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dissertation entitled

BIFURCATING PERIODIC SOLUTIONS OF A SMOOTHED PIECEWISE LINEAR DELAY DIFFERENTIAL EQUATION

presented by

Derek Graham Lane

has been accepted towards fulfillment
of the requirements for

PhD degree in Mathematics

Major professor

Date March 1, 1988

BIFURCATING PERIODIC SOLUTIONS OF A SMOOTHED PIECEWISE LINEAR DELAY DIFFERENTIAL EQUATION

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
of the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1988

ABSTRACT

BIFURCATING PERIODIC SOLUTIONS OF A SMOOTHED PIECEWISE LINEAR DELAY DIFFERENTIAL EQUATION

By

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The neighbourhood of a periodic solution of a delay differential equation $x'(t) = \alpha f(x(t-1))$ is studied. The degree of the periodic solution is shown to change at a large number of parameter values. Bifurcation of periodic orbits follows as a result, for arbitrarily large values of α.

Choosing f nearly piecewise linear simplifies formal calculations of α where the solution has neutral stability. The degree of the periodic orbit is studied through the spectrum of the linearization of the flow around the periodic orbit. Using ideas of Walther this is reduced to a set of ordinary differential equations with boundary conditions. A two part argument based on the special choice of f gives the approximate positions of α where the periodic solution can change its degree. The ordinary differential equation is studied by estimating the effect of smoothing, and calculations of the motion of a critical eigenvalue are obtained by matching arguments for maps related to the ordinary differential equation. The calculations are carried out by hand and checked symbolically by computer.

ACKNOWLEDGEMENTS

I would like to acknowledge the assistance and patience of my advisor Professor Shui Nee Chow over the prolonged preparation of this work. My thanks go also to Professor John Mallet-Paret for his aid in exploring differential equations, and to Professor Hans Otto Walther for encouragement and suggestions.

I am grateful to Professors Sheldon Axler, Charles Seebeck Jr., and Clifford Weil for their patient reading of my thesis and attending my defence.

I wish to thank Ms. Margaret Memory for proofing an early draft of the thesis, and Ms. Barbara Miller, the secretary of the graduate office of the Department of Mathematics for her help with things organizational.

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INTRODUCTION

The equation

$$x'(t) = g(x(t-1)) \tag{0.1}$$

with a continuous function $g : R \rightarrow R$ satisfying $g(0) = 0$ and $g'(0) < 0$ has been studied as a model of physiological control processes. For this application see Mackey and Glass [5], for other applications see, for example, the references in Walther [11]. Numerical simulation of shows that there are useful choices of g with complex behaviour.

An enlightening way of studying equation (0.1) is to choose $g = \alpha f$ where f satisfies $f'(0) = -1$ and α varies over the positive reals. There is a Hopf bifurcation at $\alpha = \pi/2$, giving rise to a continuum of periodic solutions of equation (0.1); with extra conditions on f these solutions are stable for α close to $\pi/2$, and there are conditions on f which imply global stability among an important class of solutions.

Other approaches to the study of equation (0.1) exist. One may choose f carefully to simplify the study of solutions to equation (0.1). Choosing f as an odd function allows the discussion of special period 4 solutions branching from the Hopf point as solutions of an ordinary differential equation with boundary conditions. see Kaplan and Yorke [4]. One may study the equations for large α, searching for fix points of cone maps using Schauder's fix point Theorem, as in Nussbaum [10]. Finally, one may choose f so simple that all solutions may be explicitly calculated, hoping for complex behaviour even with a choice of f close to a step function, see, for example, Walther [12].

We have chosen an odd function which is nearly piecewise linear to gain insight into a neighbourhood of the bifurcating periodic orbit that can be tracked using techniques from Kaplan and Yorke [4]. For large α we obtain many new periodic solutions by a degree argument, these solutions being very different from those found by Nussbaum [10]. The choice of f is tailored to simplify the ordinary differential equations Walther uses to study the degree of the special period 4 solution.

A difficulty with this choice of f is the smoothing necessary for the applicability of the standard theory, which requires smooth f for local analysis around the periodic orbit. All of the arguments are affected by this problem, some more than others.

Chapter 1 is devoted to setting up enough estimates on smoothing's effects to get possible linearized neutral stability for the periodic solution. In Section 3 we show that some techniques from Walther [11] give us a unique periodic solution satisfying the equations of Yorke [4]. We also find estimates of the size of various ies quantities needed later for the study of change of degree. Section 4 introduces a natural recasting of the problem in terms of simple maps, and a simple argument available in this case provides an alternative to part of Walther's analysis of the problem. In Sections 5 and 6 enough properties of the maps are derived to find many α's where the periodic solution could have linearized neutral stability. We use the special choice of f heavily here. In particular, a fortunate property of f makes the search for points of possible neutral stability split into two parts, simplifying the search noticeably. This part of the thesis is much less affected by smoothing f than is Chapter 2.

Chapter 2 is devoted to showing that the degree of the periodic orbit does indeed change near the α's found in Chapter 1; it is here that the main technical difficulties arise. The first problem, solved in Section 4, is to find an expression for the contribution of the smoothing to the rate of change of an appropriate eigenvalue. The second problem, solved for technical reasons in Section 3, is to find a key quantity in the calculation of Section 4. This requires a matching argument in the three natural maps defined in Chapter 1, Section 4, and estimates from Chapter 1 to justify the matching. In Section 4 of Chapter 2 the main Theorem of the thesis is proved, quoting Chapter 3, the straightforward calculations needed to finish the proof.

The last few pages of Chapter 3 are a record of a symbolic calculation of the relevant quantities by machine. This was done to check on hand calculations. Numeric calculations are restricted in application for large α's since quantities appear in the analyses of exponentially small order in α.

Chapter 1

FINDING BIFURCATION PARAMETERS FOR $x'(t) = \alpha f(x(t-1))$

1. Introduction

Walther [11] has studied a class of equations of the form

$$x'(t) = \alpha f(x(t-1)) \tag{1.1}$$

and has shown that a periodic solution of this equation undergoes at least one bifurcation as α varies. For particular choices of f close to a piecewise linear function which almost satisfy the requirements of Walther's theorems we are able to show the periodic solution undergoes a large number of bifurcations as α varies. In the next four sections we introduce the notation necessary to study the periodic solution and to study its stability. The major objects of interest are the periodic solution $x^\alpha(t)$, and a Poincare map P with an associated degree found by a study of equations (4.3) and (4.4). These equations are a two-parameter two-dimensional linear time dependent set of ordinary differential equations with boundary conditions. The notation and framework of study in these sections follows Walther[11], which uses ideas of Kaplan and Yorke [4] to find the periodic solutions, and special properties of the solution to simplify the study of its stability.

Our choice of f further simplifies the analysis of stability enough so that in sections 5 and 6 we are able to find choices of α where x^α has linearized neutral stability. Change of stability and hence bifurcation is studied in chapter 2.

2. Definition of Flow and Solutions

We consider a one-parameter family of delay differential equations:

$$x'(t) = \alpha f(x(t-1)) \tag{2.1}$$

where α, $x \in R$ and α, the parameter, is chosen positive.

We would like to study this family with f equal to the piecewise linear function f^* whose graph is given in Figure 1. This f^* is simple enough so that formal calculations related to linearized stability of a periodic solution are tractable.

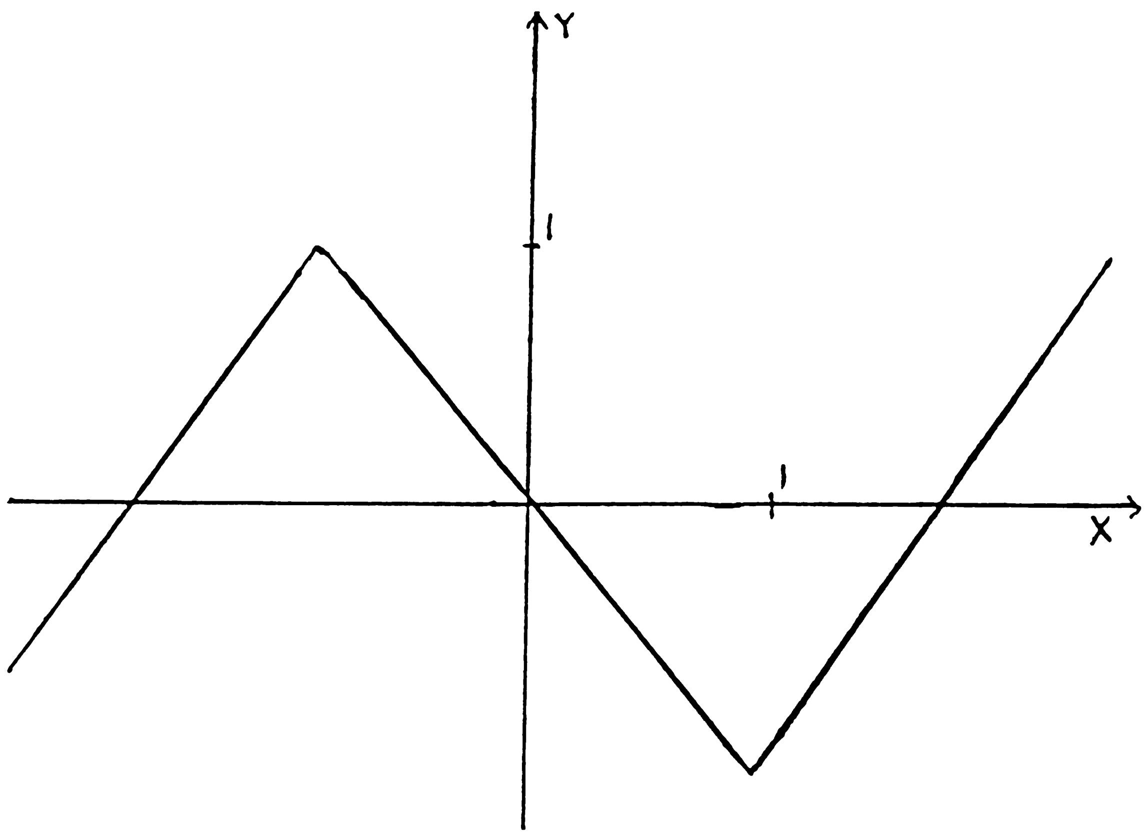

Figure 1

Unfortunately as f^* is not C^1 we know of no linearized stability theory for the flow given by equation (2.1). We approximate f^* by f_σ, a C^1 function to which the theory in, e.g. Hale [3] is directly applicable. We were guided by the formal computations for f^* throughout our work, however. f_σ's graph is given in Figure 2.

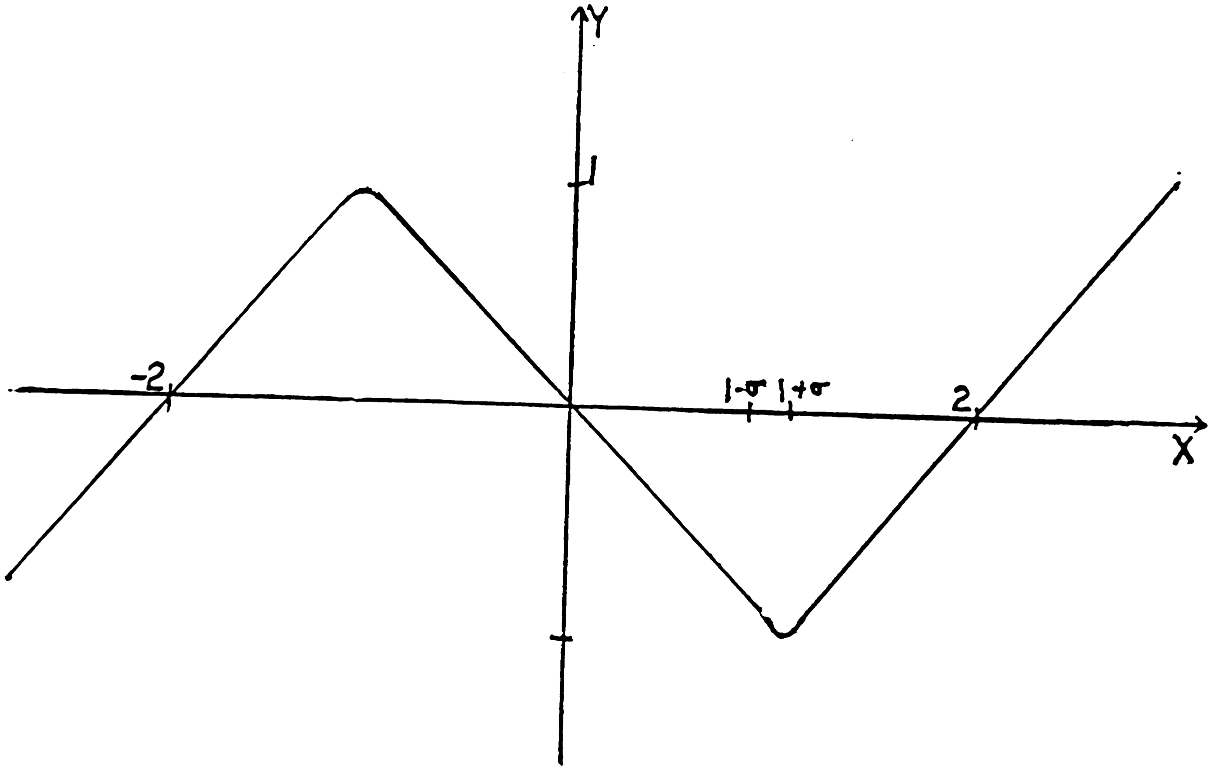

Figure 2

For definiteness we define f_σ by:

$$f_\sigma(x) = \begin{cases} -x, & \text{if } |x| < 1-\sigma \\ x-2, & \text{if } x \geq 1-\sigma \\ x+2, & \text{if } x \leq -1+\sigma \\ \frac{-1}{2\sigma}x^2 - \frac{1}{\sigma}x + 1 - \frac{\sigma}{2} - \frac{1}{2\sigma}, & \text{if } x \in [-1-\sigma, -1+\sigma] \\ \frac{1}{2\sigma}x^2 - \frac{1}{\sigma}x + 1 - \frac{\sigma}{2} - \frac{1}{2\sigma}, & \text{if } x \in [1-\sigma, 1+\sigma]. \end{cases}$$

We have used a quadratic function to smooth f^*. Our theorems will apply when σ is small and positive, and we will drop the dependence of f_σ on σ in our notation where possible. A solution of equation (2.1) with initial condition $\phi \in C[-1,0]$ is a function $x : [-1,T] \to \mathrm{R}$ satisfying $x|[-1,0] = \phi$ and $x'(t) = \alpha f(x(t-1))$ for all $t \in [0, T)$, for some $T > 0$.

Because $xf(x) \leq 0$ for $x \in (-2,2)$ the slowly oscillating solutions defined next have particular importance. Kaplan and Yorke [4], Chow, Diekmann and Mallet-

Paret[1], Chow and Walther[2], Mallet-Paret[5], Nussbaum[10]and others have studied this class of solutions in other contexts.

Definition 1.A slowly oscillating solution x is a solution of equation (2.1) where if $x(t_1) = x(t_2) = 0$ and $t_1 \neq t_2$, then $|t_1 - t_2| > 1$.

3. Construction and Study of a Periodic Solution

In this section we show that equation (1.1) has a one-parameter family of periodic solutions, one for each α. We introduce and study the linearized stability of these periodic solutions as α varies. We do this by constructing P, a Poincare map as in Walther[11], p.272.

Let α be given. We search for solutions x of equation (1.1) satisfying

$$x(t-2) = -x(t). \tag{3.1}$$

These solutions, if they exist, are periodic of period 4. Defining $y(t) = x(t-1)$ we find that if x is a solution of equation (1.1) satisfying equation (3.1) then it also satisfies:

$$\begin{aligned} x'(t) &= \alpha f(y(t)), \\ y'(t) &= -\alpha f(x(t)), \end{aligned} \tag{3.2}$$

with boundary conditions

$$\begin{aligned} x(0) &= y(1), \\ y(0) &= -x(1). \end{aligned} \tag{3.3}$$

See Walther[11] and his references for proofs and history of this method. The periodic solutions of the delay equation(1.1) satisfying the symmetry defined by equation (1) are shown by Walther[11] pp.277-278 to also satisfy equation (3.2) and equation (3.3). Conversely, in a class of functions close to f, Walther has shown that any solution of equation (3.2) and equation (3.3) is a periodic solution of equation (1.1). Kaplan and Yorke[4] show with slightly different hypotheses on f that there is at most one solution of equations (3.2) and (3.3) satisfying $y(0) = 0$ and $2 > x(0) > 0$. We next modify their existence proofs for our case, starting with some notation and lemmas. Let η be any element of R. Let $\bar{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\bar{x}(\alpha, \eta, t)$ be $\bar{x}(t)$ where x and y satisfy equations (3.2) and (3.4) below:

$$\begin{aligned} x(0) &= 2 - \eta, \\ y(0) &= 0. \end{aligned} \tag{3.4}$$

Lemma 1. For all α, η, t

$$\bar{x}(\alpha, \eta, t) = \bar{x}(1, \eta, \alpha t)$$

Proof: $\bar{x}(1, \eta, \alpha t)$ satisfies equation (3.2) and has the same initial values as $\bar{x}(\alpha, \eta, t)$. Since f is Lipshitz, these solutions must be identical. ∎

We would like to show that η determined by the condition

$$x(\alpha, \eta, 1) = 0 \qquad x(\alpha, \eta, t) > 0 \quad \text{if } t \epsilon\ (0, 1) \tag{3.5}$$

is a function of α, and to bound $\frac{\partial \eta}{\partial \alpha}$ away from zero independently of σ. We do this in the next set of lemmas. We define $\tilde{T} = \{\bar{x} : x + y \leq 1, x, y \geq 0\}$. For $x \in \tilde{T}$ we define $\theta(\bar{x})$ by $\tan \theta(\bar{x}) = \frac{y}{x}$. We give a result implying monotonicity of η as a function of α :

Theorem 2. Let $f(x)$ be an odd, continuous function such that $xf(x) < 0$ on [0,2] and $f(x) = -x$ for $|x| \leq b$. Assume $f(x)x^{-1}$ is strictly monotonic increasing for $x > 1$. If $\bar{x}_1$ and $\bar{x}_2$ satisfy equation (3.2), $\bar{x}_1(t_1) = r\bar{x}_2(t_1)$ for some $r > 1$ and $\bar{x}_i(t_1) \epsilon \tilde{T}$, then

$$\theta(\bar{x}_1(t)) \leq \theta(\bar{x}_2(t))$$

for all $t > t_1$ such that $\bar{x}_i(t) \in \tilde{T}$.

Proof: Nussbaum's proof in [10], pp.27-29 is applicable. ∎

Corollary 3. $x(\alpha, \eta_1, 1) = x(\alpha, \eta_2, 1) = 0$ implies $\eta_1 = \eta_2$ if η_2 and η_1 are greater than $1 + \sigma$ and the $\bar{x}$'s remain in $\tilde{T}$ for $t \epsilon [0, 1]$.

Proof: Take $\eta_1 > 1 + \sigma$, and assume $\eta_2 > \eta_1$. Theorem 2's hypotheses on f are satisfied by f_σ with $b = 1 - \sigma$. By explicit calculation $\theta(\bar{x}(\alpha, \eta_1, \epsilon)) > \theta(\bar{x}(\alpha, \eta_2, \epsilon))$

for ϵ small enough. Let $\bar{x}_1(t) = \bar{x}(\alpha, \eta_1, t)$ and $\bar{x}_2(t) = \bar{x}(\alpha, \eta_2, t + \Delta t)$ where $\Delta t > 0$ is chosen so that $\bar{x}_1(\epsilon) = \bar{x}_2(\epsilon)$. Theorem 2 now applies to $\bar{x}_1, \bar{x}_2$ (choose $t_1 = \epsilon$). Therefore $\theta(\bar{x}(\alpha, \eta_2, 1+\Delta t)) \leq \frac{\pi}{2}.Hence\theta(\bar{x}(\alpha, \eta_2, 1+\Delta t)) \leq \frac{\pi}{2}ver2$ Hence $x(\alpha, \eta_1, 1) > 0$, a contradiction. ■

Corollary 4. Equation (3.5) defines η as a function of α if $\eta > 1 + \sigma$.

Proof: There must be at least one η which satisfies $x(\alpha, \eta, 1) = 0$ since by explicit calculation for all $\alpha \geq \frac{\pi}{2}$, $\theta(\bar{x}(\alpha, 1, 1)) > \frac{\pi}{2}$ and $\lim_{\eta \to 0} \theta(\bar{x}(\alpha, \eta, 1)) = 0$. Continuity of θ outside a neighborhood of the origin implies the existence of a choice η_0 such that $\theta(\bar{x}(\alpha, \eta_0, 1)) = \frac{\pi}{2}$. The infinum over all such strictly positive η's will be defined to be $\eta(\alpha)$. This candidate satisfies Corollary 3 by minimality and hence is unique. It satisfies equation (3.5) by its construction. ■

We next bound $\frac{\partial x}{\partial \eta}$ from above:

Lemma 5. For all $\alpha > \frac{\pi}{2}$

$$|\frac{\partial x}{\partial \eta}(\alpha, \eta, 1)| < 2e^{2\alpha}. \tag{3.6}$$

Proof: The variational equation for the vector $\frac{\partial \bar{x}}{\partial \eta}$ is:

$$\frac{\partial \bar{x}'}{\partial \eta} = \alpha \begin{bmatrix} 0 & f'_\sigma(x^\alpha(t)) \\ f'_\sigma(y^\alpha(t)) & 0 \end{bmatrix} \frac{\partial \bar{x}}{\partial \eta}$$

where

$$\frac{\partial \bar{x}}{\partial \eta}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since

$$|\alpha \begin{bmatrix} 0 & f'_\sigma(x^\alpha(t)) \\ f'_\sigma(y^\alpha(t)) & 0 \end{bmatrix}| < 2\alpha$$

for all η, α, t, Gronwall's inequality implies

$$|\frac{\partial \bar{x}}{\partial \eta}(t)| < e^{2\alpha t} + 1 < 2e^{2\alpha t}.$$

Therefore

$$|\frac{\partial \bar{x}}{\partial \eta}(1)| \leq 2e^{2\alpha}.$$

Since

$$|\frac{\partial x}{\partial \eta}(\alpha, \eta, 1)| \leq |\frac{\partial \bar{x}}{\partial \eta}(1)|,$$

we have

$$|\frac{\partial x}{\partial \eta}(\alpha, \eta, 1)| < 2\epsilon^{2\alpha}.\blacksquare$$

We bound $|\frac{\partial x}{\partial \eta}|$ from below next. Let $a_{1\alpha} = \inf_{t>0}\{t : x(\alpha, \eta(\alpha), t) = 1 + \sigma\}$, $a_{1\alpha}$ exists and is unique for each choice of $\eta > 1$ if σ is chosen small (depending on α). We introduce a function $K : R \to R$ a map bounded on compact sets of R as a notational convenience for this lemma and later results. It will be clear that at least one such function exists making all conclusions of the lemmas true.

Lemma 6. Fix $\alpha_0 > \frac{\pi}{2}$. There is a $\sigma_0 = \sigma_0(\alpha_0)$ such that for all $\alpha > \alpha_0$ and $\sigma < \sigma_0$

$$|\frac{\partial x}{\partial \eta}(\alpha, \eta, 1)| > |K(\alpha_0)\alpha f(\eta)\cosh(\alpha a_{1\alpha})|. \tag{3.7}$$

Proof: By explicit calculation

$$\frac{\partial x}{\partial \eta}(\alpha, \eta, a_{1\alpha}) = -\cosh(\alpha a_{1\alpha}) \tag{3.8}$$

$$\frac{\partial y}{\partial \eta}(\alpha, \eta, a_{1\alpha}) = \sinh(\alpha a_{1\alpha}).$$

Here we used piecewise linearity of f_σ twice; once to show the flow stays in the region where f is linear for $t < a_{1\alpha}$, and once to calculate $x(\alpha, \eta, a_{1\alpha})$. Choosing σ_0 small enough so that $a_{1\alpha} > 0$ we find

$$\sec^2\theta\frac{\partial\theta}{\partial\eta} = \frac{x\frac{\partial y}{\partial \eta} - y\frac{\partial x}{\partial \eta}}{x^2}.$$

Therefore

$$\begin{aligned}\frac{\partial\theta}{\partial\eta} &= cos^2\theta\frac{x\frac{\partial y}{\partial \eta} - y\frac{\partial x}{\partial \eta}}{x^2} \\ &\geq \frac{-y\frac{\partial x}{\partial \eta}cos^2\theta}{x^2}.\end{aligned}$$

Since $\alpha > \alpha_0$ implies $y(\alpha,\eta,a_{1\alpha}) > k(\alpha_0)$, $|x| < 1$ and $\cos^2\theta > k(\alpha_0)$ for some $k(\alpha_0) > 0$ we have, using equation (3.8),

$$\frac{\partial\theta}{\partial\eta} \geq K(\alpha_0)\cosh(\alpha a_{1\alpha}). \tag{3.9}$$

Choose $t_0 \in R$ depending on $\Delta\eta > 0$ so that

$$\theta(\bar{x}(\alpha,\eta-\Delta\eta,t_0)) = \theta(\bar{x}(\alpha,\eta,a_{1\alpha})) \tag{3.10}$$

We have, using equation (3.9)

$$|t_0 - a_{1\alpha}| \geq \frac{K(\alpha_0)}{\alpha}\Delta\eta\cosh(\alpha a_{1\alpha}). \tag{3.11}$$

Defining

$$\bar{x}^*(t) = \bar{x}(\alpha,\eta-\Delta\eta,t+t_0-a_{1\alpha}),$$

we note that the hypotheses of Theorem 2 are satisfied with $\bar{x}_1 = \bar{x}(\alpha,\eta,t)$ and $\bar{x}_2 = \bar{x}^*$. Therefore

$$\theta(\bar{x}^*(t)) \leq \theta(\bar{x}(t))$$

for $t \leq 1$. Therefore

$$x^*(1) = x(\alpha,\eta-\Delta\eta,1+t_0-a_{1\alpha}) \geq 0 \tag{3.12}$$

since at t $= 1$ $\theta(\bar{x}(t)) = \frac{\pi}{2}$. Now for some $\chi \in [1, 1+t_0-a_{1\alpha}]$ the Mean Value Theorem implies

$$x(\alpha,\eta-\Delta\eta,1+t_0-a_{1\alpha}) - x(\alpha,\eta-\Delta\eta,1) = \alpha f(y(\alpha,\eta-\Delta\eta,\chi)(t_0-a_{1\alpha})$$

or

$$x(\alpha,\eta-\Delta\eta,1) = x(\alpha,\eta-\Delta\eta,1+t_0-a_{1\alpha}) - \alpha f(y(\alpha,\eta-\Delta\eta,\chi)(t_0-a_{1\alpha}). \tag{3.13}$$

Equations (3.12) and (3.13) together imply

$$\frac{x(\alpha,\eta-\Delta\eta,1) - x(\alpha,\eta,1)}{\Delta\eta} \geq -\alpha f(y(\alpha,\eta-\Delta\eta,\chi))\frac{(t_0-a_{1\alpha})}{\Delta\eta}. \tag{3.14}$$

Taking the limit as $\Delta\eta \rightarrow 0$ of (14) and using (11) we find

$$\frac{\partial x}{\partial \eta}(\alpha, \eta, 1) \leq \alpha f(y(\alpha, \eta, 1)\frac{K(\alpha_0)}{\alpha}\cosh(\alpha a_{1\alpha}).$$

Therefore

$$|\frac{\partial x}{\partial \eta}(\alpha, \eta, 1)| > K(\alpha_0)|f(\eta)|\cosh(\alpha a_{1\alpha}). \blacksquare$$

Lemma 7. For all $\alpha > \frac{\pi}{2}$

$$\frac{\partial x}{\partial \alpha}(\alpha, \eta, 1) = \alpha f(\eta). \tag{3.15}$$

Proof:

$$\frac{\partial x}{\partial \alpha}(\alpha, \eta, 1) = \frac{\partial x}{\partial \alpha}(1, \eta, \alpha) = x'(1, \eta, \alpha)$$
$$= \alpha f(y(1, \eta, \alpha) = \alpha f(y(\alpha, \eta, 1)) = \alpha f(2 - \eta) = \alpha f(\eta).\blacksquare$$

Putting these lemmas together we next have a bound on $\frac{\partial \eta}{\partial \alpha}$. Note that this bound is independent of σ.

Proposition 8. For all $\alpha > \frac{\pi}{2}$ and σ chosen as in lemmas 6 and 7, $\eta(\alpha)$ is well-defined and

$$|\alpha f(\eta)| \leq |\frac{\partial \eta}{\partial \alpha}| \leq |K(\alpha_0)\alpha\cosh(\alpha a_{1\alpha})^{-1}|. \tag{3.16}$$

Proof: The inequality follows from Lemmas 5, 6, and 7 and the relation $\frac{\partial \eta}{\partial \alpha} = \frac{\partial x}{\partial \alpha}\left(\frac{\partial x}{\partial \eta}\right)^{-1}$, a consequence of the Implicit Function Theorem. $\blacksquare$

Now uniqueness of solutions follows as in Walther[11]because $\frac{\partial \eta}{\partial \alpha} < 0$ i.e.

Theorem 9. Choosing α and σ as in lemmas 5, 6, and 7 there is at most one solution of equations (3.2), (3.3) and (3.4).

Proof: Proposition 8 implies that η is a function of α. $\bar{x}(t)$ is uniquely defined as an integral curve of a Lipshitz ordinary differential equation. Hence $x(\alpha, \eta(\alpha), t)$ is the unique solution to equations (3.2), (3.3), and (3.4). $\blacksquare$

Corollary 10. The unique solution in Theorem 9 is a periodic solution x^α of the delay differential equation (1.1).

Proof: The proof in [11] applies with small changes related to our choice of initial conditions. Define $x(t) = y^{\alpha}(1-t)$, and $y(t) = x^{\alpha}(1-t)$. Then it is easy to verify that $x(t)$ and $y(t)$ satisfy the same differential equation as $x^{\alpha}(t)$ and $y^{\alpha}(t)$ do and $x(0) = y^{\alpha}(1) = \eta(\alpha)$, $y(0) = x^{\alpha}(1) = 0$, where we used the Hamiltonian structure of equation (3.2). Uniqueness of solutions to Lipshitz ordinary differential equations now shows $y^{\alpha}(1-t) = x^{\alpha}(t)$. A similar argument shows $y^{\alpha}(-t) = -y^{\alpha}(t)$. Hence

$$\dot{x}^{\alpha}(t) = \frac{\partial}{\partial t}y^{\alpha}(1-t) = -\frac{\partial}{\partial t}y^{\alpha}(t-1) = \alpha f_{\sigma}(x^{\alpha}(t-1)) \blacksquare$$

We define the first positive zero of x, z_1^{ϕ} by $x(z_1^{\phi}) = 0$ and $x(t) > 0$ for all $t \in (0, z_1^{\phi})$. Similarly z_2^{ϕ}, the second zero of x is defined by $x(z_2^{\phi}) = 0$ and $x(t) < 0$ for all t $\in (z_1^{\phi}, z_2^{\phi})$. We next show how to choose ϕ, α, and σ to garauntee existence of these zeroes.

Proposition 11. For any interval $[\alpha_0, \alpha_N]$ with $\alpha_0 > \frac{\pi}{2}$ there is an ϵ so that if $|\phi - x^{\alpha}|_{C^0} < \epsilon$, then z_1^{ϕ}, z_2^{ϕ} exist and $|z_1^{\phi} - z_2^{\phi}| > 1$. We also have $x'(z_1^{\phi}) \neq 0, x'(z_2^{\phi}) \neq 0$.

Proof: The flow generated by equation (2.1) is uniformly continuous in the C^0 norm topology with respect to α, ϕ, and t in bounded time. We have $x^{\alpha}(0) > 1/2, x^{\alpha}(2) < -1/2, x^{\alpha}(4) > 1/2$. These two facts imply the existence of ϵ_0 such that if $|\phi - x^{\alpha}|_{C^0} < \epsilon_0$ and x has initial condition ϕ, then $x(0) > 1/2, x(2) < -1/2, x(4) > 1/2$, consequently by the intermediate value theorem z_1^{ϕ} exists and there is a $t_2 > z_2^{\phi}$ such that $x(t_1) = 0$. The existence of z_2^{ϕ} will be shown later; we have to rule out the possibility $x(t) = 0$ for all $t \epsilon (z_1^{\phi}, t_1)$. Since $|x'(t)| < 2\alpha_N$ for all t, $z_1^{\phi} > 1/\alpha_N$, consequently the t^* defined next is nonzero. Define $t^* = \inf_{|\phi - x^{\alpha}|_{C^0} < \epsilon_0} z_1^{\phi}$ and choose ϵ so that $|\phi - x^{\alpha}|_{C^0} < \epsilon$ implies $\phi(t) > 0$ if $t \in [-1 + t^*, t^*]$ (if $t^* > 1$ we are done). Now $x'(z_1^{\phi}) \neq 0$ by the definition of z_1^{ϕ} and of the flow. Therefore $z_2^{\phi} \leq t_1$ exists. The properties of z_2^{ϕ} now follow directly from the fact that $xf(x) \leq 0$ and that the solutions are slowly oscillating. ■

We define $C^*[-1,0]$ to be $\{\phi \epsilon C[-1,0] : \phi(-1) = 0\}$ and $P : B \subseteq C^*[-1,0] \rightarrow C^*[-1,0]$ by

$$P\phi(t) = x(z_2^\phi + t).$$

Here $B = \{\phi : |\phi - x^\alpha|_{C^0} < \epsilon\}$, x is a solution of equation (2.1) with initial condition ϕ and the ϵ is chosen small enough so that z_2^α is well defined. Walther[11] p.273 shows that this map is compact and differentiable in a neighbourhood of x^α. Since P is a continuous compact map from a neighbourhood of x^α to C^*, it has a local Leray-Shauder degree. See, for example Nirenberg[12] Chapter III.

Definition 12. The index of x^α is defined to be $\deg(x^\alpha, (id - P)|B, 0)$.

To calculate this index we need to define some more maps. We define U : $C^*[-1,0] \rightarrow C^*[-1,0]$ by: $U(\psi)(t) = u_\psi^\alpha(4+t)$, where u_ψ^α satisfies

$$u'(t) = \alpha f_\sigma{}'(x^\alpha(t-1))u(t-1) \tag{3.17}$$

and $u|[-1,0] = \psi$.

U determines the derivative of P at x^α. Let λ be any complex number. $M(\lambda)$ is defined to be the algebraic multiplicity of λ as an eigenvalue of U. We quote Walther[11] p. 271.

Theorem 13. The index of x^α is $(-1)^{\sum_{|\lambda|>1} M(\lambda)}$ if $M(1) = 1$.

Walther shows for a class of functions f, that the index changes at least once as $\alpha \rightarrow \infty$. We show for our f_σ that the index changes at a large finite number of large α's. Our work can be interpreted as saying something about the placement of the set of delay differential equations with bifurcating periodic orbits. Roughly our work shows that this set intersects a straight line in the space of delay differential equations in many places.

4. Linearization Around the Periodic Solution

Since P is compact and differentiable, its derivative is compact and has spectrum consisting entirely of eigenvalues. Following Walther[11] we search for eigenvalues equal to 1 by re-writing equation (3.17) as an ordinary differential equation with boundary conditions giving eigenvectors for a square root of U.

Let $W\psi(t) = u_\psi^\alpha(2+t)$ for all $t \in [-1,0]$, where u satisfies equation (3.17). Then $W^2 = U$ and the relation $(W-\lambda)(W+\lambda) = U - \lambda^2$ shows that all eigenvalues for U arise as squares of eigenvalues of W. Define $u_\lambda(t) = u(t)$ where u satisfies equation (3.17) and

$$u(t+2) = \lambda u(t). \tag{4.1}$$

Define $z(t) = u_\lambda(t+1)$. We will drop the subscript λ and write $u(t)$ for $u_\lambda(t)$ and $z(t)$ for $z_\lambda(t)$. We next define times t when the flow defining $\bar{u}$ changes its nature.

$$a_{1\alpha} = \inf\{t : t > 0 \text{ and } x^\alpha(t) < 1 - \sigma\}$$

$$a_\alpha = \inf\{t : t > 0 \text{ and } x^\alpha(t) < 1\}$$

$$a_{2\alpha} = \inf\{t : t > 0 \text{ and } x^\alpha(t) < 1 + \sigma\}$$

$$b_{1\alpha} = \inf\{t : t > 0 \text{ and } x^\alpha(t-1) > 1 - \sigma\}$$

$$b_\alpha = \inf\{t : t > 0 \text{ and } x^\alpha(t-1) > 1\}$$

$$b_{2\alpha} = \inf\{t : t > 0 \text{ and } x^\alpha(t) > 1 + \sigma\}$$

Then equation (3.17) becomes:

$$\bar{u}'(t) = \alpha \begin{cases} \begin{bmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{bmatrix} \bar{u} & \text{if t} \in [0, a_{1\alpha}] \\ G(\alpha,\lambda,t)\bar{u} & \text{if t} \in [a_{1\alpha}, a_{2\alpha}] \\ \begin{bmatrix} 0 & -\lambda^{-1} \\ 1 & 0 \end{bmatrix} \bar{u} & \text{if t} \in [a_{2\alpha}, b_{1\alpha}] \\ G(\alpha,\lambda,t)\bar{u} & \text{if t} \in [b_{1\alpha}, b_{2\alpha}] \\ \begin{bmatrix} 0 & -\lambda^{-1} \\ -1 & 0 \end{bmatrix} \bar{u} & \text{if t} \in [b_{2\alpha}, 1] \end{cases} \tag{4.2}$$

where

$$G(\alpha,\lambda,t)=\begin{bmatrix}0 & f'_\sigma(x^\alpha(t))\\ -f'_\sigma(y^\alpha(t)) & 0\end{bmatrix}.$$

Equation (4.2) can be written as

$$\bar{u}'=\alpha G(\alpha,\lambda,t)\bar{u}(t). \tag{4.3}$$

Then $\bar{u}$ satisfies:

$$\bar{u}(1)=\begin{bmatrix}\lambda z(0)\\ u(0)\end{bmatrix}. \tag{4.4}$$

Since f_σ is almost piecewise-linear, it is useful to change equation (4.2) to a statement about naturally defined maps. Letting $\bar{u}$ satisfy equation (4.3) we define the following five maps from R^2 to R^2 :

$$R_\alpha\bar{u}_0=\bar{u}(a_{1\alpha}) \qquad \text{where } \bar{u}(0)=\bar{u}_0$$

$$\Sigma_\alpha\bar{u}_0=\bar{u}(a_{2\alpha}) \qquad \text{where } \bar{u}(a_{1\alpha})=\bar{u}_0$$

$$H_\alpha\bar{u}_0=\bar{u}(b_{1\alpha}) \qquad \text{where } \bar{u}(a_{2\alpha})=\bar{u}_0$$

$$\Sigma^*_\alpha\bar{u}_0=\bar{u}(b_{2\alpha}) \qquad \text{where } \bar{u}(b_{1\alpha})=\bar{u}_0$$

$$R_{-\alpha}\bar{u}_0=\bar{u}(1) \qquad \text{where } \bar{u}(b_{2\alpha})=\bar{u}_0.$$

All these maps are C^1 in α if the $a_{1\alpha}$ and $b_{1\alpha}$ are smooth functions of α. R_α, H_α, $R_{-\alpha}$ are rotations and solutions of hyperbolic linear equations, and $|\Sigma_\alpha-1|$, $|\Sigma^*_\alpha-1|$ turn out to be negligible up to order σ.

The index of P as a function of α can change only as λ, an eigenvalue of W, goes through -1 or 1. We follow the outline of the proof in Walther [11] that λ does not pass through -1. Let $\theta(\bar{u}(t))$ be defined by $\tan(\theta(\bar{u}(t)))=u(t)/z(t)$ and the integer part of $\theta/2\pi$ is the winding number of $\bar{u}(t)$ about the origin. We will often write $\theta\bar{u}$ for $\theta(\bar{u})$ in the rest of the thesis for readability.

Lemma 2. λ = -1 and $\alpha>\alpha_0$ and $\sigma<\sigma_0(\alpha_0)$ implies

$$|\theta\bar{u}(a_{1\alpha})-\theta\bar{u}(b_{2\alpha})|<\frac{\pi}{2}$$

Proof: First note that $\bar{u}$ satisfies

$$\bar{u}' = \alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{u}$$

for $t \in [a_{2\alpha}, b_{1\alpha}]$. We have $a_{2\alpha} - b_{1\alpha} = 2(1/2 - a_{1\alpha})$. Thus

$$|\theta\bar{u}(b_{1\alpha}) - \theta\bar{u}(a_{2\alpha})| \leq 2\alpha(1/2 - a_{1\alpha}).$$

As in lemmas (2.2), (2.3), and (2.5)

$$|\theta\bar{u}(b_{2\alpha}) - \theta\bar{u}(b_{1\alpha})| \leq 16\sigma$$

$$|\theta\bar{u}(a_{2\alpha}) - \theta\bar{u}(a_{1\alpha})| \leq 16\sigma.$$

Taking α large and then σ small we have

$$\begin{aligned} |\theta\bar{u}(b_{2\alpha}) - \theta\bar{u}(a_{1\alpha})| &\leq |\theta\bar{u}(b_{1\alpha}) - \theta\bar{u}(a_{2\alpha})| + 16\sigma \\ &\leq 2\alpha(1/2 - a_{1\alpha}) < \frac{\pi}{2} \end{aligned}$$

since $\alpha(1/2 - a_{1\alpha}) \leq 16\sigma$ for sufficiently large α. ∎

Theorem 3 . For $\alpha > \frac{\pi}{2}$, σ chosen as in lemmas (3.5), (3.6), and (3.7) there is one solution to equation (4.3) with boundary conditions (4.4) having $\lambda = -1$ and this solution has algebraic multiplicity 1.

Proof:(i) We show that dim ker (W + id) = 1. Assume, for contradiction, that dim ker (W + id) $\neq$ 1. Then dim ker(W + id) = 2 since $\dot{x}^\alpha \in$ ker(W + id). If dim ker (W + id) = 2 then for any initial condition $\bar{u}(0)$ we find that

$$\bar{u}(1) = \begin{bmatrix} -z(0) \\ u(0) \end{bmatrix}.$$

In particular $\bar{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ implies $\bar{u}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. However, this implies $\bar{u}(b_{2\alpha}) = \begin{bmatrix} \xi \\ -\xi \end{bmatrix}$ for some $\xi > 0$. Similarly we have $\bar{u}(a_{1\alpha}) = \begin{bmatrix} \beta \\ \beta \end{bmatrix}$ for some $\beta > 0$. Therefore we have

$$|\theta\bar{u}(a_{1\alpha}) - \theta\bar{u}(b_{2\alpha})| = \frac{3\pi}{2}$$

However Lemma (4.1) says

$$|\theta\bar{u}(a_{1\alpha}) - \theta\bar{u}(b_{2\alpha})| \leq \frac{\pi}{2},$$

a contradiction .

(ii) Walther[11] 's proof applies without change to show that the algebraic multiplicity of $(W + \mathrm{id})$ is 1. ∎

5. Satisfaction of a Necessary Condition

Since we have seen that λ does not pass through -1, we expect the only way that the degree of P could change is by λ going through 1. (If λ crosses through the unit circle at a point where Re$\lambda \neq 0$, $\bar{\lambda}$ also does, and the pair does not change the degree of x^α). In this and the next section we show that λ must often equal 1 in the boundary value problems (4.3) and (4.4). This fact, while promising, does not in itself force bifurcation of x^α. We will only be able to show bifurcation in Chapter 2. We now consider $\lambda = 1$. The boundary conditions (4.4) split into two conditions; one which can be satisfied at any large α by a good choice of $\bar{u}(0)$, and one, studied in the next section, which restricts the possible α. A sketch of a typical solution to equation (4.3), Chapter 1 with $\lambda = 1$ is useful for motivating the remaining constructions in this section:

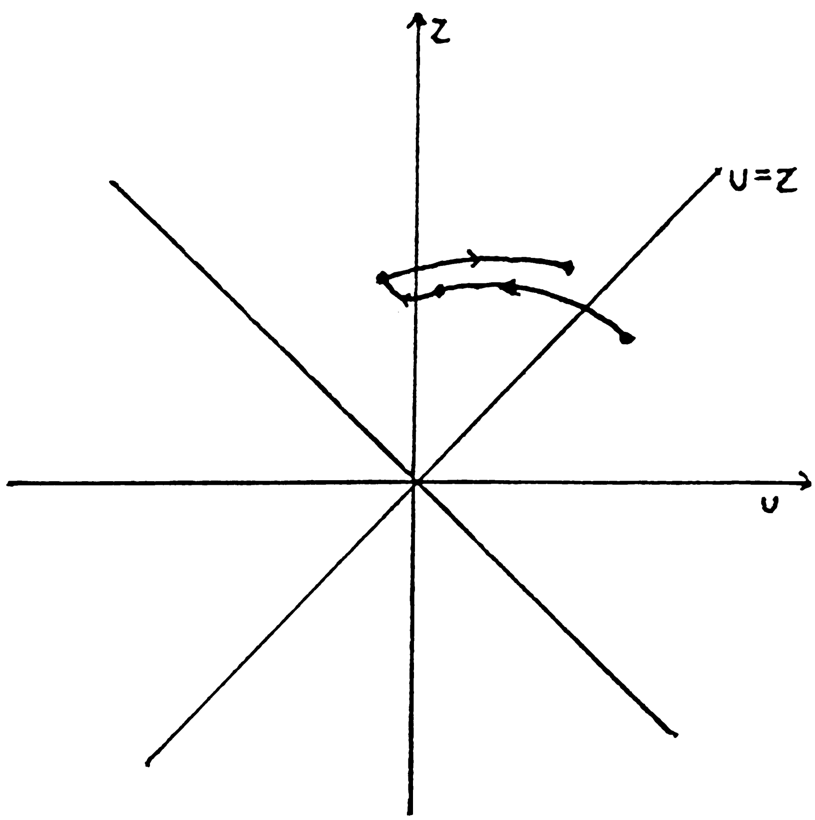

Figure 3

The boundary conditions (4.4) appear as the requirement that $\bar{u}(0)$ is the reflection through the line $u = z$ of $\bar{u}(1)$. Figure 3 suggests that a necessary condition for the boundary condition to be satisfied is :

$$| \bar{u}(a_{1\alpha}) | = | \bar{u}(b_{2\alpha}) |. \tag{5.1}$$

This is true.

Lemma 1. If $\bar{u}$ satisfies equations (4.3) and (4.4), then $\bar{u}$ satisfies equation (5.1).

Proof: Boundary condition (1.4.4) says that

$$\bar{u}(1) = \begin{bmatrix} z(0) \\ u(0) \end{bmatrix}.$$

Therefore

$$| \bar{u}(1) | = | \begin{bmatrix} z(0) \\ u(0) \end{bmatrix} | = | \bar{u}(0) |.$$

Now R_α and $R_{-\alpha}$ preserve norms and map $\bar{u}(0)$ to $\bar{u}(a_{1\alpha})$ and $\bar{u}(b_{2\alpha})$ to $\bar{u}(1)$ respectively. Hence

$$\begin{aligned} | \bar{u}(a_{1\alpha}) | &= | R_\alpha \bar{u}(0) | = | \bar{u}(0) | \\ &= | \bar{u}(1) | = | R_{-\alpha} \bar{u}(b_{2\alpha}) | = | \bar{u}(b_{2\alpha}) |. \blacksquare \end{aligned}$$

For all sufficiently large α equation (5.1) can be satisfied by some choice of $\bar{u}(0)$. To show this mainly involves explicit calculation of H_α and an appeal to the closeness of Σ_α and Σ_α^* to the identity. The next two lemmas estimate $\Sigma_\alpha - 1$ and $\Sigma_\alpha^* - 1$:

Lemma 2 . There is an α_0 such that for any $\alpha > \alpha_0$.

$$\begin{aligned} | a_{1\alpha} - a_{2\alpha} | &< \frac{4\sigma}{\alpha} & | b_{1\alpha} - b_{2\alpha} | &< \frac{4\sigma}{\alpha} \\ | a_{1\alpha} - a_\alpha | &< \frac{4\sigma}{\alpha} & | b_{1\alpha} - b_\alpha | &< \frac{4\sigma}{\alpha} \\ | a_\alpha - a_{2\alpha} | &< \frac{4\sigma}{\alpha} & | b_\alpha - b_{2\alpha} | &< \frac{4\sigma}{\alpha}. \end{aligned}$$

Proof: We estimate $|a_{1\alpha} - a_{2\alpha}|$. All the other pairs may be estimated in the same way. If $t \in [a_{1\alpha}, 1]$, then $y^\alpha(t) > y^\alpha(a_{1\alpha})$. f_σ is increasing on $[y^\alpha(a_{1\alpha}),\ y^\alpha(a_{2\alpha})]$. Thus

$$\dot{x}^\alpha(t) = \alpha f(y^\alpha(t)) \geq \alpha f(y^\alpha(a_{1\alpha}))$$

if $t \in [a_{1\alpha}, a_{2\alpha}]$. Therefore

$$x^\alpha(t) - x^\alpha(a_{1\alpha}) > \alpha f(y^\alpha(a_{1\alpha}))(t - a_{1\alpha}).$$

Setting $t = a_{2\alpha}$,

$$x^\alpha(a_{2\alpha}) - x^\alpha(a_{1\alpha}) = 2\sigma > \alpha f(y^\alpha(a_{1\alpha}))(a_{2\alpha} - a_{1\alpha}).$$

Hence

$$|a_{2\alpha} - a_{1\alpha}| < \frac{2\sigma}{\alpha f(y^\alpha(a_{1\alpha}))}$$

Choosing α_0 so that $f(y^{\alpha_0}(a_{\alpha_0})) > \frac{1}{2}$, we have

$$|a_{2\alpha} - a_{1\alpha}| < \frac{4\sigma}{\alpha}$$

if $\alpha > \alpha_0$, since $f(y^\alpha)$ is an increasing function of α. ∎

Lemma 3 . There is an α_0 such that for any $\alpha > \alpha_0$ and $\sigma < \sigma_0$

$$|\ \Sigma_\alpha - 1\ | < 16\sigma \text{ and } |\ \Sigma^*_\alpha - 1\ | < 16\sigma. \tag{5.2}$$

Proof: We will deal with Σ_α since Σ^*_α may be estimated in the same way. We have $\Sigma_\alpha \bar{u}_0 = \bar{u}(a_{2\alpha})$ where $\bar{u}' = \alpha G(\alpha, \eta, t)\bar{u}$ and $|\alpha G(\alpha, \eta, t)| < 2\alpha$. Then Gronwall's inequality implies

$$|\ \bar{u}(t) - \bar{u}(a_{1\alpha})\ | \leq |\ e^{2\alpha(t - a_{1\alpha})} - 1\ |\ |\ \bar{u}(a_{1\alpha})\ |.$$

Setting $t = a_{2\alpha}$, and choosing α_0 as in Lemma 2 we have

$$|\ \Sigma_\alpha \bar{u}_0 - \bar{u}_0\ | \leq |\ e^{2\alpha(a_{2\alpha} - a_{1\alpha})} - 1\ |\ |\ \bar{u}_0\ | \leq 4\alpha\frac{4\sigma}{\alpha}|\ \bar{u}_0\ |$$

since Lemma 2 says $|a_{2\alpha} - a_{1\alpha}| < \frac{4\sigma}{\alpha}$, and $|e^t - 1| < 2t$ if $t < 1/2$. Rewriting the last equation we have

$$| \Sigma_\alpha \bar{u}_0 - \bar{u}_0 | \leq 16\sigma | \bar{u}_0 |. \tag{5.3}$$

We re-write equation (5.3) as

$$| \Sigma_\alpha - 1 | < 16\sigma$$

and we are done. ■

We next calculate H_α.

Lemma 4.

$$H_\alpha \bar{u}_0 = \begin{bmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{bmatrix} \bar{u}_0 + K(\alpha)\sigma$$

where $t = 2\alpha(1/2 - a_{1\alpha})$.

Proof: We may solve the equations defining H_α explicitly giving the expression in Lemma 4 with $t = 2\alpha(1/2 - a_{2\alpha})$. Lemma 2 allows us to replace $a_{2\alpha}$ by $a_{1\alpha}$ yielding Lemma 4. ■

We next show that for any large α there must be a choice of $\bar{u}(0)$ satisfying equation (5.1) by using the bounds we have derived on Σ_α, Σ_α^* and the calculation of H_α.

Lemma 5. There is an $\alpha_0 > 0$ such that for each $\alpha_N > \alpha_0$ there is a σ_0 such that $\alpha \in [\alpha_0, \alpha_N]$ and $\sigma < \sigma_0$ imply $| \bar{u}(b_{2\alpha}) | = | \bar{u}(a_{1\alpha}) |$ for some choice of $\bar{u}(0)$.

Proof: We note that $\alpha(1/2 - a_{1\alpha}) \leq 8\sigma$ if $\alpha \geq \alpha_0$. This fact follows from arguments similar to those used in the proof of Lemma 2. Now let $t = 2\alpha(1/2 - a_{1\alpha})$. Then

$$\begin{aligned} H_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + K(\alpha)\sigma \\ &= (1 - t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + O(t^2) + K(\alpha)\sigma. \end{aligned}$$

Therefore

$$\Sigma_\alpha^* \, H_\alpha \, \Sigma_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1-t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + O(t^2) + K(\alpha)\sigma$$

since $|\Sigma_\alpha - 1|$ and $|\Sigma_\alpha^* - 1|$ are of order σ. Thus if $\bar{u}(a_{1\alpha}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$\bar{u}(b_{2\alpha}) = (1-t)\bar{u}(a_{1\alpha}) + O(t^2) + K(\alpha)\sigma.$$

Similarly if $\bar{u}(a_{1\alpha}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ then $\bar{u}(b_{2\alpha}) = (1+t)\bar{u}(a_{1\alpha}) + O(t^2) + K(\alpha)\sigma$. Since $t = 2\alpha(1/2 - a_{1\alpha}) \leq \sigma$, we may choose α_0 large enough and σ small enough so that $t^2 < t/3$ and then choose σ smaller, if necessary, so that the term of order σ is less than $2\alpha(1/2 - a_{1\alpha})/3$. With these choices if $\bar{u}(a_{1\alpha}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $|\ \bar{u}(b_{2\alpha})\ | < |\ \bar{u}(a_{1\alpha})\ |$. Similarly if $\bar{u}(a_{1\alpha}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ then $|\ \bar{u}(b_{2\alpha})\ | > |\ \bar{u}(a_{1\alpha})\ |$. Therefore by continuity of $\Sigma_\alpha^* H_\alpha \Sigma_\alpha : R^2 -> R^2$ there is a $\beta \in (0,2)$ so that if

$$\bar{u}(a_{1\alpha}) = \begin{bmatrix} -1+\beta \\ 1 \end{bmatrix},$$

then

$$|\ \bar{u}(b_{2\alpha})\ | = |\ \bar{u}(a_{1\alpha})\ |. \blacksquare$$

Define $\bar{u}^\alpha(t)$ by $|\ \bar{u}^\alpha(0)\ | = |\ \bar{u}^\alpha(1)\ | = 1$, $u^\alpha(0) > 0$.

Lemma 6.$\bar{u}^\alpha(t)$ is uniquely defined under the conditions in Lemma 5.

Proof: If not, by linearity of $\Sigma_\alpha^* H_\alpha \Sigma_\alpha$, all choices of $\bar{u}(0)$ would satisfy (1.4). However the proof of Lemma 5 shows that $\bar{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ never satisfies (1.4) when $\alpha > \alpha_0$ and $\sigma < \sigma_0$. $\blacksquare$

6. Determination of Bifurcation Parameters

We need to define basic geometric quantities for our later lemmas. We define $\theta_0(\alpha)$ = arclength $\{\bar{u}^\alpha(s) : a_{1\alpha} \geq s \geq 0\}$ = arclength $\{\bar{u}^\alpha(s) : 1 \geq s \geq b_{2\alpha}\}$ and $\theta_1(\alpha) = \theta\bar{u}^\alpha\,(a_{1\alpha})$.

Theorem 1. For each $N > 0$ there is an $\alpha_0 > 0$ and a $\sigma_0 > 0$ such that there is a set A of least N $\alpha's$ $\{\alpha_i\}_{i=1}^N$ which satisfy equation (1.3) and boundary conditions (1.4) with $\lambda = 1$ for any $\sigma < \sigma_0$.

Proof: The proof of Lemma 5 in the last section shows $\theta_1 - \theta_0$ that

$$\Sigma_\alpha^* H_\alpha \Sigma_\alpha \bar{u}^\alpha(a_{1\alpha}) = \bar{u}^\alpha(b_{2\alpha}).$$

Since $|\Sigma_\alpha^* - 1|$, $|\Sigma_\alpha - 1| \leq 16\sigma$, we have

$$|\; H_\alpha \bar{u}^\alpha(a_{1\alpha}) \;| \;=\; |\; \bar{u}^\alpha(b_{1\alpha}) \;| + \; K(\alpha)\sigma.$$

Hence, using properties of H_α

$$\theta\bar{u}^\alpha(a_{1\alpha}) \;+\; \theta\bar{u}^\alpha(b_{1\alpha}) \;=\; \pi \;+\; 2\;m\;\pi \;+\; K(\alpha)\sigma$$

where m is any integer. Since $|\Sigma_\alpha^* - 1|$, $|\Sigma_\alpha - 1| \leq 16\sigma$ and θ is uniformly continuous outside a neighbourhood of the origin,

$$\theta\bar{u}^\alpha(a_{1\alpha}) \;+\; \theta\bar{u}^\alpha(b_{2\alpha}) \;=\; \pi \;+\; 2m\;\pi \;+\; K(\alpha)\sigma.$$

Therefore

$$\begin{aligned} \theta\bar{u}^\alpha(1) &= \theta\bar{u}^\alpha(b_{2\alpha}) - \theta_0 \\ &= \pi - \theta_1 - \theta_0 \;+\; 2\;m\;\pi \;+\; K(\alpha)\sigma. \end{aligned}$$

The boundary condition (1.4) yields

$$\theta\bar{u}^\alpha(0) \;+\; \theta\bar{u}^\alpha(1) \;=\; \frac{\pi}{2} \;+\; 2\;m\;\pi.$$

Using $\theta\bar{u}^\alpha(0) = \theta_1 - \theta_0$, and our expression for $\bar{u}^\alpha(1)$ in terms of θ_1 and θ_0 we find

$$\begin{aligned} \theta_1 - \theta_0 + \pi - \theta_1 - \theta_0 &= \frac{\pi}{2} \;+\; 2m\;\pi \;+\; K(\alpha)\sigma \\ \pi - 2\theta_0 &= \frac{\pi}{2} \;+\; 2m\;\pi \;+\; K(\alpha)\sigma \\ \theta_0 &= \frac{\pi}{4} + m\;\pi + K(\alpha)\sigma. \end{aligned}$$

Now $\theta_0 = \alpha\, a_{1\alpha}$, an unbounded function of α since

$$\frac{\alpha}{2} < \theta_0(\alpha) < \alpha$$

for large α. To find our α_i take α_0 large enough so that there are solutions of the equation

$$\theta_0(\alpha) = \frac{\pi}{4} + 2m\pi$$

for at least N+2 integers m, this can be done because $\frac{\alpha}{2} < \theta_0(\alpha) < \alpha$ and θ_0 is continuous. Then choose

$$\sigma_0 < \frac{\pi}{2} \max_{\alpha \in [\alpha_0, \alpha_{N+2}]} K(\alpha)^{-1}.$$

With this choice we will have at least N solutions remaining of

$$\theta_0 = \frac{\pi}{4} + 2m\pi + K(\alpha)\sigma$$

by continuity of $a_{1\alpha}$ as a function of α and the Intermediate Value Theorem. ▪

Corollary 2. There are choices of σ_0, α, and α_0 such that

$$\bar{u}^\alpha(a_{1\alpha}) = \begin{bmatrix} \alpha(1/2 - a_{1\alpha}) \\ 1 \end{bmatrix} + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma$$

for $\alpha \in A$ and $\sigma < \sigma_0$.

Proof:

$$|\ H_\alpha \bar{u}^\alpha(a_{1\alpha})\ | = |\ \bar{u}^\alpha(a_{1\alpha})\ | + K(\alpha)\sigma$$

implies

$$H_\alpha \bar{u}^\alpha(a_{2\alpha}) = \pm \begin{bmatrix} -u^\alpha(a_{1\alpha}) \\ z^\alpha(a_{2\alpha}) \end{bmatrix} + K(\alpha)\sigma \tag{6.2}$$

by properties of H_α. We concentrate on the positive case. The negative case is ruled out for large α since then H_α is approximately the identity matrix. A calculation using the definition of H_α gives

$$\begin{aligned} H_\alpha \bar{u}^\alpha(a_{1\alpha}) &= \begin{bmatrix} 1 & -2\alpha(1/2 - a_{1\alpha}) \\ -2\alpha(1/2 - a_{1\alpha}) & 1 \end{bmatrix} \bar{u}^\alpha \\ &= \begin{bmatrix} u^\alpha(a_{2\alpha}) - 2\alpha(1/2 - a_{1\alpha})z^\alpha(a_{2\alpha}) \\ -2\alpha(1/2 - a_{1\alpha})u^\alpha(a_{2\alpha}) + z^\alpha(a_{2\alpha}) \end{bmatrix} \end{aligned} \tag{6.3}$$

neglecting terms of order $O((\alpha(1/2 - a_{1\alpha}))^2)$ and $K(\alpha)\sigma$. Setting the first components of equations (6.2) and (6.3) equal

$$u^\alpha(a_{1\alpha}) = \alpha(1/2 - a_{1\alpha})z^\alpha(a_{1\alpha}) + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma. \tag{6.4}$$

Using $|\ \bar{u}^\alpha(a_{1\alpha})\ |^2 = 1$ we have

$$z^\alpha(a_{1\alpha})(1 + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma) = 1.$$

Therefore

$$z^\alpha(a_{1\alpha}) = 1 + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma.$$

Hence, using equation (6.4)

$$u^\alpha(a_{1\alpha}) = \alpha(1/2 - a_{1\alpha}) + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma.$$

Using $|\Sigma_\alpha^* - 1|$, $|\Sigma_\alpha - 1| \leq 16\sigma$ we find equation (6.1). ∎

We will write Rot_x where x is any real number for the matrix

$$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Corollary 3. For choices of α and σ as in Corollary (1)

$$\bar{u}^\alpha(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma. \tag{6.5}$$

Proof: We have $\alpha a_{1\alpha} = \pi/4 \bmod 2m\,\pi$ for $\alpha \in A$. Therefore

$$\begin{aligned} \bar{u}^\alpha(0) = R_{-\alpha}\bar{u}^\alpha(a_{1\alpha}) &= (Rot_{\frac{-\pi}{4}} + K(\alpha)\sigma)\begin{bmatrix} \alpha(1/2 - a_{1\alpha}) \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma. \end{aligned}$$ ∎

Chapter 2

CHANGE OF STABILITY FOR $x'(t) = \alpha f(x(t-1))$

1. Introduction.

In this chapter we show that bifurcation occurs at all the α_i constructed in chapter 1. We do this without perturbing generically as in Mallet-Paret [6]. The techniques he uses do not allow us to perturb remaining in the class of equations of pure delay. We would need to consider equations of the form

$$x'(t) = f(x(t), x(t-1), x(t-\frac{1}{n}), x(t-\frac{2}{n}), \ldots x(t-1))$$

which leaves unanswered questions about the effect of pure delay and symmetry on this family's passage through the set of bifurcating dynamical systems. Our explicit calculations show the existence of bifurcating periodic solutions of the unperturbed family, and the calculations are algebraic in character, showing that questions of stability can be determined symbolically - an alternative and check for numerical methods for large α .

2. Definition and Discussion of a Bifurcation Function

In Chapter 1 equation (4.3) and boundary conditions (4.4) gave a way of finding the spectrum of the linearized flow around the periodic orbit of equation (1.1.1). One way of studying bifurcation near α_i is to choose a function of α and λ which vanishes when $\lambda = 1$, is differentiable and related to the flow generated by (1.1.1).

Walther [11] shows that a function $q_\alpha(\lambda)$ defined in terms of the fundamental solutions to equations (4.3) and using equation (4.4) will give us bifurcation results; we introduce functions $q_i(\alpha, \lambda)$ identical to Walther's up to multiplication by a constant. Our choice of q_i is more closely tied to the details known about the bifurcation points α_i. This specialised choice of bifurcation function is effective only because of the extra information we have in this case.

For each i an integer greater than zero define

$$q_i(\alpha, \lambda) = \det \begin{bmatrix} u(1) - \lambda z(0) & u^*(1) - \lambda z^*(0) \\ z(1) - u(0) & z^*(1) - u^*(0) \end{bmatrix}$$

where $\bar{u}(t)$ and $\bar{u}^*(t)$ satisfy equation (1.4.3) and

$$\bar{u}(0) = \begin{bmatrix} u^{\alpha_i} \\ z^{\alpha_i} \end{bmatrix} \quad \text{and} \quad \bar{u}^*(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Our $q_i(\lambda) = C_i q_\alpha(\lambda)$ where $C_i \neq 0$ and $q_\alpha(\lambda)$ is defined by Walther[11]. We quote the relevant corollary in Walther[11] as a proposition with slight changes for our situation.

Proposition 1. Suppose $q_i(\alpha, 1) \neq 1$ and the algebraic multiplicity of ker (W+id) = 1. Then the index of the fixed point x^α of P is given by

$$\text{index } x^\alpha = C_i(-1)^{\sum_{\lambda<-1} j(\lambda)}(-1)^{\sum_{1<\lambda} j(\lambda)}$$

where

$$j(\lambda) = \begin{cases} 0 & \text{if } q_i(\alpha, \lambda) \neq 0 \\ \text{multiplicity of } \lambda \text{ as a zero of } q_i(\alpha, \lambda) & \text{otherwise.} \end{cases}$$

Proof: This follows from Walther [11] section 3 and $q_i(\lambda) = C_i q_\alpha(\lambda)$.■

Proposition 2. There is an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ index $x^{\alpha_i - \epsilon} \neq$ index $x^{\alpha_i + \epsilon}$ if q_i satisfies:

$$q_i(\alpha_i, 1) = 0 \qquad (i)$$

$$\frac{\partial q_i}{\partial \alpha}(\alpha_i, 1) \neq 0 \qquad (ii)$$

$$\frac{\partial q}{\partial \lambda}(\alpha_i, 1) \neq 0 \qquad (iii)$$

Proof:(i), (ii) and (iii) imply there is a simple zero $\lambda(\alpha)$ of q_i in a neighbourhood of $\alpha = \alpha_i, \lambda = 1$. The Implicit Function Theorem, (ii) and (iii) imply $\frac{\partial \lambda}{\partial \alpha} \neq 0$. We handle the case where $\frac{\partial \lambda}{\partial \alpha} > 0$, a parallel argument works for $\frac{\partial \lambda}{\partial \alpha} < 0$.

Since $\frac{\partial \lambda}{\partial \alpha} > 0$, $\lambda(\alpha_i - \epsilon) < 1$ and $\lambda(\alpha_i + \epsilon) > 1$ for $0 < \epsilon < \epsilon_0$. We know from Theorem 3 Chapter 1 that j(-1) = 1 always, consequently the hypotheses of Proposition 1 are satisfied if $\lambda(\alpha) \neq 1$. Since $\lambda(\alpha_i + \epsilon) \neq 1$ we can use Proposition 1 to calculate the degree of x^α for $\alpha = \alpha_1 \pm \epsilon$. Rouche's Theorem implies the degree has constant contributions from any roots not equal to one at α_i. It is now clear that index $x^{\alpha_i - \epsilon} \neq$ index $x^{\alpha_i + \epsilon}$. ∎

We therefore must calculate $\frac{\partial q_i}{\partial \alpha}(\alpha_i, 1)$ and $\frac{\partial q_i}{\partial \lambda}(\alpha_i, 1)$.

3. Technical preliminaries

Let

$$N_1\bar{x}(0) = \bar{x}(a_\alpha), \qquad N_2\bar{x}(0) = \bar{x}(a_{2\alpha})$$

where $\bar{x}(a_{1\alpha}) = \bar{x}(0)$, $\bar{x}(a_\alpha) = \bar{x}(0)$ respectively and

$$\bar{x}' = \alpha \begin{bmatrix} f_\sigma(y) \\ -f_\sigma(x) \end{bmatrix}.$$

In the next two lemmas we show that these maps and their derivatives with respect to α are small. This will enable us to calculate the derivative of $a_{1\alpha}$ up to order σ. We need this derivative to find the quantities $\frac{\partial\Sigma_\alpha}{\partial\alpha}$ and $\frac{\partial\Sigma_\alpha^*}{\partial\alpha}$. To bound $\frac{\partial}{\partial\alpha}(a_{1\alpha} - a_\alpha)$ and $\frac{\partial}{\partial\alpha}(a_\alpha - a_{2\alpha})$ we need the following lemma.

Lemma 1. For all $\alpha > \alpha_0$ and $\sigma < \sigma_0$

$$|\frac{\partial}{\partial\alpha}(a_{1\alpha} - a_\alpha)| \leq K(\alpha)\sigma \tag{3.1}$$

$$|\frac{\partial}{\partial\alpha}(a_\alpha - a_{2\alpha})| \leq K(\alpha)\sigma$$

$$|\frac{\partial x^\alpha}{\partial\alpha}(t) + \frac{\partial x^\alpha}{\partial t}(a_{1\alpha})\frac{\partial a_{1\alpha}}{\partial\alpha}| \leq K(\alpha)\sigma$$

for t $\in [a_{1\alpha}, a_{2\alpha}]$.

Proof: We prove only the first case; the second is proved similarly. The third inequality is proved using similar arguments to those for the first two. We have $x^\alpha(a_{1\alpha}) = 1 + \sigma$ and $x^\alpha(a_\alpha) = 1$. The Implicit Function Theorem says that

$$\frac{\partial a_{1\alpha}}{\partial\alpha} = \frac{\partial x^\alpha}{\partial\alpha}(a_{1\alpha})(\frac{\partial x^\alpha}{\partial t}(a_{1\alpha}))^{-1} = \frac{\partial x^\alpha}{\partial\alpha}(a_{1\alpha})(\alpha f(y^\alpha(a_{1\alpha})))^{-1}$$

where $\frac{\partial a_{1\alpha}}{\partial\alpha}$ exists since the numerator is non-zero. A similar equality holds for $\frac{\partial a_{1\alpha}}{\partial\alpha}$ yielding

$$\begin{aligned}\frac{\partial}{\partial\alpha}(a_{1\alpha} - a_\alpha) &= \frac{\frac{\partial x^\alpha}{\partial\alpha}(a_{1\alpha})}{\alpha f(y^\alpha(a_{1\alpha}))} - \frac{\frac{\partial x^\alpha}{\partial\alpha}(a_\alpha)}{\alpha f(y^\alpha(a_\alpha))} \\ &\leq \frac{4}{\alpha} f(y^\alpha(a_\alpha))[\frac{\partial}{\partial\alpha}x^\alpha - \frac{\partial}{\partial\alpha}x^\alpha(a_\alpha)] \\ &\quad + \frac{4}{\alpha}\frac{\partial}{\partial\alpha}x^\alpha(a_\alpha)[f(y^\alpha(a_{1\alpha})) - f(y^\alpha(a_\alpha))].\end{aligned}$$

For the last inequality we let A = $\frac{\partial x^\alpha}{\partial\alpha}(a_{1\alpha})$, B = $\alpha f(y^\alpha(a_{1\alpha}))$, C = $\frac{\partial x^\alpha}{\partial\alpha}(a_\alpha)$, D = $\alpha f(y^\alpha(a_\alpha))$ and used the algebraic equality $A/B - C/D = (AD - BC)(BD)^{-1}$. With our choice of A,B, C and D we have B, D> 1/2, so we can use the identity AD - BC = A(D - B) + B(A - C) to justify the final step in the inequality. Using the Mean Value Theorem and the fact that $f(y^\alpha)$ has derivatives bounded by α we find

$$|\frac{\partial}{\partial\alpha}(a_{1\alpha} - a_\alpha)| \leq \frac{1}{\alpha}\frac{\partial}{\partial t}\frac{\partial x}{\partial\alpha}^\alpha(\chi)\ |a_{1\alpha} - a_\alpha| + \frac{1}{\alpha}\frac{\partial x}{\partial\alpha}^\alpha(a_\alpha)\ |\alpha|\ |a_{1\alpha} - a_\alpha| \tag{3.2}$$

where $\chi \in [a_{1\alpha}, a_\alpha]$. We now bound $\frac{\partial x}{\partial\alpha}^\alpha$ and $\frac{\partial}{\partial t}\frac{\partial x}{\partial\alpha}^\alpha$.

$$\frac{\partial x}{\partial\alpha}^\alpha(t) = \frac{\partial}{\partial\alpha}x(1,\eta,\alpha t) = \frac{\partial x}{\partial\eta}(1,\eta,\alpha t)\frac{\partial\eta}{\partial\alpha} + tx'(1,\eta,\alpha t). \tag{3.3}$$

Therefore

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial\alpha}^\alpha(t) = \frac{\partial x'}{\partial\eta}\frac{\partial\eta}{\partial\alpha} + \alpha\frac{\partial}{\partial t}\ [\ tf(y(1,\eta,\alpha t))\ ]. \tag{3.4}$$

Lemma(1.2.2) yields $\frac{\partial x}{\partial\eta} < e^{2\alpha t}$ and $\frac{\partial x'}{\partial\eta} < 2\alpha e^{2\alpha t}$ as part of its proof, and Lemma (1.2.5) yields

$$|\frac{\partial\eta}{\partial\alpha}| < K(\alpha_0)\alpha(\cosh(\alpha a_{1\alpha}))^{-1}.$$

Equation (3.3) therefore implies

$$|\frac{\partial x}{\partial\alpha}^\alpha(a_{1\alpha})| \leq 2\alpha e^{a a_{1\alpha}}K(\alpha_0)\alpha(\cosh(\alpha a_{1\alpha}))^{-1} + \alpha^2. \tag{3.5}$$

Equation (3.4) implies

$$\begin{aligned}|\frac{\partial}{\partial t}\frac{\partial x}{\partial\alpha}^\alpha(t)| &\leq 2\alpha^2K(\alpha_0)(\cosh(\alpha a_{1\alpha}))^{-1} + \alpha^2\\ &\leq 2\alpha^2K(\alpha_0)(\cosh(\frac{\alpha}{3}))^{-1} + \alpha^2.\end{aligned} \tag{3.6}$$

Substituting these bounds in equation (3.2) yields

$$\begin{aligned}|\frac{\partial}{\partial\alpha}(a_{1\alpha} - a_\alpha)| &\leq 2\alpha K(\alpha_0)(\cosh\frac{\alpha}{3})^{-1}[1 + e^{\alpha a_{1\alpha}}]\ |a_{1\alpha} - a_\alpha|\\ &\leq K(\alpha)|a_{1\alpha} - a_\alpha| \leq K(\alpha)\sigma.\blacksquare\end{aligned} \tag{3.7}$$

Lemma 2.

$$|\frac{\partial N_i}{\partial \alpha}| < K(\alpha)\sigma \qquad |N_i - 1| < K(\alpha)\sigma \qquad |DN_i - 1| \leq K(\alpha)\sigma.$$

Proof:$|DN_i - 1| \leq K(\alpha)\sigma$ and $|N_i - 1| < K(\alpha)\sigma$ follows from appropriate variational equations and Gronwall's inequality as in Chapter 1. We estimate $\frac{\partial N_1}{\partial \alpha}$. The estimation of $\frac{\partial N_2}{\partial \alpha}$ is similar. Define $\bar{x}^*(t) = \bar{x}(a_\alpha + t)$. We see that $\bar{x}^*$ satisfies the same ordinary differential equation as as $\bar{x}$ does and differentiability with respect to t and α gives

$$\frac{\partial N_1}{\partial \alpha} = \frac{\partial}{\partial \alpha}\bar{x}^*(a_{1\alpha} - a_\alpha) = \frac{\partial \bar{x}^*}{\partial \alpha}(a_{1\alpha} - a_\alpha) + \bar{x}^{*\prime}(a_{1\alpha} - a_\alpha)\frac{\partial}{\partial \alpha}(a_{1\alpha} - a_\alpha)$$

where

$$\frac{\partial \bar{x}^{*\prime}}{\partial \alpha} = \begin{bmatrix} 0 & f_\sigma'(x^\alpha(t)) \\ f_\sigma'(y^\alpha(t)) & 0 \end{bmatrix} \bar{x}^* + F,$$

$|F| < 2$, and

$$\frac{\partial \bar{x}^*}{\partial \alpha}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By Gronwall's Inequality

$$\frac{\partial \bar{x}^*}{\partial \alpha}(a_{1\alpha} - a_\alpha) < e^{2\alpha(a_{1\alpha} - a_\alpha)} - 1.$$

Therefore

$$\begin{aligned} \frac{\partial N_1}{\partial \alpha} &\leq 4(a_{1\alpha} - a_\alpha) + K(\alpha)\sigma \\ &\leq 4\frac{4\sigma}{\alpha} + K(\alpha)\sigma \leq K(\alpha)\sigma \end{aligned}$$

where we used Lemma (2.1.2) to bound $(a_{1\alpha} - a_\alpha)$. ■

If $F : R^n \to R^n$, then DF is defined to be $(\frac{\partial f_i}{\partial x_j})$ if $F = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$.

Proposition 3. If $A(\alpha)$ and $B(\alpha)$ are C^1 functions of α mapping R^n to R^n then

$$\frac{\partial}{\partial \alpha}(A(\alpha)(B(\alpha))) = DA(B(\alpha))(\frac{\partial B}{\partial \alpha}(\alpha)) + (\frac{\partial A}{\partial \alpha})(B(\alpha)).$$

Proof: This follows from the chain rule. ■

Lemma 4.

$$\alpha\frac{\partial a_{1\alpha}}{\partial\alpha}=2a_{1\alpha}\alpha(1/2-a_{1\alpha})+O(\frac{1}{2}-a_{1\alpha})+K(\alpha)\sigma$$

Proof: We have $\bar{x}^\alpha(a_{2\alpha})$ and $\bar{x}^\alpha(a_{1\alpha})$ related by $N_2N_1\bar{x}^\alpha(a_{1\alpha})=\bar{x}^\alpha(a_{2\alpha})$. Using Proposition 3 and Lemma 2 repeatedly we find

$$\frac{\partial}{\partial\alpha}\bar{x}(a_{2\alpha})=\frac{\partial}{\partial\alpha}\bar{x}(a_{1\alpha})+K(\alpha)\sigma=\frac{\partial}{\partial\alpha}\bar{x}(a_\alpha)+K(\alpha)\sigma$$

Letting $Q=\bar{x}(a_\alpha)$, $\beta=$ distance(Q, line through $(0,2)$ and $(2,0)$) and $\theta=$ angle between OQ and the line y = x we can draw Figure 4.

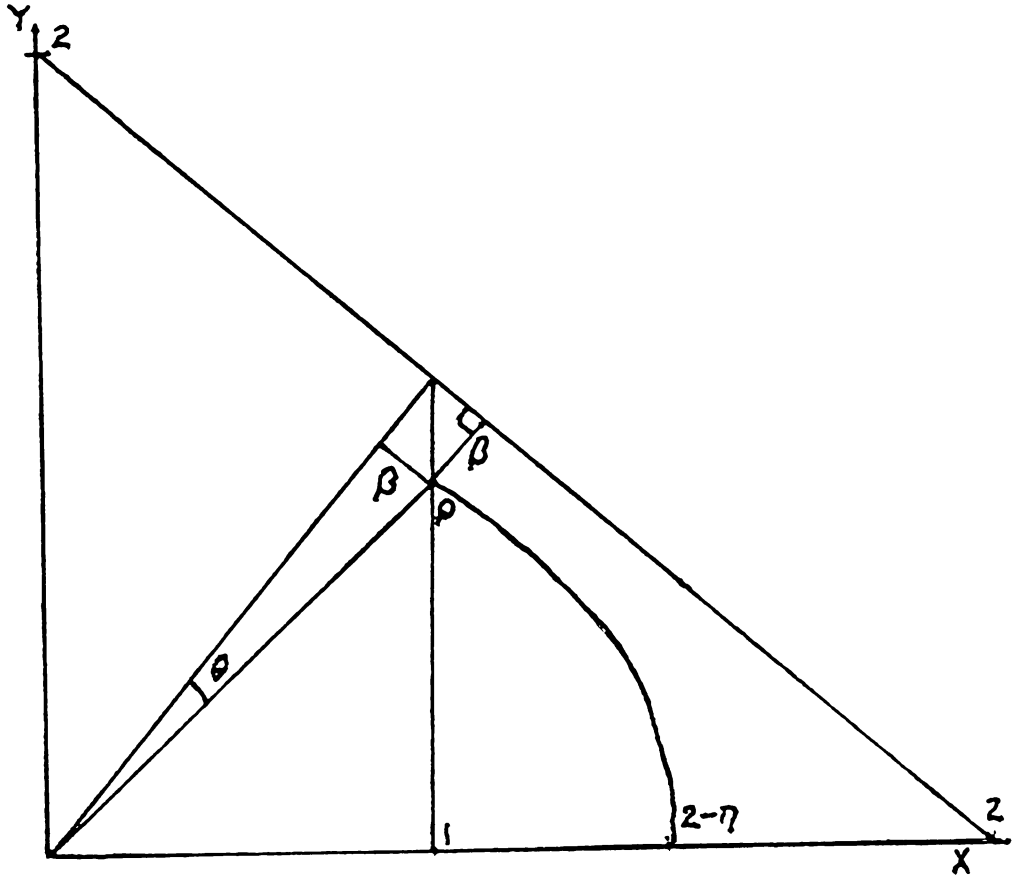

Figure 4

From Figure 4 it follows that

$$\theta=\arctan\frac{\beta}{\sqrt{2}-\beta}.$$

Therefore

$$\sec^2(\theta)\frac{\partial\theta}{\partial\beta} = \frac{\sqrt{2}}{(\sqrt{2}-\beta)^2} = \frac{1}{\sqrt{2}}(1+\frac{\beta}{\sqrt{2}}+O(\beta^2)). \tag{3.8}$$

We relate θ and β by

$$\frac{\beta}{\sqrt{2}-\beta} = \tan\theta = \int_0^\theta \sec^2 s ds \le \int_0^\theta \sec^2\theta \le \theta\sec^2\theta$$

since sec is increasing on $[0,1]$. Multiplying by $\cos^2(\theta)$ we find that

$$\frac{\beta}{2\sqrt{2}} \le \frac{\beta}{\sqrt{2}-\beta}\cos^2(\theta) = \sin\theta\cos\theta \le \theta \le \tan\theta \le \frac{\beta}{0.9\sqrt{2}}$$

for $\beta < 0.1$. Hence

$$\frac{\beta}{2\sqrt{2}} \le \theta \le \frac{\beta}{0.9\sqrt{2}}(3.9). \tag{3.9}$$

Substituting equation (3.9) in equation (3.8) and using

$$\cos^2\theta = 1 + O(\theta^2)$$

we find

$$\frac{\partial\theta}{\partial\beta} = \frac{1}{\sqrt{2}}(1+\frac{\beta}{\sqrt{2}}+O(\beta^2)) \tag{3.10}$$

$$\theta = \frac{\beta}{\sqrt{2}} + O(\beta^2).$$

Since $|N_i - 1| \le K(\alpha)\sigma$ and $|\frac{\partial N_i}{\partial\alpha}| \le K(\alpha)\sigma$, we may replace $Q = \bar{x}(a_\alpha)$ by $\bar{x}(a_{2\alpha})$ and $\bar{x}(a_{1\alpha})$ respectively in determining the relationships between β, θ, α, and all derivatives. (Here is where we use Lemmas 1 and 2.) The results of explicit calculations are

$$\theta = \alpha(1/2 - a_{1\alpha}) + K(\alpha)\sigma$$

$$\beta = \frac{\eta}{\sqrt{2}}e^{-\alpha a_{1\alpha}} + K(\alpha)\sigma$$

$$\theta = \frac{\beta}{\sqrt{2}} + O(\beta^2).$$

Removing β we find

$$\alpha(1/2 - a_{1\alpha}) = \frac{\eta}{2}e^{-\alpha a_{1\alpha}} + K(\alpha)\sigma + O(\beta^2) \tag{3.11}$$

$$\frac{\partial}{\partial\alpha}(\alpha(1/2 - a_{1\alpha})) = \frac{1}{\sqrt{2}}(1+\frac{\beta}{\sqrt{2}})\frac{\partial}{\partial\alpha}[\sqrt{2}\eta e^{-\alpha a_{1\alpha}}] + K(\alpha)\sigma + O(\beta^2). \quad (3.12)$$

This gives us relations between η, $\frac{\partial\eta}{\partial\alpha}$, and α. Expanding and simplifying equation (3.12) we find that

$$\frac{1}{2} - a_{1\alpha} - \alpha\frac{\partial a_{1\alpha}}{\partial\alpha} = \frac{1}{2}e^{-\alpha a_{1\alpha}}[-\eta\frac{\partial}{\partial\alpha}(\alpha a_{1\alpha}) + \frac{\partial\eta}{\partial\alpha}](1+O(\beta)) \quad (3.13)$$

We also have $\eta\cosh(\alpha a_{1\alpha}) = 1-\sigma$. Therefore

$$\frac{\partial\eta}{\partial\alpha}\cosh(\alpha a_{1\alpha}) + \eta\frac{\partial}{\partial\alpha}(\alpha a_{1\alpha})\sinh(\alpha a_{1\alpha}) = 0$$

which implies

$$\frac{\partial\eta}{\partial\alpha} = -\eta\frac{\partial}{\partial\alpha}(\alpha a_{1\alpha})\tanh(\alpha a_{1\alpha}). \quad (3.14)$$

Equations (3.13) and (3.14) together imply

$$\frac{1}{2} - a_{1\alpha} - \alpha\frac{\partial a_{1\alpha}}{\partial\alpha} = -e^{-\alpha a_{1\alpha}}\frac{\eta}{2}\frac{\partial}{\partial\alpha}(\alpha a_{1\alpha})(1+\tanh(\alpha a_{1\alpha})(1+O(\beta))$$

which implies

$$\begin{aligned}\alpha\frac{\partial a_{1\alpha}}{\partial\alpha} &= 2a_{1\alpha}\frac{\eta}{2}e^{-\alpha a_{1\alpha}} + O(\beta^2)\\ &= 2a_{1\alpha}\alpha(1/2 - a_{1\alpha}) + O(\beta^2) + O(\frac{1}{2} - a_{1\alpha}) \quad (3.15)\\ \alpha\frac{\partial a_{1\alpha}}{\partial\alpha} &= 2a_{1\alpha}\alpha(1/2 - a_{1\alpha}) + O(\frac{1}{2} - a_{1\alpha}) + K(\alpha)\sigma\end{aligned}$$

since $\beta \leq \alpha(1/2 - a_{1\alpha}) + K(\alpha)\sigma$. ∎

Note 5. $\alpha^2(1/2 - a_{1\alpha}) \leq \alpha e^{-\alpha a_{1\alpha}} + K(\alpha)\sigma$ by equation (11) and hence is less than $K(\alpha)\sigma$ for large α if σ is fixed.

4.Calculation of Properties of Bifurcation Functions.

We define $\mu = \lambda^{-1}$ and $A_\mu = \begin{bmatrix} 0 & \mu^{-1} \\ 1 & 0 \end{bmatrix}$. We abuse notation writing $q_i(\alpha, \mu)$ for $q_i(\alpha, \lambda^{-1})$. We define

$$\chi_1 = \bar{u}^{\alpha_i}(0) \qquad \chi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$M(\alpha, \mu) = R_{-\alpha,\mu} \Sigma^*_{\alpha,\mu} H_{\alpha,\mu} \Sigma_{\alpha,\mu} R_{\alpha,\mu}.$$

We rewrite $q_i(\alpha, \mu)$ in terms of the maps $R_{-\alpha,\mu}, \Sigma^*_{\alpha,\mu}, H_{\alpha,\mu}, \Sigma_{\alpha,\mu}, R_{\alpha,\mu}, A_\mu$ as follows, dropping dependence on μ where possible:

$$\bar{u}^\alpha(1) = R_{-\alpha} \Sigma^*_\alpha H_\alpha \Sigma_\alpha R_\alpha \bar{u}^\alpha(0) = M(\alpha, \mu) \bar{u}^\alpha(0).$$

Therefore

$$q_i(\alpha, \mu) = \det\,((M(\alpha, \mu) - A_\mu)\chi_1, ((M(\alpha, \mu) - A_\mu)\chi_2).$$

Then

$$\frac{\partial}{\partial \alpha} q_i(\alpha_i, 1) = \det(\frac{\partial}{\partial \alpha}(M(\alpha_i, 1) - A_1)\chi_1, (M(\alpha_i, 1) - A_\mu)\chi_2)$$
$$+ \det((M(\alpha_i, 1) - A_\mu)\chi_1, \frac{\partial}{\partial \alpha}(M(\alpha_i, 1) - A_1)\chi_2).$$

Using the definition of χ_1 and α_i this implies

$$\frac{\partial}{\partial \alpha} q_i(\alpha_i, 1) = \det(\frac{\partial}{\partial \alpha}(M(\alpha_i, 1) - A_1)\chi_1, (M(\alpha_i, 1) - A_1)\chi_2). \tag{4.1}$$

Similar calculations yield

$$\frac{\partial}{\partial \mu} q_i(\alpha_i, \mu) = \det\,(\frac{\partial}{\partial \mu}(M(\alpha_i, \mu) - A_\mu)\chi_1, (M(\alpha_i, 1) - A_1)\chi_2). \tag{4.2}$$

at $\mu = 1$. Explicit calculation of $(M(\alpha_i, 1) - A_1)\chi_2$ yields

$$(M(\alpha_i, 1) - A_1)\chi_2 = \sqrt{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma. \tag{4.3}$$

We replace the determinants by a dot product with an appropriate vector as this simplifies some calculations. It may be easily verified that equations (4.1) (4.2) and (4.3) imply

$$\frac{\partial}{\partial\alpha}q_i(\alpha_i,1)(1+O(\alpha(1/2-a_{1\alpha}))+K(\alpha)\sigma)=\frac{\partial}{\partial\alpha}(M(\alpha,\mu)\chi_1).\begin{bmatrix}1\\1\end{bmatrix} \qquad (4.4)$$

$$\frac{\partial}{\partial\mu}q_i(\alpha_i,1)(1+O(\alpha(1/2-a_{1\alpha}))+K(\alpha)\sigma)=M(\alpha,\mu)\chi_1-A_\mu).\begin{bmatrix}1\\1\end{bmatrix} \qquad (4.5)$$

at $\alpha=\alpha_i$ and $\mu=1$.

Lemma 1. At $\alpha=\alpha_i$

$$\frac{\partial}{\partial\alpha}(R_{-\alpha}\Sigma^*_\alpha H_\alpha\Sigma_\alpha R_\alpha\chi_1)=R_{-\alpha}[(\frac{\partial}{\partial\alpha}\Sigma^*_\alpha)H_\alpha R_\alpha+H_\alpha(\frac{\partial}{\partial\alpha}\Sigma_\alpha)R_\alpha+H_\alpha\frac{\partial}{\partial\alpha}R_\alpha$$
$$+(\frac{\partial}{\partial\alpha}H_\alpha)R_\alpha]\chi_1+(\frac{\partial}{\partial\alpha}R_{-\alpha})H_\alpha R_\alpha\chi_1+K(\alpha)\sigma.$$

Proof: This is a calculation using Lemma (3.2) repeatedly and the facts $|\Sigma_\alpha-1|,|\Sigma^*_\alpha-1|\le K(\alpha)\sigma$, $D\Sigma_\alpha=\Sigma_\alpha$, $D\Sigma^*_\alpha=\Sigma^*_\alpha$, $DH_\alpha=H_\alpha$, $DR_\alpha=R_\alpha$, and $DR_{-\alpha}=R_{-\alpha}$ (since $\Sigma_\alpha,\Sigma^*_\alpha$, H_α, and $R_{-\alpha}$ are linear). ∎

Proposition 2. For any $\bar{u}_0\in R^2$

$$\frac{\partial}{\partial\alpha}\Sigma_\alpha\bar{u}_0=-\frac{|a_{1\alpha}-a_{2\alpha}|}{\sigma}\begin{bmatrix}0\\f(y(a_{1\alpha}))u_0 2a_{1\alpha}\alpha(1/2-a_{1\alpha})\end{bmatrix}+K(\alpha)\sigma$$

$$\frac{\partial}{\partial\alpha}\Sigma^*_\alpha\bar{u}_0=-\frac{|a_{1\alpha}-a_{2\alpha}|}{\sigma}\begin{bmatrix}f(x(b_{1\alpha}))z_0 2a_{1\alpha}\alpha(1/2-a_{1\alpha})\\0\end{bmatrix}+K(\alpha)\sigma.$$

Proof: To find $\frac{\partial}{\partial\alpha}\Sigma_\alpha$ and $\frac{\partial}{\partial\alpha}\Sigma^*_\alpha$ up to order σ notice that for $t\in[a_{1\alpha},a_{2\alpha}]$, by the definition of Σ_α we must look at

$$\bar{u}'=\alpha\begin{bmatrix}0 & f'_\sigma(x^\alpha(t))\\ f'_\sigma(y^\alpha(t)) & 0\end{bmatrix}\bar{u}=\alpha\begin{bmatrix}0 & -1\\ \sigma^{-1}(x^\alpha(t)-1) & 0\end{bmatrix}\bar{u}.$$

Hence, replacing t by t - $a_{1\alpha}$

$$\frac{\partial\bar{u}'}{\partial\alpha}=\begin{bmatrix}0 & -1\\ \sigma^{-1}(x^\alpha(t)-1) & 0\end{bmatrix}\bar{u}+\frac{\alpha}{\sigma}\begin{bmatrix}0 & 0\\ \frac{\partial x^\alpha}{\partial\alpha}(a_{1\alpha}+t) & 0\end{bmatrix}\bar{u}$$
$$+\alpha\begin{bmatrix}0 & -1\\ \sigma^{-1}(x^\alpha(t)-1) & 0\end{bmatrix}\frac{\partial\bar{u}}{\partial\alpha}$$

with

$$\frac{\partial \bar{u}}{\partial \alpha}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Further dividing t by σ we find

$$\frac{\partial \bar{u}'}{\partial \alpha} = \sigma \begin{bmatrix} 0 & -1 \\ \sigma^{-1}(x^\alpha(t)-1) & 0 \end{bmatrix} \bar{u} + \alpha \begin{bmatrix} 0 & 0 \\ \frac{\partial x^\alpha}{\partial \alpha}(a_{1\alpha}+\sigma t) & 0 \end{bmatrix} \bar{u}$$
$$+ \alpha\sigma \begin{bmatrix} 0 & -1 \\ \sigma^{-1}(x^\alpha(t)-1) & 0 \end{bmatrix} \frac{\partial \bar{u}}{\partial \alpha}$$

for $t \in [0, \sigma^{-1}|a_{1\alpha} - a_{2\alpha}|] \subseteq [0,4]$. Since $|\sigma^{-1}(x^\alpha(t)-1)| \leq 1$, the last equation becomes

$$\frac{\partial \bar{u}'}{\partial \alpha} = \alpha \begin{bmatrix} 0 & 0 \\ \frac{\partial x^\alpha}{\partial \alpha}(a_{1\alpha}+\sigma t)u & 0 \end{bmatrix} \bar{u}(a_{1\alpha}) + O(\sigma).$$

The same arguments that bounded $|\Sigma_\alpha - 1|$ yield $|u - u_0| \leq K(\alpha)\sigma$. This estimate with Lemma (3.1) now implies

$$\frac{\partial \bar{u}}{\partial \alpha}(t) = \alpha \begin{bmatrix} 0 \\ \frac{\partial x^\alpha}{\partial \alpha}(a_{1\alpha})u_0 \end{bmatrix} + O(\sigma).$$

Noting by Lemma 3.1 that $\frac{\partial x^\alpha}{\partial \alpha}(a_{1\alpha}) = -\alpha f(y(a_{1\alpha})\frac{\partial}{\partial \alpha}a_{1\alpha}$ and substituting $t = \frac{(a_{2\alpha}-a_{1\alpha})}{\sigma}$ we find

$$\frac{\partial \bar{u}}{\partial \alpha}(a_{2\alpha} - a_{1\alpha}) = -\frac{|a_{1\alpha} - a_{2\alpha}|}{\sigma} \begin{bmatrix} 0 \\ f(y(a_{1\alpha}))u_0 2a_{1\alpha}\alpha(1/2 - a_{1\alpha}) \end{bmatrix} + K(\alpha)\sigma,$$

where we used continous dependence of solutions to ordinary differential equations on their right hand sides to justify the substitutions. Since

$$\frac{\partial}{\partial \alpha}\Sigma_\alpha \bar{u}_0 = \frac{\partial}{\partial \alpha}\bar{u}(a_{2\alpha} - a_{1\alpha}) = \frac{\partial \bar{u}}{\partial \alpha}(a_{2\alpha} - a_{1\alpha}) + \bar{u}'(a_{2\alpha} - a_{1\alpha})\frac{\partial}{\partial \alpha}(a_{2\alpha} - a_{1\alpha}),$$

we have

$$\frac{\partial}{\partial \alpha}\Sigma_\alpha \bar{u}_0 = -\frac{|a_{1\alpha} - a_{2\alpha}|}{\sigma} \begin{bmatrix} 0 \\ f(y(a_{1\alpha}))u_0 2a_{1\alpha}\alpha(1/2 - a_{1\alpha}) \end{bmatrix} + K(\alpha)\sigma. \qquad (4.7)$$

Similarly we find

$$\frac{\partial}{\partial \alpha}\Sigma^*_\alpha \bar{u}_0 = -\frac{|a_{1\alpha} - a_{2\alpha}|}{\sigma} \begin{bmatrix} f(x(b_{1\alpha}))z_0 2a_{1\alpha}\alpha(1/2 - a_{1\alpha}) \\ 0 \end{bmatrix} + K(\alpha)\sigma. \blacksquare \qquad (4.8)$$

The calculations to determine $\frac{\partial q_i}{\partial \alpha}$ and $\frac{\partial q_i}{\partial \mu}$ are now straightforward, we quote from the results calculated in Chapter 3 .

Proposition 3.

$$\frac{\partial q_i}{\partial \alpha} = 4\sqrt{2}a_{1\alpha}\alpha(1/2 - a_{1\alpha})(1 + O(\alpha(1/2 - a_{1\alpha}))) \tag{4.9}$$

$$\frac{\partial q_i}{\partial \mu} = -\frac{1}{\sqrt{2}} + O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma. \tag{4.10}$$

Proof: See Chapter 3, sections 1 and 2.■

The main theorem of this chapter says that for sufficiently small σ there will be a large number of bifurcation points at large α. Moreover, these bifurcation points are all isolated.

Theorem 4. For each $N > 0$ there is an $\alpha_0 > 0$, a σ_0 and a set $A = \{\alpha_i\}_{i=1}^{N}$ such that $\sigma < \sigma_0$ implies α_i are bifurcation points for equation (1.1.1) and each of the α_i is an isolated bifurcation point.

Proof: First choose α_0 and σ_0 as in Theorem (2.4.1). These choices give us a (possibly infinite) set of at least N elements $\{\alpha_i\}$ where $q(\alpha_i, 1) = 0$. With these choices of α and σ we have, by equations (4.9) and (4.10)

$$\frac{\partial q_i}{\partial \alpha} \geq 2a_{1\alpha}\alpha(\frac{1}{2} - \alpha) + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma$$

$$\frac{\partial q_i}{\partial \mu} \leq -\frac{1}{\sqrt{2}} + O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma$$

with $\alpha = \alpha_i$.

We may assume α_0 is large enough so that for all $\alpha > \alpha_0$ $O(\alpha(1/2 - a_{1\alpha})) < \frac{1}{16}$ and $O((\alpha(1/2 - a_{1\alpha}))^2) \leq \frac{1}{16}\alpha(1/2 - a_{1\alpha})$. We decrease σ_0 so that we have $\max_{\alpha \in (\alpha_0, \alpha_N)} K(\alpha)\sigma < \frac{1}{16}$. With these choices

$$\frac{\partial q_i}{\partial \alpha} \geq a_{1\alpha}\alpha_i(\frac{1}{2} - \alpha_i) \tag{4.11}$$

and

$$\frac{\partial q_i}{\partial \mu} \geq -\frac{1}{4} + \frac{1}{16} + \frac{1}{16} \geq \frac{1}{8} \tag{4.12}$$

Applying inequalities (11) and (12)

$$\frac{\partial\lambda}{\partial\alpha} = -\frac{\partial q_i}{\partial\alpha}\left(\frac{\partial q_i}{\partial\mu}\right)^{-1} \neq 0$$

at $\mu = 1$. We may therefore apply Proposition (1.2) to conclude that all a_i are bifurcation points. Furthermore $\frac{\partial\lambda}{\partial\alpha} \neq 0$ implies that the condition $q_i(\alpha, 1) = 0$ is uniquely satisfied by α_i for all α in a neighbourhood of α_i. Therefore the set A is composed of isolated points. A is a compact set (it is bounded and the preimage under a continuous map of a finite set) and hence must be finite since any compact set of isolated points is finite. ■

Note 4. $\frac{\partial\lambda}{\partial\alpha}$ is small for large α and small σ, more precisely for any ϵ there are choices of α and σ so that $\frac{\partial\lambda}{\partial\alpha} < \epsilon$ by arguments similar to those used in Theorem 4.

Note 5. Unless one has monotonicity of some type in f, determining stability of x^α (rather than change of stability) seems difficult to do analytically. A major problem in the analysis is to handle complex multipliers of the orbit. In the approach used here one has a boundary value problem in 4 real dimensions, which seems difficult to study.

Chapter 3

CALCULATIONS

1. Computation of $\frac{\partial q_1}{\partial \alpha}$

We will calculate each of the terms in Lemma (3.4.2) separately. First, the terms involving Σ_α and Σ_α^*

Lemma 1.

$$\begin{aligned} R_{-\alpha}[\frac{\partial}{\partial\alpha}\Sigma_\alpha^*(H_\alpha R_\alpha) + H_\alpha\frac{\partial}{\partial\alpha}\Sigma_\alpha(R_\alpha)]\chi_1 \\ = -R_{-\alpha}\frac{|a_{1\alpha}-a_{2\alpha}|}{\sigma}\begin{bmatrix} f(x(b1a))2a_{1\alpha}\alpha(1/2-a_{1\alpha}) \\ 0 \end{bmatrix} \\ + K(\alpha)\sigma + O((\alpha(1/2-a_{1\alpha}))^2) \end{aligned} \tag{1.1}$$

Proof: We have, by corollary (1.6.2)

$$H_\alpha R_\alpha\chi_1 = \begin{bmatrix} -\alpha(1/2-a_{1\alpha}) \\ 1 \end{bmatrix} + O((\alpha(1/2-a_{1\alpha}))^2) + K(\alpha)\sigma$$

so Proposition (2.4.3) implies

$$\begin{aligned} (\frac{\partial}{\partial\alpha}\Sigma_\alpha^*)H_\alpha R_\alpha\chi_1 = -\frac{|a_{1\alpha}-a_{2\alpha}|}{\sigma}\begin{bmatrix} f(x(b_{1\alpha}))2a_{1\alpha}\alpha(1/2-a_{1\alpha}) \\ 0 \end{bmatrix} \\ + K(\alpha)\sigma + O((\alpha(1/2-a_{1\alpha}))^2) \end{aligned} \tag{1.2}$$

Similarly

$$R_\alpha\chi_1 = \begin{bmatrix} \alpha(1/2-a_{1\alpha}) \\ 1 \end{bmatrix} + O((\alpha(1/2-a_{1\alpha}))^2).$$

Proposition (2.4.3) implies

$$\begin{aligned} H_\alpha(\frac{\partial}{\partial\alpha}\Sigma_\alpha)R_\alpha\chi_1 = H_\alpha\frac{|a_{1\alpha}-a_{2\alpha}|}{\sigma}\begin{bmatrix} 0 \\ f(x(a_{1\alpha}))2a_{1\alpha}\alpha(1/2-a_{1\alpha})^2 \end{bmatrix} \\ + K(\alpha)\sigma + O((\alpha(1/2-a_{1\alpha}))^2) \end{aligned}$$

or

$$H_\alpha(\frac{\partial}{\partial\alpha}\Sigma_\alpha)R_\alpha\chi_1 = H_\alpha O((\alpha(1/2-a_{1\alpha}))^2) + K(\alpha)\sigma + O((\alpha(1/2-a_{1\alpha}))^2)$$
$$= K(\alpha)\sigma + O((\alpha(1/2-a_{1\alpha}))^2). \tag{1.3}$$

Equations (1.2) and (1.3) together give equation (1.1) ■

Lemma 2.

$$R_{-\alpha}(H_\alpha\frac{\partial}{\partial\alpha}R_\alpha+(\frac{\partial}{\partial\alpha}H_\alpha)R_\alpha)\chi_1 = R_{-\alpha}(a_{1\alpha}\alpha(1/2-a_{1\alpha})\begin{bmatrix}4\\0\end{bmatrix}$$
$$+\frac{\partial}{\partial\alpha}(\alpha a_{1\alpha})\begin{bmatrix}-1\\3\alpha(1/2-a_{1\alpha})\end{bmatrix}) \tag{1.4}$$
$$+O((\alpha(1/2-a_{1\alpha}))^2)+O(\frac{1}{2}-a_{1\alpha})+K(\alpha)\sigma$$

Proof: First we calculate $\frac{\partial}{\partial\alpha}H_\alpha$.

$$H_\alpha = \begin{bmatrix}\cosh(2\alpha(1/2-a_{1\alpha})) & -\sinh(2\alpha(1/2-a_{1\alpha}))\\ -\sinh(2\alpha(1/2-a_{1\alpha})) & \cosh(2\alpha(1/2-a_{1\alpha}))\end{bmatrix}$$

therefore

$$\frac{\partial}{\partial\alpha}H_\alpha = 2(\frac{1}{2}-a_{1\alpha}-\alpha\frac{\partial a_{1\alpha}}{\partial\alpha})\begin{bmatrix}\sinh(2\alpha(1/2-a_{1\alpha})) & -\cosh(2\alpha(1/2-a_{1\alpha}))\\ -\cosh(2\alpha(1/2-a_{1\alpha})) & \sinh(2\alpha(1/2-a_{1\alpha}))\end{bmatrix}$$
$$= -4a_{1\alpha}\alpha(1/2-a_{1\alpha})\begin{bmatrix}2\alpha(1/2-a_{1\alpha}) & -1\\ -1 & 2\alpha(1/2-a_{1\alpha})\end{bmatrix}$$
$$+O((\alpha(1/2-a_{1\alpha}))^2)+O(\frac{1}{2}-a_{1\alpha}).$$

Using our knowledge of $\alpha(1/2-a_{1\alpha})$ we therefore have

$$\frac{\partial}{\partial\alpha}H_\alpha = 2a_{1\alpha}\alpha(1/2-a_{1\alpha})\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+O((\alpha(1/2-a_{1\alpha}))^2)+O(\frac{1}{2}-a_{1\alpha}). \tag{1.5}$$

Substituting for $R_\alpha\chi_1$ we have

$$(\frac{\partial}{\partial\alpha}H_\alpha)R_\alpha\chi_1 = a_{1\alpha}\alpha(1/2-a_{1\alpha})\begin{bmatrix}4\\0\end{bmatrix}+O((\alpha(1/2-a_{1\alpha}))^2)+O(\frac{1}{2}-a_{1\alpha}) \tag{1.6}$$

Noting that R_α is the fundamental solution for a set of differential equations and using $\alpha a_{1\alpha} = \frac{\pi}{4}$ mod 2 mπ we calculate $\frac{\partial}{\partial\alpha}R_\alpha\chi_1$:

$$\frac{\partial}{\partial\alpha}R_\alpha\chi_1 = \frac{\partial}{\partial\alpha}(\alpha a_{1\alpha})\begin{bmatrix}-1\\ \alpha(1/2-a_{1\alpha})\end{bmatrix}+O((\alpha(1/2-a_{1\alpha}))^2)+K(\alpha)\sigma$$

so

$$H_\alpha \frac{\partial}{\partial \alpha} R_\alpha \chi_1 = \frac{\partial}{\partial \alpha}(\alpha a_{1\alpha}) \begin{bmatrix} -1 \\ 3\alpha(1/2 - a_{1\alpha}) \end{bmatrix} + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma. \quad (1.7)$$

Equations (1.6) and (1.7) together give equation (1.4). ■

Lemma 3 .

$$(\frac{\partial}{\partial \alpha} R_{-\alpha}) H_\alpha R_\alpha \chi_1 = \frac{\partial}{\partial \alpha}(\alpha a_{1\alpha}) R_{-\alpha} \begin{bmatrix} 1 \\ \alpha(1/2 - a_{1\alpha}) \end{bmatrix}$$

Proof: We omit the calculation, which is similar to that in Lemma 2.

Using Lemmas 1, 2 and 3 we find, after simplification, and using the formula (2.4.4)

$$\frac{\partial}{\partial \alpha}(R_{-\alpha} \Sigma_\alpha^* H_\alpha \Sigma_\alpha R_\alpha \chi_1) = V$$
$$= \alpha(1/2 - a_{1\alpha}) a_{1\alpha} R_{-\alpha} \begin{bmatrix} \frac{2f(x(b_{1\alpha}))^{\frac{|a_{1\alpha} - a_{2\alpha}|}{\sigma}}}{4} + 4 \end{bmatrix} \quad (1.8)$$

We note that R_α is norm preserving for any α so

$$\frac{\partial q_i}{\partial \alpha} = V. \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1 + O(\alpha(1/2 - a_{1\alpha}))) = R_\alpha V. R_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1 + O(\alpha(1/2 - a_{1\alpha})))$$

We substitute for $R_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ finding

$$R_\alpha V. R_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha(1/2 - a_{1\alpha}) a_{1\alpha} \sqrt{2} \begin{bmatrix} \frac{2f(x(b_{1\alpha}))^{\frac{|a_{1\alpha} - a_{2\alpha}|}{\sigma}}}{4} + 4 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \end{bmatrix} + O((\alpha(1/2 - a_{1\alpha}))^2)$$

Therefore

$$\frac{\partial q_i}{\partial \alpha} = 4\sqrt{2} a_{1\alpha} \alpha(1/2 - a_{1\alpha}) + O((\alpha(1/2 - a_{1\alpha}))^2) \quad (1.9)$$

2. Calculation of $\frac{\partial q_i}{\partial \mu}$

For the calculation of $\frac{\partial q_i}{\partial \mu}$ we may ignore $\Sigma^*_{\alpha,\mu}, \Sigma_{\alpha,\mu}$ since $\frac{\partial}{\partial \mu}\Sigma^*_{\alpha,\mu}, \frac{\partial}{\partial \mu}\Sigma_{\alpha,\mu} \leq K(\alpha)\sigma$ by Gronwall's inequality. We therefore calculate

$$\frac{\partial}{\partial \mu}(R_{-\mu}H_\mu R_\mu - A_\mu)\chi_1 = (\frac{\partial}{\partial \mu}R_{-\mu})H_\mu R_\mu \chi_1 + R_{-\mu}(\frac{\partial}{\partial \mu}H_\mu)R_\mu\chi_1$$
$$+ R_{-\mu}H_\mu(\frac{\partial}{\partial \mu}R_\mu)\chi_1 - \frac{\partial}{\partial \mu}A_\mu\chi_1$$

at $\mu = 1$. Explicit calculation gives us

$$R_\mu = \begin{bmatrix} \cos(\alpha\sqrt{\mu}a_{1\alpha}) & \frac{1}{\sqrt{\mu}}\sin(\alpha\sqrt{\mu}a_{1\alpha}) \\ -\sqrt{\mu}\sin(\alpha\sqrt{\mu}a_{1\alpha}) & \cos(\alpha\sqrt{\mu}a_{1\alpha}) \end{bmatrix}.$$

therefore

$$\frac{\partial}{\partial \mu}R_\mu|_{\mu=1} = \frac{\alpha a_{1\alpha}}{2}\begin{bmatrix} -\sin(\alpha a_{1\alpha}) & \cos(\alpha a_{1\alpha}) \\ -\cos(\alpha a_{1\alpha}) & -\sin(\alpha a_{1\alpha}) \end{bmatrix} - \sin(\alpha a_{1\alpha})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= -\frac{\alpha a_{1\alpha}}{2}\mathrm{Rot}_{-\frac{\pi}{4}} - \sin(\alpha a_{1\alpha})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + K(\alpha)\sigma.$$

Similarly

$$\frac{\partial}{\partial \mu}R_{-\mu}|_{\mu=1} = -\frac{\alpha a_{1\alpha}}{2}Rot_{\frac{\pi}{4}} + \sin(\alpha a_{1\alpha})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + K(\alpha)\sigma$$

therefore

$$\frac{\partial}{\partial \mu}R_{-\mu}H_\mu R_\mu\chi_1 + R_{-\mu}H_\mu\frac{\partial}{\partial \mu}R_\mu\chi_1 = \frac{\alpha a_{1\alpha}}{2}(-R_\alpha H_\alpha R_\alpha\chi_1 - R_{-\alpha}H_\alpha R_{-\alpha}\chi_1)$$
$$+ \sin(\alpha a_{1\alpha})(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(H_\alpha R_\alpha\chi_1) - R_{-\alpha}H_\alpha\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\chi_1).$$

Since

$$H_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + O(\alpha(1/2 - a_{1\alpha}))$$

we have

$$\frac{\partial}{\partial \mu}R_{-\mu}H_\mu R_\mu\chi_1 + R_{-\mu}H_\mu\frac{\partial}{\partial \mu}R_\mu\chi_1$$
$$= -\alpha a_{1\alpha}(Rot_{\frac{\pi}{4}}Rot_{\frac{\pi}{4}} + Rot_{\frac{-\pi}{4}}Rot_{\frac{-\pi}{4}})\chi_1 \tag{2.2}$$
$$+ \sin(\alpha a_{1\alpha})(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}R_\alpha\chi_1 - R_{-\alpha}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\chi_1)$$
$$+ O(\alpha(1/2 - a_{1\alpha})) + K(\alpha)\sigma$$

Substituting $\chi_1 = \begin{bmatrix} \alpha(1/2 - a_{1\alpha}) \\ 1 \end{bmatrix} + O((\alpha(1/2 - a_{1\alpha}))^2) + K(\alpha)\sigma$ we find

$$\frac{\partial}{\partial\mu} R_{-\mu} H_\mu R_\mu \chi_1 + R_{-\mu} H_\mu \frac{\partial}{\partial\mu} R_\mu \chi_1$$
$$= 0 + O(\alpha^2(1/2 - a_{1\alpha})) + K(\alpha)\sigma$$

By Gronwall's inequality and the variational equation for H_α, $\frac{\partial}{\partial\mu} H_{\alpha,\mu}$ is of order $\alpha(1/2 - a_{1\alpha})$, therefore

$$\frac{\partial}{\partial\mu}|_{\mu=1}(R_\mu H_\mu R_\mu - A_\mu)\chi_1 = \frac{\partial}{\partial\mu} A_\mu|_{\mu=1}\chi_1 + O(\alpha^2(1/2 - a_{1\alpha})) + K(\alpha)\sigma \quad (2.3)$$

Therefore, using (3.4.5)

$$\frac{\partial q_i}{\partial\mu} = (\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + O(\alpha^2(1/2 - a_{1\alpha})) + K(\alpha)\sigma). \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1 + O(\alpha(1/2 - a_{1\alpha})))$$

$$\frac{\partial q_i}{\partial\mu} = -\frac{1}{\sqrt{2}} + O(\alpha^2(1/2 - a_{1\alpha})) + K(\alpha)\sigma. \blacksquare \quad (2.4)$$

3. Machine calculations

The following manipulations were done on a Sun 3/50 using macsyma, a symbol manipulation program. Pieces of the calculation as well as the final results were checked against hand calculations. There is noticeable divergence in the type of symbolic manipulations done by hand and by machine, the main cause being geometric guidance in the algebra done by hand.

/* for partial with respect to α */

/*$x = \alpha, a1 = a_1(\alpha), zz = \alpha(1/2 - a_{1\alpha})$ */

```
a1(x);zz(x);
```

/* tell the system a1 and zz are functions */

```
let(diff(zz(x),x),-2*a1(x)*zz(x));let(diff(a1(x),x),2 * a1(x) * zz(x)/x);
```

/* tell the system the derivatives of a1 and zz */

```
let(cosh(2*zz(x)),1);
let(sinh(2*zz(x)),2*zz(x));
```

```
let(cos(x*a1(x)),1/sqrt(2));

let(sin(x*a1(x)),1/sqrt(2));

/* These are true statements at α_i up to higher order

in α(1/2 - a_1α) */

h1: matrix([cosh(2* zz(x)),-sinh(2*zz(x))],

        [-sinh(2*zz(x)),cosh(2*zz(x))]);

h:letsimp(h1);

dh:letsimp(diff(h1,x));

dsigstar:matrix([0, yy*a1(x)*zz(x)],

          [0, 0 ]);

r1:matrix([cos(x*a1(x)),-sin(x*a1(x))],

        [sin(x*a1(x)), cos(x*a1(x))]);

dr:letsimp(diff(r1,x));

r:letsimp(r1);

rminus1:matrix([cos(x*a1(x)),sin(x*a1(x))],

        [-sin(x*a1(x)),cos(x*a1(x))]);

drminus:letsimp(diff(rminus1,x));

rminus:letsimp(rminus1);

oneone:matrix([1,1]);

chi1 : matrix([(1+zz(x))*1/sqrt(2)],

        [(1-zz(x))*1/sqrt(2)]);

ans:oneone.(rminus. (dsigstar.h.r + h.dr + dh.r )

        +drminus.h.r

    ).chi1

factor(letsimp(ans));

/* the result: checks with hand calculation */

                8 a1(x) zz(x) (zz(x) + 1)

(d24)        -------------------
```

sqrt(2)

/* for partial with respect to μ we define

functions of μ*/

/* zz = $\alpha(1/2 - a_{1\alpha})$ */

/* x = α */

/* mu = μ */

```
h1: matrix([cosh(2* zz),-sinh(2*zz)],
        [-sinh(2*zz),cosh(2*zz)]);
r1:matrix([cos(x*a1*sqrt(mu)),-sqrt(mu)*sin(x*a1*sqrt(mu))],
        [1/sqrt(mu)*sin(x*a1*sqrt(mu)),cos(x*a1*sqrt(mu))]);
rminus1:matrix([cos(x*a1*sqrt(mu)),sqrt(mu)*sin(x*a1*sqrt(mu))],
        [-1/sqrt(mu)*sin(x*a1*sqrt(mu)),cos(x*a1*sqrt(mu))]);
```

/* we give ourselves easily checked original versions of each of our

matrices */

```
let(sin(a1*sqrt(mu)*x),sin(a1*x));
let(cos(a1*sqrt(mu)*x),cos(a1*x));
let(cosh(2*zz),1);
let(sinh(2*zz),2*zz);
let(cos(x*a1),1/sqrt(2));
let(sin(x*a1),1/sqrt(2));
let(sqrt(mu),1);
let(1/mu,1);
let(1/power ( mu,(3/2))),1);
let(1/sqrt(mu),1);
```

/* tell the system some things that are true up to order σ

and zz ^{2} */

```
drmu:letsimp(diff(r1,mu));
drminusmu:letsimp(diff(rminus1,mu));
```

```
h:letsimp(h1);

r:letsimp(r1);

rminus:letsimp(rminus1);

/* tell the system to simplify using the let rules we have given it */

oneone:matrix([1,1]);

chi1 : matrix([(1+zz)*1/sqrt(2)],
        [(1-zz)*1/sqrt(2)]);

ans:oneone.(drminusmu.h.r+rminus.h.drmu ).chi1 ;

factor(letsimp(ans));

/* factor tells the system to work harder at simplification; without
this the system produces about 2 pages of output */
```

substituting values for x and zz(x) as defined above

we get:

```
              2 zz (zz - 2 a1 x)
(d35)       - ------------------
                  sqrt(2)
```

consistent with 0 + O (alpha*alpha*(1/2-a1(alpha)).

LIST OF REFERENCES

[1] S.-N. Chow, O. Diekmann and J. Mallet-Paret, Stability, multiplicity, and global continuation of symmetric periodic solutions of a nonlinear Volterra integral equation. Japan Journal of Applied Mathematics, 2 (1985), 433-469.

[2] S.-N. Chow and H. O. Walther, Characteristic multipliers and stability of symmetric periodic solutions of x'(t) = g(x(t-1)). In preparation (1986).

[3] J. K. Hale, Theory of Functional Differential Equations . Springer 1977.

[4] J.L. Kaplan and J.A. Yorke, Ordinary differential equations which yield periodic solutions of differential delay equations. J.Math.Anal.Appl., 48 (1974),317-324.

[5] Mackey and Glass, Pathological conditions resulting from instabilities in physiological control systems. N.Y.Acad.Sci, 316 (1979),214-235.

[6] J. Mallet-Paret, Generic propeties of retarded differential equations. Bull. Amer.Math.Soc., 81 (1975), 750-751.

[7] J. Mallet-Paret, Generic properties of retarded differential equations. J.D.E., 25 (1977) 163-183.

[8] J. Mallet-Paret, Morse decompositions and global continuation for singularly perturbed delay equations. Systems of Nonlinear Partial Differential Equations, Oxford 1982, 351-360.

[9] L.Nirenberg, Topics in Nonlinear Functional Analysis. New York Courant Institute of Mathematics and Science 1973-1974.

[10] R.D.Nussbaum, Uniqueness and non-uniqueness for periodic solutions of x'(t) = -g (x(t-1)). J.D.E., 34 (1979) 25-54.

[11] H.O.Walther, Bifurcation from periodic solutions in functional differential equations. Math.Z., 182 (1983), 269-289.

[12] H.O.Walther, Homoclinic solution and chaos in $x'(t) = f(x(t-1))$. Nonlinear Anal. 5 (1981) no. 7,775-778