FREE PERIODIC VIBRATIONS OF CONTINUOUS SYSTEMS GOVERNED BY COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

TCHUOC WEI LEE

1969

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ABSTRACT

FREE PERIODIC VIBRATIONS OF CONTINUOUS SYSTEMS GOVERNED BY COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

Tchuoc Wei Lee

A perturbation method for obtaining approximate solutions of coupled nonlinear partial differential equations is developed. The nonlinear partial differential equations are first converted into a sequence of linear partial differential equations, in which the zeroth order equation corresponds to a homogeneous linear problem and can be solved by the method of separation of variables. The higher order equations correspond to inhomogeneous linear problems and are solved by suitable eigenfunction expansions.

The method is applied to study the free periodic vibrations of continuous systems such as beams, circular membranes and circular plates with immovable boundary supports. One essential feature of all these governing equations of motion is that they incorporate effects of the so-called second invariant of the middle surface strains as well as that of the in-plane inertia. These effects are usually neglected in more elementary nonlinear theories (such as under the Berger's hypothesis) so that uncoupled

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equations of motion will result. More accurate explicit solutions for the frequency-amplitude relations, the in-plane as well as the out-of-plane displacements are obtained.

Numerical results are obtained using a CDC 6500 digital computer. Comparisons and discussions of these results with those previously obtained using more elementary nonlinear continuum theories are presented.

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By

Tchuoc Wei Lee

A THESIS

Submitted to
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for the degree of

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1969

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I. INTRODUCTION

1.1. Historical Background

Problems of finite deflection of continuous systems lead to nonlinear partial differential equations. For example, the governing equations of motion of an axisymmetric circular plate executing large amplitude vibrations are a pair of coupled nonlinear partial differential equations. The interaction between the middle surface forces in the plane of the plate and the out-of-plane deflection is a main source of the nonlinearity in the system.

This type of nonlinear problem seems to be first recognized in 1910. Von Kármán [1] extended the small deflection plate equation, introduced by Lagrange in 1810, to include the straining of the middle surface of the plate. He obtained the well-known pair of nonlinear partial differential equations now bearing his name for plates under static loadings. The difficulties presented in obtaining solutions to this pair of coupled nonlinear equations have led to approximations proposed by many researchers.

In 1955, Berger [2] suggested the neglect of the strain energy due to the second invariant of the middle surface strains. He solved the static uncoupled nonlinear

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differential equations for the problem of circular plates under various boundary conditions.

Since then, the Berger's hypothesis has been used by many researchers to solve nonlinear static problems as well as dynamic problems. Nash and Modeer [3] studied the nonlinear static problem of shallow shells. They also extended the Berger equations to non-linear vibrations of plates. Sinha [4] investigated the static problem of uniformly loaded plates rested on elastic foundation. Wah [5] solved the pair of uncoupled nonlinear equations of motion for plates by a modified Galerkin method using a one term approximation. Gajendar [6] followed the same method of Wah and solved the problem of large vibrations of plates on elastic foundations. Recently, Yen and Blotter [7] studied free periodic vibrations of continuous systems governed by nonlinear partial differential equations. Both the first and second order approximations to the solutions of the uncoupled nonlinear equations were obtained by a perturbation method.*

The Berger's hypothesis in the dynamic case consists in neglecting the strain energy due to the second invariant of the middle surface strains as well as the in-plane inertia of the system. Then the pair of nonlinear partial differential equations is uncoupled. Although it is fairly easy to solve a single equation by the perturbation method or by the Galerkin method, it should be pointed out that a satisfactory justification of the Berger's hypothesis has not yet been available in the literature.

*Many additional references on related works on beams, membranes and plates may be found at the end of [7].

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1.2. Purpose of Investigation

To include the strain energy of the second strain invariant as well as the in-plane inertia effects in the formulation of dynamic problems, it is necessary to solve a pair of coupled nonlinear partial differential equations. The purpose of this thesis is to develop a perturbation method for solving this pair of coupled nonlinear partial differential equations and to apply it to the study of free, periodic vibrations of continuous systems, such as beam, circular membrane, and circular plate, which are governed by equations of the type mentioned above.

1.3. Organization of Report

In Chapter II, a method of solution of a pair of coupled nonlinear partial differential equations is developed in general terms. This may be regarded as an extension of the work carried out recently by Yen and Blotter [7] from single nonlinear partial differential equations to coupled equations. A perturbation expansion is used to convert the nonlinear partial differential equations into a sequence of linear partial differential equations. The zeroth order solution is that of the corresponding linear problem and is well-known, or, say, can be obtained by the method of separation of variables. The higher order results are solutions of the inhomogeneous linear problems which are obtained by suitable eigenfunction expansions.

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The governing partial differential equations of motion for the continuous systems studied here such as beam, circular membrane and circular plate are derived in Chapter III using energy approach without the Berger's assumption. Both first and second order approximations to frequency-amplitude relations, longitudinal and transverse displacement are then found by using the general expressions developed in Chapter II.

Numerical results are obtained using a CDC 6500 digital computer. It is found that usually the first few terms in the eigenfunction expansions are sufficient to give satisfactory results. These results are compared with those obtained using Berger's approximation and are presented in graphical form. Assessments and discussions of the results are given in Chapter IV.

Chapter V contains the conclusions.

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II. METHOD OF ANALYSIS

2.1. Equations of Motion

Consider the free, undamped, large amplitude, periodic motion of certain continuous systems governed by the following pair of nondimensional coupled equations:

(i) an equation governing the out-of-plane or transverse motion of the system

$$L_1\eta + \omega^2 \frac{\partial^2 \eta}{\partial \tau^2} + \varepsilon\, \alpha\, \mathbf{M}(\eta, \gamma, \omega^2, \varepsilon) = 0 \qquad (2.1.1a)$$

(ii) an equation governing the in-plane or longitudinal motion of the system

$$L_2\gamma + \omega^2 \frac{\partial^2 \gamma}{\partial \tau^2} + \varepsilon\, \beta\, N(\eta, \gamma, \omega^2, \varepsilon) = 0 \qquad (2.1.1b)$$

where $\eta = \eta(\zeta, \tau)$ and $\gamma = \gamma(\zeta, \tau)$ are the two dependent functions of the independent variables ζ and τ. ζ is the spatial variable which is assumed to be defined over the domain from 0 to 1 and τ is the time variable. ω^2 is a frequency parameter and ε is a small parameter which is introduced into the problem either naturally or artificially. The coefficients α and β are dependent upon ζ. L_1 and L_2 are two linear differential operators with respect to ζ of order $2m$ and $2n$ respectively. M and

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N are two nonlinear differential operators of order not exceeding $2m$ and $2n$ respectively, and defined as

$$M(\eta,\gamma,\omega^2,\varepsilon) = f(\eta,\eta',\eta'',\ldots,\overbrace{\eta''''\cdots}^{2m},\gamma,\gamma',\gamma'',\ldots,\overbrace{\gamma''''\cdots}^{2n},$$
$$\omega^2,\ddot{\eta},\ddot{\gamma},\varepsilon) \qquad (2.1.2)$$

$$N(\eta,\gamma,\omega^2,\varepsilon) = g(\eta,\eta'\eta'',\ldots,\overbrace{\eta''''\cdots}^{2m},\gamma,\gamma'',\ldots,\overbrace{\gamma''''\cdots}^{2n},$$
$$\omega^2,\ddot{\eta},\ddot{\gamma},\varepsilon) \qquad (2.1.3)$$

where f and g are polynomials of finite degree in $\eta,\eta',\eta'',\ldots,\gamma,\gamma',\gamma'',\ldots,\omega^2,\ddot{\eta},\ddot{\gamma},\varepsilon$. The primes here denote partial differentiation with respect to ζ, and the dots stand for partial derivatives with respect to τ.

It will be assumed that the linear operators L_1 and L_2 are self-adjoint for every τ in the space of functions defined by the homogeneous boundary conditions:

$$\left.\begin{array}{lll} D_i\eta(0,\tau) = 0 & & i = 1,2,\ldots,k \\ D_j\eta(1,\tau) = 0 & & j = k+1,\ldots,2m \end{array}\right. \qquad (2.1.4a)$$

$$\left.\begin{array}{lll} \bar{D}_i\gamma(0,\tau) = 0 & & i = 1,2,\ldots,k' \\ \bar{D}_j\gamma(1,\tau) = 0 & & j = k'+1,\ldots, 2n \end{array}\right. \qquad (2.1.4b)$$

where the D's and $\bar{D}$'s are linear differential operators, of order less than $2m$ and $2n$ respectively, with respect to the spatial variable ζ.

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$$\eta(\zeta,\tau) = \eta(\zeta,\tau+2\pi) \tag{2.1.5a}$$

$$\gamma(\zeta,\tau) = \gamma(\zeta,\tau+2\pi) \tag{2.1.5b}$$

$$\dot{\eta}(\zeta,0) = 0 \tag{2.1.6a}$$

$$\dot{\gamma}(\zeta,0) = 0 \tag{2.1.6b}$$

This means that the system has zero initial velocities and periodic motions are initiated by releasing the system from rest in an as yet unspecified initial configuration.

Setting $\varepsilon = 0$ in (2.1.1), the pair of coupled, nonlinear equations become linearized as

$$L_1\eta + \omega^2 \frac{\partial^2 \eta}{\partial \tau^2} = 0 \tag{2.1.7a}$$

$$L_2\gamma + \omega^2 \frac{\partial^2 \gamma}{\partial \tau^2} = 0 \tag{2.1.7b}$$

The corresponding linear solutions are easily found.

A method for solving the pair of coupled, nonlinear equations (2.1.1) in the vicinity of a set of linear solutions subject to the boundary conditions (2.1.4), periodicity conditions (2.1.5), and the initial conditions (2.1.6) will be presented in the next section.

2.2. Method of Solution

To solve the pair of coupled nonlinear partial differential equations (2.1.1) subject to the given boundary, periodicity and initial conditions, the functions η and γ as well as ω^2 are expanded into power series in ε as

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$$\eta(\zeta,\tau) = \sum_{i=0}^{\infty} \varepsilon^i \eta_i \tag{2.2.1}$$

$$\gamma(\zeta,\tau) = \sum_{i=0}^{\infty} \varepsilon^i \gamma_i \tag{2.2.2}$$

$$\omega^2 = \sum_{i=0}^{\infty} \varepsilon^i \omega_i^2 \tag{2.2.3}$$

These expansions are substituted into the pair of equations (2.1.1a) and (2.1.1b). Upon expanding f and g about $\eta = \eta_0$, $\gamma = \gamma_0$, $\omega^2 = \omega_0^2$ and $\varepsilon = 0$ as Taylor's series and collecting like powers of ε, the following set of equations results:

ε^0: $$L_1\eta_0 + \omega_0^2\ddot{\eta}_0 = 0 \tag{2.2.4a}$$

$$L_2\gamma_0 + \omega_0^2\ddot{\gamma}_0 = 0 \tag{2.2.4b}$$

ε^1: $$L_1\eta_1 + \omega_0^2\ddot{\eta}_1 = -\omega_1^2\ddot{\eta}_0 - \alpha\bar{f} \tag{2.2.4c}$$

$$L_2\gamma_1 + \omega_0^2\ddot{\gamma}_1 = -\omega_1^2\ddot{\gamma}_0 - \beta\bar{g} \tag{2.2.4d}$$

ε^2: $$\begin{aligned} L_1\eta_2 + \omega_0^2\ddot{\eta}_2 = {} & -\omega_2^2\ddot{\eta}_0 - \omega_1^2\ddot{\eta}_1 - \alpha[\eta_1\bar{f}_\eta \\ & + \eta_1'\bar{f}_{\eta'} + \eta_1''\bar{f}_{\eta''} + \ldots + \gamma_1\bar{f}_\gamma + \gamma_1'\bar{f}_{\gamma'} \\ & + \gamma_1''\bar{f}_{\gamma''} + \ldots + \omega_1^2\bar{f}_{\omega^2} + \ddot{\eta}_1\bar{f}_{\ddot{\eta}} + \ddot{\gamma}_1\bar{f}_{\ddot{\gamma}} \\ & + \bar{f}_\varepsilon] \end{aligned} \tag{2.2.4e}$$

$$L_2\gamma_2 + \omega_0^2\ddot{\gamma}_2 = -\omega_2^2\ddot{\gamma}_0 - \omega_1^2\ddot{\gamma}_1 - \beta[\eta_1\bar{g}_\eta$$

$$+ \eta_1'\bar{g}_{\eta'} + \eta_1''\bar{f}_{\eta''} + \ldots + \gamma_1\bar{g}_\gamma + \gamma_1'\bar{g}_{\gamma'}$$

$$+ \gamma_1''\bar{g}_{\gamma''} + \ldots + \omega_1^2\bar{g}_{\omega^2} + \ddot{\eta}_1\bar{g}_{\ddot{\eta}} + \ddot{\gamma}_1\bar{g}_{\ddot{\gamma}}$$

$$+ \bar{g}_\varepsilon] \qquad (2.2.4f)$$

ε^3: --

where the notations $\bar{f}$, $\bar{f}_\eta$, etc. mean that f, f_η, etc. are evaluated at $(\eta_0, \eta_0', \eta_0'', \ldots, \gamma_0, \gamma_0', \gamma_0'', \ldots, \omega_0^2, \ddot{\eta}_0, \ddot{\gamma}_0, 0)$.

The set of equations can be solved recursively for the unknowns η_i, γ_i, and ω_i^2, $i = 0,1,2,\ldots$, assuming that η_i and γ_i individually satisfy the boundary conditions as given in equations (2.1.4), the periodicity condition (2.1.5) and the initial condition (2.1.6). As usual, for $i = 0$, the first two equations of the set are homogeneous and are just the equations of motion in the linear theory as given by (2.1.7). For $i \geq 1$, equations (2.2.4c), (2.2.4d),.... have same homogeneous parts as the first two equations but with inhomogeneous terms consisting of the lower order coefficients in the expansions (2.2.1) through (2.2.3). An essential feature of the perturbation method to be developed here consists in an eigenfunction expansion procedure for solving these inhomogeneous equations for η_i, γ_i, and ω_i^2, $i \geq 1$.

Let us first consider the homogeneous equations (2.2.4a) and (2.2.4b). The equation for η_0 can be solved easily by the method of separation of variables. It has periodic solutions of the form

$$\eta_0(\zeta,\tau) = A_{1k}V_k(\zeta)\cos\tau \tag{2.2.5}$$

$$\omega_0^2 = \Omega_k^2 \tag{2.2.6}$$

$$k = 1,2,3,\ldots\ldots\ldots$$

where the constant A_{1k} is a nondimensional amplitude parameter. $V_k(\zeta)$ and Ω_k^2 are the k-th eigenfunction and eigenvalue of the following problem:

$$L_1V_k - \Omega_k^2V_k = 0 \tag{2.2.7}$$

$$D_iV_k(0) = 0, \quad i = 1,2,\ldots\ldots,h \tag{2.2.8}$$

$$D_jV_k(1) = 0, \quad j = h+1,\ldots\ldots,2m \tag{2.2.9}$$

The orthogonality condition for the set of eigenfunctions $V_k(\zeta)$ is

$$\int_0^1 r(\zeta)V_k(\zeta)V_p(\zeta)d\zeta = \delta_{kp} \tag{2.2.10}$$

where $r(\zeta)$ is some weighting function and δ_{kp} the Kronecker delta.

The equation for γ_0 may have periodic solutions of the form

$$\gamma_0(\zeta,\tau) = \Gamma_\ell(\zeta)\cos\tau \tag{2.2.11}$$

$$\omega_0^2 = \Lambda_\ell^2 \tag{2.2.12}$$

$$\ell = 1,2,3,\ldots\ldots$$

where Γ_ℓ is the ℓ-th eigenfunction of the following problem:

$$L_2\Gamma_\ell - \Lambda_\ell^2\Gamma_\ell = 0 \tag{2.2.13}$$

$$\bar{D}_i\Gamma_\ell(0) = 0, \qquad i = 1,2,\ldots\ldots,h' \tag{2.2.14}$$

$$\bar{D}_j\Gamma_\ell(1) = 0, \qquad j = h'+1,\ldots\ldots,2n \tag{2.2.15}$$

and Λ_ℓ^2 is the corresponding eigenvalue. The orthogonality condition for the set of eigenfunctions $\Gamma_\ell(\zeta)$ is

$$\int_0^1 s(\zeta)\Gamma_\ell(\zeta)\Gamma_q(\zeta)d\zeta = \delta_{\ell q} \tag{2.2.16}$$

where $s(\zeta)$ is some weighting function and $\delta_{\ell q}$ the Kronecker delta.

Now (2.2.6) determines the values of ω_0^2. With one of these values for ω_0^2, the equation (2.2.4b) for γ_0 will in general have no nontrivial solution satisfying its periodicity and boundary conditions unless $\Lambda_\ell^2 = \Omega_k^2$. It is assumed here that

$$\Lambda_\ell^2 \neq \Omega_k^2 \tag{2.2.17}$$

$$\ell = 1,2,3,\ldots\ldots$$

Therefore with the periodic solution (2.2.5) for η_0, one has

$$\gamma_0(\zeta,\tau) \doteq 0 \tag{2.2.18}$$

This implies that the oscillations take place primarily in the transverse direction of the system.

The inhomogeneous equations (2.2.4c), (2.2.4d),....., yield the higher order terms that are demanded by nonlinear interac

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interactions with the zeroth order solution. For the equations for η_i $(i \geq 1)$, solutions are sought in the form

$$\eta_i(\zeta,\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(i)} V_n(\zeta) \cos m\tau \qquad (2.2.19)$$

$$i = 1,2,3,\ldots.$$

where the coefficients of expansion $A_{mn}^{(i)}$ are as yet to be determined. It will be assumed that

$$A_{1k}^{(i)} = 0, \qquad i \geq 1 \qquad (2.2.20)$$

i.e. there is no $V_k(\zeta)\cos\tau$ in all η_i for $i \geq 1$.

Similarly, for equations for γ_i $(i \geq 1)$, solutions are sought in the form

$$\gamma_i(\zeta,\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{(i)} \Gamma_n(\zeta) \cos m\tau \qquad (2.2.21)$$

$$i = 1,2,3,\ldots.$$

where the coefficients of expansion $C_{mn}^{(i)}$ are likewise yet to be determined.

Now, to obtain the i-th order solution for η_i and ω_i^2, the procedure consists of substiuting the η_i series given in (2.2.19) into the ε^i equation given in (2.2.4). At this stage expressions for η_j in terms of $A_{mn}^{(j)}$, γ_j in terms of $C_{mn}^{(j)}$ and ω_j^2 are already known for $j \leq i-1$. The orthogonality of both the spatial eigenfunctions and trigonometric time functions is used to determine $A_{mn}^{(i)}$ and ω_i^2. The same procedure is then applied to determine γ_i in terms of $C_{mn}^{(i)}$.

In what follows, let us illustrate how to determine the solution for η_i, γ_i and ω_i^2 up to the second order by equations (2.2.4). The process of solving subsequent higher order equations can be carried as far as is desired, but the results will not be presented.

Having determined η_0, ω_0^2, and γ_0 as given in equations (2.2.5), (2.2.6) and (2.2.18), one can calculate $\bar{f}$ and $\bar{g}$ in equation (2.2.4c) and (2.2.4d). Substituting $\eta_1(\zeta,\tau)$ as given by (2.2.19) into (2.2.4c) yields

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\Omega_n^2 - m^2\Omega_k^2)A_{mn}^{(1)}V_n(\zeta)\cos m\tau =$$

$$\omega_1^2 A_{1k}V_k(\zeta)\cos\tau - \alpha\bar{f} \qquad (2.2.22)$$

To find ω_1^2 and $A_{mn}^{(1)}$, multiply each side of the above equation by $r(\zeta)V_q(\zeta)\cos p\tau$ and integrate between limits 0 to 1 with respect to ζ and between limits 0 and 2π with respect to τ. Using the orthogonality conditions one obtains the following equation:

$$(\Omega_q^2 - p^2\Omega_k^2)A_{pq}^{(1)} = \omega_1^2 A_{1k}\delta_{kq}\delta_{1p} - \frac{1}{\pi}\int_0^1\int_0^{2\pi}\alpha\bar{f}\, r(\zeta)V_q(\zeta)\cos p\tau\, d\tau d\zeta \qquad (2.2.23)$$

Setting $p = 1$, $q = k$, one determines the first order frequency-amplitude relation as follows:

$$\omega_1^2 = \frac{1}{\pi A_{1k}}\int_0^1\int_0^{2\pi}\alpha\bar{f}\, r(\zeta)V_k(\zeta)\cos\tau\, d\tau\, d\zeta \qquad (2.2.24)$$

For $p = m$ (m being some fixed integer other than 1), and $q = n$ (n being some fixed integer other than k), equation

(2.2.23) yields:

$$A_{mn}^{(1)} = \frac{1}{\pi(\Omega_n^2 - m^2\Omega_k^2)} \int_0^1 \int_0^{2\pi} \alpha\bar{f}\ r(\zeta)V_n(\zeta)\cos m\tau\ d\tau d\zeta \quad (2.2.25)$$

Thus the amplitude parameters $A_{mn}^{(1)}$ are determined provided that $\Omega_n^2 - m^2\Omega_k^2 \neq 0$.

Let us now turn to equation (2.2.4d). Upon substituting $\gamma_1(\zeta,\tau)$ as given by (2.2.21) into equation (2.2.4d), and recalling that γ_0 has only trivial solution, it follows that

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\Lambda_n^2 - m^2\Omega_k^2)C_{mn}^{(1)}\Gamma_n(\zeta)\cos m\tau = -\beta\bar{g} \quad (2.2.26)$$

Multiply both sides of the above equation by $s(\zeta)\Gamma_q(\zeta)\cos p\tau$ and integrate between limits 0 and 1 for ζ and between limits 0 and 2π for τ. Making use of the orthogonality conditions, one determines

$$C_{mn}^{(1)} = -\frac{1}{\pi(\Lambda_n^2 - m^2\Omega_k^2)} \int_0^1 \int_0^{2\pi} \beta\bar{g}\ s(\zeta)\Gamma_n(\zeta)\cos m\tau\ d\tau d\zeta \quad (2.2.27)$$

In the above expression, $C_{mn}^{(1)}$ is determined provided that $\Lambda_n^2 - m^2\Omega_k^2 \neq 0$. Thus, the first order nonlinear correction for the k-th linear mode is completely determined as

$$\eta_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(1)}V_n(\zeta)\cos m\tau \quad (2.2.28)$$

$$m \neq 1,\ n \neq k$$

$$\gamma_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{(1)}\Gamma_n(\zeta)\cos m\tau \quad (2.2.29)$$

The next or second order nonlinear correction for the k-th linear mode shape and the corresponding frequency are determined in a similar manner. The inhomogeneous terms on the right hand side of equations (2.2.4e) and (2.2.4f) are found with the known η_0, ω_0^2, γ_0, η_1, ω_1^2, and γ_1. Substituting $\eta_2(\zeta,\tau)$ as given by (2.2.19) for $i = 2$ into (2.2.4e) one obtains

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\Omega_n^2 - m^2\Omega_k^2)A_{mn}^{(2)}V_n(\zeta)\cos m\tau = \omega_2^2 A_{1k}V_k(\zeta)\cos\tau$$

$$+ \omega_1^2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}p^2A_{pq}^{(1)}V_q(\zeta)\cos p\tau - \alpha[\bar{f}_\eta\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}A_{pq}^{(1)}V_q(\zeta)\cos p\tau$$

$$+ \bar{f}_{\eta'}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}A_{pq}^{(1)}\frac{dV_q(\zeta)}{d\zeta}\cos p\tau$$

$$+ \bar{f}_{\eta''}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}A_{pq}^{(1)}\frac{d^2V_q(\zeta)}{d\zeta^2}\cos p\tau + \ldots .$$

$$+ \bar{f}_\gamma\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}C_{pq}^{(1)}\Gamma_q(\zeta)\cos p\tau$$

$$+ \bar{f}_{\gamma'}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}C_{pq}^{(1)}\frac{d\Gamma_q(\zeta)}{d\zeta}\cos p\tau + \ldots .$$

$$+ \omega_1^2\bar{f}_{\omega^2} - \bar{f}_{\ddot{\eta}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}p^2A_{pq}^{(1)}V_q(\zeta)\cos p\tau$$

$$- \bar{f}_{\ddot{\gamma}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}p^2C_{pq}^{(1)}\Gamma_q(\zeta)\cos p\tau + \bar{f}_\varepsilon] \qquad (2.2.30)$$

Multiplying (2.2.30) by $r(\zeta)V_\ell(\zeta)\cos j\tau$, integrating with respect to ζ from 0 to 1 and τ from 0 to 2π, and using the orthogonality properties one obtains

$$
\begin{aligned}
(\Omega_\ell^2 - j^2\Omega_k^2)A_{j\ell}^{(2)} = {} & \omega_2^2 A_{1k}\delta_{k\ell}\delta_{1j} + \omega_1^2 j^2 A_{j\ell}^{(1)} \\
& - \frac{1}{\pi}\int_0^1\int_0^{2\pi}[\bar{f}_\eta \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(\zeta)\cos p\tau \\
& + \bar{f}_{\eta''}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} A_{pq}^{(1)}\frac{d^2V_q(\zeta)}{d\zeta^2}\cos p\tau + \ldots\ldots\ldots \\
& + \bar{f}_\gamma \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} C_{pq}^{(1)}\Gamma_q(\zeta)\cos p\tau \\
& + \bar{f}_{\gamma'}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} C_{pq}^{(1)}\frac{d\Gamma_q(\zeta)}{d\zeta}\cos p\tau + \ldots\ldots \\
& + \omega_1^2\bar{f}_{\omega^2} - \bar{f}_{\ddot{\eta}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(\zeta)\cos p\tau \\
& - \bar{f}_{\ddot{\gamma}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} p^2 C_{pq}^{(1)}\Gamma_q(\zeta)\cos p\tau \\
& + \bar{f}_{\ddot{\zeta}}]\ \alpha\ r(\zeta)V_\ell(\zeta)\cos j\tau\, d\tau\, d\zeta \qquad (2.2.31)
\end{aligned}
$$

Setting $j = 1$, $\ell = k$, and using the conditions of $A_{1k}^{(i)} = 0$ for $i = 1$, the second order frequency-amplitude relation is

$$
\begin{aligned}
\omega_2^2 = {} & \frac{1}{\pi A_{1k}}\int_0^1\int_0^{2\pi}[\bar{f}_\eta \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(\zeta)\cos p\tau \\
& + \bar{f}_{\eta'}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} A_{pq}^{(1)}\frac{dV_q(\zeta)}{d\zeta}\cos p\tau \\
& + \bar{f}_{\eta''}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} A_{pq}^{(1)}\frac{d^2V_q(\zeta)}{d\zeta^2}\cos p\tau + \ldots\ldots\ldots
\end{aligned}
$$

$$+ \bar{f}_{\gamma} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$+ \bar{f}_{\gamma'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{pq}^{(1)} \frac{d\Gamma_q(\zeta)}{d\zeta} \cos p\tau + \cdots\cdots$$

$$+ \omega_1^2 \bar{f}_{\omega^2} - \bar{f}_{\ddot{\eta}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(\zeta) \cos p\tau$$

$$- \bar{f}_{\ddot{\gamma}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 c_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$+ \bar{f}_{\zeta}] \alpha\, r(\zeta) V_k(\zeta) \cos \tau \, d\tau \, d\zeta \qquad (2.2.32)$$

For $j = m$, and $\ell = n$, equation (2.2.31) yields

$$A_{mn}^{(2)} = \frac{1}{(\Omega_n^2 - m^2 \Omega_k^2)} [\omega_1^2 m^2 A_{mn}^{(1)}]$$

$$- \frac{1}{\pi(\Omega_n^2 - m^2\Omega_k^2)} \int_0^1 \int_0^{2\pi} [\bar{f}_{\eta} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(\zeta) \cos p\tau$$

$$+ \bar{f}_{\eta'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{dV_q(\zeta)}{d\zeta} \cos p\tau$$

$$+ \bar{f}_{\eta''} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^2 V_q(\zeta)}{d\zeta^2} \cos p\tau + \cdots\cdots\cdots$$

$$+ \bar{f}_{\gamma} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$+ \bar{f}_{\gamma'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{pq}^{(1)} \frac{d\Gamma_q(\zeta)}{d\zeta} \cos p\tau + \cdots\cdots\cdots$$

$$+ \omega_1^2 \bar{f}_{\omega^2} - \bar{f}_{\ddot{\eta}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(\zeta) \cos p\tau$$

$$- \bar{f}_{\ddot{\gamma}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 C_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$+ \bar{f}_{\varepsilon}] \alpha\, r(\zeta) V_q(\zeta) \cos m\tau \, d\tau d\zeta, \quad \begin{matrix} m \neq 1 \\ n \neq k \end{matrix} \qquad (2.2.33)$$

Similarly, substituting $\gamma_2(\zeta,\tau)$ as given by (2.2.21) into equation (2.2.4f) and using the results of (2.2.18), it follows that

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\Lambda_n^2 - m^2 \Omega_k^2) C_{mn}^{(2)} \Gamma_n(\zeta) \cos m\tau$$

$$= \omega_1^2 \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 C_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$- \beta[\bar{g}_{\eta} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(\zeta) \cos p\tau$$

$$+ \bar{g}_{\eta'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{dV_q(\zeta)}{d\zeta} \cos p\tau$$

$$+ \bar{g}_{\eta''} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^2 V_q(\zeta)}{d\zeta^2} \cos p\tau + \cdots\cdots\cdots$$

$$+ \bar{g}_{\gamma} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} C_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau$$

$$+ \bar{g}_{\gamma'} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} C_{pq}^{(1)} \frac{d\Gamma_q(\zeta)}{d\zeta} \cos p\tau + \cdots\cdots\cdots$$

$$+ \omega_1^2 \bar{g}_{\omega^2} - \bar{g}_{\ddot{\eta}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(\zeta) \cos p\tau$$

$$- \bar{g}_{\ddot{\gamma}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 C_{pq}^{(1)} \Gamma_q(\zeta) \cos p\tau + \bar{g}_{\varepsilon}] \qquad (2.2.34)$$

Multiplying the above equation by $s(\zeta)\Gamma_{\ell}(\zeta)\cos j\tau$, integrating with respect to ζ and τ over the interval (0,1) and $(0,2\pi)$, and using the orthogonality properties one determines

$$C_{mn}^{(2)} = \frac{1}{(\Lambda_n^2-m^2\Omega_k^2)} [\omega_1^2 m^2 C_{mn}^{(1)}]$$

$$- \frac{1}{\pi(\Lambda_n^2-m^2\Omega_k^2)} \int_0^1 \int_0^{2\pi} [\bar{g}_\eta \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(\zeta)\cos p\tau$$

$$+ \bar{g}_{\eta'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{dV_q(\zeta)}{d\zeta} \cos p\tau$$

$$+ \bar{g}_{\eta''} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^2V_q(\zeta)}{d\zeta^2} \cos p\tau + \ldots\ldots.$$

$$+ \bar{g}_\gamma \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} C_{qp}^{(1)} \Gamma_q(\zeta)\cos p\tau$$

$$+ \bar{g}_{\gamma'} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} C_{qp}^{(1)} \frac{d\Gamma_q(\zeta)}{d\zeta} \cos p\tau + \ldots..$$

$$+ \omega_1^2 \bar{g}_{\omega^2} - \bar{g}_{\ddot{\eta}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(\zeta)\cos p\tau$$

$$- \bar{g}_{\ddot{\gamma}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^2 C_{pq}^{(1)} \Gamma_q(\zeta)\cos p\tau$$

$$+ \bar{g}_{\ddot{\varepsilon}}]\beta\; s(\zeta)\Gamma_n(\zeta)\cos m\tau\; d\tau d\zeta \qquad (2.2.35)$$

Thus the second order nonlinear correction for the k-th linear mode is given by

$$\eta_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(2)} V_n(\zeta)\cos m\tau, \quad m \neq 1,\ n \neq k \qquad (2.2.36)$$

$$\gamma_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{(2)} \Gamma_n(\zeta)\cos m\tau \qquad (2.2.37)$$

The complete solutions to the problem up to terms ε^2 can be written by adding the zeroth, first and second approximate solutions. The final results are explicit once a knowledge of the spatial eigenfunctions of the associated linear problem, along with the linear frequencies is available. Applications of the general results will be made to problems involving structural elements such as beams, membranes, and plates in the next chapter.

III. EXAMPLES

3.1. Introduction

The energy method will be used here to derive the equations governing the nonlinear behavior of continuous systems such as beams, circular membranes and circular plates without the Berger's hypothesis.

A continuous system possesses both the strain energy and the kinetic energy for deflections of the same order of magnitude as the thickness of a prismatic beam or a circular plate. The strain energy is composed of that of bending and that of stretching, namely,

$$V = \frac{EI}{2} \int_0^L \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{S}{I} e^2 \right] dx \tag{3.1.1}$$

for a beam and

$$V = \frac{D}{2} \int_0^{2\pi} \int_0^a \Big\{ \left[(\nabla^2 w)^2 + \frac{12}{h^2} e^2 \right]$$

$$- 2(1-\mu) \left(\frac{12}{h^2} e_2 + \frac{1}{r} \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2} \right) \Big\} r \, dr d\theta \tag{3.1.2}$$

for a plate in axisymmetric motions. The strain energy of a circular membrane in axisymmetric motions is due solely to the stretching of its middle surface

$$V = \int_0^{2\pi}\int_0^a \{N_0 e_0 + N_0 e + \frac{Eh}{2(1-\mu^2)}[e^2 - 2(1-\mu)e_2]\} r\, dr d\theta \quad (3.1.3)$$

In the above expressions, E is the elastic modulus of the material, h the thickness, μ the Poisson's ratio. The origin of the x, z coordinate is located at the left hand end of the beam. The x-axis coincides with the median line and L is the undeformed length of the beam. The z-axis is normal to the median line. S is the cross-sectional area and I the second moment of area of the beam. For the axisymmetric circular membrane and plate, the origin of the r, z coordinate is at the center. The r-axis coincides with the middle surface and a is the radius. The z-axis is normal to the middle surface. N_0 and e_0 are, respectively, the initial stress and strain of the membrance. $D = \frac{Eh^3}{12(1-\mu^2)}$ denotes the flexual rigidity of the plate. $(\nabla^2)^2$ is the biharmonic operator defined as follows

$$(\nabla^2)^2 = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) \quad (3.1.4)$$

The first strain invariant expressed in x, z and r, z coordinate system is, respectively,

$$e = \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2 \quad (3.1.5)$$

and

$$e = \varepsilon_r + \varepsilon_\theta \quad (3.1.6)$$

where the strains in the radial and tangential directions are taken to be

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \tag{3.1.7}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{3.1.8}$$

Here u and w denote the components of displacement of a point in the middle surface.

The second strain invariant for membrane and plate with circular symmetry in r, z coordinate system is

$$e_2 = \varepsilon_r \varepsilon_\theta = \left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]\frac{u}{r} \tag{3.1.9}$$

It should be noted that there is no second strain invariant in the case of beam because only the bending and stretching in the x,z plane are considered in the expression (3.1.1).

The expression for the kinetic energy is

$$T = \frac{\rho S}{2} \int_0^L \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx \tag{3.1.10}$$

for a beam, and

$$T = \frac{\rho h}{2} \int_0^{2\pi} \int_0^a \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] r \, dr d\theta \tag{3.1.11}$$

for a membrane or a plate. In the above expressions, ρ denotes the mass density per unit volume and t the time.

It is now possible to form the Hamilton's integral, Λ, for beam, membrane and plate, namely,

$$\Lambda = \int_{t_1}^{t_2} \int_0^L F(u_t, w_t, u, u_x, w, w_x, w_{xx}) dxdt \tag{3.1.12}$$

and

$$\Lambda = \int_{t_1}^{t_2} \int_0^{2\pi} \int_0^a F(u_t, w_t, u, u_r, w, w_r, w_{rr}) dr\, d\theta\, dt \qquad (3.1.13)$$

Subscripts in the above expressions denote partial derivatives and the integrand, F, is defined as

$$F(u_t, w_t, u, u_x, w, w_x, w_{xx})$$

$$= \frac{1}{2}[\rho S(u_t^2 + w_t^2) - EI(w_{xx}^2 + \frac{S}{I} e^2) \qquad (3.1.14)$$

for a beam,

$$F(u_t, w_t, u, u_r, w, w_r, w_{rr})$$

$$= r\{\frac{\rho h}{2}(u_t^2 + w_t^2) - [N_0 e_0 + N_0 e$$

$$+ \frac{Eh}{2(1-\mu^2)} (e^2 - 2(1-\mu)e_2)]\} \qquad (3.1.15)$$

for a membrane, and

$$F(u_t, w_t, u, u_r, w, w_r, w_{rr})$$

$$= r\{\frac{\rho h}{2} (u_t^2 + w_t^2)$$

$$- \frac{D}{2}[(\nabla^2 w)^2 + \frac{12}{h^2} e^2 - 2(1-\mu)(\frac{12}{h^2} e_2 + \frac{1}{r} w_r w_{rr})]\} \qquad (3.1.16)$$

for a plate.

According to Hamilton's principle

$$\delta \Lambda = 0 \qquad (3.1.17)$$

The corresponding Euler equations expressed respectively in x, z and r, z coordinate system are then

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial t}\frac{\partial F}{\partial w_t} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w_x} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xx}} = 0 \qquad (3.1.18)$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} = 0 \qquad (3.1.19)$$

and

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial t}\frac{\partial F}{\partial w_t} - \frac{\partial}{\partial r}\frac{\partial F}{\partial w_r} + \frac{\partial^2}{\partial r^2}\frac{\partial F}{\partial w_{rr}} = 0 \qquad (3.1.20)$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial u_t} - \frac{\partial}{\partial r}\frac{\partial F}{\partial u_r} = 0 \qquad (3.1.21)$$

By carrying out the differentiations, equation (3.1.18) and (3.1.19) yield a pair of partial differential equations of motion for beams vibrating at large amplitudes. Similarly, equations (3.1.20) and (3.1.21) yield two pairs of partial differential equations of axisymmetric motion for circular membranes and plates. It should be pointed out that all three pairs of partial differential equations include the longitudinal inertia effects and the last two pairs include also the so-called second invariant of the middle surface strains. One characteristic feature of all these sets of equations is that they are coupled and nonlinear. The method developed in Chapter II is then used to obtain approximate solutions of these equations. Problems of beams, membranes and plates will be considered in the next three sections.

3.2. Elastic Beams with Immovable Supports

Let us first consider the large amplitude, free, undamped, periodic vibrations of an elastic beam in the x, z-plane with an extensible median line. By (3.1.14), (3.1.18) and (3.1.19) one has

$$-\frac{\partial^4 w}{\partial x^4} - \frac{\rho S}{EI}\frac{\partial^2 w}{\partial t^2} + \frac{S}{I}\left\{\frac{\partial w}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) + \frac{\partial^2 w}{\partial x^2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\} = 0 \qquad (3.2.1a)$$

$$\frac{S}{I}\frac{\partial^2 u}{\partial x^2} - \frac{\rho S}{EI}\frac{\partial^2 u}{\partial t^2} + \frac{S}{I}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} = 0 \qquad (3.2.1b)$$

To nondimensionalize the equations, let us introduce the following quantities:

$$\bar{\eta} = \frac{w}{L} \qquad \bar{\gamma} = \frac{u}{L}, \qquad \zeta = \frac{x}{L}$$
$$\tau = \bar{\omega}t, \qquad \omega^2 = \frac{\rho\lambda L^2}{E}\bar{\omega}^{-2}, \qquad \lambda = \left(\frac{L}{r}\right)^2 \qquad (3.2.2)$$

where $r = \left(\frac{I}{S}\right)^{1/2}$ is the radius of gyration of the cross section and $\frac{L}{r}$ is the slenderness ratio of the beam. Equations (3.2.1a) and (3.2.1b) then take the following form

$$\bar{\eta}^{IV} - \omega^2\ddot{\bar{\eta}} + \lambda[\bar{\eta}'(\bar{\gamma}'' + \bar{\eta}'\bar{\eta}'') + \bar{\eta}''(\bar{\gamma}' + \tfrac{1}{2}\bar{\eta}'^2)] = 0 \qquad (3.2.3a)$$

$$\lambda\bar{\gamma}'' - \omega^2\ddot{\bar{\gamma}} + \lambda\bar{\eta}'\bar{\eta}'' = 0 \qquad (3.2.3b)$$

In order to obtain a perturbation solution of the above pair of equations, the small parameter ε is now introduced into the formulation of the problem through the following change of variables:

$$\bar{\eta} = \varepsilon\eta, \qquad \bar{\gamma} = \varepsilon\gamma \qquad (3.2.4)$$

The nondimensional equations of motion then become

$$\eta^{IV} + \omega^2\ddot{\eta} - \varepsilon\lambda(\eta'\gamma'' + \eta''\gamma' + \frac{3}{2}\varepsilon\eta'^2\eta'') = 0 \qquad (3.2.5a)$$

$$-\lambda\gamma'' + \omega^2\ddot{\gamma} - \varepsilon\lambda\eta'\eta'' = 0 \qquad (3.2.5b)$$

A comparison of the above equations with (2.1.1a) and (2.1.1b) shows that they are identical provided that

$$L_1 = \frac{\partial^4}{\partial\zeta^4}, \qquad \alpha = -\lambda \qquad (3.2.6)$$

$$f(\eta', \eta'', \gamma', \gamma'', \varepsilon) = \eta'\gamma'' + \eta''\gamma' + \frac{3}{2}\varepsilon\eta'^2\eta'' \qquad (3.2.7)$$

and

$$L_2 = -\lambda\frac{\partial^2}{\partial\zeta^2} \qquad \beta = -\lambda \qquad (3.2.8)$$

$$g(\eta', \eta'') = \eta'\eta'' \qquad (3.2.9)$$

The corresponding linear equations are found by setting $\varepsilon = 0$ in (3.2.5a) and (3.2.5b), namely,

$$\eta_0^{IV} + \omega_0^2\ddot{\eta}_0 = 0 \qquad (3.2.10a)$$

$$-\lambda\gamma_0'' + \omega_0^2\ddot{\gamma}_0 = 0 \qquad (3.2.10b)$$

The solution to equation (3.2.10a) is

$$\begin{aligned} \eta_0 &= A_{1k} V_k(\zeta)\cos\tau \\ \omega_0^2 &= \Omega_k^2 \qquad (3.2.11) \\ k &= 1,2,3,\ldots\ldots. \end{aligned}$$

and the equation (3.2.10b) has only the trivial solution, i.e.

$$\gamma_0 = 0 \qquad (3.2.12)$$

Let us consider perturbation in the neighborhood of the first linear mode. By taking $\eta_0 = A_{11}V_1(\zeta)\cos\tau$, $\omega_0^2 = \Omega_1^2$, $\gamma_0 = 0$ and $\varepsilon = 0$, one can calculate $\bar{f}$ and $\bar{g}$

as follows:

$$\bar{f} = f(\eta', \eta'', \gamma', \gamma'', \varepsilon)\Big|_{\eta_0', \eta_0'', \gamma_0', \gamma_0'', 0} = 0 \qquad (3.2.13)$$

$$\bar{g} = g(\eta', \eta'')\Big|_{\eta_0', \eta_0''} = A_{11}^2 V_1' V_1'' \cos^2 \tau \qquad (3.2.14)$$

In view of equation (3.2.13) the first order frequency-amplitude relation as given by equation (2.2.24) in Chapter II follows as

$$\omega_1^2 = 0 \qquad (3.2.15)$$

One also finds that all the amplitude parameters $A_{mn}^{(1)}$ as given by equation (2.2.25) vanish identically. Consequently,

$$\eta_1(\zeta, \tau) = 0 \qquad (3.2.16)$$

Upon substitution of $\bar{g}$ as given by equation (3.2.14) into equation (2.2.27) and integrating with respect to τ, the expansion coefficients $C_{mn}^{(1)}$ are found to be

$$C_{0n}^{(1)} = \frac{A_{11}^2}{2\Lambda_n^2} I \qquad (3.2.17)$$

$$C_{2n}^{(1)} = \frac{A_{11}^2}{2(\Lambda_n^2 - 4\Omega_1^2)} I \qquad (3.2.18)$$

where

$$I = \int_0^1 \lambda V_1' \, V_1'' \, s \, \Gamma_n \, d\zeta \qquad (3.2.19)$$

and n is an interger equal to or greater than unity. All other $C_{mn}^{(1)}$ are equal to zero.

The above results immediately lead to the first order solution for the in-plane displacement

$$\gamma_1(\zeta,\ \tau) = \sum_{n=1}^{\infty} (C_{0n}^{(1)} + C_{2n}^{(1)} \cos 2\tau)\ \Gamma_n(\zeta) \qquad (3.2.20)$$

To determine the second order approximation, derivatives of the nonlinear functions f and g are needed. These are:

$$\left.\begin{aligned} \bar{f}_{\eta'} &= \gamma_0'' = 0 \\ \bar{f}_{\eta''} &= \gamma_0' = 0 \\ \bar{f}_{\gamma'} &= \eta_0'' \\ \bar{f}_{\gamma''} &= \eta_0' \\ \bar{f}_{\varepsilon} &= \frac{3}{2}\ \eta_0'^2\ \eta_0'' \end{aligned}\right\} \qquad (3.2.21)$$

$$\left.\begin{aligned} \bar{g}_{\eta'} &= \eta_0'' \\ \bar{g}_{\eta''} &= \eta_0' \end{aligned}\right\} \qquad (3.2.22)$$

Upon substitution of equation (3.2.21) into equation (2.2.32) and integrating with respect to τ, the second order frequency-amplitude relation is

$$\omega_2^2 = -\lambda\int_0^1 [\sum_{q=1}^{\infty} (C_{0q}^{(1)} + \frac{1}{2}C_{2q}^{(1)})\ (V_1''\Gamma_q' + V_1'\Gamma_q'')$$

$$+ \frac{9}{8}A_{11}^2 V_1'^2 V_1'']\, r\ V_1 d\zeta \qquad (3.2.23)$$

The amplitude parameters follow from equation (2.2.33)

$$A_{1n}^{(2)} = \frac{A_{11}}{(\Omega_n^2-\Omega_1^2)} \int_0^1 \lambda[\sum_{q=1}^{\infty} (C_{0q}^{(1)} + \frac{1}{2}\ C_{2q}^{(1)})(V_1''\Gamma_q' + V_1'\Gamma_q'')$$

$$+ \frac{9}{8}\ A_{11}^2 V_1'^2 V_1'']\, r\ V_n\ d\zeta, \qquad n \geq 1 \qquad (3.2.24)$$

$$A_{3n}^{(2)} = \frac{A_{11}^2}{(\Omega_n^2-9\Omega_1^2)} \int_0^1 \lambda [\sum_{q=1}^{\infty} \frac{1}{2} C_{2q}^{(1)} (V_1'' \Gamma_q' + V_1' \Gamma_q'')$$

$$+ \frac{3}{8} A_{11}^2 V_1'^2 V_1''] r \ V_n \ d\zeta, \quad n \geq 1 \tag{3.2.25}$$

with all other $A_{mn}^{(2)} = 0$. Thus the second order correction for the transverse displacement is of the form of equation (2.2.36) with the coefficients determined above.

Substituting (3.2.16) and (3.2.22) into (2.2.35), one finds that the expansion coefficients $C_{mn}^{(2)}$ vanish identically. Consequently,

$$\gamma_2(\zeta,\tau) = 0 \tag{3.2.26}$$

Thus the expansion coefficients for the first order longitudinal displacement, the second order frequency-amplitude relation, and the amplitude parameters for the transverse displacement are readily obtained by evaluating the integrals (3.2.19), (3.2.23), (3.2.24), and (3.2.25). The integrands of all these integrals involve the products of three or four spatial eigenfunctions. It is, of course, the boundary conditions such as hinged-hinged, clamped-clamped or clamped-hinged that determine the set of allowed eigenvalues and the corresponding eigenfunctions. A CDC 6500 digital computer is then used to perform the integrations numerically. Several particular examples involving immovable boundary conditions will be studied and the numerical results will be discussed in the next chapter.

3.3. Elastic Circular Membrane with Initial Tension and Immovable Edge

Let us next consider the free, undamped, periodic, axisymmetric vibrations of an elastic circular membrane with initial tension and immovable edge. By equations (3.1.15), (3.1.20) and (3.1.21) one obtains the following pair of coupled nonlinear partial differential equations of motion

$$\frac{1-\mu^2}{Eh}\left[N_0\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \rho h\frac{\partial^2 w}{\partial t^2}\right] + \frac{\partial w}{\partial r}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1+\mu}{r}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{2r}\left(\frac{\partial w}{\partial r}\right)^2\right] + \frac{\partial^2 w}{\partial r^2}\left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\mu}{r}u\right] = 0 \tag{3.3.1a}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 u}{\partial t^2} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1-\mu}{2r}\left(\frac{\partial w}{\partial r}\right)^2 = 0 \tag{3.3.1b}$$

The above equations may be nondimensionalized so that the membrane is of unit radius and the period of vibration is fixed at 2π by introducing

$$\bar{\eta} = \frac{w}{a}, \qquad \bar{\gamma} = \frac{u}{a}, \qquad \zeta = \frac{r}{a}$$
$$\tau = \bar{\omega}t, \qquad \omega^2 = \frac{\rho a^2}{E}\bar{\omega}^2, \qquad e_0 = \frac{N_0}{Eh}(1-\mu) \tag{3.3.2}$$

Equations (3.3.1a) and (3.3.1b) then take the form

$$(1+\mu)e_0\nabla^2\bar{\eta} - (1-\mu^2)\omega^2\ddot{\bar{\eta}} + \bar{\eta}'\left(\bar{\gamma}'' + \frac{1+\mu}{\zeta}\bar{\gamma}' + \bar{\eta}'\bar{\eta}'' + \frac{1}{2\zeta}\bar{\eta}'^2\right) + \bar{\eta}''\left(\bar{\gamma}' + \frac{\mu}{\zeta}\bar{\gamma} + \frac{1}{2}\bar{\eta}'^2\right) = 0 \tag{3.3.3a}$$

$$(\nabla^2 - \frac{1}{\zeta^2})\bar{\gamma} - (1-\mu^2)\omega^2\ddot{\bar{\gamma}} + \bar{\eta}'\bar{\eta}'' + \frac{1-\mu}{2\zeta}\bar{\eta}'^2 = 0 \qquad (3.3.3b)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial}{\partial\zeta} \qquad (3.3.4)$$

Performing the following change of variables

$$\bar{\eta} = \varepsilon\eta \quad , \qquad \bar{\gamma} = \varepsilon\gamma \qquad (3.3.5)$$

one finally obtains the two coupled, nonlinear, nondimensional equations of motion

$$- \frac{e_0}{1-\mu}\nabla^2\eta + \omega^2\ddot{\eta} - \frac{\varepsilon}{1-\mu^2}[\eta'(\gamma'' + \frac{1+\mu}{\zeta}\gamma') + \eta''(\gamma' + \frac{\mu}{\zeta}\gamma)$$

$$+ \frac{\varepsilon}{2}(3\eta'^2\eta'' + \frac{1}{\zeta}\eta'^3)] = 0 \qquad (3.3.6a)$$

$$- \frac{1}{1-\mu^2}(\nabla^2 - \frac{1}{\zeta^2})\gamma + \omega^2\ddot{\gamma} - \frac{\varepsilon}{1-\mu^2}(\eta'\eta'' + \frac{1-\mu}{2\zeta}\eta'^2) = 0 \qquad (3.3.6b)$$

which are similar to the two equations of motion described in Chapter II. The analogous quantities are

$$L_1 = - \frac{e_0}{1-\mu}\nabla^2 \quad , \qquad \alpha = - \frac{1}{1-\mu^2} \qquad (3.3.7)$$

$$f(\eta', \eta'', \gamma, \gamma', \gamma'', \varepsilon) = \eta'(\gamma'' + \frac{1+\mu}{\zeta}\gamma') + \eta''(\gamma' + \frac{\mu}{\zeta}\gamma)$$

$$+ \frac{\varepsilon}{2}(3\eta'^2\eta'' + \frac{1}{\zeta}\eta'^3) \qquad (3.3.8)$$

and

$$L_2 = - \frac{1}{1-\mu^2}(\nabla^2 - \frac{1}{\zeta^2}) \quad , \qquad \beta = - \frac{1}{1-\mu^2} \qquad (3.3.9)$$

$$g(\eta', \eta'') = \eta'\eta'' + \frac{1-\mu}{2\zeta}\eta'^2 \qquad (3.3.10)$$

The corresponding linear equations are found by setting $\varepsilon = 0$ in (3.3.6a) and (3.3.6b), namely,

$$-\frac{e_0}{1-\mu}\nabla^2\eta_0 + \omega_0^2\ddot{\eta}_0 = 0 \qquad (3.3.11a)$$

$$-\frac{1}{1-\mu^2}\left(\nabla^2 - \frac{1}{\zeta^2}\right)\gamma_0 + \omega_0^2\ddot{\gamma}_0 = 0 \qquad (3.3.11b)$$

Their solutions are then taken to be

$$\left.\begin{aligned} \eta_0 &= A_{11}\, V_1\,(\zeta)\cos\tau \\ \omega_0^2 &= \Omega_1^2 \end{aligned}\right\} \qquad (3.3.12)$$

$$\gamma_0 = 0 \qquad (3.3.13)$$

It then follows from (3.3.8) and (3.3.10) that

$$\bar{f} = f(\eta', \eta'', \gamma, \gamma', \gamma'', \varepsilon)\Big|_{\eta_0', \eta_0'', \gamma_0, \gamma_0', \gamma_0'', 0} = 0 \qquad (3.3.14)$$

$$\bar{g} = g(\eta', \eta'')\Big|_{\eta_0', \eta_0''} = A_{11}^2\left(V_1'V_1'' + \frac{1-\mu}{2\zeta}V_1'^2\right)\cos^2\tau \qquad (3.3.15)$$

By (3.3.14), the first order frequency-amplitude relation as given by equation (2.2.24) is

$$\omega_1^2 = 0 \qquad (3.3.16)$$

and the amplitude parameters as given by equation (2.2.25) are

$$A_{mn}^{(1)} = 0 \qquad (3.3.17)$$

It then follows that

$$\eta_1(\zeta, \tau) = 0 \qquad (3.3.18)$$

Upon substitution of (3.3.15) into (2.2.27) and integrating with respect to τ lead to :

$$C_{0n}^{(1)} = \frac{A_{11}^2}{2\Lambda_n^2} I, \qquad n \geq 1 \tag{3.3.19}$$

$$C_{2n}^{(1)} = \frac{A_{11}^2}{2(\Lambda_n^2 - 4\Omega_1^2)} I, \qquad n \geq 1 \tag{3.3.20}$$

where

$$I = \int_0^1 \frac{1}{1-\mu^2} (V_1' V_1'' + \frac{1-\mu}{2\zeta} V_1'^2) s \, \Gamma_n \, d\zeta, \quad n \geq 1 \tag{3.3.21}$$

All other $C_{mn}^{(1)}$ are equal to zero. The first order radial displacement is then

$$\gamma_1(\zeta, \tau) = \sum_{n=1}^{\infty} (C_{0n}^{(1)} + C_{2n}^{(1)} \cos 2\tau) \Gamma_n(\zeta) \tag{3.3.22}$$

The second order result may be determined in a similar manner. The following derivatives of f and g are first evaluated.

$$\left.\begin{aligned}
\bar{f}_{\eta'} &= \bar{f}_{\eta''} = 0 \\
\bar{f}_{\gamma} &= A_{11} \frac{\mu}{\zeta} V_1'' \cos \tau \\
\bar{f}_{\gamma'} &= A_{11} (\frac{1+\mu}{\zeta} V_1' + V_1'') \cos \tau \\
\bar{f}_{\gamma''} &= A_{11} V_1' \cos \tau \\
\bar{f}_{\omega} &= \frac{A_{11}^3}{2} (3V_1'^2 V_1'' + \frac{1}{\zeta} V_1'^3) \cos^3 \tau
\end{aligned}\right\} \tag{3.3.23}$$

and

$$\left.\begin{aligned}
\bar{g}_{\eta'} &= A_{11} (V_1'' + \frac{1-\mu}{\zeta} V_1') \cos \tau \\
\bar{g}_{\eta''} &= A_{11} V_1' \cos \tau
\end{aligned}\right\} \tag{3.3.24}$$

By the first order amplitude parameter results (3.3.17), substituting (3.3.23) into the general expression given by equation (2.2.32) and integrating with respect to time, one obtains the second order frequency-amplitude relation as

$$\omega_2^2 = -\frac{1}{1-\mu^2}\int_0^1 \{\sum_{q=1}^{\infty} (c_{0q}^{(1)} + \frac{1}{2}c_{2q}^{(1)})[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8} A_{11}^2(3V_1'^2 V_1'' + \frac{1}{\zeta} V_1'^3)\} r\, V_1\, d\zeta \qquad (3.3.25)$$

The second order amplitude parameters follow from the general expression given by equation (2.2.33)

$$A_{1n}^{(1)} = \frac{A_{11}}{(\Omega_n^2 - \Omega_1^2)}\int_0^1 \frac{1}{1-\mu^2}\{\sum_{q=1}^{\infty} (c_{0q}^{(1)} + \frac{1}{2}c_{2q}^{(1)})[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8} A_{11}^2(3V_1'^2 V_1'' + \frac{1}{\zeta} V_1'^3)\} r\, V_n\, d\zeta, \quad n > 1 \qquad (3.3.26)$$

$$A_{3n}^{(2)} = \frac{A_{11}}{(\Omega_n^2 - 9\Omega_1^2)}\int_0^1 \frac{1}{1-\mu^2}\{\sum_{q=1}^{\infty} \frac{1}{2} c_{2q}^{(1)}[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{1}{8} A_{11}^2(3V_1'V_1'' + \frac{1}{\zeta} V_1'^3)\}\, r\, V_n\, d\zeta, \quad n \geq 1 \qquad (3.3.27)$$

All other $A_{mn}^{(2)}$ are equal to zero.

The second order transverse displacement is then

$$\eta_2(\zeta,\tau) = \sum_{n=2}^{\infty} A_{1n}^{(2)} V_n(\zeta)\cos\tau + \sum_{n=1}^{\infty} A_{3n}^{(2)} V_n(\zeta)\cos 3\tau \quad (3.3.28)$$

In view of (3.3.16), (3.3.17), and (3.3.24), one concludes that equation (2.2.35) gives

$$C_{mn}^{(2)} = 0 \tag{3.3.29}$$

Consequently,

$$\gamma_2(\zeta, \tau) = 0 \tag{3.3.30}$$

A representative example of a circular membrane with a clamped immovable edge will be studied in the next chapter.

3.4. Elastic Circular Plate with Immovable Edge

Let us finally consider the free, undamped, periodic, axisymmetric vibrations of an elastic circular plate of radius a with immovable supports at the outer rim. By equations (3.1.15), (3.1.20) and (3.1.21)

$$-(\nabla^2 w)^2 - \frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} + \frac{12}{h^2}\{\frac{\partial w}{\partial r}[\frac{\partial^2 u}{\partial r^2} + \frac{1+\mu}{r}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{2r}(\frac{\partial w}{\partial r})^2]$$

$$+ \frac{\partial^2 w}{\partial r^2}[\frac{\partial u}{\partial r} + \frac{\mu}{r}u + \frac{1}{2}(\frac{\partial w}{\partial r})^2]\} = 0 \tag{3.4.1a}$$

$$(\nabla^2 - \frac{1}{r^2})u - \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 u}{\partial t^2} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1-\mu}{2r}(\frac{\partial w}{\partial r})^2 = 0 \tag{3.4.1b}$$

Nondimensionalizing this pair of coupled nonlinear partial differential equations of motion in the same manner as was done previously, namely,

$$\begin{aligned} &\bar{\eta} = \frac{w}{a}, \qquad \bar{\gamma} = \frac{u}{a}, \qquad \zeta = \frac{r}{a} \\ &\tau = \bar{\omega}t, \qquad \omega^2 = \frac{\rho h a^4}{D}\bar{\omega}^2, \qquad \lambda = 12(\frac{a}{h})^2 \end{aligned} \tag{3.4.2}$$

equations (3.4.1a) and (3.4.1b) become

$$-(\nabla^2\bar{\eta})^2 - \omega^2\bar{\eta} + \lambda[\bar{\eta}'(\bar{\gamma}'' + \frac{1+\mu}{\zeta}\bar{\gamma}' + \bar{\eta}'\bar{\eta}'' + \frac{1}{2\zeta}\bar{\eta}'^2)$$

$$+ \bar{\eta}''(\bar{\gamma}' + \frac{\mu}{\zeta}\bar{\gamma} + \frac{1}{2}\bar{\eta}'^2)] \tag{3.4.3a}$$

$$\lambda(\nabla^2 - \frac{1}{\zeta^2})\bar{\gamma} - \omega^2\ddot{\bar{\gamma}} + \lambda(\bar{\eta}'\bar{\eta}'' + \frac{1-\mu}{2\zeta}\bar{\eta}'^2) = 0 \tag{3.4.3b}$$

Let us now introduce the perturbation parameter ε . through the following change of variables

$$\bar{\eta} = \varepsilon\eta \ , \qquad \bar{\gamma} = \varepsilon\gamma \tag{3.4.4}$$

The pair of nondimensional, governing equations of motion in a final form used to obtain solutions is

$$\nabla^4\eta + \omega^2\ddot{\eta} - \varepsilon\lambda[\eta'(\gamma'' + \frac{1+\mu}{\zeta}\gamma') + \eta''(\gamma' + \frac{\mu}{\zeta}\gamma)$$

$$+ \frac{\varepsilon}{2}(3\eta'\eta'' + \frac{1}{\zeta}\eta'^3)] = 0 \tag{3.4.5a}$$

$$-\lambda(\nabla^2 - \frac{1}{\zeta^2})\gamma + \omega^2\ddot{\gamma} - \varepsilon\lambda(\eta'\eta'' + \frac{1-\mu}{2\zeta}\eta'^2) = 0 \tag{3.4.5b}$$

The identification of equations (3.4.5) with (2.1.1) becomes evident if one lets

$$L_1 = \nabla^4 = (\nabla^2)^2 = (\frac{\partial^2}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial}{\partial\zeta})^2 \ , \ \alpha = -\lambda \tag{3.4.6}$$

$$f(\eta', \eta'', \gamma, \gamma', \gamma'', \varepsilon) = \eta'(\gamma'' + \frac{1+\mu}{\zeta}\gamma') + \eta''(\gamma' + \frac{\mu}{\zeta}\gamma)$$

$$+ \frac{\varepsilon}{2}(3\eta'\eta'' + \frac{1}{\zeta}\eta'^3) \tag{3.4.7}$$

and

$$L_2 = -\lambda(\nabla^2 - \frac{1}{\zeta^2}) \ , \qquad \beta = -\lambda \tag{3.4.8}$$

$$g(\eta', \eta'') = \eta'\eta'' + \frac{1-\mu}{2\zeta}\eta'^2 \tag{3.4.9}$$

Setting $\varepsilon = 0$ in (3.4.5a) and (3.4.5b) yields the following linear equations of motion

$$\nabla^4 \eta_0 + \omega_0^2 \ddot{\eta}_0 = 0 \tag{3.4.10a}$$

$$-\lambda\left(\nabla^2 - \frac{1}{\zeta^2}\right)\gamma_0 + \omega_0^2\ddot{\gamma}_0 = 0 \tag{3.4.10b}$$

The solution of which is taken as

$$\left.\begin{aligned} \eta_0 &= A_{11}V_1(\zeta)\cos\tau \\ \omega_0^2 &= \Omega_1^2 \end{aligned}\right\} \tag{3.4.11}$$

$$\gamma_0 = 0 \tag{3.4.12}$$

i.e. perturbations in the vicinity of the first linear mode are considered. Thus $\bar{f}$ and $\bar{g}$ can be evaluated as follows:

$$\bar{f} = f(\eta', \eta'', \gamma, \gamma', \gamma'', \varepsilon)\Big|_{\eta_0', \eta_0'', \gamma_0', \gamma_0'', 0} = 0 \tag{3.4.13}$$

$$\bar{g} = g(\eta', \eta'')\Big|_{\eta_0', \eta_0''} = A_{11}^2\left(V_1'V_1'' + \frac{1-\mu}{2\zeta}V_1'^2\right)\cos^2\tau \tag{3.4.14}$$

In view of (3.4.13), equations (2.2.24) and (2.2.25) then yield respectively

$$\omega_1^2 = 0 \tag{3.4.15}$$

$$A_{mn}^{(1)} = 0 \tag{3.4.16}$$

It follows that

$$\eta_1(\zeta, \tau) = 0 \tag{3.4.17}$$

Upon substituting (3.4.14) into (2.2.27) and integrating with respect to τ, one obtains

$$C_{0n}^{(1)} = \frac{A_{11}^2}{2\Lambda_n^2} I , \qquad n \geq 1 \tag{3.4.18}$$

$$C_{2n}^{(1)} = \frac{A_{11}^2}{2(\Lambda_n^2 - 4\Omega_1^2)} I , \qquad n \geq 1 \tag{3.4.19}$$

where

$$I = \int_0^1 \lambda (V_1' V_1'' + \frac{1-\mu}{2\zeta} V_1'^2) \; s \; \Gamma_n \, d\zeta, \; n \geq 1 \tag{3.4.20}$$

All other $C_{mn}^{(1)}$ are equal to zero.

The first order result for the radial displacement then follows as

$$\gamma_1(\zeta,\tau) = \sum_{n=1}^{\infty} (C_{0n}^{(1)} + C_{2n}^{(1)} \cos 2\tau) \; \Gamma_n(\zeta) \tag{3.4.21}$$

To determine the second order solution, the following derivatives of f and g are evaluated.

$$\left.\begin{aligned}
\bar{f}_{\eta'} &= \bar{f}_{\eta''} = 0 \\
\bar{f}_{\gamma} &= A_{11} \frac{\mu}{\zeta} V_1'' \cos \tau \\
\bar{f}_{\gamma'} &= A_{11} (\frac{1+\mu}{\zeta} V_1' + V_1'')\cos \tau \\
\bar{f}_{\gamma''} &= A_{11} V_1' \cos \tau \\
\bar{f}_{\varepsilon} &= \frac{A_{11}^2}{2} (3V_1'^2 V_1'' + \frac{1}{\zeta} V_1'^3)\cos^3 \tau
\end{aligned}\right\} \tag{3.4.22}$$

$$\left.\begin{aligned}
\bar{g}_{\eta'} &= A_{11}(V_1'' + \frac{1-\mu}{\zeta} V_1')\cos \tau \\
\bar{g}_{\eta''} &= A_{11} V_1' \cos \tau
\end{aligned}\right\} \tag{3.4.23}$$

Inserting equation (3.4.22) into (2.2.32) and upon integrating with respect to τ, the second order frequency-amplitude relation follows as

$$\omega_2^2 = -\lambda \int_0^1 \sum_{q=1}^{\infty} \{(C_{0q}^{(1)} + \frac{1}{2} C_{2q}^{(1)})[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+(\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8} A_{11}^2(3V_1'^2V_1'' + \frac{1}{\zeta} V_1'^3)\}r\ V_1 d\zeta \qquad (3.4.24)$$

Similarly, the amplitude parameters for the second order correction are obtained from equation (2.2.33)

$$A_{1n}^{(2)} = \frac{A_{11}}{(\Omega_n^2 - \Omega_1^2)} \int_0^1 \lambda\{\sum_{q=1}^{\infty} (C_{0q}^{(1)} + \frac{1}{2}C_{2q}^{(1)})[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8} A_{11}^2(3V_1'^2V_1'' + \frac{1}{\zeta} V_1'^3)\}r\ V_n d\zeta, \quad n > 1 \qquad (3.4.25)$$

$$A_{3n}^{(2)} = \frac{A_{11}}{(\Omega_n^2 - 9\Omega_1^2)} \int_0^1 \lambda\{\sum_{q=1}^{\infty} \frac{1}{2} C_{2q}^{(1)}[\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{1}{8} A_{11}^2(3V_1'V_1'' + \frac{1}{\zeta} V_1'^3)\}r\ V_n d\zeta, \quad n \geq 1 \qquad (3.4.26)$$

All other $A_{mn}^{(2)}$ are equal to zero.

The second order result for the transverse displacement then follows as

$$\eta_2(\zeta,\tau) = \sum_{n=2}^{\infty} A_{1n}^{(2)} V_n(\zeta)\cos\tau + \sum_{n=1}^{\infty} A_{3n}^{(2)} V_n(\zeta)\cos 3\tau \qquad (3.4.27)$$

In view of (3.4.15), (3.4.16), and (3.4.23), equation (2.2.35) yields

$$C_{mn}^{(2)} = 0 \tag{3.4.28}$$

It follows that

$$\gamma_2(\zeta,\tau) = 0 \tag{3.4.29}$$

An illustrative example for a circular plate with specific immovable boundary conditions will also be presented in the next chapter.

IV. NUMERICAL RESULTS AND COMPARISON STUDIES

4.1. Introduction

In the preceding chapters, general expressions for frequency-amplitude relations as well as for the mode shapes of vibrations for a wide class of nonlinear vibration problems have been presented. These expressions apply to beams, circular membranes and circular plates subject to a variety of boundary conditions. In this chapter, problems with specific boundary conditions are solved.

When the boundary conditions are given, the spatial eigenfunctions V_k, Γ_n and the corresponding eigenvalues Ω_k^2, Λ_n^2 appearing in the general expressions are therefore known. As is noted previously, the determination of the frequency-amplitude relations and of the coefficients of the series expansions for the mode shapes of vibrations is then reduced to the evaluation of integrals involving such known eigenfunctions. The integrals are evaluated numberically on a CDC 6500 digital computer. The computer program proceeds as follows:

The interval of integration (0,1) is divided initially into NP = 10 equal intervals and a six-point Newton-Cotes formula is applied to each of the NP subdivisions. A sum over NP represents an approximation to the desired integral over (0,1). NP is then doubled and the process repeated until the relative error between two approximations is less than or equal to 10^{-6}.

An additional subroutine is required for calculating Bessel functions which arise as the spatial eigenfunctions of the circular membrane and the circular plate problems. This subroutine was obtained from the Program Library of Michigan State University Computer Laboratory.

Numerical results are presented in the next three sections. These results are then discussed and compared with those previously obtained by solving single nonlinear partial differential equations.

4.2. Beams with Various Boundary Conditions

Prismatic beams with immovable supports as described in Section 3.2 are considered here for hinged-hinged, clamped - clamped, and clamped-hinged boundary conditions.

Let us first consider a beam having both ends hinged. The transverse displacement, the bending moment, and the longitudinal displacement are zero at either end. The boundary conditions then follow as

$$\begin{aligned} \eta(0,\tau) &= \eta''(0,\tau) = 0 \\ \eta(1,\tau) &= \eta''(1,\tau) = 0 \end{aligned} \tag{4.2.1}$$

$$\gamma(0,\tau) = \gamma(1,\tau) = 0 \tag{4.2.2}$$

The differential equation (2.2.7) together with boundary conditions (4.2.1) are identical to that of the Euler-Bernoulli beam. By the orthogonality condition (2.2.10) with $r(\zeta) = 1$, the normalized transverse spatial eigenfunctions and its corresponding eigenvalues are

$$\left.\begin{aligned} & V_k(\zeta) = \sqrt{2}\sin k\pi\zeta \\ & \Omega_k^2 = (\beta_k)^4 = (k\pi)^4 \\ & k = 1,2,3,\ldots\ldots \end{aligned}\right\} \quad (4.2.3)$$

Also, the differential equation (2.2.13) together with boundary conditions (4.2.2) are identical to that of a beam executing longitudinal vibrations. By the orthogonality condition (2.2.16) with $s(\zeta) = 1$, the normalized longitudinal spatial eigenfunctions and their corresponding eigenvalues are

$$\left.\begin{aligned} & \Gamma_n(\zeta) = \sqrt{2}\sin n\pi\zeta \\ & \Lambda_n^2 = \lambda(n\pi)^2 \\ & n = 1,2,3,\ \ldots\ldots \end{aligned}\right\} \quad (4.2.4)$$

Substituting of V_1, Ω_1^2, Γ_n and Λ_n^2 into equations (3.2.17), (3.2.18) and integrating with respect to ζ give respectively

$$C_{02}^{(1)} = -\frac{\pi}{16}\sqrt{2}\,A_{11}^2 \quad (4.2.5)$$

$$C_{22}^{(1)} = -\frac{\pi}{16\left(1-\frac{\pi^2}{\lambda}\right)}\sqrt{2}\,A_{11}^2 \quad (4.2.6)$$

with all other $C_{mn}^{(1)}$ being equal to zero. It is seen that $C_{22}^{(1)}$ depends on the parameter λ which is defined as $(\frac{L}{r})^2$. Numerical results of these two coefficients are given in Table 4.2.1 as a function of $\frac{L}{r}$.

Table 4.2.1. Numerical values of the expansion coefficients $C_{02}^{(1)}$ and $C_{22}^{(1)}$ of hinged-hinged beams*.

$\frac{L}{r}$	$C_{02}^{(1)}$	$C_{22}^{(1)}$
40	-0.27768	-0.27940
80	-0.27768	-0.27811
120	-0.27768	-0.27787
160	-0.27768	-0.27779
200	-0.27768	-0.27775

*All values multiplied by A_{11}^2.

The second order frequency-amplitude relation is found from equation (3.2.23) to be

$$\omega_2^2 = \frac{3}{8}\beta_1^4 F(\lambda)(\frac{L}{r} A_{11})^2 \tag{4.2.7}$$

where

$$F(\lambda) = \frac{1}{2(\pi^2-\lambda)}(\lambda - \frac{2}{3}\pi^2) + \frac{3}{2} \tag{4.2.8}$$

and

$$\beta_1 = \pi.$$

Numverical results of $F(\lambda)$ over a range of $\frac{L}{r}$ are shown in Table 4.2.2.

Table 4.2.2. Numerical values of $F(\lambda)$ of hinged-hinged beams.

$\frac{L}{r}$	40	80	120	160	200
$F(\lambda)$	0.99897	0.99974	0.99989	0.99994	0.99996

Two other beam examples of practical interest are a beam having both ends clamped and a beam clamped at one end $(\zeta = 0)$ and hinged at the other end $(\zeta = 1)$. The transcendental equation for the eigenvalues of the clamped-clamped beam is

$$\cosh \beta \cos \beta = 1 \tag{4.2.9}$$

and the transcendental equation for the eigenvalues of the clamped-hinged beam is

$$\tanh \beta = \tan \beta \tag{4.2.10}$$

The normalized transverse spatial eigenfunctions and their corresponding eigenvalues for both cases follow in the form

$$V_k(\zeta) = \cosh \beta_k \zeta - \cos \beta_k \zeta - \frac{\cosh \beta_k - \cos \beta_k}{\sinh \beta_k - \sin \beta_k}(\sinh \beta_k \zeta - \sin \beta_k \zeta)$$

$$\Omega_k^2 = (\beta_k)^4 \tag{4.2.11}$$

$$k = 1,2,3,\ldots\ldots$$

Numerical values of β_k for both end conditions are tabulated in [8].

As the boundary conditions for the longitudinal displacement here are the same as in the hinged-hinged beam case. The normalized longitudinal spatial eigenfunctions Γ_n and their corresponding eigenvalues Λ_n^2 remain the same as those given in equation (4.2.4).

After substituting V_1, Ω_1^2, Γ_n and Λ_n^2 into (3.2.17), (3.2.18) and performing the integrations on the computer the results of the first nine expansion coefficients for both cases are given in Tables 4.2.3 through 4.2.6.

One may note that in the hinged-hinged and the clamped-clamped cases, the product of the derivatives, $V_1'V_1''$, in equation (3.2.19) is anti-symmetrical about the mid-section of the beam and that the shapes $\sin 2\pi\zeta$, $\sin 4\pi\zeta$, etc. are also anti-symmetrical about this mid-section. By constrast the shapes $\sin \pi\zeta$, $\sin 3\pi\zeta$, etc. are symmetrical about the mid-section. It is then obvious that coefficients like $C_{01}^{(1)}$, $C_{03}^{(1)}$, ..., $C_{21}^{(1)}$, $C_{23}^{(1)}$, ..., etc. must vanish for these two cases. For the case of clamped-hinged beam, of course, these symmetry considerations are no longer applicable.

Using equation (3.2.23) one obtains the second order frequency-amplitude relation as

$$\omega_2^2 = \frac{3}{8}\beta_1^4 F(\lambda)\left(\frac{L}{r} A_{11}\right)^2 \tag{4.2.12}$$

where

Table 4.2.3. Numerical values of the expansion coefficients $C_{0n}^{(1)}$ of clamped-clamped beams with any $\frac{L}{r}$.*

n	2	4	6	8	10	12	14	16	18
$C_{0n}^{(1)}$	-0.06120	0.16260	0.01725	0.00398	0.00130	0.00052	0.00024	0.00012	0.00007

*All values multiplied by A_{11}^2.

Table 4.2.4. Numerical values of the expansion coefficients $C_{2n}^{(1)}$ of clamped-clamped beams.*

$C_{2n}^{(1)}$ $\frac{L}{r}$ \ n	2	4	6	8	10	12	14	16	18
40	-0.06320	0.16390	0.01731	0.00399	0.00130	0.00052	0.00024	0.00012	0.00007
80	-0.06168	0.16292	0.01727	0.00399	0.00130	0.00052	0.00024	0.00012	0.00007
120	-0.06141	0.16274	0.01726	0.00398	0.00130	0.00052	0.00024	0.00012	0.00007
160	-0.06132	0.16268	0.01726	0.00398	0.00130	0.00052	0.00024	0.00012	0.00007
200	-0.06127	0.16265	0.01726	0.00398	0.00130	0.00052	0.00024	0.00012	0.00007

*All values multipled by A_{11}^2.

Table 4.2.5. Numerical values of the expansion coefficients $C_{0n}^{(1)}$ of clamped-hinged beams with any $\frac{L}{r}$.*

n	1	2	3	4	5	6	7	8	9
$C_{0n}^{(1)}$	0.34843	-0.20705	0.16891	0.02802	0.00861	0.00339	0.00156	0.00079	0.00044

*All values multiplied by A_{11}^2.

Table 4.2.6. Numerical values of the expansion coefficients $C_{2n}^{(1)}$ of clamped-hinged beams.*

$C_{2n}^{(1)}$ n / $\frac{L}{r}$	1	2	3	4	5	6	7	8	9
40	0.37076	-0.21011	0.17005	0.02813	0.00864	0.00340	0.00156	0.00079	0.00044
80	0.35376	-0.20784	0.16920	0.02805	0.00862	0.00339	0.00156	0.00079	0.00044
120	0.35078	-0.20740	0.16904	0.02803	0.00862	0.00339	0.00156	0.00079	0.00044
160	0.34975	-0.20725	0.16898	0.02803	0.00862	0.00339	0.00156	0.00079	0.00044
200	0.34927	-0.20718	0.16896	0.02802	0.00862	0.00339	0.00156	0.00079	0.00044

*All values multipled by A_{11}^2.

$$F(\lambda) = - \int_0^1 \left[\frac{8}{3\beta_1^4 A_{11}^2} \sum_{q=1}^{\infty} (C_{0q}^{(1)} + \frac{1}{2} C_{2q}^{(1)})(V_1'' \Gamma_q' \right.$$

$$\left. + V_1' \Gamma_q'') + \frac{3}{\beta_1^4} V_1'^2 V_1'' \right] V_1 \, d\zeta \qquad (4.2.13)$$

and

$$\beta_1 = 4.7300408 \qquad (4.2.14)$$

for clamped-clamped,

$$\beta_1 = 3.9266023 \qquad (4.2.15)$$

for clamped-hinged end conditions. It is to be noted that the integration of (4.2.13) involves a series. A sufficient number of longitudinal spatial eigenfunctions must be taken to insure convergence. Tables 4.2.7 and 4.2.8 show how closely the truncated series represents the function $F(\lambda)$ for different values of N where N represents the number of terms used in the truncation.

With the information of $F(\lambda)$ from Table 4.2.2 for hinged-hinged, 4.2.7 for clamped-clamped, and 4.2.8 for clamped-hinged beams, nonlinear frequency-amplitude relations including the second order corrections are now in the form

$$\omega^2 = \omega_0^2 \left[1 + \frac{3}{8} F(\lambda) \left(\frac{L}{r} \varepsilon A_{11}\right)^2\right] \qquad (4.2.16)$$

Blotter [7] and Evensen [9] have studied nonlinear vibrations of beams with various end conditions using a single nonlinear partial differential equation. Before one compares (4.2.16) with the corresponding results of Blotter and Evensen, it will be of interest to rederive

Table 4.2.7. Convergence of $F(\lambda)$ of clamped-clamped beams.

$\frac{L}{r}$ \ $F(\lambda)$ \ N	3	6	9
40	0.30235	0.30196	0.30196
80	0.30266	0.30227	0.30227
120	0.30271	0.30233	0.30232
160	0.30273	0.30235	0.30234
200	0.30274	0.30236	0.30235

Table 4.2.8. Convergence of $F(\lambda)$ of clamped-hinged beams.

$\frac{L}{r}$ \ $F(\lambda)$ \ N	3	6	9
40	0.56483	0.55496	0.55485
80	0.56687	0.55701	0.55689
120	0.56723	0.55737	0.55726
160	0.56735	0.55750	0.55738
200	0.56741	0.55756	0.55744

their equation of motion using the energy approach, so as to see what simplifications have been made. Application of equations (3.1.16) and (3.1.17) to (3.1.12) yields

$$-\rho S \frac{\partial^2 w}{\partial t^2} + ES\left(e \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial e}{\partial x}\right) - EI \frac{\partial^4 w}{\partial x^4} = 0 \qquad (4.2.17)$$

$$-\rho S \frac{\partial^2 u}{\partial t^2} + ES \frac{\partial e}{\partial x} = 0 \qquad (4.2.18)$$

When the longitudinal inertia term is neglected, equation (4.2.18) becomes

$$\frac{\partial e}{\partial x} = 0 \qquad (4.2.19)$$

Thus, e is independent of x. It is recalled that the first strain invariant, e, is defined as

$$e = \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (3.1.5)$$

In view of the assumed constancy of e with respect to x, it is possible to multiply by dx and integrate over the length of the beam to find

$$\int_0^L e\,dx = \int_0^L \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = u\Big|_0^L + \frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2.20)$$

The vanishing of u at the boundary leads to

$$e = \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2.21)$$

Equation (4.2.17) now becomes

$$EI \frac{\partial^4 w}{\partial x^4} - T \frac{\partial^2 w}{\partial x^2} + \rho S \frac{\partial^2 w}{\partial t^2} = 0 \qquad (4.2.22)$$

where the total axial tension in the beam is defined as

$$T(t) = \frac{ES}{2L} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2.23)$$

Equation (4.2.22) is exactly the same equation of motion used by Blotter and Evensen. It was Woinowsky-Krieger [10] who first established this equation using the balance of forces approach and studied the effect of axial force on the vibration of hinged bars.

It must be pointed out that the single equation of motion (4.2.22) results as the consequence of neglecting both the effects of longitudinal inertia and that of the first spatial derivative of the first strain invariant. This theory yields only the total axial force of the beam by equation (4.2.23). To determine the distribution of the axial force in the longitudinal direction, the displacement u cannot be ignored. It is necessary to return to the full pair of coupled nonlinear partial differential equations of motion as given by (3.2.1a) and (3.2.1b).

The frequency-amplitude relationships as given by (4.2.16), along with those of Blotter and Evensen, are shown in Figures 4.2.1 through 4.2.3. In these figures A is related to A_{11} by

$$A = A_{11} \; | \, V_1 \, |_{max} \qquad (4.2.24)$$

All these curves show the same feature that the nonlinear frequencies increase with increasing amplitudes. It is to be stressed that the results of Blotter and Evensen do not depend on the slenderness ratio $\frac{L}{r}$ while the results of

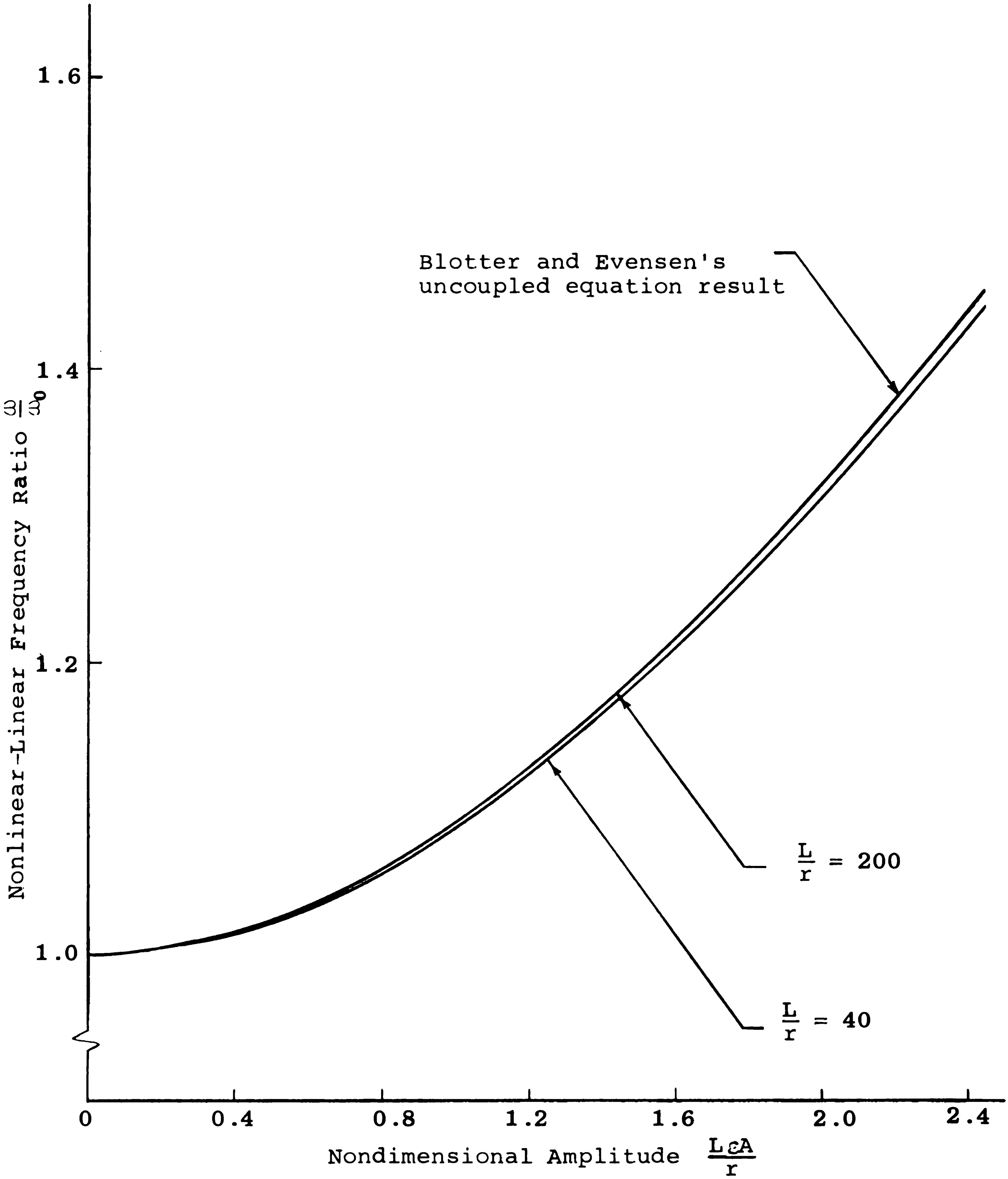


Figure 4.2.1. Frequency-amplitude curves for hinged-hinged beams.

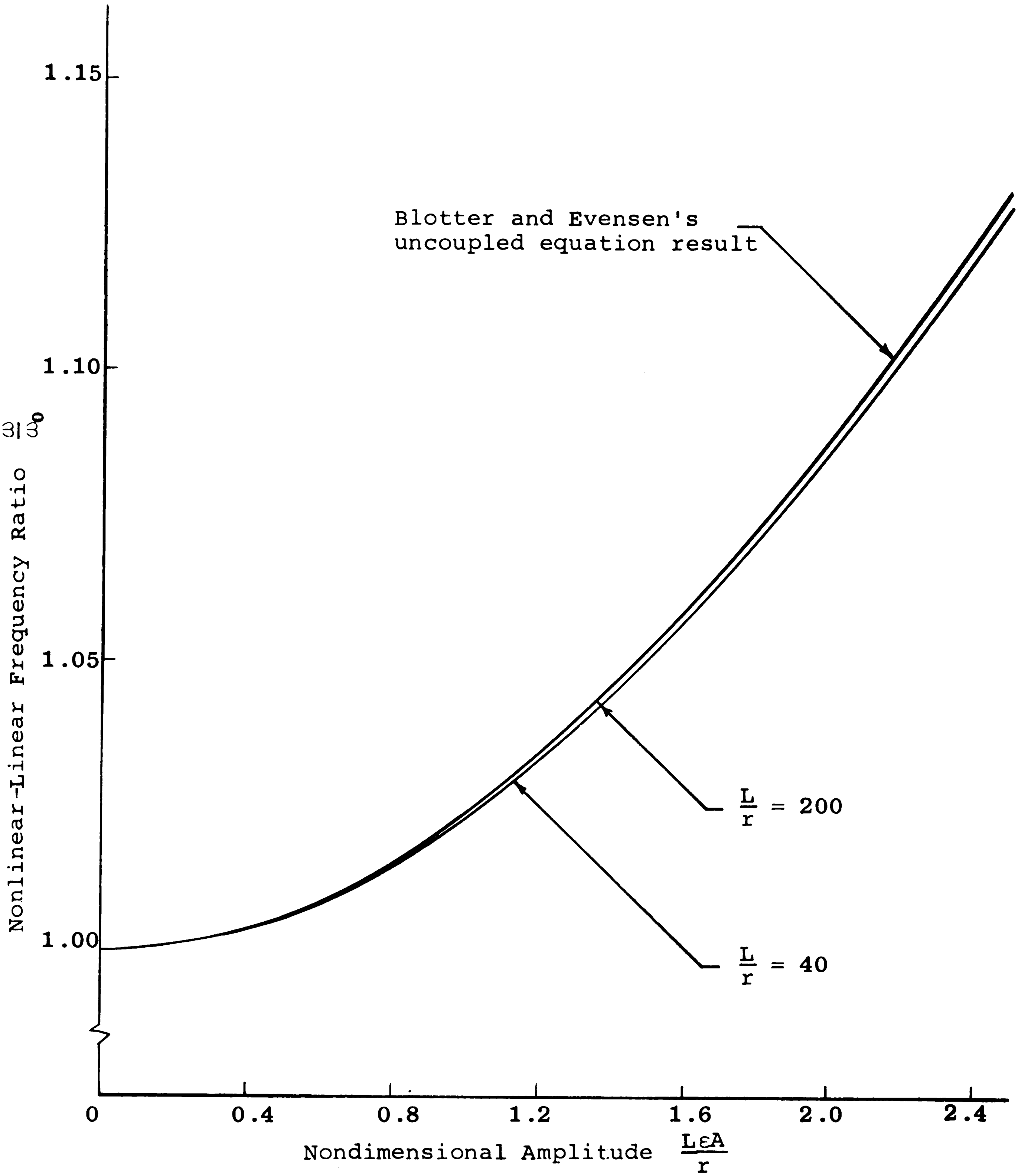


Figure 4.2.2. Frequency-amplitude curves for clamped-clamped beams.

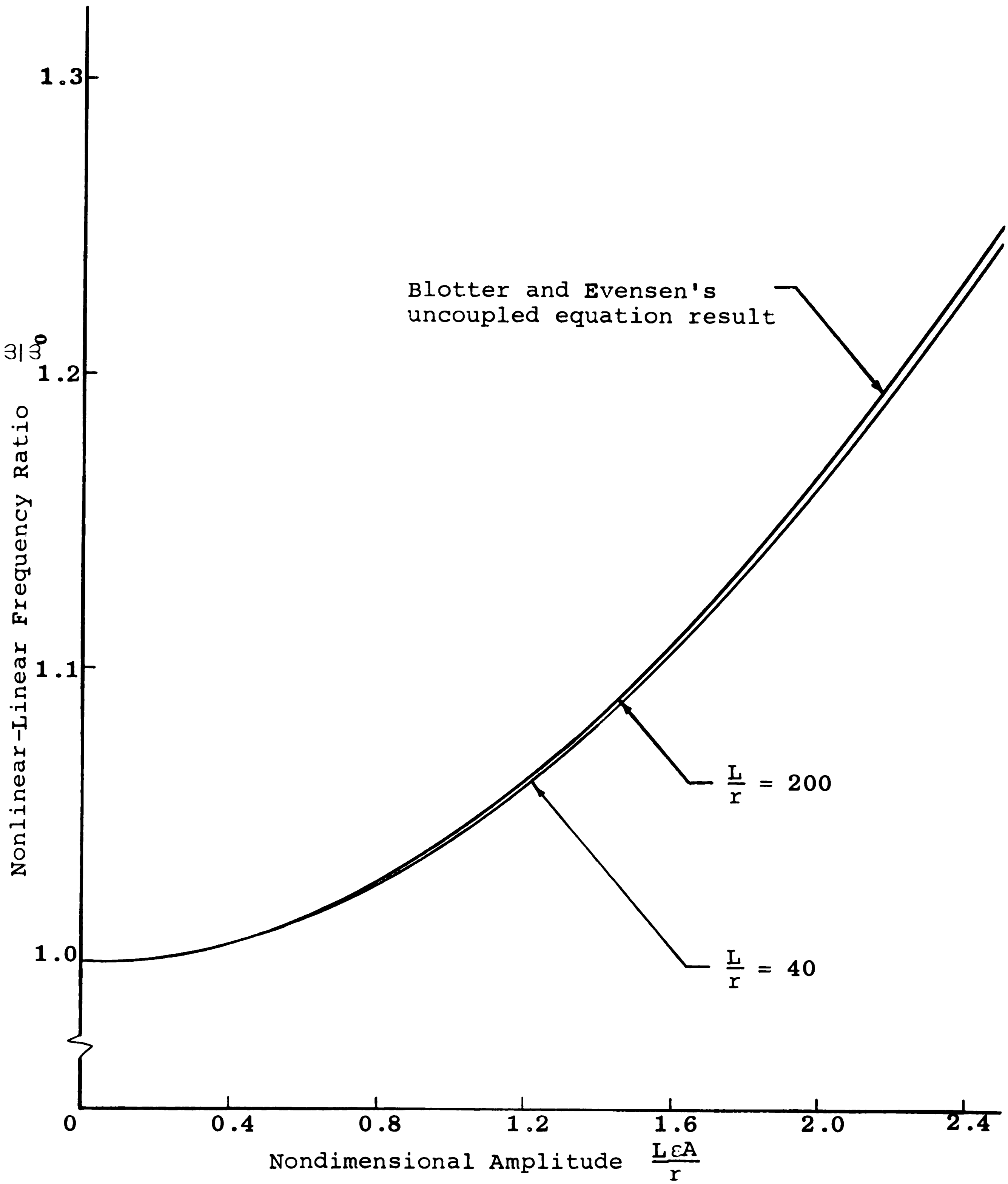


Figure 4.2.3. Frequency-amplitude curves for clamped-hinged beams.

(4.2.16) do. In all three cases, results based on (4.2.16) are shown for $\frac{L}{r} = 200$, representing a thin beam and for $\frac{L}{r} = 40$, representing neither a thin nor a thick beam. It is to be noted that for small $\frac{L}{r}$ values, curves based on the more accurate analysis lie below those of Blotter and Evensen. It is interesting to note also that Blotter and Evensen's results coincide with those of the present analysis only for beams with large slenderness ratio $(\frac{L}{r} \simeq 200)$. This fact can also be established analytically as follows. For large λ, the expansion coefficients for the longitudinal displacement of a hinged-hinged beam as given in equations (4.2.5) and (4.2.6) can be approximated by

$$C_{02}^{(1)} = C_{22}^{(1)} = -\frac{\pi}{16}\sqrt{2}\,A_{11}^{2} \qquad (4.2.25)$$

Also, $F(\lambda)$ as given in equation (4.2.8) approaches unity. The second order frequency-amplitude relation of a hinged-hinged beam as given in equation (4.2.7) becomes

$$\omega_2^2 = \frac{3}{8}\beta_1^4\left(\frac{L}{r}A_{11}\right)^2 \qquad (4.2.26)$$

which is identical with Evensen's expression.

On the basis of the above discussion, the following conclusion can therefore be drawn:

It is reasonable to neglect the longitudinal inertia effect and to assume the first strain invariant independent of the spatial variable of the beam only for slender beams vibrating transversely principally.

4.3. Clamped Circular Membrane with Initial Tension

As an example for circular membranes with initial tension and immovable edge, let us consider a circular membrane clamped at the boundary $\zeta = 1$. This implies that both the transverse and the radial displacements at the rim are zero. The boundary conditions are therefore

$$\left.\begin{aligned} &\eta \text{ is bounded as } \zeta \rightarrow 0_+ \\ &\eta(1,\tau) = 0 \end{aligned}\right\} \quad (4.3.1)$$

$$\left.\begin{aligned} &\gamma \text{ is bounded as } \zeta \rightarrow 0_+ \\ &\gamma(1,\tau) = 0 \end{aligned}\right\} \quad (4.3.2)$$

By the orthogonality condition (2.2.10) with the weighting function $r(\zeta)$ equal to ζ, the normalized transverse spatial eigenfunctions and their corresponding eigenvalues satisfying the differential equation (2.2.7) and the boundary conditions (4.3.1) follow as

$$\left.\begin{aligned} &V_k(\zeta) = \frac{\sqrt{2}}{J_1(j_{0k})} J_0(j_{0k}\zeta) \\ &\Omega_k^2 = \frac{e_0}{1-\mu} j_{0k}^2 \\ &k = 1,2,3,\ldots.. \end{aligned}\right\} \quad (4.3.3)$$

where $J_0(j_{0k}\zeta)$ and $J_1(j_{0k})$ are the Bessel functions of the first kind, of order zero and order one respectively. j_{0k} is the k-th positive zero of $J_0(j_{0k}) = 0$.

Similarly by the orthogonality condition (2.2.16) with the weighting function $s(\zeta)$ equal to ζ, the normalized radial spatial eigenfunctions and their corresponding eigenvalues satisfying the differential equation (2.2.13) and the boundary conditions (4.3.2) are

$$\left.\begin{aligned} \Gamma_n(\zeta) &= \frac{\sqrt{2}}{J_2(j_{1n})} J_1(j_{1n}\zeta) \\ \Lambda_n^2 &= \frac{1}{1-\mu^2} j_{1n}^2 \\ n &= 1, 2, 3, \ldots\ldots \end{aligned}\right\} \quad (4.3.4)$$

where j_{1n} is the n-th positive zero of $J_1(j_{1n}) = 0$.

Numerical values of j_{0k} and j_{1n} have been calculated and tabulated in [11].

Upon substituting equations (4.3.3) and (4.3.4) into (3.3.19) and (3.3.20), and performing the integrations on the computer, one obtains the first order expansion coefficients for the radial displacement. The results of the first nine coefficients are tabulated as a function of e_0 in Tables 4.3.1 and 4.3.2.

From equation (3.3.25), the second order frequency-amplitude relation follows as

$$\omega_2^2 = \omega_0^2 \, F(e_0) \left(\frac{A_{11}}{\sqrt{e_0}}\right)^2 \quad (4.3.5)$$

where

Table 4.3.1. Numerical values of the expansion coefficients $C_{0n}^{(1)}$ of clamped circular membranes with any initial strain.*

n	1	2	3	4	5	6	7	8	9
$C_{0n}^{(1)}$	0.29345	-0.02751	-0.00637	-0.00256	-0.00129	-0.00075	-0.00047	-0.00032	-0.00022

*All values multiplied by A_{11}^2.

Table 4.3.2. Numerical values of the expansion coefficients $C_{2n}^{(1)}$ of clamped circular membranes.*

$C_{2n}^{(1)}$ n / e_0	1	2	3	4	5	6	7	8	9
0.01	0.29959	-0.02768	-0.00639	-0.00256	-0.00129	-0.00075	-0.00047	-0.00032	-0.00022
0.1	0.36904	-0.02930	-0.00656	-0.00260	-0.00131	-0.00075	-0.00047	-0.00032	-0.00022
0.4	1.62403	0.03641	-0.00721	-0.00274	-0.00135	-0.00077	-0.00048	-0.00032	-0.00023
0.5	-12.16003	0.03961	-0.00746	-0.00279	-0.00137	-0.00078	-0.00048	-0.00032	-0.00023
1	-0.27994	0.07072	-0.00898	-0.00308	-0.00145	-0.00081	-0.00050	-0.00033	-0.00023
10	-0.01506	0.00538	0.00334	0.00368	0.01190	-0.00341	-0.00112	-0.00057	-0.00035
100	-0.00144	0.00046	0.00023	0.00016	0.00013	0.00011	0.00010	0.00010	0.00009

*All values multiplied by A_{11}^2.

$$F(e_0) = -\frac{1}{(1+\mu)j_{01}^2}\int_0^1 \{\frac{1}{A_{11}^2}\sum_{q=1}^{\infty}(C_{0q}^{(1)} + \frac{1}{2}C_{2q}^{(1)})[\frac{\mu}{\zeta}V_1''\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta}V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8}(3V_1'^2V_1'' + \frac{1}{\zeta}V_1'^3)\}\zeta V_1 \, d\zeta \qquad (4.3.6)$$

Table 4.3.3 shows how closely the truncated series represents the function $F(e_0)$ with the Poisson's ratio μ equal to 0.3.

Table 4.3.3. Convergence of $F(e_0)$ of clamped circular membranes.

e_0 \ $F(e_0)$ \ N	3	6	9
0.01	0.35245	0.35225	0.35223
0.1	0.34164	0.34144	0.34141
0.4	0.14744	0.14724	0.14722
0.5	2.27625	2.27605	2.27602
1	0.43980	0.43960	0.43957
10	0.40288	0.40292	0.40289
100	0.40046	0.40034	0.40032

From equation (2.2.3), the frequency-amplitude relations including terms up to ε^2 are given by

$$\omega^2 = \omega_0^2 \, [1 + F(e_0)\,(\frac{A_{11}}{\sqrt{e_0}})^2] \qquad (4.3.7)$$

The ratio of the nonlinear to the linear period then follows as

$$\frac{T^*}{T} = \frac{1}{\left[1 + \frac{F(e_0)}{|V_1|^2_{max}}\left(\frac{\varepsilon A}{\sqrt{e_0}}\right)^2\right]^{1/2}} \tag{4.3.8}$$

where

$$A = A_{11} \; |V_1|_{max} \tag{4.3.9}$$

Equation (4.3.8) again shows that the nonlinear frequencies increase (nonlinear periods decrease) with increasing amplitudes.

Chobotov and Binder [13] have obtained the pair of coupled nonlinear equations of motion for an axisymmetric clamped membrane with initial tension by summing the changes in the membrane forces due to radial and transverse displacements in the respective directions. Without considering the effects due to the radial inertia term, a perturbation procedure and the Ritz-Galerkin technique was employed to solve approximately the pair of coupled nonlinear partial differential equations. Their results can be used for the purpose of comparison with the present analysis. Also, it is interesting to investigate how the Berger's hypothesis will effect these results. Upon neglecting terms containing the second strain invariant e_2, equation (3.1.15) is modified as follows

$$F(u_t, w_t, u, u_r, w, w_r) = r\{\frac{\rho h}{2}(u^2{}_t + w_t^2)$$

$$-[N_0 e_0 + N_0 e + \frac{Eh}{2(1-\mu^2)} e^2]\} \tag{4.3.10}$$

Application of equations (3.1.20) and (3.1.21) to (4.3.10) yields

$$\nabla^2 w - \frac{\rho h}{N_0}\frac{\partial^2 w}{\partial t^2} + \frac{Eh}{(1-\mu^2)N_0}\left[e\,\nabla^2 w + \frac{\partial w}{\partial r}\frac{\partial e}{\partial r}\right] = 0 \qquad (4.3.11)$$

$$-\frac{\rho(1-\mu^2)}{E}\frac{\partial^2 u}{\partial t^2} + \frac{\partial e}{\partial r} = 0 \qquad (4.3.12)$$

When the radial inertia term is neglected, equation (4.3.12) becomes

$$\frac{\partial e}{\partial r} = 0 \qquad (4.3.13)$$

Thus, the first strain invariant, e, is independent of r. It is recalled that e for an axisymmetric membrane is defined as

$$e = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} \qquad (3.1.6)$$

Multiplying both sides of equation (3.1.6) by $rdrd\theta$ and integrating over the area, one obtains

$$e = \frac{2}{a^2}\, ru\Big|_0^a + \frac{1}{a^2}\int_0^a r\left(\frac{\partial w}{\partial r}\right)^2 dr \qquad (4.3.14)$$

The vanishing of u at the boundary leads to

$$e = \frac{1}{a^2}\int_0^a r\left(\frac{\partial w}{\partial r}\right)^2 dr \qquad (4.3.15)$$

In view of equations (4.3.13) and (4.3.15), equation (4.3.11) now becomes uncoupled as follows

$$\nabla^2 w - \frac{\rho h}{N_0}\frac{\partial^2 w}{\partial t^2} + \frac{Eh}{(1-\mu^2)N_0 a^2}\nabla^2 w\int_0^a r\left(\frac{\partial w}{\partial r}\right)^2 dr = 0 \qquad (4.3.16)$$

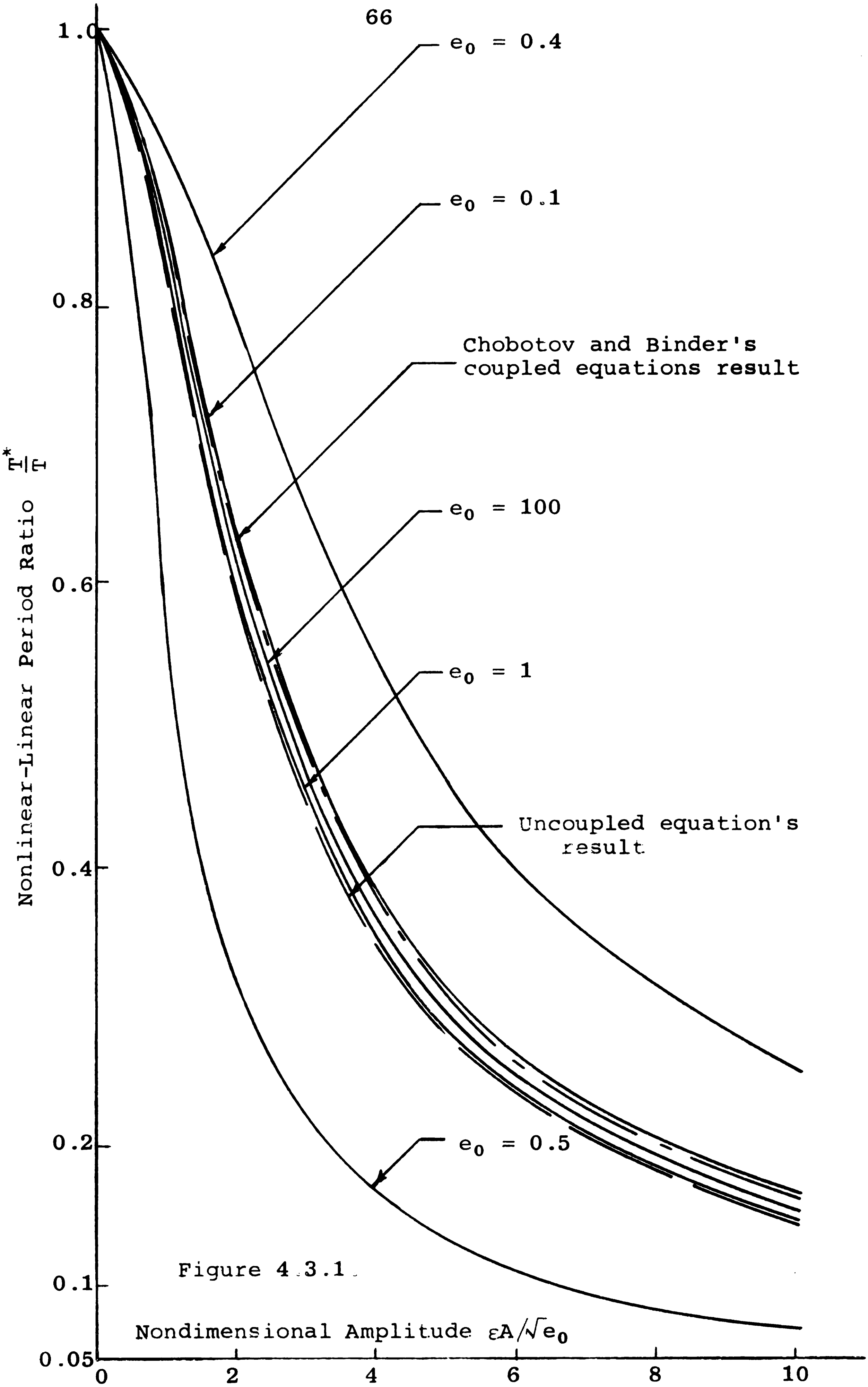
Although (4.3.16) is much easier to solve than the fully coupled pair of equations (3.3.1), the hypothesis that the term containing e_2 in equation (3.1.15) may actually be neglected lacks a physical justification. It is expected that, for a more accurate calculation of the membrane stresses from the displacements, the pair of coupled non-linear partial differential equations as given by equations (3.3.1a), (3.3.1b) instead of (4.3.11) and (4.3.12) must be employed.

Numerical results for the ratio of the nonlinear to the linear period are plotted in Figure 4.3.1 versus the dimensionaless amplitude $(\frac{\varepsilon A}{\sqrt{e_0}})$. With initial strain e_0 less than 0.487, the curves lie above those of Chobotov and Binder. It is seen that, as e_0 becomes small, they approach the latter. Physically, this is fairly easy to justify. The results of Chobotov and Binder were obtained on neglecting the radial inertia term. Since $e_0 = 0.1$, say, corresponds to an axisymmetric membrane with a small initial strain. The contribution of the radial inertia term in equation (3.3.6b) is thus negligible. This fact can be established as follows. For small initial strain, the expansion coefficients $C_{2n}^{(1)}$ for the radial displacement as given in equation (3.3.20) can be approximated by

$$C_{2n}^{(1)} = \frac{A_1^2}{2\Lambda_n^2} I = C_{0n}^{(1)} \qquad (4.3.17)$$

Thus the results for the ratio of the nonlinear to the linear period obtained by (4.3.8) are found to be reasonably

Figure 4.3.1. Ratio of nonlinear to linear period vs. nondimensional amplitude for clamped circular membranes with initial strain.



$e_0 = 0.4$
$e_0 = 0.1$
Chobotov and Binder's coupled equations result
$e_0 = 100$
$e_0 = 1$
Uncoupled equation's result
$e_0 = 0.5$
Nonlinear-Linear Period Ratio $\frac{T^*}{T}$
1.0
0.8
0.6
0.4
0.2
0.1
0.05
Nondimensional Amplitude $\varepsilon A/\sqrt{e_0}$
0
2
4
6
8
10

Figure 4.3.1

close to those of Chobotov and Binder. On the other hand, the curves for e_0 greater than 0.478 lie below those of Chobotov and Binder. This is because, as e_0 becomes large, say, $e_0 = 100$, the expansion coefficients $C_{2n}^{(1)}$ for the radial displacement change sign (see Table 4.3.2) and can be approximated by

$$C_{2n}^{(1)} = -\frac{A_{11}^2}{8\Omega_1^2} I \tag{4.3.18}$$

The function $F(e_0)$ as given by (4.3.6) does not seem to be sensitive to the variations of the large initial strain (see Table 4.3.3). Thus the differences between the results obtained by (4.3.8) and those of Chobotov and Binder are about 2.5 per cent.

When $e_0 = 0.487$, it can be shown that $\Lambda_1^2 = 4\Omega_1^2$. The expansion coefficient $C_{21}^{(1)}$ for the radial displacement becomes unbounded because the provision $\Lambda_n^2 - m^2\Omega_k^2 \neq 0$ fails. Physically, this means that the frequency of the radial oscillation induced by transverse motions approaches to that of the purely radial oscillation of the membrane. The coefficients of the radial displacement become very large. This case must be treated by a modified perturbation method and will not be considered here.

The results obtained by neglecting the effects of the second strain invariant and the radial inertia are found to be reasonably close to that from the present analysis only when e_0 is near unity.

4.4. Circular Plate with Clamped Edge

In this section the problem of a circular plate with clamped edge is considered as an illustrative example of Section 3.4. The pair of coupled nonlinear partial differential equations of motion including the effects of the so-called second strain invariant as well as the radial inertia term is given by equations (3.4.5a) and (3.4.5b). The boundary conditions are

$$\left.\begin{aligned} &\eta \text{ is bounded as } \zeta \rightarrow 0_+ \\ &\eta(1,\tau) = 0 \\ &\eta'(1,\tau) = 0 \end{aligned}\right\} \qquad (4.4.1)$$

$$\left.\begin{aligned} &\gamma \text{ is bounded as } \zeta \rightarrow 0_+ \\ &\gamma(1,\tau) = 0 \end{aligned}\right\} \qquad (4.4.2)$$

The differential equation (2.2.7) together with the boundary conditions (4.4.1) yield the following transcendental equation for the eigenvalues:

$$J_1(\beta) + \frac{J_0(\beta)}{I_0(\beta)} I_1(\beta) = 0 \qquad (4.4.3)$$

where J_n and I_n are the Besel and the modified Bessel functions of the first kind. The subscript n refers to the order of these functions. By the orthogonality condition (2.2.10) with the weighting function $r(\zeta)$ equal to ζ, the normalized transverse spatial eigenfunctions and their corresponding eigenvalues are

$$\left.\begin{aligned} V_k(\zeta) &= C\ [J_0(\beta_k\zeta) - \frac{J_0(\beta_k)}{I_0(\beta_k)} I_0(\beta_k\zeta)] \\ \Omega_k^2 &= \beta_k^4 \\ k &= 1,2,3,\ \ldots\ldots \end{aligned}\right\} \quad (4.4.4)$$

where the normalized constant C is defined as

$$C = \frac{\sqrt{2}}{\left[J_1^2(\beta_k) + 2J_0^2(\beta_k) - \frac{J_0^2(\beta_k)}{I_0^2(\beta_k)} I_1^2(\beta_k)\right]^{1/2}} \quad (4.4.5)$$

Numerical values of β_k have been calculated and tabulated in [14].

The differential equation (2.2.13) together with the boundary conditions (4.4.2) yield the following transcendental equation for the eigenvalues:

$$J_1(j) = 0 \quad (4.4.6)$$

By the orthogonality condition (2.2.16) with the weighting function $s(\zeta)$ equal to ζ, the normalized radial spatial eigenfunctions and their corresponding eigenvalues are

$$\left.\begin{aligned} \Gamma_n(\zeta) &= \frac{\sqrt{2}}{J_2(j_{1n})} J_1(j_{1n}\zeta) \\ \Lambda_n^2 &= \lambda j_{1n}^2 \\ n &= 1,2,3,\ \ldots\ldots \end{aligned}\right\} \quad (4.4.7)$$

Upon introducing equations (4.4.4) and 4.4.7) into (3.4.18) and 3.4.19), and performing the integrations on the computer, one obtains the first order expansion coefficients for the radial displacement. The results of the first nine coefficients are tabulated as a function of $\frac{a}{h}$ and μ in Tables 4.4.1 and 4.4.2.

From equation (3.4.24), the second order frequency-amplitude relations follow as

$$\omega_2^2 = \omega_0^2 \, F(\tfrac{a}{h}, \mu)\lambda A_{11}^2 \tag{4.4.8}$$

where

$$F(\tfrac{a}{h},\mu) = -\frac{1}{\omega_0^2}\int_0^1 \{ \frac{1}{A_{11}^2} \sum_{q=1}^{\infty} (C_{0q}^{(1)} + \frac{1}{2} C_{2q}^{(1)}) [\frac{\mu}{\zeta} V_1'\Gamma_q$$

$$+ (\frac{1+\mu}{\zeta} V_1' + V_1'')\Gamma_q' + V_1'\Gamma_q'']$$

$$+ \frac{3}{8}(3V_1'^2 V_1'' + \frac{1}{\zeta} V_1'^3) \} \zeta V_1 \, d\zeta \tag{4.4.9}$$

Table 4.3.3 shows how closely the truncated series represents the function $F(\frac{a}{h},\mu)$.

Using equation (2.2.3) and recalling that λ is equal to $12(\frac{a}{h})^2$, the frequency-amplitude relations up to terms including ε^2 are given by

$$\omega^2 = \omega_0^2 \, [1 + 12F(\tfrac{a}{h},\mu)(\tfrac{a}{h} \varepsilon A_{11})^2] \tag{4.4.10}$$

The ratio of the nonlinear to the linear period then follows as

$$\frac{T^*}{T} = \frac{1}{\left[1 + 12\frac{F(\frac{a}{h},\mu)}{|V_1|_{max}^2} (\frac{a}{h} \varepsilon A)^2\right]^{1/2}} \tag{4.4.11}$$

where

$$A = A_{11} |V_1|_{max} \tag{4.4.12}$$

Table 4.4.1. Numerical values of expansion coefficients $C_{0n}^{(1)}$ of clamped circular plates with any $\frac{a}{h}$.*

$C_{0n}^{(1)}$ μ \ n	1	2	3	4	5	6	7	8	9
0.1	0.16170	-0.23173	-0.02068	-0.00501	-0.00170	-0.00070	-0.00033	-0.00017	-0.00010
0.2	0.12288	-0.22871	-0.02037	-0.00493	-0.00167	-0.00069	-0.00033	-0.00017	-0.00010
0.3	0.08405	-0.22568	-0.02006	-0.00486	-0.00165	-0.00068	-0.00032	-0.00017	-0.00009
0.4	0.04522	-0.22266	-0.01975	-0.00478	-0.00162	-0.00067	-0.00032	-0.00017	-0.00009

*All values multiplied by A_{11}^2.

Table 4.4.2. Numerical values of the expansion coefficients $C_{2n}^{(1)}$ of clamped circular plates.*

$C_{2n}^{(1)}$ n / $\frac{a}{h}$	1	2	3	4	5	6	7	8	9
μ=0.1									
10	0.16563	-0.23338	-0.02075	-0.00502	-0.00170	-0.00070	-0.00033	-0.00017	-0.00010
100	0.16174	-0.23174	-0.02068	-0.00501	-0.00170	-0.00070	-0.00033	-0.00017	-0.00010
μ=0.2									
10	0.12586	-0.23033	-0.02044	-0.00494	-0.00167	-0.00069	-0.00033	-0.00017	-0.00010
100	0.12291	-0.22872	-0.02037	-0.00493	-0.00167	-0.00069	-0.00033	-0.00017	-0.00010
μ=0.3									
10	0.08609	-0.22729	-0.02013	-0.00487	-0.00165	-0.00068	-0.00032	-0.00017	-0.00009
100	0.08407	-0.22570	-0.02006	-0.00486	-0.00165	-0.00068	-0.00032	-0.00017	-0.00009
μ=0.4									
10	0.04632	-0.22425	-0.01982	-0.00479	-0.00162	-0.00067	-0.00032	-0.00017	-0.00009
100	0.04523	-0.22268	-0.01975	-0.00478	-0.00162	-0.00067	-0.00032	-0.00017	-0.00009

*All values multiplied by A_{11}^2.

Table 4.4.3. Convergence of $F(\frac{a}{h},\mu)$ of clamped circular plates.

$F(\frac{a}{h},\mu)$ \ N $\frac{a}{h}$	3	6	9
μ=0.1			
10	0.02908	0.02905	0.02905
100	0.02913	0.02910	0.02910
μ=0.2			
10	0.03031	0.03028	0.03028
100	0.03035	0.03032	0.03032
μ=0.3			
10	0.03129	0.03127	0.03127
100	0.03133	0.03130	0.03130
μ=0.4			
10	0.03204	0.03202	0.03202
100	0.03207	0.03205	0.03205

Wah [5] and Blotter [7] have both studied nonlinear vibrations of circular plates using the Berger's hypothesis. Their results along with those calculated from equation (4.4.11) are plotted in Figures 4.4.1 through 4.4.4. The curves again reveal the general feature that the nonlinear periods decrease with increasing amplitudes. It should be noted that the results of the present analysis are displayed in terms of the Poisson's ratio μ whereas those of Wah and Blotter are independent of this ratio. This is because the last term $(1-\mu)Dw_r w_{rr}$ in equation (3.1.16) has no contribution to the equation governing the transverse motion of

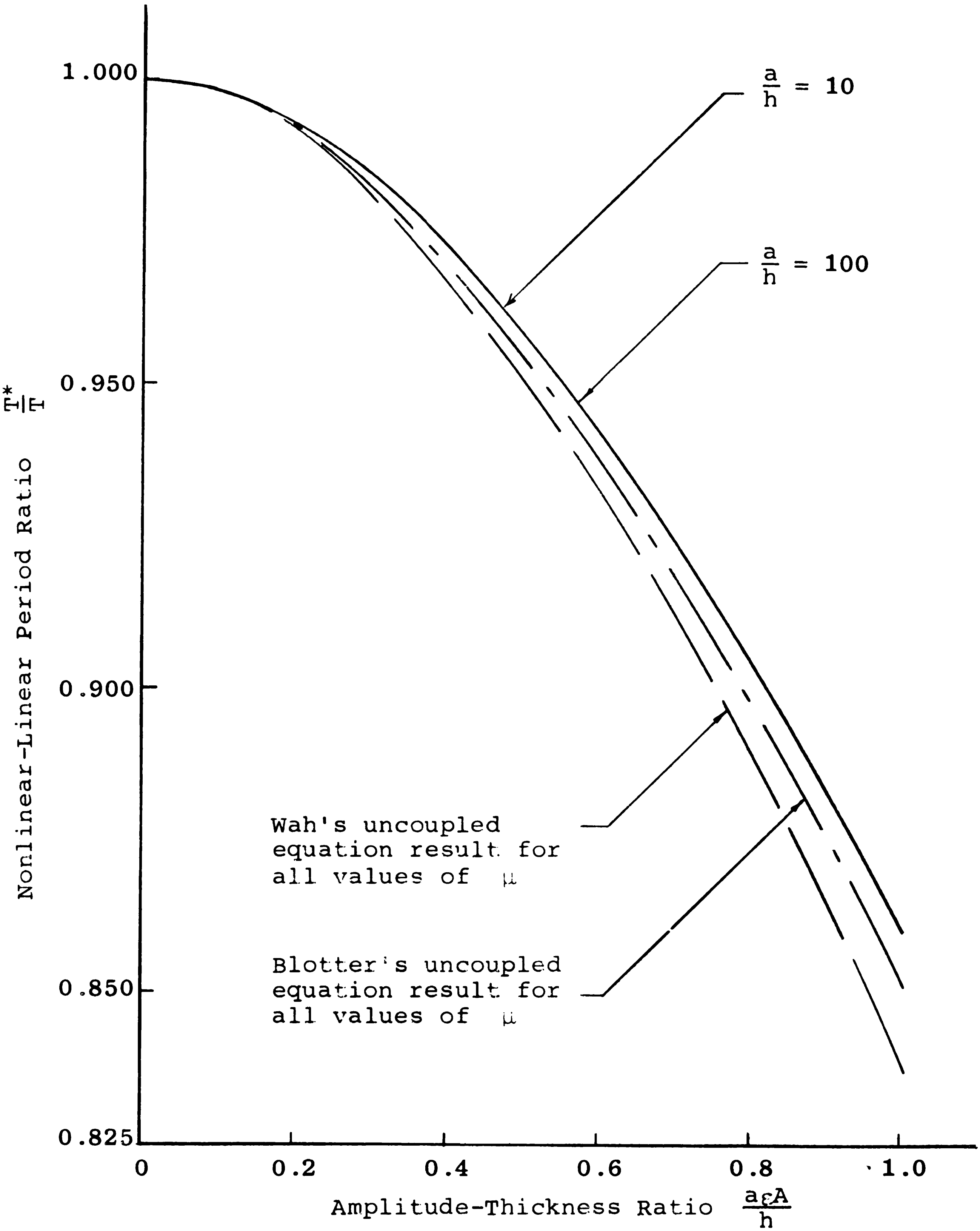


Figure 4.4.1. Ratio of nonlinear to linear period vs. ratio of amplitude to thickness for clamped circular plates with $\mu = 0.1$.

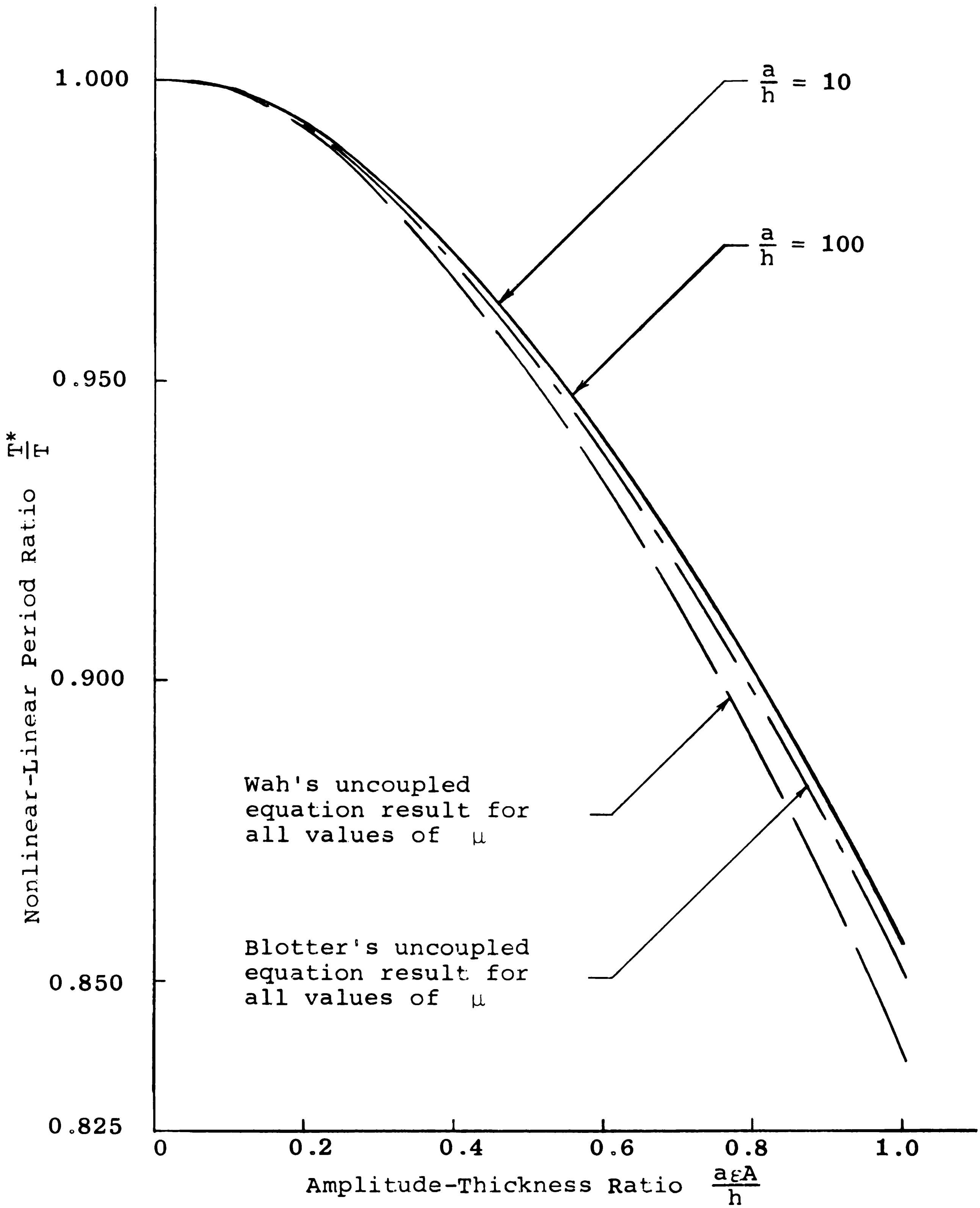


Figure 4.4.2. Ratio of nonlinear to linear period vs. ratio of amplitude to thickness for clamped circular plates with $\mu = 0.2$.

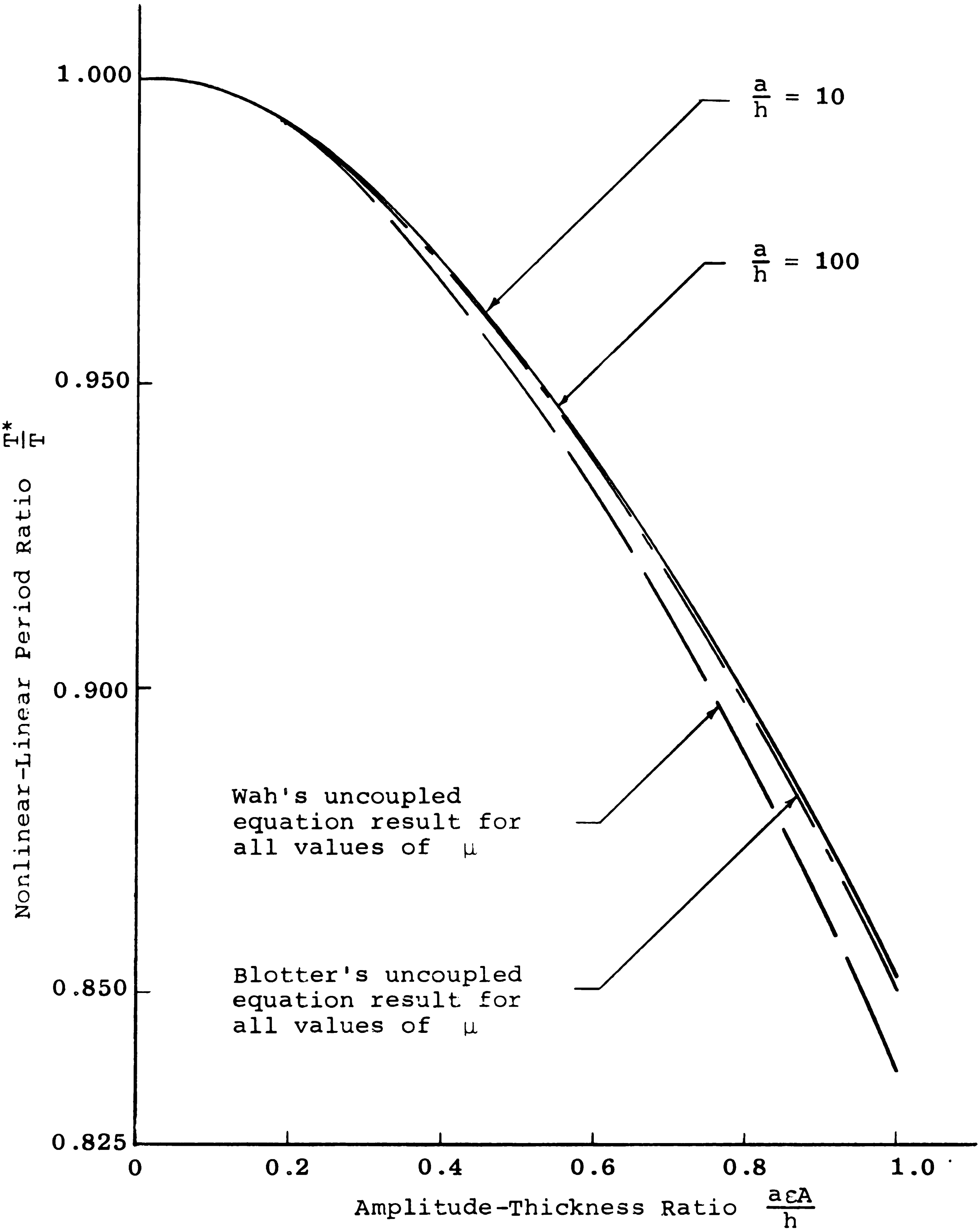


Figure 4.4.3. Ratio of nonlinear to linear period vs. ratio of amplitude to thickness for clamped circular plates with $\mu = 0.3$

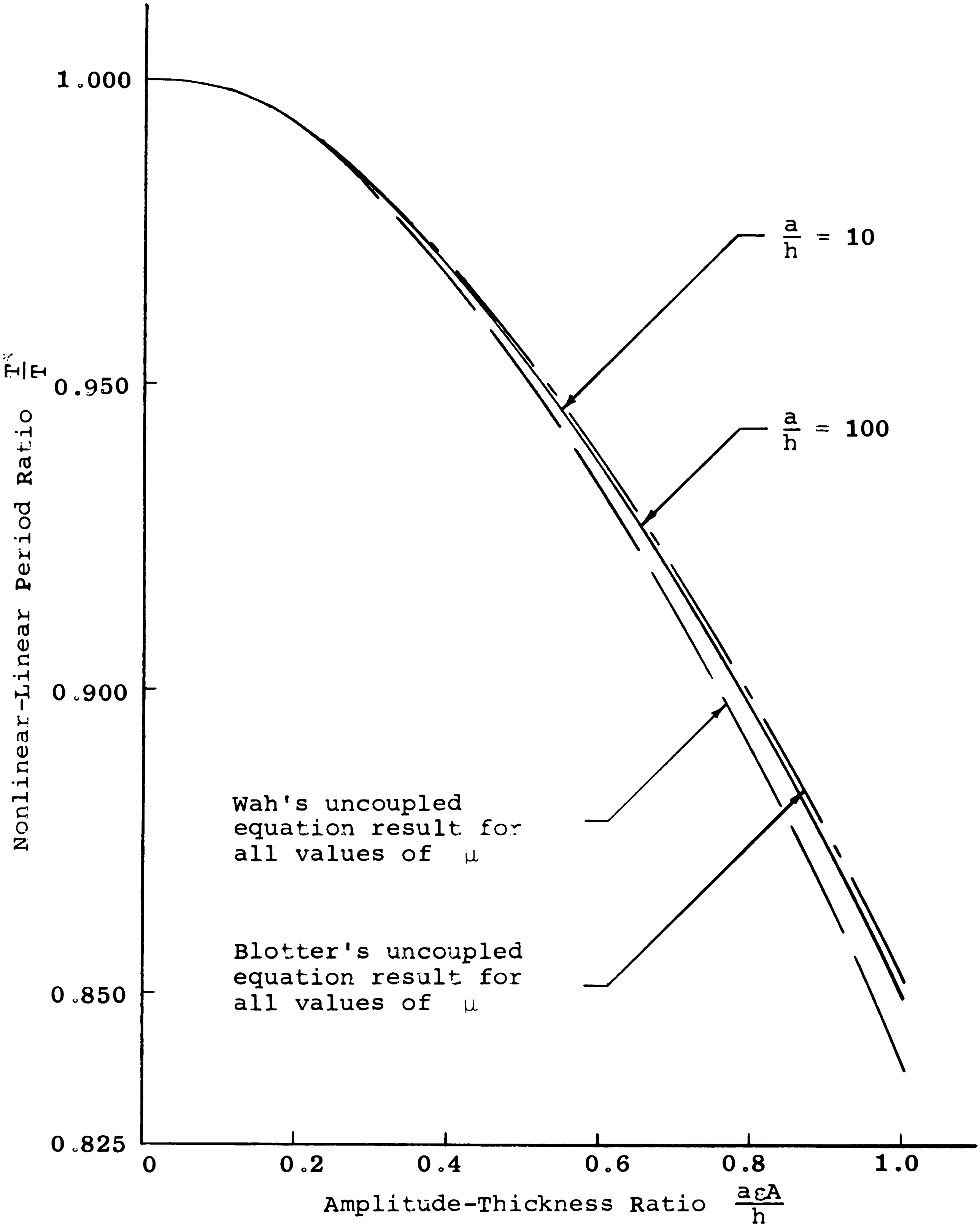


Figure 4.4.4. Ratio of nonlinear to linear period vs. ratio of amplitude to thickness for clamped circular plates with $\mu = 0.4$

the clamped circular plate. The neglecting of the second strain invariant of the middle surface leads their ratios of the nonlinear to the linear period to be independent of μ. In addition, the term corresponding to the radial inertia is ignored, their first strain invariant is constant throughout the plate. This yields only the approximate sum of the membrane stresses. To determine the variations of the membrane stress in the radial direction, the pair of coupled nonlinear partial differential equations as given by (3.4.1a) and (3.4.1b) must then be used.

The ratio $\frac{a}{h} = 100$ represents a very thin circular plate while $\frac{a}{h} = 10$ corresponds to a plate that is neither too thin nor too thick. The frequency-amplitude relations do not seem to be sensitive to the variations of this ratio. This fact is seen in Table 4.4.3. Also, the results of Wah and Blotter do not depend on this ratio. The curves for clamped plates with $\frac{a}{h} = 10$ and $\frac{a}{h} = 100$ are undistinguishable and are plotted in Figures 4.4.1 through 4.4.4. For μ equal to 0.1, 0.2, or 0.3, they lie above those of Wah and Blotter. For μ equal to 0.4, they lie above those of Wah but below those of Blotter. The periods obtained by Blotter are reasonably close to the present ones only when the Poisson's ratio μ is near 0.3 to 0.4.

V. SUMMARY AND CONCLUSIONS

In this thesis a method of solution of two coupled nonlinear partial differential equations is presented. It is one form of the perturbation technique. The pair of coupled nonlinear partial differential equations are first converted into a system of linear partial differential equations. The linear equations are then solved recursively using the method of eigenfunction expansions. The method is an extension of the one used in [7] for single partial differential equations and is motivated by the fact that many vibration problems for continuous structures are actually governed by coupled nonlinear partial differential equations, unless simplifying assumptions are made. Equations governing free vibrations of nonlinear continuous systems such as beams, circular membranes and circular plates are derived by means of Hamilton's principle. This results in equations that are both coupled and nonlinear and are of the type considered here. Using the method developed, approximate solutions for frequency-amplitude relations, out-of-plane as well as in-plane displacements up to second order are then obtained.

It is shown that, in general, uncoupling of these equations of motion is achieved by neglecting the effects of both the second invariant of the middle surface strains and the in-plane inertia. With the availability of the present results obtained from the solutions of the coupled equations, it is possible to assess the accuracy and validity of the various previously published results, especially those contained in [7], on beams, circular membranes and circular plates obtained from solving simplified, uncoupled equations of motion.

In the case of elastic beams with large slenderness ratio, the frequency-amplitude relations predicted by [7] are identical to those of the present analysis, while small deviations exist in the case of moderately thick beams ($\frac{L}{r} = 40$). Thus there is a maximum error of only 0.027 per cent for hinged-hinged beam (see Figure 4.2.1), a maximum error of only 0.014 per cent for clamped-clamped beam (see Figure 4.2.2), and a maximum error of only 0.081 per cent for clamped-hinged beam (see Figure 4.2.3). These observations tend to confirm the correctness of the assumptions of [7].

In the case of the elastic clamped membrane with small initial strain, say $e_0 = 0.1$ or smaller, the results agree with those of Chobotov and Binder. For large initial strain, say $e_0 = 100$ or larger, there is a difference of 2.5 per cent. For e_0 between 0.4 and 0.5, the results deviate substantially from those of Chobotov and Binder. The present perturbation method does not apply. The results obtained by using Berger's assumption are found to be reasonably close to that of the present analysis only when e_0 is near unity.

In the case of the thin elastic clamped plate, the results of [7] do not depend on the Poisson's ratio nor the ratio of $\frac{a}{h}$, while the results of the present more accurate analysis do. The periods predicted by [7] are reasonably close to the present ones when the Poisson's ratio μ is near 0.3 to 0.4. For μ equal to 0.1, there is a maximum error of 1.07 per cent.

The present work represents an extension of [7], in which the same perturbation method was applied only to a single nonlinear partial differential equation. The method is extended here to solve pairs of coupled nonlinear partial differential equations. It is rather remarkable that such an extension is accomplished without requiring too greater an effort than that required for solving single nonlinear partial differential equations. The pair of coupled nonlinear partial differential equations enable us to take into account the second invariant of the middle surface strains as well as the effects of the in-plane inertia of the continuous systems and thus lead to more accurate results.

Whereas the nonlinear frequency-amplitude relationships obtained here do not seem to deviate substantially from those previously obtained using simplified theories in general, it is clear, however, that solutions of the coupled equations do reveal more information on the motions of the continuous systems, such as that on the radial displacements and the distributions of the in-plane stresses in the radial direction.

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