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STUDENTS' CONCEPTIONS OF VARIABLES AND THEIR USES FOR GENERALIZATION OF MATHEMATICAL PATTERNS

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STUDENTS' CONCEPTIONS OF VARIABLES AND THEIR USES FOR GENERALIZATION OF MATHEMATICAL PATTERNS

By

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ABSTRACT

STUDENTS' CONCEPTIONS OF VARIABLES AND THEIR USES FOR GENERALIZATION OF MATHEMATICAL PATTERNS

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Statement of the Problem

This study investigated students' conceptions of variables as tools for generalizing patterns. The specific research questions addressed were:

I. What types of mathematical patterns can students recognize?

II. What types of patterns can students generalize by using variables?

III. What meaning do generalizations using variables have for students?

Methods

The research questions were addressed by in-depth interviews with seventh graders and high school algebra students (n = 13 for each group). The interviews consisted of five tasks.

Findings

One conclusion of the research was that the smaller the number of surface features built into the groups of problems the more successful students were at recognizing the deep patterns (concepts) which existed within the problem. One result of this fact was that patterns in tables were easier for students to recognize than patterns

in groups of expressions, sentences and sequences. Patterns which were built on one operation (i.e., x + 5 and 3x) were more easily recognized than patterns involving two operations (i.e., 5x + 3 and 5x - 4).

Once algebra students and seventh graders recognized a pattern they were usually able to generalize it using variables. It was harder for students to recognize patterns than it was for them to generalize those same patterns. Incorrect generalizations made by the algebra students usually contained too many variables. The errors made by the seventh graders indicate that some of them might be confused by the difference between a constant and a variable.

Working backwards from a generalization containing variables to a mathematical situation that fits the generalization was not easy for students. One of the techniques that students used to try to make sense of the generalizations was the use of the variable to represent the beginning initial of the object the problem was about. Another technique used by students was to assign the variable a constant value and to talk about it as if it were constant. A technique which was only used by seventh graders was to take expressions and change them into sentences by setting them equal to some value.

DEDICATION

This dissertation is dedicated
to my husband
Craig Stuart Ericksen
and to my grandmother
Marie Tiffany Donner.

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TABLE OF CONTENTS

		Page
LIST OF TABLES		viii
LIST OF FIGURES		x
Chapter		
I.	INTRODUCTION	1
	Background	3
	The Difference Between Arithmetic and Algebra	3
	The Historical Development of Variables	4
	The Nature of the Difficulty With Variables	5
	The Historical Versus the Modern Presentation of Variables	7
	The Research Problem	10
	Research Questions	11
	Overview of the Research Design	12
	Assumptions	13
	Limitations	13
	Overview of the Dissertation	14
II.	REVIEW OF THE LITERATURE	16
	Introduction	16
	Articles Concerning Students' Conceptions of Symbols in General and Variables in Particular	18
	Research on, And Articles About, Teaching Models Designed to Enhance Students' Understanding of Variables	34
	Summary	42
III.	METHODS	46
	Overview	46
	Subjects	47
	Data	48
	Development of the Interview	48
	The Interview	49
	The Card Sort and Generalization of Card Sort Cards	49

Chapter | Page

Recognition and Generalization of Patterns in Tables 55
Interpretation of Expression, Sentence and Sequence Generalizations by Development of Word Problems 57
Data Collection Procedure 58
Initial Meeting Between Participating Teachers and Principals 58
Gathering Information on Student Ability . 58
The Interview Schedule 59
Pilot Interviews With The Students . . . 59
The Data Collection Interviews 60
Data Analysis Procedures 60
Transcription of the Audio Tapes 60
Analysis of the Transcripts 61
Summary 64

IV. RESULTS AND DISCUSSION 65

Introduction 65
Results 68
Research Question One: What Types of Deep Patterns Can Students Recognize? . . 68
Issue One: What Deep Patterns Did Students Recognize? 69
Issue Two: What Makes a Pattern Difficult of Easy For Students to Recognize? 73
Types of Numbers Used 75
Properties of numbers 75
Common Features of more than one Expression or Sentence 80
Ways of Representing Expressions, Sentences and Sequences 80
Mathematical Symbols and Notations Used 80
Issue Three: What Types of Mistake do Different Patterns Lead Students to Make? 87
Conclusion 88
Research Question Two: What Types of Patterns Can Students Generalize by Using Variables? 91
Issue One: What Patterns Can Students Generalize and What is the Connection Between the Ability to Generalize Patterns and the Ability to Recognize Patterns 91
Issue Two: What Types of Errors Do Students Make When Generalizing Expressions and Sentences? 97
Conclusion 99

Chapter Page

Research Question Three: What Meaning Do Generalizations Using Variables Have For Students? 101
Issue One: What Strategies Do Students Use To Obtain Meaning From Generalizations? 102
Conclusion 106
Summary 108

V. SUMMARY AND IMPLICATIONS 109

Overview 109
Summary 109
Subjects 111
Data 112
Data Analysis 112
Results 113
Conclusion 116
Implications for Teaching 117
Implications for Future Research 121
Chapter Summary 123

APPENDICES . . . .

Appendix A: The Interview Tasks 124
Appendix B: A Copy of the Permission Slip 127

REFERENCES 129

LIST OF TABLES

Table		Page
3.1	Example of a Set of Interview Cards	51
4.1	Relationships Among Research Questions, Analysis Issues and Interview Tasks	67
4.2	Students' Ability to Recognize Patterns as Demonstrated by the Card Sort.	69
4.3	Results From Recognition of Patterns in Tables Task	72
4.4	What Surface Features Students Paid Attention to When Sorting the Cards Representing the Deep Patterns of Multiplicative Identity and Commutative Property of Multiplication . .	76
4.5	What Surface Features Students Paid Attention to When Sorting the Cards Representing the Deep Patterns of an Arithmetic Sequence with the common Difference of Two and a Geometric Sequence With a Common Ratio of Two . . .	77
4.6	What Surface Features Students Paid Attention to When Sorting the Cards Representing the Deep Patterns of a Area and Perimeter	78
4.7	Types of Surface Features	79
4.8	When Sorting the Cards Representing the Deep Patterns of Multiplicative Identity and Commutative Property of Multiplication Those Cards Which Students Placed Together in More Than Fifty Percent of Their Total Sorts . .	82
4.9	When Sorting the Cards Representing the Deep Patterns Arithmetic Sequences With a Common Difference of Two and Geometric Sequences with a Common Ratio Of Two Those Cards Which Students Placed Together in More Than Fifty Percent of Their Total Sorts . .	83
4.10	When Sorting the Cards Representing the Deep Patterns of Area and Perimeter Those Cards Which Students Placed Together in More Than Fifty Percent of Their Total Sorts . .	84

Table | Page

4.11 Ability to Generalize Patterns Found on Card Sort Cards Seventh Graders 92

4.12 Results of Recognition and Generalization of Patterns in Tables Tasks 96

4.13 Strategies of Students Who Incorrectly Generalized Patterns Found on Card Sort Cards 97

4.14 Ability to Generate Word Problems From Generalized Statements 103

4.15 Using the Variables as the Initial of the Subject that the Problem is About 105

LIST OF FIGURES

Figure		Page
3.1	Flowchart of the Research Interview	50
4.1	Flowchart of the Process Involved in Sorting Interview Cards	88

CHAPTER ONE

INTRODUCTION

The mastery of algebraic concepts is a prerequisite for further study for many academic majors. High school graduation requirements across the United States are changing from one to as many as three years of mathematics for students. In some cases passing Algebra I is part of the requirement. At the International congress of Mathematics Educators (I.C.M.E. 5) held in Adelaide, South Australia, in August 1984, members of the Algebra Theme Group expressed their common concerns for upgrading the teaching of algebra. They also agreed that the most difficult concept for high school algebra students is the concept of variable (Carss, 1986).

Variables are the key for opening the door between arithmetic and algebra. They give students the power to talk not about numbers but about properties of numbers and number systems. Historically variables were used as tools for generalizing patterns. Today a limited picture of variable, as place holder for solutions to equations, is taught to students in classrooms. By painting such a narrow view of variables it is questionable if students recognize the large frame within which variables exist. Research studies (Kuchemann, 1978; Clement, 1982; Wagner,

1978) suggest that students are unable to use variables for the many mathematical applications for which they are needed.

The link between arithmetic and algebraic understanding and the role which variables play in making this link is important in terms of students' comprehension of the field of mathematics as a whole. Hiebert and Lefevre (1986) write about the need for investigating the relationship between conceptual and procedural knowledge. Students can solve algebraic problems manipulatively without realizing that these problems represent generalizations from arithmetic. What is frequently lacking is the underlying conceptual knowledge. These students may come to understand the formal language and rules of algebra, but they lack the conceptual knowledge which gives these symbols and procedures meaning. For these students variables become symbols to be manipulated rather than a means of generalizing patterns from arithmetic situations. Within this context algebra is not an extension of previously learned mathematics, but a totally new and separate branch of mathematical study.

Conceptual knowledge of variables implies a depth of understanding which is not characterized by superficial manipulation of symbols. This type of understanding is defined by Lesh, Post and Behr (1987) who describe understanding in part as the ability to do three things:

1. be able to recognize a concept in a variety of representational systems,

2. manipulate flexibly the idea within a given representational system and

3. to correctly translate the idea from one system to another.

Lesh's et al. (1987) definition of understanding centers around recognition of concepts. Their definition also involves the ability to express the concept within different representational systems. The idea of representational systems will be expanded later in this chapter.

Background

The Difference Between Arithmetic and Algebra

Pettitto (1979) describes algebra as the subject which takes the arithmetic that students have previously learned and organizes it into a formal system. In algebra classes students are not expected to manipulate numbers as they were in arithmetic classes. Instead the goal of algebraic instruction is to teach students to manipulate formal statements involving arithmetic operations. The numerical content of such manipulations is irrelevant.

Bertrand Russell (1938) also sums up the nature of the difference between arithmetic and algebra when he writes

> "Elementary Arithmetic, as taught to children, is characterized by the fact that the numbers occurring in it are constants; the answer to any schoolboy's sum is obtainable without propositions concerning any number. But the fact that this is the case can only be proved by the help of propositions about any

> number, and thus we are led from schoolboy's Arithmetic to the Arithmetic which uses letters for numbers and proves general theorems."

The arithmetic which Russell refers to which uses letters is algebra. The letters which are used to prove the general theorems are called variables. It is the introduction of variables and their use which extends arithmetic into algebra.

The Historical Development of Variables

The historical development of algebra is considered to have gone through three stages of development: rhetorical algebra, syncopated algebra and symbolic algebra (Eves, 1976). Rhetorical algebra represents the period of algebra when a problem was written completely without symbols, but rather as prose. A problem during this period might be written out like this, "What three successive numbers added together to produce a new number equal to 24?" The next stage of development, syncopated algebra, saw the abbreviation of frequently occurring quantities and operations. During this stage the above problem might have been written with the words "added together" and equals replaced with symbols to reflect these commonly occurring words and phrases. At this time symbols were used simply as a shorthand method of writing problems. The final stage of development, bringing algebra to modern times, was the symbolic stage of algebra. This is present day algebra where the above problem would be written $(y - 1) + (y) + (y + 1) = 24$ and accompanying directions would be to "solve for y". In this stage the variable "y" is standing for the middle of the three consecutive numbers in the problem.

The introduction of the use of variables is attributed to Diophantas (Breslich, 1939). About two thousand years ago in his book Arithmetica Diophantas introduced variables as place holders for unknowns. Variables were easier to work with than prose. As Lamon (1972) writes, we could say "The cube of a sum of two numbers is the sum of the cubes of these numbers added to the triple of the product of the sum of the numbers multiplied by their product" but it is far more convenient to write $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

Variables provide the bridge between arithmetic and higher levels of mathematics because they allow mathematicians to talk about whole classes of numbers without specific reference to numbers at all. Unfortunately, these tools of convenience for mathematician often do not function as bridges, but rather barriers for students as they move from arithmetic to algebra.

The Nature of the Difficulty With Variables

Bertrand Russell (1960) writes,

> "When we come to algebra, and have to operate with x and y, there is a natural desire to know what x and y really are. That at least, was my feeling: I always thought the teacher knew what they really were, but would not tell me".

Russell is summing up a problem which still exists with algebra students. Students are confused by variables.

This is not surprising considering the abstract nature of variables and the many uses of letters in mathematics.

Wagner (1983, 1980) talks about some of the roles which letters can play in mathematics. There are several different letters which are used in mathematics as variables, others which are always constants, still others which, for example in formulas, stand for abbreviations of words. If the variable x is used in the equation $x + 2 = 2 + 3x$ the variable represents an unknown. However, if it is used in the equation $x + 2 = 2 + x$, the variable represents a generalized number. Depending on the context in which variables are used mathematicians are able to determine which usage of a letter is intended. Students are not always as flexible in their thinking.

Skemp (1982) writes

> "those who understand mathematics - who can attach correct mathematical meaning to its symbols - pay little attention to the symbols themselves as they pass beyond them to the associated mathematical ideas. But those who do not understand mathematics do not get beyond its symbols, which rightly or wrongly they regard as one of their main sources of difficulty".

Thus, there are not only the problems associated with the abstract nature of variables, students further must deal with the many uses of letters in mathematics, not all of which are letters as variables. If students do not see the mathematical picture in which variables are embedded they will become frustrated with mathematics and in particular the variables themselves.

The Historical Versus the Modern Presentation of Variables

North (1965) writes:

> Algebra is a generalization of arithmetic. When the operations of arithmetic- addition, subtraction, multiplication and division - are performed upon numbers, the numbers combine to form new numbers, according to some particular scheme or pattern. This pattern is an abstraction, being independent of the particular numbers. The same set of operations could be performed on a different set of numbers, but the pattern of the calculations would be the same in each case. The study and analysis of such patterns is one of the objects of algebra. When the pattern of a calculation has been analyzed, it is often possible to generalize it so that the original calculation can be applied in new ways. To analyze the pattern and to see its structure and symmetry it is necessary to eliminate the numbers, so that the pattern behind them is made apparent. Algebra does this by replacing individual numbers by letters, each letter standing for a number in the arithmetical calculation.

Looking closely at North's quote two definitions implicitly appear, that of pattern and that of variable.

Pattern - recognizable set of numerical expressions or equations which remain invariant regardless of numbers used.

Variable - mathematical device for generalizing patterns and arithmetic statements. (Traditionally thought of as the historical definition of variable.)

These two definitions describe patterns and variables in a way which is consistent with North's quote. They also describe the concept of variable and patterns in view of the historical development of algebra and variable, algebra as generalized arithmetic. Taken in terms of Lesh's et al. (1987) three components of understanding, a pattern is an instance of what they describe as a concept. Variables are one representational mode used for expressing concepts.

Based on the presentation of variable in current algebra textbooks variables are introduced to students as place holders for unknowns. In textbooks instruction about the historical development of variables has been bypassed. Students are not taught to see variables in the historical sense, as a powerful symbolic tool to represent generalizations of patterns.

The clash between introducing students to the concept of variable in a historical development sense (based on a transition from arithmetic to algebra) and the end product of such development is expressed by Wheeler.

Wheeler (1986) writes:

> The school curriculum, by insisting that algebra is about numbers or quantities, while bypassing the stages of long historical development of the algebraic language, virtually forces pedagogy to oscillate inconsistently between presenting algebra as a universal arithmetic <u>and</u> a formal linguistic system with arbitrary rules.

The formal linguistic system with arbitrary rules which Wheeler refers to is the concept of variable as Diophantas and textbooks today present it. It is a system which, a review of literature shows, leaves students with a weak concept of the meaning of variable. This system asks students to solve equations using what they have come to see as an arbitrary set of rules.

This definition of variable is different from the definition of variable which we saw from North. In this latter case variables are not seen as tools for generalizing patterns. They are seen as place holders for solutions to equations. The power of algebra is limited if students only see variables in this way.

The process of generalization from arithmetic to algebra has two aspects, on the recognition of arithmetic patterns which can be generalized and the ability to use variables and operations to generalize patterns. Letters have many uses in mathematics. Letters are used as constants (i.e., π), as variables, as abbreviations (i.e., ft. for feet) and as symbols of operations (i.e., x meaning multiplication). Students need to be knowledgeable about the different uses of letters if they are to successfully generalize patterns. They also have to be flexible enough in their thinking about mathematics and variable to be able to correctly interpret expressions containing letters. Each expression, sentence or sequence that contains variables is part of a much larger group of expressions, sentences and sequences. For example, the expression $x + 5$ could represent a fraction or a decimal or an integer increased by five. In other words $x + 5$ represents any quantity which is to be increased by five. Students' conceptions of variables are limited if they are unable to interpret generalizations containing variables. Students should be able to see the larger group which the generalization represents. If students only know variables as place holders for solutions to equations, it is questionable whether they will be able to (1) recognize mathematical patterns, (2) use variables to generalize these patterns and (3) be able to draw meaning from generalized expressions, sentences and sequences.

The Research Problem

Research studies have investigated students' interpretation of variables as place holders for solutions to equations. No study has attempted to address the larger question of students' conceptions of variables as tools for generalizing patterns. What has been previously studied is students' ability to manipulate variable. Students understanding of variables is still open for investigation.

As already stated Lesh et al. (1987) describe understanding in part as the ability to do three things: (1) be able to recognize a concept in a variety of representational systems, (2) manipulate flexibly the idea within a given representational system and (3) to correctly translate the idea from one system to another.

Lesh et al. (1987) describe five representational systems with-in the mathematical system:

1. knowledge organized around real world events,
2. manipulative models,
3. pictures or diagrams,
4. spoken language and
5. written symbols.

Within each of the five representational systems are different modes of representation. For example, within the written symbol representational system there exist (among other modes) numerical symbols, variable symbols and operational symbols. Manipulating an idea flexibly within a given representational systems requires in part that students can translate between several modes of representation within one operational system. Traditionally research on variables has focused on students' ability to manipulate literal symbols. By focusing on one mode of representation (that of variables) within one representational system such research does not consider the flexibility one needs to operate within a representational system. Such research also does not investigate students' ability to recognize the concept in a variety of representational systems or to correctly translate the idea from one system to another.

The research proposed here is designed to overcome some of the gaps in existing research on students' conceptions of variables. The focus of the research is on how well students handle tasks which require them to demonstrate skills which correlate with Lesh's et al. (1987) three aspects of understanding. In particular the ability to recognize patterns (concepts) in a variety of representational systems and the ability to translate ideas from one system to another is investigated.

Research Questions

This study investigates students' conceptions of variables as tools for generalizing patterns. It is designed to focus on students at the point in their mathematical experience just prior to and just after the formal transition between arithmetic and algebra. The specific research questions to be addressed are:

I. What types of mathematical patterns can students recognize?

II. What types of patterns can students generalize by using variables?

III. What meaning do generalizations using variables have for students?

By looking at the broad questions of students' conceptions of variables as tools for generalizing patterns these three questions, along with the tasks designed to investigate them, focus on all three aspects of Lesh's et al. (1987) definition of what it means to understand. Using the terminology of Hiebert and Lefevre (1986) these questions are designed to investigate students' conceptual rather than their procedural knowledge of variables.

Overview of the Research Design

The research questions were addressed by in-depth interviews with seventh graders and High school algebra students (n=13 for each group). The interviews were designed to investigate how students thought about and answered questions involving variables used to generalize mathematical patterns. The interviews consisted of five tasks which investigated student conceptions of variables. Each interview lasted approximately 45 minutes. Not all students encountered all tasks; however, certain tasks were common to all interviews.

Assumptions

This study is based on the assumption that it is important for students to make the transition from thinking about mathematics as dealing with specific numbers to thinking about mathematics as the study of numbers in general. It further assumes that this transition may not occur in a formal way until a student begins the study of algebra. Another assumption of the study is that viewing variables only as place holders for solutions of equations limits the ability for students to make this transition.

It is also assumed that this study will have implications concerning the current practice of introducing variables as place holders for solutions to equations. Some researchers have suggested that the introduction of variables needs to draw more upon the prior experiences students have had with arithmetic (Adda, 1982; Lampert, 1985; Herscovics and Kieran, 1980). It is assumed that this research will speak to that issue.

A third assumption is that in volunteering to take part in this research students approached all questions and tasks with the intention of answering to the best of their ability.

Limitations

The purpose of this study was to be begin to understand students' conceptions of variables as tools for generalizing patterns. Students at the seventh grade and algebra levels were targeted as subjects for the study.

Although data obtained from the two groups was to be discussed independently attempts were made to try to match the two groups in ability. There were certain problems in obtaining this result. None of the same standardized test scores were available for both sets of students. The teachers were asked to assess the ability of the students based on a three point scale of below average, average and above average. The algebra teacher expanded the scale by writing in the descriptions, average to below average and average to above average. Both of the seventh grade teachers had the students for mathematics, English and history. They felt that their ratings were based not only on mathematics but on their reflection of the students across all three subjects. So although the groups of students used are similar, as rated by the teacher, there are problems with these ratings. This limitation is minimal, however, given that the intent of the study was to develop qualitative descriptions of the students' thinking rather than to make quantitative comparisons among them.

Overview of the Dissertation

This dissertation consists of five chapters. The first chapter provides the background for the study. It also contains the research problems and research questions, an overview of the research design as well as assumptions and limitations of the study.

Chapter Two includes a review of literature. The literature is divided into four parts. The first part talks about mathematical knowledge and what it means to understand a concept. The second part describes recent essays concerning viewpoints about symbolism in general. The third part focuses on specific research concerning variable. The last part reviews literature about the role of patterns in mathematics.

Chapter Three presents the research methods used. The subjects and sites are described. A detailed explanation of the clinical interview is given. The data recording and analysis processes are also described.

Chapter Four describes the connection between the research questions, analysis issues and interview tasks. For each research question, the corresponding analysis issues, interview tasks and results of the research are presented. Immediately after the data for each question a conclusion which ties this information together is presented.

Chapter Five contains an overall summary statement relating all of the findings of the research. A discussion of these results follows. Implications for teaching and implications for research are presented.

CHAPTER TWO

REVIEW OF THE LITERATURE

Introduction

One concern of many mathematics teachers and researchers is how well students perform on tasks such as solving problems and taking tests. Recently, however, mathematics educators at all levels have begun to realize that how well students perform on mathematics problems is not always an indication of how well they understand mathematics. Sometimes a gap exists between the process of solving problems and the understanding of these problems. Hiebert (1984) addresses this issue in an article which discusses the link between "form" and "understanding".

Form as Hiebert describes it is the syntax of the mathematical system. Form indicates the symbols and numerals of mathematics. Form is also the rules which are used to join the symbols and numerals of mathematics. Understanding, on the other hand, is the intuitions and thoughts which mathematicians of all levels have about how mathematics work.

In order to discuss mathematical knowledge about any concept a clear understanding of what is meant by mathematical knowledge is needed. Hiebert defined mathematical knowledge as the link between form and understanding or the relationship between conceptual and procedural knowledge. These definitions still leave unanswered the questions of what is conceptual knowledge and what does it mean to understand.

As already described in Chapter one Lesh, Post and Behr (1987) describe understanding as the ability to: (a) recognize a concept in a variety of representational systems, (b) manipulate flexibly the idea within a given representational system and to (c)correctly translate the idea from one system to another. They also describe five representational systems with-in the mathematical system:

1. knowledge organized around real world events,
2. manipulative models,
3. pictures or diagrams,
4. spoken language and
5. written symbols.

Similar systems are described by Dufour-Janvier, Bednarz and Belanger (1987) when they discuss internal and external representations. Janvier (1987) also describes similar ideas which he calls multiple embodiments.

The research proposed in this study is designed to investigate students' understanding of variables. Understanding is defined as outlined in Lesh and Post's definition which is given above.

The research questions for this study are:

I. What types of mathematical patterns can students recognize?

II. What types of patterns can students generalize by using variables?

III. What meaning do generalizations using variables have for students?

These questions are concerned with all three skills which Lesh et al. state are necessary indicators of understanding.

The subsequent review of literature will start with a look at articles and research studies concerning students' conception of symbols in general and variables in particular. This section will be primarily directed at highlighting what questions have been and what questions have not been answered about students' understanding of variables.

Following this section will be a review of research on, and articles about, teaching models directed primarily toward enhancing students' understanding of variable. Throughout this review of literature particular attention will be paid to what the articles have to say concerning how well students understand the concept of variable as a tool for generalizing patterns. Lesh's et al. (1987) definition of understanding will help to serve as the backbone for comparison. In the final section a summary will be given relating all of the works.

Articles Concerning Students' Conceptions of Symbols in General and Variables in Particular

Bernard (1984) expressed a need for research to address the issue of the use of symbolism. He reports that researchers have considered when to introduce symbolism. He stated, however, that clinical studies and teaching experiments need to take place in order to help to develop guidelines for using symbolism. One area that he specifically targeted for investigation was the ways in which students construct meaning and understanding from symbols.

This area is one which has not been sufficiently addressed by current research about variables. Little research concerning variable goes beyond students' ability to solve equations. A focus on how students conceive of variables and how they construct meaning from variables as tools for generalizing mathematics is needed. Such a study would need to start with a basic definition, such as Lesh's et al. (1987), of what is meant by understanding.

Only one article could be found which dealt with research concerning how students conceived of symbols. Iwuoha (1985) wrote a doctoral dissertation concerning eighth grade students' conceptions of algebraic symbols. Outlined were four levels of understanding of algebraic symbols: (1) Fully, (2) Mathematically, (3) Pragmatic Mechanistic and (4) Functionally. The difference between understanding fully and mathematically was that students in both categories could explain how a symbol worked, but only when they fell into the first category could they explain why a symbol worked. Students who operated under the category labeled pragmatic mechanistic used symbols mechanically. They felt that some higher authority had designated the use of the symbols. Those students who operated functionally viewed symbols with little or no level of understanding. The term symbol encompasses many mathematical symbols besides variables (i.e., operational signs, symbols of comparisons). Although Iwuoha's work presents nice neat categories for classifying students' conceptions of symbols in general, his work does not necessarily represent students' conceptions of variable in particular. Iwuoha's work was hard to analyze in light of Lesh's et al. (1987) definition of understanding as only the dissertation abstract was easily accessible for review.

Before discussing the specific research concerning students' conceptions about variables it is necessary to emphasize briefly one distinction between variables as they appear in algebra and what some people misconceive of as variables in arithmetic. Both Woodrow (1982) and Herscovics and Kieran (1980) point out the difference between expressions such as 3 + [] = 5 and 3 + x = 5. When students encounter the first type of equation in arithmetic they are usually asked to fill in the box. This simply means that the students must remember an arithmetic fact that will make the statement true. The second type of equation is usually accompanied with directions to "solve for x". Thus, the second type of equation calls for the student to consider the x as a variable and to act on it accordingly. Referring to Lesh's et al. (1987) definition of understanding the first type of equation requires students to regard an expression written only in a numerical mode of representation.

It is the second type of equation which was under consideration in the majority of the subsequently discussed research. This second type of equation requires only that students manipulate written statements involving numerical and variable representations. The goal of such manipulations is to arrive at a new statement which involves a numerical expression being equated with a variable statement. Students are not challenged to translate or transfer between or within different representational systems, such as would be required for understanding as defined by Lesh et al. (1987). It is a third use of variable, variable as used to generalize patterns and as part of generalized statements, which incorporates all skills necessary for understanding as defined by Lesh and Post, which research is not addressing. For this reason research which currently exists focuses primarily on manipulation of variable statements not understanding of the concept of variable.

Building partly on the work of Collis (ref. Bell, Costello and Kuchemann, 1983) the study "Concepts in Secondary School Mathematics and Science (CSMS)" at Chelsea College, London was developed. One topic incorporated into this study was students' (ages 13 - 15+) conceptions of Algebra. Three thousand students, drawn from 15 different schools, took a half - hour paper and pencil test.

The steps taken in writing the test as well as the results of the test have been published in several different reports (Booth, 1984a; Kuchemann, 1978, 1981 & 1984). The research revealed that students conceive of variables in six ways: (1) letter evaluated, (2) letter not used, (3) letter as object, (4) letter as specific unknown, (5) letter as generalized number and (6) letter as variable. It needs to be noted, however, that when the authors arrived at these categories of conception they were focusing on variables as placeholders for solutions of equations. Such a narrow definition of variable focuses on procedural knowledge of one use of variables. This definition does not consider conceptions as used in generalizing arithmetic.

The first category is when a student in looking at an algebraic expression or equation assigns the variable a numerical value from the onset. This category also refers to problems where children are asked to find a specific value for an unknown (for example, "What can you say about k if k + 7 = 9?"). The second category occurs when a student just ignores the variable (for example, if e + g = 8, e + f + g = ?, students would incorrectly ignore the variables and say 12 because, 4 + 4 + 4 = 12). The third category refers to a student regarding the variable as shorthand for an object, for example, 5m means 5 meters, not 5 times the quantity m). A student who treats variables as in category four is taking variables to represent one specific unknown (for example, "What can you say about c if c + d = 10?") An incorrect response from a child which would fall into this category is that c = 4." Category five refers to the belief and carrying out of calculations based on the idea that the variable is able to take several values rather than just one (for example, a child answering the problem "What can you say about c if c + d = 10" and saying c = 1,2,3,4 would be giving a response that would be classified as category five). In the sixth category a student treats the variable as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values. This final category is the level of understanding desirable for students to achieve. Students who respond in ways consistent with category four (taking letters to represent one specific unknown) understand variables as placeholders for solutions. Again it needs to be stressed that understanding as it is referred to here is only a procedural not a conceptual understanding. Responses which fall into category four would be expected of students who are thinking about variables as they are traditionally presented in textbooks.

These categories not only represent ways of classifying students' responses to problem situations, the categories also represent types of problems (for example, a problem of category four would be a problem that requires students to use a variable as a specific unknown in order to solve the problem.) Thus, these categories can be used to classify types of problems as well as correct and incorrect responses to problems.

Kuchemann considered these six categories as a hierarchy of understanding. His work revealed that as items on a test become more difficult for students, forcing them to look at variables in one of the ways represented by the later categories, students would often chose an inappropriate technique (for example, ignoring the variable) which lead to an incorrect response.

As a result of this study Kuchemann thought that students operated at four levels of generalized arithmetic understanding. It is important to note that what Kuchemann considered as levels of generalized arithmetic is not understanding of generalized arithmetic as presented in Chapter One. Chapter One described the ability to generalize arithmetic statements as involving conceptual knowledge of variables. Kuchemann focused on procedural knowledge.

At levels one and two children could only solve problems which do not require them to operate on variables as unknowns. The main distinction between the two levels is in the degree of difficulty of the problems (for example, a child who is able to solve problems at level one would be able to correctly solve the problem, "What does x equal if x + 5 = 8, but would not be able to solve the level two problem, "What can you say about m if m = 3n + 1 and n = 4?"). Children operating at levels 3 and 4 are able to solve problems that required them to treat variables as specific unknowns or as generalized numbers or variables. Again, the main distinction between the two levels relies on problem difficulty. Of 15 year olds, 5% operated below level 1, 30% operated at level 1, 23% operated at level 2, 31% operated at level 3 and only 9% were able to operate at level 4. The majority of students were operating at level one or two which required no understanding of algebra.

While Kuchemann's work supports the contention that current efforts to teach students about variable are not successful, it would be hard to agree that his six levels provide a complete framework for describing students' conceptions of variables. The definition of variable which he bases his work on is the narrow view of variable as place holder for solutions to equations. In regard to Lesh's et al. (1987) five representational systems it is clear that Kuchemann's work is only focusing in on one mode of representation within the representational system of written symbols. Kuchemann's study does not focuses on how well students can transform variables within this system. He also does not question how well students can recognize the variables in other representational systems. Students' ability to translate from one system to another is also not investigated. A more accurate picture of students' conceptions of variables would have to take into account students' conceptions of the role of variable as used to generalize patterns and as part of generalized statements. These two areas of investigation incorporate all three skills which constitute understanding. Kuchemann's work investigates variables procedurally as textbooks present them. His research needs to be expanded with further research which focuses on understanding of variable, in particular variables as they were historically developed.

Other research findings support Kuchemanns's claims that children have difficulty with the notion of variable. Wheeler (1986) writes "That students do not find the arithmetic-algebra connection transparent emerges in our research (and is well known to practitioners anyway)". He further writes "As soon as letters are written for numbers, mathematical language changes. In spite of the apparent parallelism between, say, writing 17 + 32 and a + b, where a and b are undetermined numbers, the two expressions belong to two different orders of reference. Students experience (or may experience) puzzlement about writing the "sum" of two numbers which cannot be summed because their identity is not known". The ability to recognize this parallelism requires that students are able to transfer knowledge between numerical systems and variable systems. Research has not yet formally addressed the question as to how flexibly can students transfer variable conceptions from one representational mode to another within a given representational system, namely as in these observations the written symbol system.

Following along with this last idea was a finding by Adda (1982). She found that in problems where the directions stated "let n be a number" a typical reaction of students was to reply "n is not a number, it is a letter." This exclamation by students might explain why some students, as in Kuchemann's level two simply ignored variables in problems. If variables are regarded as other than representing numbers it is no wonder that they get ignored when students solve problems.

A study by Tonnessen (1980) also discussed the level of attainment of students' conception of variable. Eight criteria were formulated to assess attainment of the concept. Four of the criteria dealt with concept acquisition. The other four dealt with concept use. The study involved 202 algebra, 178 trig. and 186 calculus students enrolled at the University of Wisconsin. Tonnessen's work revealed that the levels of attainment of the concept of variable were low. A further conclusion was that the level of acquisition of the concept of variable could not be used as a predictor of the level of use of the variable. Unfortunately, only a dissertation abstract for Tonnessen's work was easily available so it is impossible to fully explore exactly to what extent his work further's understanding about variable, with understanding modeled by Lesh's et al. (1987) definition.

Tonnessen's work focused only on variable as place holder for solutions to equations. Research is needed which assess the attainment level of variable as tool for generalizing patterns. In other words, as already noted, research is needed which focuses on conceptual rather than procedural knowledge of variables.

A problem about which much has been written (Rosnick and Clement, 1980; Rosnick, 1981; Clement, 1982; Clement, Lockhead & Monk, 1981; Wollman, 1983; Sims-Knight and Kaput, 1983; Kaput, 1986 and Lockhead, 1980) and which illustrates Kuchemann's "letter as object category" is the student-professor problem. The problem is as follows:

> There are six times as many students as professors. Using P to represent professors and S to represent students write an algebraic expression to represent this situation.

Where Kuchemann's focus was on procedural knowledge this problem begins to look, using one example, at Lesh's et al. (1987) third requirement for understanding. This problem requires that students translate a problem from the spoken language system to the written symbol system. The majority of students given this problem write the incorrect equation 6S = P instead of the correct solution 6P = S. Clement, Lockhead & Monk (1981) listed two strategies students used when solving the problem. The first involved "word order matching" or direct mapping of the words in the problem to the equation. The second was called the "static - comparison" method. Students who used this method could draw pictures to illustrate that they understood the relationship correctly, but they still represented it incorrectly. Their conclusion was that students were taking the letters as a direct label meaning students rather than as a variable meaning number of students. In other words, for this example, students were unable to translate the idea from one representational system to another.

Booth (1984) conducted a research project designed to provide further insights into Kuchemann's six levels. She interviewed students to try to find the cause of their errors. Fifty students, with approximately 16 from each of the second, third and fourth years of high school, were interviewed. These students made the types of errors described by Kuchemann, based on the CSMS tests.

The results of these interviews revealed that errors might result from three things: (1) the meaning children attach to variables, (2) the process of operating with variables and (3) difficulties with notation and convention. One of the errors that students made which was classified as an error caused by the meaning students attach to variables was a confusion with variable representing number as opposed to letter representing an object. This is the same type of misconception which results in the student - professor problem mix-up. A second error which students make is in assuming that different variables represent different numbers. Other mistakes included assuming that variables represent only whole numbers and in ignoring the meaning of the variable.

Students who ignored the meaning of a variable used three different strategies in simplify problems of the type (2x + 8y + 3x). One approach was to add up all of the numbers, then put down each variable that occurs (once only). That approach would lead to the incorrect simplification 13xy. A second approach was when the student would add up all the numbers, then put the variable for every time it occurs in the expression. This approach would lead to the incorrect solution 13xyx. The last faulty approach involved adding up all of the numbers, then putting down the variable that occurred most often. An answer of 13x was the result of using this method. Students who made errors of this type had trouble separating the meaning of variable representations versus numerical representations.

In other research Assab (1978) found that students had what was labeled a "single letter fixation". His research revealed that students were unable to accept algebraic expressions containing more than a single variable as representing real numbers. Such a finding might reveal why students over simplified expressions such as (2x + 8y + 3x) and particularly why 13x would be a common simplification.

The final types of errors that Booth labeled as errors involving the meaning children attached to numbers were errors when the child assumed a "pattern" between a variable and the number it was suppose to represent. For example assuming that x,y,z meant 2,3,4 or 10,20,30.

Another error of this type was saying that y is "higher" than p.

Sigrid Wagner (1977, 1980, 1981, 1983) has written several papers about this same idea of y being "higher" than p. She referred to this as the childs' inability to conserve letter. She found that when children were presented with equations such as 7 x w + 22 = 109 and 7 x n + 22 = 109 and ask which was larger they would respond with such answers as "w because it is further in the alphabet." These students had trouble working within the variable representational mode within the written symbols representational system. The tasks she presented to students involved looking at equations and functions. She labeled students who gave the above response as nonconserving. Students who gave the response "you can't tell without solving the equation" were said to be transitional. Those students who gave the correct answers were said to be conservers of equation and function under transformations of variables. She found that the majority of students were either nonconserving or transitional in their responses.

Booth's second category of types of errors were caused by what she called difficulty concerning the process of operating with variables. She found that students quite often needed to see a problem worked with numbers before they were able to consider it without numbers. She felt that often if students were unable to work with an arithmetic process then they were unable to do a similar problem involving variables. Another common response that falls into this category would be responses such as "I can't do it, I don't know what the letters are."

The third cause of errors that Booth reported were errors that dealt with difficulty involving notation. Errors that fell into this category were saying that a + m was the same as am; k + 2 = k2; 4m = m,m,m,m (four m's not 4 times m); 2 lots of x = 2x; 2 lots of 7 = 27 and xy, if x = 3 and y = 2, equalled 32. She also found confusion with the idea of exponents. She further found that students felt that brackets were unnecessary since you should perform operations in the order in which they occur in the equation. Several of these same results were found by Matz (1980). She felt that these errors could be attributed to misuse of existing rules or the students making up their own rules. These three types of errors resulted when students look procedurally at using variables as place holders for solutions to equations. It would be interesting to note whether similar errors result when students are questioned on tasks which require them to regard variables conceptually.

Booth conducted a second set of interviews to further test her hypothesis involving the three causes of students errors. This second set of interviews involved 17 students from the first interview phase and 7 new students. Again her results were quite similar.

As a by product of research that Nelson (1986) was doing concerning children and Logo he found that students were often confused when a diagram was kept the same, but given different variable dimensions. An example of this problem would be to let a square of total given area of 25 cm. be labeled with sides "b" cm. If students solved for "b" they'd find that "b" represents 5. Teachers often save themselves some work by using the same diagram but changing the total given area, for example letting it be 64 cm. In this later problem the "b" would represent an 8. He said that it was erroneous to assume that all students understand this demonstration since students see dimensions as fixed. In other words the students could not operate flexibly using variables to represent the dimensions.

Harper (1980) interviewed 12 pupils in years one through five of High School. He drew two lines a long red line which he labeled "b" cm and a shorter green line which he labeled "a" cm. He asked students questions such as, "which line is longer? Could they ever be the same in length? When would they be the same in length?" He found two types of responses. Some children focused only on the physical length of the line. Others talked about the variables a and b. Of the year one students 11 times as many were the first type. In years, two, three and four there were about 3 times as many of the first type of students than the second. By year five there were only twice as many students who focused on the physical length as opposed to the variable measurements. These two research studies reveal two "buggy algorithms" which might affect students' overall conceptions of variables. Detailed research which takes into account Lesh's et al. (1987) overall model for understanding needs to be undertaken in order to see just how these misconceptions fit into a larger picture of understanding.

Research on, And Articles About, Teaching Models Designed to Enhance Students' Understanding of Variables

Research and subsequent instructional interventions which approach variable as a tool for generalizing patterns are lacking. Research needs to be done which assess students' understanding of variable as part of a system of generalized arithmetic. Such research and subsequent teaching would incorporate all three skills necessary for understanding as described by Lesh et al. (1987). The majority of articles described in this section of the paper focus on student instruction of the procedural use of variables. One exception is an article by James and Mason (1982) which looks at introducing variables conceptually as tools for generalizing patterns.

James and Mason (1982), Skemp (1982) and Freudenthal (1973) discuss the need to be careful when introducing symbols to children. Freudenthal argues that textbooks used to be very careful when introducing symbolism to children. With the introduction of New Mathematics he feels that the emphasis was so heavy on introducing the symbols that little attention was paid to how they were introduced. In the language of Lesh et al. (1987) Freudenthal is stating that the emphasis was placed too heavily on variable as a representational mode rather than variable as a concept which can express ideas from several representational systems.

Both Skemp (1982) and James and Mason (1982) talk about the need to draw from the experience of a child when introducing symbols. Skemp offers four steps for helping students to integrate symbols in mathematics. These steps are to: (1) give children as many physical embodiments as possible (2) assimilate new concepts - not allowing students to memorize steps of symbolic manipulation (3) stay longer with verbal mathematics and (4) allow students to use their own symbols in order to experience advantages and disadvantages. Skemp stated that children needed to be pushed to use many physical embodiments. Skemp is arguing for the type of symbol education which is consistent with Lesh's et al. (1987) skills necessary for understanding.

Based on difficulty concerning the student-professor problem Rosnick and Clement (1980) attempted to tutor students on the problem. The tutoring methods they tried included: (1) telling the students that the reversal is incorrect, (2) telling the students that the variable should be thought of as "number of students," not "students", (3) pointing out with pictures that the student group is bigger and that the need to multiply by 6 to get equal sets, (4) having students test equations by plugging in numbers (5) having students draw graphs and tables (6) showing students how to set up a proportion to solve the problem and (7) demonstrating a correct solution using an analogous problem. Despite their efforts seven out of the nine students they tutored persisted with their misconceptions.

Booth (1984 a.) as part of the final stage of her research also included a teaching phase where students were instructed with worksheets which were designed to reveal their mistakes concerning variables to them. For example they might be asked to find the area of a square after which they would discuss the difference between a x a and a x b. These worksheets were aimed directly at instructing students in using variables as expressed by Kuchemann's level six within the context of variables as placeholders for solutions to equations. The conclusion after the teaching was that the program had been effective in improving students' conceptions about variable, but the gains that occurred were not overly impressive.

It is possible that the lack of success with teaching variables based on the emphasis of variable as a tool for solving equations is a result of the fact that this instructional approach is based on an incomplete conception of children's problems in understanding and using variables. This approach does not focus on students' prior knowledge about mathematics. Using this approach children are only required to operate within one representational system. In order to achieve conceptual understanding children should also be required to recognize variables in other representational systems. They should also be encouraged to translate among different representational systems.

Herscovics and Kieran (1980) attempted to teach students about variables in a way that would improve their conceptions about variable use. Their approach was to start first with the childrens' ideas about arithmetic and from there to move into algebra. They did this by first having students look at equations without variables. The students used their fingers to cover numbers in the identities. They then had the students replace numbers with letters. They worked with having the students hide more than one number and replacing their fingers with a variety of letters. From this point the students built different identities by replacing the letters with numbers. No statistical information was given to suggest whether or not this instructional sequence led to improved conceptions about variables, but the authors felt that they were seeing positive results. The approach suggested by Herscovics and Kieran is consistent with Lesh's et al. (1987) skills of recognizing a concept in a variety of representational systems and correctly translating from one representational system to another.

James and Mason talk about three activities designed to help students draw from their wealth of experience when learning about symbols. One of the activities is particularly interesting because it deals with the introduction of variable by using patterns. Students first built frames around various size pictures. They learned that a 2 x 2 picture's frame used 2 blocks on each side with one block in each corner. The 3 x 3 block used three blocks on each side and one block in each corner in order to build its frame. Students continued building frames and noticing the pattern. Eventually the students wrote out the following pattern which they observed:

1 in the corner	becomes 1
size of the picture	becomes []
add one onto the size and take it three times	becomes ([] + 1) x 3

Then add all of these together: ([] + 1) x 3 + [] + 1
(Note: Where I have used a [] the children actually used a cloud. They chose the cloud to represent the thinking cloud often used in childrens' comic books.)

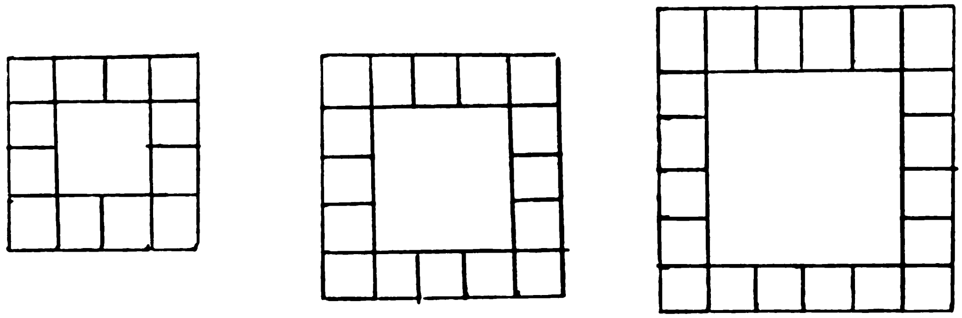
The students were eventually led by the teacher to substitute an n for the [] (cloud). The final expression became 3(n + 1) + n + 1 equals the number of cubes needed to surround a square picture of size n. The article did not address the question of why the students were led to this equation as opposed the more simplified 4([] + 1). The initial importance of such a teaching model is not what generalization students eventually arrive at but rather, the fact that this example is one instance where students were beginning to investigate variables conceptually as opposed to procedurally. This article is a good example of a demonstration of a task which requires a student to translate a pattern using pictures to a parallel statement using written symbols. Research articles are missing which discuss students' success at tasks which require them to recognize and generalize patterns with variables. Such research would fit directly with Lesh's et al. (1987) model of understanding.

Niegenann and Parr (1986) conducted a study which involved presenting students with thirty three word problems of which there were two or three problems each of various types (i.e., mixture, transportation, saving money etc...). The problems were selected from ninth grade algebra and physic books. The students were told not to solve the problems, but to sort the problems into those piles that could be solved in the same way.

Niegenann and Paar found that students sorted the problems according to five classes of criteria: structural, surface, difficulty, physical principles and a final group labeled other. They further found that the number of surface categories increased from 21% for experts, to 38% for college students to a high of 53% for high school students.

This study provides information concerning the types of features which students rely on when viewing patterns in story problems. It does not provide any information concerning how variables would be used by the students to generalize patterns they had identified.

One article which does investigate students' ability to recognize patterns is an article by Lopez-Real (1984). She described a situation where students were using multilink cubes to build frames with different size edges. The students built frames such as the following:



They then filled in the following table:

Edge of Square	3	4	5	6
Number of Cubes	8	12	16	20

The two girls working on this problem were pushed by the teacher to describe a pattern. They realized that the number of cubes needed to build the squares were multiples of four. The teacher realizing that recognition of patterns is a first step toward proof pushed the students to explain why the number of cubes was always a multiple of four. What the teacher did not do was to push the students to generalize the pattern with variables and to use this generalization to predict how many cubes were needed to build different size squares. In fact, no research was found that reported students' success and failure at generalizing patterns with variables. In other words, students were not questioned concerning how skillful they were at recognizing patterns in various representational systems. They also were not questioned concerning their ability to generalize variables using patterns.

One last article which discussed the need for students to be able to recognize and articulate symbolically patterns was a report by Burton (1984). Burton defined mathematical thinking as making sense out of mathematical situations. He said that the study of relationships is central for doing mathematics. He further stated that mathematical thinking involved three steps (1) manipulating (2) getting a sense of pattern and (3) articulating that pattern symbolically. He considered articulation of a pattern to be either verbal, diagrammatic or symbolic. Burton's definition of manipulations is much different than the definition used in most of the research and teaching methods presented in this chapter. His definition reflects a conceptual look at manipulation which is consistent with Lesh's et al. (1987) definition of understanding, not a procedural approach. Despite their importance, however, these areas do not appear to be explored by existing research. Few existing teaching methods focus on helping students to be successful with these three skills. It is not enough to say that students need to be able to successfully handle all three tasks. Research needs to address the quality of students performance on tasks of this nature.

Summary

The three research questions outlined at the beginning of this chapter are all built around the overall question of how students conceive of variables as tools for generalizing mathematical patterns. The ability to generalize patterns involves all three skills which Lesh et al. (1987) state are necessary for conceptual understanding.

A review of the literature reveals the difficulty students have with mathematical symbolism and in particular with variables. Several misconceptions and erroneous uses of variable were revealed through numerous research projects. These errors led Kuchemann to identity six levels at which students can operate with variables. Booth furthered Kuchemann's work by identifying three causes for why students made the type of errors which Kuchemann had identified.

Although Kuchemann and Booth, as well as other researchers, talk about levels of attainment of generalized arithmetic in students, their definition of generalized arithmetic is narrow. Of Lesh's et al (1987) five representational systems only the written symbol system is investigated extensively. Students' abilities to transfer within and translate between systems as well as their abilities to recognize variables in various representational systems are only considered slightly in existing literature. As explained earlier Woodrow, Herscovics and Kieran point out a difference between having students solve the equations 3 + [] = 5 and 3 + x = 5. A similar distinction between generalized arithmetic as described in this review of literature and generalized arithmetic as described in Chapter One needs to be made.

Levels of generalized arithmetic as discussed in this literature section means having the students move from solving equations without variables to solving equations containing variables. This definition provides a limited view of what it means to have students use variables to generalize arithmetic. North's definition and the definition given in Chapter One is much broader.

Students who can use variables to generalize arithmetic as described in Chapter One, and in particular as described by North, understand generalized arithmetic in a much broader sense. They not only can recognize and flexibly use variables within one representational system they are also able to recognize variables in other systems. They further are able to translate variables to these other systems. This view of generalized arithmetic involves (1) recognizing that patterns exist in groups of equations, expressions and sequences and (2) using variables to generalize those patterns. Very little of the research reported investigated the process of children generalizing arithmetic using variables in this broader sense.

Research concerning students' ability to recognize patterns as relationships between sentences, equations and formulas is almost non-existent. Niegenann and Parr's study is one exception. Their research investigated the types of features which students pay attention to when looking for patterns. Their study did not, however, address the issues of what patterns students could recognize, nor was the question of whether or not students could use variables to generalize the patterns which they recognized addressed.

Lopez-Real and James & Mason illustrated the importance of using patterns to motivate the concepts of variables and proof. Burton defined mathematical thinking as the process of making sense out of mathematical situations. He identified getting a sense of the pattern and articulating the pattern symbolically as being key to mathematical thinking. Being able to recognize and generalize patterns is the essence of doing mathematics. Research studies which only focus on procedural understanding of a concept are only investigating superficial understanding. How well students understand a concept can not be measured by how well students perform on traditional tasks. Students' conceptions need to be thoroughly investigated by focusing on tasks which require them to move beyond specific problems to the generalizations of those problems. Such problems involve all three aspects of Lesh's et al (1987) definition of understanding.

CHAPTER THREE

METHODS

Overview

The purpose of the study is to investigate students' conceptions of variables as tools for generalizing patterns. The transitional step from arithmetic to generalized arithmetic formally occurs when students begin to study algebra. In order to understand students' conceptions both before and after this point subjects at the seventh grade and algebra levels were interviewed.

Students were interviewed for thirty to sixty minutes using a clinical interview. The interview was designed around the following three tasks: (1) sorting cards (2) looking for and generalizing patterns in tables and (3) interpreting generalized expressions, sentences and sequences. These tasks were designed to illuminate students' understanding of variables as tools for generalizing patterns. Understanding is defined according to Lesh, Post and Behr's (1987) definition which is given in Chapters One and Two.

The data was analyzed using seven analysis issues. Tables were generated to organize the findings. Instances of student responses support the arguments. The subjects, research design, development of data collection instruments and analysis procedures are explained in further detail within this chapter.

Subjects

The subjects for this study were drawn from four seventh grade and two algebra classes. The seventh graders attended a middle school in a small rural town. The algebra students came from the neighboring high school in the same community. The four seventh grades were taught by two different teachers. Each teacher taught two classes. The two algebra classes were both taught by the same teacher. The teachers and classrooms involved in the study were selected because of the teachers' willingness to participate in the study. The students were selected on a volunteer basis.

From the students who volunteered to participate in the study an attempt was initially made to select fifteen students at the seventh grade and fifteen students at the algebra level. These fifteen at each level were to be divided into three groups; five below average, five average and five above average in mathematics.

Problems arose with trying to obtain standardized test scores for the two groups. No set of the same standardized test scores were available for both the seventh graders and algebra students. It was decided to ask the teachers to assess the students' mathematical achievement based on a three point scale of below average, average and above average. The algebra teacher expanded the scale by writing in the descriptions, average to below average and average to above average. Both of the seventh grade teachers had the students for mathematics, English and history. They felt that their ratings were based not only on mathematics but on their reflection of the students across all three subjects. The two group of students were matched as closely as possible based on the teachers' ratings. The final number of students used from both levels was reduced to thirteen per group because of time constraints. These thirteen students at each level fell into the groups of 4 below average, 4 average and 5 above average in achievement. Eight students at the seventh grade level and seven algebra students also were interviewed during the pilot stage of the research.

Data

Development of the Interview

The interview tasks were developed to reflect the proposed research concerns. At no point during the interview were students asked to evaluate expressions or sentences containing variables. The role of variable as tool for solving equations was not under consideration. Instead tasks were developed that required the students to use variables as tools for generalizing patterns and interpreting generalizations. During pilot interviews the data collected was analyzed and used to develop and polish the final interview.

The Interview

The interview consisted of five basic tasks: sorting cards, generalizing the card sort, looking for patterns in tables, generalizing patterns in tables and interpretations of expression, sentence and sequence generalizing by creating a story to match the situation. A flowchart of the interview structure is given in Figure 3.1.

The Card Sort and Generalization of Card Sort Cards

Seven sets of cards were developed for the card sort. An example set of cards is given in Table 3.1. The complete set of cards is shown in appendix A. Six of the sets were used for data collection purposes. One set was developed to use in explaining the type of task to students. The explanatory set of cards does not model the structure of the actual card sorts. Each of the other sets of cards in the card sort contained twelve to fourteen expressions, sentences or sequences printed individually on cards. The cards were organized around three surface features on which students might focus when sorting the cards. A surface feature is defined as a physical characteristic of an expression, sentence or sequence which is noticeable by looking at one card in isolation from the other cards. For example the sentence: $8.2 + 3 = 11.2$ has the following surface features, it is an addition problem of a decimal plus a whole number.

Two surface features which were present in all the sets were four expressions containing whole numbers and

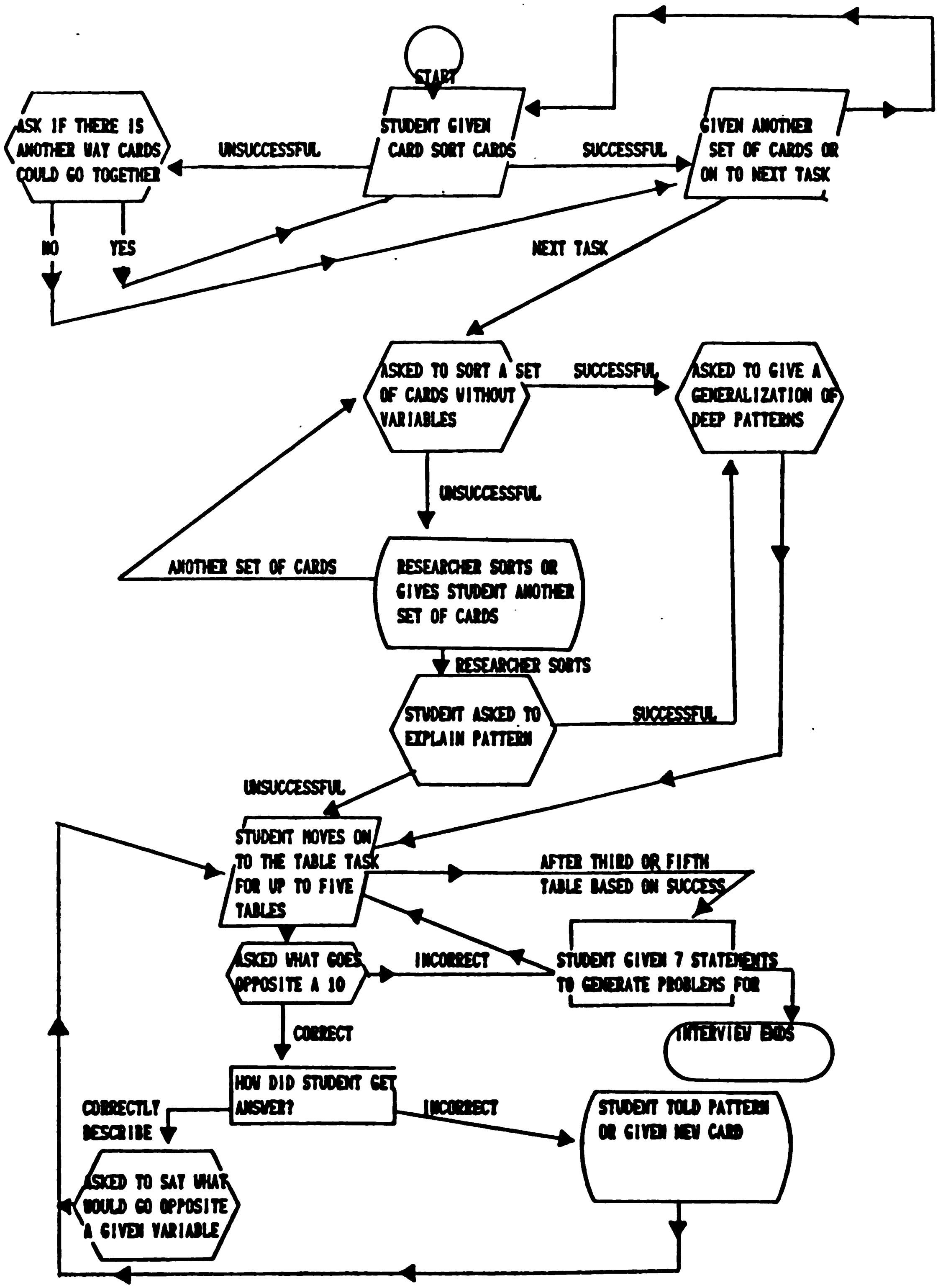
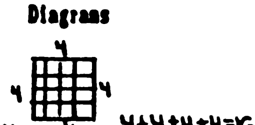
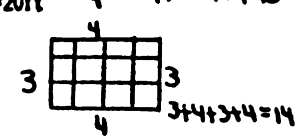
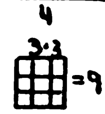
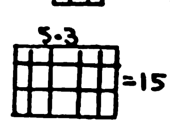


Figure 3.1: Flowchart of The Research Interview

four or five expressions containing variables. These two surface features represented two different representational modes within the written symbols representational system. The third surface feature changed from card set to card set. The third surface feature might present an idea within the written symbols representational system or it might represent the idea within a different representational system. For example, the orange geo cards which are pictured below illustrate the concepts of area and perimeter within both the written symbols representational system and the picture and diagram representational system. The whole number cards and variable/formula cards are examples of cards which illustrate the numerical and variable representational modes within the written symbols representational system.

Table 3.1
Example of a Set of Interview Cards

Orange Geo (Used In Every Interview)

Deep Patterns	Surface Features		
	Whole Numbers	Diagrams	Variables/Formulas
Perimeter	4 4ft = 16ft		p = 2b + 2h
	(2 4ft) + (2 6ft)=20ft		p = 2w + 2l
			p = 4s
Area	5ft 5ft = 25ft^2		A = lw
	9in 3in = 27 in^2		A = bh
			A = s^2

Also built into the set of cards were two deep patterns. Deep patterns are the patterns which require looking at a whole set of expressions, sentences or sequences.

For example the set of cards:

$$5 + 0 = 5$$
$$8.2 + 0 = 8.2$$
$$6 + 0 = 6$$
$$3.9 + 0 = 3.9$$

all have the deep pattern that can be expressed by the generalization $u + 0 = u$. A summary of all of the deep and surface features of all of the sets of cards is given in Appendix A.

During the card sort the students were given the twelve or fourteen cards which go together as illustrated in Appendix A. Each of the cards was laid out separately by the researcher. All of the cards were placed on the table so that no cards overlapped. The students were asked to place the cards into piles according to what they felt went best together. They were then asked to explain why they sorted the cards in the way that they had. If they correctly sorted and explained the cards by the deep patterns during the first sort a new set of cards was given to them. The directions and sorting process were repeated with this new set of cards. If students were unsuccessful at sorting the cards according to the deep patterns they were asked "If I were to mix up the cards and ask you to sort them again is there another way that they can be put together?" This process continued until one of three things happened (1) the student successfully sorted and explained the cards according to the deep patterns (2) the student could no longer see any more sorts or (3) the researcher felt that the student was not able to see the deep pattern or was confused by the cards. After one of these events occurred the student was either given a new set of cards to be sorted or the next task. Consequently students have from one to seven attempts at sorting the cards recorded for them.

The order in which the card sets were given to the students and the actual sets that each student was given to sort depended in part on their success at the previous sort. For example, if a student had trouble with a set of cards they would have been given another similar set later on in the interview. The only two exceptions were if it was apparent that the student was too frustrated by the task to continue seriously thinking about the sorts or if there were no more available sets of cards. Certain sets of cards were given to all of the students to sort, but not all sets were given to every student.

The next task was to have the students sort a set of cards from the card sets in Appendix A with the cards containing variables removed. If students successfully sorted these cards by the deep patterns they were then asked to generalize the patterns they saw using a given variable. For example the researcher said, "can you describe the pattern that you see on the cards in this pile by using a "t"?" If students were unsuccessful at sorting the cards they would either be given another set of cards to sort or the cards would be sorted by the researcher. The researcher only sorted the cards for the students if one of two things happened (1) the students were so frustrated by the task that they were becoming discouraged and reluctant to sort the cards themselves (2) the students had already failed to sort one set of cards without variable generalizations and could see no more sorts in a second set of cards. After the researcher sorted the cards, students were then asked if they saw the pattern. If they could explain why the researcher put the cards in the specific piles they then were asked to generalize the pattern. If the student could not see the deep pattern the task ended.

All of the cards which made up the deep patterns used in this task contained either expression or sentences written using only the numerical representational mode within the written symbols representational system. Successful completion of this part of the card sort task involved two skills: (1) the recognition of deep patterns which were written within the numerical representational mode of the written symbols representational system and (2) the translation from the numerical representational mode to the variable representational mode within the written symbols representational system.

Recognition and Generalization of Patterns in Tables

The next task was to show the students the tables, which are given in Appendix A, one at a time. Each time that the students saw a table they were first asked, "what would go after a ten in this table?" This question was designed to see if the students recognized the pattern. If the students gave a correct reply they would then be asked why they gave the answer that they did. If the students stated how they found the answer in a way that illustrated that they recognized the pattern they were then asked to generalize the pattern using a certain variable. This was done by asking the students what would be on the other side of the table from a "b" (for example). At this point the next card was shown to the students. If the students could not say what would go after the ten the researcher would ask the students how they found their answers. The researcher then moved on to the next card.

If the students did not successfully see any of the patterns in the first three tables that they were shown (tables 1, 2 and 3 in appendix A) the task was ended. If the students saw some of the patterns but not all of them then the researcher might elect to tell the student the pattern. The decision to tell a student a pattern was based on three factors (1) the level of frustration of the student with the task (2) curiosity by students after unsuccessfully looking for any patterns themselves or (3) as a way of helping students, who after several minutes of looking for patterns were unwilling to end the task until they had seen a pattern. This usually happened when the students were stumped by tables four and five (as numbered in Appendix A). The pattern was revealed to the students in a way which did not give away the variable generalization. For example, looking at the table

1	8
2	13
3	18
4	23

the researcher said to the students, "one of the other students told me that they could find the number that went on the other side of the table by multiplying by five and adding three. Do you think that that student was right?" The students then would check the rule and in all cases discovered that it was true. After the students successfully showed that they recognized the pattern by saying what went after several numbers they than were asked to generalize the pattern by telling what would go on the other side of the table after a specific variable. This task involved seeing if students could recognize patterns which were written within the numerical representational mode of the written symbols representational system, but which were presented within the tables, pictures and diagrams representational system. After investigating students' ability to recognize these patterns students were questioned to see if they could translate the patterns to the variable representational mode of the written symbols representational system.

Interpretation of Expression, Sentence and Sequence Generalizations by Development of Word Problems

Next the researcher moved to the expressions, sentences and sequences that are given in Appendix A. Each of these were printed separately on cards and the students were shown them one at a time. The task was introduced to the students by the researcher showing the students one of the cards and saying, "you know how you have story problems in your math books that you have to write equations for? Here are some equations, but the story problem is missing. Can you make up a story problem that would fit the equation on the card?" All of the students were shown all of the expressions, sentences and sequences printed on the cards despite their success or failure on any given card. This was done since the cards differed significantly in their structure and success or failure on one card would not necessarily predict success or failure on the next card.

This task differs considerably from the last two tasks. Both of the last two tasks were designed partly to see if students could translate deep patterns to the variable representational mode within the written symbols representational system. The focus of this task was an opposite form of translation. For this task students were given generalized expressions, sentences and sequences which were written using the variable representational mode of the written symbols representational system. The students were asked to translate these generalizations into stories within the spoken language representational system which fit the generalizations. The stories the students created were taken as reflections of their knowledge of how the generalized statements represent real world events. This task was designed to illustrate an overlapping of how the students saw the generalizations of concepts within the spoken language and real world representational systems.

Data Collection Procedures

The Initial Meeting Between Participating Teachers and Principals

The first step in data collection was to set up a meeting with the teacher(s) and principal in the schools where the study was to take place. During the initial meeting with the teacher(s) and principals the expected commitment of the teacher(s) and the principal was explained. The teachers provided the researcher with a list of students in the class. They also collected permission slips from the students. The teachers released one student per period to be interviewed. The principals were asked to provide a room where the interview could take place. In one case a conference room was arranged. In the other the vice principal's office was used.

Gathering Information on Student Ability

Immediately after the permission slips were turned in the researcher attempted to record standardized test scores for the students. The problem of not having the same standardized scores for both the seventh grade and algebra students was encountered. The researcher decided instead to have the teachers rate the students on a scale of below average, average and above average. The problems with this process have already been described earlier in this chapter and in Chapter One.

The Interview Schedule

The teachers at the seventh grade level met weekly to discuss their curriculum with each other. They also used the same textbook and covered the same material at the same time. For this reason it was felt that no distinction would be made between the students in one teacher's classroom versus the students in the other teacher's classroom. For the pilot and actual interviews students were drawn from either classroom depending on which teacher would be least interrupted by having the students leaving and entering the room. With the algebra students, the interviews took place on days that tests weren't being given.

Pilot Interviews With The Students

The first seven interviews with the algebra students and the first eight interviews with seventh graders were considered pilot interviews. Audio tapes of each of these interviews were quickly transcribed by the researcher. They were then analyzed to provide the researcher information concerning how well the students were understanding the tasks, what kind of questions were providing useful information and which task needed to be changed. Review of these transcripts helped to fine tune the pilot interview until it was ready to be used for actual data collection. This phase of data collection took approximately three weeks.

The Data Collection Interviews

After the pilot interviewing was completed and the interview protocol was revised the interviews for the study began. During the interview students were provided with scrap paper and pencil. Students were asked to think out loud as they worked on a task. The tasks which students were given were in part determined by their success on various parts of the interview. Each interview had core tasks which were common to all of the interviews. These are designated as such in Appendix A. While the student was being interviewed the researcher kept notes on a separate sheet of paper. These notes along with the audio tapes combined to provide the transcripts of the interviews. In all cases the interviews lasted from thirty minutes to an hour, averaging forty five minutes.

Data Analysis Procedures

Transcription of the Audio Tapes

All verbal comments made by the students were transcribed as close as possible to verbatim. If a part of the tape was hard to hear the researcher's notes were used to supplement the audio tape. The researcher's comments were only transcribed if they differed in any way from the standard interview directions and explanations. For example, if a student asked the researcher a question during the interview the reply, along with the question, was transcribed. Also, any time that a student had trouble with a task and the researcher provided additional information beyond the initial directions the researcher's comments were transcribed.

Analysis of the Transcripts

The transcripts were analyzed in much the same way as they were recorded. The first level of analysis was to categorize responses as a success or failure on a certain task. At this point questions such as "how many students successfully sorted the blue sets of cards?" were answered. The data for the seventh graders and algebra students were always separated.

Next the transcript for each individual student was analyzed internally as well as being compared with other students. This was done in order to determine if there were certain groups of students both on the interview as a whole and on separate parts of the interview which fell into similar groups. For example, was there a group of students who were never successful on the tasks, a group who were part of the time successful and a group who were always successful. It was hoped that profiles of say a successful versus an unsuccessful student could be developed. Unfortunately, no such groups emerged. No one task predicted success or failure on another task. Similarly a student might fail to recognize the patterns that the majority of students recognized yet would recognize a pattern that no other student recognized. No student profiles could be developed without producing individual charts on each student along with results on every tasks. The patterns that existed seemed to be patterns of the group of students as a whole or of the seventh graders versus the algebra students.

At this point tables were developed to illustrate the students' responses to the various tasks. Up to this point success and failures on tasks or generalizations of patterns were considered without specific reference to the transcribed reasons why the students completed tasks in the way that they did. The next step in the analysis was to analyze the explanations students gave along with their answers to a specific task. Each transcript was coded two separate times.

For the card sort all of the explanations that students gave for why they sorted cards the way they did were compiled. These responses were then assigned a number. The transcripts then were read through on two separate occasions. Each time a student sorted or explained that they had sorted according to a certain feature the code for that feature was marked down on a sheet. For example, sorting all of the sentences containing fractions into one pile would be coded with a ten if the student explained that the presence of fractions is why they sorted the cards in the way that they did. The coding of the data in the tables was done by assigning of colors to indicate specific correct or incorrect error patterns. The story problems that the students developed as well as the comments they made while doing this task were coded in the same way that the card sorts were coded.

After examining the explanations students gave along with their performance on specific tasks analysis issues began to emerge. These analysis issues served as a link between the research questions and the interview tasks. For example, the sorting of cards and the identification of patterns in tables were designed to provide information for the first research question (What types of patterns can students recognize?). The analysis issues functioned as a way that data from these two tasks could be organized in order to address the first research question. Table 4.1 illustrates the relationship among research questions, analysis issues and interview tasks.

Once the analysis issues were identified all of the data in the transcripts were reexamined. As patterns emerged more tables were developed to explain the findings. These findings were highlighted by student comments which illustrate students reactions to tasks. Not only was the question of <u>how</u> did the students perform on the tasks examined, but also examined was the question <u>why</u> did they answer in the way that they did. The final step in the data analysis was to relate this how and why in summary tables which explained the relation between the two. In answer to both questions student responses were cited in order to illustrate the summaries presented in the tables.

Summary

This chapter has described the research methods and analysis procedures used in this study. A detailed description of the clinical interview was given. This chapter also described the population involved in the study. The data collection and analysis procedures were also examined in detail.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

The purpose of this study is to address the question of students' understanding of variables as used to generalize patterns. The following three research questions served as the structure for the research:

I. What types of patterns can students recognize?

II. What types of patterns can students generalize by using variables?

III. What meaning does generalizations using variables have for students?

These questions were addressed through clinical interviews. The clinical interview was designed so that students' conceptions about variables used to generalize patterns was explored. Students' conceptions of variables as part of generalized expressions, sentences and sequences were also explored. Analysis of the interview involved reviewing student responses to tasks while keeping Lesh, Post and Behr's (1987) definition of understanding in mind. Results of the interview will be presented in light of this definition.

Within the framework of the three research questions several analysis issues emerged. In the same way that different interview tasks were designed to address specific research questions individual issues which emerged were also most closely identified with a specific research question. This relationship among the research questions, analysis issues and interview tasks is summarized in Table 4.1.

Research question one is elaborated through the following questions:

1. What patterns from the interview were students able to recognize?

2. What makes a deep pattern difficult or easy for students to recognize?

3. What types of mistakes do different patterns lead students to make in the process of recognition of those patterns?

These specific issues, as well as the general research question, were addressed through the card sort and the identification of patterns from tables task.

The second research question is "What types of patterns can students generalize by using variables?". This question was addressed through the following issues:

1. What types of patterns can students generalize and what is the connection between the ability to generalize patterns and the ability to recognize patterns?

2. What types of errors do students make when generalizing expressions and sentences?

These specific issues along with the general research question were investigated through two tasks. These tasks included the generalization of grouped cards in the card sort and the generalization of patterns in tables.

Table 4.1

Relationships Among Research Questions, Analysis Issues and Interview Tasks

Research Questions	Analysis Issues	Interview Tasks	Type of Skills Associated With Understanding Task Investigated
1. What Types of Patterns Can Students Recognize?	1. What Patterns From the Interviews Were Students Able To Recognize? 2. What Makes a Pattern Difficult or Easy for Students to Recognize? 3. What Types of Mistakes Do Different Patterns Lead Students To Make When They Try to Recognize Patterns?	Card Sort, Identification of Patterns From Tables	Recognition of Deep Patterns within the Numerical Representational Mode of the Written Symbols Representational System, Recognition of Patterns Written Within The Numerical Representational Mode of the Written Symbols Representational System Which Were Presented Within The Pictures and Diagram Representational System.
2. What Types of Patterns Can Students Generalize By Using Variables?	1. What Types of Patterns Can Students Generalize and What is the Connection Between the Ability to Generalize Patterns and The Ability to Recognize Patterns? 2. What Types of Errors do Students Make When Generalizing Expressions and Sentences?	Generalization of Groups of Cards Within a Sort, Generalized Statements of Patterns in Tables	Translation From The Numerical Representational Mode to the Variable Representational Mode Within the Written Symbols Representational System, Translation Of Patterns to the Variable Representational Mode Of the Written Symbols Representational System.
3. What Meaning Do Generalizations Using Variables Have For Students?	1. What Strategies do Students Use to Obtain Meaning From Expressions, Sentences and Sequences Containing Variables?	Making up of Story Problems to Fit Expressions and Sentences	Translation of Statements Written Using The Variable Representational Mode Of The Written Symbol Representational System Based on Knowledge of How The Generalizations Reflect Real World Events.

The final research question is "What meaning do generalizations using variables have for students?" This question was investigated through analysis of the following issue:

1. What strategies do students use to obtain meaning from expressions, sentences and sequences containing variables?

This issue along with the third specified research question were investigated through the interview task which asked students to generate story problems to fit given expressions, sentences and sequences.

Results

Research Question One: What Types of Deep Patterns Can Students Recognize?

During the card sort all of the students were given several different sets of cards to sort. Three sets of cards were common to all interviews. These three sets contained the following pairs of deep patterns:

1. The multiplicative identity and the commutative property of multiplication.

2. An arithmetic sequence with the rule of each term being two greater than the previous term and a geometric sequence with the rule that each term was twice the previous term.

3. Cards illustrating through formulas, diagrams and numeric sentences the concepts of area and perimeter.

All of these deep patterns are representative of what Lesh et al. (1987) label as concepts. As described in detail in Chapter Three each set of cards was also organized around three surface features. Surface features are physical characteristics of the expressions, sentences or sequences.

Issue One: What Deep Patterns Did Students Recognize?

Table 4.2 summarizes the success of seventh graders and algebra students respectively on each of the sorts listed above. Overall both the seventh graders and algebra students were mostly unsuccessful at these tasks. One exception was the algebra students' ability to recognize the multiplicative identity and to a lesser degree the commutative property of multiplication. The students were unsuccessful at recognizing the deep patterns in the cards containing sequences as well as in the area and perimeter cards.

Table 4.2

Students' Ability To Recognize Patterns as Demonstrated By the Card Sort

n = 13

	Multiplicative Identity		Commutative Property of Multiplication		Sequence: x,x+2,x+4, x+6,x+8...		Sequence: x,2x,4x,8x 16x...		Perimeter		Area	
	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.
Correctly Sorted On First Try Based On Deep Pattern	3	9	1	8	0	0	1	0	0	0	0	0
Correctly Sorted On Later Sort By Deep Pattern	1	2	3	0	0	1	2	1	1	0	2	2
Saw Deep Pattern With Numerical Rep. But Not Variable Rep.	0	0	1	1	2	0	1	0	0	0	0	0
All Sorts Based on Surface Features	9	2	8	4	11	12	9	12	12	13	11	11

Examples of two different students' sorts, one correct, one incorrect, for the set of cards containing the multiplicative identity and the commutative property of multiplication are given here.

Frank (7th grader):

Cards Sorted Together	Reason Why Frank Placed the Cards Together
3/4 • 1 = 3/4 18/19 • 1 = 18/19	They look pretty much the same. The way they're set up. They're fractions and then they have equals another fraction.
74 • 4 = 4 • 75 754 • 1 = 754 83 • 1 = 83 11 • 3 = 3 • 11	They all have a number and then an equal sign and then another number.
7/20 • 1/4 = 1/4 • 7/20 2/3 • 5/7 = 5/7 • 2/3	They have a fraction and then a multiplication sign and then another fraction and then an equal sign. Then the same fraction just set up in a different order.
a • 1 = a b • c = c • b j • m = m • j p • 1 = p	They all have a letter that you multiply by. The first thing is a letter.

Frank saw no other ways that the cards could be sorted.

Opal (algebra student)

Cards sorted Together	Reason Opal Placed The Cards Together
3/4 • 1 = 3/4 18/19 • 1 = 18/19 754 • 1 = 754 83 • 1 = 83 a • 1 = a p • 1 = p	Because anything times one is itself and so, all of those are just times one. So it's always itself.
75 • 4 = 4 • 75 11 • 3 = 3 • 11 7/20 • 1/4 = 1/4 • 7/20 2/3 • 5/7 = 5/7 • 2/3 j • m = m • j b • c = c • b	Because they both have the same answers. c • b is b • c. So all of them are like that b • c = c • b You always get the same answers in both ones so it is equal.

The above examples illustrate both a correct and incorrect sort. As indicated in Table 4.2, the majority of sorts were incorrect.

Given the students' lack of success on these tasks a question raised is whether or not the presence of variables influenced students inability to recognize most of the deep patterns. This question can be answered on two levels. On one level the answer is yes. Students were affected by the presence of cards containing variables. This effect will be discussed further under the issue of what makes a pattern difficult or easy to recognize. On the other hand, one way that the students might have chosen to sort the cards was to first sort all of the cards containing variables together. They then would have been able to sort the remaining cards without paying attention to the cards containing variables. Table 4.2 indicates that few students on each task successfully employed this strategy. The cards containing variables did not provide clues helping students to recognize patterns. These cards also did not stand directly in the way of deep pattern recognition.

The second task which investigated students' ability to recognize patterns was the identification of patterns in tables task. Tables 4.3 summarizes the results of this task. Both groups were more successful at identifying these patterns than they were at identifying the patterns in the card sorts.

Table 4.3

Results From Recognition of Patterns in Tables Task

	x^2		x + 5		3x		5x + 3		5x - 4	
	7th grd. n = 13	alg. sts. n = 13	7th grd. n = 13	alg. sts. n = 13	7th grd. n = 12	alg. sts. n = 13	7th grd. n = 5	alg. sts. n = 11	7th grd. n = 2	alg. sts. n = 6
Pattern Recognized	13	13	8	11	3	12	2	1	0	0

If we look at just the results of having students recognize the patterns x + 5 and 3x the algebra students were more successful at recognizing these patterns than were the seventh graders. Almost all of the algebra students recognized these patterns. All of the seventh graders and algebra students successfully recognized the pattern x^2. Only eight out of thirteen seventh graders identified the pattern x + 5. For the pattern described by the expression 3x the success rate for seventh graders was only three out of twelve.

The table representing the pattern 5x + 3 was only shown to five seventh graders as opposed to eleven algebra students. The decision to not continue the task with the majority was based on their lack of success with the x + 5 and 3x patterns. Pilot interviews had revealed that if students were unsuccessful at seeing the patterns with these cards they would also fail at seeing the 5x + 3 and 5x - 4 patterns. Of the five seventh graders who saw the pattern 5x + 3 two successfully identified it. Of the eleven algebra students who saw the pattern only one was able to identify it.

Issue Two: What Makes a Pattern Difficult or Easy For Students to Recognize?

The next question of what makes a pattern difficult or easy to recognize follows logically from a look at which patterns students did or did not identify. In addressing this issue it was necessary to go beyond the success or failure of students on a task to the analysis of what reasons students gave for sorting cards in the ways that they did. From the responses students gave for why they sorted cards into the groups that they did, different features of the expressions, sentences, sequences, and diagrams which affected how students sorted cards emerged.

As described in Chapter Three each of the sets of cards in the card sort were organized around three surface features which students might focus on when sorting the cards. The students often saw many more. Responses on the three sets of cards which were common to all of the interviews will illustrate this point.

The cards which were to be sorted into two piles representing multiplicative identity and commutative property of multiplication had the built in surface features of whole numbers, fractions and variables. Table 4.4 illustrates the number of students who sorted at one point using these surface features. Each student might have sorted the set of cards one or more times. If a student sorted the cards five different times and in three of the sorts made reference to cards being together because they were fractions this student was only credited once with having been influenced by the surface feature of fractions. In this way a number was obtained to reflect how many students were affected by each surface feature.

Looking at Table 4.4 it is apparent that students found the presence of whole numbers versus fractions versus variables to be an important distinction. A couple of students were also influenced by the fact that some of the sentences contained the same numbers. Another influencing feature for a couple of the seventh graders and algebra students was the physical appearance of the sentence itself. Physical structure here would be statements such as the following:

"Cause it's got one number times another number equals the same number on the other side." (Xavier, algebra student)

"These are numbers times numbers."(Stan, algebra student)

"Cause they're both like, they have a number and then a dot (a multiplication sign) and then a number." (Grace, 7th grader)

The emphasis is not on the numbers themselves, but on the way in which they are operationally joined.

Tables 4.5 and 4.6 illustrate other surface features which students paid attention to which were or were not intentionally built into the respective card sorts. By looking at these two tables and the transcripts from other card sorts that the students did, a list of types of surface features to which students pay attention emerged.

Students paid attention to surface features which fell into the following five groups:

1. Types of numbers used.
2. Properties of numbers.
3. Common features of more than one expression or sentence.
4. Ways of representing expressions, sentences and sequences.
5. Mathematical symbols and notation used.

These five broad categories along with types of surface features which fell under each heading are summarized in table 4.7.

Some examples from the transcripts which illustrate each of the five types are as follows:

Types of Numbers Used

For the card sort containing the deep patterns of multiplication by 5 and multiplication by 1 Frank, a 7th grader, put the following cards together: 4/8 • 5, 4/6 • 1, 5 • 8/9, 1 • 6/8. His reason for grouping these cards as such was "They all have a fraction that you multiply by a whole number."

Properties of Numbers

In sorting the same set of cards which Frank (above) sorted, Umed, an algebra student, placed the cards 1 • 53, 5 • 87 and 317 • 5 together. His reason for sorting the cards in this fashion was :Because it's a prime number times another prime number."

Table 4.4

What Surface Features Students Paid Attention to When Sorting The Cards Representing The Deep Patterns of Multiplicative Identity and Commutative Property of Multiplication (Blue Cards)

Surface Features Built in By The Researcher	Number of Seventh Graders Who Sorted Paying Attention to the Particular Surface Feature n = 13	Number of Algebra Students Who Sorted Paying Attention to the Particular Surface Feature n = 13
Whole Numbers	6	4
Fractions	9	5
Variables	11	5
Other Surface Features The Students Identified		
Because they contained a Certain Number	2	0
The Physical Structure of the Problem	3	2

Table 4.5

What Surface Features Students Paid Attention to When Sorting The Cards Representing The Deep Patterns of an Arithmetic Sequence with the common Difference of Two and a Geometric Sequence With a Common Ratio of Two (Yellow Sequence Cards)

Surface Features Built in By The Researcher	Number of Seventh Graders Who Sorted Paying Attention to the Particular Surface Feature n = 13	Number of Algebra Students Who Sorted Paying Attention to the Particular Surface Feature n = 13
Whole Numbers	5	5
Fractions	10	9
Variables	6	6
Other Surface Features The Students Identified		
Certain Number in Common	0	1
Mixed Fractions	4	4
Parenthesis	5	2
Same Denominators	1	3
Same Numerators	1	2
Odd Numbers	2	3
Even Numbers	3	4
Size of Numbers	1	2

Table 4.6

What Surface Features Students Paid Attention to When Sorting The Cards Representing The Deep Patterns of Area and Perimeter (Orange Geo. Cards)

Surface Features Built in By The Researcher	Number of Seventh Graders Who Sorted Paying Attention to the Particular Surface Feature n = 13	Number of Algebra Students Who Sorted Paying Attention to the Particular Surface Feature n = 13
Whole Numbers	2	1
Diagrams	5	6
Variables/Formulas*	0/7	4/5
*Students discussed Variables and Formulas Differently		
Other Surface Features The Students Identified		
Equals P	1	4
Equals A	2	4
Square Numbers	1	2
Presence of S	2	1
Feet & Inches (measurements)	2	4
Feet	2	3
Inches	1	1
Odd Numbers	0	3
Even Numbers	0	3
Multiplication Problems	3	7
Addition Problems	2	6
Problems with Length and Width	1	2
Problems with Base and Height	1	2
Certain Number in Common	1	1
One Letter Equal to Two Others	3	3

Table 4.7

Types of Surface Features

TYPES OF NUMBERS	PROPERTIES OF NUMBERS	REPRESENTATION OF EXPRESSIONS, SENTENCES AND SEQUENCES
Fractions	Prime Numbers	Pictorial Representations
Mixed Fractions	Odd Numbers	Numerical Representations
Decimals	Even Numbers	Variable Generalizations
Variables	Size of Numbers Used	
	Perfect Squares	
	Number of digits in numbers	

COMMON FEATURES BETWEEN GROUPS OF EXPRESSIONS OR SENTENCES

Presence of certain numbers
Use of certain letters
in formulas a's for area, p's for perimeter
lengths such as b = base, h = height
Measurement Units Used
Operations Used (+, -, x)
Physical Structure of Problems
Perceive Difficulty of Problems if They Were to be Solved

MATHEMATICAL SYMBOLS AND CONVENTIONS USED

Parenthesis
3j versus 3 x j versus 3 • j

Common Features of More than one Expression or Sentence

When sorting the cards containing the deep patterns of commutative property of multiplication and multiplicative identity Mary, a seventh grader, placed $75 \cdot 4 = 4 \cdot 75$, $754 \cdot 1 = 754$, $7/20 \cdot 1/4 = 1/4 \cdot 7/20$, $2/3 \cdot 5/7 = 5/7 \cdot 2/3$ together. Her reason for placing these four cards together was "Cause they both have something that has like 7 in it."

Ways of Representing Expressions, Sentences and Sequences

When Yoyo, an algebra student, sorted the set of cards containing the deep patterns of arithmetic sequence with a common difference of two and geometric sequences with a common ratio of two he placed the cards containing the sequences: n,2n,4n,8n,16n....
r, (r + 2), (r + 4), (r + 6), (r + 8) ...
1/5 a, 2/5 a, 4/5 a, 8/5 a, 16/5 a ... and
(s - 4), (s - 2), s, (s + 2), (s + 4) ...

together. His reason was "These ones have variables." The other pile he made were all of the cards which did not have variables on them.

Mathematical Symbols and Notations Used

Zoe, an algebra student, when shown the cards containing the same deep pattern as the set Yoyo sorted placed: r, (r + 2), (r + 4), (r + 6), (r + 8)... and (s - 4), (s - 2), s, (s + 2), (s + 4) ... together. Her reason for putting these two cards together was "because they were all in parenthesis."

Each of these previous vignettes are just one of many examples which could be used to illustrate the five groups of surface features which students paid attention to. By paying attention to surface features, of the sequences, expressions and sentences, students often failed to see the deep underlying patterns that existed. Thus the surface features became noise or static which prevented the students from receiving the picture of the deep patterns.

Evidence of the interference caused by surface features is obvious when looking at the cards which students placed together when sorting the various groups of cards. Those cards that were grouped together in more than fifty percent of the total number of sorts made by the students is summarized in Tables 4.8, 4.9 and 4.10.

Students often sorted a particular set of cards more than one time. The resulting total number of sorts is higher than 26. Once a student successfully sorted a group of cards the sort ended. These tables include all features that were salient for students previous to and including successful sorts.

Table 4.8

When Sorting The Cards Representing The Deep Patterns Of Multiplicative Identity and Commutative Property of Multiplication Those Cards Which Students Placed Together In More Than Fifty Percent Of Their Total Sorts (Blue Cards)

n = 42

Percent of Sorts Placed Together	Cards	Correct Or Incorrect Match Based on Deep Pattern
71.4	p • 1 = p a • 1 = a	correct
73.8	83 • 1 = 83 754 • 1 = 754	correct
90.5	75 • 4 = 4 • 754 11 • 3 = 3 • 11	correct
90.5	2/3 • 5/7 = 5/7 • 2/3 7/20 • 1/4 = 1/4 • 7/20	correct
92.9	3/4 • 1 = 3/4 18/19 • 1 = 18/19	correct
95.2	j • m = m • j b • c = c • b	correct

Table 4.9

When Sorting The Cards Representing The Deep Patterns Arithmetic Sequences With a Common Difference of Two and Geometric Sequences with a Common Ratio Of Two Those Cards Which Students Placed Together In More Than Fifty Percent of Their Total Sorts (Yellow Sequence Cards)

n = 49

Percent of Sorts Placed Together	Cards	Correct Or Incorrect Match Based on Deep Pattern
51.0	201,203,205,207,209... 100,200,400,800,1600... 2,4,6,8,10... 3,6,12,24,48...	incorrect
55.1	3,6,12,24,48... 201,203,205,207,209...	incorrect
55.1	100,200,400,800,1600... 2,4,6,8,10...	incorrect
63.3	201,203,205,207,209... 2,4,6,8,10...	correct
65.3	2,4,6,8,10... 3,6,12,24,24...	incorrect
67.3	201,203,205,207,209... 100,200,400,800,1600...	incorrect
67.3	100,200,400,800,1600... 3,6,12,24,48...	correct
75.5	r,(r+2),(r+4),(r+6),(r+8)... (s-4),(s-2),s,(s+2),(s+4)...	correct
81.6	20 3/4, 22 3/4, 24 3/4, 26 3/4, 28 3/4... 5 2/3, 7 2/3, 9 2/3, 11 2/3, 13 2/3...	correct
98.0	3/5,6/5,12/5,24/5,48/5,56/5... 1/4,2/4,4/4,8/4,16/4...	correct

Table 4.10

When Sorting The Cards Representing The Deep Patterns of Area and Perimeter
Those Cards Which Students Placed Together In More Than Fifty Percent of Their Total Sorts
(Orange Geo. Cards)

n = 53

Percent of Sorts Placed Together	Cards	Correct Or Incorrect Match Based on Deep Pattern
50.9	$p = 4s$ $p = 2b + 2h$ $p = 2w + 2l$	correct
52.8	$(2 \cdot 4ft) + (2 \cdot 6ft) = 20ft$ $4 \cdot 4ft = 16ft$	correct
52.8	$p = 4s$ $p = 2b + 2h$	correct
68.0	$9in \cdot 3in = 27in^2$ $4 \cdot 4ft = 16ft$ $5ft \cdot 5ft = 25ft^2$	incorrect
71.7	$A = s^2$ $A = lw$ $A = bh$	correct
81.1	Perimeter Diagrams	correct
83.0	$p = 2w + 2l$ $p = 2b + 2h$	correct
85.9	$A = lw$ $A = bh$	correct
86.8	Area Diagrams	correct
90.6	$4 \cdot 4ft = 16ft$ $5ft \cdot 5ft = 25ft^2$	incorrect

Table 4.8 shows that when sorting the cards containing the multiplicative identity or commutative property of multiplication the following cards were placed together more than fifty percent of the time:

1.	$j \cdot m = m \cdot j$	and	$b \cdot c = c \cdot b$
2.	$3/4 \cdot 1 = 3/4$	and	$18/19 \cdot 1 = 18/19$
3.	$75 \cdot 4 = 4 \cdot 75$	and	$11 \cdot 3 = 3 \cdot 11$
4.	$2/3 \cdot 5/7 = 5/7 \cdot 2/3$	and	$7/20 \cdot 1/4 = 1/4 \cdot 7/20$
5.	$83 \cdot 1 = 83$	and	$754 \cdot 1 = 754$
6.	$p \cdot 1 = p$	and	$a \cdot 1 = a$

While it is encouraging that all of these cards are correctly placed together it is discouraging that students saw a difference between sets one, three and four, as numbered above, in more than half of the sorts. It is equally discouraging that more than half of the students failed to recognize that sets two, five and six contained the same deep pattern.

Similar results are seen in Tables 4.9 and 4.10 which contain the information for the sorts of the cards representing the arithmetic and geometric sequences and the area and perimeter cards respectively. Here however, not only weren't cards that should have gone together placed together, but cards were incorrectly placed together as well. An example of two cards placed incorrectly together would be the following two cards: $4 \cdot 4ft = 16ft$ and $5ft \cdot 5ft = 25ft^2$. The first card illustrates the finding of the perimeter of a square while the second card illustrates the finding of the area of a square. A typical reason that would accompany a sort like this was "because they're all perfect squares" (Robin, algebra student) or "it is a number times itself." (Andy, seventh grader) Only in one instance did any student (a seventh grader) make any comment that would indicate that they were aware of the difference between feet and feet squared.

It appears that the more potential noise, or surface features, a group of problems contains affects how difficult or easy it will be for students to recognize the deep patterns within these sets of cards. This may explain why students were more successful at recognizing the patterns in tables rather in groups of cards. The very way in which a table is set up almost eliminates any surface features. Thus, not having to deal with surface features when looking at the tables students were free to concentrate on finding the deep patterns. This explanation also sheds light on why students were most successful at identifying the multiplicative identity. The multiplication by 1 could easily have been recognized in all equations much as a surface feature is identified. It was impossible to separate those students who recognized that all of the equations contained a one from those students who realized the deeper property of one, when one of two factors is 1 the product always equals the other factor.

Issue Three: What Types of Mistakes do Different Patterns Lead Students to Make?

The third issue that was identified under research question one was the types of mistakes which different patterns lead students to make. This question is tied very closely with the last question. If a group of equations contain several potential surface features students often will focus only on these features. Any deep patterns will be completely ignored. Thus the students fall in a trap of focusing only on the types of issues outlined in the previously given Table 4.7.

Figure 4.1 illustrates a flowchart of the process involved in sorting cards. Many students are unable to recognize the path that leads to the left. These students get caught in the process of sorting cards by surface features. They are unable to look beyond the surface features to the deep patterns. If this flowchart were three dimensional the deep pattern path would have to be illustrated somewhere hidden behind the surface feature path out of view for most students.

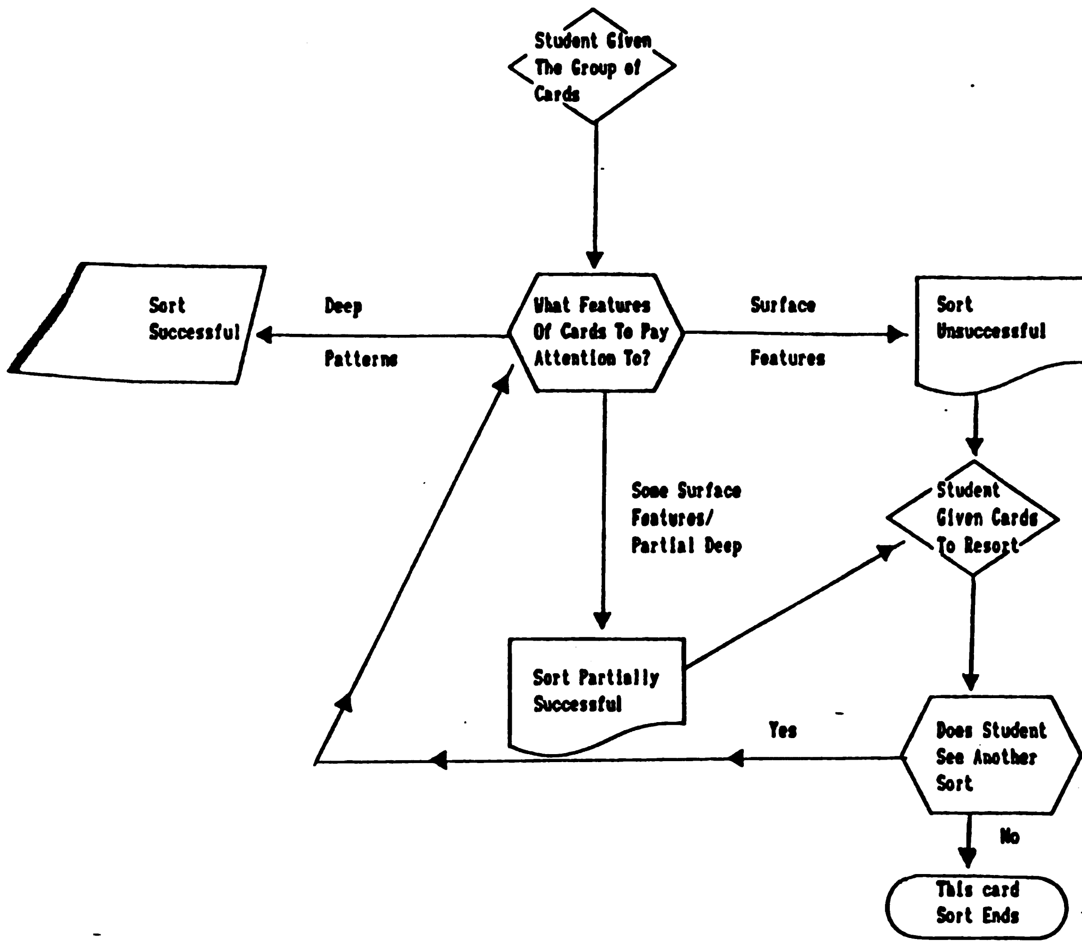


Figure 4.1: Flowchart of the Process Involved in Sorting Interview Cards

Conclusion

The overall research question which was addressed in this section was the question of what types of deep patterns will students recognize? Two tasks, the card sort and identification of patterns from tables task, were designed to answer this research question. In terms of

Lesh's et al. (1987) definition of understanding the card sort task investigated students' ability to identify concepts (deep patterns) which were expressed using at least two different representational modes within the written symbols representational system. Two constant representational modes of expression were the numerical and variable representational modes. One set of cards, namely the cards organized around the deep patterns of area and perimeter, also expressed the concepts using the pictures and diagrams representation system.

The identification of patterns from the table task was structured so as to see if students could recognize patterns which were written within the numerical representational mode of the written symbols representational system but which were presented within the pictures and diagrams representational system. One conclusion of the research was that the smaller the number of surface features built into the groups of problems the more successful students were at recognizing the deep patterns (concepts) which existed within the problem. One result of this fact was that patterns in tables were easier for students to recognize than patterns in groups of expressions, sentences and sequences because they contain few or no surface features. Within the tables the nature of the pattern to be generalized also affected the ease of recognizing certain patterns. For example, the pattern x^2 was easily recognized by students. This is not surprising considering that the sequence represented by this pattern seemed familiar to most students. Patterns which were built on one operation (i.e., x + 5 and 3x) were also more easily recognized than patterns involving two operations (i.e., 5x + 3 and 5x - 4).

When groups of expressions, sentences and sequences contain many potential surface features seventh graders and algebra students were equally and highly unsuccessful at recognizing these patterns. When potential surface features were minimized both algebra students and seventh graders were more successful at recognizing deep patterns. In this later case, however, the algebra students were more successful than the seventh graders. The presence of the generalized form of a pattern did not help the students by providing clues to the structure of the deep pattern.

These results indicated that students had quite a bit of difficulty recognizing deep patterns. Students not only had trouble recognizing patterns which were expressed within different representational systems, such as was the case with the area and perimeter cards, but they also had trouble recognizing patterns which were expressed with one representational system, as was the case with all but the area and perimeter cards. The fact that concepts were expressed in all card sorts using both the numerical mode and variable mode within the written symbols representational systems did not seem to either hinder or help the students to recognize deep patterns. In terms of

Lesh's et al. (1987) three criteria for understanding students in this study were usually unable to recognize deep patterns which were expressed within one or more representational systems.

Research Question Two: What Types of Patterns Can Students Generalize By Using Variables?

Issue One: What Patterns Can Students Generalize and What is the Connection Between the Ability to Generalize Patterns and the Ability to Recognize Patterns?

One of the tasks which took place during the card sort phase of the interview was to have the students sort a group of cards where the variable generalizations were not present. If the students successfully sorted the cards they were then asked to give a generalized statement, using a given variable for the pattern they saw. Since each set of cards contained two deep patterns it was possible for a student to correctly sort out one deep pattern, but not the other. In this case the student was asked only to generalize with a variable the pattern they correctly saw. If a student was unsuccessful at sorting the cards they were then sorted correctly for him/her by the researcher. If the student could explain why the researcher sorted the cards the way she did then the student was asked to generalize the cards. If the student could not successfully explain why the cards were sorted in the way that they were, the task ended. The results of this task are given in Table 4.11.

Table 4.11

Ability To Generalize Patterns Found on Card Sort Cards
Seventh Graders

	Commutative Property Of Addition n = 9	Additive Identity n = 12	Multiplication Of A Single Digit Number By a Variable n = 5
Number of Students Who Were Successful	6	5	2
	Algebra Students		
	n = 9	n = 13	n = 8
Number of Students Who Were Successful	8	8	2

Table 4.11 indicates that not all students were given the same set of cards to generalize. It depended on the students success at various points in the interview as to what set of cards they were asked to generalize. This table also shows that the seventh graders and algebra students were about equally successful at this task. At this phase of the interview each student was only asked to generalize one set of cards.

Table 4.12 is an expanded version of Table 4.3. This table indicates which of the patterns in the tables students were able to generalize. The numbers of students who could produce generalizations with variables is in some cases larger than the number who recognized those patterns.

That is because, using the method described in chapter three, some students who did not recognize the pattern were told how the pattern worked (without any reference to variables) and then asked if they could generate an appropriate generalization with a variable.

Table 4.12 indicates that more algebra students were successful at recognizing the patterns in the tables. This fact has already been discussed in greater detail earlier in the chapter. As Table 4.12 indicates many of the seventh graders who did not recognize the patterns for x + 5 and 3x were not told what the patterns were and then asked to generalize them. This is because only the patterns for 5x + 3 and 5x - 4 were usually told to students unless the student expressed interest on their own in knowing the patterns for x + 5 and 3x. Table 4.12 shows that the majority of algebra students who recognized patterns could also generalize them. It also shows that if the algebra students did not recognize the patterns they still were able, in most cases, to generalize the pattern with variables if they were told what the pattern was. The algebra students' ability to generalize with variables equations seems to exceed their ability to recognize patterns.

An example of an interaction which illustrates this point is the following:

Zoe, an algebra student, correctly identified and generalized the patterns in the table which were generalized by the equations m + 5, 3k and j^2. When presented with the table which looked like this:

1	8
2	13
3	18
4	23

Zoe said " Oh, I know, it also alternate, 8 and 3 and 8 and 3." By following this alteration and also noting that each number in the right column was increased by 5, Zoe was able to correctly say what would go opposite a 10 in the table. I pointed out to her that if we wanted to figure out what went after 100 in the table, using her method would be time consuming. I asked her if she could think up a formula, or shortcut way, like she had used in the previous three tables, to find out what would go opposite 100. She was unable to do this so I eventually said to her, "One other student told me that they thought that if you multiplied the number in the left column by 5 and then added 3 you could find the number that went in the right hand column. Do you think that that student was correct?" Zoe tried a few examples and decided that the student had given a correct rule. I next asked Zoe what would go opposite a p in the table. She wrote down " p(5 + 3)." She then said "wouldn't those two together it would be 8 • p. No it wouldn't work. Add those two together it would be 8 • p. It's 5p + 3." So by hearing the fictitious other student's rule and using a bit of testing of the first written generalization Zoe was able to generalize with a variable a pattern which she was unable to recognize on her own.

Only patterns for 5x + 3 and 5x - 4 were revealed to the students. For the majority of seventh graders the task had ended before this point so it is hard to make a similar statement about them. Looking at the data for the seventh grade students who saw the patterns indicated by the expressions 5x + 3 and 5x - 4 it is clear that with these patterns seventh graders' ability to generalize patterns with variables exceeded their ability to recognize the pattern. Therefore, for the seventh graders the data tends to indicate that their ability to generalize patterns exceeds their ability to recognize patterns, but it is impossible to state this as a fact based on the available data. Further research needs to be done before a definite statement about the relationship between the ability of seventh graders to recognize patterns and their ability of generalize these patterns with variables can be made.

Table 4.12 indicates that the patterns that students can generalize are dependent on the patterns that they recognize. Students' ability to generalize patterns is not dependent on their ability to recognize those patterns on their own. Recognition of a pattern is a prerequisite for generalization of that pattern. The data indicates, however, that if students were more successful at recognizing patterns they would be equally more successful at generalizing those same patterns.

Table 4.12

Results of Recognition and Generalization of Patterns in Tables Tasks

Seventh Graders

Pattern	x2 n = 13	x + 5 n = 13	3x n = 12	5x + 3 n = 5	5x - 4 n = 2
Pattern Recognized	13	8	3	2	0
Pattern Correctly Generalized	5	5 (1 student told the pattern)	4 (1 student told the pattern)	5 (3 students told the pattern)	2 (2 students told the pattern)

Algebra Students

Pattern	x^2 n = 13	x + 5 n = 13	3x n = 13	5x + 3 n = 11	5x - 4 n = 6
Pattern Recognized	13	11	12	1	0
Pattern Correctly Generalized	12	12 (1 student told the pattern)	12	10 (10 students told the pattern)	5 (5 students told the pattern)

<u>Issue Two: What Types of Errors Do Students Make When Generalizing Expressions and Sentences?</u>

Table 4.13 looks at the types of errors students made when trying to generalize the cards in the card sort. This table shows that for seventh graders the most common type of error was to generalize by making the variable parts of the patterns constants and the constant parts of the equations variables. For example, given the group of equations:

$$17 + 0 = 17$$
$$8.2 + 0 = 8.2$$
$$3 + 0 = 3$$
$$99.3 + 0 = 99.3$$

Table 4.13

Strategies of Students Who Incorrectly Generalized Patterns Found on Card Sort Cards

	Incorrect Strategy Of Making The Constant The Variable	Incorrect Strategy Of Use of Too Many Variables
	Seventh Graders n = 10	
Number of Students Who Employed The Strategy	6	0
	Algebra Students n = 13	
Number of Students Who Employed The Strategy	3	5

The variable parts of these sentences are the numerals, 17, 8.2, 3 and 99.3. A correct generalization of these sentences indicates this variation among sentences by use of a variable. A correct generalization would be, $b + 0 = b$. The zero is constant in all of the sentences. A sample of the incorrect generalizations given by some of the seventh graders were:

8.2	+	p	=	8.2	(Andy)
3	+	x	=	3	(Grace)
99.3	+	b	=	99.3	(Hank)

Fewer algebra students than seventh graders make this error when generalizing equations. As Table 4.13 shows algebra students had a tendency to err when generalizing by using too many variables or by making the generalization overly complex by adding in unnecessary numerals. Some of the incorrect generalizations they gave for the previously given set of sentences were:

a	+	b	=	a	(Robin)
a	+	b	=	c	(Tom)
3b	+	0	=	3b	(Zoe)

When generalizing the patterns in the tables the algebra students were quite successful with their generalizations. One error pattern consistent with Wagner's (1981) finding that students had difficulty conserving variables emerged within the seventh grade group. Four students who attempted to generalize the pattern indicated by the expression $t + 5$ erred in their generalizations. Of the four, three used the incorrect technique of thinking of the variable as having a numerical equivalence. When asked what would go on the other side of a table opposite the variable "a" one student (Andy, seventh grader) replied "f cause it's 5 down from it." This student was thinking a, b, c, d, e, f. "F" is the fifth letter from "a". Similarly, one student (Karen, seventh grader) said that opposite a "k" would be a "p". Again thinking, k, l, m, n, o, p. The fifth letter after "k" is "p". Finally, the last student who applied this same error gave a slightly different reply. When asked what went of the other side of the table from a "k" this student (Mary, seventh grader) replied "16". When asked why 16 she wrote A,B,C,D,E,F,G,H,I,J,K 11 + 5 = 16. "K" is the eleventh letter in the alphabet. Even when it was pointed out to the students that the variable represented a number in the table they still gave these same replies.

Conclusion

In this section the research question "What types of patterns will students be able to generalize?" was addressed. The generalization of groups of cards within a sort and generalized statements of patterns in tables task were designed to investigate this question. The generalization of groups of cards within a sort task required that students be able to do two things (1) recognize deep patterns within the numerical representational mode of the written symbols representational system and (2) to translate concepts from the numerical representational mode to the variable representational mode within the written symbols representational system. The second task required students to translate patterns in tables which were written within the numerical representational mode of the written symbols representational system to the variable representational mode also within the written symbols representational system.

The results of these tasks indicated that once algebra students realized what the pattern was their generalizations of that pattern was usually correct. One exception to this last statement occurred in the card sort when incorrect generalizations usually contained too many variables. One reason that this error probably did not show up in the tables task is that the patterns which were being generalized resulted in expressions as answers rather than sentences.

The errors made by the seventh graders when generalizing patterns indicate that some of them might be confused by the difference between a constant and a variable. When students incorrectly generalized the additive identity as, for example, 5 + a = 5 rather than a + 0 = a they were exhibiting confusion between constants and variables. The assigning of a specific numerical value associated with position in the alphabet to variables also indicated the confusion. This type of error showed that these seventh graders had trouble distinguishing between the written symbols and variable representational modes of the written symbols representational system.

It can not be denied that a prerequisite step to generalizing a pattern is recognizing the pattern. Data indicates, however, that it is harder for students to recognize patterns than it is for them to generalize those same patterns. The tasks in this section indicated that students have more trouble recognizing concepts within a representational system than they did translating concepts from one mode of representation to another.

Research Question Three: What Meaning Do Generalizations Using Variables Have For Students?

Each expression, sentence or sequence that contains variables is a generalization of an infinite group of expressions, sentences and sequences. The last task of the interview was designed to see whether or not students could work backwards from a generalization and provide one real life example from the infinite group of examples which could fit the generalization.

To work backwards from generalizations to situations which fit those generalizations was hard for students. For example, no student saw w, $w + 2$, $w + 4$, $w + 6...$ as a series of consecutive numbers differing by 2. This sequence was also not viewed by any student as infinite. A few students interpret the pattern by focusing on w, $w + 2$, $w + 4$ and $w + 6$ as independent numbers or as descriptions of some numerical event over time. Some examples from the transcripts are:

> "Well like you have w number of bugs in your house and then you get two more to come along and then you get 4 more to come along and then 6 more come along." (Irene, 7th grader)
>
> "This rich guy, he has w houses and he goes out and gets bored and he takes a trip around the world so he goes and, he already has w houses so he goes and buys 2 more houses, then he gets bored so he buys 4 more houses. Now he has a lot of houses and he's still bored and he marries a lady and he needs more houses so he goes out and buys six more houses." (Penny, algebra student)
>
> "Jenny had w amount of stickers and her friend Becky had the same amount plus 2 and Jody had the same amount as Jenny had plus 4 and their neighbor had 6 more than Jenny had." (Zoe, algebra student)

Similarly, no student saw the sentence $b + (b + 3) + (b + 6) = 735$ as the sum of three numbers differing by a value of three which equal 735. After reviewing the many ways in which students viewed the expressions, sentences and sequences the interesting question of how did students try to make sense of this task emerged.

<u>Issue One: What Strategies Do Students Use To Obtain Meaning From Generalizations</u>?

Table 4.14 illustrates one of the techniques which students used to obtain meaning from the generalizations. What some students did to make sense of the generalizations was to assign and use numeric values in their problems. Some examples, taken from the transcripts for the generalization $c + 0 = c$, which illustrates the use of this technique are:

TABLE 4.14

Ability to Generate Word Problems From Generalized Statements

Seventh Graders

Generalized Statement	$c + 0 = c$ $n = 13$	$t + 8$ $n = 13$	$w, w+2, w+4, w+6 \ldots$ $n = 13$	$m - 9$ $n = 13$	$b+(b+3)+(b+6)=735$ $n = 13$	$8 + g = 70$ $n = 12$	$55 - k = 7$ $n = 11$
Generated Word Problems By Assigning Constant Values To The Variables	5	2	1	3	0	1	1

Algebra Students

Generalized Statements	$c + 0 = c$ $n = 13$	$t + 8$ $n = 13$	$w, w+2, w+4, w+6 \ldots$ $n = 13$	$m - 9$ $n = 13$	$b+(b+3)+(b+6)=735$ $n = 13$	$8 + g = 70$ $n = 13$	$55 - k = 7$ $n = 11$
Generated Word Problems By Assigning Constant Values To The Variables	2	2	0	1	1	2	0

"There are six cats and you added zero which would equal 6 cats again." (Cathy, 7th grader)

"If Joe has 13 marbles and he doesn't have the money to buy any more he will still have 13 marbles." (Frank, 7th grader)

"One cucumber plus zero cucumbers equals one cucumber." (Vladimir, algebra student)

"You have ten cookies in a jar and you add zero to it. How many cookies do you have?" (Stan, algebra student)

These students found it easier to make sense of the generalizations if they thought of the variable as if it were a constant. Or in other words if they looked at a specific instance of the variable.

Another technique which students used to make sense of the generalizations was to think of the variable as being the first initial of the object the generalization was about. The number of students who used this technique are summarized in Table 4.15. Some examples from the transcripts for the generalization w, w + 2, w + 4, w + 6 ... are:

"You go to a car wash and you wash your window and when you wash 2 and then you wash 4 and then 8." (Andy, 7th grader)

"Um, you have like, you have ten gallons of water there and someone brings you two more gallons and you're adding 4 more, 6 more." (Karen, 7th grader)

"Uh, let's see. If Bill had uh, w number of wheels and he added an even consecutive number, he'd have, something like that." (Yoyo, algebra student)

Table 4.15

Using The Variables As The Initial of The Subject That The Problem Is About

	c + 0 = c		t + 8		v,v+2,v+4, v+6...		m - 9		b+(b+3)+ (b+6)=735		8 + g = 70		55 - k = 7	
	n=13	n=13	n=13	n=13	n=13	n=13	n=13	n=13	n=13	n=13	n=12	n=13	n=11	n=11
	4	4	1	2	3	2	1	2	1	1	2	2	0	0
	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.	7th grd.	alg. sts.
Totals	8		3		5		3		2		4		0	

During the interview many students paused when they had to come up with an object that the problem was about. It sometimes took them a little while to match the variable to a word starting with the correct letter or to give up and generate the problem about an "initially" unrelated object. For some students the need to match the variable to an object that the problem was about was one of the goals of the task. Cathy, a seventh grader, made this explicit when she said " I'm using the first letter for words."

One technique used by four seventh grade students on one expression each was to change the expression to an equation. By doing so they forced the variable into a situation where it represented a specific unknown. For example for the generalization t + 8 the following word problems were given:

> "If you had 12 cats, how many would you have to add to 8 to get 12?" (Cathy, seventh grader)

> "There's a certain number of suckers and you add 8 and you have to equal 13." (Mary, seventh grader)

For the generalization m - 9 the following were given:

> "Like Mary has so many apples and she gave away nine and she has nine left." (Debbie, seventh grader)

> "If Joe has blank marbles and he lost 9 now he has 8." (Frank, seventh grader)

This technique was not used on both expressions by any of the four seventh graders. For the expression m - 9 Cathy replied that you couldn't do it "because there's no answer." Mary came up with a correct situation for m - 9 without setting the problem equal to anything. For the expression t + 8 both Debbie and Frank said, "I can't think of one for that." None of the algebra students used this technique for making sense of the generalizations.

Conclusion

The third research question addressed by the study was "What meaning does generalizations using variables have for students?" The making up of story problems to fit expressions and sentences task was designed to investigate this question. The task required students to translate statements written using the variable representational mode of the written symbol representational system to the spoken language representational system based on their knowledge of how the generalizations reflect real world events.

The meaning generalizations have for students is measured in part by the ability of the students to interpret those generalizations. One type of interpretation is the ability of students to fit the generalizations into their schema of mathematical situations. It is clear from the data presented in this section that working backwards from a generalization to a mathematical situation that fits the generalization was not easy for students.

One of the techniques that students used to try to make sense of the generalizations was the use of the variable to represent the beginning initial of the object the problem was about. Another technique used by students was to assign the variable a constant value and to talk about it as if it were constant. A technique which was only used by seventh graders was to take expressions and change them into sentences by setting them equal to something. No other recognized techniques were used to help the students interpret the generalizations.

A test of the meaning that students give to generalizations is their ability to use the generalizations to create mathematical situations. It is obvious that few students are seeing this connection. The link from generalizations to real life mathematical situations to fit the generalizations is weak for most students. In other words, students were unable to correctly translate statements from one system (the written symbol system) to another system (knowledge organized around real- world events.)

Summary

The results of this study indicate that both the seventh graders and the algebra students had trouble identifying patterns among groups of expressions, sentences and sequences. Patterns in tables were more easily identified by both groups, although the algebra students had more success with these tasks than did the seventh graders. The ability of students to generalize patterns seems to exceed their ability to recognize patterns. Both groups of students had quite a bit of trouble generating word problems from generalizations with variables.

CHAPTER FIVE

SUMMARY AND IMPLICATIONS

Overview

This chapter has three sections. The first section provides a summary of the dissertation along with a summary of the results of the research. Secondly, is a section discussing implications for teaching. Lastly implications for future research are presented.

Summary

The purpose of this research was to address the question concerning students' difficulties in using variables. Students' conceptions of variables used to generalize patterns and as part of generalizations were investigated. This question arose out of consideration for the historical development of variables. Historically variables were developed as a part of a shorthand system for generalizing patterns. Today this historical development is bypassed in schools. Students are taught to think about variables as place holders for solutions to equations.

By focusing instruction on variables as static place holders for solutions to equations students are not being presented with a picture of variables which would lead to conceptual understanding of the uses of variables to describe dynamic situations. Lesh, Post and Behr (1987) describe three skills which are necessary for conceptual understanding of a concept. These are:

(1) be able to recognize a concept in a variety of representational systems.

(2) manipulate flexibly the idea within a given representational system and

(3) to correctly translate the idea from one system to another.

There are five representational systems which Lesh et al. (1978) identify:

(1) knowledge organized around real world events,
(2) manipulative models,
(3) pictures or diagrams,
(4) spoken language and
(5) written symbols.

In contrast to presenting variables as place holders for solutions to equations the historical development of variables involved all three of Lesh's et al. (1987) abilities associated with understanding.

Existing research reveals that students have difficulty understanding and using variables. Kuchemann (1978, 1981 & 1984) concluded from his research that students fell into six levels of understanding concerning variables. These levels are discussed fully in Chapter Two. Booth (1984a) extended Kuchemann's work with further research. He classified the types of errors which students make when working with variables. Other researchers also revealed various misconceptions and difficulties which students have with the concept of variable.

In order to research and discuss students' conceptions of variables it was important to focus on a clear definition of what it means to understand a concept. Lesh's et al. (1987) definition of understanding provided the theoretical backbone for the analysis. Results of this study concerning students' conceptions of variables were regarded in terms of this definition of understanding.

A further consideration of this study was the way in which variables were defined. Past research has focused on variable as a placeholder for solutions to equations. A broader definition of variable has to consider variable as a tool for generalizing patterns and as part of generalizations. This broad definition formed the structure for this research.

Subjects

Because the move from arithmetic to generalized arithmetic is supposed to take place with the introduction of algebra it was decided to interview seventh graders and algebra students for this study. Seventh graders have had the highest level of instruction possible without yet being introduced to formal algebra. Such would not be the case with eighth graders since prealgebra is often taught at the eighth grade level. Algebra students on the other hand will have had a formal introduction to algebra. Twelve students at each level were interviewed.

Data

The data for the research were collected through the use of clinical interview. The clinical interview was designed to answer the following three research questions:

(1) what types of deep patterns can students recognize

(2) what types of patterns will students be able to generalize and

(3) what meaning does a generalization have for students?

After conducting pilot work with students at the seventh grade and algebra levels the research interview was put into finalized form. This finalized interview consisted of five tasks: sorting cards, generalizing cards in the card sort, looking for patterns in tables, generalizing patterns in tables and interpreting expressions, sentences and sequences by creating story settings. The card sorts were organized around three surface features and two deep patterns. The whole interview was audio taped and later transcribed.

Data Analysis

The transcribed interviews were viewed qualitatively with the three research questions in mind. Data were analyzed around six issues which elaborated the three research questions. A table relating the research questions, analysis issues, types of understanding investigated and interview tasks can be found in Chapter Four (Table 4.1). The analysis issues arose from the analysis of data with the research questions in mind.

Tables and supporting vignettes were presented to summarize the research results.

Results

While Lesh's et al. (1987) three characteristics of mathematical understanding provided the structure of the study, the three research questions reflecting these skills structured the interviews. The research questions and interview tasks based on the questions were designed to probe students' conceptions of variables.

The analysis of the research revealed several results concerning students' conceptions of variables. Both the seventh graders and algebra students were weak at identifying patterns in the card sort. When students were presented with patterns in tables, both groups identified these patterns more easily. On this task, however, the algebra students were more successful than the seventh graders. While data were inconclusive for the seventh graders, the algebra students were more successful at using variables to generalize patterns than they were at recognizing patterns. Both groups found the task of providing real life situations for existing generalizations difficult.

While the above set of findings revealed how well students performed on tasks, other findings helped to explain why they were successful or unsuccessful on these tasks. One factor which was quite influential in distracting students from seeing deep patterns was the presence of surface features in the problems. Although only three surface features were deliberately built into problems, students often identified many more. These surface features fell into five groups: (1) types of numbers used (2) properties of numbers (3) common features of more than one expression or sentence (4) ways of representing expressions, sentences and sequences and (5) mathematical symbols and notation used. These surface features directly affected the way in which students sorted the cards and the ways in which they viewed the patterns.

Another factor which influenced the students' ability to see deep patterns was the structure of the patterns themselves. For example, patterns requiring only one operation (i.e., x^2, $3x$, $x + 5$) were easier for students to recognize than patterns involving two operations (i.e., $5x + 3$, $5x - 4$). Numerical patterns, such as those represented in tables, were also more easily recognized than patterns in expressions, sentences and sequences.

Students did not use the cards containing variable generalizations to obtain clues concerning the deep patterns. This result ties in with the fact that students were unable to generate real life situations from expressions or equations containing variables. It is clear that students have limited knowledge of what a generalization means and limited ability to translate among representational systems.

In trying to obtain meaning from generalizations students used different techniques. Both the seventh graders and algebra students sometimes assigned constant values to the variables. They then generated their problems around this constant value. Assigning constant values to the variables helped the students to make more sense of the generalizations. Students often thought of the variable as designating the first initial of the object that their problem should be about. A few of the seventh graders were uncomfortable with the expressions. The need to make situations that gave a precise answer seemed to influence students to change the expressions into sentences in order to make sense of them. No algebra students used this technique. This change is significant since variables in equalities only have a limited domain of possibilities which they can assume in order to represent an equality which is true. Variables in expressions can assume an infinite number of values.

Another place where algebra students and seventh graders answered differently involved their generalizations. When generalizing with variables the cards in the card sort the seventh graders were most likely to err by taking the constant parts of the equations and identifying them with variables. Similarly they took the variable parts of the equations and gave them constant values. A few algebra students used this technique. The majority of algebra students who erred were more likely to make mistakes which involved using more variables then were needed or making their generalizations too complex by adding numbers which were unnecessary. The seventh graders' confusion between variables and constants also showed up when they were asked to generalize the patterns in the tables. A few of the seventh graders on this task associated the variables with their position in the alphabet. Thus, "a" had a numeric value of 1, "b" had a numeric value of 2 with the pattern continuing for all of the letters of the alphabet. Again, this error was not seen in the algebra students. The algebra students had a clearer conception of the difference between a variable and constant than did the seventh graders.

Conclusion

Lesh's et al. (1987) definition of understanding requires that students recognize a concept in a variety of representational systems, manipulate flexibly the idea within a given representational system and correctly translate the idea from one system to another. Students in this study were largely unsuccessful with the first and third aspects of understanding of variables as tools for generalizing patterns. The second skill associated with understanding was not investigated.

Implications for Teaching

In order for teachers to help students to become knowledgeable about variables so that they can perform successfully on skills associated with all three aspects of Lesh's et al. (1987) criteria for understanding they need to have an idea of the present extent and structure of knowledge. The results of this study cannot be generalized to all groups of seventh graders and algebra students. This study, however, begins to shed some light on students' conceptions of variables as tools for generalizing patterns. Emphasizing the results of the study will help teachers to begin to get a feel of how poorly students currently understand variables, of the naive conceptions students have, and of the lack of flexibility in interpretation of situations that characterizes students knowledge. The results further speak for the need of an alternative approach toward the teaching of variables, such as variables as used to generalize patterns, which would incorporate all of Lesh's et al. (1987) aspects of understanding. The process of using variables to generalize real world situations and using those generalizations to solve problems requires that students: (1) recognize a pattern (2) generalize that pattern (3) correctly interpret what the generalization means and (4) substitute numbers into the generalization to solve problems related to specific situations. These four skills are somewhat parallel to Lesh's et al. (1987) three aspects of understanding.

The results of the card sorts and tables task revealed that students were largely unsuccessful at recognizing patterns. One exception to this statement was the ability of seventh graders and algebra students to recognize patterns consisting of one arithmetic operation presented using the numerical mode of representation within the pictures and diagrams representational system. Teachers need to start by expanding students' knowledge about table patterns. Since students were successful at recognizing patterns using one arithmetic operation a logical next branch of expansion would be to give students increased experience with patterns in tables consisting of two operations. When students begin to become successful with the recognition of these types of patterns it is time to present students with different types of patterns. One example would be series of equations all illustrating the same concepts such as those which were used in the card sort. Opportunities to explore a wide variety of pattern situations many help students develop strategies for recognizing patterns. They will begin to realize that patterns are an integral part of mathematics.

From kindergarten on up students need to be guided toward seeing similarities and differences between expressions, sentences and sequences. They need to be led from interpreting mathematical statements in isolation to thinking about the total mathematical system within which such statements exist. This study revealed that students are unsuccessful at making connections between expressions, sentences and sequences which share common structural patterns. As one example, of many, most students were unable to recognize that 2,4,6,8,10... and

201,203,205,207,209... are both sequences with a common difference of two. Teachers need to make such links explicit to students. Teachers also need to provide students with opportunities which lead them to experience the power of identifying patterns to help them to make predictions about events in their day to day life.

Once teachers have helped students to recognize patterns they need to guide them in generalizing these patterns. The results of this study indicate that teachers should experience success in teaching students to generalize a recognized pattern. It was much harder for students to recognize patterns than it was for them to generalize those patterns with variables.

Although students can be expected to learn to generalizing patterns with variables, teachers need to pay careful attention to how they organize and set tasks for students that help them develop a deep understanding of what instructing students as to what generalizations with variables mean. This study revealed that students had trouble with the third aspect of using variables to generalize patterns: working backwards from generalizations written with variables to generate situations which would fit those generalizations. Teachers should help students to be able to look at an expression such as "k + 5" and be successful at the following tasks:

- writing other examples with and without variables that fit the expression,
- illustrating the expression pictorially,

- deriving the expression from situations,
- writing a real life situation which fits the expression.

This last task was very difficult for students in this study.

This research revealed that compounded with students conceptions of variables is their conceptions of other mathematical ideas. If student view fractions as something other than numbers than they are going to have a hard time knowing that fractions can be substituted into expressions just as easily as whole numbers. Similarly if students don't understand certain mathematical conventions, such as writing two variables side by side to indicate multiplication, then they will become bogged down trying to make sense of them. An example from this research illustrating this problem was that when sorting the cards with deep patterns of area and perimeter 90.6 percent of the sorts involved having the cards $4 \cdot 4ft = 16ft$ and $5ft \cdot 5ft = 25ft^2$ together. The fact that the numbers were squared was more salient than the deep patterns. Students did not recognize that $5ft \cdot 5ft = 25ft^2$ had to be an example of finding an area whereas $4 \cdot 4ft = 16ft$ illustrated a length of 4ft occurring 4 times. The reason that students sorted so often by surface features is in part because they still have trouble fully understanding the language of expressions, sentences and sequences. It is not enough to focus instruction on improving students' conceptions of variables. Teachers must be tuned in to all aspects of how students make sense of mathematics.

Implications for Future Research

Lesh et al. (1987) describe aspects of understanding as well as five representational systems. This study focused on students' conceptions of variables as demonstrated in three areas: (1) students' ability to recognize patterns in two representational systems, (2) students' ability to translate patterns from the numerical representational mode within the written symbols representational system to the variable representational mode within the written symbols representational system and (3) students' ability to translate variable representations within the written symbols representational system to spoken language representations based on knowledge organized around real world events. Future research needs to focus on all three aspects of understanding and the interactions between them as well as all five representational systems.

This study focused mainly on students' ability to recognize and represent patterns using variables. Future research needs to focus on students' ability to recognize and represent patterns with manipulatives, pictures and diagrams, spoken language and numerical representations. One such task would be to present students with variable expressions and let them generate numerical examples to fit the expression. Another task might be to present students with tables already generalized and let them generate entries for the tables. These tasks are extensions of tasks which were used in this study. Showing students manipulatives illustrating a pattern and having them verbalize the pattern could be another investigation task. In general, more tasks which focus on all five representational systems need to be incorporated into future studies.

The flexibility of students in manipulating variable expressions within one representational system was not investigated by this study. Finding out what types of problems students can solve with given generalizations will give input as to what power generalizations hold for students.

Finally, this study was based on the belief that the ability to recognize patterns and generalize them using variables were prerequisite skills for student understanding the power behind variable expressions. An alternative belief could be that by having students solve several different problems using variables as place holders for solutions to equations they are gaining a feel for the power variable expressions have. Teaching experiments need to be conducted. Subsequent research should begin to look at the strengths of using the two different approaches based on students'' understanding variables, using Lesh's et al. (1987) definition of understanding. Only after students' conceptions of variables are more fully understood can curriculum materials be developed to address misconceptions. Such developed curriculum would require classroom studies and further research designed at continued study of students' conceptions of variables.

Chapter Summary

This chapter provided an overall summary of the dissertation along with results of the research. Educational implications based on Lesh's et al. (1987) definition of understanding were presented. Lastly, implications for future research, also based on Lesh's et al. (1987) definition for understanding, were presented.

APPENDICES

APPENDIX A
THE INTERVIEW TASKS

Card Sort Cards

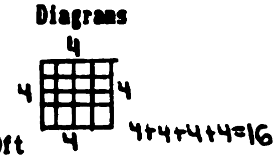
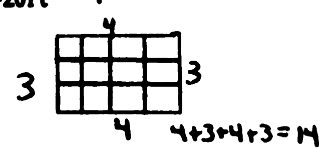
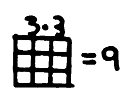
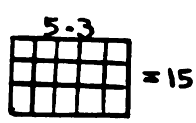
Blue Set (Used in Every Interview)

Deep Patterns	Surface Features: Whole Numbers	Fractions	Variables
Commutative Property of Multiplication	$11 \cdot 3 = 3 \cdot 11$	$2/3 \cdot 5/7 = 5/7 \cdot 2/3$	$b \cdot c = c \cdot b$
	$75 \cdot 4 = 4 \cdot 75$	$7/20 \cdot 1/4 = 1/4 \cdot 7/20$	$j \cdot m = m \cdot j$
Multiplicative Identity	$83 \cdot 1 = 83$	$18/19 \cdot 1 = 18/19$	$p \cdot 1 = p$
	$754 \cdot 1 = 754$	$3/4 \cdot 1 = 3/4$	$a \cdot 1 = a$

Yellow Sequece Set (Used in Every Interview)

Deep Patterns	Surface Features: Whole Numbers	Fractions	Variables
Arithmetic Sequence (+ 2)	201,203,205,207,209...	$5\frac{2}{3}, 7\frac{2}{3}, 9\frac{2}{3}, 11\frac{2}{3}, 13\frac{2}{3}$...	r,(r+2),(r+4),(r+6),(r+8)...
	2,4,6,8,10...	$20\frac{3}{4}, 22\frac{3}{4}, 24\frac{3}{4}, 26\frac{3}{4}, 28\frac{3}{4}$.	(s-4),(s-2),s,(s+2),(s+4)...
Geometric Sequence (x 2)	3,6,12,24,48...	3/5,6/5,12/5,24/5,48/5...	$\frac{1}{5}a, \frac{2}{5}a, \frac{4}{5}a, \frac{8}{5}a, \frac{16}{5}a$...
	100,200,400,800,1600...	1/4,2/4,4/4,8/4,16/4...	a,2a,4a,8a,16a...

Orange Geo (Used in Every Interview)

Deep Patterns	Surface Features: Whole Numbers	Diagrams	Variables/Formulas
Perimeter	$4 \cdot 4ft = 16ft$		$p = 2b + 2h$
	$(2 \cdot 4ft) + (2 \cdot 6ft) = 20ft$		$p = 2w + 2l$
			$p = 4s$
Area	$5ft \cdot 5ft = 25ft^2$		$A = lw$
	$9in \cdot 3in = 27\ in^2$		$A = bh$
			$A = s^2$

Yellow Set

Deep Patterns	Surface Features		
	Whole Numbers	Fractions	Variables
Multiplication	317 • 5	4/8 • 5	5 • s
By 5	5 • 87	5 • 8/9	v • 5
Multiplication	66 • 1	4/6 • 1	1 • c
By 1	1 • 53	1 • 6/8	1 • h

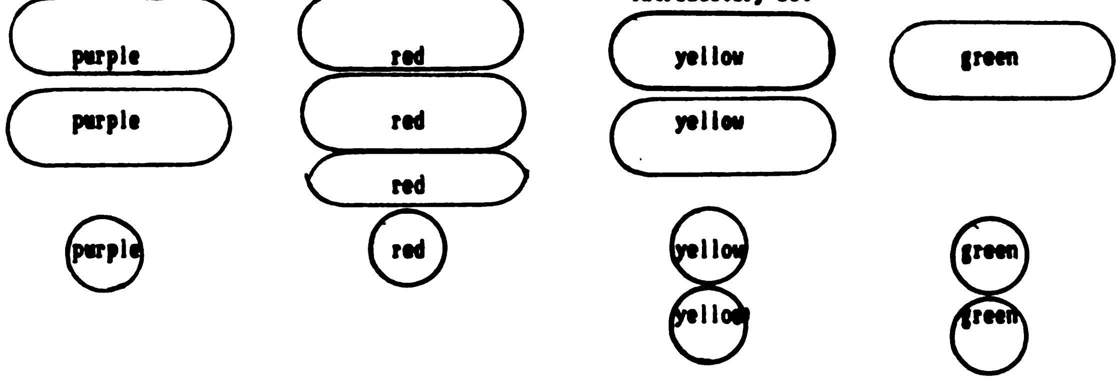
Pink Set

Deep Patterns	Surface Features		
	Whole Numbers	Decimals	Variables
Commutative Property	3 + 1 = 1 + 3	11.5 + 3.2 = 3.2 + 11.5	b + x = x + b
Of Addition	12 + 2 = 2 + 12	66.4 + 8.9 = 8.9 + 66.4	m + k = k + m
Additive Identity	17 + 0 = 17	99.3 + 0 = 99.3	y + o = y
	3 + 0 = 3	8.2 + 0 = 8.2	a + 0 = a

Green Set

Deep Pattern	Surface Features		
	Whole Numbers	Large Whole Numbers	Variables
Multiplication	4 • 3 = 12	5312312 • 3 = 15936936	3 • v = 3v
By 3	1 • 3 = 3	2473 • 3 = 7419	r • 3 = 3r
Multiplication	7 • 2 = 14	2499 • 2 = 4998	z • 2 = 2z
By 2	5 • 2 = 10	8323421 • 2 = 16646842	2 • g = 2g

Introductory Set



Tables

Orange #1

1	6
2	7
3	8
4	9

Yellow #2

1	1
2	4
3	9
4	16

Green #3

1	3
2	6
3	9
4	12

Blue #4

1	8
2	13
3	18
4	23

Pink #5

1	1
2	6
3	11
4	16

Generalized Statements

$$c + 0 = c$$

$$t + 8$$

$$w,\ w + 2,\ w + 4,\ w + 6 \ldots$$

$$m - 9$$

$$b + (b + 3) + (b + 6) + (b + 9) = 735$$

$$8 + g = 70$$

$$55 - k = 7$$

APPENDIX B
A COPY OF THE PERMISSION SLIP

Donna E. Bird
252 Erickson
Michigan State University
East Lansing, Michigan
March, 1987

Dear Parent,

As part of my doctoral research at Michigan State University I am interested in investigating students' conceptions of mathematical patterns and variables. Your childs' teacher and principal recognize the potential benefits of this research for mathematics instruction. They have agreed to allow me to interview students during class time.

Your child is a valuable component of my research. In order to investigate the ways in which students think about mathematical concepts it is necessary to talk with them one on one in a relaxed atmosphere.

The research is to consist of an interview during which the students will be asked to think about mathematical situations. The students responses to these situations will be tape recorded. After the study is completed the tapes will be destroyed. The interview will last approximately the time of one class period.

In order to conduct these interviews with students it is necessary that parents give their consent. At any time during the interview either the student or the parent can withdraw their consent.

All results of the research will be treated with strict confidence. Subjects will remain anonymous. Within these restrictions results will be made available to subjects if requested.

This research is an important step toward answering the question of "how do students think about mathematics?" If you have any questions before giving your consent please feel free to contact me at the below phone number.

Thank you for you cooperation,

Donna E. Bird

Donna E. Bird
349 - 8286

Student Consent To Participate in the Study of Students' Conceptions of Mathematics

By

Donna E. Bird

I agree to allow my child to participate in the doctoral research study being conducted by Donna E. Bird. This project is part of an approved research program at Michigan State University. A description of the rationale and design of the study has been provided separately.

I understand that as a participant in the study, my child will be expected to:

1. Participate in an interview taking approximately one mathematics period.

I understand that the following precautions will be taken to protect against abuse of my childs' confidence or the data from this study:

1. All data collected during this study will be kept confidential and the study will be reported without the identification of individual students, their teachers, or schools.

2. I may request data on my child (and a group) and review it with the research.

3. My child may withdraw from the study at any time without recrimination.

Signature______________________________

Printed name___________________________

Date_________________________________

Printed name of child_____________________

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