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Farzad Esfandiari

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Hassan Khalil
Major professor

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By

Farzad Esfandiari

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ABSTRACT

OUTPUT FEEDBACK STABILIZATION OF FULLY-LINEARIZABLE SYSTEMS

By

Farzad Esfandiari

In this work, we study the problem of output feedback control of nonlinear systems which are fully-linearizable via static state feedback, left-invertible, and minimum-phase. The output feedback controller proposed is an observer-based control, whose state feedback component consists of two parts: An inner loop to cancel the nonlinearities (either exactly or approximately), and an outer loop which is a robust stabilizing control law such as variable structure control, or min-max control.

To implement such state feedback controllers using an observer-based control, the observer should be designed to reject disturbances caused by model uncertainties, as well as by estimation error. Observer designs with such a disturbance rejection property are high-gain observers, where certain observer gains are pushed asymptotically towards infinity to locate some observer poles far to the left in the complex plane. When observer poles are assigned in this way, the trajectory of the closed-loop system exhibits an impulsive-like behavior, which is known as the *peaking phenomenon*. The peaking phenomenon which is generally present in systems of relative degree higher than one, has a destabilizing effect on the behavior

of the closed-loop system.

In this work, we design such high-gain observers using a singular perturbation approach. In this approach peaking exhibits itself through certain scalings which are dependent on the singular perturbation parameter. We prove a new singular perturbation result on the behavior of the closed-loop system in the presence of such scalings. Then, as a corollary of this reult, we show that presence of saturating nonlinearities at the plant input eliminates the destabilizing effect of peaking, since it provides a buffer that prevents the impulsive-like behavior of the observer from passing to the plant.

To my mother, Parvin Rastegar

and to the memory of my father, Mohammad Mehdi Esfandiari

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Table of Contents

List of Figures

Nomenclature

$[v]_i$	the i-th component of vector v
$\lVert x \rVert$	the 2-norm of x
$\lVert x \rVert_1$	the 1-norm of x
$\lVert x \rVert_\infty$	the infinite norm of x
$\lambda_{\max}(P)$	the maximum eigenvalue of matrix P
$\lambda_{\min}(P)$	the minimum eigenvalue of matrix P
$u_1 \equiv u_2$	$u_1(t)=u_2(t)$ for all t

1 Introduction

In the last two decades exact linearization of nonlinear systems via feedback have received considerable attention in nonlinear control community [Isidori (1989)]. Roughly speaking, feedback-linearizable systems are those classes of nonlinear systems which can be made to behave linearly under the effect of an appropriate state feedback and a possible change of coordinates. The problem of state feedback control of feedback-linearizable systems have been extensively studied in the literature (refer to Isidori's book (1989) for a survey. For more recent results, refer to the work of Sussmann-Kokotovic (1989)). However, few results are available on the problem of *output feedback* control of linearizable systems. we briefly go over the available results:

In the work of Marino (1985) a high-gain static output feedback control is proposed for stabilization of single-input single-output linearizable systems which have a relative degree of one. Assuming that the relative degree of the system is one, excludes most of the physical problems of interest. For instance, the equation of a robotic arm has a relative degree of two (Refer to Example 2.1 in Chapter 2). The case of systems whose relative degree is higher than one has been studied by Khalil-Saberi (1987), Isidori-Byrnes (1990) and Isidori (1989), where lead-lag compensators are proposed for *local* stabilization of linearizable systems. In this work, we address the problem of *nonlocal* output feedback stabilization of a class of linearizable systems, namely systems which are fully-linearizable via static state feedback, left-invertible, and minimum-phase. In chapter 2, we study this class of nonlinear systems in detail. Since most of the physical problems to which techniques of exact linearization have been applied are fully-linearizable systems for a meaning choice of output variables (Refer to Chapter 2), we focus on the class of fully-linearizable systems, rather than the more general class of input-output linearizable systems.

The proposed output feedback controller is an observer-based control, whose state feedback component consists of two parts: An inner loop to cancel the nonlinearities (either exactly or approximately), and an outer loop which is a robust stabilizing control law such as variable structure control, or min-max control. In Chapter 3, we study the problem of designing such a state feedback control. Then, in chapter 4, we study the problem of observer design. To ensure that the observer-based control preserves the stability properties of the state feedback control, the observer should be designed to reject the effects of model uncertainties and estimation errors. Observer designs with such a disturbance rejection property are high-gain observers, where certain observer gains are pushed asymptotically towards infinity to locate some observer poles far to the left in the complex plane. In section 4.3, we will design such an observer by transforming the system into a canonical form that exhibits the finite and infinite zero structure of the linearized system. Then, in section 4.4, we perform the closed-loop stability analysis using singular perturbation theory.

The closed-loop stability results of section 4.4 are *local* results in most cases. In chapter 5, we will argue that the locality of the stability results is due to what is known in linear system theory as *peaking phenomenon*. When some of the observer poles are located far to the left in the complex plane, the trajectory of the closed-loop system exhibits an impulsive-like behavior, which is known as the peaking phenomenon. Singular perturbation theory provides an elegant framework for the analysis of peaking phenomenon, since peaking exhibits itself through certain scalings which are dependent on the infinite zero structure of the system and the singular perturbation parameter. In Chapter 6, we prove a singular perturbation result on the behavior of the closed-loop system in the presence of such scalings. As a corollary of this result, we show that the presence of saturating nonlinearities at the plant input eliminates the peaking phenomenon, and hence the local nature of the stability results which is caused by peaking.

2 Full Linearization

Consider the nonlinear system

$$\begin{cases} \dot{\xi} = f(\xi) + g(\xi)\, u + g(\xi) \left[\hat{\Delta}_f(\xi,t) + \hat{\Delta}_g(\xi,t)\, u + \hat{\Delta}_{ed}(\xi,t) \right] \\ y = h(\xi) \end{cases} \tag{2.1}$$

where $\xi \in \mathbf{R}^p$, $u \in \mathbf{R}^q$, and $y \in \mathbf{R}^r$ are state, input and output vectors, respectively. $\hat{\Delta}_f(.,.)$, and $\hat{\Delta}_g(.,.)$ represent parametric uncertainties in $f(.)$ and $g(.)$, respectively, while $\hat{\Delta}_{ed}(.,.)$ represents exogenous disturbances. Note that all the uncertainties and disturbances satisfy the matching condition, i.e., they enter the right-hand side of the state equation at the same point as the input. In this chapter, we define the class of nonlinear systems which is under study in this work. Consider the following nominal model for (2.1), obtained by setting $\hat{\Delta}_f$, $\hat{\Delta}_g$, and $\hat{\Delta}_{ed}$ to zero:

$$\begin{cases} \dot{\xi} = f(\xi) + g(\xi)\, u \\ y = h(\xi) \end{cases} \tag{2.2}$$

Definition 2.1: [Cheng, et. al., (1988)] System (2.2) is said to be *fully-linearizable*, if there exist an open connected set $\Psi \subset \mathbf{R}^p$ containing the origin, a diffeomorphic transformation $T : \Psi \rightarrow \mathbf{R}^p$, smooth mappings $\hat{\alpha} : \Psi \rightarrow \mathbf{R}^q$,

$\hat{\beta} : \Psi \rightarrow \mathbf{R}^q \times \mathbf{R}^q$, with $\hat{\beta}(\xi)$ invertible for all $\xi \in \Psi$, such that the state feedback control $u = \hat{\alpha}(\xi) + \hat{\beta}(\xi)v$ and the change of coordinates $z = T(\xi)$ transform system (2.2) into a controllable linear system:

$$\begin{cases} \dot{z} = Az + Bv \\ y = Cz \end{cases} \tag{2.3}$$

Local necessary and sufficient conditions for full linearization are given in the work of Cheng-et. al. (1988).

Definition 2.2: [Hirschorn (1979)] Let $y(t,\xi_0,u(t))$ be the output of system (2.2) for the initial condition ξ_0 and the input $u(t)$. System (2.2) is said to be *left-invertible on* $\Psi \subset \mathbf{R}^p$, if for all $\xi_0 \in \Psi$

$$y(t,\xi_0,u_1(t)) = y(t,\xi_0,u_2(t)) \text{ for all } t \geq 0 \;\Rightarrow\; u_1(t) = u_2(t) \text{ for all } t \geq 0$$

Definition 2.3: [Isidori-Moog (1986)] Suppose there exists a set $\Psi \subset \mathbf{R}^p$ containing the origin and a smooth submanifold N_0 of Ψ containing the origin with the following properties:

i) $N_0 \subset Ker\ h(\xi)$

ii) There exists a state feedback control $u = \gamma(\xi)$, defined on Ψ, such that $f^*(\xi) := f(\xi) + g(\xi)\gamma(\xi)$ is tangent to N_0.

iii) N_0 is maximal, i.e., any submanifold of Ψ which contains the origin and satisfies conditions (i) and (ii) is contained in N_0.

Then the vector field of N_0 defined as the restriction of f^* to N_0 is said to be a *local zero dynamics* of system (2.2).

Definition 2.4: [Isidori-Moog (1986)] System (2.2) is said to be *minimum-phase*, if the vector field of N_0 of Definition 2.3 is asymptotically stable.

Proposition 2.1: Suppose that system (2.2) is fully-linearizable. Then system (2.2) is left-invertible and minimum-phase over the domain Ψ if and only if the linear system (2.3) is left-invertible, minimum-phase, and detectable.

Proof of Proposition 2.1:

Sufficiency: The coordinate transformation $z = T(\xi)$ transforms system (2.2) into

$$\begin{cases} \dot{z} = Az + B\ \beta^{-1}(z) \left[u - \alpha(z) \right] \\ y = Cz \end{cases} \tag{2.4}$$

where

$$\alpha := \hat{\alpha} \circ T^{-1}\ , \ \ \beta := \hat{\beta} \circ T^{-1}$$

So, without loss of generality, we prove that system (2.4) is left-invertible and minimum-phase, if system (2.3) is left-invertible, minimum-phase, and detectable.

Let $z(t, z_0, u(t))$ denote the solution of the state equation of (2.4) for the initial condition vector z_0, and the input function $u(t)$. Similarly, let $x(t, z_0, v(t))$ denote the solution of the state equation of (2.3) for the initial condition vector z_0, and the input vector $v(t)$.

i) System (2.4) is left-invertible on $\mathbf{R}^p$.

Proof of (i): Suppose that (2.4) is not left-invertible on $\mathbf{R}^p$. Then there exist an initial condition vector $z_0 \in \mathbf{R}^p$, input functions $u_1(t)$ and $u_2(t)$, such that $u_1 \not\equiv u_2$, but

$$C\ z(t, z_0, u_1) \equiv C\ z(t, z_0, u_2) \tag{2.5}$$

Define

$$z_1(t) := z(t, z_0, u_1)$$

$$z_2(t) := z(t, z_0, u_2)$$

$$v_1(t) := \beta^{-1}(z_1)\left[u_1(t) - \alpha(z_1)\right]$$

$$v_2(t) := \beta^{-1}(z_2)\left[u_2(t) - \alpha(z_2)\right]$$

$$x_1(t) := x(t, z_0, v_1)$$

$$x_2(t) := x(t, z_0, v_2)$$

It is easy to see that

$$z_1 \equiv x_1 \,, \text{ and } z_2 \equiv x_2 \tag{2.6}$$

Therefore,

$$\begin{aligned} C\ x_1 &\equiv C\ z_1 \\ &\equiv C\ z_2 && \text{by (2.5)} \\ &\equiv C\ x_2 \end{aligned}$$

Since (2.3) is left-invertible, the last equality implies that

$$v_1 \equiv v_2 \tag{2.7}$$

which in turn implies that

$$x_1 \equiv x_2 \tag{2.8}$$

Therefore, by (2.6)

$$z_1 \equiv z_2 \tag{2.9}$$

Going back to the definition of v_1, and v_2, it is easy to see that (2.7), (2.9), together with invertibility of $\beta(.)$ imply that

$$u_1 \equiv u_2$$

which is a contradiction. This concludes the proof of (i).

ii) System (2.4) is minimum-phase.

Proof of (ii): To prove (ii), we transform system (2.3) into the special coordinate basis of Saberi-Sannuti (1987) (For more information on this transformation, refer to section 4.2). It has been proved by Saberi-Sannuti (1987) that, due to left-invertibility of (2.3), there exist nonsingular transformations Γ, Γ_{out}, Γ_{in}, integers K, q_a, q_b, q_i, r_i, $i=1,\ldots,K$ such that the transformation

$$z = \Gamma \begin{bmatrix} \tilde{z}_a \\ \tilde{z}_b \\ \tilde{z}_f \end{bmatrix}, \quad y = \Gamma_{out} \begin{bmatrix} \tilde{y}_f \\ \tilde{y}_s \end{bmatrix}, \quad v = \Gamma_{in}\, \tilde{v}(.) \tag{2.10}$$

transforms system (2.3) into the following form:

$$\begin{cases} \dot{\tilde{z}}_a = A_{aa}\tilde{z}_a + A_{af}\tilde{y}_f + A_{as}\tilde{y}_s \\ \dot{\tilde{z}}_b = A_{bb}\tilde{z}_b + A_{bf}\tilde{y}_f \\ \dot{\tilde{z}}_f = A_f\tilde{z}_f + M_f\tilde{y}_f + B_f\left[D_a\tilde{z}_a + D_b\tilde{z}_b + D_f\tilde{z}_f + \tilde{v}\right] \end{cases} \tag{2.11}$$

$$\begin{cases} \tilde{y}_f = C_f\tilde{z}_f \\ \tilde{y}_s = C_s\tilde{z}_b \end{cases}$$

where the dimensions of $\tilde{z}_a$, $\tilde{z}_b$, $\tilde{z}_f$, $\tilde{y}_f$, and $\tilde{y}_s$ are q_a, q_b, $\sum_{i=1}^{K} iq_i$, $\sum_{i=1}^{K} q_i$, and $r - \sum_{i=1}^{K} q_i$, respectively. Moreover, invariant zeros of (C, A, B) are the eigenvalues of A_{aa}, (C_s, A_{bb}) is observable, and A_f, B_f, C_f, A_{bb}, and C_s have the following canonical structure,

$$A_f := Block\ Diag\ (A_{1f}, \ldots, A_{Kf})$$

$$B_f := Block\ Diag\ (B_{1f}, \ldots, B_{Kf})$$

$$C_f := Block\ Diag\ (C_{1f}, \ldots, C_{Kf})$$

$$A_{bb} := Block\ Diag\ (A_{1bb}, \ldots, A_{Kbb})$$

$$C_s := Block\ Diag\ (C_{1s}, \ldots, C_{Ks})$$

where $A_{1f} = 0_{q_1 \times q_1}$, $B_{1f} = I_{q_1}$, $C_{1f} = I_{q_1}$, $A_{1bb} = 0_{r_1 \times r_1}$, $C_{1s} = I_{r_1}$, if $i=1$,

while

$$A_{if} = \begin{bmatrix} 0_{l_i \times q_i} & I_{l_i} \\ 0_{q_i \times q_i} & 0_{q_i \times l_i} \end{bmatrix}, \quad B_{if} = \begin{bmatrix} 0_{l_i \times q_i} \\ I_{q_i} \end{bmatrix}, \quad C_{if} = \begin{bmatrix} I_{q_i} & 0_{q_i \times l_i} \end{bmatrix}$$

$$A_{ibb} = \begin{bmatrix} 0_{m_i \times r_i} & I_{m_i} \\ 0_{r_i \times r_i} & 0_{r_i \times m_i} \end{bmatrix}, \quad C_{is} = \begin{bmatrix} I_{r_i} & 0_{r_i \times m_i} \end{bmatrix}$$

$$l_i = (i-1) \times q_i \ , \quad m_i = (i-1) \times r_i$$

for $i > 1$.

Now,

$$\begin{aligned} \tilde{v} &= \Gamma_{in}^{-1}\, v \\ &= \Gamma_{in}^{-1}\, \beta^{-1}(z) \left[u - \alpha(z) \right] \\ &= \Gamma_{in}^{-1}\, \beta^{-1}(\Gamma \tilde{z}) \left[u - \alpha(\Gamma \tilde{z}) \right] \end{aligned}$$

$$:= \tilde{\beta}^{-1}(\tilde{z})\left[u - \tilde{\alpha}(\tilde{z})\right]$$

Therefore, transformation (2.10) transforms the nonlinear system (2.4) into the following form:

$$\begin{cases} \dot{\tilde{z}}_a = A_{aa}\tilde{z}_a + A_{af}\tilde{y}_f + A_{as}\tilde{y}_s \\ \dot{\tilde{z}}_b = A_{bb}\tilde{z}_b + A_{bf}\tilde{y}_f \\ \dot{\tilde{z}}_f = A_f\tilde{z}_f + M_f\tilde{y}_f + B_f\left[D_a\tilde{z}_a + D_b\tilde{z}_b + D_f\tilde{z}_f + \tilde{\beta}^{-1}(\tilde{z})\left[u - \tilde{\alpha}(\tilde{z})\right]\right] \end{cases} \tag{2.12}$$

$$\begin{cases} \tilde{y}_f = C_f\tilde{z}_f \\ \tilde{y}_s = C_s\tilde{z}_b \end{cases}$$

Let

$$N_0 := span \;\; \tilde{z}_a$$

Due to the canonical structure of A_f, C_f, A_{bb}, and C_s,

$$N_0 = Ker \;\; Cz$$

Let

$$u(\tilde{z}) := \tilde{\alpha}(\tilde{z}) - \tilde{\beta}(\tilde{z})\, D_a\; \tilde{z}_a \tag{2.13}$$

On N_0, the closed-loop system (2.12) and (2.13) is

$$\begin{cases} \dot{\tilde{z}}_a = A_{aa}\tilde{z}_a \\ \dot{\tilde{z}}_b = 0 \\ \dot{\tilde{z}}_f = 0 \end{cases} \tag{2.14}$$

Therefore, the direction of the vector field on N_0 is tangent to it. Moreover, N_0 is clearly maximal. Therefore, system (2.14) defines the zero dynamics of system (2.4).

Since system (2.3) is minimum-phase and detectable, the invariant zeros of (C, A, B) (which are the eigenvalues of A_{aa}) are in the open left-half complex plane. Therefore, system (2.14) is globally asymptotically stable, which implies that system (2.4) is minimum-phase. This concludes the proof of (ii). The necessity proof is very similar to the sufficiency proof, and hence is deleted.

Assumption 2.G1: System (2.2), i.e., the nominal system, is fully-linearizable via state feedback, left-invertible, and minimum-phase on $S \subset \mathbf{R}^p$, where S is an open connected set containing the origin.

It may be argued that Assumption 2.G1 is restrictive. However, techniques of exact linearization have been applied to a number of interesting physical problems in robotics, control of electric power system, and flight control (refer to references given later in this section). Most of these problems satisfy Assumption 2.G1 for a meaningful choice of output variables.

Example 2.1: Motion of a robotic arm may be described by the following dynamic equation [Brady, et.al. (1982)],

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = D^{-1}(\xi_1) \left[u - E(\xi_1, \xi_2) \right] \end{cases} \tag{2.15}$$

where $\xi_1 \in \mathbf{R}^n$, and $\xi_2 \in \mathbf{R}^n$ are the angular position and speed of the joints, respectively. $u \in \mathbf{R}^n$ denotes the driving torques of the joints, $E(\,.\,,\,.\,)$ represents coriolis, centrifugal, and gravitational forces, and $D(\,.\,)$ is the inertia matrix. Assuming that all the states of the system are available for feedback, the nonlinear terms in (2.15) can be canceled by

$$u = E(\xi_1, \xi_2) + D(\xi_1)\, v \tag{2.16}$$

Cancelling the nonlinear terms as is done by (2.16) is known in the robotics literature as the method of computed torque. To control the system without using measurements of angular speeds of the joints, define the output vector

$$y = \xi_1 \tag{2.17}$$

Applying (2.16) to system (2.15) and (2.17) results in the linear system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = v \\ y = \xi_1 \end{cases} \tag{2.18}$$

Therefore, system (2.15) and (2.17) is fully-linearizable via state feedback. Moreover, system (2.18) is invertible and has no zero dynamics, which implies that the nonlinear system (2.15) and (2.17) is invertible and has no zero dynamics (by Proposition 2.1).

Example 2.2: As another example, consider the following model for nonlinear excitation control of two interconnected synchronous generators, studied by Ilic-Mak (1989).

$$\begin{cases} \dot{\delta}_k = \omega_k - \omega_0 \\ \dot{\omega}_k = \dfrac{\omega_0}{2H_k} [\, -E_{qk}' i_{qk} - \dfrac{D_k}{\omega_0}(\omega_k - \omega_0) + T_{mk} \,] \\ \dot{E}_{qk}' = \dfrac{1}{T'_{dok}} [\, -E_{qk}' - (L_{dk} - L_{dk}')i_{dk} + E_{fdk} \,] \end{cases} \tag{2.19}$$

for $k=1, 2$, where δ_k is the rotor angle, ω_k is the rotor speed, E_{qk}' is a voltage proportional to damper winding flux linkage, and the currents i_{qk} and i_{dk} are defined by nonlinear functions of δ_k and E_{qk}'. The control variables in (2.19) are E_{fdk}'s, which are the field voltages. Refer to [Ilic-Mak (1989)] for the details. It is shown in [Ilic-Mak (1989)] that the change of variables $z_{1k}=\delta_k$, $z_{2k}=\dot{\delta}_k$, and $z_{3k}=\dot{\omega}_k$ transforms (2.19) into the form

$$\begin{cases} \dot{z}_{1k} = z_{2k} \\ \dot{z}_{2k} = z_{3k} \\ \dot{z}_{3k} = \rho_k(z) + \beta_k(z)\, u_k \end{cases} \tag{2.20}$$

where $\beta_k(.)$'s are invertible. Therefore, assuming that all the states of the system are available for feedback, the nonlinear terms can be canceled by the control $u_k = \beta_k^{-1}(z)\, [\, -\rho_k(z) + v_k\,]$. This requires measurement of the rotors' angles, speeds, and accelerations. To control the system without using acceleration measurements, define the output vector

$$y = [z_{11}, z_{21}, z_{12}, z_{22}]'$$

The dynamics of the system can be represented by

$$\begin{cases} \dot{x} = Ax + B\ [\, \rho(x) + \psi(x)u\,] \\ y = Cx \end{cases} \tag{2.21}$$

where $x = [\, z_{11}, z_{21}, z_{31}, z_{12}, z_{22}, z_{32}\,]'$, $A = diag\ [\, A_1, A_2\,]$, $B = diag\ [\, B_1, B_2\,]$, $C = diag\ [\, C_1, C_2\,]$, $\rho = [\, \rho_1, \rho_2\,]'$, $\psi = diag\ [\, \beta_1, \beta_2\,]$, and $u = [\, u_1, u_2\,]'$, where the matrices A_k, B_k, and C_k are given by

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ B_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.22}$$

The 4×2 transfer function $C(sI_6 - A)^{-1}B$ is block diagonal, with the diagonal blocks $[\frac{1}{s^3}, \frac{1}{s^2}]'$. This transfer function is left-invertible and has no transmission zeros; hence it is minimum-phase.

Other physical examples which are left-invertible, minimum-phase and fully-linearizable for a meaningful set of output variables include the helicopter model of Meyer-Su-Hunt (1984) and the spacecraft with gas-jet actuator model of Dwyer (1984). In the case of the helicopter model, the 12th order system is fully-linearizable with measurement of four state variables, three of which define the position and the forth is one of the three attitude angles (r_1, r_2, r_3, and ϕ_3 in the notation of Meyer-Su-Hunt (1984)). In the case of the spacecraft model, the 6th order model is fully-linearizable with measurement of three state variables which determine the attitude of the body with respect to an inertial reference frame (γ_1, γ_2, and γ_3 in the notation of Dwyer (1984)).

3 State Feedback Control

3.1 Introduction

In this chapter we study the problem of state feedback stabilization of the class of nonlinear systems defined in Chapter 2, i.e., system (2.1) under Assumption 2.G1. To motivate the discussion, let us see how the nominal system (2.3) can be stabilized. System (2.2) is fully-linearizable, i.e., the transformation $z = T(\xi)$ and the control

$$u = \hat{\alpha}(\xi) + \hat{\beta}(\xi)v \tag{3.1}$$

transform (2.2) into the linear system (2.3). Since (A, B) is controllable, one can find a gain matrix K such that $A + BK$ is Hurwitz. Therefore, the following control

$$u = \hat{\alpha}(\xi) + \hat{\beta}(\xi)\, K\; T(\xi) \tag{3.2}$$

renders the origin of the nominal system (2.2) asymptotically stable.

To design a state feedback controller for system (2.1), let us transform (2.1) into the z-coordinates via $z = T(\xi)$. This transformation transforms (2.1) into

$$\begin{cases} \dot{z} = A\ z + B\ \beta^{-1}(z)\left[u + \Delta_f(z,t) + \Delta_g(z,t)\ u + \Delta_{ed}(z,t) - \alpha(z) \right] \\ y = C\ z \end{cases} \tag{3.3}$$

where

$$\alpha(z) := \hat{\alpha}(\xi)$$

$$\beta(z) := \hat{\beta}(\xi)$$

$$\Delta_f(z,t) := \hat{\Delta}_f(\xi, t)$$

$$\Delta_g(z,t) := \hat{\Delta}_g(\xi, t)$$

$$\Delta_{ed}(z,t) := \hat{\Delta}_{ed}(\xi, t)$$

Similar to the case of the nominal system, control law (3.1) can be applied in this case to cancel the nonlinear terms $\alpha(.)$, and $\beta(.)$. However, in practice, exact cancellation of the nonlinear terms are usually either undesirable due to their complexity, or impossible due to parametric uncertainties. Therefore, instead of exact cancellation, the following control may be used

$$u = \overline{\alpha}(z) + \overline{\beta}(z)\ v \tag{3.4}$$

where $\overline{\alpha}(.)$ and $\overline{\beta}(.)$ are nominal or simplified versions of $\alpha(.)$ and $\beta(.)$. Applying (3.4) to system (3.3) results in

$$\begin{cases} \dot{z} = Az + Bv + B\,\delta(z, v, t) \\ y = Cz \end{cases} \tag{3.5}$$

where

$$\delta(z, v, t) := \beta^{-1}(z)\left[(\Delta_\alpha + \Delta_g\overline{\alpha} + \Delta_f) + (\Delta_\beta + \Delta_g\overline{\beta})\ v + \Delta_{ed} \right] \tag{3.6}$$

and

$$\Delta_\alpha := \bar{\alpha}(z) - \alpha(z)$$
$$\Delta_\beta := \bar{\beta}(z) - \beta(z)$$

The effect of the uncertainties and the simplification of α and β have appeared in (3.5) as a disturbance term. Therefore, the control v should be designed to stabilize (3.5) in the presence of $\delta(z, v, t)$. Since the disturbance term $\delta(z, v, t)$ satisfies the matching condition, such a stabilizing control can be designed under an assumption on the growth of $\delta(z, v, t)$,

Assumption 3.G2: The following inequalities are satisfied for all $z \in S$, $t \in \mathbf{R}^+$

$$\| \beta^{-1} (\Delta_\alpha + \Delta_g \bar{\alpha} + \Delta_f) \| \le k_1 \| z \|$$

$$\| \beta^{-1} (\Delta_\beta + \Delta_g \bar{\beta}) \| \le k_2, \quad k_2 < \theta$$

$$\| \beta^{-1} \Delta_{ed} \| \le k_3 \| z \| + k_4$$

where $S \subset \mathbf{R}^p$ is an open connected set containing the origin, k_i's are nonnegative constants, and θ is a constant that depends on the robust control technique being used. Later in this chapter, we will say more about θ.

There are several methods in the literature for designing such a stabilizing state feedback control. In particular, variable structure control [Utkin (1987)], min-max control [Corless-Leitmann (1981)], or linear high-gain control [Barmish-Corless-Leitmann (1983)] can be used to stabilize system (3.5). Using such techniques, one finds a state feedback control

$$v = \phi(z), \qquad \phi(0) = 0 \tag{3.7}$$

together with a quadratic Lyapunov function

$$W(z) = z'Pz, \quad P \text{ symmetric positive definite} \tag{3.8}$$

such that, under Assumption 3.G2, the derivative of W along the trajectory of the closed-loop system (3.5) and (3.7) satisfies the following inequality for all $z \in S$

$$\frac{dW}{dt} \le -\gamma_2 \| z \|^2 + \gamma_1 \| z \| + \gamma_0 \tag{3.9}$$

where $\gamma_2 > 0$, $\gamma_1 \ge 0$, and $\gamma_0 \ge 0$.

Example 3.1 (Linear High-Gain Control): As an example of the kind of state feedback control that we are interested in, we quickly go over the linear high-gain control, introduced in the work of Barmish-Corless-Leitmann (1983). In this technique, one starts by choosing K such that $\hat{A} := A - BK$ is Hurwitz. Then the state feedback control is chosen to be

$$v = -Kz - \frac{1}{\zeta} B' P z \tag{3.10}$$

where $\zeta > 0$ is a constant to be chosen, and P is the symmetric positive definite solution of the Lyapunov equation $P\hat{A} + \hat{A}'P = -I_p$. Consider the Lyapunov function candidate $W(z) = z'Pz$. The derivative of W along the trajectory of the closed-loop system (3.5) and (3.10) is

$$\dot{W} = -\| z \|^2 - \frac{2}{\zeta} \| B'Pz \|^2 + 2z'PB\,\delta(z, v, t) \tag{3.11}$$

$$\le -\| z \|^2 - \frac{2}{\zeta} (1-k_2) \| B'Pz \|^2$$

$$+ 2 \| B'Pz \| \left[(k_1 + k_3 + k_2 \| K \|) \| z \| + k_4 \right] \tag{3.12}$$

by Assumption 3.G2. It can be seen that to preserve the second term on the right-hand side of (3.12) as a negative quadratic term k_2 should be strictly less than 1 (θ of Assumption 3.G2 is 1 in this technique). It can be shown from (3.12) that

$$\dot{W} \leq -\left[\, 1 - \frac{\zeta}{2(1-k_2)}(k_1+k_3+k_2\,\| K \|)^2 \,\right] \| z \|^2$$

$$+ \zeta \frac{k_4}{1-k_2}(k_1+k_3+k_2\,\| K \|)\,\| z \| + \zeta \frac{k_4^2}{2(1-k_2)} \tag{3.13}$$

where we have used the fact that

$$- ay^2 + b\ y \leq \frac{b^2}{4a} \quad \text{for} y \geq 0,\, b \geq 0, \text{ and } \ a > 0$$

Choose ζ such that

$$1 - \frac{\zeta}{2(1-k_2)}(k_1+k_3+k_2\,\| K \|)^2 \geq \frac{1}{2}$$

Then,

$$\dot{W} \leq -\frac{1}{2}\,\| z \|^2 + \zeta \frac{k_4}{1-k_2}(k_1+k_3+k_2\,\| K \|)\,\| z \|$$

$$+ \zeta \frac{k_4^2}{2(1-k_2)} \tag{3.14}$$

which is inequality (3.9).

In inequality (3.9), if γ_1 and γ_2 are zero, then (3.9) implies that the origin of the closed-loop system (3.5) and (3.7) is asymptotically stable. In general, when γ_1 and γ_2 are not zero, inequality (3.9) implies that the trajectory of the closed-loop system converges to a neighborhood of the origin. This property is known as uniform ultimate boundedness, which is defined in the following way:

Definition 3.1: Consider

$$\dot{x} = F(x, t) \tag{3.15}$$

where $x \in \mathbf{R}^k$, and let $N_\mu(\Omega)$ denote the μ-neighborhood of set Ω ,i.e.,

$$N_\mu(\Omega) := \left\{ z \in \mathbf{R}^k \mid inf_{y \in \Omega} \| z-y \| < \mu \right\}$$

System (3.15) is said to be *uniformly ultimately bounded (U.U.B.) with respect to the set* $\Omega \subset \mathbf{R}^k$ *with* $\Sigma \subset \mathbf{R}^k$ *inside the region of attraction*, if for every $x_0 \in \Sigma$ and $\mu > 0$ there exists $T \geq 0$ such that the solution $x(.) : [t_0, \infty) \rightarrow \mathbf{R}^k$ of (3.15) with $x(t_0) = x_0$ satisfies the following for all $t_0 \in \mathbf{R}$:

$$x(t) \in N_\mu(\Omega), \quad \text{for all } t \geq t_0 + T$$

Remark 3.1: Definition 3.1 is a modified version of the conventional definition of UUB found in the work of Corless-Leitmann (1981). The modification allows us to present our stability results in a concise way.

The following proposition gives an estimate of the set with respect to which the closed-loop system (3.5) and (3.7) is UUB, when inequality (3.9) is satisfied. The estimate given in this proposition is a special case of the one given by Leitmann (1981). Nevertheless, we have included the proof in Appendix A, since it contains certain technicalities that arise due to Definition 3.1.

Proposition 3.1: Let

$$\Omega_c := \left\{ z \in \mathbf{R}^p \mid W(z) \leq c \right\} \tag{3.16}$$

$$\sigma := \lambda_{max}(P) \left\{ \frac{\gamma_1}{2\gamma_2} + \left[\frac{\gamma_1^2}{4\gamma_2^2} + \frac{\gamma_0}{\gamma_2} \right]^{\frac{1}{2}} \right\}^2 \tag{3.17}$$

and suppose there exists $r > \sigma$ such that $\Omega_r \subset S$. Then, inequality (3.9) implies that the closed-loop system (3.5) and (3.7) is U.U.B. with respect to Ω_σ with Ω_r inside the region of attraction.

Example 3.1 (Continued): In Example 3.1 we found that $\dot{W}$ satisfies inequality (3.14). If k_4 in Assumption 3.G2 is zero (which corresponds to the case when Δ_{ed} vanishes at the origin), then (3.14) implies that the origin of the closed-loop system (3.5) and (3.10) is asymptotically stable. When k_4 is not zero, the closed-loop system is UUB with respect to the set Ω_σ as given by Proposition 3.1. Note that σ goes to zero, as ζ goes to zero. In other words, the set Ω_σ can be made arbitrarily small by increasing the gain of the second term in (3.10).

The state feedback control of this chapter will be used as the state feedback component of our observer-based control. The analysis of the next chapter (refer to section 4.4) shows that in order to use the state feedback controller in this context, the control law (3.7) has to satisfy a Lipschitz condition (Assumption 4.G3 in chapter 4). The linear high-gain control of Example 3.1 satisfies a global Lipschitz condition. However, min-max control and variable structure control are discontinuous control laws that do not satisfy any Lipschitz condition. Therefore, we have to use continuous approximations of such control laws. Continuous approximations of min-max control has been introduced by Corless-Leitmann (1981). Following the development of Corless-Leitmann (1981), one can come up with a continuous control law (3.7) and a quadratic Lyapunov function (3.8) that satisfy (3.9). Continuous approximations of variable structure control has been discussed by Slotine-Sastry (1983), Slotine (1984), and Ryan-Corless (1984). However, There is no result in the literature on how to obtain a Lyapunov function of the form (3.8) to satisfy inequality (3.9). Therefore, in the next two sections, we study the problem of finding such a Lyapunov function. Variable Structure Control can be applied to a very large class of nonlinear system, namely those systems that can be transformed into the regular form. Feedback linearizable systems is only a small subset of this class. Therefore, in section 3.2 we present the stability results in the general framework of variable structure control, since these results are of interest on their own. Then, in section 3.3, we specialize the results of section 3.2 to the case of fully-linearizable systems.

3.2 Continuous Implementation of Variable Structure Control

To design a variable structure control law for system (2.1), first (2.1) is transformed via a smooth change of coordinates

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \tilde{T}(\xi), \quad x_1 \in \mathbf{R}^{p-q}, x_2 \in \mathbf{R}^q, \tilde{T}(0) = 0 \tag{3.18}$$

into the following so-called regular form [Utkin (1987)], [DeCarlo-Zak-Mathews (1988)]:

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2,t) \\ \dot{x}_2 = f_2(x_1,x_2,t) + B(x_1,x_2,t)\,u \\ \qquad + B(x_1,x_2,t)\left[\Delta_f(x_1,x_2,t) + \Delta_g(x_1,x_2,t)\,u + \Delta_{ed}(x_1,x_2,t)\right] \end{cases} \tag{3.19}$$

where $B(x_1,x_2,t)$ is nonsingular for all $x_1 \in \mathbf{R}^{p-q}$, $x_2 \in \mathbf{R}^q$ and $t \in \mathbf{R}$. The arguments x_1, x_2 and t are deleted for the sake of brevity, whenever no confusion is likely to arise. A special case of (3.19) that was treated by Slotine-Sastry (1983) and Ryan-Corless (1984) is the case when

$$f_1(x_1, x_2, t) = A_{11}x_1 + A_{12}x_2$$

and B is a constant matrix. In this case the nominal state equation is linearizable via state feedback control.

After transforming system (2.1) into the regular form (3.19), a function $\rho(.)$ is found to satisfy the following assumption,

Assumption 3.1: There exists a continuously differentiable function $\rho : \mathbf{R}^{p-q} \to \mathbf{R}^q$ such that $\rho(0) = 0$ and the system

$$\dot{x}_1 = f_1(x_1, \rho(x_1), t) \tag{3.20}$$

has a globally uniformly asymptotically stable equilibrium point at $x_1 = 0$.

We also need to assume that the uncertainty in the input distribution matrix Δ_g is small enough,

Assumption 3.2: $\| B \; \Delta_g \; B^{-1} \| \le \delta_u < 1$

where δ_u is a nonnegative constant.

Then, under Assumptions 3.1 and 3.2, the variable structure control is chosen to be

$$u = -\, \phi(x,t) \, B^{-1} \, sgn\,(s) \tag{3.21}$$

where

$$s = x_2 - \rho(x_1) \tag{3.22}$$

$$[sgn\,(s)]_i := sgn\; s_i \qquad i = 1, \ldots, q \tag{3.23}$$

and $\phi(.,.,.)$ is a scalar-valued function that satisfies the following inequality for all $x \in \mathbf{R}^p$, $t \in \mathbf{R}$ and any arbitrary positive constant α :

$$\phi(x,t) \ge \frac{1}{1-\delta_u} \left[\alpha + \| f_2 + B\Delta_f + B\Delta_{ed} - \frac{\partial \rho}{\partial x_1} f_1 \|_\infty \right] \tag{3.24}$$

The surface $s = 0$ is known as the sliding surface.

The stability analysis of the closed-loop system (3.19) and (3.21)-(3.24) is done in the following way: It can be easily shown that, under Assumption 3.2, the derivative of the function $\frac{1}{2} s's$ along the trajectory of the closed-loop system satisfies the following inequality,

$$\frac{d}{dt}\left(\frac{1}{2}s's\right) = s'\dot{s} \leq -\alpha \| s \|_1 \tag{3.25}$$

Due to (3.25), the trajectory reaches the sliding surface in finite time and on the sliding surface Assumption 3.1 implies uniform asymptotic stability of the origin. Note that in this argument, no Lyapunov function is obtained for the closed-loop system.

Now let us replace the signum function in (3.21) by a saturation function of the form [Slotine-Sastry (1983)]:

$$[sat_\zeta(s)]_i := \begin{cases} sgn\ s_i & \text{if } |s_i| \geq \zeta \\ \dfrac{s_i}{\zeta} & \text{otherwise} \end{cases} \qquad i = 1, \ldots, q \tag{3.26}$$

i.e., we are considering the following continuous approximation of (3.21),

$$u = -\phi(x,t)\, B^{-1}\, sat_\zeta(s) \tag{3.27}$$

The control law (3.27) causes the trajectory of the closed-loop system to converge to a boundary-layer set

$$\Omega_2 = \left\{ x \in \mathbf{R}^p \ :\ |s_i| \leq \zeta,\ i=1, \ldots, q \right\}$$

in finite time. Since within the boundary layer Ω_2 the trajectory is not necessarily confined to the sliding surface $s = 0$, we are forced to work with a perturbed version of equation (3.20) rather than equation (3.20) itself, namely,

$$\dot{x}_1 = f_1(x_1, \rho(x_1), t) + \left[f_1(x_1, x_2, t) - f_1(x_1, \rho(x_1), t) \right] \tag{3.28}$$

To preserve the stability properties of the unperturbed system (3.20), we need to impose a growth assumption on the perturbation term. To state this assumption we use standard converse Lyapunov theorems, e.g., [Hahn (1967)], which, under

Assumption 3.1, guarantee the existence of a Lyapunov function $V(x_1, t)$ and functions α_1, α_2 and α_3 of class K_∞ such that for all $x_1 \in \mathbf{R}^{p-q}$ and $t \in \mathbf{R}$,

$$\alpha_1(\|x_1\|) \le V(x_1, t) \le \alpha_2(\|x_1\|) \tag{3.29}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(x_1, \rho(x_1), t) \le -\alpha_3(\|x_1\|) \tag{3.30}$$

A function $\gamma : \mathbf{R}^+ \to \mathbf{R}^+$ is said to be of class K_∞, if it is continuous, strictly increasing, $\gamma(0) = 0$, and $\gamma(r) \to \infty$ as $r \to \infty$. Note that if γ is of class K_∞, then γ^{-1} is of class K_∞, and if γ_1 and γ_2 are of class K_∞, then $\gamma_1 \circ \gamma_2$ is also of class K_∞. The fact that α_3 in (3.30) is of class K_∞ is not shown in [Hahn (1967)], but has been shown recently by Sontag (1989).

Assumption 3.3: The Lyapunov function $V(x_1, t)$ of Inequalities (3.29) and (3.30) satisfies the following inequality for all $x \in \mathbf{R}^p$ and all $t \in \mathbf{R}$,

$$\frac{\partial V}{\partial x_1}\Big[f_1(x_1, x_2, t) - f_1(x_1, \rho(x_1), t)\Big] \le$$

$$\alpha_4(\|x_1\|)\, \alpha_5(\|x_2 - \rho(x_1)\|) \tag{3.31}$$

where $\alpha_4(r)\alpha_6(r) \le \alpha_3(r)$ and $\alpha_4(.)$, $\alpha_5(.)$ and $\alpha_6(.)$ are of class K_∞.
When the origin of (3.20) is globally exponentially stable, Assumption 3.3 reduces to the requirement that $f_1(.,.,.)$ be globally Lipschitzian in x_2. This follows from the well-known result [Hahn (1967)] that in the case of exponential stability $V(x_1, t)$ can be chosen such that $\alpha_i(r) = K_i\, r^2$, i=1, 2, 3 and $\left\|\frac{\partial V}{\partial x_1}\right\| \le K \|x_1\|$. Then $\alpha_4(.)$, $\alpha_5(.)$, and $\alpha_6(.)$ take the form $\alpha_i(r) = K_i\, r$, i=4, 5, 6.

Theorem 3.1: Under Assumptions 3.1-3.3, there exists a class K_∞ function $\beta(.)$ such that the closed-loop system (3.19), (3.24), and (3.26)-(3.27) is G.U.U.B.

with respect to the closed ball

$$\Omega_\zeta = \left\{ x \in \mathbf{R}^p \;\mid\; \| x \| \le \beta(\zeta) \right\}.$$

Proof of Theorem 3.1:

Equations (3.19), (3.22) and (3.27) imply that

$$\dot{s} = f_2 - B\left[I_q + \Delta_g\right]\phi B^{-1} sat_\zeta(s) + B\,\Delta_f + B\,\Delta_{ed} - \frac{\partial\rho}{\partial x_1} f_1$$

$$\Rightarrow\; s_i\,\dot{s}_i = -\phi\, s_i\,[sat_\zeta(s)]_i + s_i\,[f_2 - \phi B\,\Delta_g\,B^{-1} sat_\zeta(s)$$

$$+ B\,\Delta_f + B\,\Delta_{ed} - \frac{\partial\rho}{\partial x_1} f_1]_i$$

$$\le -\phi\, s_i\,[sat_\zeta(s)]_i + |s_i| \left[\left\| f_2 + B\Delta_f + B\Delta_{ed} - \frac{\partial\rho}{\partial x_1} f_1 \right\|_\infty \right.$$

$$\left. + \phi \left\| B\Delta_g B^{-1} sat_\zeta(s) \right\|_\infty \right]$$

$$\le -\phi\, s_i\,[sat_\zeta(s)]_i + |s_i| \left[\phi - \alpha\right] \qquad \text{by Assumption 3.2 and (3.24)}$$

$$\le -\alpha\,|s_i| \quad \text{if } |s_i| \ge \zeta \qquad \text{by (3.26)}$$

which implies that Ω_2 is an invariant set and any trajectory starting outside Ω_2 reaches it in finite time.

Let us calculate the derivative of $V(x_1, t)$ along the trajectory of the closed-loop system.

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(x_1, x_2, t)$$

$$= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(x_1, \rho(x_1), t)$$

$$+ \frac{\partial V}{\partial x_1} \Big[f_1(x_1, x_2, t) - f_1(x_1, \rho(x_1), t) \Big]$$

$$\le -\alpha_3(\| x_1 \|) + \alpha_4(\| x_1 \|)\, \alpha_5(\| s \|)$$

by (3.30) and (3.31). Inside Ω_2, $\| s \| \le K\zeta$, where K depends on the type of norm. Hence

$$\dot{V} \le -\alpha_3(\| x_1 \|) + \alpha_4(\| x_1 \|)\, \alpha_5(K\zeta)$$

$$\le -\frac{1}{2}\alpha_3(\| x_1 \|) - \alpha_4(\| x_1 \|) \left[\frac{1}{2}\, \alpha_6(\| x_1 \|) - \alpha_5(K\zeta) \right]$$

which shows that $\dot{V} \le -\frac{1}{2}\alpha_3(\| x_1 \|)$ for $\| x_1 \| \ge \alpha_6^{-1}[2\alpha_5(K\zeta)]$. Let $\beta_1 = \alpha_6^{-1} \circ [2\alpha_5]$, $\beta_2 = \alpha_2 \circ \beta_1$ and $\beta_3 = \alpha_1^{-1} \circ \beta_2$ and define the sets Ω_1, M_1, and M_2 by

$$\Omega_1 = \left\{ x \in \mathbf{R}^p \;\mid\; V(x_1, t) \le \beta_2(K\zeta) \right\}$$

$$M_1 = \left\{ x \in \mathbf{R}^p \;\mid\; \| x_1 \| \le \beta_1(K\zeta) \right\}$$

$$M_2 = \left\{ x \in \mathbf{R}^p \;\mid\; \| x_1 \| \le \beta_3(K\zeta) \right\}$$

The set Ω_1 is dependent on t, but using (3.29), it can be verified that $M_1 \subset \Omega_1 \subset M_2$, uniformly in t.

Now any trajectory starting outside Ω_2 must enter Ω_2 in finite time and remain thereafter. Moreover on the set $\Omega_2 - M_1$, $\dot{V} \le -\frac{1}{2}\alpha_3(\| x_1 \|)$. Thus, the trajectory must enter the set $\Omega_1 \cap \Omega_2$ in finite time, and it remains in the set for all t thereafter, since $\dot{V}$ is negative on the boundary of Ω_1. Hence, there exists a finite time T such that

$$x(t) \in M_2 \cap \Omega_2 \text{ for all } t \geq T$$

Since $\rho(x_1)$ is continuously differentiable for all $x_1 \in \mathbf{R}^{p-q}$, $\| \rho(x_1) \| \leq \gamma(\| x_1 \|)$, where $\gamma(.)$ is of class K_∞. Setting

$$\beta(r) = \beta_3(Kr) + Kr + \gamma(\beta_3(Kr))$$

completes the proof of the theorem, since $M_2 \cap \Omega_2 \subset \Omega_\zeta$. □

We illustrate, via an example, that a growth condition like Assumption 3.3 is indeed needed.

Example 3.2: Consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + (x_1^2 + 1) x_2 \\ \dot{x}_2 = u + \Delta_{ed}(t) \end{cases} \tag{3.32}$$

where $|\Delta_2(t)| \leq 1.0$ is a disturbance term. Choose $\rho(x_1) = 0$. It can be easily verified that Assumptions 3.1 and 3.2 are satisfied. The discontinuous VSC law $u = -2\, sgn\,(x_2)$ yields $\dot{x}_2 x_2 \leq -|x_2|$. Hence, the trajectory reaches the sliding surface $x_2 = 0$ in finite time, and on the surface $x_2 = 0$, the motion is governed by $\dot{x}_1 = -x_1$, and the origin is globally asymptotically stable. Now consider the continuous VSC law $u = -2\, sat_\zeta\,(x_2)$. Taking $x_2(0)=1$, it can be verified that $x_2(t) \geq \frac{\zeta}{2}$ for all $t \geq 0$. Taking $x_1(0) = \frac{2}{\zeta}$ and using $x_2(t) \geq \frac{\zeta}{2}$, it can be verified that $x_1(t) \to \infty$ as $t \to \infty$. Thus, the system is not *globally* uniformly ultimately bounded.

Assumption 3.3 is dependent on the regular form in which the nonlinear system has been expressed. If system (2.1) is state-equation-linearizable, we can always transform (2.1) into a regular form for which Assumption 3.3 is satisfied ‡ . For instance for Example 3.2 the following change of coordinates

$$\begin{cases} z_1 = x_1 \\ z_2 = -x_1 + (x_1^2 + 1)\, x_2 \end{cases} \tag{3.33}$$

transforms system (3.32) into

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \dfrac{(z_1^2 + 2z_1z_2 - 1)\, z_2}{z_1^2+1} + (z_1^2+1)\,(u + \Delta_{ed}(t)) \end{cases} \tag{3.34}$$

For (3.34), let the sliding surface be $s = z_1 + z_2$. Following the procedure outlined earlier, the ideal variable structure control is

$$\begin{aligned} u &= -\frac{1}{z_1^2+1}\,\phi(z)\, sgn\,(s) \\ \phi(z) &= \frac{2|z_1z_2(z_1+z_2)|}{z_1^2+1} + 2 + z_1^2 \end{aligned} \tag{3.35}$$

However, system (3.34) clearly satisfies Assumption 3.3.

If system (2.1) is not state-equation linearizable, it may still be possible to satisfy Assumption 3.3 by a change of coordinates. For instance, consider the system

‡ This point was made by Professor J.J. Slotine of MIT in a personal discussion.

$$\begin{cases} \dot{x}_1 = -x_1 + x_1^2 x_2 \\ \dot{x}_2 = u + \Delta_{ed}(t) \end{cases} \tag{3.36}$$

where $\Delta_{ed}(t)$ is as in Example 3.2. If variable structure control is designed in the present coordinates, similar to Example 3.2, one can prove that continuous approximation of variable structure control would not have the global uniform ultimate boundedness property of the ideal vsc. Moreover, system (3.36) is not linearizable at the origin (To see this point, check the necessary and sufficient conditions for state-equation linearization in [Isidori (1989)]). However, we can use approximate linearization ideas of Hauser-Sastry-Kokotovic (1989) to design a variable structure control for which Assumption 3.3 is satisfied. To this end, consider the change of coordinates

$$\begin{cases} z_1 = x_1 \\ z_2 = -x_1 + (x_1^2 + a)x_2 \end{cases} \tag{3.37}$$

where a is an arbitrary positive constant. Transformation (3.37) transforms (3.36) into

$$\begin{cases} \dot{z}_1 = z_2 - \dfrac{a(z_1+z_2)}{z_1^2+a} \\ \dot{z}_2 = \left[-1 + \dfrac{2z_1(z_1+z_2)}{z_1^2+a} \right] \left[z_2 - \dfrac{a(z_1+z_2)}{z_1^2+a} \right] + (z_1^2+a)\,(\,u + \Delta_{ed}(t)\,) \end{cases} \tag{3.38}$$

Let the sliding surface be $s = z_1 + z_2$. Following the procedure outlined earlier, the variable structure control is designed to be

$$u = -\frac{1}{z_1^2+a}\left[1 + z_1^2 + a + \frac{2z_1^2\ |s|\ |z_1z_2-a|}{(z_1^2+a)^2}\right] sgn\ (s) \tag{3.39}$$

It can be easily checked that Assumption 3.3 is satisfied for this system.

Asymptotic Stability in the

Absence of Persistent Disturbance

Theorem 3.1 only shows G.U.U.B. with respect to Ω_ζ. Although Ω_ζ can be made arbitrarily small by choosing ζ small enough, the origin does not have to be asymptotically stable (refer to Example 3.3, below). In fact the origin might not be an equilibrium point at all. However, if external disturbances vanish at the origin, i.e., $\Delta_{ed}(0,0,t) = 0$, one might expect that the continuous implementation of VSC would stabilize the origin. It turns out that this is indeed the case, due to the fact that inside the boundary layer Ω_2, control (3.27) would act as a high-gain feedback control which stabilizes the origin, provided ζ is sufficiently small. To prove such a result we make the following assumption.

Assumption 3.4: There exists a Lyapunov function $W(x_1, t)$ such that the following inequalities hold for all $t \in \mathbf{R}$, and all x in a domain Ω containing the origin.

$$c_1 \| x_1 \|^2 \leq W(x_1,t) \leq c_2 \| x_1 \|^2 \qquad c_1 > 0\ , c_2 > 0 \tag{3.40}$$

$$\left\| \frac{\partial W}{\partial x_1} \right\| \leq c_3 \| x_1 \| \qquad c_3 \geq 0 \tag{3.41}$$

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_1} f_1(x_1,\rho(x_1),t) \leq -c_4 \| x_1 \|^2 \qquad c_4>0 \tag{3.42}$$

$$\| f_1(x_1,\xi_1,t) - f_1(x_1,\xi_2,t) \| \leq c_5 \| \xi_1 - \xi_2 \| \qquad c_5 \geq 0 \tag{3.43}$$

$$\| f_2 + B\,\Delta_f + B\,\Delta_{ed} - \frac{\partial \rho}{\partial x_1} f_1 \| \leq c_6 \| x \| \qquad c_6 \geq 0 \tag{3.44}$$

$$\| \rho(x_1) \| \leq c_7 \| x_1 \| \qquad c_7 \geq 0 \tag{3.45}$$

Assumption 3.4 is a mild one, since it is required to hold only locally. In fact, Assumption 3.4 is implied by the smoothness assumptions made earlier, together with the assumption that $\Delta_{ed}(0,0,t)=0$ and that $\dot{z} = A(t)\, z$ is uniformly asymptotically stable, where $A(t) = \frac{\partial}{\partial x_1} f_1(x_1,\rho(x_1),t) \Big|_{x_1=0}$.

Theorem 3.2: Under Assumptions 3.1-3.4, there exists $\zeta^*>0$ such that for all $\zeta < \zeta^*$, the origin is globally uniformly asymptotically stable.

Proof: Choose ζ_1^* small enough such that $\Omega_\zeta \subset \Omega$ for all $\zeta<\zeta_1^*$. In the proof of Theorem 3.1 it was shown that the trajectory enters the set $\Omega_1 \cap \Omega_2 \subset \Omega_\zeta$ in finite time and does not leave it thereafter. Inside this set, $u = -\phi B^{-1} \frac{s}{\zeta}$. Therefore

$$s'\dot{s} = -\phi \frac{s's}{\zeta} + s' \left[f_2 - \phi B\,\Delta_g B^{-1} \frac{s}{\zeta} + B\Delta_f + B\Delta_{ed} - \frac{\partial \rho}{\partial x_1} f_1 \right]$$

$$\leq -\phi(1-\delta_u) \frac{\| s \|^2}{\zeta} + c_6 \| s \| \| x \|$$

$$\leq -\frac{\alpha}{\zeta} \| s \|^2 + c_6 \| s \| (\| x_1 \| + \| s \| + c_7 \| x_1 \|)$$

Let $v(x_1, t) := W(x_1,t) + \frac{1}{2}s's$, then

$$\dot{v} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_1} f_1(x_1,x_2,t) + s'\dot{s}$$

$$\leq -c_4 \| x_1 \|^2 + c_3 c_5 \| x_1 \| \| s \| - \frac{\alpha}{\zeta} \| s \|^2$$

$$+ c_6 \| s \| \left[\| x_1 \| + \| s \| + c_7 \| x_1 \| \right]$$

$$\leq - \begin{bmatrix} \| x_1 \| & \| s \| \end{bmatrix} \begin{bmatrix} c_4 & -a \\ -a & \frac{\alpha}{\zeta} - c_6 \end{bmatrix} \begin{bmatrix} \| x_1 \| \\ \| s \| \end{bmatrix}$$

where $a = \frac{1}{2}[c_3 c_5 + c_6(1+c_7)]$. Thus $\dot{v}<0$ for $\zeta<\zeta_2^* =: \frac{c_4 \alpha}{2(c_4 c_6 + a^2)}$. Take $\zeta^* = \min\left\{\zeta_1^*, \zeta_2^*\right\}$. For all $\zeta<\zeta^*$, every trajectory enters $\Omega_1 \cap \Omega_2$ where $\dot{v}$ satisfies $\dot{v} \leq -cv$ for some $c>0$. Since the trajectory can not leave the set $\Omega_1 \cap \Omega_2$, it can be easily seen that it approaches the origin as $t \to \infty$. □

One important difference between Theorems 3.1 and 3.2 is that the conclusions of Theorem 3.1 holds for any $\zeta>0$, while the conclusion of Theorem 3.2 is guaranteed to hold only for sufficiently small ζ. The following example shows that if ζ is not small enough the origin may not be asymptotically stable, while the system is G.U.U.B. with respect to Ω_ζ.

Example 3.3: Consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = a\,x_1 + u\;, \qquad 0 < a \le 1 \end{cases}$$

Take $\rho(x_1) = 0$ and $\phi = \alpha + |x_1|$. Inside the boundary layer Ω_2, $u = -\frac{1}{\zeta}(\alpha + |x_1|)\,x_2$, and the closed-loop system is given by

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = a\,x_1 - \frac{1}{\zeta}\,(\alpha + |x_1|)\,x_2 \end{cases}$$

The Jacobian of the right-hand side at $x = 0$ is given by

$$A = \begin{bmatrix} -1 & 1 \\ a & -\frac{\alpha}{\zeta} \end{bmatrix}$$

It can be verified that when $\zeta > \frac{\alpha}{a}$, one of the eigenvalues of A is in the open right-half plane. Hence, the origin is unstable. On the other hand the set $\Omega_1 \cap \Omega_2$, given by

$$\Omega_1 \cap \Omega_2 = \left\{ x \in \mathbf{R}^2 \;\mid\; |x_1| \le \zeta,\; |x_2| \le \zeta \right\}$$

is an invariant set and every trajectory of the closed-loop system reaches it in finite time, irrespective of the value of ζ.

3.3 VSC Design for Fully-linearizable Systems

In this section, we focus on designing a continuous approximation of variable structure control for stabilization of (3.5). We will closely follow the development of last section, with some modifications in the control design. The first step of the control design is the choice of a sliding surface. Choose the sliding surface

$$s := G\,z = 0 \tag{3.46}$$

where G is a $q \times (p-q)$ matrix such that

i) GB is nonsingular.

ii) $G(sI - A)^{-1}B$ is minimum-phase.

iii) (G, A) is detectable.

Choosing G to satisfy (i)-(iii) essentially guarantees that Assumption 3.1 of section 3.2 is satisfied. Consider the following continuous approximation of variable structure control

$$v = -\,\phi(z)\,(GB)^{-1}\,sat_{\zeta}(s) \tag{3.47}$$

where $sat_{\zeta}(.)$ is given by (3.26), and $\phi(z)$ will be chosen later. To obtain a quadratic Lyapunov function for the closed-loop system (3.5) and (3.47), we use the same idea that was used in the proof of Theorem 3.2, i.e., first system (3.5) is transformed into a regular form. In the new coordinates, two Lyapunov functions are defined, one to characterize the motion of the closed-loop system on the sliding surface, and the other to characterize the motion of the closed-loop system towards the sliding surface. Then, a weighted sum of these two Lyapunov functions is considered as a Lyapunov function candidate for the overall closed-loop system. The following lemma formalizes the ideas mentioned above, providing us with a Lyapunov function candidate for the closed-loop system.

Lemma 3.1: Let

$$\hat{A} = A - B(GB)^{-1}GA - \mu B(GB)^{-1}G \tag{3.47}$$

where μ is an arbitrary positive constant. If the sliding surface $s = Gx$ is chosen such that (i)-(iii) are satisfied, then

a) $\hat{A}$ as given by Equation (3.47) is Hurwitz.

b) There exist symmetric positive definite matrices P and $Q \in \mathbf{R}^{p \times p}$ such that

$$P\hat{A} + \hat{A}'P = -Q$$

$$PB = G'GB.$$

Remark 3.2 The second term in (3.47) is precisely what is known in the literature as the equivalent control [Utkin (1987)]. Intuitively one can see the reason for the introduction of the third term in $\hat{A}$. The second term of (3.47) places $p-q$ of the eigenvalues of the closed-loop system at the invariant zeros of (G, A, B) and the rest at the origin. So the purpose of the third term is to shift the eigenvalues which are at the origin into the open left-half plane.

Proof of Lemma 3.1: There exists a similarity transformation T [Young-Kokotovic-Utkin (1977)] such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad TB = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$$

$$GT^{-1} = (\,0 \quad GB\,)$$

where A_{11} is Hurwitz, due to (ii) and (iii). Then

$$T\hat{A}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & -\mu I_q \end{bmatrix}$$

which is clearly Hurwitz. Let $\bar{Q}_1 \in \mathbf{R}^{p-q \times p-q}$ be symmetric positive definite, then there exists $\bar{P}_1$ symmetric positive definite such that $\bar{P}_1 A_{11} + A_{11}'\bar{P}_1 = -\bar{Q}_1$. Let

$$\bar{Q} := \begin{bmatrix} \alpha\bar{Q}_1 & -\alpha\bar{P}_1 A_{12} \\ -\alpha A_{12}'\bar{P}_1 & 2\mu(GB)'GB \end{bmatrix}$$

Choose $\alpha>0$ small enough such that $\bar{Q}$ is positive definite.

Let $\quad P = T' \begin{bmatrix} \alpha\bar{P}_1 & 0 \\ 0 & (GB)'GB \end{bmatrix} T \quad$ and $\quad Q = T'\,\bar{Q}\,T$

Then straight-forward computation shows that

$$P\hat{A} + \hat{A}'P = -Q$$

$$PB = G'GB \qquad \square$$

Now suppose that $\delta(z, v, t)$ in (3.5) satisfies Assumption 3.G2 with

$$\theta = \frac{1}{\sqrt{q}\ \| GB \|\ \| GB \|^{-1}}$$

and choose positive constants ρ_1 and ρ_0 such that the following inequality is satisfied in the domain of interest,

$$\rho_1 \| z \| + \rho_0 \geq \frac{(\| GA+\mu G \| + (k_1+k_3) \| GB \|) \| z \| + k_4 \| GB \|}{1-\sqrt{q}k_2 \| GB \| \| (GB)^{-1} \|}$$

and let $\phi(.)$ in (3.47) be

$$\phi(z) := \rho_1 \| z \| + \rho_0$$

The derivative of $W(z)=z'Pz$ along the trajectory of the closed-loop system (3.5) and (3.47)

$$\Rightarrow \dot{W} = z'(P\hat{A} + \hat{A}'P)z + 2z'PB(GB)^{-1}\left[-\phi(z)\, sat_{\zeta} s + GB\,\delta(z,v,t) + GAz + \mu Gz\right]$$

$$= -z'Qz + 2z'G'\left[-\phi(z)\, sat\, Gz + GB\,\delta(z,v,t) + GAz + \mu Gz\right]$$

$$= -z'Qz - 2\,\phi(z)\sum_{i=1}^{q} s_i [sat_{\zeta} s]_i + 2s'\left[GB\,\delta(z,v,t) + GAz + \mu Gz\right]$$

Let

$$I := \left\{ i \in \mathrm{N} \mid 1 \le i \le q, \quad |s_i| > \zeta \right\}$$

$$I' := \left\{ i \in \mathrm{N} \mid 1 \le i \le q, \quad |s_i| \le \zeta \right\}$$

$$\Rightarrow \dot{W} = -z'Qz - 2\phi(z)\sum_{i \in I} s_i\, sgn\, s_i - 2\phi(z)\sum_{i \in I'} s_i \frac{s_i}{\zeta}$$

$$+ 2s'\left[GB\,\delta(z,v,t) + GAz + \mu Gz\right]$$

$$= -z'Qz - 2\phi(z)\| s \|_1 + 2\phi(z)\sum_{i \in I'}\left[|s_i| - \frac{s_i^2}{\zeta}\right]$$

$$+ 2s'\left[GB\,\delta(z,v,t) + GAz + \mu Gz\right]$$

$$\le -\lambda_{\min}(Q)\| z \|^2 - 2\phi(z)\| s \|_1 + \frac{q\zeta}{2}\phi(z)$$

$$+ 2\| s \|_1 \| GB\,\delta(z,v,t) + GAz + \mu Gz \|$$

$$\le -\lambda_{\min}(Q)\| z \|^2 + \frac{q\zeta}{2}\phi(z)$$

$$= -\lambda_{\min}(Q)\| z \|^2 + \frac{q\zeta\rho_1}{2}\| z \| + \frac{q\zeta\rho_0}{2}$$

which, by Proposition 3.1, implies uniform ultimate boundedness of the closed-loop system. Note that, similar to the linear high-gain control of Example 3.1, the estimate of the set of uniform ultimate boundedness given by Proposition 3.1 shrinks as ζ goes to zero.

4 Observer-Based Control

4.1 Introduction

In this chapter, we propose an observer-based controller for stabilization of system (2.1), under the assumption that the nominal system (2.2) is fully-linearizable, left-invertible, and minimum-phase. In chapter 3, we studied the first step of such a design process, which is design of an appropriate state feedback controller for system (2.1). To design the state feedback controller, we first transformed system (2.1) into system (3.3). Then, we found the state feedback control (3.4) and (3.7), along with the quadratic Lyapunov function (3.8) such that, under Assumption 3.G2, the derivative of W along the trajectory of the closed-loop system (3.3), (3.4), and (3.7) satisfies inequality (3.9), which in general implies uniform ultimate boundedness of the closed-loop system. The next step of the control design is to design an observer to estimate the state z of system (3.3). We design the observer based on the linear part of (3.3) independent of the (possibly uncertain) nonlinear terms. Let us consider the following observer-based control

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + B\ \phi(\hat{z}) + L\ (y - C\hat{z}) \\ u = F(\hat{z}) := \bar{\alpha}(\hat{z}) + \bar{\beta}(\hat{z})\ \phi(\hat{z}) \end{cases} \tag{4.1}$$

where $\bar{\alpha}(.)$, $\bar{\beta}(.)$, and $\phi(.)$ are given by (3.4) and (3.7), and L is the observer gain to be designed later in section 4.3. Let $e := z - \hat{z}$ be the estimation error, then the error equation is

$$\dot{e} = (A - LC)\, e + B\, \Delta(z, \hat{z}, t) \tag{4.2}$$

where

$$\begin{aligned}\Delta(z,\hat{z},t) &:= \beta^{-1}(z)\Big[\bar{\alpha}(\hat{z}) - \alpha(z) + \Big[\bar{\beta}(\hat{z}) - \beta(z)\Big]\phi(\hat{z}) \\ &\qquad + \Delta_f(z) + \Delta_{ed}(z) + \Delta_g(z)\Big[\bar{\alpha}(\hat{z}) + \bar{\beta}(\hat{z})\phi(\hat{z})\Big]\Big] \\ &= \delta(z, \phi(z), t) + \beta^{-1}(z)\,(I_q + \Delta_g)\Big[F(\hat{z}) - F(z)\Big] \\ &\qquad + \phi(z) - \phi(\hat{z})\end{aligned} \tag{4.3}$$

Note that the term $\Delta(.,.,.)$ has been created by three different sources:

1) Uncertainties and disturbances in system (2.1)

2) Simplification of the nonlinear terms $\alpha(.)$ and $\beta(.)$.

3) Estimation error

Therefore, even if there is neither uncertainties in (2.1), nor any simplification in the cancellation of the nonlinear terms, the disturbance term (4.3) will still be present in the error equation (4.2). In other words, output feedback control of the nominal system (2.2) is as difficult a problem as that of system (2.1).

It is well known that in the presence of the term $\Delta(z,\hat{z},t)$ in (4.2), choosing L to locate the eigenvalues of $(A - LC)$ in the open left-half complex plane does not ensure stability of the closed-loop system. Instead, the observer should be designed such that the disturbance term is decoupled (either exactly or asymptotically) from the error equation. In robust control of linear systems, such robust observers could be designed via loop transfer recovery techniques (for a survey, refer to Stein-Athans (1987)),which consist of asymptotic methods that use Riccati equations and transfer function manipulations. Such transfer function manipulation can not be extended to nonlinear systems. Therefore, we design the observer through a singular perturbation approach which has been recently developed by Saberi-Sannuti (1990) and Esfandiari-Khalil (1989). The conceptual idea of this approach is that to determine the amount of required gain at each element of the observer gain matrix L, we need to know the finite and infinite zero structure of the linear system (2.3). Therefore, we first transform system (2.3) into a canonical form that explicitly shows its finite and infinite zero structure. Then, in this canonical form, the observer is designed via *asymptotic pole placement* to reject the effect of the disturbance term (4.3) on the error equation (4.2). Esfandiari-Khalil (1989) use this approach to design nonlinear output feedback controllers for uncertain linear systems, while Saberi-Sannuti (1990) use it to design multiple-time-scale observers for loop transfer recovery.

4.2. A Special Coordinate Basis

Consider the linear system (2.3) which was obtained by exact linearization of the nominal system (2.2). In this section, we transform (2.3) into the special coordinate basis of Saberi-Sannuti (1987), which explicitly shows the finite and infinite zero structure of (2.3). The infinite zero structure of a linear system is closely related to the number of inherent integrations that exist between its inputs and outputs. Therefore, the idea behind the special coordinate basis of Saberi-Sannuti (1987) is to linearly combine and partition the input vector v, as well as the output vector y, such that the inherent number of integrations between certain parts of v and corresponding parts of y are exhibited clearly in the coordinate basis. To this end, it has been proved by Saberi-Sannuti (1987) that, since (2.3) is left-invertible (by Assumption 2.G1 and Proposition 2.1), there exist nonsingular transformations

Γ, Γ_{out}, Γ_{in}, integers K, q_a, q_b, q_i, $i=1,...,K$ such that the transformation

$$z = \Gamma \begin{bmatrix} \tilde{z}_a \\ \tilde{z}_b \\ \tilde{z}_f \end{bmatrix}, \quad y = \Gamma_{out} \begin{bmatrix} \tilde{y}_f \\ \tilde{y}_s \end{bmatrix}, \quad v = \Gamma_{in} \tilde{v} \tag{4.4}$$

transforms system (2.3) into the following form:

$$\begin{cases} \dot{\tilde{z}}_a = A_{aa}\tilde{z}_a + A_{af}\tilde{y}_f + A_{as}\tilde{y}_s \\ \dot{\tilde{z}}_b = A_{bb}\tilde{z}_b + A_{bf}\tilde{y}_f \\ \dot{\tilde{z}}_f = A_f\tilde{z}_f + M_f\tilde{y}_f + B_f\left[D_a\tilde{z}_a + D_b\tilde{z}_b + D_f\tilde{z}_f + \tilde{v}\right] \end{cases} \tag{4.5}$$

$$\begin{cases} \tilde{y}_f = C_f\tilde{z}_f \\ \tilde{y}_s = C_s\tilde{z}_b \end{cases}$$

where the dimensions of $\tilde{z}_a$, $\tilde{z}_b$, $\tilde{z}_f$, $\tilde{y}_f$, and $\tilde{y}_s$ are q_a, q_b, $\sum_{i=1}^{K} iq_i$, $\sum_{i=1}^{K} q_i$, and $p - \sum_{i=1}^{K} q_i$, respectively. Moreover, invariant zeros of (C, A, B) are the eigenvalues of A_{aa}, (C_s, A_{bb}) is observable, and A_f, B_f, and C_f have the following canonical structure:

$$A_f := Block\ Diag\ (A_{1f}, \ldots, A_{Kf})$$

$$B_f := Block\ Diag\ (B_{1f}, \ldots, B_{Kf})$$

$$C_f := Block\ Diag\ (C_{1f}, \ldots, C_{Kf})$$

where † $A_{1f} = 0_{q_1 \times q_1}$, $B_{1f} = I_{q_1}$, $C_{1f} = I_{q_1}$, if $i=1$, while

$$A_{if} = \begin{bmatrix} 0_{l_i \times q_i} & I_{l_i} \\ 0_{q_i \times q_i} & 0_{q_i \times l_i} \end{bmatrix}, \quad B_{if} = \begin{bmatrix} 0_{l_i \times q_i} \\ I_{q_i} \end{bmatrix}$$

---

† $0_{m \times n}$ denotes the $m \times n$ zero matrix, and I_m denotes the $m \times m$ identity matrix.

$$C_{\text{if}} = \begin{bmatrix} I_{q_i} & 0_{q_i \times l_i} \end{bmatrix}, \quad l_i = (i-1) \times q_i$$

for $i > 1$.

Partitioning $\tilde{v}$ and $\tilde{y}_f$ into

$$\tilde{v} = \begin{bmatrix} \tilde{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{v}_K \end{bmatrix}, \quad \tilde{y}_f = \begin{bmatrix} \tilde{y}_{1f} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}_{Kf} \end{bmatrix}$$

where $\tilde{v}_i$, and $\tilde{y}_{\text{if}}$ are q_i-dimensional vectors, one can see that the variables $\tilde{v}_i$ controls the output y_{if} through a stack of i integrators.

The vectors $\tilde{z}_a$, $\tilde{z}_b$, and $\tilde{z}_f$ span some well-known invariant subspaces of geometric theory of linear systems [Wonham (1979)]:

1) $\tilde{z}_a$ spans the largest (A,B)-invariant subspace contained in the Kernel of C.

2) $\tilde{z}_b$ spans the largest (A',C')-controllability subspace contained in the Kernel of B'.

3) $\tilde{z}_f$ spans the smallest (A,C)-invariant subspace containing the Range of B.

Remark 4.1: The transformations Γ, Γ_{in}, and Γ_{out} are obtained by Saberi-Sannuti (1987) via a modification of the structural algorithm of Silverman (1969). A numerical algorithm which is based on the procedure of Saberi-Sannuti (1987) is available in Linear Algebra and Systems (LAS) package. Note that many physical problems of interest to us are already in the form (4.5). For instance, Examples 2.1 and 2.2 of Chapter 2 are already in the form (4.5). Also in Appendix B, we give an alternative way of arriving at (4.5) for a subclass of linear systems under study, namely for linear systems which are square, invertible, minimum-phase, and have a left diagonal interactor. The advantage of the algorithm given in Appendix B over that of Saberi-Sannuti (1987) is its simplicity.

4.3 Observer Design

The problem of observer design to reject the effect of the disturbances modeled by $\Delta(z,\hat{z},t)$ becomes an asymptotic pole placement problem in the special coordinate basis (4.5). Let us choose L_b and L_{if}, $i=1,\ldots,K$, such that $A_{bb}-L_bC_s$ and $A_{\mathrm{if}}-L_{\mathrm{if}}C_{\mathrm{if}}$ are Hurwitz. Let

$$L_f := diag\ (L_{1f}, L_{2f}, \ldots, L_{Kf})$$

$$M(\varepsilon) := diag\ (M_1(\varepsilon), \ldots, M_K(\varepsilon))$$

$$M_i(\varepsilon) := diag\ \left(\frac{I_{q_i}}{\varepsilon}, \frac{I_{q_i}}{\varepsilon^2}, \ldots, \frac{I_{q_i}}{\varepsilon^i}\right)$$

and choose the observer gain to be

$$L(\varepsilon) := \Gamma \begin{bmatrix} A_{af} & A_{as} \\ A_{bf} & L_b \\ M_f + M(\varepsilon)L_f & 0 \end{bmatrix} \Gamma_{out}^{-1} \tag{4.6}$$

Then it is easy to show that

$$\Gamma^{-1}(A-LC)\Gamma = \begin{bmatrix} A_{aa} & 0 & 0 \\ 0 & A_{bb}-L_bC_s & 0 \\ B_fD_a & B_fD_b & A_f - M(\varepsilon)L_fC_f + B_fD_f \end{bmatrix} \tag{4.7}$$

Note that (4.7) has a block triangular structure. The eigenvalues of the first two diagonal blocks are $O(1)$, while the eigenvalues of the last diagonal block are

$O(\frac{1}{\varepsilon})$. Define

$$\begin{bmatrix} e_s \\ \underline{e}_f \end{bmatrix} := \Gamma^{-1} e \tag{4.7}$$

$$A_s := Block\ Diag\ (A_{aa}, A_{bb} - L_b C_s),\ D_s := (D_a, D_b)$$

where e is the estimation error, and dimensions of e_s, and $\underline{e}_f$ are $q_a + q_b$ and $\sum_{i=1}^{K} i q_i$, respectively. Then the error equation (4.2) is transformed into

$$\begin{cases} \dot{e}_s = A_s e_s \\ \dot{\underline{e}}_f = [A_f - M(\varepsilon) L_f C_f]\, \underline{e}_f + B_f\, [D_s e_s + D_f \underline{e}_f + \Gamma_{in}^{-1} \Delta(z, \hat{z}, t)] \end{cases} \tag{4.8}$$

Scale $\underline{e}_f$ in the following way

$$e_f := N^{-1}(\varepsilon) \underline{e}_f \tag{4.9}$$

$$N(\varepsilon) := Block\ Diag\ (N_1(\varepsilon), \ldots, N_K(\varepsilon)) \tag{4.10}$$

$$N_i(\varepsilon) := Block\ Diag\ (\varepsilon^{i-1} I_{q_i}, \varepsilon^{i-2} I_{q_i}, \ldots, I_{q_i}) \tag{4.11}$$

Then it is easy to show that

$$N^{-1}(\varepsilon) B_f = B_f \tag{4.12}$$

$$N^{-1}(\varepsilon)\, [A_f - M(\varepsilon) L_f C_f]\, N(\varepsilon) = \frac{1}{\varepsilon} [A_f - L_f C_f] \tag{4.13}$$

Using (4.12) and (4.13), it can be shown that the scaling (4.9) transforms (4.8) into the following form:

$$\begin{cases} \dot{e}_s = A_s\, e_s \\ \varepsilon\, \dot{e}_f = [A_f - L_f C_f]\, e_f + \varepsilon\, B_f\, [D_s e_s + D_f N(\varepsilon) e_f + \Gamma_{in}^{-1} \Delta(z, \hat{z}, t)] \end{cases} \tag{4.14}$$

The desired disturbance rejection property of the observer can be explicitly seen in (4.14), since the slow part of (4.14) is completely decoupled from $\Delta(z, \hat{z})$, and the effect of the term $\Delta(z, \hat{z})$ on the fast part of (4.14) decreases, as ε tends to zero. It is also clear from (4.14) that in the course of achieving such a disturbance rejection property, we have to locate some of the eigenvalues of $(A - LC)$ far in the left-half complex plane.

4.4 Closed-loop Stability Analysis

By the development in section 4.3, the closed-loop system (3.3) and (4.1), with L given by (4.6) can be written as

$$\begin{cases} \dot{z} = Az + B\phi(z) + B\,\delta(z, \phi(z), t) + B\beta^{-1}(z)(I_q + \Delta_g)\Big[F(\hat{z}) - F(z)\Big] \\ \dot{e}_s = A_s\, e_s \\ \varepsilon\, \dot{e}_f = (A_f - L_f C_f)\, e_f + \varepsilon\, B_f \Big[D_s e_s + D_f N(\varepsilon) e_f + \Gamma_{in}^{-1} \Delta(z, \hat{z}, t)\Big] \end{cases} \tag{4.15}$$

where $\delta(z, v, t)$ is given by (3.6), $\Delta(z, \hat{z}, t)$ by (4.3),

$$F(z) = \bar{\alpha}(z) + \bar{\beta}(z)\,\phi(z) \tag{4.16}$$

and by (4.7) and (4.9)-(4.11), $\hat{z}$, which is the estimate of the state z, can be written as

$$\hat{z} = z - e = z - \Gamma \begin{bmatrix} e_s \\ N(\varepsilon)\, e_f \end{bmatrix} = z - \Gamma_1 e_s - \Gamma_2 N(\varepsilon) e_f \tag{4.17}$$

Note that $N(\varepsilon)$ is a polynomial matrix in ε; hence it is bounded for small ε. System (4.15) is a standard singularly perturbed system, with (z, e_s) as the slow variable and e_f as the fast variable. The slow and fast subsystems of (4.15) are respectively

$$\begin{cases} \dot{z} = Az + B\phi(z) + B\,\delta(z,\phi(z),t) \\ \qquad\qquad + B\,\beta^{-1}(z)\,(I_q+\Delta_g)\left[F(z-\Gamma_1 e_s) - F(z)\right] \\ \dot{e}_s = A_s\, e_s \end{cases} \tag{4.18}$$

and

$$\frac{de_f}{d\tau} = (A_f - L_f C_f)\, e_f \tag{4.19}$$

where $\tau = t/\varepsilon$.

Uniform Ultimate Boundedness

In this part, we argue that the uniform ultimate boundedness property of the state feedback control is preserved by our output feedback controller, under certain conditions. Suppose that we have designed the state feedback control (3.4) and (3.7) to ensure uniform ultimate boundedness of the closed-loop system (3.3), (3.4), and (3.7), i.e., there exists a quadratic Lyapunov function (3.8) such that inequality (3.9) is satisfied along the trajectory of the closed-loop system (3.3), (3.4), and (3.7). To state the theorem on the uniform ultimate boundedness of the output feedback control, we make the following assumption:

Assumption 4.G3: For all $z, \hat{z} \in S$

$$|\phi(z) - \phi(\hat{z})| \le k_5 |z - \hat{z}|$$

$$|\beta^{-1}(z)\,(I_q+\Delta_g)\left[F(z) - F(\hat{z})\right]| \le k_6 |z - \hat{z}|$$

where k_5, and k_6 are nonnegative constants, and S is the set of Assumption 3.G2.

Theorem 4.1: Under Assumptions 2.G1, 3.G2, and 4.G3, suppose that the state feedback control (3.4) and (3.7) is designed such that Lyapunov equation (3.8) satisfies inequality (3.9), and let σ and r be the constants of Proposition 3.1. Consider system (4.15), and let P_s and P_f be the symmetric positive definite solutions of the Lyapunov equations

$$P_s A_s + A_s' P_s = -I_{q_a+q_b}$$

$$P_f (A_f - L_f C_f) + (A_f - L_f C_f)' P_f = -I_{n-q_a-q_b} \,.$$

Suppose that $\sigma < \dfrac{\lambda_{\min}(P)}{\lambda_{\max}(P)} \dfrac{r}{9}$. Then there exist positive constants α_f, r_s, and $\bar{\varepsilon}$, and a continuous function $g : (0,\bar{\varepsilon}) \rightarrow \mathbf{R}^+$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$ system (4.15) is uniformly ultimate bounded with respect to Φ_ε, with Σ inside the region of attraction, where

$$\Phi_\varepsilon = \left\{ (z', e_s', e_f')' \in \mathbf{R}^{2n} \mid e_s = 0,\ W(z) + \alpha_f\, e_f' P_f e_f < g(\varepsilon) \right\}. \tag{4.20}$$

$$\Sigma = \left\{ (z', e_s', e_f')' \in \mathbf{R}^{2n} \mid e_s' P_s e_s < r_s, \right.$$

$$\left. W(z) + \alpha_f\, e_f' P_f e_f < \frac{\lambda_{\min}(P) r}{9 \lambda_{\max}(P)} \right\} \tag{4.21}$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = \sigma \tag{4.22}$$

Remark 4.2: (4.22) shows that the z-projection of the set of uniform ultimate boundedness in the case of our observer-based control approaches Ω_σ of Proposition 3.1 as $\varepsilon \rightarrow 0$.

Proof of Theorem 4.1: The arguments ε and t are dropped whenever it causes no confusion. Let

$$\alpha_f := \frac{\lambda_{\min}(P)\,|\Gamma_2|^2}{\lambda_{\min}(P_f)}$$

$$\alpha_s := \frac{\lambda_{\min}(P_s)r}{9\lambda_{\max}(P)\,|\Gamma_1|^2}$$

where Γ_1 and Γ_2 are given by (4.17). Let

$$V_s := e_s' P_s e_s$$

$$V := W(z) + \alpha_f e_f' P_f e_f$$

$$\tilde{\Sigma} := \left\{ (z', e_s', e_f')' \in \mathbf{R}^{2n} \;\middle|\; V_s < \alpha_s,\; V < \frac{\lambda_{\min}(P)r}{9\lambda_{\max}(P)} \right\}$$

Let $\varepsilon \in (0,1)$. Then $|N(\varepsilon)| < 1$, and it can be easily shown that for all $(z, e_s, e_f) \in \tilde{\Sigma}$, z, and $\hat{z}$ belong to the set $\Omega_r \subset S$, where S is the set defined in Assumptions 3.G2 and 4.G3, and Ω_r is defined in Proposition 3.1.

Along the trajectory of the closed-loop system, $\dot{V}_s$, and $\dot{V}$ satisfy the following inequalities on $\tilde{\Sigma}$:

$$\dot{V}_s \le -\frac{1}{\lambda_{\max}(P_s)} V_s \tag{4.23}$$

$$\begin{aligned} \dot{V} \le\; & -\gamma_2 |z|^2 + \gamma_1 |z| + \gamma_0 \\ & + 2k_6 |PB|\,|z|\,(|\Gamma_1 e_s| + |\Gamma_2 N(\varepsilon) e_f|) - \frac{\alpha_f}{\varepsilon}|e_f|^2 \\ & + 2\alpha_f |P_f B_f D_s|\,|e_s|\,|e_f| + 2\alpha_f |P_f B_f D_f N(\varepsilon)|\,|e_f|^2 \\ & + 2\alpha_f |e_f|\,|P_f B_f \Gamma_{in}^{-1}|\,|\Delta(z, \hat{z}, t)| \end{aligned}$$

Let

$$a_0 := \gamma_2$$

$$a_1 := \gamma_1 + 2k_6 |PB| |\Gamma_1| \left[\frac{V_s}{\lambda_{\min}(P_s)}\right]^{\frac{1}{2}}$$

$$a_2 := \gamma_0$$

$$a_3 := 2k_6 |PB| |\Gamma_2| + 2\alpha_f (k_1+k_2k_5+k_3) |P_f B_f \Gamma_{in}^{-1}|$$

$$a_4 := \alpha_f \left[\frac{1}{\varepsilon} - 2|P_f B_f D_f| - 2(k_5+k_6)|P_f B_f \Gamma_{in}^{-1}| |\Gamma_2| \right]$$

$$a_5 := 2\alpha_f \left[|P_f B_f D_s| \left[\frac{V_s}{\lambda_{\min}(P_s)}\right]^{\frac{1}{2}} + k_4 |P_f B_f \Gamma_{in}^{-1}| \right.$$

$$\left. + (k_5+k_6)|\Gamma_1| |P_f B_f \Gamma_{in}^{-1}| \left[\frac{V_s}{\lambda_{\min}(P_s)}\right]^{\frac{1}{2}} \right]$$

$$h(z,V_s,e_f) := -a_0|z|^2+a_1|z|+a_2+a_3|z||e_f|$$

$$-a_4|e_f|^2+a_5|e_f|$$

Then

$$\dot{V} \le h(z,V_s,e_f) \quad \text{on } \tilde{\Sigma} \text{ and } \varepsilon \in (0,1) \tag{4.24}$$

Note that $a_3^2-4a_0a_4 < 0$ for all $\varepsilon \in (0,\varepsilon_1)$, where $\varepsilon_1 \le 1$ is a positive constant. For $\varepsilon \in (0,\varepsilon_1)$, let $c_z(\varepsilon,V_s)$ and $c_e(\varepsilon,V_s)$ be the unique real positive roots of the following polynomials

$$(a_3^2-4a_0a_4)c_z^2 + (2a_3a_5+4a_1a_4)c_z + a_5^2+4a_2a_4 = 0 \tag{4.25}$$

$$(a_3^2-4a_0a_4)c_e^2 + (2a_1a_3+4a_0a_5)c_e + a_1^2+4a_0a_2 = 0 \tag{4.26}$$

These roots satisfy

$$\lim_{\varepsilon\to 0} c_z^2(\varepsilon,0) = \frac{\sigma}{\lambda_{\max}(P)}, \quad \lim_{\varepsilon\to 0} c_e(\varepsilon,0) = 0. \tag{4.27}$$

We will show that $h(z,V_s,e_f)$ is negative if $|z| > c_z(\varepsilon,V_s)$ or $|e_f| > c_e(\varepsilon,V_s)$. Considering $h(z,V_s,e_f)$ as a quadratic term in $|e_f|$ implies that

$$h(z,V_s,e_f) \le \frac{(a_3|z|+a_5)^2}{4a_4} + a_2 + a_1|z| - a_0|z|^2$$

$$= \frac{1}{4a_4}\Big[(a_3^2-4a_0a_4)|z|^2$$

$$+ (2a_3a_5+4a_1a_4)|z| + a_5^2+4a_2a_4\Big] \tag{4.28}$$

Comparing (4.25) with (4.28) shows that $h(z,V_s,e_f) < 0$ if $|z| > c_z(\varepsilon,V_s)$.

Similarly one can prove that $h(z,V_s,e_f) < 0$ if $|e_f| > c_e(\varepsilon,V_s)$.

Let $G(\varepsilon,V_s) := \lambda_{\max}(P)\Big[c_z(\varepsilon,V_s)\Big]^2 + \lambda_{\max}(P_f)\alpha_f\Big[c_e(\varepsilon,V_s)\Big]^2$. Then

$$h(z,V_s,e_f) < 0 \qquad \text{if } V > G(\varepsilon,V_s) \text{ and } \varepsilon<\varepsilon_1 \tag{4.29}$$

Let $g(\varepsilon) := G(\varepsilon,0)$. By (4.27), $\lim_{\varepsilon\to 0} g(\varepsilon) = \sigma < \frac{\lambda_{\min}(P)r}{9\lambda_{\max}(P)}$. Therefore there exists $\bar{\varepsilon}\le\varepsilon_1$ such that $g(\varepsilon)<r$ for all $\varepsilon \in (0,\bar{\varepsilon})$. Let $K(V_s) := G(\bar{\varepsilon},V_s)$. By differentiating (4.25) and (4.26) with respect to V_s, it can be shown that $K(.)$ is monotonically increasing, hence $K^{-1}(.)$ is well defined. Choose $r_s>0$ to be strictly less than $\min\Big\{\alpha_s, K^{-1}(r)\Big\}$. Let Σ be as given by (4.21) and $\varepsilon \in (0,\bar{\varepsilon})$. By (4.23) and (4.29), considering the direction of the vector field on the boundary of Σ, it can be shown that Σ is an invariant set of the trajectory.

Next we will show uniform ultimate boundedness with respect to Φ_ε, as given by (4.20). Given $\mu>0$, suppose, without loss of generality, that $N_\mu(\Phi_\varepsilon) \subset \Sigma$. By a simple topological argument it can be shown that there exists $\delta>0$ such that

$$R_\delta := \left\{ (z', e_s', e_f')' \in \mathbf{R}^{2n} \mid V_s \le \delta \, , \, V \le \delta + G(\varepsilon, \delta) \right\} \subset N_\mu(\Phi_\varepsilon) \qquad (4.30)$$

and R_δ is an invariant set. Let

$$F := \left\{ (z', e_s', e_f')' \in \mathbf{R}^{2n} \mid V_s \le \delta \, , \; \delta + G(\varepsilon, \delta) \le V \le r \right\}$$

By (4.29), there exists $\alpha>0$ such that $\dot{V} \le -\alpha$ on F. Using this fact, (4.23), and the fact that R_δ is an invariant set, it is easy to show that

$$(z(t)', e_s(t)', e_f(t)')' \in R_\delta \text{ for all } t \ge T$$

where

$$T := \lambda_{\max}(P_s) \log \frac{r_s}{\delta} + \frac{1}{\alpha} [\, r - \delta - G(\varepsilon, \delta) \,]$$

This, together with (4.30), proves uniform ultimate boundedness with respect to Φ_ε.

□

Corollary 4.1: If Assumptions 2.G1, 3.G2, and 4.G3 hold globally, and the state feedback control (3.3) and (3.7) is designed such that inequality (3.9) holds globally, then there exist positive constants α_f and $\bar{\varepsilon}$, and a continuous function $g : (0, \bar{\varepsilon}) \to \mathbf{R}^+$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, system (4.15) is globally uniformly ultimately bounded with respect to Φ_ε, given by (4.20).

Asymptotic Stability

Next, we prove that if the state feedback control (3.4) and (3.7) renders the origin of system (3.3) asymptotically stable, then so does our observer-based control.

Theorem 4.2: Under Assumptions 2.G1, 3.G2, and 4.G3, suppose that $k_4=0$, and the state feedback control (3.4) and (3.7) is designed such that Lyapunov function (3.8) satisfies inequality (3.9) with $\gamma_1=\gamma_0=0$. Let r, P_s, P_f, and Σ be as in Theorem 4.1. Then there exist positive constants α_f, r_s, and $\bar{\varepsilon}$ such that the origin of system (4.15) is an asymptotically stable equilibrium point with set Σ inside the region of attraction.

Proof of Theorem 4.2: The proof follows as a corollary of the proof of Theorem 4.1.

5 Peaking Phenomenon

Theorems 4.1 and 4.2 imply that some of the poles of the observer (namely, those associated with e_f) have to be placed far in the left-half complex plane, in order to achieve stabilization of system (4.15). In general, placing poles far in the left-half complex plane causes an impulsive-like behavior which is known as the peaking phenomenon. The effect of the peaking phenomenon on stabilization of nonlinear systems via state feedback has been recently studied by Sussmann-Kokotovic (1989).

The effect of the peaking phenomenon on the output feedback control proposed in this paper can be explained in the following way: If the observer gain $L(.)$ is chosen such that the real part of some of the eigenvalues of $(A-LC)$ have $O(\frac{1}{\varepsilon})$ magnitudes, then the state transition matrix $e^{(A-LC)t}$ contains terms like

$$\frac{1}{\varepsilon^i}\, e^{-\alpha t/\varepsilon} \quad \text{where } \alpha>0 \text{ and } i \in \mathrm{N}$$

in general. Therefore, if the initial conditions of the error are $O(1)$, the transient behavior of the error $e(t)$ contains overshoots of order $O(\frac{1}{\varepsilon^i})$. Since the error equation is coupled to the state equation, these excessively large overshoots are transmitted to the states of the nonlinear system, causing peaking to appear in these states as well. The following example illustrates this phenomenon:

Example 5.1: Consider the problem of stabilization of an inverted pendulum, whose motion is described by the following equation (See Figure 5.1),

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + b_1 \sin\theta = u(t) - \frac{\nu(t)}{\lambda}\cos\theta \tag{5.1}$$

where θ is the angular position of the pendulum measured versus the stable equilibrium point, $u(t)$ is the control moment applied to the pivot point, $\nu(t)$ is the horizontal acceleration of the pivot point, and λ is the length of the pendulum. Assume that only the angle θ is available for measurement, and that the only information available about b_1 and $\nu(t)$ are the following bounds,

$$|b_1| < 1\,, \quad |\nu(t)| < \lambda \quad \text{for all } t \tag{5.2}$$

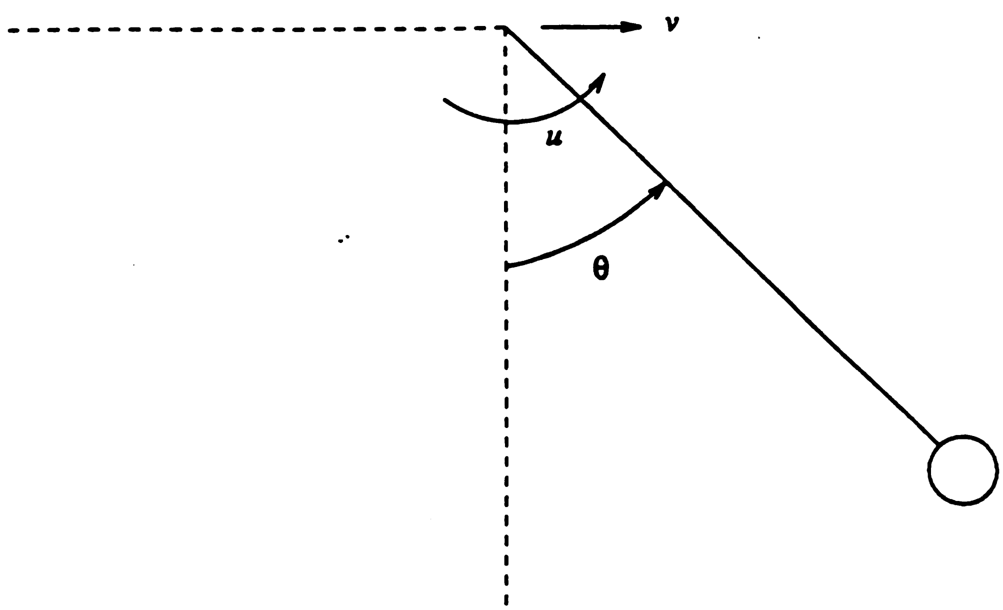


Figure 5.1- Pendulum of Example 5.1

Let

$$y := x_1 := \theta - \pi , \quad x_2 := \frac{d\theta}{dt} \tag{5.3}$$

where x_1 and x_2 are the state variables and y is the output. Then the state equation of the pendulum is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u(t) + \delta(x_1, t) \end{cases} \tag{5.4}$$

where

$$\delta(x_1, t) := b_1 \sin x_1 + \frac{v(t)}{\lambda} \cos x_1 \tag{5.5}$$

Following the development of section 3.3, we designed the following variable structure control for system (5.4),

$$u = -x_1 - x_2 - 2.0\, sat_\zeta(x_1+x_2) , \quad \zeta = 0.01 \tag{5.6}$$

which renders system (5.4) globally uniformly ultimately bounded with respect to a small neighborhood of the origin (Refer to Figure 5.2a for phase plane trajectory of the closed-loop system for the case when $x_1(0)=1.0$, and $x_2(0)=0.0$). Following the algorithm of section 4.3, the observer-based control was designed as

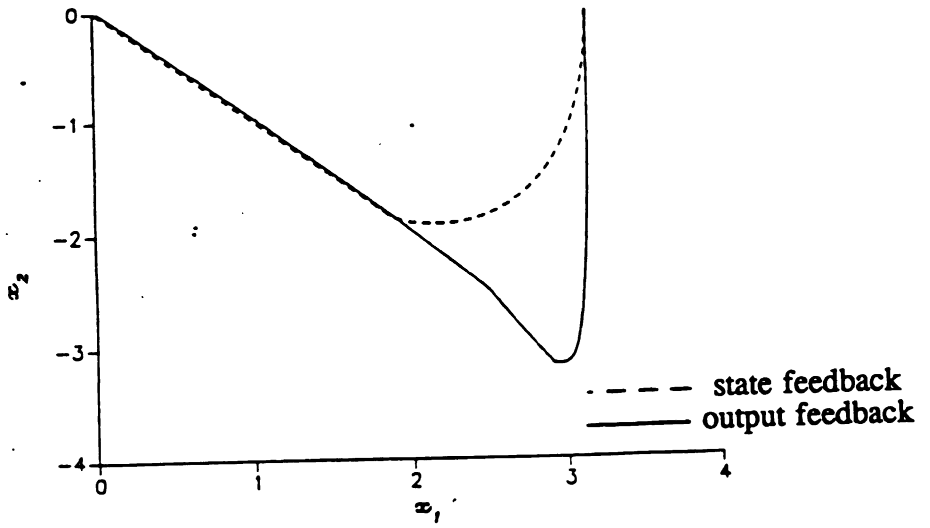
$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon} (y - \hat{x}_1) \\ \dot{\hat{x}}_2 = u + \frac{1}{\varepsilon^2} (y - \hat{x}_1) \\ u = -\hat{x}_1 - \hat{x}_2 - 2.0\, sat_\zeta(\hat{x}_1+\hat{x}_2) , \quad \zeta = 0.01 \end{cases} \tag{5.7}$$

It can be easily checked that Assumptions 2.G1, 3.G2, and 4.G3 hold globally in this case, and hence by Corollary 4.1 the closed-loop system (5.4) and (5.7) is globally uniformly ultimately bounded, for sufficiently small ε. Figure 5.2 shows results of the simulations with

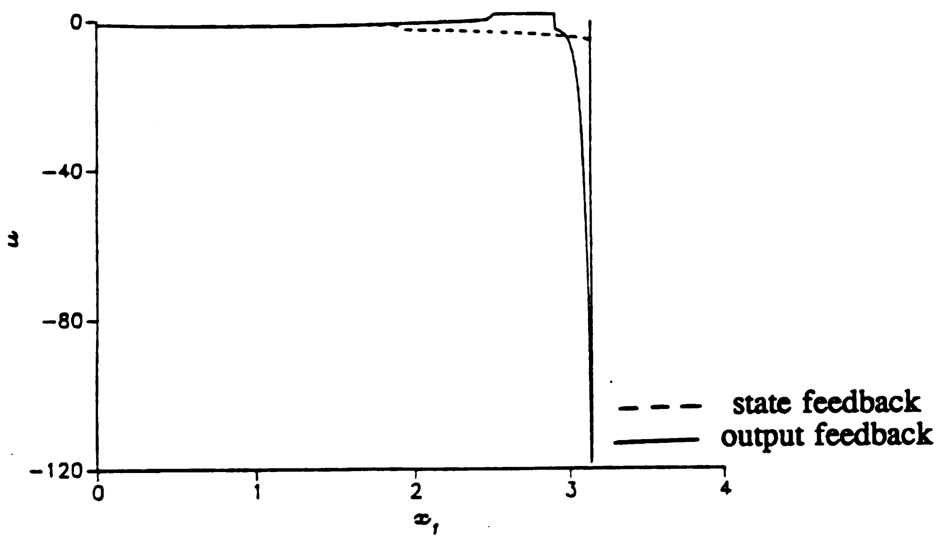
$$\varepsilon = 0.01\ ,\ x_1(0) = 1.0\ ,\ x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0.0$$

Figures 5.2b and 5.2c show the peaking in the input u and the estimation error, respectively. Note that the input peaks to an $O(\frac{1}{\varepsilon})$ quantity.

(a)



(b)



(c)

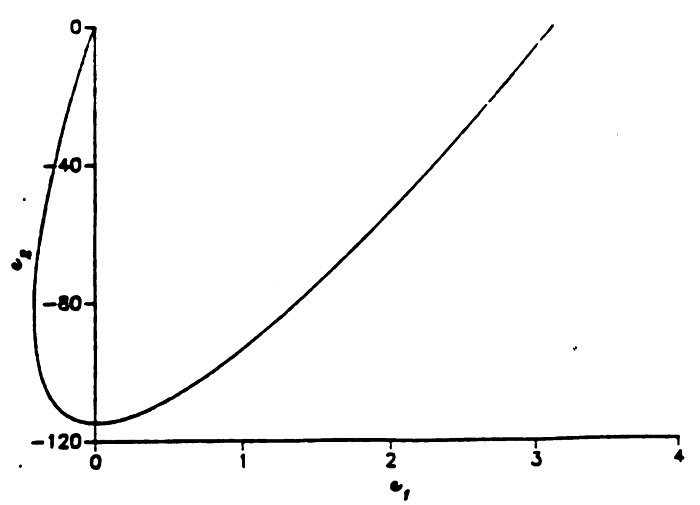


Figure 5.2- Simulation Results of Example 5.1

Example 5.1 clearly shows that peaking in the transient behavior of the system is undesirable. Peaking might even destabilize the closed-loop system as we decrease ε. The following example illustrates the destabilizing effect of the peaking phenomenon.

Example 5.2: Consider the second order system,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 3\,(1+\theta)\,x_2^3 + u \\ y = x_1 \end{cases} \tag{5.8}$$

where θ is an unknown parameter whose nominal value is zero. Let the state feedback control be

$$\begin{cases} u = -3\,x_2^3 + v \\ v = -2\,x_1 - x_2 \end{cases} \tag{5.9}$$

which ensures asymptotic stability of the origin. Figure 5.3 shows the phase portrait of the closed-loop system (5.8) and (5.9), with θ=0.1. The unstable limit cycle is the boundary of the region of attraction of the closed-loop system. Following the algorithm of section 4.3 the observer-based control is designed as

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \dfrac{2}{\varepsilon}\,(y - \hat{x}_1) \\ \dot{\hat{x}}_2 = v + \dfrac{1}{\varepsilon^2}\,(y - \hat{x}_1) \\ u = -3\,\hat{x}_2^3 + v \\ v = -2\,\hat{x}_1 - \hat{x}_2 \end{cases} \tag{5.10}$$

Figure 5.4 show the results of simulations for the closed-loop system (5.8) and (5.10), with θ=0.1, ε=0.014, and the following initial conditions,

$$x_1(0)=0.01, \quad x_2(0)=\hat{x}_1(0)=\hat{x}_2(0)=0.0$$

The closed-loop system is asymptotically stable in this case, but we can see the large overshoot in the state of the system. Note that x_2 reaches a maximum of 2.1, while the only nonzero initial state is 0.01. Figure 5.5 shows the results of simulation for the closed-loop system (5.8) and (5.10), with ε=0.013, and all the other constants and initial conditions the same as in Figure 5.4. Figure 5.5 shows that the closed-loop system is unstable in this case. As a matter of fact, results of our simulations show that for all ε less than 0.013 the closed-loop system is unstable. This is due to the peaking phenomenon which is present in the observer. The impulsive-like behavior of the observer state variables is passed to the states of the system. After an $O(\varepsilon)$ time, the estimation error has decayed to a very small value. However, the initial jump in x_2 takes the trajectory out of the region of attraction, resulting in instability.

It should be emphasized that the instability we have seen in this example does not contradict Theorems 4.1 and 4.2. The theorems estimate the region of attraction by the set Σ. Notice that Σ is defined using the scaled estimation error e_f. For the initial state of the closed-loop system to belong to Σ, $e_f(0)$ should be order $O(1)$. From the scaling equations (4.9)-(4.11) we can see that for $e_f(0)$ to be order of one, some components of $\underline{e}_f(0)$ must be of order $O(\varepsilon^{\beta})$ for some β>0 that is determined by (4.11). In the above example the initial condition of the unscaled estimation error $x_1(0) - \hat{x}_1(0)$ is 0.01. When scaled by ε, the initial condition of the scaled estimation error becomes $\frac{0.01}{\varepsilon}$, which (for sufficiently small ε) places the initial state of the closed-loop system outside the estimate Σ of Theorems 4.1 and 4.2.

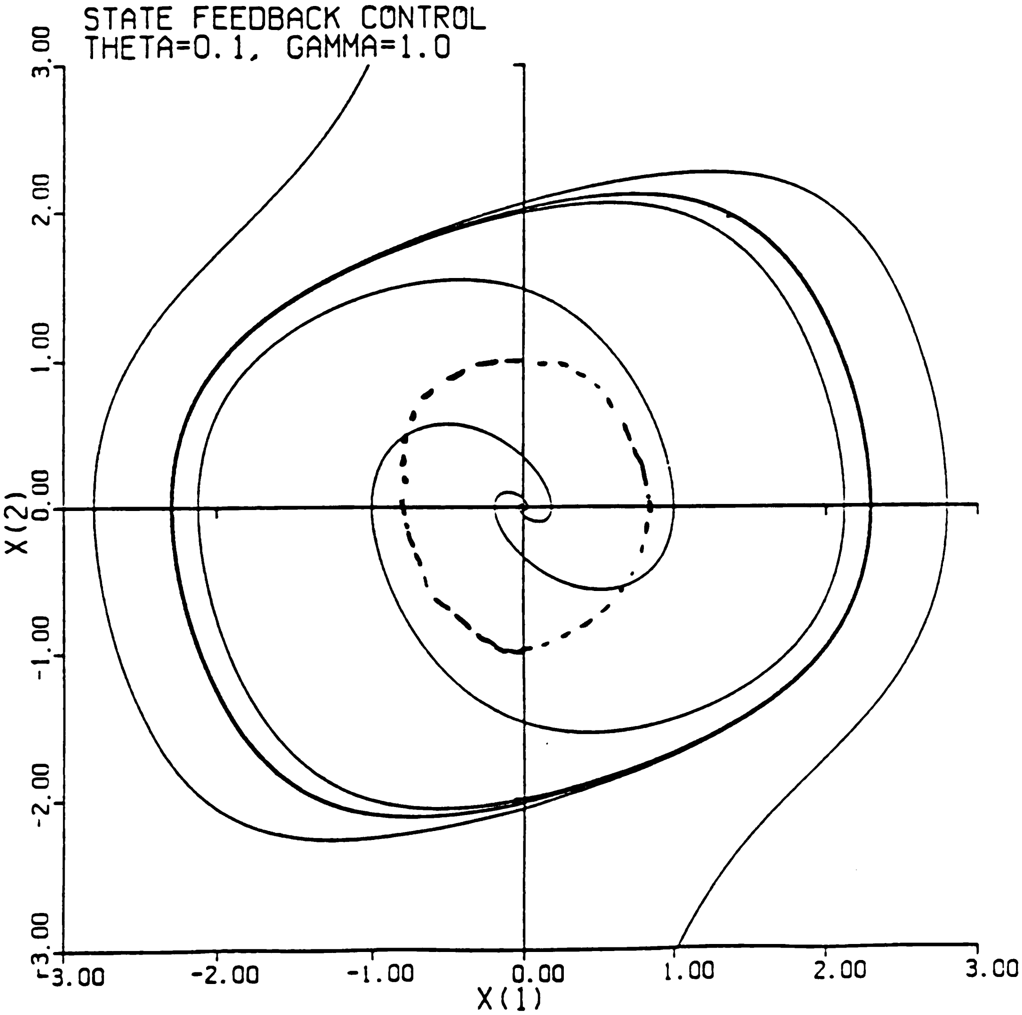


Figure 5.3- Phase Portrait of state feedback control of Example 5.2

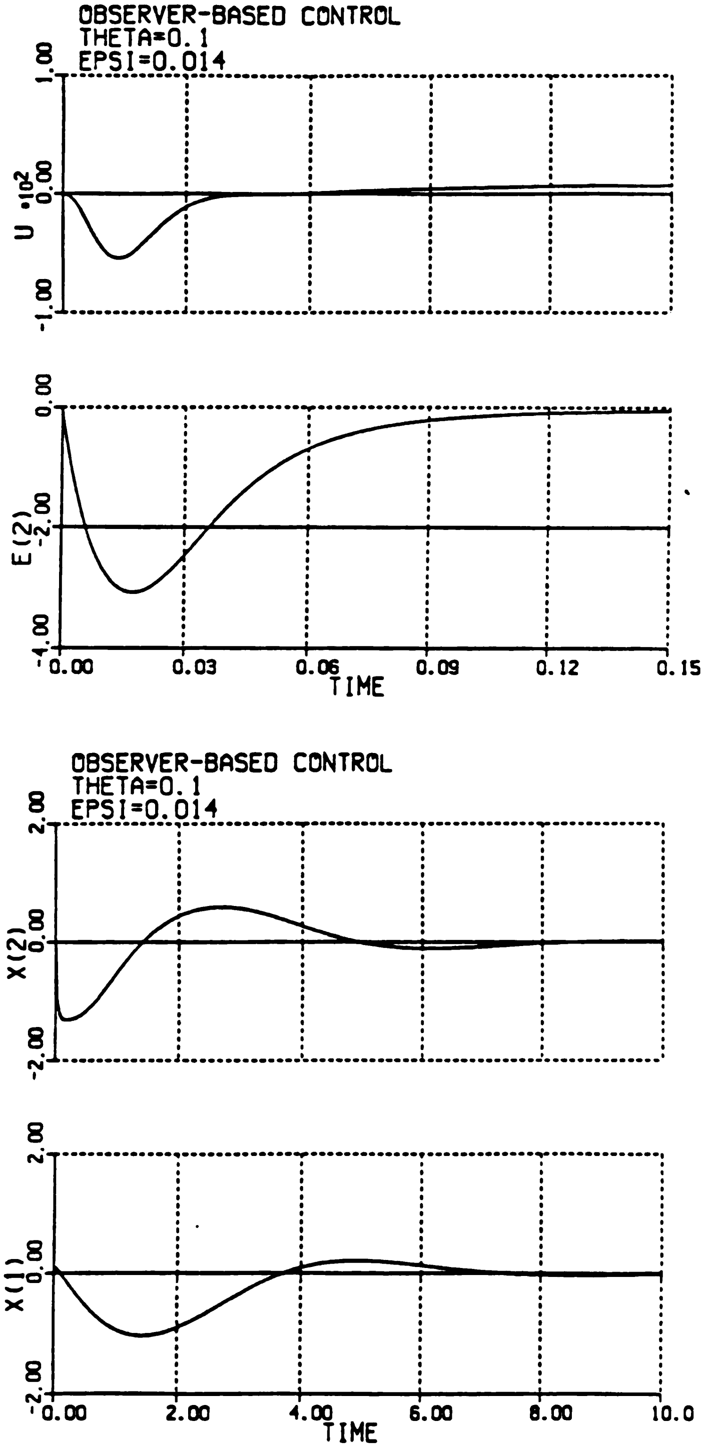


Figure 5.4- Time profiles of state variables for output feedback control of Example 5.2, with $\varepsilon = 0.014$.

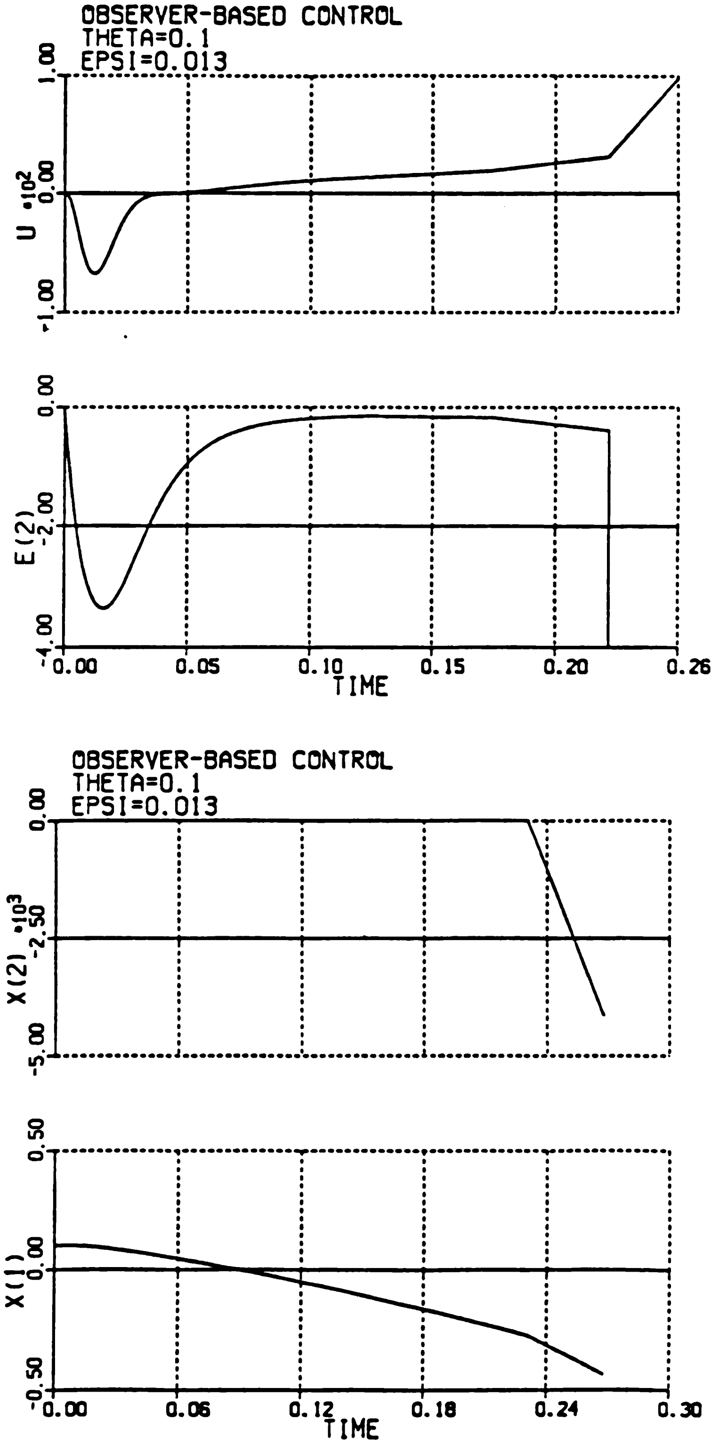


Figure 5.5- Time profiles of state variables for output feedback control of Example 5.2, with $\varepsilon = 0.013$.

6 Globally Bounded Control

6.1 Introduction

In Chapters 5, we studied the effects of peaking on the behavior of the closed-loop system. In this chapter, we argue that the if the state feedback component of the observer-based control is designed to be globally bounded, the states of the nonlinear system will not exhibit peaking, and consequently the destabilization phenomenon associated with peaking will not take place. In order to prove this point, we first present a new singular perturbation result in section 6.2. Then, in section 6.3, as a corollary of the result of section 6.2, we present a result on the stability of the closed-loop system, for the case when the control is globally bounded. Finally, in section 6.4 we apply bounded control to the examples of chapter 5 and present simulations to show that the undesirable effects of peaking are indeed eliminated.

6.2 Singular Perturbation Result

The closed-loop system (4.15) is a standard two-time-scale singularly perturbed system which can be written in the following form,

$$\begin{cases} \dot{x} = f(x) + \tilde{f}(x, N(\varepsilon) y), & x(0) = x_0, \quad x \in \mathbf{R}^n \\ \varepsilon \dot{y} = A y + \varepsilon \tilde{g}(x, N(\varepsilon) y), & y(0) = y_0, \quad y \in \mathbf{R}^m \end{cases} \tag{6.1}$$

where

$$x := \begin{bmatrix} z \\ e_s \end{bmatrix}, \; y := e_f ,$$

A is Hurwitz, $f(0) = 0$, and $\tilde{f}(x, 0) = 0$ for all $x \in \mathbf{R}^n$. The slow subsystem is obtained by setting $\varepsilon = 0$ in (6.1) and dropping the initial condition, $y(0) = y_0$ to get

$$\dot{x} = f(x), \qquad x(0) = x_0. \tag{6.2}$$

Assume that $f, \tilde{f}$ and $\tilde{g}$ are smooth enough to ensure existence and uniqueness of the solution of (6.1) and (6.2). Denote the solution of (6.1) by $(x_\varepsilon(t), y_\varepsilon(t))$ and the solution of (6.2) by $x_s(t)$. Moreover assume that the origin is the unique equilibrium point of (6.1) and (6.2).

In this section, we study the asymptotic behavior of system (6.1). First, we recall some known results from singular perturbation theory which are relevant to our problem. Then we study a case that arises in our problem, namely when the initial condition of the fast variable, y_0, is $O(\varepsilon^{-\beta})$ (β is a positive integer).

By Tikhonov's Theorem [Kokotovic, et.al. (1986)], $x_\varepsilon(t) \to x_s(t)$ as $\varepsilon \to 0^+$ uniformly on compact time intervals. Hoppensteadt (1966) generalized Tikhonov's result to infinite time intervals. We quote a result of Saberi-Khalil (1984) which is a nonlocal version of Hoppensteat's result. For the sake of clarity, we will closely follow the notation of Saberi-Khalil (1984).

Assumption 6.1: There exists a Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}_+$ such that the following inequalities hold for all $x \in \Omega$:

$$\frac{\partial V}{\partial x} f(x) \leq -\beta_0 \| x \|^2 \tag{6.3}$$

$$\| \frac{\partial V}{\partial x} \| \leq \beta_1 \| x \| \tag{6.4}$$

$$\beta_2 \| x \|^2 \leq V(x) \leq \beta_3 \| x \|^2 \tag{6.5}$$

where β_0, β_1, β_2 and β_3 are positive constants, and $\Omega \subset \mathbf{R}^n$ is an open connected set containing the origin.

Assumption 6.2: The following inequalities hold for all $x \in \Omega$, $y \in \Sigma$ and $\varepsilon \in [0, \bar{\varepsilon}]$:

$$\| \tilde{f}(x, N(\varepsilon)y) \| \leq \beta_4 \| y \| \tag{6.6}$$

$$\| \tilde{g}(x, N(\varepsilon)y) \| \leq \beta_5 \| x \| + \beta_6 \| y \| \tag{6.7}$$

where β_4, β_5 and β_6 are nonnegative constants, $\bar{\varepsilon}$ is a positive constant, and $\Sigma \subset \mathbf{R}^m$ is an open connected set containing the origin.

Let P be the symmetric positive definite solution of the Lyapunov equation $PA + A'P = -I_m$ and $W(y) := y'Py$. Choose $v_0 > 0$ and $w_0 > 0$ such that

$$L_R := \left\{ x \in \mathbf{R}^n \mid V(x) \leq v_0 \right\} \subset \Omega \quad \text{and} \tag{6.8}$$

$$L_B := \left\{ y \in \mathbf{R}^m \mid W(y) \leq w_0 \right\} \subset \Sigma \tag{6.9}$$

and define the set L as

$$L := \left\{ (x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid \frac{V(x)}{v_0} + \frac{W(y)}{w_0} \leq 1 \right\} \tag{6.10}$$

Theorem 6.1‡ [Saberi-Khalil (1984)]: Under Assumptions 6.1 and 6.2, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the equilibrium point $(x = 0, y = 0)$ of (6.1) is asymptotically stable with L inside the region of attraction. Moreover for every initial condition $(x_0, y_0) \in L$, $x_\varepsilon(t) \to x_s(t)$ as $\varepsilon \to 0^+$ uniformly in t on $[0, \infty)$.

In Theorem 6.1, the initial condition (x_0, y_0) is bounded, uniformly in ε, i.e., it is $O(1)$. However, from (3.8)-(3.10), when the initial condition of the estimation error $e(0)$ is $O(1)$,the initial conditions of some of the components of e_f are in general $O(\frac{1}{\varepsilon^{K-1}})$. The following example shows that in such cases Assumptions 6.1 and 6.2 are not sufficient for the convergence of $x_\varepsilon(t)$ towards $x_s(t)$ as $\varepsilon \to 0$.

Example 6.1: Consider the singularly-perturbed system

$$\begin{cases} \dot{x} = -x + y & x_0 = 1 \\ \varepsilon \dot{y} = -y & y_0 = \dfrac{1}{\varepsilon^2} \end{cases} \tag{6.11}$$

which is asymptotically stable for all $\varepsilon > 0$ and satisfies Assumptions 6.1 and 6.2 globally. The solution of (5.3) is

‡ The statement of this theorem can be strengthened to $x_\varepsilon(t) - x_s(t) = O(\varepsilon)$, using results of Hoppensteadt (1966). However, the extension will take some space that might divert attention from the main point of this thesis.

$$\begin{cases} x_\varepsilon(t) = e^{-t} + \dfrac{1}{\varepsilon(1-\varepsilon)}\,(\,e^{-t} - e^{-t/\varepsilon}\,) \\ y_\varepsilon(t) = \dfrac{1}{\varepsilon^2}\,e^{-t/\varepsilon} \end{cases}$$

while the solution of the slow subsystem $\dot{x} = -x$, $x(0) = 1$ is $x_s(t) = e^{-t}$. It can be easily seen that on any compact subset of $(0, \infty)$, $x_\varepsilon(t) - x_s(t)$ diverges as $\varepsilon \to 0^+$.

Therefore, we need to develop a trajectory approximation result for system (6.1), in the case when the initial condition of the fast variable, y, is $O(\varepsilon^{-\beta})$. From the previous example, it is clear that some additional conditions must be imposed, if a trajectory approximation result is to hold in the case when y_0 is $O(\varepsilon^{-\beta})$.

Assumption 6.3: The following inequalities hold for all $x \in \Omega$, $y \in \mathbf{R}^m$ and $\varepsilon \in [0, \bar{\varepsilon}]$:

$$\| f(x) \| + \| \tilde{f}(x, N(\varepsilon)y) \| \le \beta_7 \| x \| + \beta_8 \tag{6.12}$$

$$\| \tilde{g}(x, N(\varepsilon)y) \| \le \beta_9 \| x \| + \beta_{10} \| y \| + \beta_{11} \tag{6.13}$$

where β_8, β_9, β_{10} and β_{11} are nonnegative constants and β_7 is a positive constant.

Inequality (6.12) is a restrictive requirement, because the right-hand-side is independent of y. This is a requirement that one would not expect to hold in a general singularly perturbed system, but it holds in our application when the control is bounded. Let

$$\tilde{L}_R := \left\{ x \in L_R \ \mid \ V(x) \le v_1 \right\} \tag{6.14}$$

$$\tilde{L}_B := \left\{ y \in \mathbf{R}^m \ \mid \ \| y \| \le \frac{\gamma}{\varepsilon^\beta} \right\} \tag{6.15}$$

where $v_1 \in (0, v_0)$, $\gamma \in (0, \infty)$ and $\beta \in \mathbf{N}$ are chosen arbitrarily.

Theorem 6.2: Under Assumptions 6.1-6.3, there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ the equilibrium point $(x = 0, y = 0)$ of (6.1) is asymptotically stable with $\tilde{L}_R \times \tilde{L}_B$ inside the region of attraction. Moreover for every initial condition $(x_0, y_0) \in \tilde{L}_R \times \tilde{L}_B$, $x_\varepsilon(t) \to x_s(t)$ as $\varepsilon \to 0^+$ uniformly in t on $[0, \infty)$.

Proof of Theorem 6.2: Since A is a Hurwitz matrix, there exist positive constants K and α_1 such that

$$\| e^{At/\varepsilon} \| \le K \, e^{-\alpha_1 t/\varepsilon} \ \text{for all} \ t \in \mathbf{R}_+. \tag{6.16}$$

Claim 1: For every $\eta > 0$, there exist $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ the origin of (6.1) is asymptotically stable with $\tilde{L}_R \times \tilde{L}_B$ inside the region of attraction. Moreover,

$$\| y_\varepsilon(t) \| \le \eta \ \ \text{for all } t \ge \frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}} \right]$$

Proof of Claim 1: Given $\eta > 0$, let

$$\zeta := \min \left\{ \eta^2 \lambda_{\min}(P) \, , \, \frac{w_0}{2}(1 - \frac{v_1}{v_0}) \right\} \tag{6.17}$$

and

$$S := \left\{ (x, y) \in L \mid W(y) = y'Py \leq \zeta \right\} \tag{6.18}$$

where P, ν_0, w_0, and L were introduced after Assumption 6.2, and $\nu_1 \in (0,\nu_0)$ was introduced in (6.14). Then it is easy to see that

$$(x, y) \in S \Rightarrow \| x \| \leq \sqrt{\frac{\nu_0}{\beta_2}} \quad \text{and} \quad \| y \| \leq \eta \tag{6.19}$$

Moreover, for all $(x, y) \in S$ and $\varepsilon \in [0,\bar{\varepsilon}]$

$$\dot{W} = 2y'P\left[\frac{A}{\varepsilon} y + \tilde{g}(x, N(\varepsilon)y) \right]$$

$$\leq -\frac{1}{\varepsilon} \| y \|^2 + 2 \| P \| \| y \| (\beta_5 \| x \| + \beta_6 \| y \|)$$

$$\leq -\frac{W}{\varepsilon\lambda_{max}(P)} + 2\beta_5 \| P \| \eta\sqrt{\frac{\nu_0}{\beta_2}} + 2\beta_6 \| P \| \eta^2$$

$$= -\frac{\zeta}{\varepsilon\lambda_{max}(P)} + 2\beta_5 \| P \| \eta\sqrt{\frac{\nu_0}{\beta_2}} + 2\beta_6 \| P \| \eta^2 \qquad \text{on } H$$

where

$$H := \left\{ (x, y) \in S \mid W(y) = \zeta \right\}$$

Therefore, $\dot{W}<0$ on H for sufficiently small ε. By Theorem 6.1, the set L is an invariant set for sufficiently small ε. Therefore, the set S is an invariant set for sufficiently small ε. Moreover, by Theorem 6.1, the origin of the closed-loop system is asymptotically stable with S inside the region of attraction. To conclude the proof of Claim 1, it suffices to show that

$$\left[x_\varepsilon\left(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]\right), y_\varepsilon\left(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]\right) \right] \in S \tag{6.20}$$

for sufficiently small ε. The idea is to show that for all initial states in the set $\tilde{L}_R \times \tilde{L}_B$, $y_\varepsilon(t)$ decays rapidly towards S, while during the same time $x_\varepsilon(t)$ can not grow out of S, due to inequality (6.12). To show this, we start by calculating a worst-case bound on the growth of $x_\varepsilon(t)$. By (6.1), (6.3), (6.4), (6.5) and (6.12), we have

$$\begin{aligned} \frac{d}{dt}\sqrt{V} &= \frac{\dot{V}}{2\sqrt{V}} = \frac{1}{2\sqrt{V}} \frac{\partial V}{\partial x}\left[f + \tilde{f}\right] \\ &\leq \frac{\beta_1\beta_7}{2\beta_2}\sqrt{V} + \frac{\beta_1\beta_8}{2\sqrt{\beta_2}} \quad \text{on } L_R \times \mathbf{R}^m \end{aligned} \tag{6.21}$$

Solving the above differential inequality for $\sqrt{V}$ yields

$$\sqrt{V(t)} \leq \left[\sqrt{V(0)} + a\right] e^{bt} - a \tag{6.22}$$

where

$$a = \frac{\beta_8\sqrt{\beta_2}}{\beta_7}, \quad b = \frac{\beta_1\beta_7}{2\beta_2}$$

which implies that

$$\sqrt{V(t)} \leq \left[\sqrt{\mathrm{v}_1} + a\right] e^{bt} - a \tag{6.23}$$

It can be easily seen that

$$\sqrt{V(t)} \leq \sqrt{0.5\,(\mathrm{v}_0 + \mathrm{v}_1)}, \quad \text{for all } t \leq \psi \tag{6.24}$$

where

$$\psi = \frac{1}{b} \ln \left[\frac{a + \sqrt{0.5(v_0+v_1)}}{a + \sqrt{v_1}} \right] \tag{6.25}$$

By (6.5) and (6.24), we conclude that

$$\| x_\varepsilon(t) \|^2 \le \frac{1}{2\beta_2} (v_0+v_1) \quad \text{for all } \ t \le \psi \tag{6.26}$$

which by (6.13) implies

$$\| \tilde{g}\,(x_\varepsilon(t)\,, N(\varepsilon)\, y_\varepsilon(t)\,) \| \le \beta_{10} \| y_\varepsilon(t) \| + \beta_{12} \quad \text{for all } \ t \le \psi \tag{6.27}$$

where

$$\beta_{12} := \beta_9 \left[\frac{v_0+v_1}{2\beta_2} \right]^{1/2} + \beta_{11} \tag{6.28}$$

Now, the solution of (6.1) is given by

$$y_\varepsilon(t) = e^{At/\varepsilon} y_0 + \int_0^t e^{A(t-\tau)/\varepsilon}\, \tilde{g}\,(x_\varepsilon(\tau), N(\varepsilon) y_\varepsilon(\tau)\,)\, d\tau \tag{6.29}$$

Therefore, by (6.16), we obtain

$$\| y_\varepsilon(t) \| \le K \| y_0 \| e^{-\alpha_1 t/\varepsilon} + \int_0^t K\, e^{-\alpha_1 (t-\tau)/\varepsilon} \left[\beta_{10} \| y_\varepsilon(\tau) \| + \beta_{12} \right] d\tau \tag{6.30}$$

for all $t \in [0,\psi]$. Multiply (6.30) by $e^{\alpha_1 t/\varepsilon}$ and let

$$z(t) := e^{\alpha_1 t/\varepsilon} \| y_\varepsilon(t) \| \tag{6.31}$$

Then,

$$z(t) \le K \| y_0 \| + \frac{\varepsilon K \beta_{12}}{\alpha_1} (e^{\alpha_1 t/\varepsilon} - 1) + \int_0^t K \beta_{10} z(\tau) d\tau \tag{6.32}$$

Application of the generalized Bellman-Gronwall inequality [Hahn (1967)] to (6.32) and use of (6.31) imply that

$$\| y_\varepsilon(t) \| < K \| y_0 \| e^{(K\beta_{10} - \frac{\alpha_1}{\varepsilon}) t} + \frac{K \beta_{12} \varepsilon}{\alpha_1 - \varepsilon K \beta_{10}} [1 - e^{(K\beta_{10} - \frac{\alpha_1}{\varepsilon})t}] \tag{6.33}$$

for all $t \in [0,\psi]$. Let

$$\varepsilon_2 := \frac{\alpha_1}{2K\beta_{10}}$$

Then

$$\| y_\varepsilon(t) \| \le K \gamma \varepsilon^{-\beta} e^{-\alpha_1 t/2\varepsilon} + \frac{2K\beta_{12}\varepsilon}{\alpha_1} \tag{6.34}$$

for all $\varepsilon \in (0,\varepsilon_2)$ and $t \in [0,\psi]$. Furthermore, since $\varepsilon \ln \left[\frac{1}{\varepsilon^{\beta+1}} \right] \to 0$ as $\varepsilon \to 0$, there exists $\varepsilon_3 \in (0,\varepsilon_2)$ such that

$$\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}} \right] \le \psi \tag{6.35}$$

and

$$K\gamma\varepsilon + \frac{2K\beta_{12}\varepsilon}{\alpha_1} \le \left[\frac{\zeta}{\lambda_{max}(P)} \right]^{\frac{1}{2}} \tag{6.36}$$

for all $\varepsilon \in (0,\varepsilon_3)$. Therefore,

$$\| y_\varepsilon(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]) \| \le K\gamma\varepsilon + \frac{2K\beta_{12}\varepsilon}{\alpha_1} \le \left[\frac{\zeta}{\lambda_{max}(P)}\right]^{\frac{1}{2}} \tag{6.37}$$

for all $\varepsilon \in (0,\varepsilon_3)$. By (6.24) and (6.35)

$$V(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]) \le 0.5(v_0+v_1) \tag{6.38}$$

for all $\varepsilon \in (0,\varepsilon_3)$. Finally, from (6.37) and (6.38), we can see that

$$\left[x_\varepsilon(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]) , y_\varepsilon(\frac{2\varepsilon}{\alpha_1} \ln \left[\frac{1}{\varepsilon^{\beta+1}}\right]) \right] \in S$$

for all $\varepsilon \in (0,\varepsilon_3)$, which concludes the proof of Claim 1.

The first part of Theorem 6.2 clearly follows from Claim 1. To prove the uniform convergence result, let (ε_n) be a positive sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. By Claim 1, there exists $\underline{\varepsilon}>0$ such that for all $\varepsilon \in (0,\underline{\varepsilon})$, $x_\varepsilon(t) \to 0$ as $t \to \infty$, and $\frac{d}{dt}[\frac{V}{v_0} + \frac{W}{w_0}]$ is negative definite along the trajectory uniformly in ε. Therefore, given $\xi>0$, there exists $M \in \mathbf{N}$ and $T>0$ such that

$$\| x_{\varepsilon_n}(t) - x_s(t) \| < \xi \tag{6.39}$$

for all $t \ge T$ and $n \ge M$. Next, we will show that $x_{\varepsilon_n}(t) \to 0$ as $n \to \infty$ uniformly on $[0, T]$.

Claim 2:

$$\int_0^t \tilde{f}(x_{\varepsilon_n}(\tau), N(\varepsilon_n) y_{\varepsilon_n}(\tau))\, d\tau \to 0 \text{ as } n \to \infty, \text{ uniformly on } [0,T].$$

Proof of Claim 2: Given $\xi>0$, by Claim 1, there exists $N_1 \in \mathbf{N}$ such that for all $n \ge N_1$, the origin is asymptotically stable and

$$\| y_{\varepsilon_n}(t) \| \le \frac{\xi}{2T\beta_4} \quad \text{for all } t \ge \frac{2\varepsilon_n}{\alpha_1} \ln\left[\frac{1}{\varepsilon_n^{\beta+1}}\right] \tag{6.40}$$

Therefore,

$$\| \int_0^t \tilde{f}(x_{\varepsilon_n}(\tau), N(\varepsilon_n) y_{\varepsilon_n}(\tau))\, d\tau \| \le \int_0^{T_n} \left[\beta_7 \| x_{\varepsilon_n} \| + \beta_8 \right] d\tau$$

$$+ \int_{T_n}^t \beta_4 \| y_{\varepsilon_n}(\tau) \| \, d\tau$$

$$\le T_n \left[\beta_7 \sqrt{\frac{\nu_0}{\beta_2}} + \beta_8 \right] + \frac{T\beta_4 \xi}{2T\beta_4}$$

where

$$T_n := \frac{2\varepsilon_n}{\alpha_1} \ln\left[\frac{1}{\varepsilon_n^{\beta+1}}\right]$$

and we used (6.12) on $[0, T_n]$ and (6.6) on $[T_n, t]$, to arrive at the above inequality.

There exists $N_2 \in \mathbf{N}$ such that

$$\frac{2\varepsilon_n}{\alpha_1} \ln\left[\frac{1}{\varepsilon_n^{\beta+1}}\right] \left[\beta_7 \sqrt{\frac{\nu_0}{\beta_2}} + \beta_8 \right] < \frac{\xi}{2} \quad \text{for all } n \ge N_2$$

Therefore,

$$\| \int_0^t \tilde{f}(x_{\varepsilon_n}(\tau), N(\varepsilon_n) y_{\varepsilon_n}(\tau))\, d\tau \| < \xi$$

for all $n \ge \max\left\{ N_1, N_2 \right\}$ and all $t \in [0, T]$, which concludes the proof of Claim 2.

By Claim 2 and continuous dependence of solutions of differential equations [Hahn (1967)] we can show that

$$x_{\varepsilon_n}(t) \rightarrow x_s(t) \ \text{ uniformly on } [0,T]$$

This fact together with (6.39) show that $x_\varepsilon(t) \rightarrow x_s(t)$ as $\varepsilon \rightarrow 0^+$ uniformly in t on $[0, \infty)$. □

6.3 Stability Result

In this section we apply Theorem 6.2 to prove that global boundedness of the state feedback component of the observer-based control prevents the destabilizing effect of the peaking phenomenon.

Theorem 6.3: Suppose that Assumptions 2.G1, 3.G2, and 4.G3 are satisfied, and that $\bar{\alpha}(.)$, $\bar{\beta}(.)$, and $\phi(.)$ are globally bounded. Then there exist positive constants d_1, d_2, and $\bar{\varepsilon}$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ the origin of the closed-loop system (3.3) and (4.1), with $L(.)$ given by (4.6), is asymptotically stable with the set

$$R := \left\{ (z, e) \in \mathbf{R}^{2n} \mid W(z) \le d_1, \ \| e \| \le d_2 \right\} \tag{6.41}$$

inside the region of attraction.

Proof of Theorem 6.3: The proof follows as a corollary of Theorem 6.2. Since $W(z)$ is a quadratic Lyapunov function,

$$\eta_1 \| z \|^2 \le W(z) \le \eta_2 \| z \|^2$$

$$\left\| \frac{\partial W}{\partial z} \right\| \le \eta_3 \| z \|$$

for some positive constants η_1, η_2, and η_3. The set S of Assumptions 3.G2, and 4.G3 is an open set containing the origin. Therefore, there exist $r>0$ such that

$$\left\{ z \in \mathbf{R}^p \mid \| z \| \le r \right\} \subset S \tag{6.42}$$

Let

$$\Sigma_1 := \left\{ z \in \mathbf{R}^p \mid W(z) \le \eta_1 \frac{r^2}{9} \right\} \tag{6.43}$$

$$\Sigma_2 := \left\{ e_s \in \mathbf{R}^l \mid e_s' P_s e_s \le \lambda_{\min}(P_s) \frac{r^2}{9 \, | \Gamma_1 |^2} \right\} \tag{6.44}$$

$$V(z, e_s) := W(z) + d \; e_s' P_s e_s \qquad d>0 \tag{6.45}$$

There exist $\nu_0>0$ and $\bar{d}>0$ such that for all $d \ge \bar{d}$

$$\Omega := \left\{ (z, e_s) \in \mathbf{R}^{p+l} \mid V(z, e_s) \le \nu_0 \right\} \subset \Sigma_1 \times \Sigma_2 \tag{6.46}$$

The derivative of V along the trajectory of the slow subsystem (4.18) is

$$\frac{dV}{dt} = \frac{\partial W}{\partial z} \left[Az + B\phi(z) + B\beta^{-1}(z) \left[\Delta\alpha + \Delta\beta \, \phi(z) \right] \right.$$

$$\left. + B\beta^{-1}(z) \left[F(z-\Gamma_1 e_s) - F(z) \right] \right] - d \, | e_s |^2 \tag{6.47}$$

By (3.9), Assumption 4.G3, and (6.46)

$$\frac{dV}{dt} \le -\gamma_2 | z |^2 + \eta_3 k_6 | B | \, | \Gamma_1 | \, | z | \, | e_s |$$

$$- d \, | e_s |^2 \quad \text{on } \Omega$$

$$\le -\frac{\alpha_0}{2} | z |^2 - \frac{d}{2} | e_s |^2$$

for sufficiently large d. Therefore, for sufficiently large d

$$\frac{dV}{dt} \leq -\min\left\{ \frac{\alpha_0}{2}, \frac{d}{2} \right\} \left\| \begin{bmatrix} z \\ e_s \end{bmatrix} \right\|^2 \quad \text{on } \Omega \tag{6.48}$$

which is inequality (6.3) of Assumption 6.1. It can be easily seen that inequalities (6.4) and (6.5) of Assumption 6.1 are also satisfied. To prove that Assumption 6.2 is also satisfied, let

$$\Sigma := \left\{ e_f \in \mathbf{R}^{n-l} \mid e_f{}' P_f e_f \leq \omega_0 \right\} \tag{6.49}$$

$$\omega_0 := \lambda_{\min}(P_f) \, \frac{r^2}{9 \, \| \Gamma_2 \|^2}$$

The sets Σ_1, Σ_2, and Σ_3 are chosen such that for all $(z, e_s, e_f) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$, $\hat{z}$ satisfies the bound:

$$\begin{aligned} \| \hat{z} \| &= \| z - \Gamma_1 e_s - \Gamma_2 N(\varepsilon) e_f \| \\ &\leq \| z \| + \| \Gamma_1 \| \, \| e_s \| + \| \Gamma_2 \| \, \| e_f \| \\ &\leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r \end{aligned}$$

where we have assumed that $\varepsilon \leq 1$, so that $\| N(\varepsilon) \| \leq 1$. Then, by Assumption 4.G3, (6.46), and (6.49)

$$\begin{aligned} \| B \beta^{-1}(z) \, (I_q + \Delta_g) \Big[& F(z - \Gamma_1 e_s - \Gamma_2 N(\varepsilon) e_f) \\ & - F(z - \Gamma_1 e_s) \Big] \| \leq k_6 \, \| B \| \, \| \Gamma_2 N(\varepsilon) e_f \| \end{aligned}$$

which implies inequality (6.6) of Assumption 6.2. To prove inequality (6.7), note that $\tilde{g}$ of (6.1) is given by

$$\tilde{g}(\,(z, e_s), N(\varepsilon) e_f\,) = B_f \left[D_s e_s + D_f N(\varepsilon) e_f + \Gamma_{in}^{-1} \Delta(z, \hat{z}) \right] \tag{6.50}$$

By Assumptions 3.G2 and 4.G3, (6.46), and (6.49)

$$\| \Delta(z,\hat{z}) \| \le (k_1+k_3+k_2k_5) \| z \| + (k_5+k_6) \| \Gamma_1 e_s + \Gamma_2 N(\varepsilon) e_f \| \tag{6.51}$$

for all $(z, e_s) \in \Omega$ and $e_f \in \Sigma$. This, together with (6.50), proves inequality (6.7). Therefore, Assumptions (6.1) and (6.2) are satisfied. It remains to show that Assumption 6.3 is also satisfied for system (5.1), under the assumption of global boundedness of $\bar{\alpha}(.)$, $\bar{\beta}(.)$, and $\phi(.)$. To show that Assumption 6.3 is satisfied, note that by Assumptions 2.G1 and G3

$$\| Az + B\phi(z) + B\delta(z, \phi(z), t) + B\beta^{-1}(z)(I_q + \Delta_g)\left[F(z-\Gamma_1 e_s) - F(z)\right] \| \le \beta_7 \left\| \begin{bmatrix} z \\ e_s \end{bmatrix} \right\|$$

and by boundedness of $F(.)$

$$\| B\beta^{-1}(z)\ (I_q + \Delta_g)\left[F(z-\Gamma_1 e_s-\Gamma_2 N(\varepsilon)e_f) - F(z-\Gamma_1 e_s)\right] \| \le \beta_8$$

for all $(z, e_s) \in \Omega$, and $e_f \in \mathbb{R}^{n-l}$, for some positive constants β_7 and β_8, which implies inequality (6.12). Inequality (6.13) follows similarly from Assumption 3.G2, (6.50), (4.3), and boundedness of $F(.)$ and $\phi(.)$. Therefore, all the conditions of Theorem 6.2 are satisfied. Choose $\nu_1 \in (0, \nu_0)$, $\gamma>0$, and $\beta=K-1$, where K is the integer introduced along with transformation (4.4) (K can be viewed as the relative degree of system (2.2)). Then, by Theorem (6.2), there exists $\bar{\varepsilon}>0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ the origin is an asymptotically stable equilibrium point with the set

$$\tilde{L}_R \times \tilde{L}_B = \left\{ \begin{bmatrix} z \\ e_s \\ e_f \end{bmatrix} \in \mathbf{R}^{2n} \;\middle|\; V(z, e_s) \le \nu_1, \;\; \| e_f \| \le \frac{\gamma}{\varepsilon^{K-1}} \right\}$$

inside the region of attraction. By (4.7), (4.9)-(4.11), and (6.45), it can be easily seen that there exist $d_1 > 0$, $d_2 > 0$ such that

$$R \subset \tilde{L}_R \times \tilde{L}_B$$

which completes the proof. □

Example 5.1 (Continued): Let us apply the following globally bounded control to the pendulum example of Chapter 5,

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \dfrac{2}{\varepsilon} (y - \hat{x}_1) \\ \dot{\hat{x}}_2 = u + \dfrac{1}{\varepsilon^2} (y - \hat{x}_1) \\ u = - 2.0 \, sat_\zeta(\hat{x}_1 + \hat{x}_2) \, , \;\; \zeta = 0.01 \end{cases}$$

Figure 6.1 shows results of the simulations with $\varepsilon = 0.01$, and all the initial conditions the same as that of Figure 5.2. Note that although the peaking is present in the estimation error (Figure 6.1c), there is no peaking in input u or the states of the plant.

Example 5.2 (Continued): Now let us apply a globally bounded control to Example 5.2 of Chapter 5. We use a saturation nonlinearity to bound u and v. The observer-based control in this case is

$$
\left\{
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= v + \frac{1}{\varepsilon^2}(y - \hat{x}_1) \\
u &= sat_\sigma[-3\,\hat{x}_2^3\,] + v \\
v &= sat_\sigma[-2\,\hat{x}_1 - \hat{x}_2\,]
\end{aligned}
\right.
\tag{6.52}
$$

Figure 6.2 and 6.3 shows the results of simulation for the closed-loop system with θ=0.1 , ε=0.001, σ=1.0, and the same initial conditions as in Figure 5.5. The closed-loop system is stabilized in this case, due to the fact that the saturation non-linearity acts as a buffer protecting the plant from the impulsive-like behavior of the observer. Figure 6.2 shows the behavior of the closed-loop system within the boundary layer.

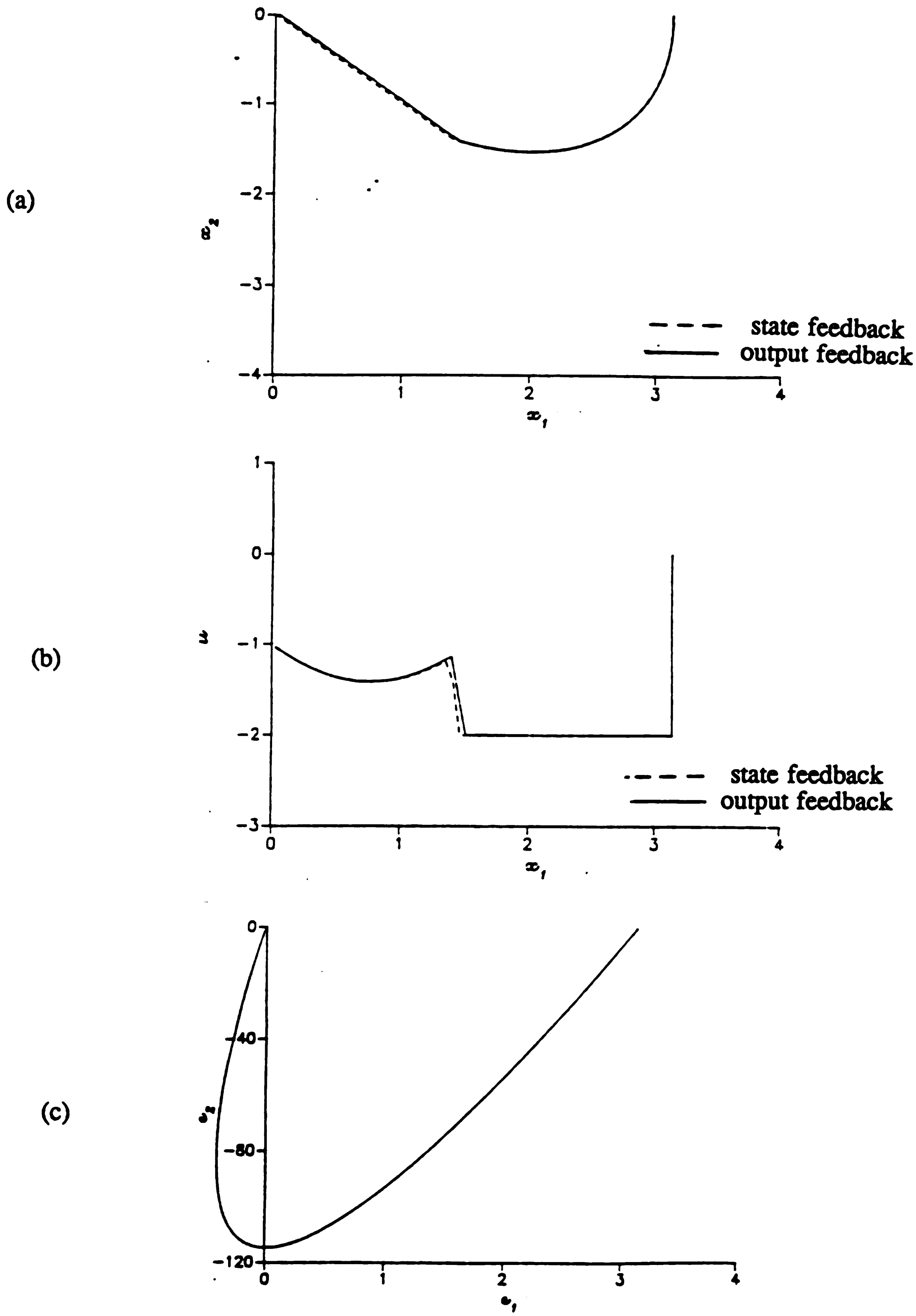


Figure 6.1- Example 5.1 with bounded control

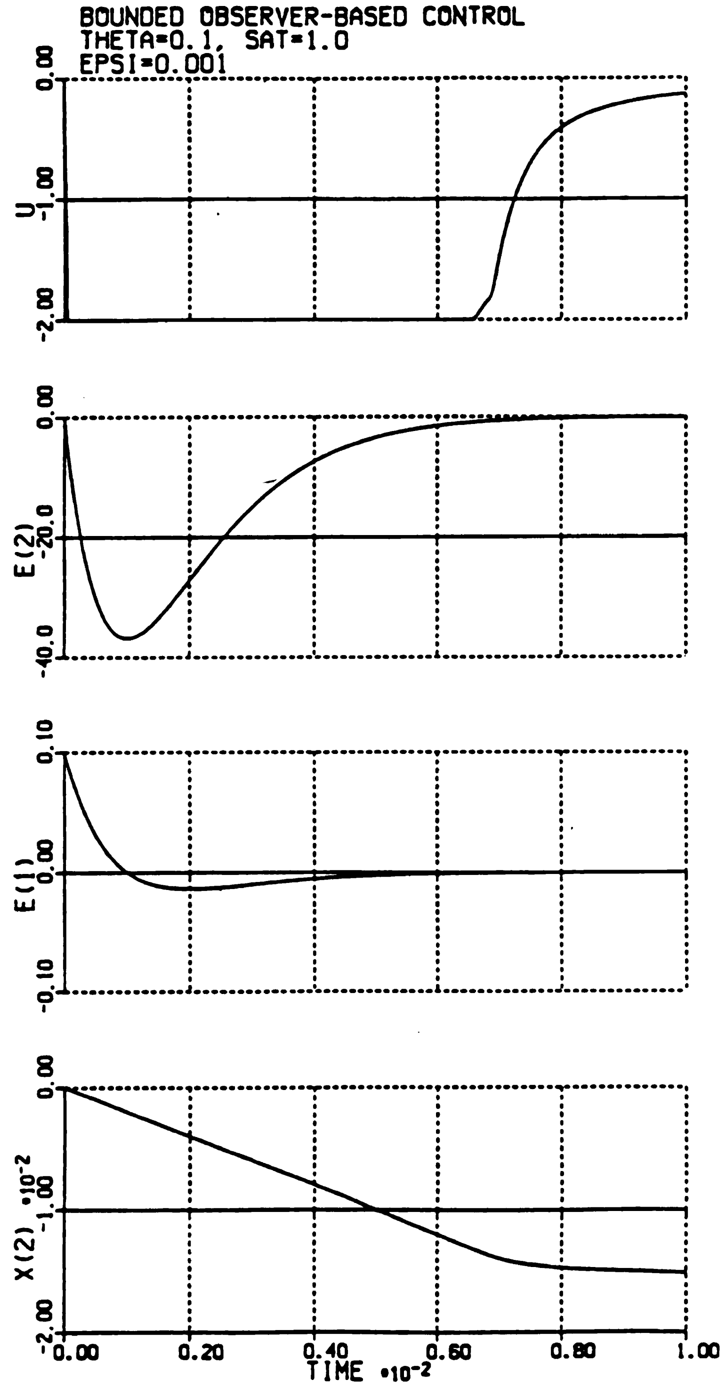


Figure 6.2-Time profiles of the state variables for Example 5.2 with bounded control in the boundary layer

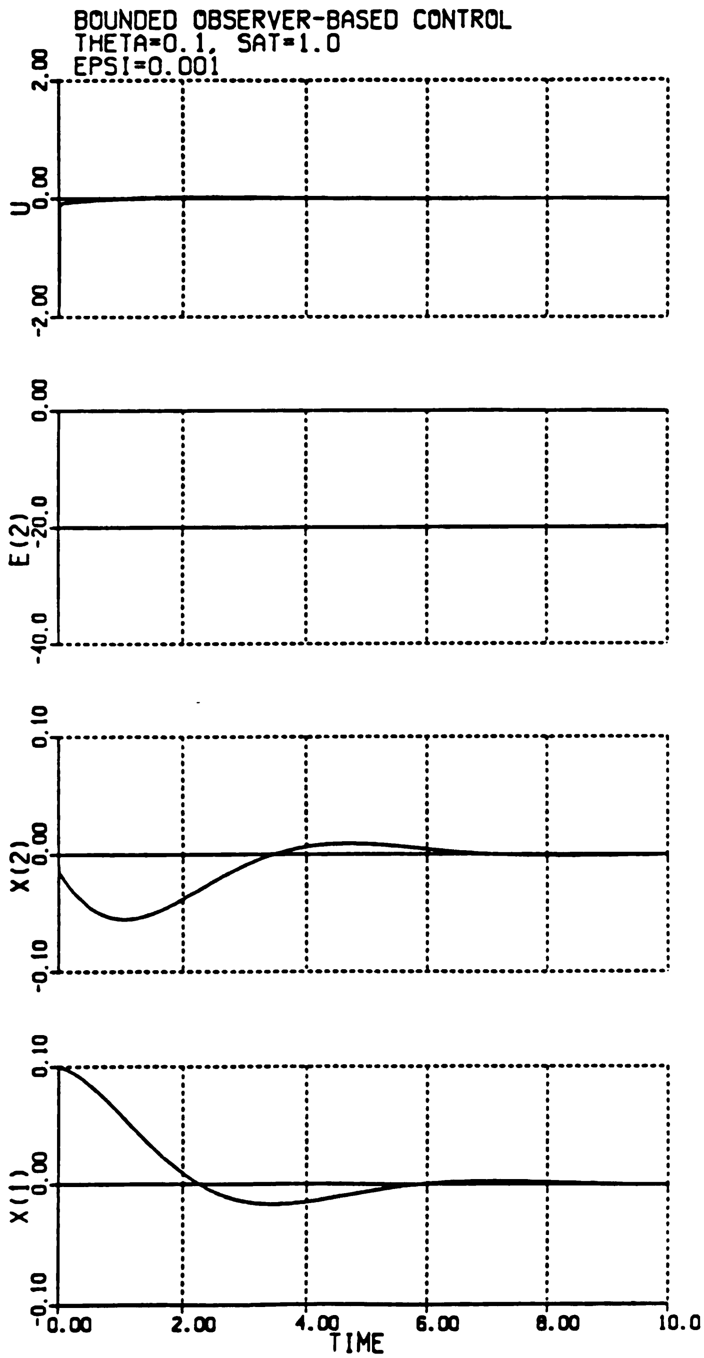


Figure 6.3- Time profile of the state variables for Example 5.2 with bounded control

7 Future Work

There are a number of performance and robustness issues that could be the subject of future work on this problem. We briefly go over them.

Robustness to Unmatched Uncertainties

Our output feedback control scheme is robust with respect to parametric uncertainties that satisfy the matching condition. An important robustness issue is assessing the performance of the closed-loop system, in the presence of other kinds of modeling uncertainties, such as unmatched parametric uncertainties, and unmodeled high-frequency dynamics. Since the closed-loop stability results of this work were proved using Lyapunov theory, it is clear that unmatched uncertainties and unmodeled high-frequency dynamics would not destroy closed-loop stability as long as they are sufficiently small. Therefore the purpose of such a performance assessment should be to find quantitative bounds on how small such uncertainties should be.

Semiglobality

The notion of semiglobality was first introduced in Sussmann-Kokotovic (1989) in connection with stabilization of nonlinear systems via state feedback control. In the work of Sussmann-Kokotovic (1989), the definition of semiglobality is motivated by the fact that the conditions that are needed for global stabilization are usually very stringent. On the other hand, when a nonlinear system is locally stabilized, the designer in general has no control on the region of attraction of the system. The notion of semiglobality is a compromise between these two extremes: Given an *arbitrary* bounded set B in the state space, under what conditions is it possible to find a controller that renders the origin of the closed-loop system asymptotically stable, with the set B inside the region of attraction?

In the case of our problem, the question of semiglobality can be raised in the following context: If Assumptions 2.G1, 3.G2, and 4.G3 are globally satisfied, then Theorem 4.2 ensures global asymptotic stability of the origin. However, It is very restrictive to assume that functions $F(.)$ and $\phi(.)$ are globally Lipschitzian. On the other hand, by assuming that Assumption 4.G3 is only satisfied on a compact set around the origin (which boils down to assuming sufficient smoothness of $F(.)$ and $\phi(.)$), Theorem 4.3 gives an estimate of the region of attraction which shrinks as ε tends to zero, due to the peaking phenomenon. In Theorem 6.3, we isolated the peaking form the plant, and hence, were able to obtain a region of attraction which was O(1) large. Now, suppose that we have a static state feedback control that renders the origin of the closed-loop system globally asymptotically stable. Given any bounded set $B \subset \mathbf{R}^{2p}$, is it possible to find an observer-based control that renders the origin of the closed-loop system asymptotically stable, with B inside the region of attraction? We believe that the singular perturbation result of section 6.2 can give conditions under which semiglobality can be obtained, at least in the case when there is no zero dynamics.

Input/output Linearizable Systems

Most of the ideas of this work can be generalized to the class of input-output linearizable systems. At the moment, the main obstacle to such results seems to be the lack of appropriate normal forms for input-output linearizable systems.

APPENDICES

Appendix A

Proof of Proposition 3.1

Given ε, without loss of generality, assume that

$$N_\varepsilon(\Omega_\sigma) \subset \Omega_r$$

Claim : There exists $\delta>0$ such that $\Omega_{\sigma+\delta} \subset N_\varepsilon(\Omega_\sigma)$.

Proof of Claim: Suppose not. Then there exists $\varepsilon>0$ such that for all $n \in N$, there exist $z_n \in \Omega_{\sigma+1/n}$ such that

$$z_n \notin N_\varepsilon(\Omega_\sigma) \Rightarrow \quad d(z_n, \Omega_\sigma) \geq \varepsilon \text{ , for all } n \tag{A.1}$$

Moreover $z_n \in \Omega_{\sigma+1/n} \subset \Omega_{\sigma+1}$. $\Omega_{\sigma+1}$ is compact $\Rightarrow$ there exists a subsequence of (z_n), say (z_{n_k}), such that $z_{n_k} \rightarrow z$, for some z.

Now since $z_n \in \Omega_{\sigma+1/n} - \Omega_\sigma$, it follows that $\sigma < W(z_n) \leq \sigma + \frac{1}{n}$. Hence

$W(z) = \lim\limits_{k \rightarrow \infty} W(z_{n_k}) = \sigma \Rightarrow d(z, \Omega_\sigma) = 0.$

But by (A.1) $\varepsilon \leq \lim\limits_{k \rightarrow \infty} d(z_{n_k}, \Omega_\sigma) = d(z, \Omega_\sigma) = 0$; contradiction.

By Assumption 2

$\dot{W} \leq -\gamma_2 \| z \|^2 + \gamma_1 \| z \| + \gamma_0 := g(z)$

Considering $g(z)$ as a quadratic term in $\| z \|$, it is easy to see that

$$g(z) < 0 \text{ if } \quad \| z \| > \frac{\gamma_1}{2\gamma_2} + \left[\frac{\gamma_1^2}{4\gamma_2^2} + \frac{\gamma_0}{\gamma_2} \right]^{\frac{1}{2}}$$

$$\Rightarrow \; g(z) < 0 \text{ for all } z \in \left\{ z \mid W(z) > \sigma \right\} = \Omega_\sigma^c \tag{A.2}$$

Let $E := \Omega_\zeta - \Omega_{\sigma+\delta}$.

If $E=\emptyset$, then $z(t) \in \Omega_{\sigma+\delta}$ for all $t \geq t_0$, since $\Omega_{\sigma+\delta}$ is an invariant set of the trajectory.

So suppose $E \neq \emptyset$. Let $F := \left\{ g(z) \mid z \in \bar{E} \right\}$. Since $\bar{E} \subset \Omega_\sigma^c$, $\sup F \leq 0$.

<u>Claim</u> : $\sup F < 0$.

Proof of Claim: Suppose not. Then $\sup F = 0$. $\bar{E}$ is compact and $g(.)$ is continuous $\Rightarrow$ F is compact $\Rightarrow$ $0 \in F$ $\Rightarrow$ there exists $z \in \bar{E} \subset \Omega_\sigma^c$ such that $g(z) = 0$ which contradicts (A.2).

So $\dot{W} \leq -\alpha$ for some $\alpha > 0$. Let $T + t_0$ be the first time the trajectory enters $\Omega_{\sigma+\delta}$. Then

$$\int_\zeta^{\sigma+\delta} dW \leq - \int_{t_0}^{T+t_0} \alpha dt \;\; \Rightarrow \; T \leq \frac{\zeta - \sigma - \delta}{\alpha}$$

Moreover since $\Omega_{\sigma+\delta}$ is an invariant set of the trajectory

$$z(t) \in \Omega_{\sigma+\delta} \subset N_\varepsilon(\Omega_\sigma) \subset N_\mu(\Omega_\sigma) \qquad \text{for all } t \geq T + t_0 \quad \square$$

Appendix B

More on Special Coordinate Basis of Section 4.2

In this appendix, we give a simple explanation of how to arrive at (4.5) for a special class of systems. Suppose that the transfer function matrix $P(s) := C\ (sI - A)^{-1} B$ is square, invertible, minimum-phase, and has a left diagonal interactor $D(s) := diag\ (s^{\alpha_1}, \ldots, s^{\alpha_m})$, i.e.,

$$\lim_{s \to \infty} D(s) P(s) = L \tag{B.1}$$

where α_i's are nonnegative integers and L is nonsingular. Write $P^{-1}(s)$ as $P^{-1}(s) = Q(s) + R(s)$, where $Q(s)$ is a polynomial matrix and $R(s)$ is a strictly proper transfer function matrix. $P(s)$ can be written as

$$P = Q^{-1} (I + RQ^{-1})^{-1}$$

which implies that $P(s)$ can be represented by the negative feedback connection of $Q^{-1}(s)$ and $R(s)$ with $Q^{-1}(s)$ in the feedforward path and $R(s)$ in the feedback path.

By (B.1), $\lim_{s \to \infty} Q(s) D^{-1}(s) = L^{-1}$. Hence, $Q(s)$ is column-reduced, α_i's are the column degrees of $Q(s)$ and $Q(s)$ can be written in the following form

$$Q(s) = L^{-1} D(s) + Q_{lc} K(s)$$

where

$$K(s) = Block\ Diag\ [\ (\ 1, s, \ldots, s^{\alpha_i - 1}\ )'\ ]$$

Therefore, a controllable-form realization of $Q^{-1}(s)$ can be obtained by the coprime fraction method [Chen (1984)], in the following form

$$\begin{cases} \dot{x}_f = (A_{co} - B_{co} L Q_{lc}) x_f + B_{co} L u \\ y = C_{co} x_f \end{cases}$$

where

$$A_{co} := Block\ Diag\ [\ \begin{bmatrix} 0 & I_{\alpha_i - 1} \\ 0 & 0 \end{bmatrix}\ ]$$

$$B_{co} := Block\ Diag\ [\ (0, \ldots, 0, 1)'\ ]$$

$$C_{co} := Block\ Diag\ [\ (1, 0, \ldots, 0)\ ]$$

Let $(\ C_s, A_s, B_s\ )$ be a minimal realization of $R(s)$ (Since $P(s)$ is minimum-phase, A_s is Hurwitz). Then $P(s)$ has the following realization,

$$\begin{cases} \dot{x}_s = A_s x_s + B_s y \\ \dot{x}_f = (A_{co} - B_{co} L Q_{lc}) x_f + B_{co} L u - B_{co} L C_s x_s \end{cases}$$

$$y = C_{co} x_f$$

Now it is easy to see that if the components of x_f are interchanged such that integrator chains of the same length appear in the same block, this realization takes the form of (4.5), as a special case where $\tilde{x}_b$ does not exist and $M_f = 0$. □

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