in general. The generalized dimensionless time for the plate is,

$$t_x^* = \frac{\alpha\,(t - \tau)}{L_x^2}. \tag{5.5.1}$$

In the y-direction, the dimensionless time based on the length in the y direction is,

$$t_y^* = \frac{\alpha\,(t - \tau)}{L_y^2}. \tag{5.5.2}$$

Defining an aspect ratio (r^+) as,

$$(r^+) = L_y/L_x, \tag{5.5.3}$$

the generalized dimensionless time in the y-direction can be written as,

$$t_y^* = t_x^*\,(r^+)^2. \tag{5.5.4}$$

The time region is split into three components. When the aspect ratio (r^+) is greater than one, see Figure 5.6a, the first time region is defined as the X_sY_s region because the Green's function for this area, in both the x and y direction, are for small time. The dimensionless time goes from zero to t^*_{1x}. The second time region, $X_\ell Y_s$, is when the Green's function in the x-direction is for large times and the Green's function in the y-direction is for small times. The time region for region 2 goes from t^*_{1x} to t^*_{2x}. The third time region is valid for times greater than t^*_{2x} and is designated $X_\ell Y_\ell$. Both the Green's functions, in the x and y direction, for the third region are based on the large time.

When the aspect ratio is less than one, see Figure 5.6b, a change in the Green's function occurs in the second time region. The Green's function in the x-direction is for small times and in the y-direction is for large times, or X_sY_ℓ.

Once a partition time is chosen, the values of t^*_{1x} and t^*_{2x} may be calculated. These values depend on the aspect ratio and the boundary condition applied to the plate. When the aspect ratio is greater than one, see Figure 5.6a,

$$t^*_{1x} = t^*_1 \tag{5.5.5}$$

and,

$$t^*_{2x} = t^*_1 \cdot (r^+)^2, \tag{5.5.6}$$

where t^*_1 is the dimensionless partition time. The partition time is defined as the time when the small time Green's function can be expressed with less than four reflection terms.

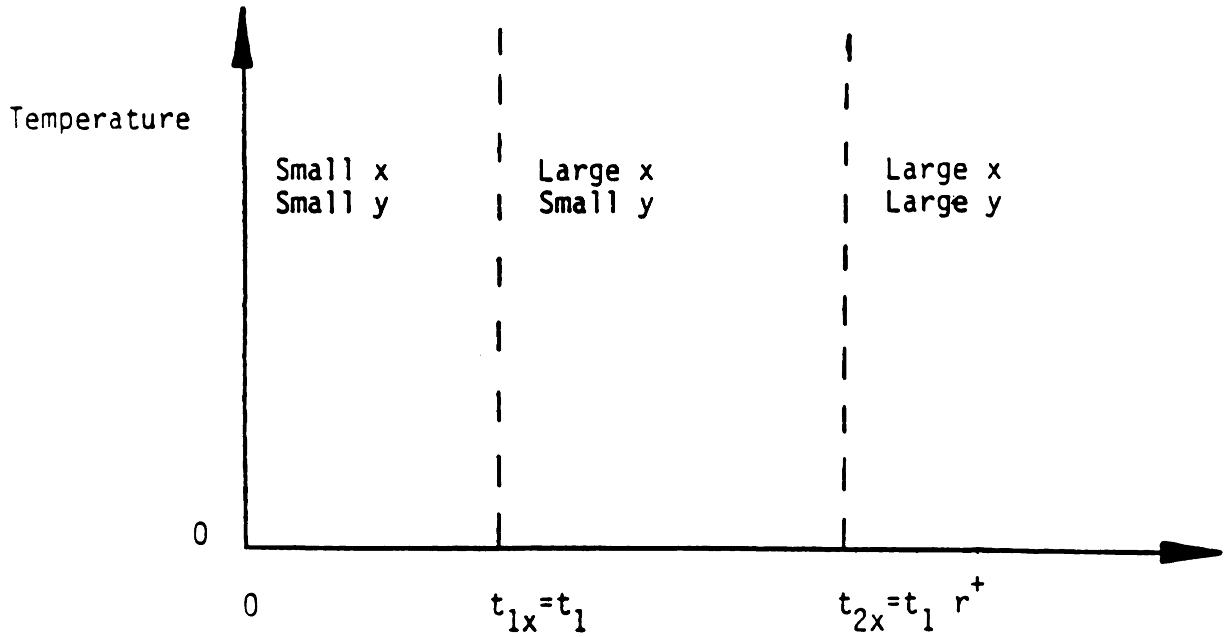

Figure 5.6a Time Partitioning for an Aspect Ratio Greater Than One for a Two Dimensional Body

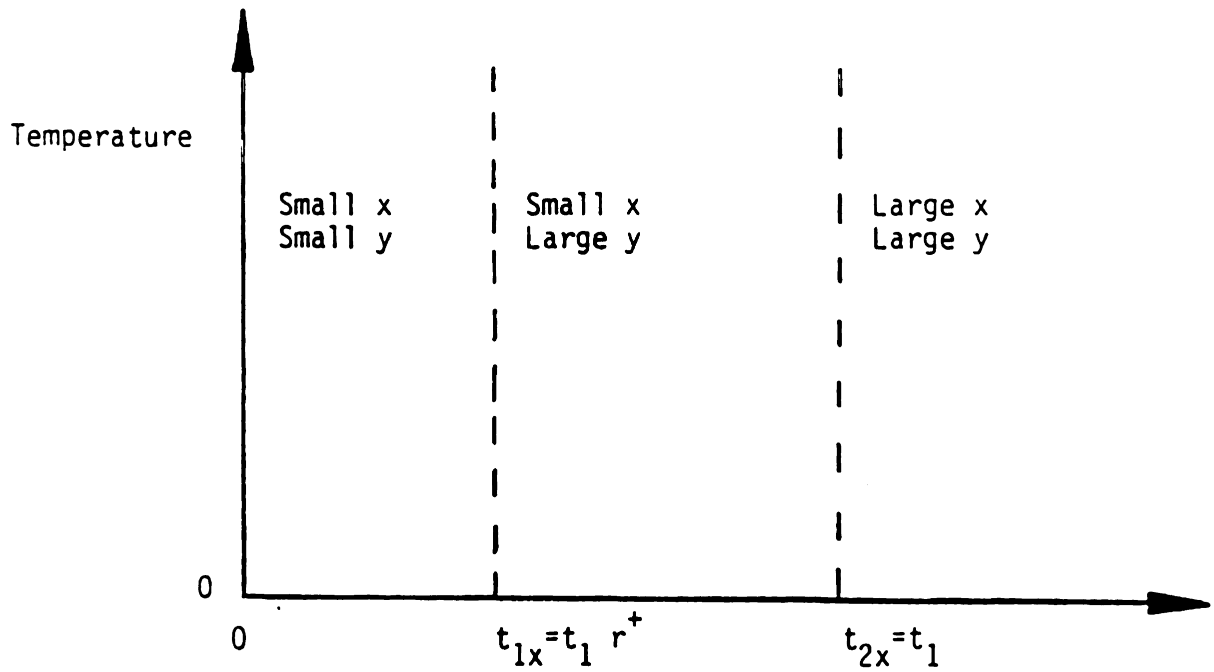

Figure 5.6b Time Partitioning for an Aspect Ratio Less Than One for a Two Dimensional Body

When the aspect ratio is less than one, see Figure 5.6b,

$$t_{1x}^* = t_1^* \cdot (r^+)^2, \qquad (5.5.7)$$

and,

$$t_{2x}^* = t_1^*. \qquad (5.5.8)$$

When the aspect ratio is equal to one, i.e. the plate is square,

$$t_{1x}^* = t_{2x}^* = t_1^*, \qquad (5.5.9)$$

and only two time regions appear.

The time regions for the three dimensional case follow in the same manner as the two dimensional case. For three dimensional cases, it is convenient to make the assumptions that the direction of the shortest length of the body is in the x direction, the next shortest length is in the y direction, and the longest length is in the z direction. Assigning these coordinate directions leads to the following three relationships for the aspect ratios,

$$\frac{L_y}{L_x} = (r^+)_{yx} > 1, \qquad (5.5.10)$$

$$\frac{L_z}{L_x} = (r^+)_{zx} > 1, \quad \text{and} \qquad (5.5.11)$$

$$\frac{L_z}{L_y} = (r^+)_{zy} > 1. \qquad (5.5.12)$$

If, for example, the partition time in the x direction is defined to be 0.1, and the lengths of the body in the x, y and z direction respectively are $L_x = 1$, $L_y = 2$, and $L_z = 4$, the aspect ratios for the previous equations are $(r^+)_{yx} = 2$, $(r^+)_{zx} = 4$, and $(r^+)_{zy} = 2$. Choosing the x direction as the characteristic dimension yields,

$$\frac{\alpha(t-\tau)}{L_x^2} = 0.1 \ , \qquad (5.5.13)$$

as the definition of the partition time. When the time in the characteristic direction is 0.1, the associated time in the y and z direction is,

$$\frac{\alpha(t-\tau)}{L_y^2} = \frac{\alpha(t-\tau)}{L_y^2}\frac{L_x^2}{L_x^2} = \frac{\alpha(t-\tau)}{L_x^2}\frac{L_x^2}{L_y^2} = 0.1\left(\frac{1}{4}\right) = 0.025, \qquad (5.5.14)$$

and,

$$\frac{\alpha(t-\tau)}{L_z^2} = \frac{\alpha(t-\tau)}{L_z^2}\frac{L_x^2}{L_x^2} = \frac{\alpha(t-\tau)}{L_x^2}\frac{L_x^2}{L_z^2} = 0.1\left(\frac{1}{16}\right) = 0.00625 \qquad (5.5.15)$$

This means that when the time based on the characteristic direction x forces the program to switch the x direction Green's function from small to large time, the local time in the other two directions are still small and the small time Green's functions are appropriate. A description of this example is shown in Figure 5.7. Notice that no more than four time regions may appear for the three dimensional problems. The number of time regions is reduced by one if the length of two of the sides of a three dimensional body are equal. If the three dimensional body is square, only two time regions appear.

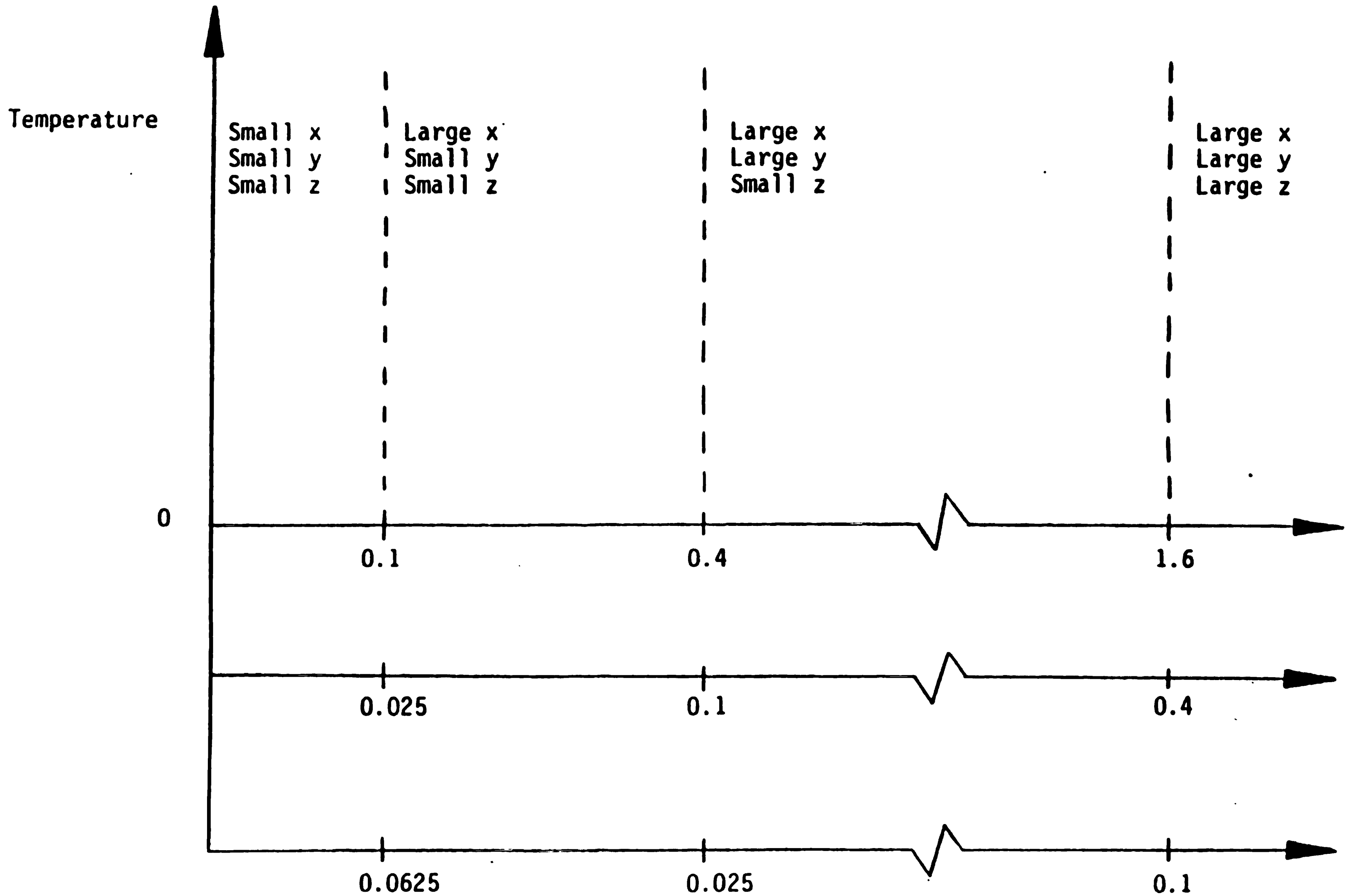

Figure 5.7 Time Partition Regions for a Three Dimensional Body Where $r^+_{yx} > 1$, $r^+_{zx} > 1$, and $r^+_{zy} > 1$.

5.6 Summary

A program that successfully generates symbolic temperature distributions for plate geometries for three types of boundary conditions is presented. Two example problems were chosen and calculated by the CANSS2D algorithm. Some parts of the two example problems appear in the literature and the CANSS2D program matches the analytical solutions exactly. Some integrals that appear in the calculation of the temperature distribution of a two dimensional plate are discussed extensively. The concept of time partitioning in two or three dimensions is discussed.

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1 Summary

A computer algebra program called SMP is applied to transient heat diffusion problems using a Green's function approach to obtain analytical temperature distributions in one and two dimensions. The method restricts the describing partial differential equation, initial condition, boundary conditions, and volume energy heat source to be linear. The symbolic algorithm is written in the SMP programming language for the cartesian coordinate system.

The Green's function approach is an ideal candidate for computer algebra programs. The formalism for the approach, developed in Chapter 2, relies on the complex mathematical concepts of differentiation and integration. The CANSS programs applies the symbolic integration and differentiation procedures that are internal to SMP and that have been developed for this thesis to the Green's function formalism and calculates symbolic temperatures. The computer aids the calculation by repeated application of the Green's function and the integration knowledge base to the formalism described in Chapter 2.

The Green's function approach operates on transient, linear, multidimensional heat diffusion equations and is developed in Chapter 2. The boundary conditions are linear and include the traditional boundary conditions of the first, second and third kinds. Three additional boundary condition types are discussed in this chapter. The three additional types of boundary conditions are a boundary condition of the zeroth kind, or a "natural" condition, a boundary condition of the fourth kind, a non-convective thin film condition, and a boundary condition of the fifth kind, a convective thin film.

A major difficulty of the Green's function approach is obtaining Green's functions for small dimensionless times that converge quickly. A new analytical technique for generating Green's functions efficient for small times is given in Chapter 3. An equation and a group of tables are presented in Chapter 3 to generate the Laplace transformed small time Green's functions. A small table of Laplace transforms (Appendix A) is used to transform these functions back into real time.

The Green's functions that are efficient for large times are developed by the use of the separation of variables technique and have been investigated in many texts and papers for boundary conditions of the first, second and third kinds.

The effect of different types of boundary conditions on the surface of a semi-infinite body in one dimension is described in Chapter 3. Boundary conditions of the zeroth, first and second kinds are examined and boundary conditions of the third, fourth, and fifth kinds with different values of their associated properties are examined. The figures show that when the location of the source and point of interest coincide and the dimensionless time is small ($t^* \leq 0.05$), the Green's function is the same for all boundary conditions.

A one dimensional program called CANSS which uses the techniques of the Green's function method to generate symbolic temperature distributions for a variety of boundary conditions, initial conditions and volume energy heat sources is presented in Chapter Four. Three one dimensional example problems are examined. The first example problem is well documented and the CANSS distribution matches the known solution exactly. The second and third example problems have not been examined previously and represent new solutions. Two types of integrals that appear during the calculation of the one dimensional temperature distribution, called the Dawson and convoluted exponential integral are examined. The efficient evaluation of these integrals are described and shown in the figures.

Sixteen distinct cases of the geometry can be accessed by the CANSS program. Boundary conditions that are functions of polynomials in time (i.e. 1, t, t^2 ...), initial conditions that are functions of the spatial coordinates (i.e. 1, x, x^2 ...), and volume energy heat sources that are polynomials in both time and space can be split up by superposition, calculated independently, and added to obtain temperature distributions because of the linearity of the Green's function approach.

Time partitioning of the one dimensional problems is necessary for the efficient evaluation of the temperature distribution. The discussion and figures in Chapter Four describe the method to be used for partitioning the time variable and suggests the point of time for the partition. A flowchart for the one dimensional CANSS routine is presented.

Symbolic temperature distributions for problems in two dimensions using the Green's function approach are presented in Chapter Five. A symbolic algorithm called CANSS2D calculates the temperature distribution boundary conditions of the zeroth, first, and second kind. The solutions to many two dimensional problem in the literature and texts are often left in terms of the initial functions, but the CANSS2D program calculates the symbolic solutions in terms of the coordinates and an appropriate range of time. There are a minimum of ninety three distinct cases for the CANSS2D program. This number is lower than actually possible because the forcing function on the boundaries may extend over any portion of the surface which would lead to an infinite number of cases handled.

Two example problems are presented in Chapter Five. The symbolic temperature distribution for a portion of the first example can be found in the literature. The temperature distribution output from the CANSS2D program matches the known distribution exactly. New expressions for the efficient calculation of temperature distribution are calculated for the small and medium time ranges. These solutions have never appeared in the literature.

The second example problem in Chapter 5 shows the strength of the CANSS2D program. Exact temperature distributions are generated for a plate with only partial heating occurring at one surface. None of the solutions for this case have appreaed in the literature.

Two integrals that often appear in calculating the solutions to two dimensional problems are presented. Each integral is described and examined to obtain solutions that are efficient.

Time partitioning in two and three dimensions is explained and the technique used in the CANSS2D program is shown in some figures. Some insight to the extension of the time partitioning method is given for three dimensional problems.

6.2 Conclusions

The use of a symbolic manipulation program to generate a knowledge based expert system is a new development for a computer algebra system.

The SMP language was used to develop a knowledge base of Green's functions and integrals to be used in conjunction with two programs called CANSS and CANSS2D. A numbering system associated with the Green's functions [Beck and Litkouhi, 1983] has been used extensively by the CANSS program based on the Green's function formalism. The numbering system eliminates the need in the CANSS program for calculation of the appropriate Green's function based on the problem geometry and boundary conditions The Green's function data base contains all the possible combinations of one dimensional cartesian Green's functions. The strength of the CANNS program is not in the controlling program, but in the knowledge bases over which it has control.

A symbolic manipulation program operates well in an problem environment which includes a rigid structure. The mathematical formalism of the Green's function approach is particularly appropriate for use in a computer algebra system. The symbolic manipulation program takes advantage of the structure by eliminating the need for calculating the appropriate Green's function from the homogeneous Green's function equation. Instead, a data base of Green's functions is used to determine the Green's function. Also, a small data base of integrals is included that keeps the program from searching through extensive external libraries, wasting memory space and computer time.

It is important to note here that the smaller the data base, the faster the calculation of the temperature distribution. In most computer algebra systems, any defined function, such as a Green's function, is loaded into the memory and retained until the system is halted or the definition is removed. This extra baggage has a detrimental effect on the speed of the calculations because larger pieces of memory will be swapped to the storage device. The number of page faults (the number of times the memory is filled and must be dumped to a storage device) can become enormous if a check is not kept on the data base size. For example, in the CANSS environment, the Green's function data base is split into three libraries. Each library can be loaded into the CANSS program but since all the definitions are not necessary, a large amount of dead storage is not carried with the program and swapped in and out of memory core.

Computer algebra programs can also be useful for calculations that do not include a broad class of problems. SMP was use to calculated and collect coefficients from the partial fraction expansion. These calculations are straightforward, but are very repetitious and tedious when performed by pencil-and-paper. The probability of human error in these computations is high when done by hand but low when delegated to a computer algebra program.

The advantage of using SMP over other computer algebra software is the small amount of memory initially allocated loading the program. For small and simple problems, many users may interact with SMP simultaneously. The kernel of functions in SMP is small when compared with other computer algebra systems. External files, which contain special functions and procedures written in the SMP language, are only loaded when necessary. This causes the SMP basic kernel of functions to grow by adding to the memory and storage, thus slowing down the calculation speed.

The program SMP is written in the C language which makes it portable to many hardware systems. Other computer algebra systems are written in the computer language LISP and are machine and hardware dependent.

SMP has a very poor user interface. The natural form of equations is two dimensional, but SMP can input only strings of text. This weakness in the user interface is common to all computer algebra systems and much work is necessary to strengthen the area. Some recommendations regarding the computer algebra user interface are given later in this chapter.

Most computer algebra software developers claim a simple yet effective language for their system. For many simple problems, the interactive nature of the software is direct and efficient. When it is necessary to program complicated problems in mathematical physics, the interactive language may retard the progress of the solution to the problem. This is due to the lack of a debug function that could catch major flaws in the input of the procedures and functions.

Good program documentation is essential for developing functions and procedures in new programming languages. Documentation of the capabilities of computer algebra systems and, most importantly, limitations of the program are poorly described. The documentation available for computer algebra systems typically consists of a library of functions available to the user. A few vague examples of simple functions are examined in the reference guide and for simple or small problems, this type of documentation is sufficient.

Examples of well documented procedures written in the SMP language cannot be found in the SMP reference guide or primer. The SMP reference and primer direct the user to poorly documented external files in the SMP library for instruction. This may be compared to learning French armed with only a dictionary and a simple set of phrases. The structure and intent of a command may be different from what is expected even though the words may be correct.

The use of computer algebra software for the analysis of problems in heat transfer is appealing because of the ability of the software to obtain symbolic solutions. Numerical solutions to heat transfer problems are acceptable as long as the solution at a particular coordinate and time is desired. Even so, numerical instabilities of the method may cause the numerical solutions to deviate or, in a worse case, to oscillate from the correct solution.

The computer algebra program CANSS can be useful identifing areas in conduction heat transfer problems that cannot be solved in closed form. Identification of these areas can lead to detailed examination of the types of integrals which must be evaluated to obtain symbolic solutions.

The new concept of time partitioning is used to optimize the symbolic temperature distributions calculated by the two CANSS programs. These symbolic solutions are quickly convergent for the time regions of interest. This is the first application of the time partitioning scheme introduced by Keltner and Beck [1985] for symbolic temperature distributions.

Expert systems using computer algebra systems are best suited for fields that exhibit narrow, specialized domains. Almost every technical area can benefit from the use of computer algebra based on this description. For example, the field of acoustics is a narrow, specialized domain of a broad class of wave mechanics. Sommerfield [1949, Section 27, Appendix II] shows the Green's function approach can be applied to problems in acoustics and results in a formalism that is similar to the formalism for heat transfer described in this thesis. The Green's functions for the wave equations can be cataloged and stored in a knowledge base. Special integrals that occur during the calculation of the solutions may also be stored in the knowledge base. The formalism and the knowledge base can lead to an expert system for wave equation problems in acoustics.

A floppy disk containing the CANSS and CANDD2D programs can be obtained from the author. All libraries, procedures and sub-procedures can be loaded to a VAX/VMS microcomputer and run with SMP.

6.3 Recommendations

1. The small time Green's function method should be applied to other orthogonal coordinate systems such as the radial, spherical, and elliptical systems. The approximations in the radial coordinate system are similar to the approximations made in the cartesian coordinate system except the functions that are approximated are different. Care must be exercised in the application of the Green's function approach to other coordinate systems. In the radial coordinate system for example, only a limited number of one, two, and three dimensional cases can use the Green's function formalism because of the differences in the partial differential equations between the rectangular and radial coordinate systems. A data base of small and large time Green's functions need to be calculated for the radial coordinate system. Beck [1986] is continuing to catalog small and large time Green's functions for rectangular, radial, and spherical geometries for various boundary conditions.

2. The kernel or influence functions developed in the small time Green's function approach should be applied in unsteady surface element [Keltner and Beck, 1981] and finite element methods. In the unsteady surface method, small time Green's functions are particularly important because they allow calculations at small and large times are made speedily. The small time influence functions developed can be used as accurate trial functions in finite element methods which will improve

the convergence of the solution. Green's function based influence functions can be used for trial functions for non-linear problems. The efficiency of both numerical procedures will be enhanced through the use of functions that are defined for specific ranges of time.

3. The integration procedures used in SMP should be extended to recognize and include more functions. The work of Cherry [1985] and Knowles [1986] in the area of symbolic integrators should be examined closely and incorporated into the internal SMP integration routines. Their work extends the algorithm of Risch [1969] to include specified logarithmic and exponential functions.

4. The two dimensional CANSS program should be extended to include additional types of boundary conditions, initial conditions, and volume energy heat sources. An additional boundary condition that can be readily applied to the CANSS2D program is the boundary condition of the third kind. Initial conditions that are polynomial and transcendental function can be investigated. Volume energy heat sources that are functions of the spatial coordinates and time can be examined.

5. The one dimensional CANSS program should be extended to include boundary conditions of the fifth kinds, convective thin films. Due to the non-symbolic nature of the Green's functions associated with boundary conditions of the fifth kinds, more variables will be generated, which will cause a shortage of storage space, and greatly slow down the calculation of the temperature distribution.

6. The Green's function approach to the solution of heat diffusion problems is not serial but parallel. The formalism and linearity

of the method allows each piece of the solution to be calculated independently. Each processor in the parallel computer could be assigned a portion of the problem and independently and simultaneously execute the operations necessary for the solution. The SMP program has anticipated the move into parallel processing by including functions that takes advantage of the parallelism.

7. A major flaw in computer algebra systems is the inability of the program to interface with the user. Equations are two dimensional objects with structure but computer algebra system treat them as strings of text. A user interface in needed for both the input and output that can make use of bit-mapped video screens, menus, and a mouse. Bit-mapped displays allow the user to point and pick objects such as equations and portions of equations. Also, bit-mapped screens can draw the special symbols that appear in mathematical equations. Draw down menus can speed the input of complicated expression by drawing mathematical symbols, along with the mathematical operation, to the input line.

8. Output expression display should make use of windows or horizontal scrolling. The expressions generated by a typical computer algebra problem are, in many cases, longer than the 80 characters available on a normal video screen. Windows would allow the user to split the input or output expressions into small portions. Horizontal scrolling would be an alternative method of displaying large output expressions.

9. An on-line status area is recommended for computer algebra systems. The on-line status area can inform the user of the complexity

of a calculation by indicating the amount of time in the calculation, the amount of storage space and memory used by the calculation, and an indication of the progress of the calculation.

10. When an integral in the CANSS environment is not evaluated, it may represent a new type of function. Numerical integration could completely describe the unknown integral. The new expression for the unknown could be entered into the integration procedures of the CANNS environment and the solution could be determined.

11. A data base of convolution type integrals associated with diffusion problems is recommended. The Risch-base integration procedures cannot evaluated these integral at the present time. Computer algebra systems can use their expertise in handling Laplace transforms and inverse Laplace transforms to begin this data base.

12. The output of symbols and equations from the computer algebra systems should be in a form that is natural to the investigator. The computer algebra output should be able to generate symbols such as integral signs, summation signs, and partial derivitive symbols to name a few. More work using the graphical capabilities of bit-mapped screens is necessary for the natural output of symbolic expressions.

APPENDICES

APPENDIX A

LAPLACE TRANSFORMS FOR SMALL TIMES

A short table of Laplace transforms is given below. See Abranowitz and Steguns [1959, pp. 1021-1026] for a comprehensive table of Laplace transforms.

f(s)	F(t)	
$\frac{1}{s}$	1	(A.1)
$\frac{1}{s^2}$	t	(A.2)
$\frac{\Gamma(\nu)}{s^\nu} \quad (\nu > 0)$	$t^{\nu-1}$	(A.3)
$\frac{1}{s + a}$	e^{-at}	(A.4)
$\frac{1}{s - a}$	e^{at}	(A.5)
$\frac{\sqrt{s}}{s + a^2}$	$\frac{1}{\Gamma(1/2)\ t^{1/2}} - \frac{2\ a}{\Gamma(1/2)} \mathrm{Daw}(a \sqrt{t})$	(A.6)
$\frac{1}{\sqrt{s}\ (s + a^2)}$	$\frac{2\ a}{\Gamma(1/2)} \mathrm{Daw}(a \sqrt{t})$	(A.7)

$$e^{-a\sqrt{s}} \qquad \frac{a}{2}\ \frac{e^{-a^2/(4\ t)}}{\Gamma(1/2)\ t^{3/2}} \qquad \text{(A.8)}$$

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} \qquad \frac{e^{-a^2/(4\ t)}}{\Gamma(1/2)\ t^{1/2}} \qquad \text{(A.9)}$$

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}\ (b + \sqrt{s})} \qquad e^{-b^2 t + a\ b}\ \mathrm{erfc}\left(\frac{a}{\sqrt{(4\ t)}} + b\sqrt{t} \right) \qquad \text{(A.10)}$$

$$\frac{e^{-a\sqrt{s}}}{s\ (b + \sqrt{s})} \qquad \frac{1}{b}\ \mathrm{erfc}\left(\frac{a}{\sqrt{(4\ t)}} \right)$$

$$- \frac{1}{b}\ e^{-b^2 t + a\ b}\ \mathrm{erfc}\left(\frac{a}{\sqrt{(4\ t)}} + b\sqrt{t} \right) \qquad \text{(A.11)}$$

APPENDIX B

SOME USEFUL INTEGRALS

B.1 Introduction

The purpose of this Appendix is to collect and present special integrals important in diffusion problems. Many of the integrals found in this appendix are used in the CANSS programs. A definition that appears in following sections is,

$$K(x-x',t-\tau) = K(-x+x',t-\tau) = (4\ \pi\ \alpha\ (t-\tau))^{-1/2}\ e^{\frac{-(x-x')^2}{4\alpha(t\ -\ \tau)}} \tag{B.1}$$

B.2 Integration Over Space with Respect to x' for Small Time Functions

B.2.1 Integral Equation

$$\int_a^b K(x-x',t-\tau)\ F(x')\ dx' = \int_a^b K(-x+x',t-\tau)\ F(x')\ dx'$$

F(x')	Integral
$\delta(x_0-x')$	$K(x-x_0,t-\tau)$, $a < x_0 < b$, otherwise zero (B.2)
1	$\frac{1}{2}\left\{ \operatorname{erfc}\left[\frac{x-b}{(4\alpha\ (t-\tau))^{1/2}} \right] - \operatorname{erfc}\left[\frac{x-a}{(4\alpha\ (t-\tau))^{1/2}} \right] \right\}$ (B.3)
$\frac{x'}{L}$	$\frac{x}{2\ L}\left\{ \operatorname{erfc}\left[\frac{x-b}{(4\alpha\ (t-\tau))^{1/2}} \right] - \operatorname{erfc}\left[\frac{x-a}{(4\alpha\ (t-\tau))^{1/2}} \right] \right\} + \frac{2\alpha\ (t-\tau)}{L^2}\left\{ K(x-a,t-\tau) - K(x-b,t-\tau) \right\}$ (B.4)
$\left(\frac{x'}{L}\right)^2$	$\left\{ \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{\alpha(t-\tau)}{L^2} \right\} \cdot \left\{ \operatorname{erfc}\left[\frac{x-b}{(4\alpha\ (t-\tau))^{1/2}} \right] - \operatorname{erfc}\left[\frac{x-a}{(4\alpha\ (t-\tau))^{1/2}} \right] \right\} + \frac{2\alpha\ (t-\tau)}{L^2}\left\{ (x+a)K(x-a,t-\tau) - (x+b)K(x-b,t-\tau) \right\}$ (B.5)

B.2.2 Integral Equation

$$\int_a^b K(x+x',t-\tau)\ F(x')\ dx' = \int_a^b K(-x-x',t-\tau)\ F(x')\ dx'$$

F(x')	Integral
$\delta(x_0-x')$	$K(x+x_0,t-\tau)$, $a < x_0 < b$, otherwise zero (B.6)
1	$\frac{1}{2}\left\{ \mathrm{erfc}\left(\frac{x+a}{(4\alpha\ (t-\tau))^{1/2}} \right) - \mathrm{erfc}\left(\frac{x+b}{(4\alpha\ (t-\tau))^{1/2}} \right) \right\}$ (B.7)
$\frac{x'}{L}$	$\frac{x}{2\ L}\left\{ \mathrm{erfc}\left(\frac{x+b}{(4\alpha\ (t-\tau))^{1/2}} \right) - \mathrm{erfc}\left(\frac{x+a}{(4\alpha\ (t-\tau))^{1/2}} \right) \right\} + \frac{2\alpha\ (t-\tau)}{L^2}\left\{ K(x+a,t-\tau) - K(x+b,t-\tau) \right\}$ (B.8)
$\left(\frac{x'}{L}\right)^2$	$\left\{ \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{\alpha(t-\tau)}{L^2} \right\} \cdot \left\{ \mathrm{erfc}\left(\frac{x+a}{(4\alpha\ (t-\tau))^{1/2}} \right) - \mathrm{erfc}\left(\frac{x+b}{(4\alpha\ (t-\tau))^{1/2}} \right) \right\} + \frac{2\alpha\ (t-\tau)}{L^2}\left\{ (x-b)K(x+b,t-\tau) - (x-a)K(x+a,t-\tau) \right\}$ (B.9)

B.3 Exponential and Error Integrals

B.3.1 Integral Expression

$$\int_a^b \theta^{n/2}\, e^{-C^2\theta}\, d\theta$$

n	Integral
-3	$(4\pi)^{1/2}\left[\frac{1}{\sqrt{a}}\,\mathrm{ierfc}(C\sqrt{a}) - \frac{1}{\sqrt{b}}\,\mathrm{ierfc}(C\sqrt{b})\right]$ (B.10)
-1	$(\pi/C^2)^{1/2}\left[\mathrm{erfc}(C\sqrt{a}) - \mathrm{erfc}(C\sqrt{b})\right]$ (B.11)
0	$\frac{1}{C^2}\left[e^{-C^2 a} - e^{-C^2 b}\right]$ (B.12)
1	$\left[\frac{\sqrt{\theta}}{C^2}\, e^{-C^2\theta} + \frac{\sqrt{\pi}}{2C^3}\,\mathrm{erfc}(C\sqrt{\theta})\right]_{\theta=b}^{a}$ (B.13)
3	$\left[\left(\frac{3\,\theta^{3/2}}{2\,C^4} + \frac{\theta^{3/2}}{C^2}\right) e^{-C^2\theta} + \frac{3\sqrt{\pi}}{4C^5}\,\mathrm{erfc}(C\sqrt{\theta})\right]_{\theta=b}^{a}$ (B.14)

B.3.2 Integral Expression

$$\int_a^b \theta^{n/2} e^{-C^2/\theta} d\theta$$

n	Integral
-3	$(\pi/C^2)^{1/2} \left[\mathrm{erfc}(C/\sqrt{b}) - \mathrm{erfc}(C/\sqrt{a}) \right]$ (B.15)
-1	$(4\pi)^{1/2} \left[(b)^{1/2} \mathrm{ierfc}(C\sqrt{b}) - (a)^{1/2} \mathrm{ierfc}(C\sqrt{a}) \right]$ (B.16)
1	$\frac{2}{3} \left[\theta^{3/2} e^{-C^2/\theta} + 2C^2 \sqrt{(\pi\theta)}\, \mathrm{ierfc}(C/\sqrt{\theta}) \right]_{\theta=a}^{b}$ (B.17)
3	$\frac{2}{15} \left[\theta^{3/2} (3\theta - 4C^2) e^{-C^2/\theta} + 4C^4 \sqrt{(\pi\theta)}\, \mathrm{ierfc}(C/\sqrt{\theta}) \right]_{\theta=a}^{b}$ (B.18)

B.3.3 Integral Expression

$$\int_a^b \theta^{n/2}\, e^{-C_1^2 \theta \; -C_2^2/\theta}\, d\theta$$

n	Integral
-3	$\frac{\sqrt{\pi}}{2C_2}\left[e^{2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) - e^{-2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} - \frac{C_2}{\sqrt{\theta}}) \right]_{\theta=a}^{b}$ (B.19)
-1	$\frac{\sqrt{\pi}}{2C_1}\left[e^{2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) + e^{-2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} - \frac{C_2}{\sqrt{\theta}}) \right]_{\theta=a}^{b}$ (B.20)
0	$\frac{\sqrt{\pi}}{4C_1}\left[e^{2C_1C_2} \operatorname{erfc}(C_1\theta + \frac{C_2}{\theta}) + e^{-2C_1C_2} \operatorname{erfc}(C_1\theta - \frac{C_2}{\theta}) \right]_{\theta=a}^{b}$ (B.21)
1	$\frac{\sqrt{\pi}}{2C_1^2}\left[\left(\frac{1}{2C_1} - C_2\right) e^{2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} + \frac{C_2}{\sqrt{\theta}}) + \left(\frac{1}{2C_1} - C_2\right) e^{-2C_1C_2} \operatorname{erfc}(C_1\sqrt{\theta} - \frac{C_2}{\sqrt{\theta}}) + 2\frac{\sqrt{\theta}}{\sqrt{\pi}} e^{-C_1^2\theta - C_2^2/\theta} \right]_{\theta=b}^{a}$ (B.22)

3

$$\frac{\sqrt{\pi}}{2C_1^3}\left[\left(C_2^2+\frac{3}{4C_1^2}-\frac{3C_2}{2C_1}\right)e^{2C_1C_2}\operatorname{erfc}\left(C_1\sqrt{\theta}+\frac{C_2}{\sqrt{\theta}}\right)\right.$$

$$+\left(C_2^2+\frac{3}{4C_1^2}-\frac{3C_2}{2C_1}\right)e^{-2C_1C_2}\operatorname{erfc}\left(C_1\sqrt{\theta}-\frac{C_2}{\sqrt{\theta}}\right)$$

$$\left.+\frac{\sqrt{\theta}}{\sqrt{\pi}}\left(\frac{3}{C_1}+2\,\theta\,C_1\right)e^{-C_1^2\theta\,-\,C_2^2/\theta}\right]_{\theta=b}^{a}$$

(B.23)

B.3.4 Integral Expression

$$\int_a^b \theta^n \operatorname{erfc}(C\,\theta)\, d\theta$$

n	Integral
-1	$\left[\ell n(\theta) \operatorname{erfc}(C\theta) \right]_{\theta=a}^{b} + \frac{2C}{\sqrt{\pi}} \int_a^b \ell n(\theta)\, e^{-C^2\theta^2}\, d\theta$ (B.24)
0	$\frac{1}{C} \left[\operatorname{ierfc}(C\,a) - \operatorname{ierfc}(C\,b) \right]$ (B.25)
1	$\frac{1}{4C^2} \left[\operatorname{erf}(C\,\theta) - 2\,\theta\,C \operatorname{ierfc}(C\,\theta) \right]_{\theta=a}^{b}$ (B.26)

B.3.5 Integral Expression

$$\int_a^b \theta^n \operatorname{erfc}(C/\sqrt{\theta})\, d\theta$$

n	Integral
0	$\left[\operatorname{erfc}(C/\sqrt{\theta}) - 2\, C\sqrt{\theta}\ \operatorname{ierfc}(C/\sqrt{\theta}) \right]_{\theta=a}^{b}$ (B.27)
1	$\left[\frac{\theta^2}{2} \operatorname{erfc}(C/\sqrt{\theta}) + \frac{2C^2\sqrt{\theta}}{3} \operatorname{ierfc}(C/\sqrt{\theta}) - \frac{C}{3} (\theta^3/\pi)^{1/2}\, e^{-C^2/\theta} \right]_{\theta=a}^{b}$ (B.28)

B.3.6 Integral Expression

$$\int_a^b e^{n\theta}\, \mathrm{erfc}(C_1\sqrt{\theta} + C_2/\sqrt{\theta})\, d\theta$$

n	Integral
0	$\left[\frac{1}{4C_1^2} \left(\mathrm{erfc}(C_1\sqrt{\theta} + C_2/\sqrt{\theta}) + e^{-4C_1C_2}\, \mathrm{erfc}(C_1\sqrt{\theta} - C_2/\sqrt{\theta}) \right) + \frac{\sqrt{\theta}}{C_1}\, \mathrm{ierfc}(C_1\sqrt{\theta} - C_2/\sqrt{\theta}) \right]_{\theta=b}^{a}$ (B.29)
$(C_1^2 - C_3^2)$	$\frac{1}{2(C_1^2 - C_3^2)} \left[2\, e^{-(C_1^2 - C_3^2)\theta}\, \mathrm{erfc}(C_1\sqrt{\theta} + C_2/\sqrt{\theta})\, d\theta - (1 + \frac{C_1}{C_2})\, e^{-2(C_1 - C_3)C_2}\, \mathrm{erfc}(C_3\sqrt{\theta} + C_2/\sqrt{\theta})\, d\theta + (1 - \frac{C_1}{C_2})\, e^{-2(C_1 + C_3)C_2}\, \mathrm{erfc}(C_3\sqrt{\theta} - C_2/\sqrt{\theta})\, d\theta \right]_{\theta=a}^{b}$ (B.30)

B.4 Integration Over Time for Some Error Function Integrals

An integral which occurs often in two dimensional problems is,

$$I = \frac{1}{2\,\alpha} \int_{w=1/(4\alpha t)^{1/2}}^{\infty} w^{-3}\ \mathrm{erfc}[x\ w]\ \mathrm{erfc}[y\ w]\ dw \qquad (B.31)$$

Integrating this equation by parts, letting,

$$v = \mathrm{erfc}[x\ w]\ \mathrm{erfc}[y\ w]$$

$$du = w^{-3} dw$$

yields,

$$I = \frac{-1}{4\,\alpha} \left(w^{-2}\ \mathrm{erfc}[x\ w]\ \mathrm{erfc}[y\ w] \right)_{1/(4\alpha t)^{1/2}}^{\infty}$$

$$- \frac{x}{2\alpha\sqrt{\pi}} \int_{w=1/(4\alpha t)^{1/2}}^{\infty} w^{-2}\ \exp[-(xw)^2]\ \mathrm{erfc}[y\ w]\ dw$$

$$- \frac{y}{2\alpha\sqrt{\pi}} \int_{w=1/(4\alpha t)^{1/2}}^{\infty} w^{-2}\ \exp[-(yw)^2]\ \mathrm{erfc}[x\ w]\ dw \qquad (B.32)$$

Note that the first and second integrals in the above equation has the same form but different parameters.

Substuting $u = x\ w$ in the first integral of equation (B.32), the integral can be written as,

$$II = \frac{2\ t}{\sqrt{\pi}} X^2 \int_{X}^{\infty} u^{-2}\ e^{-u^2}\ \mathrm{erfc}[u/p]\ du \qquad (B.33)$$

where $X^2 = x^2/(4\alpha t)$, $Y^2 = y^2/(4\alpha t)$, and $p = x/y$ or X/Y.

Litkouhi [1982] shows that this integral can be expressed as,

$$II = \frac{2\ t}{\sqrt{\pi}} X^2 \left(\frac{\sqrt{\pi}}{X} \operatorname{ierfc}[X] - e^{-X^2} \operatorname{erf}[Y]/X \right.$$

$$\left. - \frac{1}{p\sqrt{\pi}} E_1(X^2 + Y^2) + 2p \int_X^\infty e^{-(p\ w)^2} \operatorname{erf}[w]\ dw \right) \quad (B.34)$$

The second integral in equation (B.32) can also be placed in the form of equation (B.34) as,

$$III = \frac{2\ t}{\sqrt{\pi}} Y^2 \left(\frac{\sqrt{\pi}}{Y} \operatorname{ierfc}[Y] - e^{-Y^2} \operatorname{erf}[X]/Y \right.$$

$$\left. - \frac{p}{\sqrt{\pi}} E_1(X^2 + Y^2) + \frac{2}{p} \int_Y^\infty e^{-(w/p)^2} \operatorname{erf}[w]\ dw \right) \quad (B.35)$$

Litkouhi replaces the integrals in equations (B.34) and (B.35) with a function called $H(\cdot,\cdot)$ which is examined in a text by Rosser [1948]. The solution to the following four integrals below use a similar technique and functions.

$$\int_{\tau=0}^{t} \mathrm{erf}\left(\frac{x}{\sqrt{4\ \alpha\ (t-\tau)}} \right) \mathrm{erf}\left(\frac{y}{\sqrt{4\ \alpha\ (t-\tau)}} \right) d\tau =$$

$$2\ t \left[\frac{1}{2} - \frac{\mathrm{erfc}(X)}{2} - \frac{\mathrm{erfc}(Y)}{2} + \frac{\mathrm{erfc}(X)\ \mathrm{erfc}(Y)}{2} \right.$$

$$\left(X\ \mathrm{ierfc}(X) + Y\ \mathrm{ierfc}(Y) \right)$$

$$\left. - \left(X\ \mathrm{zerfc}(X,Y) + Y\ \mathrm{zerfc}(Y,X) \right) / \sqrt{\pi} \right] \quad (B.36)$$

$$\int_{\tau=0}^{t} \mathrm{erfc}\left(\frac{x}{\sqrt{4\ \alpha\ (t-\tau)}} \right) \mathrm{erfc}\left(\frac{y}{\sqrt{4\ \alpha\ (t-\tau)}} \right) d\tau =$$

$$2\ t \left[\frac{\mathrm{erfc}(X)\ \mathrm{erfc}(Y)}{2} \right.$$

$$\left. - \left(X\ \mathrm{zerfc}(X,Y) + Y\ \mathrm{zerfc}(Y,X) \right) / \sqrt{\pi} \right] \quad (B.37)$$

where $X = \frac{x}{\sqrt{4\alpha t}}$, $Y = \frac{y}{\sqrt{4\alpha t}}$, $\mathrm{zerfc}(X,Y) = X \int_{X}^{\infty} u^{-1}\ e^{-u^2}\ \mathrm{erfc}(\frac{Y\ u}{X})du$

or $\mathrm{zerfc}(X,Y) = \sqrt{\pi}\ \mathrm{ierfc}(X) - e^{-X^2} \mathrm{erf}(Y) - \frac{Y}{\sqrt{\pi}} E_1(X^2 + Y^2) + \sqrt{\pi}\ X\ H(Y,X/Y)$

and where H[•,•] is the integral discussed in Litkouhi [1982] and Rosser [1948].

$$\int_{\tau=0}^{t} \tau \, \mathrm{erf}\left(\frac{x}{\sqrt{4\,\alpha\,(t-\tau)}} \right) \mathrm{erf}\left(\frac{y}{\sqrt{4\,\alpha\,(t-\tau)}} \right) d\tau =$$

$$2\,t^2 \left[\frac{1}{4} - 8\,(i^4\mathrm{erfc}(X) + i^4\mathrm{erfc}(Y)\,) + \frac{\mathrm{erfc}(X)\,\mathrm{erfc}(Y)}{4} \right.$$

$$- \frac{X}{\sqrt{\pi}}\,(1 + \frac{X^2}{3})\,\mathrm{zerfc}(X,Y) - \frac{Y}{\sqrt{\pi}}\,(1 + \frac{Y^2}{3})\,\mathrm{zerfc}(Y,X)$$

$$+ \frac{X\,Y}{3\pi}\left[(X^2 + Y^2)\,E_1(X^2 + Y^2) - e^{-(X^2 + Y^2)} \right]$$

$$\left. + \frac{1}{6\,\sqrt{\pi}}\left[X\,e^{-X^2}\mathrm{erf}(Y) + Y\,e^{-Y^2}\mathrm{erf}(X) \right] \right] \qquad (B.38)$$

$$\int_{\tau=0}^{t} \tau \, \mathrm{erfc}\left(\frac{x}{\sqrt{4\,\alpha\,(t-\tau)}} \right) \mathrm{erfc}\left(\frac{y}{\sqrt{4\,\alpha\,(t-\tau)}} \right) d\tau =$$

$$2\,t^2 \left[\frac{\mathrm{erfc}(X)\,\mathrm{erfc}(Y)}{4} \right.$$

$$- \frac{X}{\sqrt{\pi}}\,(1 + \frac{X^2}{3})\,\mathrm{zerfc}(X,Y) - \frac{Y}{\sqrt{\pi}}\,(1 + \frac{Y^2}{3})\,\mathrm{zerfc}(Y,X)$$

$$+ \frac{X\,Y}{3\pi}\left[(X^2 + Y^2)\,E_1(X^2 + Y^2) - e^{-(X^2 + Y^2)} \right]$$

$$\left. + \frac{1}{6\,\sqrt{\pi}}\left[X\,e^{-X^2}\mathrm{erf}(Y) + Y\,e^{-Y^2}\mathrm{erf}(X) \right] \right] \qquad (B.39)$$

where X, Y, zerfc(X,Y), and H(X,Y/X) have been defined on the previous page.

B.5 Some Integrals for Small Time Green's Functions

$$\int_{\tau=0}^{t} \tau^n \; K(x,t-\tau) \; d\tau = \frac{1}{2\sqrt{\alpha}} \; \Gamma\left(\frac{n}{2} + 1 \right) \; (4 \; t)^{(n+1)/2} \; i^{n+1}\mathrm{erfc}\left[\frac{|x|}{(4\alpha t)^{1/2}} \right]$$

$$n = 1-,0,1,2,... \qquad \text{(B.40)}$$

$$\int_{\tau=0}^{t} \tau^n \; \frac{\alpha(t-\tau)}{L^2} \; K(x,t-\tau) \; d\tau = \frac{2^n x}{L^2} \; \Gamma\left(\frac{n}{2} + 1 \right) \; t^{\frac{n+2}{2}} \; \cdot$$

$$\left\{ \left(\frac{4\alpha t}{x^2} \right)^{1/2} \; i^{n+3}\mathrm{erfc}\left[\frac{|x|}{(4\alpha t)^{1/2}} \right] + i^{n+2}\mathrm{erfc}\left[\frac{|x|}{(4\alpha t)^{1/2}} \right] \right\}$$

$$n=-2,-1,0,1,2,... \qquad \text{(B.41)}$$

$$\int_{\tau=0}^{t} \tau^n \; \mathrm{erfc}\left[\frac{|x|}{(4\alpha(t-\tau))^{1/2}} \right] = \Gamma\left(\frac{n}{2} + 1 \right) \; (4t)^{\frac{n+2}{2}} \; i^{n+2}\mathrm{erfc}\left[\frac{|x|}{(4\alpha t)^{1/2}} \right]$$

$$n=-1,0,1,2,... \qquad \text{(B.42)}$$

$$\int_{\tau=t-\Delta t}^{t} K(w,t-\tau) \; d\tau = \left(\frac{\Delta t}{\alpha} \right)^{1/2} \; \mathrm{ierfc}\left[\frac{|w|}{(4\alpha \; \Delta t)^{1/2}} \right]$$

$$\text{(B.43)}$$

$$2 \int_{\tau=t-\Delta t}^{t} \alpha(t-\tau) \; K(w,t-\tau) \; d\tau = \int_{\tau=t-\Delta t}^{t} \left(\frac{\alpha(t-\tau)}{\pi} \right)^{1/2} \; e^{\frac{w^2}{4\alpha(t-\tau)}} \; d\tau$$

$$= \frac{w^3}{6\,\alpha} \left[\operatorname{erfc}\left(\frac{w}{(4\alpha\,\Delta t)^{1/2}} \right) - \left(\frac{4\alpha\,\Delta t}{\pi\,w^3} \right)^{1/2} e^{\frac{w^2}{4\alpha\,\Delta t}} \left(1 - \frac{2\alpha\,\Delta t}{w^2} \right) \right] \tag{B.44}$$

$$\int_{\tau=t-\Delta t}^{t} \operatorname{erfc}\left(\frac{w}{(4\alpha(t-\tau))^{1/2}} \right) d\tau = 4\,\Delta t\; i^2\operatorname{erfc}\left(\frac{w}{(4\alpha\,\Delta t)^{1/2}} \right) \tag{B.45}$$

$$\int_{\tau=t-\Delta t}^{t} (t-\tau)\, \operatorname{erfc}\left(\frac{w}{(4\alpha(t-\tau))^{1/2}} \right) d\tau =$$

$$4(\Delta t)^2 \left\{ i^2\operatorname{erfc}\left(\frac{w}{(4\alpha\,\Delta t)^{1/2}} \right) - 4\; i^4\operatorname{erfc}\left(\frac{w}{(4\alpha\,\Delta t)^{1/2}} \right) \right. \tag{B.46}$$

$$\int_{\tau=t-\Delta t}^{t} \left(\frac{\alpha\;w}{[4\pi\,(\alpha(t-\tau))^3]^{1/2}} \right) e^{\frac{w^2}{4\alpha(t-\tau)}}\, d\tau = \operatorname{erfc}\left(\frac{|w|}{(4\alpha\,\Delta t)^{1/2}} \right) \tag{B.47}$$

APPENDIX C

CANSS AND CANSS2D PROGRAM EXAMPLES

```
 Welcome to Paul Zang's version of SMP.

SMP 1.5.0
28-FEB-1987 13:54:49.05


#I[1]::  <"canss.prg"


The CANSS procedure takes about 30 seconds to load!!!!
Please wait a moment......


 The [zang.smp.c2d]grab.int function is loaded.
 The [zang.smp.c2d]exp.int library is loaded.
 The CANSS utilities are loaded.



      CCCCCCCCC      AA          N    N      SSSSSS       SSSSSS
     C              A A         NN    N     S            S
    C              A  A        N N   N       S            S
   C              AAAAA       N  N N          SS            SS
  C              A    A      N   NN             S             S    by
 CCCCCCCCC      A     A     N    N       SSSSSSS      SSSSSSS    P. H. Zang


    This procedure will calculate the temperature distribution in a one
 dimensional slab that has boundary conditions of the first through
 fourth kind at any surface.  Constant heat generation may occur
 and the initial temperature of the slab is a polynomial function of x.



Press <Enter> or <Return> key to continue:
 The CANSS left boundary input routines are loaded.
 The CANSS right boundary input routines are loaded.
 The CANSS IC & HGT and Status are loaded.
 The CANSS function to generate GF's is loaded.


NOTE  ::
         Small dimensionless time is defined as being < 0.025
         Assume the forcing functions on the surfaces begin at tau = 0
```

```
Press <Enter> or <Return> key to continue:

            Left Boundary Condition



 What type of boundary condition do you have:

            0 - Semi-infinite Condition
            1 - Temperature Condition
            2 - Heat Flux Condition
            3 - Convective
            4 - Non-convective Thin Film



Enter : 2
Form: dT/dx = constant*time^(n/2)   n = -1,0,1,2,...
Enter the constant value.Qo
Enter the value for n. n can equal -1,0,1,2,...  1

            Right Boundary Condition



 What type of boundary condition do you have:

            0 - Semi-infinite Condition
            1 - Temperature Condition
            2 - Heat Flux Condition
            3 - Convective
            4 - Non-convective Thin Film



Enter : 2
Form: dT/dx = constant*time^(n/2)   n = -1,0,1,2,...
Enter the constant value.0
Enter the value for n.0

            Initial Condition
```

```
The initial condition term is constant with respect to time.


Form :: F(x) = Constant * x^$n , where $n = positive integer.

Enter the constant value for the initial condition.0

Enter the value for $n.0


Enter the constant value for the heat generation.0



  Status of the CANSS routine at this time.

You are dealing with the X(    2       2       ) case.

The initial condition is equal to      0

The heat generation is constant and equal to   0

Loading the [zang.smp.gflibs]eigen.con library.
Hang on....   This takes about 10 seconds!!
Press <Enter> or <Return> key to continue:
Loading the [zang.smp.gflibs]gfld.lib library.
Hang on....   This takes about 30 seconds!!
The [zang.smp.gflibs]gfld.lib library is loaded!!
The small time Green's function is ::
                                 2                          2
        -0.25 (2nn + x - xp)          -0.25 (2nn + x + xp)
0.5(Exp[--------------------] + Exp[--------------------])
                theta                         theta
------------------------------------------------------------
                            0.5      0.5
                     theta     Lx Pi
The large time Green's function is ::
                                               2        2
1 + 2Cos[mm x Pi] Cos[mm xp Pi] Exp[- mm  theta Pi ]
----------------------------------------------------
                         Lx
Press <Enter> or <Return> key to continue:
Integral # 3
 The integral of       0       is not in the library.
 We will try the internal integrator.
 The small time boundary solution is ::

             2                       0.5(2nl + x)
       4tl Lx  Qo Gamma[1.5] IErfc[------------,2]
                                          0.5
```

```
                                       t1
        ------------------------------------------
                             0.5
                        alpha    k
NOTE:: n1 or n2 are summation indexes that may go from
       minus infinity to plus infinity.


Press <Enter> or <Return> key to continue:
                           0.5
The integral of        tau      is not in the library.
We will try the internal integrator.
The integral of        0        is not in the library.
We will try the internal integrator.
The large time boundary solution is ::

                                                         0.5
                           -DawsonsInt[m1 Pi (-t1 + t2)    ]

                                          2      2
                                * Exp[- m1  t1 Pi ]
          2Cos[m1 x Pi] (--------------------------------
                                     m1 Pi

                                          2      2            0.5
    2                           + Exp[- m1  t1 Pi ] (-t1 + t2)    )
   Lx  Qo (-------------------------------------------------------
                                      2   2
                                    m1  Pi

                              1.5              1.5
                - 0.666667 t1    + 0.666667 t2    )
        ----------------------------------------------------------
                                    0.5
                               alpha    k
NOTE:: m1 or m2 are summation indexes that may go from 1 to infinity.


Press <Enter> or <Return> key to continue:
The integral of        0        is not in the library.
We will try the internal integrator.
The initial condition solution is ::    0
NOTE:: m1 or m2 are summation indexes that may go from 1 to infinity.


Press <Enter> or <Return> key to continue:
The integral of        0        is not in the library.
We will try the internal integrator.
The small time solution for the heat generation term is:       0
NOTE:: n1 or n2 are summation indexes that may go from
       minus infinity to plus infinity.


Press <Enter> or <Return> key to continue:
Err[61.35,0]

*I[2]::  <end>
```

```
Welcome to Paul Zang's version of SMP.

SMP 1.5.0
28-FEB-1987 13:59:01.05

#I[1]::  <"canss2d.prg"

Loading the XRepexplg external file...
Loading the [zang.smp.c2d]grab.int external file...
The [zang.smp.c2d]grab.int function is loaded.
Loading the [zang.smp.c2d]exp.int external file...
The [zang.smp.c2d]exp.int library is loaded.
Loading the [zang.smp.c2d]z2d.int external file...
Loading the [zang.smp.z]z2d.int library
The [zang.smp.c2d]z2d.int library is loaded.


      CCCCCCCCC      AA          N    N      SSSSSS      SSSSSS
     C              A A        NN   N      S           S
    C              A   A       N N  N       S           S
   C              AAAAA      N  N N         SS          SS
  C              A     A    N    NN           S           S   by
 CCCCCCCCC      A       A  N     N      SSSSSSS     SSSSSSS   P. H. Zang

   This procedure will calculate the temperature distribution in a two
dimensional plate that has boundary conditions of the first or
second kind at any surface.  No heat generation will occur in the plate
and the initial temperature of the plate is zero.  You may either use a
constant forcing function at the surfaces or zero at the surfaces. The
forcing functions may extend to any percentage of surface length.


Press <Enter> or <Return> key to continue:


Some values must be given before we can continue.  The default values
of the constants are:
     thermal conductivity (k)          =    1
     thermal diffusivity  (alpha)      =    1

NOTE  ::
        Small dimensionless time is defined as being < 0.025
        Assume the forcing functions on the surfaces begin at tau = 0

Do you wish to use the default values?  (Y  or  N)
Enter : y
```

```
 Now we determine the boundary conditions.  The options are ::
               0    -    Semi-infinite boundary
               1    -    Temperature Condition  (Dirichlet)
               2    -    Heat Flux Condition    (Neumann)


"What type of condition occurs on the bottom? (0,1,2)"2
"What type of condition occurs on the left side? (0,1,2)"2
"What type of condition occurs on the top? (0,1,2)"1
"What type of condition occurs on the right side? (0,1,2)"1

 This is the X( 2       1       )Y(     2       1       ) case


 This is the input data routine.  You will be asked
 to enter the length of the plate in the x and y
 direction.  Then you will be asked to input the
 forcing function that occurs at the boundaries.
 Please continue.....


What is the length of the plate in the x-direction?1
What is the length of the plate in the y-direction?2


Is the forcing function on the bottom zero?
Enter : y


Is the forcing function on the left side zero?
Enter : n
 What is the value of the       heat flux        forcing function
 on the          left side       of the plate?
Forcing function on this surface = Qo
Where does it start? Dimensionlessly (0 - 1) = 0
Where does it end?   Dimensionlessly (0 - 1) = .5


Is the forcing function on the top zero?
Enter : y


Is the forcing function on the right side zero?
Enter : y


Loading the [zang.smp.gflibs]gf.lib library.
The [zang.smp.gflibs]gf.lib library is loaded!!
*******************************************************************
**                                                               **
**      You have chosen a two-dimensional problem                **
**                                                               **
*******************************************************************
 We begin by setting up the 2-D Green's function for time region 1.
```

$$0.5\,(-1)^{n1 + n2}\ Qo\ \mathrm{Exp}\left[\frac{-0.25\,(2n1 + x)^{2}}{theta}\right]$$

```
                                      2                        2
                    - (2n2 + y - yp)           - (2n2 + y + yp)
            * (Exp[-----------------] + Exp[-----------------])
                          theta                    theta
{0,-------------------------------------------------------------,0,0}
                             theta Pi
The GF for time region 1 is completed


 Begin setup of GF for time region 2.
       n2                                2               2
   (-1)    Qo Cos[x Pi (-0.5 + m1)] Exp[-theta Pi  (-0.5 + m1) ]

                                     2                        2
                 - (2n2 + y - yp)           - (2n2 + y + yp)
         * (Exp[-----------------] + Exp[-----------------])
                       theta                    theta
{0,-----------------------------------------------------------,0,0}
                              0.5   0.5
                         theta     Pi
The GF for time region 2 is completed


 Begin setup of GF for time region 3.
{0,2Qo Cos[x Pi (-0.5 + m1)] Cos[0.5y Pi (-0.5 + m2)] Cos[0.5yp Pi (-0.5 +
m2)]

                     2            2                       2            2
      * Exp[-theta Pi  (-0.5 + m1) ] Exp[-0.25theta Pi  (-0.5 + m2) ], 0,0}
The GF for time region 3 is completed


Now we have finished entering the data.  We move to integration
over space...
Now I am calculating the spatial integration for region 1....
                                              2
         n1 + n2        -0.25 (2n1 + x)
0.25 (-1)        Qo Exp[----------------]
                             theta

                 -0.5(1 - 4n2 - 2y)         0.5(1 + 4n2 + 2y)
           * (Erfc[------------------] - Erfc[-----------------])
                            0.5                        0.5
                       theta                      theta
-----------------------------------------------------------------
                                0.5   0.5
                           theta     Pi
Finished with region 1!!!



Now I am calculating the spatial integration for region 2....
         n2                                              2            2
0.5 (-1)    Qo Cos[x Pi (-0.5 + m1)] Exp[-theta Pi  (-0.5 + m1) ]

                -0.5(1 - 4n2 - 2y)         0.5(1 + 4n2 + 2y)
          * (Erfc[------------------] - Erfc[-----------------])
                           0.5                        0.5
```

```
                      theta                            theta
 Finished with region 2!!!


Now I am calculating the spatial integration for region 3....
                                                                       2                 2
4Qo Cos[x Pi (-0.5 + m1)] Cos[0.5y Pi (-0.5 + m2)] Exp[-theta Pi  (-0.5 + m1)
]

                        2               2
   * Exp[-0.25theta Pi  (-0.5 + m2) ] Sin[0.25Pi (-0.5 + m2)]
-----------------------------------------------------------------------------
-
                                Pi (-0.5 + m2)
Finished with region 3!!!


 Here is the temperature for time region 1 ...
         n1 + n2         0.5   0.5
0.25 (-1)        Qo (2 t1    Pi

                                                                        2
                                     -0.5(1 - 4n2 - 2y)      -0.25 (2n1 + x)
                                -Erf[------------------] Exp[---------------]
                                            0.5                     t1
                                          t1
                             * (--------------------------------------------
                                                   0.5
                                                 Pi

                                                              2
                                          0.25 (2n1 + x)
                                    0.5Expi[1,--------------
                                                    t1

                                                       (1 - 4n2
                                                              2
                                                  0.25     - 2y)
                                                + -------------]
                                                        t1

                                        * (2n1 + x)
                                    - ---------------------------------
                                                  0.5
                                              t1     Pi

                                        0.5(2n1 + x)  -(2n1 + x)
                                    0.5HH[------------,-------------]
                                              0.5      1 - 4n2 - 2y
                                            t1

                                        * (2n1 + x)
                                    + -------------------------------
                                                   0.5
                                                 t1
```

```
                                      0.5(2nl + x)
                               + IErfc[------------])
                                          0.5
                                        tl

          0.5   0.5
   - 2 tl    Pi

                                                                          2
                                0.5(1 + 4n2 + 2y)       -0.25 (2nl + x)
                           -Erf[-----------------] Exp[----------------]
                                       0.5                   tl
                                     tl
                         * (-------------------------------------------------
                                                0.5
                                              Pi

                                                              2
                                          0.25 (2nl + x)
                                 0.5Expi[1,---------------
                                                 tl

                                                               (1 + 4n2
                                                                       2
                                                        0.25     + 2y)
                                                      + --------------]
                                                               tl

                                           * (2nl + x)
                               - ---------------------------------
                                                  0.5
                                                tl    Pi

                                        0.5(2nl + x)    2nl + x
                                 0.5HH[------------,------------]
                                             0.5     1 + 4n2 + 2y
                                           tl

                                           * (2nl + x)
                               + ---------------------------------
                                                   0.5
                                                 tl

                                        0.5(2nl + x)
                               + IErfc[------------]))
                                            0.5
                                          tl
-------------------------------------------------------------------------------------
                                          0.5
                                        Pi
Here is the temperature for time region 2 ...
         n2
0.5 (-1)    Qo Cos[x Pi (-0.5 + ml)]

                    -0.5(1 - 4n2 - 2y)            2               2
             -(-Erfc[-----------------] Exp[-tl Pi  (-0.5 + ml) ]
                              0.5
                            tl

                             -0.5(1 - 4n2 - 2y)      0.5
```

```
              - 0.5Erfc[----------------- + t1    Pi (-0.5 + m1)]
                                0.5
                              t1

                      * Exp[-Pi (-0.5 + m1) (1 - 4n2 - 2y)]

                        0.5(1 - 4n2 - 2y)     0.5
              + 0.5Erfc[----------------- + t1    Pi (-0.5 + m1)]
                                0.5
                              t1

                      * Exp[Pi (-0.5 + m1) (1 - 4n2 - 2y)])
* (-----------------------------------------------------------------
                            2            2
                          Pi  (-0.5 + m1)

                   -0.5(1 - 4n2 - 2y)             2            2
             -Erfc[------------------] Exp[-t2 Pi  (-0.5 + m1) ]
                           0.5
                         t2

                                 -0.5(1 - 4n2 - 2y)
                       - 0.5Erfc[-----------------
                                         0.5
                                       t2

                                             0.5
                                         + t2    Pi (-0.5 + m1)]

                              * Exp[-Pi (-0.5 + m1)

                                       * (1 - 4n2 - 2y)]

                                 0.5(1 - 4n2 - 2y)
                       + 0.5Erfc[----------------
                                         0.5
                                       t2

                                             0.5
                                         + t2    Pi (-0.5 + m1)]

                              * Exp[Pi (-0.5 + m1) (1 - 4n2 - 2y)]
           + ------------------------------------------------------
                                 2            2
                               Pi  (-0.5 + m1)

                   0.5(1 + 4n2 + 2y)             2            2
             -Erfc[-----------------] Exp[-t1 Pi  (-0.5 + m1) ]
                           0.5
                         t1

                                 -0.5(1 + 4n2 + 2y)
                       + 0.5Erfc[-----------------
                                         0.5
                                       t1

                                             0.5
                                         + t1    Pi (-0.5 + m1)]

                              * Exp[-Pi (-0.5 + m1)
```

```
                                                * (1 + 4n2 + 2y)]

                                           0.5(1 + 4n2 + 2y)
                                - 0.5Erfc[----------------
                                                  0.5
                                                t1

                                                      0.5
                                                  + t1    Pi (-0.5 + m1)]

                                       * Exp[Pi (-0.5 + m1) (1 + 4n2 + 2y)]
                    + ----------------------------------------------------
                                          2            2
                                        Pi  (-0.5 + m1)

                            0.5(1 + 4n2 + 2y)              2            2
                      -Erfc[-----------------] Exp[-t2 Pi  (-0.5 + m1) ]
                                    0.5
                                  t2

                                          -0.5(1 + 4n2 + 2y)
                                + 0.5Erfc[------------------
                                                  0.5
                                                t2

                                                      0.5
                                                  + t2    Pi (-0.5 + m1)]

                                       * Exp[-Pi (-0.5 + m1)

                                                * (1 + 4n2 + 2y)]

                                          0.5(1 + 4n2 + 2y)
                                - 0.5Erfc[----------------
                                                  0.5
                                                t2

                                                      0.5
                                                  + t2    Pi (-0.5 + m1)]

                                       * Exp[Pi (-0.5 + m1) (1 + 4n2 + 2y)]
                     - ----------------------------------------------------)
                                          2            2
                                        Pi  (-0.5 + m1)
 Here is the temperature for time region 3 ...
4Qo Cos[x Pi (-0.5 + m1)] Cos[0.5y Pi (-0.5 + m2)]

                   2            2          2            2
   * (-Exp[t2 (- Pi  (-0.5 + m1)  - 0.25 Pi  (-0.5 + m2) )]

                             2            2          2            2
               + Exp[t3 (- Pi  (-0.5 + m1)  - 0.25 Pi  (-0.5 + m2) )])

   * Sin[0.25Pi (-0.5 + m2)]
---------------------------------------------------------------------
                         2            2          2            2
     Pi (-0.5 + m2) (- Pi  (-0.5 + m1)  - 0.25 Pi  (-0.5 + m2) )
DDD....DDDDD......DDDDDDats' all folks'!!!!!!
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