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THE PEANO DERIVATIVES

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Clifford E. Weil
Major professor

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THE PEANO DERIVATIVES

by

Hajrudin Fejzić

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ABSTRACT
THE PEANO DERIVATIVES
by
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Let f be a function defined on an interval $[a, b]$ and that $k \in \mathbf{N}$. We say that f has a k-th Peano derivative at $x \in [a, b]$ if there exist coefficents $f_1(x), \ldots, f_k(x)$ such that $f(x+h) = f(x) + hf_1(x) + \ldots + \frac{h^k}{k!} f_k(x) + h^k \epsilon_k(x, h)$ where $\lim_{h \to 0} \epsilon_k(x, h) = 0$. The coefficent $f_k(x)$ is called the k-th Peano derivative of f at x. The existence of a k-th ordinary derivative, $f^{(k)}(x)$, implies the existence of $f_k(x)$ and $f_k(x) = f^{(k)}(x)$, but the converse is not true for $k \geq 2$.

Let Δ' be the class of all derivatives, and let $[\Delta']$ be the vector space generated by Δ' and O'Malley's class B_1^*. S. Agronsky, R. Biskner, A. Bruckner and J. Mařík have showed that every function $[\Delta']$ has the form $g' + hk'$, where g,h and k are differentiable. They also proved that $f \in [\Delta']$ if and only if there is a sequence of derivatives $\{v_n\}$ and closed sets $\{A_n\}$ such that $\cup_{n=1}^{\infty} A_n = \mathbf{R}$ and $f = v_n$ on A_n. The sets A_n and corresponding functions v_n are called a decompositon of f. The question they posed is whether every Peano derivative belongs to this class of functions. In the first part of this thesis a positive answer to this question is given. Also it is shown that for Peano derivatives the sets A_n can be chosen to be perfect. Moreover it is shown that every k-th Peano derivative is the composite derivative of the $(k-1)$-th Peano derivative relative to the sequence $\{A_n\}$.

A. Bruckner, R. O'Malley and B. Thomson introduced the notation of path derivatives. They showed that path derivatives have many of the properties possessed by ordinary derivatives. In the second part of this thesis it is shown that Peano derivatives are also path derivatives and hence they have all the properties possessed by path derivatives. This gives another proof of the many properties possessed by Peano

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derivatives and also answers the question posed by the above authors.

The third part of this thesis shows that a k-th Peano derivative is a selective derivative of the $(k-1)$-th Peano derivative, and hence gives a positive answer to the question posed by C. Weil regarding Peano and selective derivatives.

Finally the last part of this thesis shows that these results are still true if we replace Peano derivatives with generalized Peano derivatives, introduced by M. Laczkovich, and studied by C. Lee.

To my brother-in-law MIDHAT DRINA, who was killed by Serbian irregulars on June 22, 1992.

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Clifford Weil, for all his help, encouragement and advice. His knowledge and enthusiasm were invaluable.

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INTRODUCTION

The definition of the k-th ordinary derivative of a real valued function is iterative in nature and thus easily comprehended if one initially understands what a first derivative is. This nice feature can present a problem, however, because in order to find the k-th derivative of a function f at a point x, one must know all the previous derivatives, not only at x, but at every point in some neighborhood of x. One type of generalized k-th order differentiation, having Taylor's theorem as its motivation, attempts to skirt this drawback. This kind of differentiation is called Peano differentiation.

Definition 0.0.1 *A function f is said to have a k-th Peano derivative at x if there exist numbers $f_1(x)$, $f_2(x), \ldots, f_k(x)$ such that*

$$f(x+h) = f(x) + hf_1(x) + \cdots + \frac{h^k}{k!}\left(f_k(x) + \epsilon_k(x,h)\right) \tag{1}$$

where $\epsilon_k(x,h) \to 0$ as $h \to 0$. The number $f_k(x)$ is called the k-th Peano derivative of f at x. It will be convenient to denote $f(x)$ by $f_0(x)$. With this notation (1) *becomes*

$$f(x+h) = \sum_{j=0}^{k} \frac{h^j}{j!} f_j(x) + \frac{h^k}{k!}\epsilon_k(x,h).$$

This concept was presented in 1891 by the italian mathematician G. Peano. Peano introduced this type of derivative, obtained a product rule, a quotient rule, and pointed out that if a function f has an ordinary k-th derivative at x, $f^{(k)}(x)$, then it must have a k-th Peano derivative at x and $f_k(x) = f^{(k)}(x)$. The converse is not true for $k \geq 2$ as can be seen from the following example. Let

$$f(x) = x^{k+1}\sin\frac{1}{x} \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

It is easy to see that $f_k(0) = 0$ but $f^{(k)}$ at 0 doesn't exist. Thus the k-th Peano derivative is a true generalization of the ordinary k-th derivative although obviously there is no difference for $k = 1$.

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In 1954 H. W. Oliver published the first extensive work devoted exclusively to exhibiting properties of k-th Peano derivatives. (See [10].) He showed that such a derivative has several of the properties known to be possessed by an ordinary derivative. Oliver established that *if f_k exists for all x in some interval I, then f_k is of Baire class one; i.e., f_k can be written as a pointwise limit of a sequence of continuous functions* **(A)**. (Denjoy had obtained this result earlier in the more general setting where f_k is defined relative to a perfect set H.) Oliver also showed that f_k *must have the Darboux property; i.e., that for any interval $[a, b] \subset I$ if y is a point between $f_k(a)$ and $f_k(b)$, then there is $c \in (a, b)$ so that $f_k(c) = y$* **(B)**, another property well known and easily verified for ordinary derivatives. Moreover, he showed that if f_k *is bounded above or below on some $[a, b] \subset I$, then f_k is the ordinary k-th derivative of f on $[a, b]$* **(C)**. In particular, this yields *the monotonicity theorem which states that if $f_k \geq 0$ on $[a, b]$, then f_{k-1} is nondecreasing and continuous on $[a, b]$* **(D)**. Combining this with the fact that f_k is of Baire class one, it follows that f_k *is an ordinary k-th derivative on an open, dense subset of I* **(E)**. R. J. O'Malley and C. E. Weil showed that *if f_k attains both values $-M$ and M on some interval $[a, b] \subset I$, then there is an open interval $J \subset [a, b]$, on which $f_k = f^{(k)}$ and $f^{(k)}$ attains both values $-M$ and M on J* **(F)**. (See [12].)

If g is an ordinary derivative on I, then for any open interval, (a, b), $g^{-1}(a, b)$ either is empty or has positive Lebesgue measure, a result first proved by Denjoy. A function having this property is said to have the Denjoy property. Oliver showed that f_k possesses the Denjoy property on I **(G)**. (See [10].)

Z. Zahorski proved that the following property is possessed by every ordinary derivative.

Definition 0.0.2 *A function g is said to have the Zahorski property if for each open interval (a, b), for each $x \in g^{-1}(a, b)$, and for each sequence of intervals $\{I_n\}$ converging to x, (The end points of the I_n converge to x but x belongs to no I_n.) with*

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$m(g^{-1}(a,b)\cap I_n)=0$ for every n, implies $\lim_{n\to\infty}\frac{m(I_n)}{dist(x,I_n)}=0$, where $m(I_n)$ denotes Lebesgue measure of I_n and $dist(x,I_n)$ denotes the distance between x and I_n **(H)**.

C. E. Weil showed that a k-th Peano derivative also has the Zahorski property, and he introduced a property somewhat stronger that the Zahorski property, which he called property Z. (See [18].)

Definition 0.0.3 *A function g defined on an interval I is said to have property Z if for every $\epsilon>0$, each $x\in I$, and each sequence of intervals $\{I_n\}$ converging to x such that $g(y)\geq g(x)$ on I_n or $g(y)\leq g(x)$ on I_n for each n, we have*

$$\lim_{n\to\infty}\frac{m(\{y\in I_n\,:\,|g(y)-g(x)|\geq\epsilon\})}{m(I_n)+dist(x,I_n)}=0.$$

Weil showed that this property is strictly stronger than the Zahorski property, yet still is possessed by every k-th Peano derivative.

In [3] the authors introduced the concept of a path derivative as a unifying approach to the study of a number of generalized derivatives. Namely since many other generalized derivatives like approximate derivative, possess most of the properties mentioned above that are possessed by Peano derivatives, the authors in [3] where looking for a framework within which all of these derivatives could be presented.

The perspective they chose was to consider just those derivatives of a function F at a point x which can be obtained as

$$\lim_{y\in E_x,\,y\to x}\frac{F(y)-F(x)}{y-x}$$

for appropriate choices of sets E_x. One generalized derivative, then, differs from another only by the choice of the family of sets $\{E_x : x\in\mathbf{R}\}$ through which the difference quotient passes to its limit. For example, an approximately differentiable function F permits a choice of sets $\{E_x : x\in\mathbf{R}\}$ so that each E_x has density 1 at x; for a Dini derivative each set may consist only of a sequence converging to x. This framework includes any generalized derivative for which the derivative at a point is a derived number of the function at that point. Since Weil has proved that $f_k(x)$ is a derived number of f_{k-1} at a point x, we see that this concept of path derivatives also includes k-th Peano derivatives.

But in order to get some properties for path derivatives, like those possessed by Peano or approximate derivatives, we require that the family of sets $\{E_x : x \in \mathbf{R}\}$ satisfy various "thickness" conditions. These conditions relate to the "thickness" of each of the sets E_x and the way in which two of the sets intersect. The authors proved that path derivatives with certain type of conditions imposed on the family of sets $\{E_x : x \in \mathbf{R}\}$, have many of the properties possessed by approximate and Peano derivatives.

We will show that Peano derivatives are path derivatives with $\{E_x : x \in \mathbf{R}\}$ satisfying some of the intersection conditions introduced by the authors mentioned above. This will give a positive answer to the question posed in [3]. In proving this assertion, we won't use any known results for Peano derivatives. So this can be regarded as a new approach to studying Peano derivatives. Namely all of the properties **(B)**, **(C)**, **(D)**, **(E)**, **(F)**, **(G)** and **(H)**, that we mentioned before, we will get for Peano derivatives directly from the corresponding properties of path derivatives. The main tool will be a decomposition of Peano derivatives which we will discuss next.

Let C be the family of all continuous functions on $\mathbf{R}$, Δ the family of all differentiable functions on $\mathbf{R}$ and Δ' the family of all derivatives on $\mathbf{R}$. If Γ is a family of functions defined on $\mathbf{R}$, then by $[\Gamma]$ we denote the family of all functions f on $\mathbf{R}$ with the following property: for each $n \in \mathbf{N}$ there exist $v_n \in \Gamma$ and a closed set A_n such that $f = v_n$ on A_n and $\cup_{n=1}^{\infty} A_n = \mathbf{R}$. In [1] (Theorem 2) it is shown that the following four conditions are equivalent:

(i) There are g, h and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.

(ii) There is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.

(iii) The function $f \in [\Delta']$.

(iv) There is a dense open set T such that f is a derivative on T and f is a derivative on $\mathbf{R} \setminus T$ with respect to $\mathbf{R} \setminus T$.

The statement (ii) implies that $[\Delta']$ is the vector space generated by Δ' and $[C]$. In [1] (Theorem 3) it is shown that each approximate derivative, each approximately continuous function and each function in $B_1^* = [C]$ belongs to the class $[\Delta']$. In [10] O'Malley showed that for approximate derivatives, the sets A_n from the definition of $[\Delta']$ can be chosen to be perfect. The following question is raised in [1]. "Does every Peano derivative belong to $[\Delta']$?". We will give a positive answer to this question, plus we will prove even more.

Definition 0.0.4 *Let f be a function defined on $\mathbf{R}$. If there exist a function g, and closed sets A_n, $n = 1, 2, \ldots$ such that $\cup_{n=1}^{\infty} A_n = \mathbf{R}$ and $g|'_{A_n}(x) = f(x)$ for $x \in A_n$, then we say that f is a composite derivative of g.*

We will prove that f_k is a composite derivative of f_{k-1} with respect to the sets $\overline{P}_{1/n}$, where for $\epsilon > 0$, $\delta > 0$ we define

$$P_\delta = P(f, \epsilon, \delta) = \{x : |\epsilon_k(x, h)| < \epsilon \text{ for } |h| < \delta\}.$$

These sets were first introduced by A. Denjoy. He showed that with respect to these sets for $0 \leq l < i$, $i = 1, \ldots, k-1$, f_i is an l-th Peano derivative of f_{i-l}, with $(f_l|_{\overline{P}_\delta})_{(i-l)}(x) = f_i(x)$ for $x \in \overline{P}_\delta$, where the $(l-i)$-th Peano derivative is computed relative to $\overline{P}_\delta$. Using different techniques, we are able to improve his result. Namely, we show that the result also holds for the case $i = k$. Since $\cup_{n=1}^{\infty} P_n = \mathbf{R}$, we have that f_k is a composite derivative of f_{k-1}. This gives a positive answer to a question raised by C. Weil. (See [19].) From this result it is easy to conclude that $f_k \in [\Delta']$. We just need to recall the fact that for any function g defined on a closed set A, such that at every point $x \in A$, a derivative $g'(x)$ computed relative to A exists, there exists a function G differentiable at every point $x \in \mathbf{R}$ so that $G|_A = g$ and $G'|_A = g'$.

We can enlarge the sets $\overline{P}_n$, so that they are perfect and that f_k is still a composite derivative of f_{k-1} with respect to these perfect sets. Therefore one more property possessed by approximate derivatives is also possessed by Peano derivatives.

Because every composite derivative is a Baire 1 function, we see that f_k is a Baire 1 function. Although this property is very easy to establish for Peano derivatives, for generalized Peano derivatives it is not so easy. We will discuss these derivatives in Chapter IV, but using some techniques similar to those that we use for Peano derivatives we will prove that generalized Peano derivatives are also composite derivatives and therefore, they are Baire 1 functions.

Another immediate corollary is that f_k is the approximate derivative of f_{k-1} almost everywhere. This result was first proved by Zygmund and Marcinkiewicz. (See [20] page 75.) We will generalize their result showing that f_k is the l-th approximate Peano derivative of f_{k-l} with $(f_{k-l})_{(ap-l)}(x) = f_k(x)$ almost everywhere, where $(ap-l)$ denotes l-th approximate Peano derivative.

The sequence of sets $\{P_n\}$ satisfies the condition $P_n \subset P_{n+1}$. This fact together with results already established will enable us to construct a system of paths $\{E_x : x \in \mathbf{R}\}$ satisfying the I.C. property (as it was defined in [3]), so that f_k is a path derivative of f_{k-}

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of f_{k-1} with respect to this system. Using an induction argument and known results for path derivatives with such a system of paths, we get that Peano derivatives possess properties **(B)**, **(C)**, **(D)**, **(E)**, and **(G)**. Since there is a nonporous system of paths $\{E_x : x \in \mathbf{R}\}$, such that f_k is the path derivative of f_{k-1} with respect to that system, (a fact established in [3]), we get that Peano derivatives possess also properties **(F)** and **(H)**.

Finally we show that there is a system of paths $\{E_x : x \in \mathbf{R}\}$ satisfying the I.I.C. condition (as it is defined in [3]), with f_k the path derivative of f_{k-1} with respect to that system. This implies that f_k is the selective derivative of f_{k-1}.

Definition 0.0.5 *If for a given function F there is a function p of two variables called a selection, satisfying $p(x,y) = p(y,x)$ and $p(x,y) \in (x,y)$, so that*

$$\lim_{y \to x} \frac{F(p(x,y)) - F(x)}{p(x,y) - x} \tag{2}$$

exists, we say that F is selectively differentiable at x, and the limit in (2) we call the selective derivative of F at the point x and denote it by $F'_p(x)$.

Selective differentiation was introduced by R. O'Malley. Motivation for introducing selective differentiation was the fact that approximate derivatives are selective derivatives, which was proved by O'Malley. Showing that f_k is a selective derivative of f_{k-1} we give a positive answer to a question raised by C. Weil. (See [19].) So Peano derivatives possess one more property possessed by approximate derivatives.

Generalized Peano derivatives were introduced by C. Lee. (See [9].) He showed that every absolute Peano derivative on a compact interval is a generalized Peano derivative. Absolute Peano derivatives were introduced by M. Laczkovich. (See [7].)

Definition 0.0.6 *Let f be defined in a neighborhood of x. We say that the absolute Peano derivative of f at x exists and is A (in symbols $f^*(x) = A$) if there is a function g, a nonnegative integer k, and a $\delta > 0$ such that*

$$(i)\ g_k = f \text{ on } (x-\delta, x+\delta) \text{ and } \quad (ii)\ g_{k+1}(x) = A.$$

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Laczkovich showed that this concept is unambiguously defined, that if f^* exists on an interval, it is a function of Baire class one, it has the Darboux property, and if f^* is bounded above or below on an interval, then $f^* = f'$ on that interval.

Definition 0.0.7 *Let F be a continuous function defined on $\mathbf{R}$, and let $n \in \mathbf{N}$. We say that F is n-th generalized Peano differentiable at $x \in \mathbf{R}$, if there is a positive integer q, and coefficients $F_{[i]}(x)$, $i = 1, \ldots, n$ such that*

$$F^{(-q)}(x+h) = \sum_{j=0}^{q-1} h^j \frac{F^{(-q+j)}(x)}{j!} + \sum_{j=0}^{n} h^{q+j} \frac{F_{[j]}(x)}{(q+j)!} + h^{q+n} \epsilon_{q+n}^{[q]}(x,h) \qquad (3)$$

where $\lim_{h \to 0} \epsilon_{q+n}^{[q]}(x,h) = 0$.

Here $F_{[0]}(x) = F(x) = F^{(0)}(x)$ and $F^{(-j)}(x) = \int^x F^{(-j+1)}(t)\,dt$; i.e. $F^{(-j)}$ is an indefinite Riemann integral of the continuous function $F^{(-j+1)}$ for $j = 1, \ldots, q$. Note that the definitions of $F_{[i]}(x)$, $i = 0, 1, \ldots, n$ and of $\epsilon_{q+n}^{[q]}(x,h)$ don't depend on which q-fold indefinite Riemann integral $F^{(-q)}$ of the continuous function F, is taken because any two differ by a polynomial of a degree less than q. The above definition is the same as the definition of $(q+n)$-th Peano derivative of a function $F^{(-q)}$ at the point x. Note that every n-th Peano derivative is also a n-th generalized Peano derivative, but the converse is not true. Namely M. Laczkovich has constructed an absolute Peano derivative on an interval which is not an ordinary Peano derivative of any order.

C. Lee showed that all properties **(A)**, **(B)**, **(C)**, **(D)**, **(E)**, **(F)**, **(G)**, **(H)** and Weil's Z property are possessed by generalized Peano derivatives and in particular they are possessed by absolute Peano derivatives. (See [8] and [9].)

We will take a different approach to studying generalized Peano derivatives than that taken by C. Lee. Our approach will be similar to the one we used in studying Peano derivatives, so many results that we established for Peano derivatives will hold also for generalized Peano derivatives. In particular we will obtain that generalized Peano derivatives are composite derivatives and hence belong to $[\Delta']$. Also we will show that generalized Peano derivatives are path derivatives with respect to a bilateral, nonporous system of paths satisfying I.I.C. condition. Therefore generalized Peano derivatives are also selective derivatives.

show that generalized Peano derivatives are path derivatives with respect to a bilateral, nonporous system of paths satisfying I.I.C. condition. Therefore generalized Peano derivatives are also selective derivatives.

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CHAPTER I

Throughout this theses all the properties will be established for functions defined on $\mathbf{R}$. But it can be easily seen that $\mathbf{R}$ can be replaced by any connected subset of $\mathbf{R}$.

1.1 Decomposition of Peano derivatives

Let C be the family of all continuous functions on $\mathbf{R}$, Δ the family of all differentiable functions on $\mathbf{R}$ and Δ' the family of all derivatives on $\mathbf{R}$. If Γ is a family of functions defined on $\mathbf{R}$, then by $[\Gamma]$ we denote the family of all functions f on $\mathbf{R}$ with the property that for every $n \in \mathbf{N}$ there exist $v_n \in \Gamma$ and a closed set A_n such that $f = v_n$ on A_n and $\cup_{n=1}^{\infty} A_n = \mathbf{R}$. In [1] (Theorem 2) it is shown that the following four conditions are equivalent:

(i) There are g, h and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.

(ii) There is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.

(iii) The function $f \in [\Delta']$.

(iv) There is a dense open set T such that f is a derivative on T and f is a derivative on $\mathbf{R} \setminus T$ with respect to $\mathbf{R} \setminus T$.

The statement (ii) implies that $[\Delta']$ is the vector space generated by Δ' and $[C]$. In [1] (Theorem 3) it is shown that each approximate derivative, each approximately continuous function and each function in $B_1^* = [C]$ belongs to the class $[\Delta']$. The main goal of this chapter is to show that every Peano derivative is in $[\Delta']$. We will prove even more. Namely we will prove that every Peano derivative is a composite derivative.

Definition 1.1.1 *Let f be a function defined on* $\mathbf{R}$. *If there exist a function g, and for $n \in \mathbf{N}$ there is a closed set A_n with $g|'_{A_n}(x) = f(x)$ $\forall x \in A_n$ and $\cup_{n=1}^{\infty} A_n = \mathbf{R}$, then we say that f is the composite derivative of g.*

The following result due to Mařík says that every composite derivative is in $[\Delta']$. (See [14].)

Theorem 1.1.2 *Let a function g be defined on a closed set H. If g' exists on H, where g' is computed relative to H, then there is a function G differentiable on* $\mathbf{R}$ *so that $G|_H = g$ and $G'|_H = g'$.*

O'Malley proved that every approximate derivative is the composite derivative of its primitive. (See [11].) In this chapter we will prove that a k-th Peano derivative is the composite derivative of the $(k-1)$-th Peano derivative. Thus we will get that every Peano derivative is in $[\Delta']$, and hence possesses all the properties possessed by functions in $[\Delta']$. We will start with an elementary lemma.

Lemma 1.1.3 *For $m \in \mathbf{N}$ we have*

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^i = \left\{ \begin{array}{cl} 0 & \textit{if } i = 0, \ldots, m-1 \\ m! & \textit{if } i = m \\ \frac{m}{2}(m+1)! & \textit{if } i = m+1 \ . \end{array} \right.$$

Proof: Let $B_m^i = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^i$. Then $B_1^0 = 0$, $B_1^1 = 1$, and $B_1^2 = 1$. Now we will proceed by induction on m. Suppose

$$B_{m-1}^t = \left\{ \begin{array}{cl} 0 & \text{if } t = 0, \ldots, m-2 \\ (m-1)! & \text{if } t = m-1 \\ \frac{m-1}{2}m! & \text{if } t = m. \end{array} \right.$$

Note that $B_m^0 = (1-1)^m = 0$. Let $1 \le i \le m+1$. Since $i \ge 1$, $0^i = 0$. Thus

$$B_m^i = m\sum_{j=1}^{m}(-1)^{m-j}\binom{m-1}{j-1}j^{i-1}$$

$$\begin{aligned}
&= m\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(j+1)^{i-1} \\
&= m\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\sum_{r=0}^{i-1}\binom{i-1}{r}j^{i-1-r} \\
&= m\sum_{r=0}^{i-1}\binom{i-1}{r}B_{m-1}^{i-1-r}\ , \quad \text{and by the induction hypothesis} \\
&= \begin{cases} 0 & \text{if } i-1<m-1 \\ mB_{m-1}^{m-1} & \text{if } i-1=m-1 \\ m\left(\binom{m}{1}B_{m-1}^{m-1}+\binom{m}{0}B_{m-1}^{m}\right) & \text{if } i-1=m \end{cases} \\
&= \begin{cases} 0 & \text{if } i<m \\ m(m-1)! & \text{if } i=m \\ m(m(m-1)!+\frac{m-1}{2}m!) & \text{if } i=m+1 \end{cases} \\
&= \begin{cases} 0 & \text{if } i=0,\ldots,m-1 \\ m! & \text{if } i=m \\ \frac{m}{2}(m+1)! & \text{if } i=m+1. \end{cases} \qquad \square
\end{aligned}$$

Definition 1.1.4 *The Riemann difference $\Delta_t^m f(x)$ of order m of a real valued function f at a point x is defined by $\Delta_t^m f(x) = \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} f(x+jt)$.*

If f is continuous on $\mathbf{R}$, then $\Delta_t f(y)$ is continuous on $\mathbf{R}$. This is the case if the k-th Peano derivative f_k exists on $\mathbf{R}$.

The relationship between Δ_t^m and Δ_t^{m+1} is given by the following simple lemma.

Lemma 1.1.5

$$\Delta_t^{m+1} f(x) = \Delta_t^m f(x+t) - \Delta_t^m f(x).$$

Proof:

$$\Delta_t^m f(x+t) - \Delta_t^m f(x) =$$

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+(j+1)t) - \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt) =$$

$$f(x+(m+1)t) + \sum_{j=1}^{m}(-1)^{m+1-j}\left(\binom{m}{j-1}+\binom{m}{j}\right)f(x+jt) +$$

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$(-1)^{m+1}f(x)$ (since $\binom{m}{j-1}+\binom{m}{j}=\binom{m+1}{j}$ it is equal to)

$$= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}f(x+jt) = \Delta_t^{m+1}f(x). \quad \square$$

For the remainder of this chapter k will be a fixed positive integer greater than 1. If a function f at some point $x \in \mathbf{R}$ has a k-th Peano derivative $f_k(x)$, then using Lemma 1.1.3, we have the following formula for Riemman differences.

Lemma 1.1.6 *Let f be a function defined on $\mathbf{R}$. Let $x \in \mathbf{R}$ be such that $f_k(x)$ exists. Then for $0 \le m \le k$*

$$\Delta_t^m f(x) = \begin{cases} t^m f_m(x) + t^m \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} j^m \epsilon_m(x,jt) & \text{if } m = i \\ t^i \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} j^i \epsilon_i(x,jt) & \text{if } m > i. \end{cases}$$

Proof:

$$\begin{aligned} \Delta_t^m f(x) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt) \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\sum_{l=0}^{i}(jt)^l\frac{f_l(x)}{l!} + (jt)^i\epsilon_i(x,jt)\right) \\ &= \sum_{l=0}^{i} t^l\frac{f_l(x)}{l!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^l + t^i\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^i\epsilon_i(x,jt) \end{aligned}$$

and by Lemma 1.1.3 we have

$$= \begin{cases} t^m f_m(x) + t^m \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} j^m \epsilon_m(x,jt) & \text{if } m = i \\ t^i \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} j^i \epsilon_i(x,jt) & \text{if } m > i\,. \end{cases} \quad \square$$

The next theorem is an easy consequence of Lemma 1.1.6.

Theorem 1.1.7 *Let f be a function defined on $\mathbf{R}$ such that f_k exists for each $x \in \mathbf{R}$. Then f_k is a Baire 1 function.*

Proof: For each $n \in \mathbf{N}$ let $g_n(x) = n^k\Delta^k_{\frac{1}{n}}f(x)$. Then by Lemma 1.1.6 (applied with $m = i = k$) we have $\lim_{n\to\infty} g_n(x) = f_k(x)$. It remains only to notice that each

g_n is a continuous function. Therefore f_k is a pointwise limit of continuous functions; i.e., f_k is a Baire 1 function. $\square$

In order to prove that every Peano derivative is a composite derivative, we need to construct a sequence of closed sets $\{A_n\}$ whose union is $\mathbf{R}$, and with respect to which f_k is a composite derivative. The obvious candidate for a primitive is f_{k-1}. We will investigate a relationship among the Peano derivatives f_i for $i = 1, \ldots, k$ on certain sets $\overline{P}_k(f, \epsilon, \delta)$ which are defined next.

Definition 1.1.8 *Let $\epsilon > 0$ and $\delta > 0$ be given. Let*

$$P = P_k(f, \epsilon, \delta) = \{x : |\epsilon_k(x, h)| < \epsilon \text{ whenever } |h| < \delta\}.$$

These sets were first introduced by A. Denjoy. (See [5].) He proved that for $i = 1, \ldots, k-1$ and for $0 \le l \le i$, f_i is the $(i-l)$-th Peano derivative of f_l with respect to the closure of these sets with the expected values; i.e., with $(f_l|_{\overline{P}})_{(i-l)}(x) = f_i(x)$ for every $x \in \overline{P}$. (He proved that the same conclusion holds if f is defined on some perfect set H having finite index provided f_k, computed relative to this set, exists on H.) (See [5].) When f is defined on $\mathbf{R}$ so that f_k exists on $\mathbf{R}$ (or f_k exists on a perfect set H), we will show that the result also holds for $i = k$. First, using different techniques than Denjoy used, we will prove his result. We begin with some elementary formulas.

Formula 1.1.9 *Let f be a function on $\mathbf{R}$ and let $x \in \mathbf{R}$. For $l \in \mathbf{N}$, suppose that $f_l(x)$ exists. Then for each $t \in \mathbf{R}$*

$$\epsilon_{l-1}(x, t) = t\frac{f_l(x)}{l!} + t\epsilon_l(x, t).$$

Proof: The assertion follows directly from the definition of Peano derivatives. □

Formula 1.1.10 *Let f be a function on* $\mathbf{R}$ *and let $x, t \in \mathbf{R}$. For $l \in \mathbf{N}$, suppose that $f_l(x)$ and $f_l(x+t)$ exist. Then*

$$f_{l-1}(x+t) - f_{l-1}(x) - tf_l(x) = t\sum_{j=1}^{l}(-1)^{l-j}\binom{l}{j}j^l\epsilon_l(x,jt) +$$

$$\sum_{j=1}^{l-1}(-1)^{l-1-j}\binom{l-1}{j}j^{l-1}\epsilon_{l-1}(x,jt) -$$

$$\sum_{j=1}^{l-1}(-1)^{l-1-j}\binom{l-1}{j}j^{l-1}\epsilon_{l-1}(x+t,jt).$$

Proof:

By Lemma 1.1.5 $\Delta_t^l f(x) = \Delta_t^{l-1} f(x+t) - \Delta_t^{l-1} f(x)$.

Applying Lemma 1.1.6 to both sides of above equality we get

$$t^l f_l(x) + t^l \sum_{j=1}^{l}(-1)^{l-j}\binom{l}{j}j^l\epsilon_l(x,jt) =$$

$$t^{l-1}f_{l-1}(x+t) + t^{l-1}\sum_{j=0}^{l-1}(-1)^{l-1-j}\binom{l-1}{j}j^{l-1}\epsilon_{l-1}(x+t,jt) -$$

$$t^{l-1}f_{l-1}(x) - t^{l-1}\sum_{j=0}^{l-1}(-1)^{l-1-j}\binom{l-1}{j}j^{l-1}\epsilon_{l-1}(x,jt).$$

Dividing both sides of the above equality by t^{l-1} gives the desired formula. □

Formula 1.1.11 *Let f be a function on* $\mathbf{R}$ *and let $x, t \in \mathbf{R}$. For $l \in \mathbf{N}$, suppose that $f_l(x)$ and $f_l(x+t)$ exist. Then*

$$f_l(x+t) - f_l(x) = \sum_{j=0}^{l+1}(-1)^{l+1-j}\binom{l+1}{j}j^l\epsilon_l(x,jt) +$$

$$\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}j^l\epsilon_l(x,jt) - \sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}j^l\epsilon_l(x+t,jt).$$

Proof:

By Lemma 1.1.5 $\Delta_t^{l+1}f(x) = \Delta_t^l f(x+t) - \Delta_t^l f(x)$.

Applying Lemma 1.1.6 to both sides of the above equality we get

$$t^l \sum_{j=0}^{l+1} (-1)^{l+1-j} \binom{l+1}{j} j^l \epsilon_l(x, jt) =$$

$$t^l f_l(x+t) + t^l \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^l \epsilon_l(x+t, jt) -$$

$$t^l f_l(x) - t^l \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^l \epsilon_l(x, jt).$$

Dividing both sides of the above equality by t^l gives the desired formula. □

Formula 1.1.12 *Let f be a function on $\mathbf{R}$ and let $x, t \in \mathbf{R}$. For $l \in \mathbf{N}$, suppose that $f_l(x)$ and $f_l(x+t)$ exist. Then*

$$\frac{f_{l-1}(x+t) - f_l(x)}{t} - f_l(x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^l \epsilon_l(x, jt) +$$

$$\frac{l-1}{2}(f_l(x) - f_l(x+t)) + \sum_{j=0}^{l-1} (-1)^{l-1-j} \binom{l-1}{j} j^l (\epsilon_l(x, jt) - \epsilon_l(x+t, jt)).$$

Proof: From Formulas 1.1.10 and 1.1.9 we get

$$f_{l-1}(x+t) - f_l(x) - t f_l(x) = t \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^l \epsilon_l(x, jt) +$$

$$\sum_{j=0}^{l-1} (-1)^{l-1-j} \binom{l-1}{j} j^{l-1} \left(jt \frac{f_l(x)}{l!} + jt \epsilon_l(x, jt) \right) -$$

$$\sum_{j=0}^{l-1} (-1)^{l-1-j} \binom{l-1}{j} j^{l-1} \left(jt \frac{f_l(x+t)}{l!} + jt \epsilon_l(x+t, jt) \right).$$

Dividing both sides of the above equality by t, and applying Lemma 1.1.3 we get the desired formula. □

Theorem 1.1.13 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. There is a positive constant M such that $\forall\ \epsilon > 0$ and $\delta > 0$ if x and y are in P with $|x - y| < \frac{\delta}{k+1}$, then*

$$|f_k(y) - f_k(x)| \leq M\epsilon \tag{1}$$

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right| \leq M\epsilon. \tag{2}$$

Moreover for $|h| < \delta$ $l = 1, \ldots, k$ f_l and $\epsilon_l(\cdot, h)$ are bounded on $P \cap [a, b]$ independent of h, for any interval $[a, b]$.

Proof: Let x and y be in P such that $|y - x| < \frac{\delta}{k+1}$, and let $t = y - x$. Set $B = \sum_{j=1}^{k+1} \binom{k+1}{j} j^k$. Then the left hand sides of the equalities in Formulas 1.1.11 and 1.1.12 are bounded by $3B\epsilon$ and $3B\epsilon + 3B\epsilon\frac{k-1}{2}$ respectively. Hence (1) and (2) follows for $M = 3B\frac{k+1}{2}$.

Let $[a, b]$ be any interval. From (1) we see that f_k is bounded on $P \cap [a, b]$. From Formula 1.1.9 (applied with $l = k$) it follows that for $|h| < \delta$ $|\epsilon_{k-1}(\cdot, h)|$ is bounded on $P \cap [a, b]$ independent of h. Now from Formula 1.1.10 (applied with $l = k$) we see that f_{k-1} is bounded on $P \cap [a, b]$, and again going back to Formula 1.1.9 (applied with $l = k-1$) we see that for $|h| < \delta$ $|\epsilon_{k-2}(\cdot, h)|$ is bounded on $P \cap [a, b]$ independent of h. Continuing we can deduce that there is a constant C so that $|f_l(y)| \leq C$ and $|\epsilon_l(y, h)| \leq C$ whenever $y \in P \cap [a, b]$, $|h| < \delta$ and $1 \leq l \leq k$. $\square$

The next theorem says that if we replace P by $\overline{P}$, then the conclusion of Theorem 1.1.13 still holds.

Theorem 1.1.14 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then $\overline{P} \subseteq P(f, 3\epsilon, \delta)$.*

Proof: Let $x \in \bar{P}$, and let $\{x_n\} \in P$ be a sequence such that $\lim_{n\to\infty} x_n = x$. Let $[a,b]$ be such that $\{x_n\} \subset P \cap [a,b]$. From Theorem 1.1.2 we see that f_l for $1 \le l \le k$ is bounded on $P \cap [a,b]$. Therefore we can choose a subsequence $\{x_{n_j}\}$ converging to x such that for $i = 1 \cdots k$ the sequence $\{f_i(x_{n_j})\}$ is convergent. Let these sequences converge to $F_i(x)$, $i = 1, \ldots, k$ respectively.

Let h with $|h| < \delta$ be given. Suppose that $|h + x - x_{n_j}| < \delta$ for every $j \in \mathbb{N}$. Therefore $|\epsilon_k(x_{n_j}, h+x-x_{n_j})| < \epsilon$ so we can also suppose that this sequence converges. (If not, then extract a convergent subsequence.) Denote its limit by $E(h)$. Since $f_k(x_{n_j})$ exists,

$$f(x+h) = f(x_{n_j}) + (h + x - x_{n_j})f_1(x_{n_j}) + \cdots + \frac{(h+x-x_{n_j})^{k-1}}{(k-1)!} f_{k-1}(x_{n_j})$$

$$+ (h + x - x_{n_j})^{k-1}(h + x - x_{n_j}) \left(\frac{f_k(x_{n_j})}{k!} + \epsilon_k(x_{n_j}, h + x - x_{n_j}) \right). \tag{3}$$

Letting $j \to \infty$ in (3) we get

$$f(x+h) = f(x) + hF_1(x) + \cdots + \frac{h^{k-1}}{(k-1)!} F_{k-1}(x) + h^{k-1}\, h \left(\frac{F_k(x)}{k!} + E(h) \right).$$

Since $\lim_{h\to 0} h(\frac{F_k(x)}{k!} + E(h)) = 0$, by the uniqueness of Peano derivatives we have $F_i(x) = f_i(x)$ for $1 \le i \le k-1$ and

$$\frac{f_k(x)}{k!} + \epsilon_k(x,h) = \frac{F_k(x)}{k!} + E(h). \tag{4}$$

Since $|E(h)| \le \epsilon$, from (4) we have that

$$\left| \frac{f_k(x) - F_k(x)}{k!} \right| = |E(h) - \epsilon_k(x,h)| \le \epsilon + |\epsilon_k(x,h)|. \tag{5}$$

The left hand side of (5) doesn't depend on h so letting $h \to 0$ in the right hand side of (5) we get $\left|\frac{f_k(x) - F_k(x)}{k!}\right| \le \epsilon$.

Finally this estimate and the formula $\epsilon_k(x,h) = E(h) + \frac{F_k(x) - f_k(x)}{k!}$ gives $|\epsilon_k(x,h)| \le 2\epsilon < 3\epsilon$ for $|h| < \delta$. Hence $x \in P(f, 3\epsilon, \delta)$ and the theorem is proved. $\square$

In order to prove that Theorem 1.1.7 holds for $i = k$, we need a formula that involves more than two variables. We will derive a formula, (Theorem 1.1.17 below) involving three independent variables. The proof of the formula is elementary, but the formula itself is the crux in what follows.

Lemma 1.1.15 *Let $0 \leq s \leq k-1$, $x_1 \in \mathbf{R}$ and let a function f be defined on $\mathbf{R}$, having a k-th Peano derivative at x_1. Then*

$$\Delta_t^s f(x_1) = t^s f_s(x_1) + \sum_{l=s+1}^{k} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^l \frac{t^l f_l(x_1)}{l!} +$$

$$t^k \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^k \epsilon_k(x_1, jt).$$

Proof:

$$\begin{aligned}
\Delta_t^s f(x_1) &= \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} \left(\sum_{l=0}^{k} (jt)^l \frac{f_l(x_1)}{l!} + (jt)^k \epsilon_k(x_1, jt) \right) \\
&= \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} \sum_{l=0}^{k} j^l t^l \frac{f_l(x_1)}{l!} + t^k \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^k \epsilon_k(x_1, jt) \\
&= \sum_{l=0}^{k} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^l \frac{t^l f_l(x_1)}{l!} + t^k \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^k \epsilon_k(x_1, jt) \\
&\text{by Lemma 1.1.3} \\
&= \sum_{l=s}^{k} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^l \frac{t^l f_l(x_1)}{l!} + t^k \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^k \epsilon_k(x_1, jt). \\
&\text{Applying Lemma 1.1.3 once more} \\
\Delta_t^s f(x_1) &= t^s f_s(x_1) + \sum_{l=s+1}^{k} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^l \frac{t^l f_l(x_1)}{l!} + \\
&\quad t^k \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^k \epsilon_k(x_1, jt). \quad \square
\end{aligned}$$

The next lemma gives a different formula for $\Delta_t^s f(x_1)$ when we suppose that the k-th Peano derivative of a function f on $\mathbf{R}$, exists at a point $x \neq x_1$.

Lemma 1.1.16 *Let $0 \le s \le k-1$, x_1 and x two points in* $\mathbf{R}$. *Suppose that a function f defined on* $\mathbf{R}$, *has a k-th Peano derivative at x. Then*

$$\Delta_t^s f(x_1) = \sum_{l=s}^{k}(x_1 - x)^{l-s} t^s \frac{f_l(x)}{(l-s)!} +$$

$$\sum_{l=s+1}^{k} \sum_{i=s+1}^{l} \binom{l}{i}(x_1 - x)^{l-i} t^i \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} j^i \frac{f_l(x)}{l!} +$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1 - x + jt)^k \epsilon_k(x, x_1 - x + jt).$$

Proof:

$$\begin{aligned}
\Delta_t^s f(x_1) &= \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{l=0}^{k}(x_1 - x + jt)^l \frac{f_l(x)}{l!} + \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1 - x + jt)^k \epsilon_k(x, x_1 - x + jt) \\
&= \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{l=0}^{k}\left(\sum_{i=0}^{l}\binom{l}{i}(x_1 - x)^{l-i} j^i t^i\right)\frac{f_l(x)}{l!} + \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1 - x + jt)^k \epsilon_k(x, x_1 - x + jt) \\
&= \sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1 - x)^{l-i} t^i \left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} j^i\right)\frac{f_l(x)}{l!} + \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1 - x + jt)^k \epsilon_k(x, x_1 - x + jt) \\
&\quad \text{which by Lemma 1.1.3} \\
&= \sum_{l=s}^{k}\sum_{i=s}^{l}\binom{l}{i}(x_1 - x)^{l-i} t^i \left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} j^i\right)\frac{f_l(x)}{l!} + \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1 - x + jt)^k \epsilon_k(x, x_1 - x + jt) \\
&= \sum_{l=s}^{k}\binom{l}{s}(x_1 - x)^{l-s} t^s \left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} j^s\right)\frac{f_l(x)}{l!} +
\end{aligned}$$

Lemma 1.1.16 *Let $0 \le s \le k-1$, x_1 and x two points in* $\mathbf{R}$*. Suppose that a function f defined on* $\mathbf{R}$*, has a k-th Peano derivative at x. Then*

$$\Delta_t^s f(x_1) = \sum_{l=s}^{k}(x_1-x)^{l-s}t^s\frac{f_l(x)}{(l-s)!}+$$

$$\sum_{l=s+1}^{k}\sum_{i=s+1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\frac{f_l(x)}{l!}+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt).$$

Proof:

$$\begin{aligned}
\Delta_t^s f(x_1) &= \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!}+ \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt) \\
&= \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{l=0}^{k}\left(\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}j^it^i\right)\frac{f_l(x)}{l!}+ \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt) \\
&= \sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\right)\frac{f_l(x)}{l!}+ \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt) \\
&\quad \text{which by Lemma 1.1.3} \\
&= \sum_{l=s}^{k}\sum_{i=s}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\right)\frac{f_l(x)}{l!}+ \\
&\quad \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt) \\
&= \sum_{l=s}^{k}\binom{l}{s}(x_1-x)^{l-s}t^s\left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^s\right)\frac{f_l(x)}{l!}+
\end{aligned}$$

$$\sum_{l=s+1}^{k}\sum_{i=s+1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\frac{f_l(x)}{l!}+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt)$$

applying Lemma 1.1.3 once more yields

$$\begin{aligned}\Delta_t^s f(x_1) &= \sum_{l=s}^{k}\binom{l}{s}(x_1-x)^{l-s}t^s\frac{s!}{l!}f_l(x)+\\ &\sum_{l=s+1}^{k}\sum_{i=s+1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\frac{f_l(x)}{l!}+\\ &\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt)=\\ &\sum_{l=s}^{k}(x_1-x)^{l-s}t^s\frac{f_l(x)}{(l-s)!}+\\ &\sum_{l=s+1}^{k}\sum_{i=s+1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\frac{f_l(x)}{l!}+\\ &\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x+jt)^k\epsilon_k(x,x_1-x+jt). \square\end{aligned}$$

Putting Lemmas 1.1.15 and 1.1.16 together we get a formula that is the crux of the proof of Theorems 1.1.20 and 1.1.27.

Theorem 1.1.17 *Let $0 \le s \le k-1$, $0 \neq t$ and x_1 and x two points in $\mathbf{R}$. Suppose that for a function f defined on $\mathbf{R}$ $f_k(x_1)$ and $f_k(x)$ exist. Then*

$$f_s(x_1)-\sum_{l=s}^{k}\frac{(x_1-x)^{l-s}}{(l-s)!}f_l(x)=$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}t^{i-s}\left(\sum_{l=i}^{k}\frac{(x_1-x)^{l-i}}{(l-i)!}f_l(x)-f_i(x_1)\right)+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1-x+jt)^k}{t^s}\epsilon_k(x,x_1-x+jt)-$$

$$t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,jt).$$

Proof: By Lemma 1.1.15 and Lemma 1.1.16

$$f_s(x_1)-\sum_{l=s}^{k}\frac{(x_1-x)^{l-s}}{(l-s)!}f_l(x)=$$

$$\sum_{l=s+1}^{k}\sum_{i=s+1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^{i-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^i\frac{f_l(x)}{l!}+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1-x+jt)^k}{t^s}\epsilon_k(x,x_1-x+jt)-$$

$$\sum_{l=s+1}^{k}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^lt^{l-s}\frac{f_l(x_1)}{l!}-$$

$$t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,jt)\quad\text{(changing the order of summation)}$$

$$=\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}\sum_{l=i}^{k}\binom{l}{i}(x_1-x)^{l-i}t^{i-s}j^i\frac{f_l(x)}{l!}-$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}j^it^{i-s}\frac{f_i(x_1)}{i!}+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1-x+jt)^k}{t^s}\epsilon_k(x,x_1-x+jt)-$$

$$t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,jt)$$

$$=\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}t^{i-s}\left(\sum_{l=i}^{k}\binom{l}{i}\frac{(x_1-x)^{l-i}i!}{l!}f_l(x)-f_i(x_1)\right)+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1-x+jt)^k}{t^s}\epsilon_k(x,x_1-x+jt)-$$

$$t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,jt)$$

$$=\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}t^{i-s}\left(\sum_{l=i}^{k}\frac{(x_1-x)^{l-i}}{(l-i)!}f_l(x)-f_i(x_1)\right)+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1-x+jt)^k}{t^s}\epsilon_k(x,x_1-x+jt)-$$
$$t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,jt).\ \square$$

The following formula is the special case of Theorem 1.1.17 where $t=x_1-x$.

Formula 1.1.18 *Under the assumptions of Theorem 1.1.17 the following formula holds:*

$$f_s(x_1)-\sum_{l=s}^{k}\frac{(x_1-x)^{l-s}}{(l-s)!}f_l(x)=$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}(x_1-x)^{i-s}\left(\sum_{l=i}^{k}\frac{(x_1-x)^{l-i}}{(l-i)!}f_l(x)-f_i(x_1)\right)+$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(x_1-x)^{k-s}(1+j)^k\epsilon_k(x,(x_1-x)(1+j))-$$

$$(x_1-x)^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}j^k\epsilon_k(x_1,j(x_1-x)).$$

The next theorem is a generalization of Theorem 1.1.13.

Theorem 1.1.19 *Let $x,x_1\in\overline{P}(f,\epsilon,\delta)$ with $|x_1-x|<\frac{\delta}{k+1}$. Then there is a constant M not depending on ϵ or δ such that*

$$\left|f_s(x_1)-\sum_{l=s}^{k}\frac{(x_1-x)^{l-s}}{(l-s)!}f_l(x)\right|\leq|x_1-x|^{k-s}M\epsilon$$

for $s=1,\ldots,k$.

Proof: The proof is by induction on s. The case $s = k$ follows from Theorems 1.1.13 and 1.1.14. Suppose it is true for all $s+1 \le i \le k$. Therefore there is a constant M_1 so that

$$\left| f_i(x_1) - \sum_{l=i}^{k} \frac{(x_1 - x)^{l-i}}{(l-i)!} f_l(x) \right| \le |x_1 - x|^{k-i} M_1 \epsilon \tag{6}$$

for $i = s+1, \ldots, k$. Then Formula 1.1.18, Theorem 1.1.14 and the induction hypothesis (6) yield

$$\begin{aligned}
&\left| f_s(x_1) - \sum_{l=s}^{k} \frac{(x_1 - x)^{l-s}}{(l-s)!} f_l(x) \right| \le \\
&\sum_{j=0}^{s} \binom{s}{j} \sum_{i=s+1}^{k} \frac{j^i}{i!} |x_1 - x|^{i-s} |x_1 - x|^{k-i} M_1 \epsilon + \\
&\sum_{j=0}^{s} \binom{s}{j} |x_1 - x|^{k-s} (1+j)^k 3\epsilon + \quad (\text{since } \overline{P}(f, \epsilon, \delta) \subset P(f, 3\epsilon, \delta)) \\
&|x_1 - x|^{k-s} \sum_{j=0}^{s} \binom{s}{j} j^k 3\epsilon \;\; (\text{since } \overline{P}(f, \epsilon, \delta) \subset P(f, 3\epsilon, \delta)) \\
&= |x_1 - x|^{k-s} M \epsilon
\end{aligned}$$

where M is a constant that does not depend on ϵ or δ. The induction is complete as is the proof. □

Now we are ready to prove our main theorem.

Theorem 1.1.20 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Suppose $0 < s < k$ and $\delta > 0$. Then f_s is $(k-s)$ times Peano differentiable with respect to $\overline{P}(f, 1, \delta)$ with the expected value; i.e., $(f_s|_{\overline{P}})_{k-s}(x) = f_k(x)$.*

Proof: Let $x \in \bar{P}(f, 1, \delta)$ and let $1 > \epsilon > 0$ be given. Then there is a $0 < \eta < \delta$ such that $|\epsilon_k(x, h)| < \epsilon$ whenever $|h| < \eta$. Let M be the constant from Theorem 1.1.19. Let $x_1 \in \overline{P}(f, 1, \delta)$ so that $|x_1 - x| < \frac{\eta}{k+1}$. Let $t = (x_1 - x)\epsilon^{\frac{1}{k}}$. Then for $j = 0, 1, \ldots, k-1$ we have

$$|x_1 - x + jt| = (1 + j\epsilon^{\frac{1}{k}})|x_1 - x| < k|x_1 - x| < \eta. \text{ Hence} \tag{7}$$

$$|\epsilon_k(x, x_1 - x + jt)| \le \epsilon \text{ and} \tag{8}$$

$$|\epsilon_k(x_1, jt)| \le 3. \text{ (Since } \overline{P}(f,1,\delta) \subset P(f,3,\delta) \text{ and } k|t| \le k|x_1 - x| < \eta < \delta.) \tag{9}$$

By Theorem 1.1.17

$$\left|f_s(x_1) - \sum_{l=s}^{k} \frac{(x_1 - x)^{l-s}}{(l-s)!} f_l(x)\right| =$$

$$\left|\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} \sum_{i=s+1}^{k} \frac{j^i}{i!} t^{i-s} \left(\sum_{l=i}^{k} \frac{(x_1 - x)^{l-i}}{(l-i)!} f_l(x) - f_i(x_1)\right) + \right.$$

$$\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\frac{(x_1 - x + jt)^k}{t^s}\epsilon_k(x, x_1 - x + jt) -$$

$$\left. t^{k-s}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j} j^k \epsilon_k(x_1, jt)\right| \quad \text{and by Theorem 1.1.19}$$

$$\left|f_s(x_1) - \sum_{l=s}^{k} \frac{(x_1 - x)^{l-s}}{(l-s)!} f_l(x)\right| \le$$

$$\sum_{j=0}^{s}\binom{s}{j} \sum_{i=s+1}^{k} \frac{j^i}{i!}|x_1 - x|^{i-s}\epsilon^{\frac{i-s}{k}} M|x_1 - x|^{k-i} +$$

$$\sum_{j=0}^{s}\binom{s}{j}|x_1 - x|^{k-s}\frac{(1 + j\epsilon^{1/k})^k}{\epsilon^{\frac{s}{k}}}|\epsilon_k(x, x_1 - x + jt)| +$$

$$|x_1 - x|^{k-s}\epsilon^{\frac{k-s}{k}}\sum_{j=0}^{s}\binom{s}{j} j^k |\epsilon_k(x_1, jt)| \quad \text{and by (8) and (9)}$$

$$\le \sum_{j=0}^{s}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}|x_1 - x|^{k-s}\epsilon^{\frac{i-s}{k}} M + \sum_{j=0}^{s}\binom{s}{j}|x_1 - x|^{k-s}\frac{(1+j)^k}{\epsilon^{\frac{s}{k}}}\epsilon +$$

$$|x_1 - x|^{k-s}\epsilon^{\frac{k-s}{k}}\sum_{j=0}^{s}\binom{s}{j} j^k 3 = |x_1 - x|^{k-s} M \sum_{j=0}^{s}\binom{s}{j}\sum_{i=s+1}^{k}\frac{j^i}{i!}\epsilon^{\frac{i-s}{k}} +$$

$$|x_1 - x|^{k-s}\sum_{j=0}^{s}\binom{s}{j}(1+j)^k\epsilon^{\frac{k-s}{k}} + |x_1 - x|^{k-s} 3\sum_{j=0}^{s}\binom{s}{j} j^k \epsilon^{\frac{k-s}{k}}.$$

Since ϵ was arbitrary, we have that

$$\lim_{x_1 \in \overline{P}, x_1 \to x} \frac{f_s(x_1) - \sum_{l=s}^{k} \frac{(x_1 - x)^{l-s}}{(l-s)!} f_l(x)}{(x_1 - x)^{k-s}} = 0.$$

Therefore the assertion of the theorem is proved. □

Theorem 1.1.20, together with the simple observation that $\cup_{n=1}^{\infty} P_k(f,1,1/n) = \mathbf{R}$ has many applications.

Corollary 1.1.21 *Let f be a function defined on $\mathbf{R}$ such that f_k exists for each $x \in \mathbf{R}$. Then f_k is the composite derivative of f_{k-1}.*

Definition 1.1.22 *A function f is said to have a k-th approximate Peano derivative at x if there exist numbers $f_{ap(1)}(x)$, $f_{ap(2)}(x), \ldots, f_{ap(k)}(x)$ and a set V with density 1 at 0, such that*

$$f(x+h) = f(x) + hf_{ap(1)}(x) + \cdots + \frac{h^k}{k!} f_{ap(k)}(x) + \epsilon_k(x,h) \tag{10}$$

where $\epsilon_k(x,h) \to 0$ as $h \in V$, $h \to 0$.

For $k = 1$ we have the definition of the approximate derivative.

Corollary 1.1.23 *Let f be a function defined on $\mathbf{R}$ such that f_k exists for each $x \in \mathbf{R}$. Then f_s is almost everywhere $(k-s)$ times approximately Peano differentiable with the expected values; i.e., $(f_s)_{ap(k-s)}(x) = f_k(x)$ for $s = 1, \ldots, k-1$.*

Proof: Let x be a point of density of $P = P_k(f,1,\frac{1}{n})$. By Theorem 1.1.20, $(f_s|_P)_{k-s}(x) = f_k(x)$. Since x is a point of density of P, we see that $(f_s)_{ap(k-s)}$ exists at x and equals $f_k(x)$. Finally the Lebesgue Density Theorem and the fact that $\cup_{n=1}^{\infty} P_k(f,1,\frac{1}{n}) = \mathbf{R}$ proves the corollary. □

The case $s = k-1$ was proved by Zygmund and Marcinkiewicz. (See [20], page 77.) Corollary 1.1.23 can be regarded as a generalization of their result.

Corollary 1.1.24 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then $f_k \in [\Delta']$, and hence*

(i) there are g, h and q in Δ such that $h', q' \in [C]$ and $f_k = g' + hq'$,

(ii) there is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f_k = \varphi + \psi$,

(iii) there is a dense open set T such that f_k is a derivative on T and f_k is a derivative on $\mathbf{R} \setminus T$ with respect to $\mathbf{R} \setminus T$.

We will end this chapter with a different decomposition of $\mathbf{R}$ into closed sets so that f_k is the composite derivative of f_{k-1}.

Definition 1.1.25 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$ and let*

$$H(f, M, \delta) = \{x : \left|\frac{k-1}{2} f_k(x) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \epsilon_k(x, jt)\right| \leq M \text{ for } |t| < \delta\}$$

where M and δ are positive constants.

Lemma 1.1.26 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then for any $\delta > 0$ we have $\cup_{M=1}^{\infty} H(f, M, \delta) = \mathbf{R}$.*

Proof: The assertion follows from the fact that $\epsilon_k(x, jt)$ is a continuous function of t for $j = 0, 1, \ldots, k-1$. $\square$

Theorem 1.1.27 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then $H = H(f, M, \delta)$ is closed and f_{k-1} is differentiable on H relative to H with $f_{k-1}|'_H(x) = f_k(x)$, also $|f_k(x)| \leq 2M$ for every $x \in H$.*

Proof: Let $x \in \overline{H}$. Let $1 > \epsilon > 0$ be given. There is $0 < \eta < \delta$ such that $|\epsilon_k(x,h)| < \epsilon$ whenever $|h| < \eta$. Let $x_n \in H$ so that $|x_n - x| < \frac{\eta}{k}$. Then for $t = (x_n - x)\epsilon^{\frac{1}{k}}$ we have $|t| < \delta$ and $|x_n - x + jt| < \eta$, for $j = 0, 1, \dots, k-1$. By the formula from Theorem 1.1.17 applied with $s = k-1$

$$f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x)f_k(x) = t\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}f_k(x) +$$
$$\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\frac{(x_n - x + jt)^k}{t^{k-1}}\epsilon_k(x, x_n - x + jt) -$$
$$t\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k(f_k(x_n) + \epsilon_k(x_n, jt)).$$

By Lemma 1.1.3 we have

$$\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x) = \frac{t}{x_n - x}\frac{k-1}{2}f_k(x) +$$
$$\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\frac{(x_n - x + jt)^k}{t^{k-1}(x_n - x)}\epsilon_k(x, x_n - x + jt) - \tag{11}$$
$$\frac{t}{x_n - x}\left(\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\epsilon_k(x_n, jt)\right).$$

Since $\frac{t}{x_n - x} = \epsilon^{\frac{1}{k}}$, from (11) we have

$$\left|\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x)\right| \le \epsilon^{\frac{1}{k}}\frac{k-1}{2}|f_k(x)| +$$
$$\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1 + j\epsilon^{\frac{1}{k}})^k}{\epsilon^{\frac{k-1}{k}}}|\epsilon_k(x, x_n - x + jt)| +$$
$$\epsilon^{\frac{1}{k}}\left|\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\epsilon_k(x_n, jt)\right|. \tag{12}$$

Since $|\epsilon_k(x, x_n - x + jt)| < \epsilon$ and since $x_n \in H(f, M, \delta)$, the left hand side of (12) is

$$\le \epsilon^{\frac{1}{k}}\frac{k-1}{2}|f_k(x)| + \sum_{j=0}^{k-1}\binom{k-1}{j}(1 + j\epsilon^{\frac{1}{k}})^k\epsilon^{\frac{1}{k}} + \epsilon^{\frac{1}{k}}M.$$

Therefore if $\{x_n\}$ is a sequence in H converging to x, then

$$\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x) \to 0 .$$

Let $x \in \overline{H}$, $\{x_n\}$ a sequence in H so that $x_n \to x$ and let $0 \neq |t| < \delta$. Then (11) yields

$$\left| t\left(\frac{k-1}{2} f_k(x) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \frac{(x_n - x + jt)^k}{t^k} \epsilon_k(x, x_n - x + jt)\right)\right| \leq$$

$$\left| t\left(\frac{k-1}{2} f_k(x_n) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \epsilon_k(x_n, jt)\right)\right| +$$

$$|f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x) f_k(x)| \leq$$

$$|t|M + |f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x) f_k(x)|. \tag{13}$$

Letting $n \to \infty$ the left hand side of (13) becomes

$$\left| t\left(\frac{k-1}{2} f_k(x) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \epsilon_k(x, jt)\right)\right|$$

while the right hand side of (13) is $|t|M$. Hence $x \in H$.

That $|f_k(x)| \leq 2M$ on H follows from Definition 1.1.25 taking $t = 0$. $\square$

Theorem 1.1.27 and Lemma 1.1.26 combine to say that f_k is the composite derivative of f_{k-1}, a result that we already established. But this can be regarded as a simpler proof of that result, because the only tool we used was a special case of Theorem 1.1.17, whose proof is even more elementary than the proof of Theorem 1.1.17. On the other hand the sets $H(f, M, \delta)$ are already closed.

CHAPTER II

2.1 Peano and path derivatives

We will start this chapter with the notion of a path derivative that was introduced in [3].

Definition 2.1.1 *Let $x \in \mathbf{R}$. A path leading to x is a set $E_x \subset \mathbf{R}$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths is a collection $E = \{E_x : x \in \mathbf{R}\}$ such that each E_x is a path leading to x.*

Definition 2.1.2 *Let $F : \mathbf{R} \to \mathbf{R}$ and let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths. If*

$$\lim_{y \in E_x, y \to x} \frac{F(y) - F(x)}{y - x} = f(x)$$

is finite, then we say that F is E-differentiable at x and write $F'_E(x) = f(x)$. If F is E-differentiable at every point x, then we say simply that F is E-differentiable; we call F an E-primitive and f an E-derivative.

Definition 2.1.3 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths.(If E has any of these properties at each point, then we say that E has that property.)*

E is said to be bilateral at x if x is a bilateral point of accumulation of E_x.

E is said to be nonporous at x if E_x has left and right porosity 0 at x.

The basic definition of porosity of a set E at x from the right (left) is the value $\limsup_{r \to 0+} l(x, r, E)/r$, where $l(x, r, E)$ denotes the length of the largest interval contained in the set $(x, x + r) \cap (\mathbf{R} \setminus E)$ $((x - r, x) \cap (\mathbf{R} \setminus E))$. Porosity 0 at x means both right and left porosity 0. Note that a nonporous system is necessarily bilateral.

Definition 2.1.4 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths. E will be said to satisfy the condition listed below if there is associated with E a positive function δ on $\mathbf{R}$ so that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$, the sets E_x and E_y intersect in the stated fashion:*

i) intersection condition I.C.: $E_x \cap E_y \cap [x, y] \neq \emptyset$;

ii) internal intersection condition I.I.C.: $E_x \cap E_y \cap (x, y) \neq \emptyset$;

iii) external intersection condition E.I.C.:

$E_x \cap E_y \cap (y, 2y - x) \neq \emptyset$ and $E_x \cap E_y \cap (2x - y, x) \neq \emptyset$

We will prove that for every k-th Peano derivative f_k there is a nonporous bilateral system of paths E satisfying I.C. and I.I.C. conditions, for which f_k is the E-derivative of f_{k-1}. In this chapter we will prove that E satisfies only the I.C. condition. To show this first we will prove the following theorem due to Mařík. (See [13].)

Theorem 2.1.5 *Let $k \in \mathbf{N}$, $x \in \mathbf{R}$. Suppose that a function f has a k-th Peano derivative at x. Define $P(y) = \sum_{i=0}^{k}(y - x)^i \frac{f_i(x)}{i!}$ $(y \in \mathbf{R})$. Let $\epsilon > 0$, $\eta > 0$. Then there is a $\delta > 0$ such that if I is a subinterval of $(x - \delta, x + \delta)$, j an integer with $0 < j \leq k$ and if either $f^{(j)} \leq P^{(j)}$ on I or $f^{(j)} \geq P^{(j)}$ on I, then $m(\{y \in I : |f^{(j)}(y) - P^{(j)}(y)| \geq \epsilon|y - x|^{k-j}\}) \leq \eta \cdot (m(I) + d(x, I))$. (Here m denotes Lebesgue measure and $d(x, I)$ denotes the distance from x to I.)*

In order to prove Theorem 2.1.5 we need two lemmas.

Lemma 2.1.6 *Let f be a monotone differentiable function on a bounded interval I. Let $\epsilon > 0$, $\beta > 0$ and let $m\{x \in I : |f'(x)| \geq \epsilon\} \geq \beta$. Then there is an interval $J \subset I$ such that $m(J) = \beta/4$ and that $|f| \geq \epsilon\beta/4$ on J.*

Proof: We may suppose that $f' \geq 0$ on I. Let (a,b) be the interior of I. There is a $c \in [a,b]$ such that $f \leq 0$ on (a,c) and $f \geq 0$ on (c,b). Set $B = \{x \in I : f'(x) \geq \epsilon\}$. If $m(B \cap (c,b)) \geq \beta/2$ and if $x \in (b - \beta/4, b)$, then $f(x) \geq \int_c^x f' \geq \epsilon \cdot m(B \cap (c,x)) \geq \epsilon \cdot (m(B \cap (c,b)) - (b-x)) \geq \epsilon \cdot (\beta/2 - \beta/4) = \epsilon\beta/4$. If $m(B \cap (a,c)) \geq \beta/2$, then, analogously, $f \leq -\epsilon\beta/4$ on $(a, a+\beta/4)$. $\square$

Lemma 2.1.7 *Let I be a bounded interval and let j be a natural number. Let g be a function such that either $g^{(j)} \geq 0$ on I or $g^{(j)} \leq 0$ on I. Let $\epsilon > 0$, $\beta > 0$ and let $m\{x \in I : |g^{(j)}(x)| \geq \epsilon\} \geq \beta$. Then there is an interval $J \subset I$ such that $m(J) = \beta/4^j$ and that $|g| \geq \epsilon\beta^j/4^{1+\cdots+j}$ on J.*

Proof: The assertion follows by induction from Lemma 2.1.6. $\square$

Proof of Theorem 2.1.5:

Let $g = f - P$, and let $\alpha = 4^{1+\cdots+k}$. There is a $\delta > 0$ such that for each $y \in (x-\delta, x+\delta)$ we have

$$3^k \alpha |g(y)| \leq \epsilon \eta^k |y - x|^k. \tag{1}$$

Now let I be a subinterval of $(x-\delta, x+\delta)$ and let j be an integer, $0 < j \leq k$. Let $B = \{y \in I : |g^{(j)}(y)| \geq \epsilon |y-x|^{k-j}\}$, $\beta = \frac{1}{3}m(B)$. Suppose that $\beta > 0$. Let $C = B \setminus (x - \beta, x + \beta)$. Now $|g^{(j)}| \geq \epsilon\beta^{(k-j)}$ on C and $m(C) \geq \beta$. If either $g^{(j)} \geq 0$ on I or $g^{(j)} \leq 0$ on I, then by Lemma 2.1.7, there is an interval $J \subset I$ such that $m(J) = \beta/4^j$ and that

$$|g| \geq \frac{1}{\alpha}\epsilon\beta^{k-j} \cdot \beta^j = \frac{1}{\alpha}\epsilon\beta^k \text{ on } J. \tag{2}$$

Together (1) and (2) yield $(3\beta)^k \leq \eta^k |y-x|^k$ for every $y \in J$. Hence $m(B) \leq \eta d(x, I)$.

$\square$

Definition 2.1.8 *A real valued function f defined on an interval I is said to have the intermediate value property if whenever x_1 and x_2 are in I, and y is any number between $f(x_1)$ and $f(x_2)$, there is a number x_3 between x_1 and x_2 such that $f(x_3) = y$. A function having the intermediate value property is called a Darboux function.*

It is known that a k-th Peano derivative, f_k, is a Darboux function. Also it is known that if f_k is bounded either from above or below, then the k-th ordinary derivative, $f^{(k)}$, exists with the obvious equality, $f^{(k)} = f_k$. In the next theorem we will only assume that these two properties hold for any l-th Peano derivative where $0 \le l \le k-1$. We know that any continuous function is Darboux, so for $k = 1$ the above assumptions trivially hold.

Theorem 2.1.9 *Let k, $l \in \mathbf{N}$, with $l \le k-1$. Assume for each function g defined on an interval I having an l-th Peano derivative, g_l, on I, g_l is Darboux and if $g_l \ge 0$ on I then $g_l = g^{(l)}$ on I. Suppose f is a function defined on $\mathbf{R}$ so that f_k exists for each $x \in \mathbf{R}$. Then there is a bilateral nonporous system of paths $E = \{E_x : x \in \mathbf{R}\}$ satisfying the I.C. condition such that f_k is the E-derivative of f_{k-1}.*

We will need some lemmas before we prove this theorem.

Lemma 2.1.10 *Suppose that the assumptions of Theorem 2.1.9 hold. Then for every $\epsilon > 0$ and $\eta > 0$ there is a $\delta > 0$ such that if I is a closed subinterval of $(x-\delta, x+\delta)$ with x not in I such that*

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| \ge \epsilon \tag{3}$$

for all $y \in I$, then $m(I) \le \eta d(x, I)$.

Proof: Let δ be chosen according to Theorem 2.1.5 applied with η replaced by $\eta_1 = \eta/(1+\eta)$ and with $j = k-1$. Let I be as above, and let $g(y) = f(y) -$

$y^{k-1}\frac{f_{k-1}(x)}{(k-1)!} - (y-x)^k\frac{f_k(x)}{k!}$. Then g has a $(k-1)$-th Peano derivative and $g_{k-1}(y) = f_{k-1}(y) - f_{k-1}(x) - (y-x)f_k(x)$. So by assumptions g_{k-1} is a Darboux function. By (3) $|g_{k-1}(y)| \geq \epsilon|y-x|$ on I. Since x is not in I, $|g_{k-1}(y)| > 0$ on I and since g_{k-1} is a Darboux function, we have either $g_{k-1} > 0$ on I or $-g_{k-1} > 0$ on I. Hence by the assumptions, g_{k-1} is the $(k-1)$-th ordinary derivative of g on I. Therefore f is $(k-1)$ times ordinarily differentiable on I and by the uniqueness of Peano derivatives, $f^{(k-1)} = f_{k-1}$ on I. Now we can apply Theorem 2.1.5 with $j = k-1$, which gives that $m(I) \leq \eta_1 \cdot (m(I) + d(x,I))$. Hence $m(I) \leq \eta d(x,I)$. □

Next we will prove a lemma using ideas from the proof of 3.6.1 in [3].

Lemma 2.1.11 *Under the assumptions of Theorem 2.1.9, for each point $x \in I$ there is a path E_x leading to x and nonporous at x so that*

$$\lim_{y \in E_x, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} = f_k(x).$$

Proof: For each $\epsilon > 0$ let $\delta(\epsilon)$ be as in Lemma 2.1.10 applied with $\eta = \epsilon/2$ and let $\{\delta_l\}$ be a sequence so that $0 < \delta_l \leq \delta(1/l)$ and $\delta_{l+1} < \delta_l/2$. Set

$$E'_x = \{x\} \cup \bigcup_{l=1}^{\infty} \{y > \delta_{l+2} : \left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| < 1/l\}.$$

It is certainly true that

$$\lim_{y \in E'_x, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} = f_k(x).$$

(Although this assertion is true if x is not a point of accumulation of E'_x, in fact we will prove that E'_x is nonporous from the right at x.)

Suppose E'_x is porous from the right at x. Then there must exist a number $1/2 < \theta < 1$ and a sequence of numbers $h_l \downarrow 0$ with $[x + \theta h_l, x + h_l] \cap E'_x = \emptyset$ for every index l. Choose an integer l_0 larger that $(1-\theta)^{-1}$ (i.e. so that if $l \geq l_0$, then $1 - 1/l > \theta$) and let j_0 be the first index for which $h_{j_0} < \delta_{l_0}$. Fix l so that $\delta_{l+1} \leq h_{j_0} < \delta_l$, and note that $l \geq l_0$.

Since $h_{j_0} < \delta_l \leq \delta(1/l)$, by Lemma 2.1.10, there must be a point z with $x + h_{j_0}(1 - 1/l) \leq z \leq x + h_{j_0}$ such that $\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z-x} - f_k(x)\right| < 1/l$.

We then have the inequalities,

$$x + \delta_{l+2} < x + \frac{1}{2}\delta_{l+1} < x + \theta\delta_{l+1} \leq x + \theta h_{j_0} < x + (1 - 1/l)h_{j_0} \leq z \leq x + h_{j_0}.$$

From this, then, we see that $\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z-x} - f_k(x)\right| < 1/l$ and $x + \delta_{l+2} < z$, so $z \in E'_x$. But also $x + \theta h_{j_0} < z \leq x + h_{j_0}$, so $z \in [x + \theta h_{j_0}, x + h_{j_0}]$. This contradicts the fact that $E'_x \cap [x + \theta h_j, x + h_j] = \emptyset$ for all j. Similarly we define a path E''_x leading to x that is nonporous from the left at x. The path $E_x = E'_x \cup E''_x$ has the desired properties. $\square$

Now we are ready to prove Theorem 2.1.9.

Proof: For each $x \in \mathbf{R}$ let E'_x be a path satisfying the conclusions of Lemma 2.1.11. We will define the system of paths $E = \{E_x : x \in \mathbf{R}\}$ as follows:

For $x \in \mathbf{R}$ let $E_x = E'_x \cup P(f, 1, \delta(x))$ where $\delta(x)$ is such that $x \in P(f, 1, \delta(x))$. That E is nonporous (therefore bilateral) follows directly from Lemma 2.1.11. Also Lemma 2.1.11 and Theorem 1.1.19, imply that f_{k-1} is E differentiable with $f_{k-1}|'_E(x) = f_k(x)$ for every $x \in \mathbf{R}$. It remains to prove that E satisfies the intersection condition I.C.. We will prove that for any two distinct points x and y, $E_x \cap E_y \cap [x, y] \neq \emptyset$ which is stronger than the I.C. condition.

Let x and y be any two distinct points. If $\delta(x) \leq \delta(y)$, then $P(f, 1, \delta(y)) \subset P(f, 1, \delta(x))$ and hence $y \in E_x$. If $\delta(x) \geq \delta(y)$, then $P(f, 1, \delta(x)) \subset P(f, 1, \delta(y))$ and hence $x \in E_y$. Therefore $E_x \cap E_y \cap [x, y] \neq \emptyset$. Hence E satisfies the I.C. condition. This completes the proof of the theorem . $\square$

Now we will prove that we can drop the assumptions concerning the arbitrary function g from Theorem 2.1.9. In order to do that we list some theorems from [3] about path derivatives.

Theorem 2.1.12 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths that is bilateral and satisfies the I.C. condition. If f is an exact E-derivative and is Baire 1, then f has the Darboux property.*

Proof: This is Theorem 6.4 in [3]. $\square$

Theorem 2.1.13 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths that is bilateral and satisfies the I.C. condition. If $\underline{F}'_E \geq 0$ on $[a,b]$, then F is nondecreasing on the interval $[a,b]$.*

Proof: See 4.7.1 in [3]. $\square$

Theorem 2.1.14 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths and suppose F is monotonic. If E is nonporous at a point x, then*

$$\underline{F}'_E(x) = \underline{F}'(x) \text{ and } \overline{F}'_E(x) = \overline{F}'(x).$$

Proof: See Theorem 4.4.3 in [3]. $\square$

Theorem 2.1.15 *Let f be a function defined on $\mathbf{R}$ and let $k \in \mathbf{N}$. Suppose $f_k(x)$ exists for each $x \in \mathbf{R}$. Then there is a bilateral nonporous system of paths $E = \{E_x : x \in \mathbf{R}\}$ satisfying the I.C. condition such that f_k is the E-derivative of f_{k-1}.*

Proof: The proof is by induction on k. For $k = 1$ there is nothing to prove. Let $1 \le l \le k-1$, and let a function g defined on some closed interval I have a l-th Peano derivative on I. Suppose the assertion of the theorem is true for every $1 \le j \le k-1$, and every function h defined on some closed interval J, so that h_j exists on J. (Note that we can restrict ourselves only to closed subintervals because we can always extend h to $\mathbf{R}$ so that h_j exists on $\mathbf{R}$. For example if $J = [a, b]$, then, we can define $h(y) = \sum_{i=0}^{j}(y-x)^i \frac{f_i(a)}{i!}$ for $y \in (-\infty, a)$ and $h(y) = \sum_{i=0}^{j}(y-x)^i \frac{f_i(b)}{i!}$ for $y \in (b, \infty)$.) By Theorem 1.1.7 g_l is a Baire 1 function. By the induction hypothesis and Theorem 2.1.12, g_l is a Darboux function. Suppose that $g_l \ge 0$ on I. Again by the induction hypothesis but now using Theorem 2.1.13, g_{l-1} is nondecreasing on I. By Theorem 2.1.14 $g'_{l-1} = g_l$ on I. Also there is an α such that $g_{l-1} - \alpha \ge 0$ on I. Let $h(x) = g(x) - \alpha\frac{x^{l-1}}{(l-1)!}$. Then $h_{l-1} = g_{l-1} - \alpha$ and hence $h_{l-1} \ge 0$ on I. Proceeding as before $h'_{l-2} = h_{l-1}$ on I. This implies $g'_{l-2} = g_{l-1}$ on I. Continuing in this fashion one can deduce that $g^{(l)}$ exists on I. Now we can apply Theorem 2.1.9. □

Corollary 2.1.16 *Let f be a function defined on $\mathbf{R}$ such that f_k exists for each $x \in \mathbf{R}$. Then f_k is a Darboux function.*

Proof: The assertion follows directly from Theorems 2.1.15, 2.1.12 and Theorem 1.1.7. □

Definition 2.1.17 *A perfect road of a function f at a point x is a perfect set P such that*

(1) x is a bilateral point of accumulation of P

(2) $f|P$ is continuous at x.

The assertion of the next corollary follows directly from the properties of Baire 1, Darboux functions. (See [2].)

Corollary 2.1.18 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then*

(1) For each x, there exist sequences $x_n \nearrow x$ and $y_n \searrow x$ such that $f_k(x) = \lim_{n\to\infty} f_k(x_n) = \lim_{n\to\infty} f_k(y_n)$.

(2) For each x

$$f_k(x) \in [\liminf_{z\to x^-} f_k(z), \limsup_{z\to x^-} f_k(z)] \cap [\liminf_{z\to x^+} f_k(z), \limsup_{z\to x^+} f_k(z)].$$

(3) For each real number a, the sets $\{f_k \leq a\}$ and $\{f_k \geq a\}$ have compact components.

(4) The graph of f_k is connected.

(5) The function f_k has a perfect road at each point.

(6) Each of sets $\{f_k < a\}$ and $\{f_k > a\}$ is bilaterally c-dense in itself. (See [2].)

(7) Each of sets $\{f_k < a\}$ and $\{f_k > a\}$ is bilaterally dense in itself.

Definition 2.1.19 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths and F a function on* $\mathbf{R}$. *We say that F has the monotonicity property relative to E if for any interval $[a, b]$ the conditions $F'_E(x)$ exists a.e. in $[a, b]$ and $F'_E(x) \geq \alpha$ a.e. in $[a, b]$ (resp. $F'_E(x) \leq \alpha$) imply that the function $F(x) - \alpha x$ (resp. $\alpha x - F(x)$) is nondecreasing on $[a, b]$.*

Theorem 2.1.20 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths and let F be a function. If E is bilateral and satisfies the intersection condition, and F is E-differentiable, then F has the monotonicity property relative to E.*

Proof: See Theorem 6.6.1 in [3]. □

Corollary 2.1.21 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Let $[a, b]$ be an interval, and α be any constant. If $f_k \geq \alpha$ (or $f_k \leq \alpha$) on $[a, b]$, then*

a) $f_{k-1}(x) - \alpha x$ $(\alpha x - f_{k-1}(x)\ )$ is nondecreasing and continuous on $[a, b]$

b) $f^{(k)}$ exists and $f^{(k)} = f_k$ on $[a, b]$.

Proof: The assertion follows directly from Theorems 2.1.15, 2.1.20 and 2.1.14. □

Corollary 2.1.21 was first proved by Oliver in [10] and Corominas in [4]. See also Verblunsky [15].

Definition 2.1.22 *Let f be a function defined on* $\mathbf{R}$. *If for any interval (a, b), $f^{-1}(a, b) \neq \emptyset$ implies $m(\{x : f(x) \in (a, b)\}) > 0$, then we say that f has the Denjoy property.*

Theorem 2.1.23 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths and let F be an E-differentiable function that has the monotonicity property relative to E. If F'_E is Darboux Baire 1, then F'_E has the Denjoy property.*

Proof: This is Theorem 6.7 in [3]. □

Corollary 2.1.24 *Let f be a function defined on* $\mathbf{R}$ *such that f_k exists for each $x \in \mathbf{R}$. Then f_k has the Denjoy property.*

Proof: The assertion follows directly from Theorems 2.1.15, 2.1.20, 2.1.23 and Corollary 2.1.16. □

Corollary 2.1.21 first was proved by Weil in [17].

Theorem 2.1.25 *Let $E = \{E_x : x \in \mathbf{R}\}$ be a nonporous system of paths satisfying the intersection condition. Suppose that F is an E-differentiable function with F'_E Baire 1. If F'_E attains the values M and $-M$ on an interval I_0, then there is a subinterval I of I_0 on which F is differentiable and F' attains both values M and $-M$.*

Proof: This is Theorem 8.1 in [3].

An immediate consequence of Theorems 2.1.25 and 2.1.15 is the following corollary first proved by O'Malley and Weil in [12].

Corollary 2.1.26 *Suppose $f_k(x)$ exists for all x in I_0 and let $M \geq 0$. If f_k attains both M and $-M$ on I_0, then there is a subinterval I of I_0 on which $f_k = f^{(k)}$ and $f^{(k)}$ attains both M and $-M$ on I.*

This corollary has some nice and immediate applications. The reader is referred to [12] for the details and proofs.

CHAPTER III

In Chapter I we have shown that for any k-th Peano derivative, f_k, defined on $\mathbf{R}$, there is a countable decomposition $\{H_n\}$ of $\mathbf{R}$ into closed sets and a sequence of differentiable functions $\{v_n\}$ so that for each $n \in \mathbf{N}$, $v_n'|_{H_n} = f_k$. In [10] O'Malley showed that the same holds for approximate derivatives. Moreover he proved that for any approximate derivative there is a decomposition of $\mathbf{R}$ into perfect sets with the above property. In this chapter we will show that the same holds for Peano derivatives. Also we will show that any k-th Peano derivative, f_k, is a path derivative of f_{k-1} with respect to a system of paths that is bilateral, nonporous and that satisfies the internal intersection condition I.I.C.. This will enable us to give a positive answer to the question posed by C. Weil, regarding the relationship between Peano and selective derivatives, (See [19].) namely, the last result of this chapter is that f_k is a selective derivative of f_{k-1}.

3.1 Relationship between f_k and f_{k-1}

We will begin this section with a very well known lemma.

Lemma 3.1.1 *Let $n \in \mathbf{N}$ and let f and g be functions on $\mathbf{R}$ having n-th Peano derivatives $f_n(x)$ and $g_n(x)$ at some point x. Then the function fg has n-th Peano derivative at x and*

$$(fg)_n(x) = \sum_{j=0}^{n} \binom{n}{j} f_j(x) g_{n-j}(x).$$

Proof: Let $x \in [a,b]$ be such that $f_n(x)$ and $g_n(x)$ exist. Thus the following formulas hold

$f(x+h) = f(x) + hf_1(x) + \cdots + \frac{h^n}{n!} f_n(x) + h^n \epsilon_n(x,h)$

$g(x+h) = g(x) + hg_1(x) + \cdots + \frac{h^n}{n!} g_n(x) + h^n \bar{\epsilon}_n(x,h)$

where $\epsilon_n(x,h)$ and $\bar{\epsilon}_n(x,h)$ tend toward 0 as $h \to 0$. Then

$$f(x+h)g(x+h) = \sum_{j=0}^{n} \frac{h^j}{j!} \sum_{i=0}^{j} j! \frac{f_i(x)}{i!} \frac{g_{j-i}(x)}{(j-i)!} +$$
$$h^n \epsilon_n(x,h) g(x+h) + h^n \bar{\epsilon}_n(x,h)(f(x+h) - \epsilon_n(x,h)) =$$
$$\sum_{j=0}^{n} \frac{h^j}{j!} \sum_{i=0}^{j} \binom{j}{i} f_i(x) g_{j-i}(x) + h^n \epsilon'_n(x,h)$$

where $\epsilon'_n(x,h) = \epsilon_n(x,h)g(x+h) + \bar{\epsilon}_n(x,h)(f(x+h) - \epsilon_n(x,h))$

Since obviously $\lim_{h\to 0} \epsilon'_n(x,h) = 0$ we have

$(fg)_n(x)$ exists and $(fg)_n(x) = \sum_{j=0}^{n} \binom{n}{j} f_j(x) g_{n-j}(x)$.□

Lemma 3.1.2 *Let f and g be functions on* $\mathbf{R}$ *such that the n-th Peano derivative, $f_n(x)$, and the n-th ordinary derivative, $g^{(n)}(x)$, exist at some point x. Then*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) = f_n(x)g(x).$$

Proof: By Lemma 3.1.1

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) =$$
$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x) (g^{(j)})_{(n-j-i)}(x) =$$
$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x) g^{(n-i)}(x) =$$
$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{j} \binom{n-j}{i} f_i(x) g^{(n-i)}(x) =$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\binom{n}{i}f_i(x)g^{(n-i)}(x) =$$
$$\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f_i(x)g^{(n-i)}(x) =$$
$$f_n(x)g(x) + \sum_{i=0}^{n-1}\binom{n}{i}(1-1)^{n-i}f_i(x)g^{(n-i)}(x) = f_n(x)g(x). \square$$

Lemma 3.1.3 *Let H be a continuous function in an interval $[a, b]$ containing y. Suppose that H is n times Peano differentiable at each $x \in [a, b]$ and that H_n is m times Peano differentiable at y. Then H is $(n + m)$ times Peano differentiable at y, and $H_{(n+m)}(y) = (H_n)_m(y)$.*

This lemma was first proved by Corominas in [4]. We will use ideas of his proof to prove this lemma, but before we give the proof we need some other properties of Peano derivatives that are known. The following definitions and Lemma 3.1.5 are due to Oliver. (See [10]). We will give a simpler proof of Lemma 3.1.5, than is given in [10].

Definition 3.1.4 *If f has an n-th Peano derivative at each point of an interval $[a, b]$, we say that f satisfies the mean value theorems M_n^k, $k = 0, 1, \ldots, n-1$ (or that $f \in M_n^k$), if for each x and $x + h \in [a, b]$, there is an x' between x and $x + h$ such that:*

$$\frac{f_k(x+h) - f_k(x) - hf_{k+1}(x) - \cdots - \frac{h^{n-k-1}}{(n-k-1)!}f_{n-1}(x)}{\frac{h^{n-k}}{(n-k)!}} = f_n(x'). \qquad (1)$$

When $n = 1$ and $k = 0$, we have the ordinary mean value theorem for first derivatives. The mean value theorem (Lemma 3.1.5 below) is that if f_n exists on some interval $[a, b]$, then $f \in M_n^k$, $k = 0, 1, \ldots, n-1$.

The special case of M_n^k when the left hand side of (1) equals 0, we refer to as Rolle's Theorem, R_n^k. In the usual manner, M_n^k follows from R_n^k by adding a suitable polynomial to f. If f has an n-th Peano derivative on $[a,b]$ and if y and $y+h$ are given in $[a,b]$, for each $x \in [a,b]$ we set

$$g(x) = f(x) - \frac{f_k(y+h) - \sum_{j=k}^{n-1} \frac{h^{j-k}}{(j-k)!} f_j(y)}{h^{n-k}/(n-k)!} \cdot \frac{(x-y)^n}{n!}.$$

It follows immediately that

$$g_k(y+h) - g_k(y) - hg_{k+1}(y) - \cdots - \frac{h^{n-k-1}}{(n-k-1)!} g_{n-1}(y) = 0,$$

i.e., that g satisfies the hypothesis of R_n^k; and that

$$g_n(x) = f_n(x) - \frac{f_k(y+h) - \sum_{j=k}^{n-1} \frac{h^{j-k}}{(j-k)!} f_j(y)}{h^{n-k}/(n-k)!}.$$

Applying the conclusion of R_n^k to g, i.e., replacing x by x' and $g_n(x')$ by 0, the conclusion of M_n^k follows for f.

It is also possible to deduce R_n^k, $k = 0, 1, \ldots, n-2$, from R_n^{n-1} and M_{n-1}^k, as follows. We may write

$$\frac{f_k(x+h) - f_k(x) - \cdots - \frac{h^{n-k-2}}{(n-k-2)!} f_{n-2}(x) - \frac{h^{n-k-1}}{(n-k-1)!} f_{n-1}(x)}{h^{n-k}/(n-k)!} = 0$$

in the form

$$\frac{\frac{f_k(x+h) - f_k(x) - \cdots - \frac{h^{n-k-2}}{(n-k-2)!} f_{n-2}(x)}{h^{n-k-1}/(n-k-1)!} - f_{n-1}(x)}{h/(n-k)} = 0$$

and replace the first ratio, using M_{n-1}^k, to obtain $f_{n-1}(x'') - f_{n-1}(x) = 0$ for some x'' between x and $x+h$. We use R_n^{n-1} to deduce from this last equation the existence of x' between x and x'' for which $f_n(x') = 0$, the conclusion required by R_n^k.

Lemma 3.1.5 *Let $n \in \mathbb{N}$ and let f be a function defined on some interval $[a,b]$ so that f_n exists on $[a,b]$. Then f_n satisfies the mean value theorems M_n^k, $k = 0, 1, \ldots, n-1$, on $[a,b]$.*

Proof: The proof is by induction on n. Since the assertion holds for $n = 1$, the remarks before the lemma show that the induction will be completed by proving $f \in R_n^{n-1}$. So it is enough to prove that if $f_{n-1}(x+h) = f_{n-1}(x)$, then there is an x' between x and $x+h$ for which $f_n(x') = 0$.

We may assume $h > 0$, because a proof for the case $h < 0$ is similar. If f_n is positive on $[x, x+h]$, then, by Corollary 2.1.21, $f^{(n)}$ exists and hence by Rolle's Theorem for derivatives there is an x' between x and $x+h$ so that $f^{(n)}(x') = 0$. Similarly if f_n is negative. If f_n takes on both positive and negative values, then since f_n is Darboux, f_n attains the value zero. The induction is complete. □

Now we are ready to prove Lemma 3.1.3.

Proof: For each $x \in [a, b]$ let

$$T(x) = H(x) - \sum_{j=0}^{n}(x-y)^j \frac{H_j(y)}{j!} - \sum_{i=1}^{m}(x-y)^{n+i}\frac{(H_n)_i(y)}{(n+i)!}. \tag{2}$$

Then $T(y) = T_1(y) = \cdots = T_{n-1}(y) = 0$, $T_n(x) = H_n(x) - \sum_{i=0}^{m}(x-y)^i \frac{(H_n)_i(y)}{i!}$ and $T_n(y) = (T_n)_1(y) = \cdots = (T_n)_m(y) = 0$. Since $T \in M_n^0$, for each $x \in [a, b]$ there is a $c_x \in [a, b]$ between x and y such that

$$T(x) = \frac{(x-y)^n}{n!} T_n(c_x). \tag{3}$$

Since $T_n(y) = (T_n)_1(y) = \cdots = (T_n)_m(y) = 0$, we have

$$T_n(x) = (x-y)^m \epsilon_m(y, x-y) \text{ where } \epsilon_m(y, x-y) \to 0 \text{ as } x \to y. \tag{4}$$

Combining (3) and (4) we get

$$T(x) = (x-y)^{n+m} \frac{(c_x - y)^m}{n!(x-y)^m} \epsilon_m(y, c_x - y).$$

Now (2) becomes

$$H(x) = \sum_{j=0}^{n}(x-y)^j \frac{H_j(y)}{j!} + \sum_{i=1}^{m}(x-y)^{n+i}\frac{(H_n)_i(y)}{(n+i)!} + (x-y)^{n+m}\epsilon'_{n+m}(y, x-y)$$

where $\epsilon'_{n+m}(y, x-y) = \frac{(c_x-y)^m}{n!(x-y)^m}\epsilon_m(y, c_x - y) \to 0$ as $x \to y$. This proves that $H_{n+m}(y)$ exists and equals $(H_n)_m(y)$. $\square$

Lemma 3.1.6 *Let f be defined in an interval $[a,b]$ containing 0. Suppose that the k-th Peano derivative of f at 0 exists, and that the l-th Peano derivative of f exists on $[a,b]$, where k and l are positive integers with $l \le k-1$. Also suppose that $f(0) = f_1(0) = \cdots = f_k(0) = 0$. Let $g(y) = y^{-(k-l)}$. Then the function h defined by*

$$h(y) = \binom{l}{0} f(y)g(y) - \binom{l}{1}\int_0^y f(t)g'(t)\,dt + \cdots +$$

$$(-1)^l \binom{l}{l}\int_0^y \int_0^{x_2} \cdots \int_0^{x_l} f(t)g^{(l)}(t)\,dt \cdots dx_2 \quad \text{for } y \neq 0,$$

and $h(0) = 0$ has an l-th Peano derivative on $[a,b]$.

Moreover

$$h_l(y) = \begin{cases} \frac{f_l(y)}{y^{k-l}} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0\,. \end{cases}$$

Proof: By assumption $f(y) = y^k \epsilon_k(0, y)$. Consequently all of the above integrals are integrals of continuous functions. Hence h is well defined. Moreover for $y \neq 0$, $y \in [a,b]$ $H(y) = \int_0^y \int_0^{x_2} \cdots \int_0^{x_i} f(t)g^{(i)}(t)\,dt \cdots dx_2 \quad i = 1, \ldots, l$ is i times ordinarily differentiable and $H^{(i)}(y) = f(y)g^{(i)}(y)$ for $i = 1, \ldots, l$. By Lemma 3.1.1, $fg^{(i)}$ is $l-i$ times Peano differentiable at y. Therefore by Lemma 3.1.3, H is l times Peano differentiable at y and $H_l(y) = (H^{(i)})_{l-i}(y) = (f(y)g^{(i)}(y))_{(l-i)}$. Hence h is l times Peano differentiable at y and

$$h_l(y) = \sum_{j=0}^{l}(-1)^j \binom{l}{j}(fg^{(j)})_{(l-j)}(y)$$

and by Lemma 3.1.2, $h_l(y) = f_l(y)g(y)$.

It remains to prove that $h_l(0)$ exists and that $h_l(0) = 0$. For $y \neq 0$

$$\frac{h(y)}{y^l} = \frac{1}{y^l}\Big\{\binom{l}{0}y^l\epsilon_k(0,y) + (k-l)\binom{l}{1}\int_0^y t^{l-1}\epsilon_k(0,t)\,dt + \cdots +$$
$$(k-l)(k-l+1)\cdots(k-1)\binom{l}{l}\int_0^y\int_0^{x_2}\cdots\int_0^{x_l}\epsilon_k(0,t)\,dt\cdots dx_2\Big\}.$$

Hence $\lim_{y\to 0}\dfrac{h(y)}{y^l} = 0$. Therefore $h(0) = h_1(0) = \cdots = h_l(0) = 0$. $\square$

Now suppose that f has an l-th Peano derivative in an interval $[a,b]$ containing x, and that $f_k(x)$ exists. Consider a function

$$T(y) = f(y) - f(x) - (y-x)f_1(x) - \cdots - (y-x)^k\frac{f_k(x)}{k!}$$

and its translate $G(t) = T(x+t)$. Then G satisfies the hypothesis of Lemma 3.1.6 and by that lemma the function H defined by

$$H(y) = \binom{l}{0}G(y)g(y) - \binom{l}{1}\int_0^y G(t)g'(t)\,dt + \cdots +$$
$$(-1)^l\binom{l}{l}\int_0^y\int_0^{x_2}\cdots\int_0^{x_l}G(t)g^{(l)}(t)\,dt\cdots dx_2 \quad \text{for } y\neq 0$$

and $H(0) = 0$ has an l-th Peano derivative on $x - [a,b]$. Moreover by the same lemma, $H_l(y) = \begin{cases} \frac{G_l(y)}{y^{k-l}} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0\,. \end{cases}$

But $G_l(t) = T_l(t+x) = f_l(t+x) - f_l(x) - tf_{l+1}(x) - \cdots - t^{k-l}\frac{f_k(x)}{(k-l)!}$. Therefore we have proved the following theorem.

Theorem 3.1.7 *Suppose that a function f defined on an interval $[a,b]$ containing a point x has an l-th Peano derivative on $[a,b]$ and a k-th Peano derivative at x, where $0 \leq l \leq k$. Then the function F defined on $[a,b]$ by*

$$F(y) = \begin{cases} \dfrac{f_l(y) - \sum_{j=0}^{k-l}\frac{(y-x)^j}{j!}f_{l+j}(x)}{(y-x)^{k-l}} & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

is an l-th Peano derivative.

Corollary 3.1.8 *Suppose that a function f defined on an interval $[a,b]$ containing a point x has a $(k-1)$-th Peano derivative on $[a,b]$ and k-th Peano derivative at x. Then there exists a perfect set $P \subset [a,b]$ of positive measure such that x is a bilateral point of accumulation of P and*

$$\lim_{y \in P, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} = f_k(x).$$

Proof: The function F from Theorem 3.1.7, applied with $l = k-1$ is a $(k-1)$-th Peano derivative and hence Baire 1, Darboux and has the Denjoy property. Therefore, by Corollary 2.1.18 there is a perfect set H such that F is continuous at x with respect to H. Since F has the Denjoy property there is a perfect set P of positive measure, containing H, so that F is still continuous at x with respect to P. The set P satisfies the assertion of the corollary. □

3.2 Peano derivatives and Property Z

Property Z was introduced in [18] by Weil. He proved, that f_k has the property Z at every point of $\mathbf{R}$. In [13] Mařík gives a different proof of this fact. Moreover he proved there that a k-th approximate Peano derivative has property Z. Also he generalized this result to an assertion which when specialized to k-th Peano derivatives is the following theorem.

Theorem 3.2.1 *Let j and k be integers, $1 \le j \le k$. Let $x \in \mathbf{R}$ and let f be a function such that $f_k(x)$ exists. Define $P(y) = \sum_{i=0}^{k}(y-x)^i \frac{f_i(x)}{i!}$ for $y \in \mathbf{R}$. Let $\epsilon > 0$, $\eta > 0$. Then there is a $\delta > 0$ with the following property: If I is a subinterval of $(x-\delta, x+\delta)$ such that f_j exists on I and that $|f_j(y) - P^{(j)}(y)| \ge \epsilon |y-x|^{k-j}$ for all $y \in I$, then $m(I) \le \eta d(x, I)$.*

For the special case of k-th Peano derivatives, the proof of Mařík's result is simpler than for approximate k-th Peano derivatives. Moreover the proof given here concludes the case $1 \le j \le k-1$ as a consequence of the case $j = k$. The case $j = k$, using Theorem 3.1.7, is an immediate consequence of property Z for Peano derivatives.

Proof of Theorem 3.2.1: Case $j = k$. Let $g(y) = f(y) - P(y)$, and let $\delta > 0$ be such that

$$|g(y)| < \epsilon(\frac{\eta}{1+\eta})^k \frac{|y-x|^k}{\alpha}. \tag{5}$$

Let I be a subinterval of $(x-\delta, x+\delta)$ such that

$$|g_k(y)| \ge \epsilon \text{ for } y \in I. \tag{6}$$

By the Darboux property, either $g_k(y) \ge \epsilon$ on I, or $g_k \le \epsilon$ on I. By Corollary 2.1.21, $g^{(k)}$ exists on I, and hence by Lemma 2.1.7, there is a subinterval J of I such that

$$|g| \ge \epsilon \cdot \frac{(m(I))^k}{\alpha}. \tag{7}$$

Combining (5) and (7) give $m(I) < \frac{\eta}{1+\eta}|y-x|$ for every $y \in J$. Therefore $m(I) \le \frac{\eta}{1+\eta} \cdot (m(I) + d(x, I))$, hence $m(I) \le \eta \cdot d(x, I)$.

Case $j < k$. By Theorem 3.1.7, the function $h_j(y) = \frac{(f(y)-P(y))_j}{(y-x)^{k-j}}$ for $y \neq x$ and $h_j(x) = 0$, is a j-th Peano derivative, and by what was just proved, for any $\epsilon > 0$ and $\eta > 0$ there is a δ such that whenever I is a subinterval of $(x-\delta, x+\delta)$ and such that $|hj(y) - h_j(x)| \ge \epsilon$ for $y \in I$, then $m(I) \le \eta \cdot d(x, I)$. This is exactly the claim of the theorem for $j < k$. □

This theorem enables us to prove the following analogous of Theorem 2.1.15.

Theorem 3.2.2 *Let $k \in \mathbb{N}$ and let f be a function defined on $\mathbb{R}$ with $f_k(x)$ existing for all $x \in \mathbb{R}$. Then for each integer $1 < r \le k-1$ there is a bilateral nonporous system of paths $E = \{E_x : x \in \mathbb{R}\}$ satisfying the I.C. condition such that*

$$f_k(x) = \lim_{y \in E_x, y \to x} \frac{f_r(y) - \sum_{j=0}^{k-r-1} \frac{(y-x)^j}{j!} f_{r+j}(x)}{(y-x)^{k-r}}.$$

The proof of this theorem is similar to the proof of Lemma 2.1.11 and Theorem 2.1.9. Proof:

Let $\epsilon > 0$ and let $\delta(\epsilon)$ be the δ from Theorem 3.2.1 applied with $\eta = \epsilon/2$. Let $\{\delta_l\}$ be a sequence so that $0 < \delta_l \leq \delta(1/l)$ and $\delta_{l+1} < \delta_l/2$, and define the set E'_x by

$$E'_x = \{x\} \cup \bigcup_{l=1}^{\infty} \{z > x + \delta_{l+2} : \left| \frac{f_r(z) - \sum_{j=0}^{k-r-1} \frac{(z-x)^j}{j!} f_{r+j}(x)}{(z-x)^{k-r}} - f_k(x) \right| < 1/l\}.$$

It is certainly true that

$$f_k(x) = \lim_{y \in E'_x, y \to x} \frac{f_r(y) - \sum_{j=0}^{k-r-1} \frac{(y-x)^j}{j!} f_{r+j}(x)}{(y-x)^{k-r}}.$$

Now we will prove that E'_x is nonporous on the right at x. Suppose not. Then there must exist a number $1/2 < \theta < 1$ and a sequence of numbers $h_l \downarrow 0$ with $(x + \theta h_l, x + h_l) \cap E'_x = \emptyset$ for every index l. Choose an integer l_0 larger that $(1-\theta)^{-1}$ (i.e. so that if $l \geq l_0$, then $1 - 1/l > \theta$) and let j_0 be the first index for which $h_{j_0} < \delta_{l_0}$. Fix l so that $\delta_{l+1} \leq h_{j_0} < \delta_l$, and note that $l \geq l_0$.

Since $h_{j_0} < \delta_l \leq \delta(1/l)$, by Corollary 3.1.8 there must be a point z with $x + (1 - 1/l)h_{j_0} \leq z \leq x + h_{j_0}$ such that

$$\left| \frac{f_r(z) - \sum_{j=0}^{k-r-1} \frac{(z-x)^j}{j!} f_{r+j}(x)}{(z-x)^{k-r}} - f_k(x) \right| < 1/l.$$

We then have the inequalities,

$x + \delta_{l+2} < x + \frac{1}{2}\delta_{l+1} < x + \theta\delta_{l+1} \leq x + \theta h_{j_0} < x + (1 - 1/l)h_{j_0} \leq z \leq x + h_{j_0}$.

From this we see that $\left| \frac{f_r(z) - \sum_{j=0}^{k-r-1} \frac{(z-x)^j}{j!} f_{r+j}(x)}{(z-x)^{k-r}} - f_k(x) \right| < 1/l$ and $x + \delta_{l+2} \leq z$ so $z \in E'_x$. But also $x + \theta h_{j_0} < z \leq x + h_{j_0}$, so $z \in [x + \theta h_{j_0}, x + h_{j_0}]$. This contradicts the fact that $E'_x \cap [x + \theta h_j, x + h_j] = \emptyset$ for all j. Similarly we define a path E''_x leading to x that is nonporous from the left at x.

For each $x \in \mathbf{R}$ we will define the system of paths $E = \{E_x : x \in \mathbf{R}\}$ as follows:

For $x \in \mathbf{R}$ let $E_x = E'_x \cup E''_x \cup P(f,1,\delta(x))$ where $\delta(x)$ is such that $x \in P(f,1,\delta(x))$. E is nonporous (therefore bilateral) follows from the fact that $E'_x \cup E''_x$ is nonporous at x, also Theorem 1.1.19 and what we proved in the first part assure us that

$$f_k(x) = \lim_{y \in E_x, y \to x} \frac{f_r(y) - \sum_{j=0}^{k-r-1} \frac{(y-x)^j}{j!} f_{r+j}(x)}{(y-x)^{k-r}}$$

for every $x \in \mathbf{R}$. It remains only to prove that E satisfies the intersection condition I.C.. We will prove that for any two different points x and y, $E_x \cap E_y \cap [x,y] \neq \emptyset$ which is stronger than the I.C. condition.

Let x and y be any two different points. If $\delta(x) \leq \delta(y)$, then $P(f,1,\delta(y)) \subset P(f,1,\delta(x))$ and hence $y \in E_x$. If $\delta(x) \geq \delta(y)$, then $P(f,1,\delta(x)) \subset P(f,1,\delta(y))$ and hence $x \in E_y$. Therefore $E_x \cap E_y \cap [x,y] \neq \emptyset$. Thus E satisfies the I.C. condition. This completes the proof of the theorem . $\square$

3.3 Peano and selective derivatives

Recall the sets

$$H(f,M,1) = \{x : \left| \frac{k-1}{2} f_k(x) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \epsilon_k(x,jt) \right| \leq M \text{ for } |t| < 1\}$$

where $M \in \mathbf{N}$, from Definition 1.1.25. In Chapter I we showed that these sets are closed, their union is $\mathbf{R}$, and that with respect to these sets f_{k-1} differentiates to $f_k(x)$ and $|f_k(x)| \leq 2M$ for $x \in H(f,M,1)$. (See Theorem 1.1.27.) We have seen several applications of this decomposition. Now we will prove that these sets can be enlarged so that they are perfect, and that still with respect to these enlarged sets, f_{k-1} differentiates to f_k.

Let $y \in H(f,M,1)$ be an isolated point of $H(f,M,1)$. Then there is a $1 > \delta(y) > 0$ so that $(y - 2\delta(y), y + 2\delta(y)) \cap H(f,M,1) = \{y\}$. Let P_y be a perfect set containing y so that y is a bilateral point of accumulation of P_y satisfying

$$\lim_{z \in P_y, z \to y} \frac{f_{k-1}(z) - f_{k-1}(y)}{z-y} = f_k(y)$$

and

$$\left|\frac{f_{k-1}(z)-f_{k-1}(y)}{z-y}-f_k(y)\right| \leq 1 \text{ for every } z \in P_y.$$

Corollary 3.1.8 assures the existence of P_y. If $P_y \cap (y+\frac{1}{n+1}, y+\frac{1}{n}) \neq \emptyset$, for $n \in \mathbb{Z} \setminus \{-1, 0\}$, then by the Baire category theorem there is a perfect set $Q_n(y) \subset P_y \cap (y+\frac{1}{n+1}, y+\frac{1}{n})$, there is $M_n(y) \in \mathbb{N}$ with $Q_n(y) \subset H(f, M_n(y), 1)$. Let

$$Q_y = \bigcup_{n \in \mathbb{Z}\setminus\{-1,0\}} Q_n(y) \cap (y-\delta^2(y), y+\delta^2(y)) \cup \{y\}, \qquad \text{and let}$$

$H_M = H(f, M, 1) \cup \{Q_y : y \in H(f, M, 1),\ y \text{ is isolated in } H(f, M, 1)\}$

Theorem 3.3.1 *H_M is a perfect set, and f_{k-1} is differentiable on H_M relative to H_M with $(f_{k-1}|_{H_M})'(x) = f_k(x)$, for each $x \in H_M$.*

Proof: By the construction of H_M we see that no point is an isolated point. Note that each Q_y is perfect and that $Q_y \cap Q_z = \emptyset$ if $y, z \in H(f, M, 1)$ are two different isolated points of $H(f, M, 1)$. Let $\{z_n\}$ be a sequence in H_M such that $\lim_{n\to\infty} z_n = z$. If $z_n \in H(f, M, 1)$ for infinitely many n, then $z \in H(f, M, 1)$ since $H(f, M, 1)$ is closed. Assume z_n not in $H(f, M, 1)$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there is an isolated point $y_n \in H(f, M, 1)$ such that $z_n \in Q_{y_n}$. If there are only finitely many different y_n, then $z_n \in Q_y$ for infinitely many n. Since Q_y is closed, $z \in Q_y \subset H_M$. Assume there are infinitely many different y_n. Since $|z_n - y_n| < \delta(y_n) < 1$, and since $\lim_{n\to\infty} z_n = z$, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\{y_{n_j}\}$ converges. Let $y = \lim_{j\to\infty} y_{n_j}$. Then $y \in H(f, M, 1)$ and it follows that $z = \lim_{j\to\infty} z_{n_j} = y$. So $z \in H_M$. Therefore H_M is closed.

Now if $x \in H_M$ is an isolated point of $H(f, M, 1)$, then clearly f'_{k-1} at x relative to H_M, exists and is equal to $f_k(x)$. If $x \in Q_y$ for some $y \in H(f, M, 1)$ where y is an isolated point of $H(f, M, 1)$, then there is $n \in \mathbb{Z}$ so that $x \in Q_n(y) \subset H(f, M_n(y), 1)$

and by the fact that there are two numbers $a < b$ so that $(a,b) \cap H_M = Q_n(y)$, we see that f'_{k-1} at x relative to H_M exists and is equal to $f_k(x)$.

Finally let $x \in H(f,M,1)$, and x not an isolated point of $H(f,M,1)$. Let $\epsilon > 0$ be given. Then there is $\epsilon > \eta > 0$ so that

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| < \epsilon$$

whenever $y \in H(f,M,1)$ and $|y-x| < \eta$.

Let y be an isolated point of $H(f,M,1)$ and let $z \in Q_y$ with $|z-x| < \eta/2$. Since $|y-z| < \delta^2(y) < \delta(y)$ and $|y-x| > 2\delta(y)$, we have $\eta/2 > |x-z| \geq |x-y| - |y-z| > 2\delta(y) - \delta(y) = \delta(y)$. Hence $|y-x| \leq |y-z| + |z-x| < \delta(y) + \eta/2 < \eta$.

Thus

$$\begin{aligned}
&\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z-x} - f_k(x)\right| = \left|\left(\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right)\frac{y-x}{z-x} + \right. \\
&\left.\left(\frac{f_{k-1}(z) - f_{k-1}(y)}{z-y} - f_k(y)\right)\frac{z-y}{z-x} + \frac{z-y}{z-x}(f_k(y) - f_k(x))\right| \leq \\
&\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right|\left|1 - \frac{z-y}{z-x}\right| + \\
&\left|\frac{f_{k-1}(z) - f_{k-1}(y)}{z-y} - f_k(y)\right|\left|\frac{z-y}{z-x}\right| + \left|\frac{z-y}{z-x}\right|(|f_k(x)| + |f_k(y)|) \leq \\
&\epsilon\left(1 + \frac{\delta^2(y)}{\delta(y)}\right) + 1 \cdot \frac{\delta^2(y)}{\delta(y)} + \frac{\delta^2(y)}{\delta(y)}4M \leq \\
&2\epsilon + \delta(y)(1+4M) \leq 2\epsilon + \frac{\epsilon}{2}(1+4M)
\end{aligned}$$

and since ϵ was arbitrary we have that f'_{k-1} at x relative to H_M exists and equals $f_k(x)$. □

We end this chapter showing that a k-th Peano derivative is a path derivative of the $(k-1)$-th Peano derivative with a system of paths satisfying the I.I.C. condition. As a corollary to this result we will obtain that a k-th Peano derivative is a selective derivative of the $(k-1)$-th Peano derivative.

To define the system $\{E_x : x \in \mathbf{R}\}$ of paths with respect to which a given k-th Peano derivative, f_k, is the path derivative of f_{k-1}, we begin with some notation.

Notation For $x, y \in \mathbf{R}$ let $\delta(x,y) = \min\{1, \frac{|y-x|}{3}\}$. For $x \in \mathbf{R}$ and $M \in \mathbf{N}$ let

$$R_x = \bigcup\{P_y \cap [y, y+\delta^2(x,y)) : y \in H(f,M,1) \text{ and } y \text{ is right isolated from } H(f,N,1) \text{ for } N \in \mathbf{N}\} \text{ and let}$$

$$L_x = \bigcup\{P_y \cap (y-\delta^2(x,y), y] : y \in H(f,M,1) \text{ and } y \text{ is left isolated from } H(f,N,1) \text{ for } N \in \mathbf{N}\}.$$

Definition 3.3.2 *Let $x \in \mathbf{R}$. If there is an $M_x \in \mathbf{N}$ such that x is a bilateral point of accumulation of $H(f, M_x, 1)$, then let*

$$E_x = H_{M_x} \cup R_x \cup L_x.$$

If x is a right isolated point of $H(f,M,1)$ for every positive constant M but there is an M_x so that x is a left point of accumulation of $H(f,M_x,1)$, or if x is a left isolated point of $H(f,M,1)$ for every positive constant M but there is an M_x so that x is a right point of accumulation of $H(f,M_x,1)$, let

$$E_x = H_{M_x} \cup P_x \cup R_x \cup L_x.$$

Finally if x is an isolated point of $H(f,M,1)$ for every positive constant M then let $M_x = 1$ and let

$$E_x = H_{M_x} \cup P_x \cup R_x \cup L_x.$$

Definition 3.3.3 *Let E be the system of paths $\{E_x : x \in \mathbf{R}\}$.*

Lemma 3.3.4 *Let $k \in \mathbf{N}$ and let f be a function defined on $\mathbf{R}$ such that $f_k(x)$ exists $\forall x \in \mathbf{R}$. Then E is bilateral and satisfies I.I.C. condition.*

Proof: Clearly E is bilateral. We will prove a stronger condition than I.I.C. . Namely we will prove that for any two points x and y $E_x \cap E_y \cap (x,y) \neq \emptyset$. Let $x < y$ be any two points. Suppose $M_x \leq M_y$. If x is a right point of accumulation of $H(f, M_x, 1) \subset H(f, M_y, 1)$, then $E_x \cap E_y \cap (x,y) \neq \emptyset$.

If x is a right isolated point of $H(f, M_x, 1)$, then by choice of M_x, x is a right isolated point of $H(f, M, 1)$ for every $M \in \mathbf{N}$ and $x \in H(f, M_y, 1)$. Thus

$$\emptyset \neq P_x \cap [x, x + \delta^2(x,y)) \cap (x,y) \subset E_x \cap E_y \cap (x,y).$$

If $M_x > M_y$ and if y is a left point of accumulation of $H(f, M_y, 1) \subset H(f, M_x, 1)$ then $E_x \cap E_y \cap (x,y) \neq \emptyset$.

If y is a left isolated point of $H(f, M_y, 1)$, then by an argument similar to the above $E_x \cap E_y \cap (x,y) \neq \emptyset$. Therefore E satisfies the I.I.C. condition.

□

Theorem 3.3.5 *Let k and f be as in Lemma 3.3.4. Then f_{k-1} is E differentiable with $f'_{(k-1)E}(x) = f_k(x)$.*

Proof: Let $x \in \mathbf{R}$, and $\epsilon > 0$ be given. Then there is an $\epsilon > \eta > 0$ such that

$$\left| \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x) \right| < \epsilon$$

whenever $|y - x| < \eta$ and $y \in H(f, M_x, 1)$ or $y \in P_x$. Let $z \in E_x$ be such that $|z - x| < \frac{\eta}{2}$. If $z \in P_y$ for some $y \in H(f, M_x, 1)$ such that y is an isolated point of $H(f, M, 1)$ from either left or right, and for every positive constant M, then $\frac{\eta}{2} > |z - x| \geq |x - y| - |y - z| \geq 2\delta(x,y) - \delta(x,y) = \delta(x,y)$. Therefore $|y - x| \leq |y - z| + |x - z| < \delta(x,y) + \eta/2 < \eta$. Hence

$$\left| \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x) \right| < \epsilon \tag{8}$$

Thus

$$\left|\frac{f_{k-1}(z)-f_{k-1}(x)}{z-x}-f_k(x)\right|=\left|\left(\frac{f_{k-1}(y)-f_{k-1}(x)}{y-x}-f_k(x)\right)\frac{y-x}{z-x}+\right.$$

$$\left.\left(\frac{f_{k-1}(z)-f_{k-1}(y)}{z-y}-f_k(y)\right)\frac{z-y}{z-x}+\frac{z-y}{z-x}(f_k(y)-f_k(x))\right|\le$$

$$\left|\frac{f_{k-1}(y)-f_{k-1}(x)}{y-x}-f_k(x)\right|\left|1-\frac{z-y}{z-x}\right|+$$

$$\left|\frac{f_{k-1}(z)-f_{k-1}(y)}{z-y}-f_k(y)\right|\left|\frac{z-y}{z-x}\right|+\left|\frac{z-y}{z-x}\right|(|f_k(x)|+|f_k(y)|).$$

By (8), Theorem 1.1.27 and the relationship among x, y and z, the above inequality is

$$\le \epsilon\left(1+\frac{\delta^2(x,y)}{\delta(x,y)}\right)+1\cdot\frac{\delta^2(x,y)}{\delta(x,y)}+\frac{\delta^2(x,y)}{\delta(x,y)}4M_x\le$$

$$2\epsilon+\delta(x,y)(1+4M_x)\le 2\epsilon+\frac{\epsilon}{2}(1+4M_x)$$

and since ϵ was arbitrary we have that $f'_{(k-1)E}(x)$ exists and equals $f_k(x)$. □

Corollary 3.3.6 *Let k and f be as in Lemma 3.3.4. Then f_k is a selective derivative of f_{k-1}.*

Proof: Let a selection $p(x,y)$ be defined as follows:

If $x<y$ let $p(x,y)=z$, where z is any point in $E_x\cap E_y\cap(x,y)$, if $x=y$, let $p(x,x)=x$. Then for fixed point x_0 we have

$$\lim_{y\to x_0}\frac{f_{k-1}(p(x_0,y))-f_{k-1}(x_0)}{p(x_0,y)-x_0}=\lim_{z\to x_0}\frac{f_{k-1}(z)-f_{k-1}(x_0)}{z-x_0}.$$

Since $z\in E_x$ we have that the above limit exists and equals $f_k(x_0)$. □

CHAPTER IV

4.1 Decomposition of Generalized Peano derivatives

Definition 4.1.1 *Let F be a continuous function defined on* $\mathbf{R}$, *and let $n \in \mathbf{N}$. We say that F is n-th generalized Peano differentiable at $x \in \mathbf{R}$, if there is a positive integer q, and coefficients $F_{[i]}(x)$, $i = 1, \dots, n$ such that for each $h \in \mathbf{R}$*

$$F^{(-q)}(x+h) = \sum_{j=0}^{q-1} h^j \frac{F^{(-q+j)}(x)}{j!} + \sum_{j=0}^{n} h^{q+j} \frac{F_{[j]}(x)}{(q+j)!} + h^{q+n} \epsilon_{q+n}^{[q]}(x,h) \tag{1}$$

where $\lim_{h\to 0} \epsilon_{q+n}^{[q]}(x,h) = 0$.

Here $F_{[0]}(x) = F(x) = F^{(0)}(x)$ and $F^{(-j)}(x) = \int^x F^{(-j+1)}(t)\,dt$; i.e. $F^{(-j)}$ is an indefinite Riemann integral of the continuous function $F^{(-j+1)}$ for $j = 1, \dots, q$. Note that the definition of $F_{[i]}(x)$, $i = 0, 1, \dots, n$ and of $\epsilon_{q+n}^{[q]}(x,h)$ don't depend on which q-fold indefinite Riemann integral, $F^{(-q)}$, of the continuous function F, is taken because any two differ by a polynomial of a degree less than q. The above definition is the same as the definition of $(q+n)$-th Peano derivative of a function $F^{(-q)}$ at a point x. Therefore by Lemma 3.1.3 the coefficients $F_{[i]}(x)$, $i = 1, \dots, n$ don't depend on q, either. The coefficient $F_{[n]}(x)$ is called the n-th generalized Peano derivative of the function F at the point x. For the remainder of this chapter n will be a fixed positive integer, and F will be a continuous function defined on $\mathbf{R}$.

Definition 4.1.2 *For $q \in \mathbf{N}$, let A_q be the set of all $x \in \mathbf{R}$ so that* (1) *holds, and for $\epsilon > 0$, $\delta > 0$ let*

$$P_q = P_q(\epsilon, \delta) = \{x \in A_q : |\epsilon_{q+n}^{[q]}(x,h)| < \epsilon, \textit{ for } |h| < \delta\}. \tag{2}$$

Note that if $x \in A_q$, then $x \in A_p$ for every $p \geq q$. Also $x \in A_q$ iff $F^{(-q)}$ has an $(q+n)$-th Peano derivative at x with $(F^{(-q)})_{q+n}(x) = F_{[n]}(x)$.

Lemma 4.1.3 *For $q \leq p$, $P_q(\epsilon, \delta) \subset P_p(\epsilon\frac{(q+n)!}{(p+n)!}, \delta)$.*

Proof: Let $x \in P_q(\epsilon, \delta)$. Then $x \in A_q$ and for $t \in \mathbf{R}$

$$F^{(-q)}(x+t) = \sum_{j=0}^{q-1} t^j \frac{F^{(-q+j)}(x)}{j!} + \sum_{j=0}^{n} t^{q+j} \frac{F_{[j]}(x)}{(q+j)!} + t^{q+n} \epsilon_{q+n}^{[q]}(x,t). \tag{3}$$

Integrating both sides of (3) from 0 to h we get

$$F^{(-q-1)}(x+h) - F^{(-q-1)}(x) =$$

$$\sum_{j=0}^{q-1} h^{j+1} \frac{F^{(-q+j)}(x)}{(j+1)!} + \sum_{j=0}^{n} h^{q+1+j} \frac{F_{[j]}(x)}{(q+1+j)!} + \int_0^h t^{q+n} \epsilon_{q+n}^{[q]}(x,t)dt.$$

Thus $x \in A_{q+1}$. By the remark after Definition 4.1.1, we have

$$h^{q+1+n} \epsilon_{q+n+1}^{[q+1]}(x,h) = \int_0^h t^{q+n} \epsilon_{q+n}^{[q]}(x,t)\, dt \tag{4}$$

and since $x \in P_q(\epsilon, \delta)$ for $0 \neq |h| < \delta$ from (4) we have

$$|h|^{q+1+n} |\epsilon_{q+n+1}^{[q+1]}(x,h)| < \int_0^{|h|} t^{q+n} \epsilon \, dt = \epsilon \frac{|h|^{q+1+n}}{q+1+n}.$$

Hence $|\epsilon_{q+n+1}^{[q+1]}(x,h)| < \epsilon/(q+n+1)$ whenever $|h| < \delta$. Therefore

$$P_q(\epsilon, \delta) \subset P_{q+1}\left(\epsilon \frac{(q+n)!}{(q+n+1)!}, \delta\right).$$

The general result follows by induction. □

Definition 4.1.4 *For $x \in A_q$ and for $i = 1, \ldots, n$, define $\epsilon_{q+i}^{[q]}(x,h)$ by*

$$F^{(-q)}(x+h) = \sum_{j=0}^{q-1} h^j \frac{F^{(-q+j)}(x)}{j!} + \sum_{j=0}^{i} h^{q+j} \frac{F_{[j]}(x)}{(q+j)!} + h^{q+i} \epsilon_{q+i}^{[q]}(x,h). \tag{5}$$

Note that $\epsilon^{[q]}_{q+i}(x,h)$ doesn't depend on which q-fold indefinite Riemann integral, $F^{(-q)}$, of F is taken.

The following formula follows directly from Definition 4.1.4.

Formula 4.1.5 *Let $x \in A_q$. Then for $i \in N$ with $2 \le i \le n$ we have*

$$\epsilon^{[q]}_{q+i-1}(x,t) = t\frac{F_{[i]}(x)}{(q+i)!} + t\epsilon^{[q]}_{q+i}(x,t).$$

Recall Lemma 1.1.3, Definition of Riemann difference $\Delta^m_t f(x)$ and Lemma 1.1.4 from Chapter I.

Lemma 4.1.6 *For $m \in \mathbb{N}$ the following holds:*

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^i = \begin{cases} 0 & \text{if } i = 0,\dots,m-1 \\ m! & \text{if } i = m \\ \frac{m}{2}(m+1)! & \text{if } i = m+1 \end{cases}$$

Definition 4.1.7 *For any function f defined on $\mathbf{R}$ the Riemann difference $\Delta^m_t f(x)$ at a point x, of order m is defined by*

$$\Delta^m_t f(x) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt).$$

The relationship between Δ^m_t and Δ^{m+1}_t is given by the following assertion.

Lemma 4.1.8 *For any function f defined on $\mathbf{R}$, for any $m \in \mathbb{N}$ and $t \in \mathbf{R}$ we have*

$$\Delta^{m+1}_t f(x) = \Delta^m_t f(x+t) - \Delta^m_t f(x).$$

Lemma 4.1.9 *Let $x \in A_q$. Then for each $i = 1, \ldots, n$*

$$\Delta_t^{q+m} F^{(-q)}(x) =$$

$$\begin{cases} t^{q+m} F_{[m]}(x) + t^{q+m} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+m} \epsilon_{q+m}^{[q]}(x, jt) & \textit{if } m = i \\ t^{q+i} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+i} \epsilon_i(x, jt) & \textit{if } m > i. \end{cases}$$

Proof:

$$\Delta_t^{q+m} F^{(-q)}(x) = \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} F^{(-q)}(x + jt)$$

$$= \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} \left(\sum_{l=0}^{q-1} (jt)^l \frac{F^{(-q+l)}(x)}{l!} + \right.$$

$$\left. \sum_{l=0}^{i} (jt)^{q+l} \frac{F_{[l]}(x)}{(q+l)!} + (jt)^{q+i} \epsilon_{q+i}^{[q]}(x, jt) \right)$$

$$= \sum_{l=0}^{q-1} t^l \frac{F^{(-q+l)}(x)}{l!} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^l +$$

$$\sum_{l=0}^{i} t^{q+l} \frac{F_{[l]}(x)}{(q+l)!} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+l} +$$

$$t^{q+i} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+i} \epsilon_{q+i}^{[q]}(x, jt)$$

which by Lemma 4.1.6, is equal to

$$\begin{cases} t^{q+m} F_{[m]}(x) + t^{q+m} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+m} \epsilon_{q+m}^{[q]}(x, jt) & \text{if } m = i \\ t^{q+i} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+i} \epsilon_{q+i}^{[q]}(x, jt) & \text{if } m > i. \end{cases} \square$$

Formula 4.1.10 *Let $x, x + t \in A_q$ and let $i \in \mathbb{N}$ with $1 \leq i \leq n$. Then*

$$F_{[i]}(x + t) - F_{[i]}(x) = \sum_{j=0}^{q+i+1} (-1)^{q+i+1-j} \binom{q+i+1}{j} j^{q+i} \epsilon_{q+i}^{[q]}(x, jt) +$$

$$\sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \epsilon_{q+i}^{[q]}(x, jt) - \sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \epsilon_{q+i}^{[q]}(x + t, jt).$$

Proof: By Lemma 4.1.8,

$$\Delta_t^{q+i+1}F^{(-q)}(x) = \Delta_t^{q+i}F^{(-q)}(x+t) - \Delta_t^{q+i}F^{(-q)}(x). \tag{6}$$

Applying Lemma 4.1.9, with $m = i+1$ to the left hand side

and with $m = i$ to the right hand side of (6) we get

$$t^{q+i}\sum_{j=0}^{q+i+1}(-1)^{q+i+1-j}\binom{q+i+1}{j}j^{q+i}\epsilon_{q+i}^{[q]}(x,jt) =$$

$$t^{q+i}F_{[i]}(x+t) + t^{q+i}\sum_{j=0}^{q+i}(-1)^{q+i-j}\binom{q+i}{j}j^{q+i}\epsilon_{q+i}^{[q]}(x+t,jt) -$$

$$t^{q+i}F_{[i]}(x) - t^{q+i}\sum_{j=0}^{q+i}(-1)^{q+i-j}\binom{q+i}{j}j^{q+i}\epsilon_{q+i}^{[q]}(x,jt).$$

Dividing both sides by t^{q+i} gives the above formula. □

Theorem 4.1.11 *For any interval $[a,b]$, $F_{[n]}$ is bounded on $\overline{P}_q(\epsilon,\delta)\cap[a,b]$.*

Proof: Let $[a,b]$ be an interval. Let $x,\ y \in P_q(\epsilon,\delta)\cap[a,b]$, so that for $t = y-x$ we have $|t| < \delta/(q+n+1)$, and let $B = \sum_{j=1}^{q+n+1}\binom{q+n+1}{j}j^{q+n}$. Then the right hand side of Formula 4.1.10 applied with $i = n$, is bounded by $3B\epsilon$. It follows that $F_{[n]}$ is bounded on $P_q(\epsilon,\delta)\cap[a,b]$. From Formula 4.1.5 (applied with $i = n$) it follows that for $|h| < \delta$, $|\epsilon_{q+n-1}^{[q]}(\cdot,h)|$ is bounded on $P_q(\epsilon,\delta)\cap[a,b]$. Now from Formula 4.1.10 (applied with $i = n-1$) we see that $F_{[n-1]}$ is bounded on $P_q(\epsilon,\delta)\cap[a,b]$, and again going back to Formula 4.1.5 (applied with $i = n-1$) we see that for $|h| < \delta$, $|\epsilon_{q+n-2}^{[q]}(\cdot,h)|$ is bounded on $P_q(\epsilon,\delta)\cap[a,b]$. Continuing we can deduce that there is a constant C so that $|F_{[i]}(x)| \le C$ for $1 \le i \le n$, for $x \in P_q(\epsilon,\delta)\cap[a,b]$.

Let $x \in \overline{P}_q$, and let $\{x_m\}$ be a sequence in $P_q(\epsilon,\delta)$ such that $\lim_{m\to\infty} x_m = x$. Choose $p \ge q$ such that $x \in A_p$. Let $[a,b]$ be such that $\{x_m\} \subset P_q(\epsilon,\delta)\cap[a,b]$. From the first part of the proof we see that for $1 \le i \le n$, $F_{[i]}$ is bounded on $P_q(\epsilon,\delta)\cap[a,b]$.

Therefore we can choose a subsequence $\{x_{m_j}\}$ converging to x such that $\{F_{[i]}(x_{m_j})\}$ converges for each $1 \le i \le n$. Let these sequences converge to $G_i(x)$, $i = 1, \ldots, n$ respectively.

Let $|h| < \delta$, and, as we may, suppose that $|h + x - x_{m_j}| < \delta$ for every $j \in \mathbb{N}$. Since $x_{m_j} \in P_q(\epsilon, \delta)$, by Lemma 4.1.3 we have $|\epsilon_{p+n}^{[p]}(x_{m_j}, h + x - x_{m_j})| < \epsilon\frac{(q+n)!}{(p+n)!}$. Thus we may also suppose that the sequence $\epsilon_{p+n}^{[p]}(x_{m_j}, h + x - x_{m_j})$ converges. Denote its limit by $E(h)$. Now letting $j \to \infty$ in the formula

$$F^{(-p)}(x+h) = F^{(-p)}(x_{m_j}) + (h + x - x_{m_j})F^{(-p+1)}(x_{m_j}) + \cdots +$$
$$\frac{(h + x - x_{m_j})^p}{p!}F_{[0]}(x_{m_j}) + \cdots + \frac{(h + x - x_{m_j})^{p+n-1}}{(p+n-1)!}F_{[n-1]}(x_{m_j}) +$$
$$+(h + x - x_{m_j})^{p+n}\left(\frac{F_{[n]}(x_{m_j})}{(p+n)!} + \epsilon_{p+n}^{[p]}(x_{m_j}, h + x - x_{m_j})\right)$$

we get

$$F^{(-p)}(x+h) = \sum_{j=0}^{p-1}\frac{h^j}{j!}F^{(-p+j)}(x) + \frac{h^p}{p!}F_{[0]}(x) + \frac{h^{p+1}}{(p+1)!}G_1(x) + \cdots +$$

$$\frac{h^{p+n-1}}{(p+n-1)!}G_{n-1}(x) + h^{p+n}\left(\frac{G_n(x)}{(p+n)!} + E(h)\right). \tag{7}$$

Since $\frac{G_n(x)}{(p+n)!} + E(h)$ is bounded, by the uniqueness of Peano derivatives from (7) we have $G_i(x) = F_{[i]}(x)$ for $1 \le i \le n-1$ and

$$\frac{F_{[n]}(x)}{(p+n)!} + \epsilon_{p+n}^{[p]}(x, h) = \frac{G_n(x)}{(p+n)!} + E(h). \tag{8}$$

Since $|E(h)| \le \epsilon\frac{(q+n)!}{(p+n)!}$, from (8) we have that

$$\left|\frac{F_{[n]}(x) - G_n(x)}{(p+n)!}\right| = |E(h) - \epsilon_{p+n}^{[p]}(x, h)| \le \epsilon\frac{(q+n)!}{(p+n)!} + |\epsilon_{p+n}^{[p]}(x, h)|. \tag{9}$$

The left hand side of (9) doesn't depend on h so letting $h \to 0$ in the right hand side of (9) we get

$$\left|\frac{F_{[n]}(x) - G_n(x)}{(p+n)!}\right| \le \epsilon\frac{(q+n)!}{(p+n)!}. \tag{10}$$

Finally from the first part of the proof we know that there is a constant C, so that $|F_{[n]}(y)| \leq C$, for $y \in P_q(\epsilon, \delta) \cap [a, b]$. Since $\lim_{j\to\infty} F_{[n]}(x_{m_j}) = G_n(x)$, $|G_n(x)| \leq C$. Hence by (10) $|F_{[n]}(x)| \leq \epsilon(q+n)! + C$. Note that the bound on $F_{[n]}(x)$ doesn't depend on the choice of p. □

If x_1 and x are two different points in A_q, then since $F_{[n]}(y) = (F^{(-q)})_{q+n}(y)$ for $y \in A_q$, we have a formula for generalized Peano derivatives similar to the one in Theorem 1.1.17. We will use this formula only for the case $s = k - 1$ in Theorem 1.1.17. For the sake of completeness, we will state this formula for generalized Peano derivatives as Theorem 4.1.12 below, and we will give a proof of this theorem not recalling the corresponding result for Peano derivatives; i.e. Theorem 1.1.17.

Theorem 4.1.12 *Let x, $x_1 \in A_q$ such that $x \neq x_1$ and $t \neq 0$. Then*

$$\frac{F_{[n-1]}(x_1) - F_{[n-1]}(x)}{x_1 - x} - F_{[n]}(x) = \frac{t}{x_1 - x}\frac{q+n-1}{2}F_{[n]}(x) +$$

$$\sum_{j=0}^{q+n-1} (-1)^{q+n-1-j}\binom{q+n-1}{j}\frac{(x_1 - x + jt)^{q+n}}{t^{q+n-1}(x_1 - x)}\epsilon^{[q]}_{q+n}(x, x_1 - x + jt) -$$

$$\frac{t}{x_1 - x}\left\{\frac{q+n-1}{2}F_{[n]}(x_1) + \sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+n}\epsilon^{[q]}_{q+n}(x_1, jt)\right\}.$$

To prove this theorem we will need two technical lemmas.

Lemma 4.1.13 *Let $x_1 \in A_q$. Then for any t*

$$\Delta_t^{q+n-1}F^{(-q)}(x_1) = t^{q+n-1}F_{[n-1]}(x_1) +$$

$$t^{q+n}\frac{q+n-1}{2}F_{[n]}(x_1) + t^{q+n}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+n}\epsilon^{[q]}_{q+n}(x_1, t).$$

Proof:

$$\Delta_t^{q+n-1}F^{(-q)}(x_1) = \sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\left(\sum_{l=0}^{q-1}(jt)^l\frac{F^{(-q+l)}(x_1)}{l!}+\right.$$

$$\left.\sum_{l=0}^{n}(jt)^{q+l}\frac{F_{[l]}(x_1)}{(q+l)!}+(jt)^{q+n}\epsilon_{q+n}^{[q]}(x_1,t)\right)=$$

$$\sum_{l=0}^{q-1}\left(\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^l\right)t^l\frac{F^{(-q+l)}(x_1)}{l!}+$$

$$\sum_{l=0}^{n}\left(\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+l}\right)t^{q+l}\frac{F_{[l]}(x_1)}{(q+l)!}+$$

$$t^{q+n}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+n}\epsilon_{q+n}^{[q]}(x_1,jt).$$

By Lemma 4.1.6, the above is equal to

$$t^{q+n-1}F_{[n-1]}(x_1)+t^{q+n}\frac{q+n-1}{2}F_{[n]}(x_1)+$$

$$t^{q+n}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+n}\epsilon_{q+n}^{[q]}(x_1,\ t).\ \square$$

Lemma 4.1.14 *Let $x,\ x_1\in A_q$. Then for any t*

$$\Delta_t^{q+n-1}F^{(-q)}(x_1)=t^{q+n-1}F_{[n-1]}(x)+$$

$$t^{q+n-1}(x_1-x)F_{[n]}(x)+t^{q+n}\frac{q+n-1}{2}F_{[n]}(x)+$$

$$\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}(x_1-x+jt)^{q+n}\epsilon_{q+n}^{[q]}(x,x_1-x+jt).$$

Proof:

$$\Delta_t^{q+n-1}F^{(-q)}(x_1)=\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}F^{(-q)}(x+x_1-x+jt)$$

$$=\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\cdot$$

$$\left(\sum_{l=0}^{q-1}(x_1-x+jt)^l\frac{F^{(-q+l)}(x)}{l!}+\sum_{l=0}^{n}(x_1-x+jt)^{q+l}\frac{F_{[l]}(x)}{(q+l)!}+\right.$$
$$\left.(x_1-x+jt)^{q+n}\epsilon^{[q]}_{q+n}(x,x_1-x+jt)\right)$$
$$=\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\left(\sum_{l=0}^{q-1}(x_1-x+jt)^l\frac{F^{(-q+l)}(x)}{l!}+\right.$$
$$\left.\sum_{l=0}^{n}(x_1-x+jt)^{q+l}\frac{F_{[l]}(x)}{(q+l)!}\right)+$$
$$\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}(x_1-x+jt)^{q+n}\epsilon^{[q]}_{q+n}(x,x_1-x+jt).$$

Since

$$\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\left(\sum_{l=0}^{q-1}(x_1-x+jt)^l\frac{F^{(-q+l)}(x)}{l!}+\right.$$
$$\left.\sum_{l=0}^{n}(x_1-x+jt)^{q+l}\frac{F_{[l]}(x)}{(q+l)!}\right)$$
$$=\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\left(\sum_{l=0}^{q-1}\sum_{s=0}^{l}\binom{l}{s}(x_1-x)^{l-s}(jt)^s\frac{F^{(-q+l)}(x)}{l!}+\right.$$
$$\left.\sum_{l=0}^{n}\sum_{s=0}^{q+l}\binom{q+l}{s}(x_1-x)^{q+l-s}(jt)^s\frac{F_{[l]}(x)}{(q+l)!}\right)$$
$$=\sum_{l=0}^{q-1}\sum_{s=0}^{l}\binom{l}{s}(x_1-x)^{l-s}t^s\frac{F^{(-q+l)}(x)}{l!}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^s+$$
$$\sum_{l=0}^{n}\sum_{s=0}^{q+l}\binom{q+l}{s}(x_1-x)^{q+l-s}t^s\frac{F_{[l]}(x)}{(q+l)!}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^s,$$

by Lemma 4.1.6 the above is equal to

$$\sum_{l=n-1}^{n}\sum_{s=q+n-1}^{q+l}\binom{q+l}{s}(x_1-x)^{q+l-s}t^s\frac{F_{[l]}(x)}{(q+l)!}\sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^s.$$

Applying Lemma 4.1.6 once more it is equal to

$$t^{q+n-1}F_{[n-1]}(x)+t^{q+n-1}(x_1-x)F_{[n]}(x)+t^{q+n}\frac{q+n-1}{2}F_{[n]}(x).$$

This completes the proof. □

Proof of Theorem 4.1.12: The proof follows directly from Lemma 4.1.13 and Lemma 4.1.14. □

Finally we are ready to prove the main result of this Chapter.

Theorem 4.1.15 *Suppose for each $x \in \mathbf{R}$ F is n-th generalized Peano differentiable at x. Then $F_{[n-1]}$ is differentiable on $\overline{P}_q = \overline{P}_q(\epsilon, \delta)$ relative to $\overline{P}_q$ with*

$$F_{[n-1]}|'_{\overline{P}_q}(x) = F_{[n]}(x).$$

Proof: Let $x \in \overline{P}_q$. There is a $p \geq q$ so that $x \in A_p$. Let $1 > \epsilon' > 0$ be given. There is $0 < \eta < \delta$ such that $|\epsilon^{[p]}_{p+n}(x,h)| < \epsilon'$ whenever $|h| < \eta$. Let $\{x_m\}$ be a sequence in P_q converging to x, so that $|x_m - x| < \frac{\eta}{p+n}$. By Theorem 4.1.11, there is a constant C so that $|F_{[n]}(x_m)| \leq C$, for every $m \in \mathbf{N}$. Let $t = (x_m - x)\epsilon'^{\frac{1}{p+n}}$. Then $|jt| < \delta$ and $|x_m - x + jt| < \eta$, for $j = 0, 1, \ldots, q+n-1$. Therefore

$$|\epsilon^{[p]}_{p+n}(x, x_m - x + jt)| < \epsilon' \tag{11}$$

and by Lemma 4.1.3,

$$|\epsilon^{[p]}_{p+n}(x_m, jt)| < \epsilon \text{ for every } j = 0, 1 \ldots, p+n-1 \text{ and for every } m \in \mathbf{N}. \tag{12}$$

Since $x_m \in A_p$, the formula from Theorem 4.1.12 gives

$$\left|\frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x)\right| \leq \epsilon'^{\frac{1}{p+n}}\frac{p+n-1}{2}|F_{[n]}(x)| +$$
$$\sum_{j=0}^{p+n-1}\binom{p+n-1}{j}\frac{(1+j\epsilon'^{\frac{1}{p+n}})^{p+n}}{\epsilon'^{\frac{p+n-1}{p+n}}}|\epsilon^{[p]}_{p+n}(x, x_m - x + jt)| +$$
$$\epsilon'^{\frac{1}{p+n}}\left|\frac{p+n-1}{2}F_{[n]}(x_m) + \sum_{j=0}^{p+n-1}(-1)^{p+n-1-j}\binom{p+n-1}{j}j^{p+n}\epsilon^{[p]}_{p+n}(x_m, jt)\right|.$$

By (12) and (11) together with Theorem 4.1.11 the above is

$$\leq \epsilon'^{\frac{1}{p+n}}\frac{p+n-1}{2}|F_{[n]}(x)| + \sum_{j=0}^{p+n-1}\binom{p+n-1}{j}(1+j\epsilon'^{\frac{1}{p+n}})^{p+n}\epsilon'^{\frac{1}{p+n}} +$$
$$\epsilon'^{\frac{1}{p+n}}\left\{\frac{p+n-1}{2}C + \sum_{j=0}^{p+n-1}\binom{p+n-1}{j}j^{p+n}\epsilon\right\}.$$

Since ϵ' was arbitrary we have

$$\frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x) \to 0 \text{ as } x_m \in P_q,\ x_m \to x.$$

Now for the general case let $\{x_m\}$ be a sequence in $\overline{P}_q$ such that $x_m \to x$. Let $y_m \in P_q$ be such that $|y_m - x_m| \leq \frac{1}{m}|x_m - x|$ and that

$$\left| \frac{F_{[n-1]}(y_m) - F_{[n-1]}(x_m)}{y_m - x_m} - F_{[n]}(x_m) \right| \leq 1. \tag{13}$$

By what was just proved, there is such a sequence y_m. By Theorem 4.1.11, there is a constant C such that $|F_{[n]}(x_m)| \leq C$ for every $m \in \mathbf{N}$. This and (13) give

$$\left| \frac{F_{[n-1]}(y_m) - F_{[n-1]}(x_m)}{y_m - x_m} \right| \leq C + 1. \tag{14}$$

Now

$$\frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x) = \frac{F_{[n-1]}(x_m) - F_{[n-1]}(y_m)}{x_m - y_m} \frac{x_m - y_m}{x_m - x} +$$
$$\left\{ \frac{F_{[n-1]}(y_m) - F_{[n-1]}(x)}{y_m - x} - F_{[n]}(x) \right\} \frac{y_m - x}{x_m - x} - F_{[n]}(x) \frac{x_m - y_m}{x_m - x}.$$

So by (14)

$$\left| \frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x) \right| \leq$$
$$(C+1)\frac{1}{m} + \left| \frac{F_{[n-1]}(y_m) - F_{[n-1]}(x)}{y_m - x} - F_{[n]}(x) \right| (1 + \frac{1}{m}) + C\frac{1}{m}. \tag{15}$$

Finally since $x_m \to x$, $y_m \to x$. But $y_m \in P_q$, and hence by the first part

$$\frac{F_{[n-1]}(y_m) - F_{[n-1]}(x)}{y_m - x} - F_{[n]}(x) \to 0. \tag{16}$$

Therefore by (15) and (16)

$$\frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x) \to 0 \text{ as } x_m \in \overline{P}_q,\ x_m \to x.$$

This completes the proof. $\square$

Lemma 4.1.16 *For each $\epsilon > 0$, $\bigcup_{q=0}^{\infty} \bigcup_{m=1}^{\infty} P_q(\epsilon, 1/m) = \mathbf{R}$.*

Proof: The assertion follows from Definition 4.1.2. $\square$

Corollary 4.1.17 *Suppose for each $x \in \mathbf{R}$ F is n-th generalized Peano differentiable at x. Then $F_{[n]}$ is a composite derivative of $F_{[n-1]}$.*

Corollary 4.1.18 *Suppose for each $x \in \mathbf{R}$ F is n-th generalized Peano differentiable at x. Then $F_{[n]}$ is an approximate derivative of $F_{[n-1]}$ a.e..*

Corollary 4.1.19 *Suppose for each $x \in \mathbf{R}$ F is n-th generalized Peano differentiable at x. Then $F_{[n]} \in [\Delta]'$.*

Corollary 4.1.20 *Suppose for each $x \in \mathbf{R}$ F is n-th generalized Peano differentiable at x. Then $F_{[n]}$ is a Baire 1 function.*

That $F_{[n]}$ is a Baire 1 function, was proved in [9]. The proof in that paper is not as simple as the proof for Peano derivatives. Corollary 4.1.20 gives another proof of this assertion.

4.2 Generalized Peano, path and selective derivatives

Next we will show that the following analogy of Theorem 2.1.9, holds for generalized Peano derivatives.

Theorem 4.2.1 *Let $l \in \mathbf{N}$ with $l \leq n-1$. Assume for each function g defined on a closed interval I having an l-th generalized Peano derivative, $g_{[l]}$, on I, $g_{[l]}$ is a Darboux function and if $g_{[l]} \geq 0$ on I, then $g_{[l]} = g^{(l)}$ on I. Suppose $F_{[n]}$ exists on $\mathbf{R}$. Then there is a bilateral nonporous system of paths $E = \{E_x : x \in R\}$ satisfying the I.C. condition such that $F_{[n]}$ is the E-derivative of $F_{[n-1]}$.*

We will need some lemmas before we prove this theorem.

Lemma 4.2.2 *Under the assumptions of Theorem 4.2.1 for every $\epsilon > 0$ and $\eta > 0$ there is a $\delta > 0$ such that if I is a closed subinterval of $(x - \delta, x + \delta)$, x is not in I with*

$$\left| \frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y - x} - F_{[n]}(x) \right| \geq \epsilon \tag{17}$$

for all $y \in I$, then $m(I) \leq \eta d(x, I)$.

Proof: Let δ be chosen according to Theorem 2.1.5, applied with η replaced by $\eta_1 = \eta/(1+\eta)$ and with $j = n-1$. Let I be as above, and let $g(y) = F(y) - y^{n-1}\frac{F_{[n-1]}(x)}{(n-1)!} - (y-x)^n \frac{F_{[n]}(x)}{n!}$. Then g has an $(n-1)$-th generalized Peano derivative and $g_{[n-1]}(y) = F_{[n-1]}(y) - F_{[n-1]}(x) - (y-x)F_{[n]}(x)$. So by assumptions $g_{[n-1]}$ is Darboux. By (17) $|g_{[n-1]}(y)| \geq \epsilon |y - x|$ on I. Since x is not in I, $|g_{[n-1]}(y)| > 0$ for $y \in I$ and since $g_{[n-1]}$ is Darboux, we have either $g_{[n-1]} > 0$ or $-g_{[n-1]} > 0$ on I. Hence by the assumptions, $g_{[n-1]}$ is an $(n-1)$-th ordinary derivative of g on I. Therefore F is $(n-1)$ times ordinarily differentiable on I and by the uniqueness of generalized Peano derivatives, $F^{(n-1)} = F_{[n-1]}$ on I. Now we can apply Theorem 2.1.5, with $n = k$ and $j = n-1$, which gives $m(I) \leq \eta_1 \cdot (m(I) + d(x, I))$. Hence $m(I) \leq \eta d(x, I)$. □

The statement and the proof of a next lemma follow line by line the corresponding Lemma 2.1.11 for Peano derivatives. So we will only state the lemma and omit the proof.

Lemma 4.2.3 *Under the assumptions of Theorem 4.2.1, for each point $x \in I$ there is a path E_x leading to x and nonporous at x so that*

$$\lim_{y \in E_x, y \to x} \frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y - x} = F_{[n]}(x).$$

Lemma 4.2.4 *Let $m \leq l$ be two positive integers and let $\epsilon > 0$. Then*

$$P_m(\epsilon, \frac{1}{m}) \subset P_l(\epsilon, \frac{1}{l}).$$

Proof: By Lemma 4.1.3,

$$P_m(\epsilon, \frac{1}{m}) \subset P_l(\epsilon, \frac{1}{m}), \text{ and by the Definition 4.1.2}$$

$$P_l(\epsilon, \frac{1}{m}) \subset P_l(\epsilon, \frac{1}{l}). \square$$

Now we are ready to prove Theorem 4.2.1.

Proof: For each $x \in R$ let E'_x be a path satisfying the assertions of Lemma 4.1.9. We will define the system of paths $E = \{E_x : x \in R\}$ as follows:

For $x \in R$ let $E_x = E'_x \cup P_m(1, 1/m)$ where m is a positive integer such that $x \in P_m(1, 1/m)$. That E is nonporous (therefore bilateral) follows directly from Lemma 4.2.3. Also Lemma 4.2.3 and Theorem 4.1.15, assure us that $F_{[n-1]}$ is E differentiable with $F_{[n-1]}|'_E(x) = F_{[n]}(x)$ for every $x \in R$. It remains only to prove that E satisfies the intersection condition I.C.. We will prove that for any two different points x and y, $E_x \cap E_y \cap [x, y] \neq \emptyset$ which is stronger than the I.C. condition.

Let $x \in P_m(1, 1/m)$ and $y \in P_l(1, 1/l)$ be any two distinct points. If $m \leq l$, then by Lemma 4.2.4 $P_m(1, 1/m) \subset P_l(1, 1/l)$ and hence $x \in E_y$. Similarly if $m \geq l$, then $y \in E_x$. Therefore $E_x \cap E_y \cap [x, y] \neq \emptyset$. Hence E satisfies the I.C. condition. This completes the proof of Theorem . $\square$

Next we indicate that we can drop the assumption concerning the arbitrary function from Theorem 4.2.1. The proof of that assertion follows line by line the proof of Theorem 2.1.15, which is a corresponding result for Peano derivatives. So we will only state the result.

Theorem 4.2.5 *Let F be a continuous function defined on R so that $F_{[n]}$ exists on R. Then there is a bilateral nonporous system of paths $E = \{E_x : x \in R\}$ satisfying the I.C. condition such that $F_{[n]}$ is the E-derivative of $F_{[n-1]}$.*

Now using properties of path derivatives we get the following corollaries:

Corollary 4.2.6 *Under the assumptions of Theorem 4.2.5 $F_{[n]}$ is Darboux.*

Corollary 4.2.7 *Let F be as in Theorem 4.2.5, let $[a,b]$ be an interval and $\alpha \in \mathbf{R}$. If $F_{[n]} \geq \alpha$ (or $F_{[n]} \leq \alpha$) then*

a) $F_{[n-1]}(x) - \alpha x$ $(\alpha x - F_{[n-1]}(x)$ $)$ *is nondecreasing and continuous on $[a,b]$*

b) $F^{(n)}$ *exists and* $F^{(n)} = F_{[n]}$ *on* $[a,b]$.

Corollary 4.2.8 *Under the assumptions of Theorem 4.2.5 $F_{[n]}$ has the Denjoy property.*

Corollary 4.2.9 *Suppose $F_{[n]}(x)$ exists for all x in I_0 and let $M \geq 0$. If $F_{[n]}$ attains both M and $-M$ on I_0, then there is a subinterval I of I_0 on which $F_{[n]} = F^{(n)}$ and $F^{(n)}$ attains both M and $-M$ on I.*

We end this chapter showing that every generalized Peano derivative $F_{[n]}$ is a selective derivative of $F_{[n-1]}$. The idea is very similar to the one that we used for Peano derivatives.

Definition 4.2.10 *Let P_y be a set containing y so that y is a bilateral point of accumulation of P_y,*

$$\lim_{z \in P_y, z \to y} \frac{F_{[n-1]}(z) - F_{[n-1]}(y)}{z - y} = F_{[n]}(y)$$

and

$$\left| \frac{F_{[n-1]}(z) - F_{[n-1]}(y)}{z - y} - F_{[n]}(y) \right| \leq 1 \text{ for every } z \in P_y.$$

Theorem 4.2.5 assures the existence of P_y.

To define the system $\{E_x : x \in \mathbf{R}\}$ of paths with respect to which a given n-th generalized Peano derivative, $F_{[n]}$, is the path derivative of $F_{[n-1]}$, we begin with some notation.

Notation For $x, y \in \mathbf{R}$ let $\delta(x, y) = \min\{1, \frac{|y-x|}{3}\}$. For $x \in \mathbf{R}$ and $M \in \mathbf{N}$ let

$$R_x = \bigcup\{P_y \cap [y, y + \delta^2(x, y)) : y \in \overline{P}_M(1, 1/M) \text{ and } y \text{ is right isolated from } \overline{P}_N(1, 1/N) \text{ for } N \in \mathbf{N}\} \text{ and let}$$

$$L_x = \bigcup\{P_y \cap (y - \delta^2(x, y), y] : y \in \overline{P}_M(1, 1/M) \text{ and } y \text{ is left isolated from } \overline{P}_N(1, 1/N) \text{ for } N \in \mathbf{N}\}.$$

Definition 4.2.11 *Let $x \in \mathbf{R}$. If there is an $M_x \in \mathbf{N}$ such that x is a bilateral point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, then let*

$$E_x = \overline{P}_{M_x}(1, 1/M_x) \cup R_x \cup L_x.$$

If x is a right isolated point of $\overline{P}_M(1, 1/M)$ for every positive constant M but there is an M_x so that x is a left point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, or if x is a left isolated point of $\overline{P}_M(1, 1/M)$ for every positive constant M but there is an M_x so that x is a right point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, let

$$E_x = \overline{P}_{M_x}(1, 1/M_x) \cup P_x \cup R_x \cup L_x.$$

Finally if x is an isolated point of $\overline{P}_M(1,1/M)$ for every positive constant M then let $M_x = 1$ and let

$$E_x = \overline{P}_{M_x}(1,1/M_x) \cup P_x \cup R_x \cup L_x.$$

Definition 4.2.12 *Let E be the system of paths $\{E_x : x \in \mathbf{R}\}$.*

Lemma 4.2.13 *Let $n \in \mathbf{N}$ and let F be a function defined on $\mathbf{R}$ such that $F_{[n]}(x)$ exists $\forall x \in \mathbf{R}$. Then E is bilateral and satisfies I.I.C. condition.*

Proof: Clearly E is bilateral. We will prove a stronger condition than I.I.C. . Namely we will prove that for any two points x and y $E_x \cap E_y \cap (x,y) \neq \emptyset$. Let $x < y$ be any two points. Suppose $M_x \leq M_y$. If x is a right point of accumulation of $\overline{P}_{M_x}(1,1/M_x) \subset \overline{P}_{M_y}(1,1/M_y)$, then $E_x \cap E_y \cap (x,y) \neq \emptyset$.

If x is a right isolated point of $\overline{P}_{M_x}(1,1/M_x)$, then by choice of M_x, x is a right isolated point of $\overline{P}_M(1,1/M)$ for every $M \in \mathbf{N}$ and $x \in \overline{P}_{M_y}(1,1/M_y)$. Thus

$$\emptyset \neq P_x \cap [x, x + \delta^2(x,y)) \cap (x,y) \subset E_x \cap E_y \cap (x,y).$$

If $M_x > M_y$ and if y is a left point of accumulation of $\overline{P}_{M_y}(1,1/M_y) \subset \overline{P}_{M_x}(1,1/M_x)$ then $E_x \cap E_y \cap (x,y) \neq \emptyset$.

If y is a left isolated point of $\overline{P}_{M_y}(1,1/M_y)$, then by an argument similar to the above $E_x \cap E_y \cap (x,y) \neq \emptyset$. Therefore E satisfies the I.I.C. condition. $\square$

Theorem 4.2.14 *Let F be a continuous function defined on $\mathbf{R}$ so that $F_{[n]}$ exists at every point $x \in \mathbf{R}$. Then $F_{[n-1]}$ is E differentiable with $F'_{[n-1]E}(x) = F_{[n]}(x)$.*

Proof: Let $x \in \mathbf{R}$, and $\epsilon > 0$ be given. Then there is an $\epsilon > \eta > 0$ such that

$$\left|\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x}-F_{[n](x)}\right|<\epsilon \tag{18}$$

whenever $|y-x|<\eta$ where $y \in \overline{P}_{M_x}(1,1/M_x)$ or $y \in P_x$. Let $z \in E_x$ be such that $|z-x|<\frac{\eta}{2}$. If $z \in P_y$ for some $y \in \overline{P}_{M_x}(1,1/M_x)$ such that y is an isolated point of $\overline{P}_N(1,1/N)$ from either left or right, and for every positive constant N, then $\frac{\eta}{2}>|z-x|\geq|x-y|-|y-z|\geq 2\delta(x,y)-\delta(x,y)=\delta(x,y)$. Therefore $|y-x|\leq|y-z|+|x-z|<\delta(x,y)+\eta/2<\eta$. Hence by (18)

$$\left|\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x}-F_{[n]}(x)\right|<\epsilon. \tag{19}$$

Thus

$$\left|\frac{F_{[n-1]}(z)-F_{[n-1]}(x)}{z-x}-F_{[n]}(x)\right|=\left|\left(\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x}-F_{[n]}(x)\right)\frac{y-x}{z-x}+\right.$$

$$\left.\left(\frac{F_{[n-1]}(z)-F_{[n-1]}(y)}{z-y}-F_{[n]}(y)\right)\frac{z-y}{z-x}+\frac{z-y}{z-x}(F_{[n]}(y)-F_{[n]}(x))\right|\leq$$

$$\left|\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x}-F_{[n]}(x)\right|\left|1-\frac{z-y}{z-x}\right|+$$

$$\left|\frac{F_{[n-1]}(z)-F_{[n-1]}(y)}{z-y}-F_{[n]}(y)\right|\left|\frac{z-y}{z-x}\right|+\left|\frac{z-y}{z-x}\right|(|F_{[n]}(x)|+|F_{[n]}(y)|)$$

By (19), Theorem 4.1.11 and the relationship between points x, y and z we get the above inequality

$$\leq\epsilon(1+\frac{\delta^2(x,y)}{\delta(x,y)})+1\cdot\frac{\delta^2(x,y)}{\delta(x,y)}+\frac{\delta^2(x,y)}{\delta(x,y)}4M\leq$$

$$2\epsilon+\delta(x,y)(1+4M)\leq 2\epsilon+\frac{\epsilon}{2}(1+4M)$$

where M is a constant from Theorem 4.1.11. Since ϵ was arbitrary we have that $F'_{[n-1]E}$ at x exists and equals to $F_{[n]}(x)$ $\square$

Corollary 4.2.15 *Let F be a continuous function defined on* $\mathbf{R}$ *so that $F_{[n]}$ exists at every point $x \in \mathbf{R}$. Then $F_{[n]}$ is a selective derivative of $F_{[n-1]}$.*

Proof: Let a selection $p(x, y)$ be defined as follows:

If $x < y$ let $p(x, y) = z$, where z is any point in $E_x \cap E_y \cap (x, y)$, if $x = y$, let $p(x, x) = x$. Then for fixed point x_0 we have

$$\lim_{y \to x_0} \frac{F_{[n-1]}(p(x_0, y)) - F_{[n-1]}(x_0)}{p(x_0, y) - x_0} = \lim_{z \to x_0} \frac{F_{[n-1]}(z) - F_{[n-1]}(x_0)}{z - x_0}.$$

Since $z \in E_x$ we have that the above limit exists and equals $F_{[n]}(x_0)$. □

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