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ELASTIC MODULI OF 2-D COMPOSITES WITH SLIDING INCLUSIONS - A COMPARISON OF THE EFFECTIVE MEDIUM THEORIES

By

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ABSTRACT

ELASTIC MODULI OF 2-D COMPOSITES WITH SLIDING INCLUSIONS - A COMPARISON OF THE EFFECTIVE MEDIUM THEORIES

By

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The effective elastic moduli of 2-D composites with circular sliding inclusions randomly distributed in the matrix are investigated. A sliding parameter is introduced to simulate sliding effect, which gives perfect bonding or pure sliding boundary conditions. The elastic moduli are evaluated by using four popular effective medium theories: a self-consistent method, a differential scheme, a Mori-Tanaka method and a generalized self-consistent method. These methods are modified to account for sliding. In this thesis two aspects are focused. One is the study of the effect of interface on the elastic constants of composites and the other is the comparison of the effective medium theories for the cases of both perfect bonding and sliding. A recently stated Cherkaev-Lurie-Milton theorem (CLM theorem) is used for evaluation of theses methods. This theorem gives the general relations between the effective elastic constants of 2-D composite. The results of these theories are also compared with those from the numerical simulations.

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CHAPTER 1

INTRODUCTION

The composite materials are composed of two or more constituents which have different mechanical properties from each other. Generally it is very difficult to predict the mechanical responses of the composite materials. In order to understand the mechanical properties of the composites, it is important to find the the effective elastic moduli of the composites. The effective elastic moduli C_{ijkl} are defined by the relation (Christensen, 1979; Hashin,1983))

$$\langle \sigma_{ij} \rangle = C_{ijkl} \langle \varepsilon_{kl} \rangle \tag{1.1}$$

where $\langle \sigma_{ij} \rangle$ and $\langle \varepsilon_{kl} \rangle$ are the volume averages of stress and strain fields respectively.

The exact theoretical evaluation of the effective elastic properties is very difficult because it requires the knowledge of the stress field everywhere in the composite. There have been many effective medium theories to find the effective elastic moduli of composites by employing simplified geometrical model of composite to make the problem mathematically tractable. Out of them the four popular effective medium theories are studied in this thesis; the self consistent method, the differential scheme, the Mori - Tanaka method and the generalized self consistent method. However, because of the different assumptions used, they yield different results for the effective elastic constants of composite materials. Therefore it is very important to compare these methods. Recently Christensen (1990) and Zimmerman (1991) compared the results of these effective medium theories. Their work has been done on the basis that the interface between matrix and inclusion is perfectly bonded.

However the interface in composite materials may not be perfectly bonded and sliding or debonding may exist. A number of recent studies on the effect of interfaces showed that the contribution of interface is important but the study in this area is far from complete. In this thesis the effect of sliding interfaces on the effective elastic moduli of composites are investigated and the comparison of the effective medium theories is carried out. The four above mentioned effective medium theories are modified for the effects of sliding to be included.

In this thesis the general solutions for the case of elastic inclusions are obtained and several interesting limits, such as holes, rigid inclusions, the constituents with equal shear moduli and the constituents with the same elastic properties and sliding at interfaces, are studied for the effective medium theories to be compared.

The analysis of this thesis is also emphasized on the two dimensional geometry in order for a *C*herkaev-*L*urie-*M*ilton theorem (Cherkaev et al., 1992) to be used. Although this CLM theorem holds only in two dimensions, it gives the general relations between the elastic moduli and is very powerful because it gives results which are independent of the details of microstructure. The CLM theorem also provides the general results for the stress fields which agrees with the famous results by Dundurs (1967, 1970) and Michell (1989) as investigated by Thorpe and Jasiuk(1992). The effective medium theories should satisfy the CLM theorem since this theorem places no restriction on the microstructure of composite materials. The CLM theorem has been proved only for the limit of perfect bonding so far. In this thesis it is shown that all effective medium theories studied here do satisfy the theorem in the limit of sliding as well as perfect bonding.

CHAPTER 2

EFFECTIVE MODULI OF 2D COMPOSITES CONTAINING ELASTIC INCLUSIONS WITH SLIDING INTERFACES

In this chapter the effective moduli of two dimensional composite material which is composed of matrix and elastic inclusions with sliding interfaces are obtained by using the effective medium theories. For simplicity the inclusion is considered to be a circular inclusion. The first section of this chapter is devoted to the results of dilute case that the volume fraction of inclusions is very small. The parameter which represents the degree of sliding of interfaces is introduced by boundary conditions. The other sections are for the effective medium theories.

2.1 DILUTE RESULTS

The exact solution of the effective moduli for the case that a single inclusion is in an infinite matrix can be solved by the equivalency of energy which is introduced by Eshelby (1957). All effective medium theories should show the same result as the Eshelby's solution when the volume fraction of inclusions is near zero. Therefore the solutions of dilute results are useful to test the values of effective moduli obtained by an effective medium theory.

In this thesis, the sliding condition between matrix and inclusions is introduced by the boundary conditions which involve the discontinuity of the tangential components of displacements across interface, which are written as

$$u_r^m(a,\theta) = u_r^f(a,\theta) \tag{2.1}$$

$$\sigma_{r\theta}^m(a,\theta) = \sigma_{r\theta}^f(a,\theta) = k\left[u_\theta^m(a,\theta) - u_\theta^f(a,\theta)\right] \tag{2.2}$$

$$\sigma_{rr}^m(a,\theta) = \sigma_{rr}^f(a,\theta) \tag{2.3}$$

The superscripts f and m denote inclusion and matrix respectively and a is the radius of a circular inclusion. The subscripts r, θ are for the polar coordinates. The degree of sliding at the interface is represented by the factor k which is zero when the interface is purely sliding and which goes to infinity when the interface reaches perfect-bonding. These boundary conditions to involve the slipping interfaces have been used by Jones and Whittier (1967), Lene and Leguillon (1982), Benveniste (1985), Achenbach and Zhu (1989) Hashin (1990) and many others.

As mentioned before, the exact solutions of effective moduli of composites having single inclusion can be obtained by the equavalency of the strain energy (Christensen, 1979). When the stress field σ_{ij}^0 is applied at infinity to the single inclusion Ω in a domain D, the elastic strain energy is expressed as (see Fig. 2.1)

$$W = W^0 + \frac{1}{2}\int_{|\Omega|}(\sigma_{ij}^0 n_j u_i - \sigma_{ij} n_j u_i^0)\,dS + \frac{1}{2}\int_{|\Omega|}\sigma_{ij}^0 n_j [u_i]\,dS \tag{2.4}$$

where $W^0 = \frac{1}{2}\int_D \sigma_{ij}^0 \varepsilon_{ij}^0 dV$. The superscript 0 denotes the quantities of the applied fields and the subscripts i, j denote the general coordinates. The n_j represents the unit vector which is normal to the interface and the jump of displacement at interface is defined as $[u_j] = u_j^m - u_j^i$.

Using the boundary conditions and the equivalency of strain energy, the effective area bulk modulus K and the effective shear modulus μ are given as (Thorpe and Jasiuk, 1992),

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{K^f} - \frac{1}{K^m}\right)\left[\frac{\dfrac{1}{K^m} + \dfrac{1}{\mu^m}}{\dfrac{1}{K^f} + \dfrac{1}{\mu^m}}\right] \qquad (2.5)$$

$$\frac{1}{\mu} = \frac{1}{\mu^m} + 2c\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right)\left[\left(\frac{2}{\mu^f} - \frac{1}{\mu^m} + \frac{1}{K^f}\right) + \tilde{k}\left(\frac{1}{\mu^f} - \frac{1}{\mu^m}\right)\left(\frac{1}{\mu^m} + \frac{1}{\mu^f} + \frac{2}{K^f}\right)\right]$$

$$\times\left[\frac{1}{\left(\dfrac{2}{\mu^m} + \dfrac{2}{\mu^f} + \dfrac{3}{K^m} + \dfrac{1}{K^f}\right) + k\left(\dfrac{1}{\mu^m} + \dfrac{1}{\mu^f} + \dfrac{2}{K^m}\right)\left(\dfrac{1}{\mu^m} + \dfrac{1}{\mu^f} + \dfrac{2}{K^f}\right)}\right] \qquad (2.6)$$

where $\tilde{k} = \dfrac{ka}{2}$ and c is the volume fraction of inclusions. These solutions yield two limiting cases. One is perfect bonding ($\tilde{k} \rightarrow \infty$) and the other is pure sliding ($\tilde{k} \rightarrow 0$).

The effective medium theories should obey these results when the volume fraction of inclusions approaches zero.

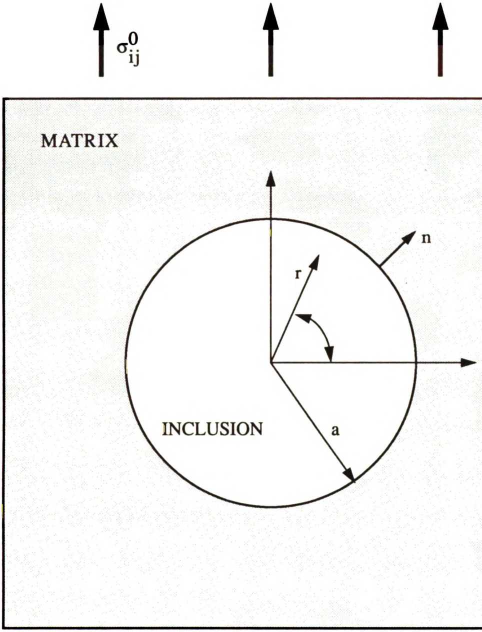

FIGURE 2.1 Single Inclusion Problem

2.2 SELF CONSISTENT METHOD

The self consistent method (Budiansky, 1965; Hill, 1965) suggests a geometrical model that an inclusion is embedded in a homogeneous effective medium (see Fig. 2.2). This model is the simple boundary value problem which can be solved by using the results of the section 2.1. The results that the self consistent method predicts are much more reliable at low volume fraction than high volume fraction.

In order to calculate the effective shear modulus, an external shear stress σ^0_{xy} should be applied and then, the volume average fields of strain ε and stress σ are given as (Mura, 1982)

$$\langle \varepsilon_{xy} \rangle = \frac{1}{V}\int_D (\varepsilon^0_{xy} + \varepsilon_{xy})\, dV + \frac{1}{V}\int_{|\Omega|} \frac{1}{2}\left([u_x]\, n_y + [u_y]\, n_x \right) dS \qquad (2.7)$$

$$\langle \sigma_{xy} \rangle = \frac{1}{V}\int_D (\sigma^0_{xy} + \sigma_{xy})\, dV \qquad (2.8)$$

where the subscripts x,y denote the rectangular coordinates. The second term of $\langle \varepsilon_{xy} \rangle$ in eq.(2.7) accounts for sliding effect and it can be calculated by the boundary conditions at the sliding interface which are introduced in the previous section. It is obtained as follows

$$\frac{1}{V}\int_{|\Omega|} \frac{1}{2}\left([u_x]\, n_y + [u_y]\, n_x \right) dS = c\left(\frac{\langle \varepsilon_{xy} \rangle}{g^c}\right)\mu^c \mu^f \left[\frac{\frac{1}{\mu} + \frac{1}{\mu^f} + \frac{2}{K^f}}{3 + 2\bar{k}\left(\frac{1}{\mu} + \frac{1}{\mu^f} + \frac{2}{K^f}\right)} \right] \qquad (2.9)$$

where the g^c is defined by

$$\langle \sigma^0_{xy} + \sigma_{xy} \rangle_\Omega = 2\left(\frac{\varepsilon_{xy}{}^0}{g^c}\right)\mu\mu^f \qquad (2.10)$$

in this problem,

$$\frac{1}{g^c} = \left[\frac{\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f}}{3+2\tilde{k}\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f})}\right]\left[\frac{(\frac{\mu}{\mu^f})\,(\frac{1}{\mu}+\frac{1}{K})}{(\frac{2}{\mu}+\frac{2}{\mu^f}+\frac{3}{K}+\frac{1}{K^f})+\tilde{k}\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K})\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f})}\right] \tag{2.11}$$

Therefore the average strain and stress field are given as

$$\langle\varepsilon_{xy}\rangle = c\left(\frac{\langle\varepsilon_{xy}\rangle}{g^c}\right)\mu + (1-c)\,\varepsilon^m + c\left(\frac{\langle\varepsilon_{xy}\rangle}{g^c}\right)\mu\mu^f\left[\frac{\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f}}{3+2\tilde{k}\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f})}\right] \tag{2.12}$$

$$\langle\sigma_{xy}\rangle = 2c\left(\frac{\langle\varepsilon_{xy}\rangle}{g^c}\right)\mu^f\mu + (1-c)\,\sigma^m \tag{2.13}$$

where σ^m, ε^m are average shear stress and strain in the original matrix. By using $\sigma^m = 2\mu^m\varepsilon^m$ and the above average strain and stress field, the effective shear modulus is computed as below

$$\frac{1}{\mu} = \frac{1}{\mu^m}+c\,(\frac{1}{\mu}+\frac{1}{K})\left[\frac{(\frac{1}{\mu}-\frac{3}{\mu^m}+\frac{4}{\mu^f}+\frac{2}{K^f})+2\tilde{k}\,(\frac{1}{\mu^f}-\frac{1}{\mu^m})\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f})}{(\frac{2}{\mu}+\frac{2}{\mu^f}+\frac{3}{K}+\frac{1}{K^f})+\tilde{k}\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K})\,(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f})}\right] \tag{2.14}$$

The area bulk modulus can be obtained in a similar way if the external loads are applied by $\sigma^0_{xx} = \sigma^0_{yy} = \sigma^0$. The result is

The final results of the effective shear and bulk modulus are highly coupled each other. Therefore it is not easy to calculate these effective moduli directly

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{K^f} - \frac{1}{K^m}\right)\left[\frac{\frac{1}{K} + \frac{1}{\mu}}{\frac{1}{K^f} + \frac{1}{\mu}}\right] \tag{2.15}$$

from the given data of the elastic constants of matrix and inclusion. Several interesting limit cases will be investigated for evaluating the results of this method in the next chapter.

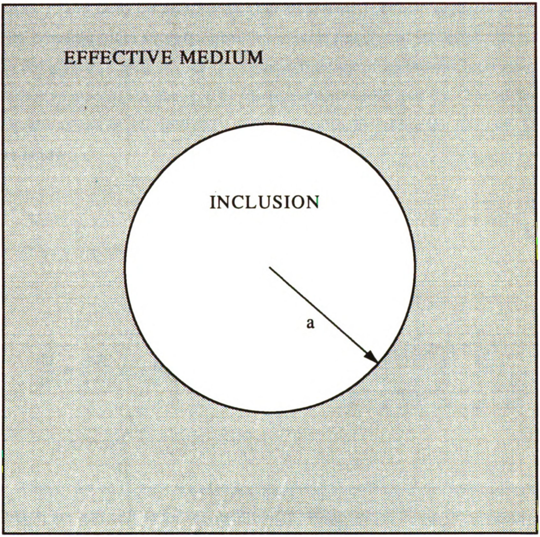

FIGURE 2.2 Geometrical Model of Self Consistent Method

2.3 DIFFERENTIAL SCHEME

The differential scheme has been developed by McLaughlin (1977) and Norris (1985).

Let us consider a homogeneous medium of which the moduli are given by eq.(2.5) and (2.6) for dilute case. The differential scheme suggests the sequential procedure that a small amount of volume fraction of inclusions δc is added to the effective medium. As the volume fraction of inclusions increases, the effective moduli are changed by the increments of δK and $\delta\mu$. The differential equations can be set up from the dilute results, by taking the limit of $\delta c \to 0$, as below

$$\frac{dK}{K} = \frac{dc}{1-c}\left(1-\frac{K}{K^f}\right)\left(\frac{\frac{1}{K}+\frac{1}{\mu}}{\frac{1}{K^f}+\frac{1}{\mu}}\right) \tag{2.16}$$

$$\frac{d\mu}{\mu} = \frac{2dc}{1-c}\left(1+\frac{\mu}{K}\right)\left[\frac{\left(\frac{1}{\mu}-\frac{2}{\mu^f}-\frac{1}{K^f}\right)+\tilde{k}\left(\frac{1}{\mu}-\frac{1}{\mu^f}\right)\left(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f}\right)}{\left(\frac{2}{\mu}+\frac{2}{\mu^f}+\frac{3}{K}+\frac{1}{K^f}\right)+\tilde{k}\left(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K}\right)\left(\frac{1}{\mu}+\frac{1}{\mu^f}+\frac{2}{K^f}\right)}\right] \tag{2.17}$$

These two equations are also highly coupled nonlinear differential equations which are difficult to be solved directly. However, in some limit cases, the solution can be easily obtained with the condition that $\mu = \mu^m$ and $K = K^m$ at $f = 0$.

2.4 MORI - TANAKA METHOD

This average field method was introduced by Mori and Tanaka (1973). Afterwards Benveniste (1987) reformulated this method to compute the effective properties of composite materials. Recently Mori and Wakashima (1990) developed a successive iteration method which is a different interpretation of Mori-Tanaka method. Shibata et al.(1990) used this iteration method to demonstrate the interfacial effect although their work was confined to the case that the matrix and the inclusion are of the same material.

In this section the results of the Mori - Tanaka method are calculated for the general composite materials of which inclusion is different material from matrix and the interface is slipping.

If an external load σ^0_{ij} is applied at infinity, the average stress field in the region of a single inclusion (in Ω) is given as

$$\sigma^0_{ij} + \langle\sigma_{ij}\rangle = C^f_{ijkl}(\varepsilon^0_{kl} + \langle\varepsilon_{kl}\rangle) = C^m_{ijkl}(\varepsilon^0_{kl} + \langle\varepsilon_{kl}\rangle - \langle\varepsilon_{kl}^*\rangle) \tag{2.18}$$

where ε_{kl}^* is the eigenstrain in the inclusion.

The effect of sliding interface is involved by considering the energy equivalency in the region of inclusion

$$\frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}dV + \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon_{ij}^*\,dV = \frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^0_{ij}dV + \frac{1}{2}\int_\Omega (\sigma^0_{ij}\varepsilon_{ij} - \sigma_{ij}\varepsilon^0_{ij})\,dV + \frac{1}{2}\int_{|\Omega|} \sigma_{ij}n_j[u_i]\,dS \tag{2.19}$$

which can be rewritten as

$$\frac{\sigma^0_{ij}}{\Omega}\int_\Omega \varepsilon_{ij}^*\,dV = \sigma^0_{ij}\langle\varepsilon_{ij}^*\rangle_\Omega = \frac{1}{\Omega}\int_\Omega (\sigma^0_{ij}\varepsilon_{ij} - \sigma_{ij}\varepsilon^0_{ij})\,dS + \frac{1}{\Omega}\int_{|\Omega|} \sigma_{ij}n_j[u_i]\,dS \tag{2.20}$$

where $\langle\varepsilon_{ij}^*\rangle_\Omega$ is the volume average of the eigenstrain in the inclusion.

The effective shear modulus is calculated when an external shear stress σ^0_{xy}

is applied to the medium. By the definition,

$$\frac{\sigma_{xy}^0}{2\mu} = \frac{\sigma_{xy}^0}{2\mu^m} + c\langle\varepsilon_{xy}^*\rangle_\Omega \tag{2.21}$$

where $\langle\varepsilon_{xy}^*\rangle_\Omega$ is the volume average of eigenstrain in the inclusion

$$\langle\varepsilon_{xy}^*\rangle_\Omega = \frac{\beta}{1+c\alpha}\sigma_{xy}^0 \tag{2.22}$$

In our analysis, α and β are defined as below,

$$\alpha = -\frac{A}{2} \tag{2.23}$$

$$\beta = \frac{A}{2}\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \tag{2.24}$$

where

$$A = \frac{\left(\frac{2}{\mu^f} - \frac{1}{\mu^m} + \frac{1}{K^f}\right) + \tilde{k}\left(\frac{1}{\mu^f} - \frac{1}{\mu^m}\right)\left(\frac{1}{\mu^m} + \frac{1}{\mu^f} + \frac{2}{K^f}\right)}{\left(\frac{2}{\mu^m} + \frac{2}{\mu^f} + \frac{3}{K^m} + \frac{1}{K^f}\right) + \tilde{k}\left(\frac{1}{\mu^m} + \frac{1}{\mu^f} + \frac{2}{K^m}\right)\left(\frac{1}{\mu^m} + \frac{1}{\mu^f} + \frac{2}{K^f}\right)} \tag{2.25}$$

After manipulating the above relations, the effective shear modulus of the Mori - Tanaka method is given as

$$\frac{1}{\mu} = \frac{1}{\mu^m} + \frac{2cA}{(1-cA)}\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \tag{2.26}$$

The effective area bulk modulus is computed directly from the formulation which is provided by Benveniste (1987) because no interfacial effect occurs when the applied load is $\sigma_{xx}^0 = \sigma_{yy}^0 = \sigma^0$.

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{K^f} - \frac{1}{K^m}\right)\left[\frac{\dfrac{1}{K^m} + \dfrac{1}{\mu^m}}{\left(\dfrac{1}{K^f} + \dfrac{1}{\mu^m}\right) + c\left(\dfrac{1}{K^m} - \dfrac{1}{K^f}\right)}\right] \tag{2.27}$$

In contrast to the first two methods, the advantage of the Mori - Tanaka method is that the effective moduli of real composite materials can be obtained by substituting the elastic constants of matrix and inclusion into the final closed forms. The Mori - Tanaka method, however, has the wider range of error at high volume fraction than the generalized self consistent method (Christensen, 1992). The Mori - Tanaka method shows several unique results at high volume fraction which are much different from the results of other methods, which will be discussed in the next chapter.

2.5 GENERALIZED SELF CONSISTENT METHOD

The generalized self consistent method is an extension of self consitent method, which is due to Christensen and Lo (1979). It modifies the self consistent method by putting matrix phase between inclusion and effective medium (see Fig.2.4).

The equivalency of elastic strain energy of the two media in Fig.2.4 leads to (Christensen and Lo, 1979)

$$W^0 = W^0 + \int (\sigma^0_{ij} n_j u_i - \sigma_{ij} n_j u^0_i)\, dS \qquad (2.28)$$

In addition to the boundary conditions of eq.(2.1) - (2.3), the new boundary conditions that $\sigma_{rr}, \sigma_{r\theta}, u_r, u_\theta$ are continuous at r=b are needed. If an external shear stress is applied, the effective shear modulus is calculated by the boundary conditions and the equivalency of energy. There are eight constants which are determined by the above boundary conditions. By the way, one of the constants vanishes by the above equation of energy equivalency. This leads to a quadratic equation for μ

$$C_1 \left(\frac{\mu}{\mu^m}\right)^2 + C_2 \left(\frac{\mu}{\mu^m}\right) + C_3 = 0 \qquad (2.29)$$

where the coefficients C_1, C_2, C_3 are given in Appendix. These coefficients contain the sliding parameter k which is defined in section 2.1. It can be easily shown that, if the parameter is taken to be infinite (perfect bonding), the solution of eq.(2.30) coincides with that of Christensen and Lo (1979) exactly. These coefficients do not contatin the unknown effective moduli, which means that the solution can be obtained only by putting the elastic constants of matrix and inclusion into the equation.

The effective area bulk modulus is computed when the applied loads are given by $\sigma^0_{xx} = \sigma^0_{yy} = \sigma^0$. This result is given as below

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{K^f} - \frac{1}{K^m}\right)\left[\frac{\dfrac{1}{K^m} + \dfrac{1}{\mu^m}}{\dfrac{1}{K^f} + \dfrac{1}{\mu^m} + c\left(\dfrac{1}{K^m} - \dfrac{1}{K^f}\right)}\right] \tag{2.30}$$

which is exactly same form as the result of the Mori - Tanaka method as expected.

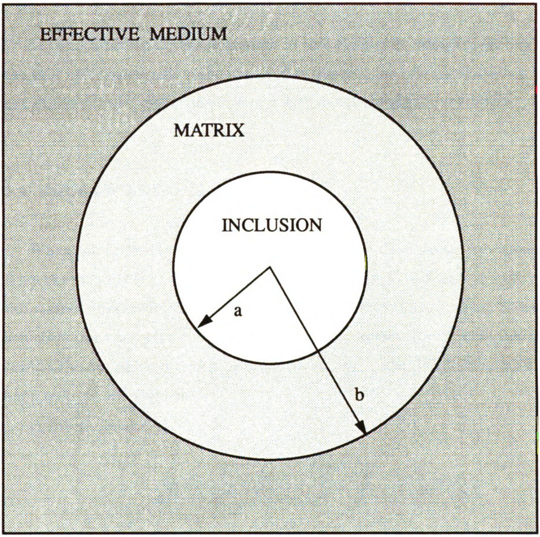

FIGURE 2.3 Geometrical Model of Generalized Self Consistent Method

CHAPTER 3

SPECIAL LIMIT CASES

Several limiting cases are studied in this chapter to compare the results of the four effective medium theories and to observe the effects of sliding interfaces on the elastic properties of two dimensional composite materials.

3.1 HOLE INCLUSIONS

When the inclusions are holes, the effective moduli are easily calculated from the results of the chapter 2 by taking the limit $K^f \to 0$ and $\mu^f \to 0$. It does not matter whether the interface is perfect bonding or pure sliding, because the interface does not give any actual contribution to the effective moduli in this limit. The normalized area bulk modulus and the normalized shear modulus of each method are given as

(1) Dilute results

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \qquad \frac{1}{\mu} = \frac{1}{\mu^m} + c\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \tag{3.1}$$

(2) Self consistent method

$$\frac{1}{K} = \frac{1}{K^m} + c\left(\frac{1}{\mu} + \frac{1}{K}\right) \qquad \frac{1}{\mu} = \frac{1}{\mu^m} + c\left(\frac{1}{\mu} + \frac{1}{K}\right) \tag{3.2}$$

(3) Differential scheme

$$\frac{1}{K} = \frac{1}{K^m}\left(\frac{1}{1-c}\right)^{-(1+\frac{K}{\mu})} \qquad \frac{1}{\mu} = \frac{1}{\mu^m}\left(\frac{1}{1-c}\right)^{-2(1+\frac{\mu}{K})} \tag{3.3}$$

(4) Mori Tanaka theorem

$$\frac{1}{K} = \frac{1}{K^m} + \frac{c}{1-c}\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \qquad \frac{1}{\mu} = \frac{1}{\mu^m} + \frac{2c}{1-c}\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \tag{3.4}$$

For the self consistent method and the differential scheme, the dilute result of the Poisson ratio

$$\nu = \nu^m - c\,(3\nu^m - 1) \tag{3.5}$$

and the general relation of two dimensional elasticity

$$\nu = \frac{K-\mu}{K+\mu} \tag{3.6}$$

should be used. The integrations of differential scheme are easily carried out with the limit conditions of section 2.3.

(5) Generalized self consistent method

$$\frac{1}{K} = \frac{1}{K^m} + \frac{c}{1-c}\left(\frac{1}{\mu^m} + \frac{1}{K^m}\right) \tag{3.7}$$

$$C_1\left(\frac{\mu}{\mu^m}\right)^2 + C_2\left(\frac{\mu}{\mu^m}\right) + C_3 = 0 \tag{3.8}$$

where $\eta^m = 1 + 2\frac{\mu^m}{K^m}$.

$$C_1 = -3c(1-c)^2 - (\eta^m + c^3)(1 + c\eta^m) \tag{3.9}$$

$$C_2 = 3c(1-c)^2 + \frac{1}{2}(1-c)(\eta^m - 1 + 2c^3) - \frac{1}{2}c(\eta^m + 1)(1 - c^3) \tag{3.10}$$

$$C_3 = -3c(1-c)^2 + (1-c)(1-c^3) \tag{3.11}$$

The closed forms of K/K^m exactly same as each other as refered in the chapter 2. Fig.(3.1) - (3.16) illustrate the normalized area bulk modulus the normalized shear modulus, the Poisson ratio and the normalized Young's modulus of effective medium which are calculated by the four effective medium theories. In all figures in this thesis, the effective moduli are denoted by superscript c.

The most interesting result arises from the Poisson ratio. Every theory shows the tendency that, as the volume fraction increases, the values of the Poisson ratio go to a fixed value which is independent of the elastic constants of matrix. However the critical volume fraction c_0 where the fixed value exists is 1/3 for the self consistent method while $c_0 = 1$ for the other methods. The fixed value itself also varies depending on the effective medium theory used. The generalized self consistent method predicts the value is 1 and the other theories 1/3.

The existence of the fixed value of the Poisson ratio has been also observed by other researchers. Day et al.(1991) have obtained the elastic moduli of a matrix containing circular holes by computer simulation techniques. Their result of the Poisson ratio for the regular honeycomb network is quite similar to that of the generalized self consistent method in the sense that the fixed value is 1. Zimmerman (1991) has also calculated the effective moduli of composites by the three popular effective medium theories. In his paper, the fixed value of the Poisson ratio for the case of hole inclusions is observed although his work is for the three dimensional composite materials.

Another important result is about the Young's modulus. All the four theories predict the coincident result that the effective Young's modulus is independent of the elastic constant of matrix, at the whole range of volume fraction. This result is closely related to the fact that the stress field in the matrix having hole

inclusions is independent of the elastic constant of matrix (Michell, 1899; Thorpe and Jasiuk, 1992). This property of the Young's modulus is discussed in the next chapter which is about the CLM theorem.

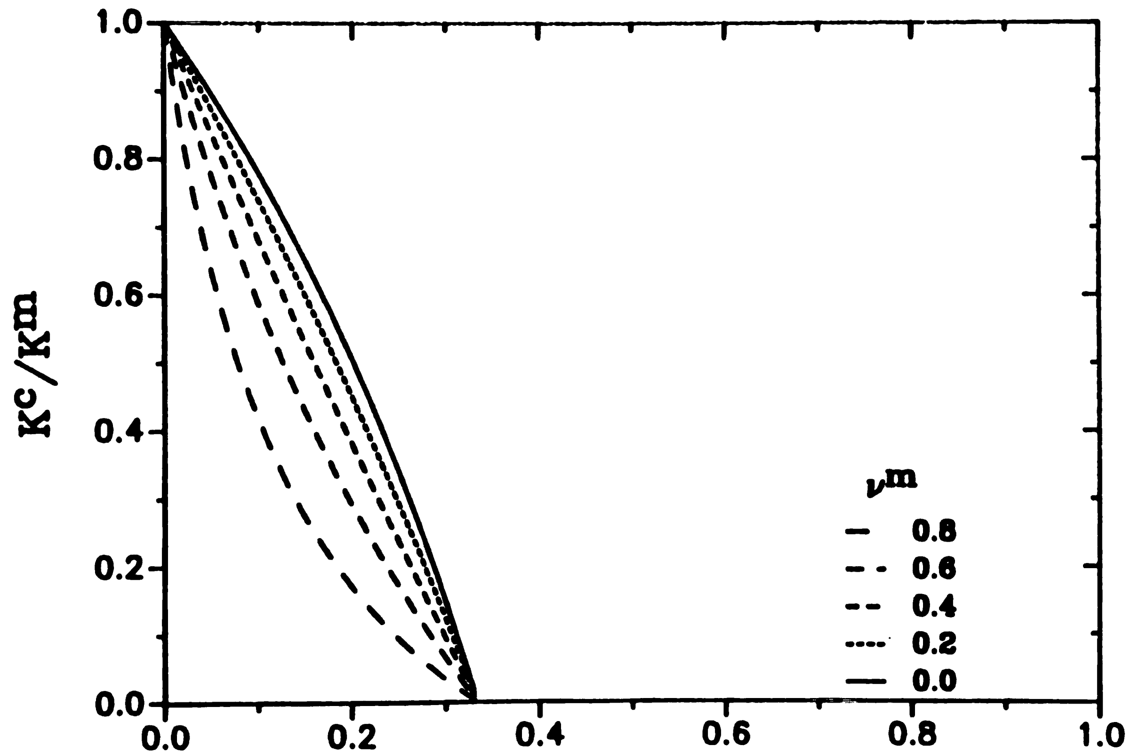

FIGURE 3.1 K^c/K^m vs. c for hole inclusions by self-consistent method

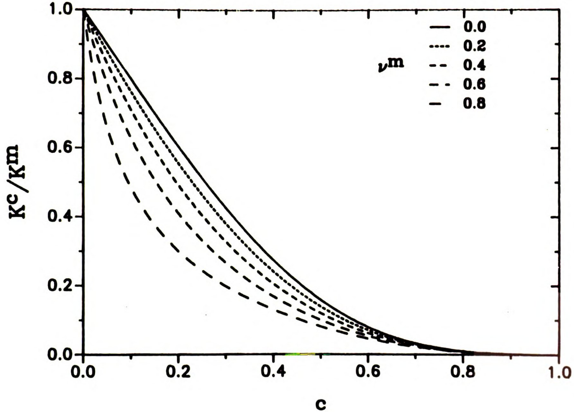

FIGURE 3.2 K^c/K^m vs. c for hole inclusions by differential scheme

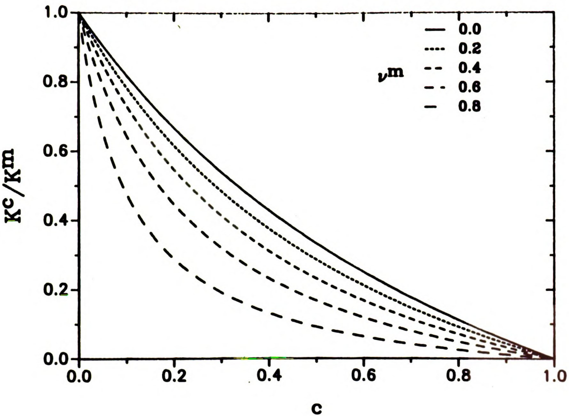

FIGURE 3.3 K^c/K^m vs. c for hole inclusions by Mori-Tanaka method

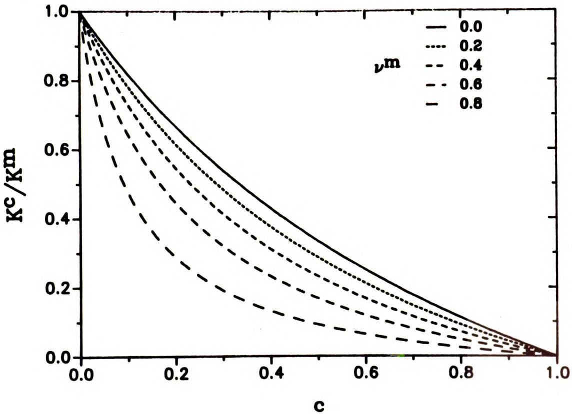

FIGURE 3.4 K^c/K^m vs. c for hole inclusions by generalized self-consistent method

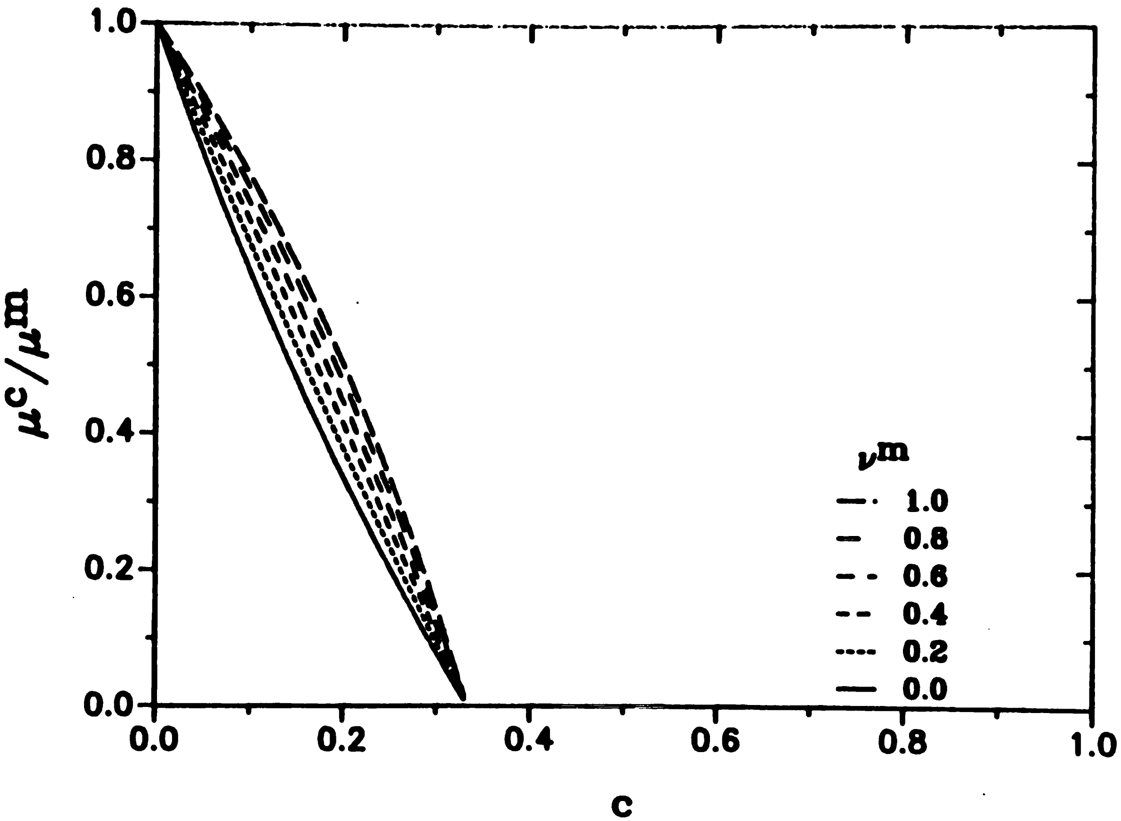

FIGURE 3.5 μ^c/μ^m vs. c for hole inclusions by self-consistent method

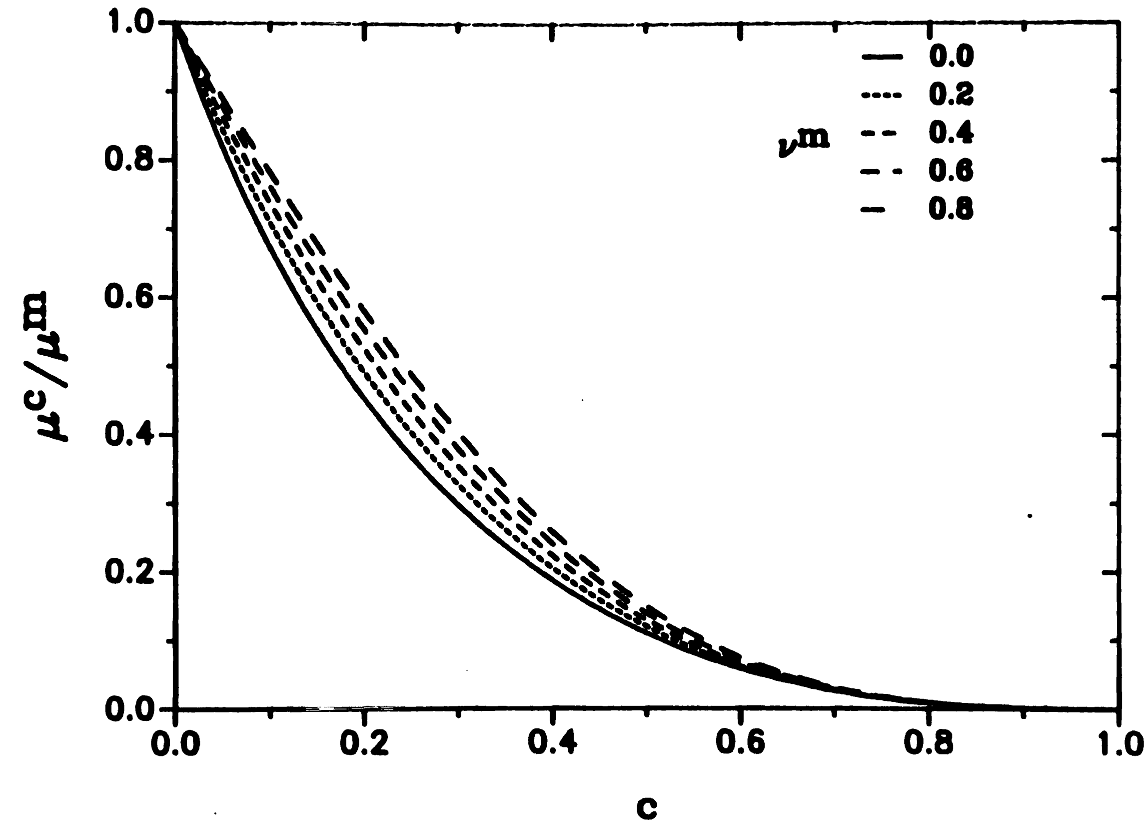

FIGURE 3.6 μ^c/μ^m vs. c for hole inclusions by differential scheme

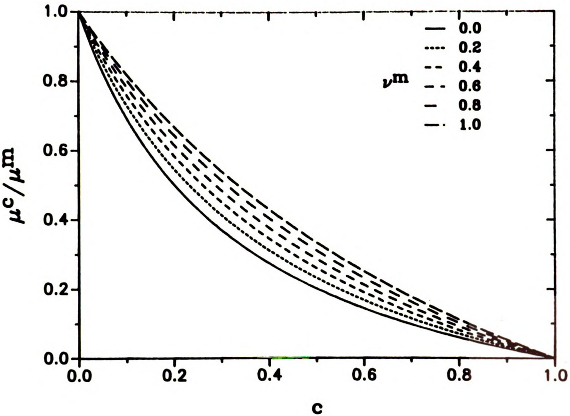

FIGURE 3.7 μ^c/μ^m vs. c for hole inclusions by Mori-Tanaka method

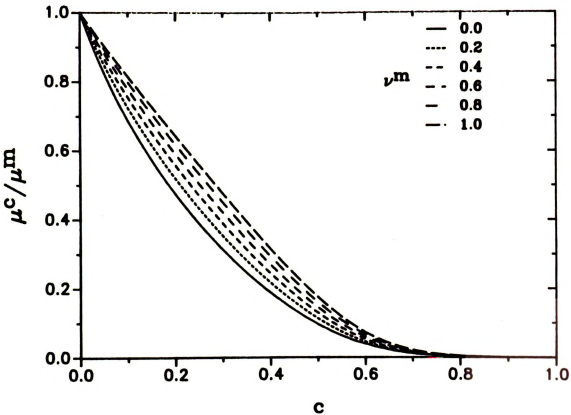

FIGURE 3.8 μ^c/μ^m vs. c for hole inclusions by generalized self-consistent method

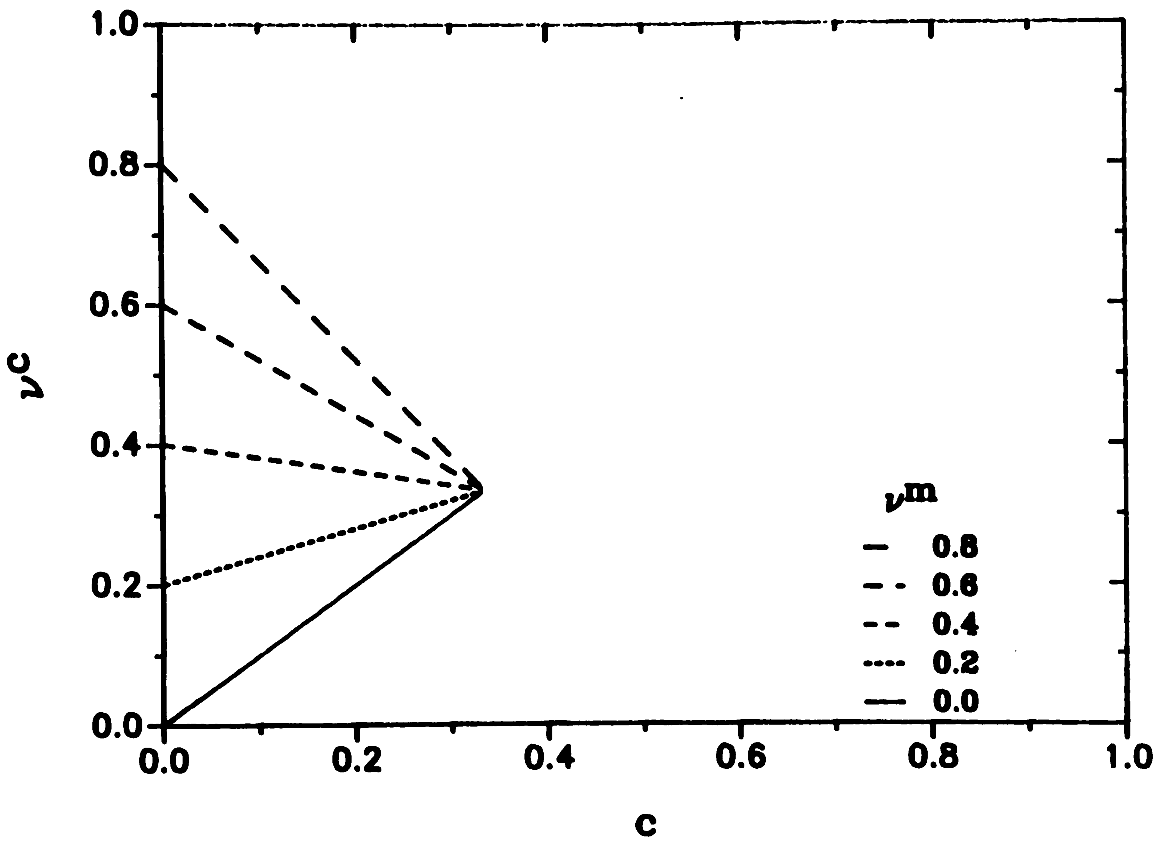

FIGURE 3.9 ν^c vs. c for hole inclusions by self-consistent method

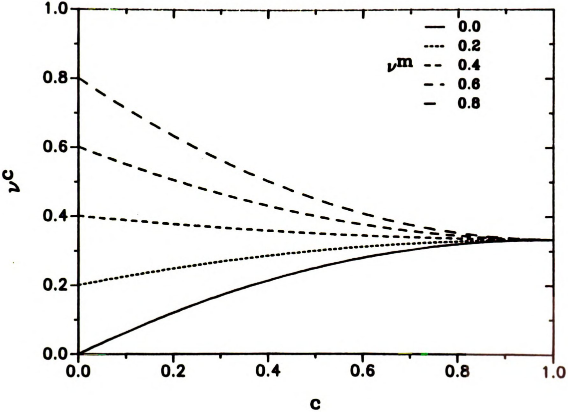

FIGURE 3.10 ν^c vs. c for hole inclusions by differential scheme

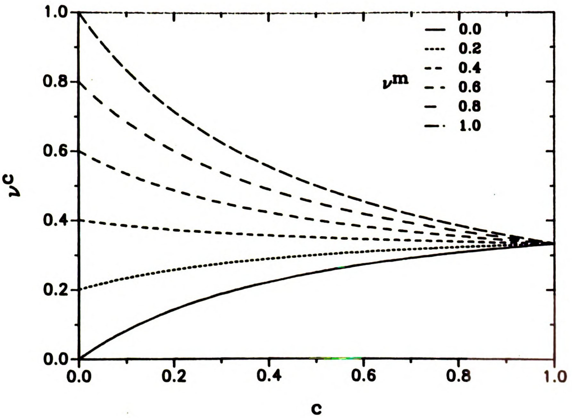

FIGURE 3.11 ν^c vs. c for hole inclusions by Mori-Tanaka method

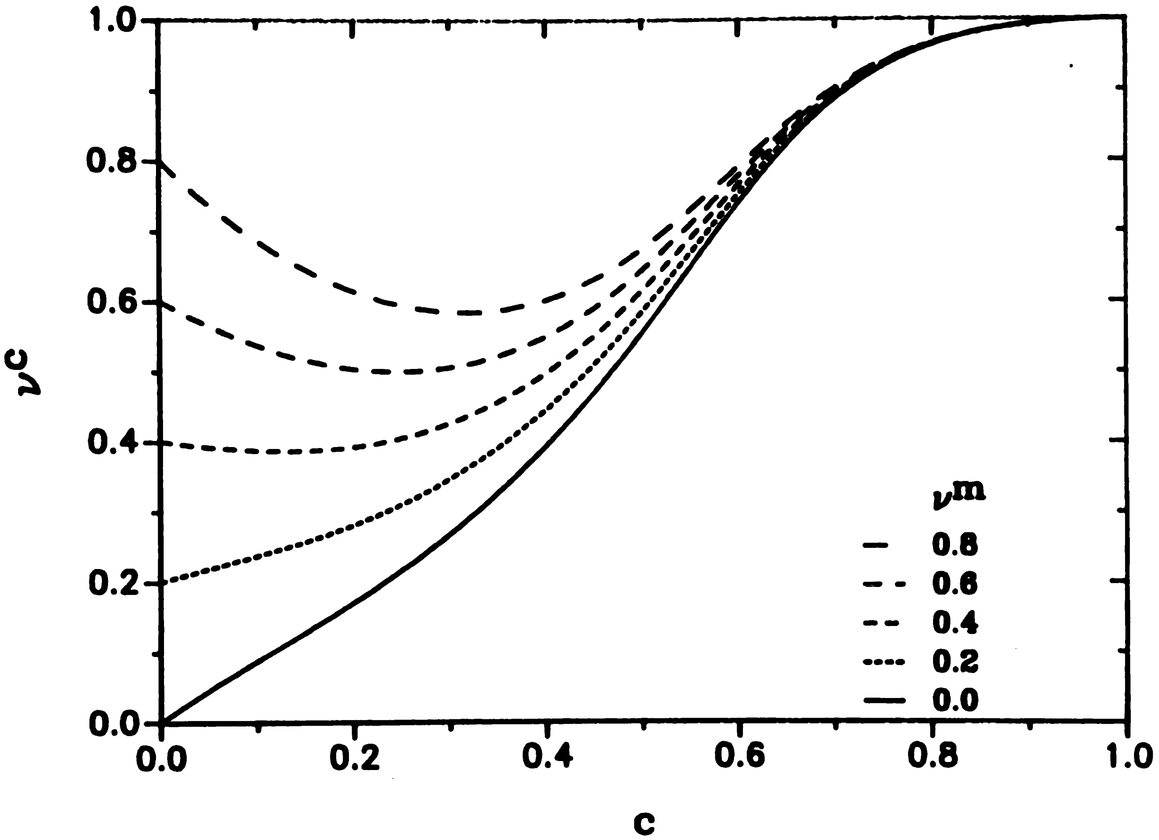

FIGURE 3.12 ν^c vs. c for hole inclusions by generalized self-consistent method

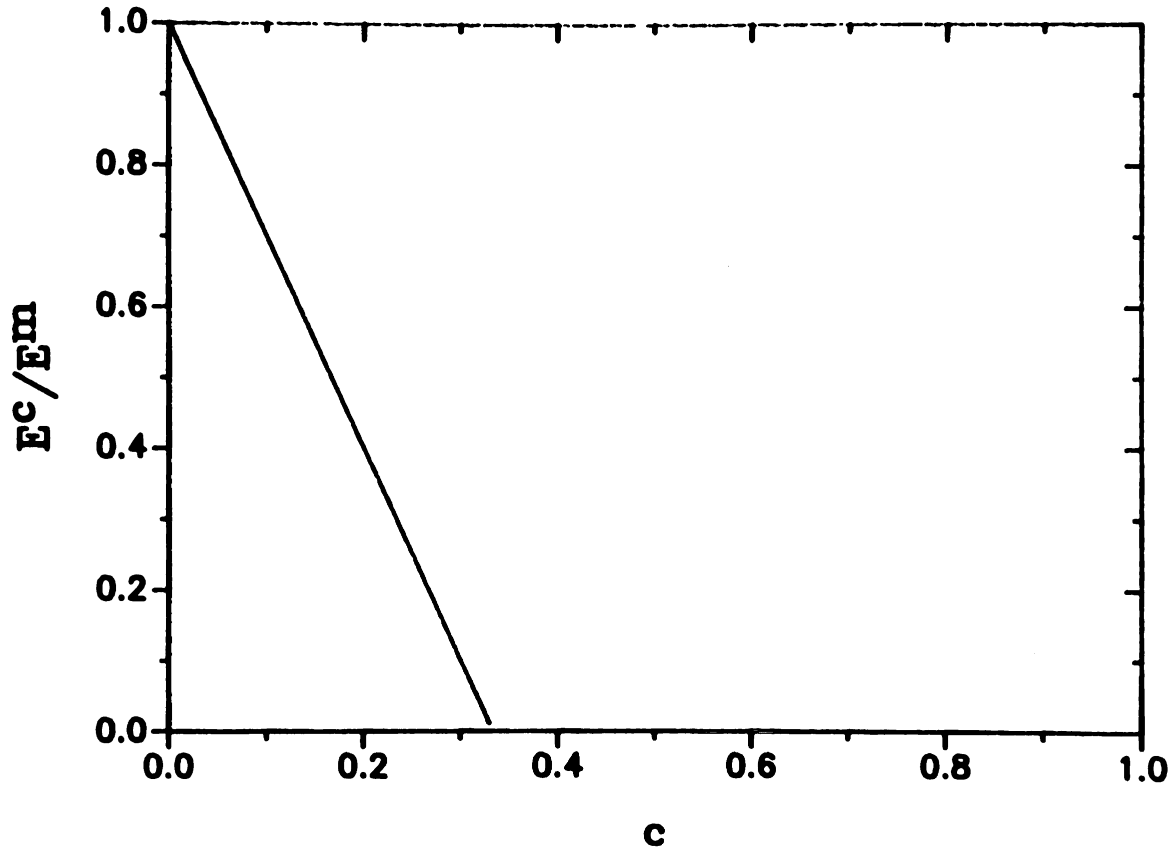

FIGURE 3.13 E^c/E^m vs. c for hole inclusions by self-consistent method

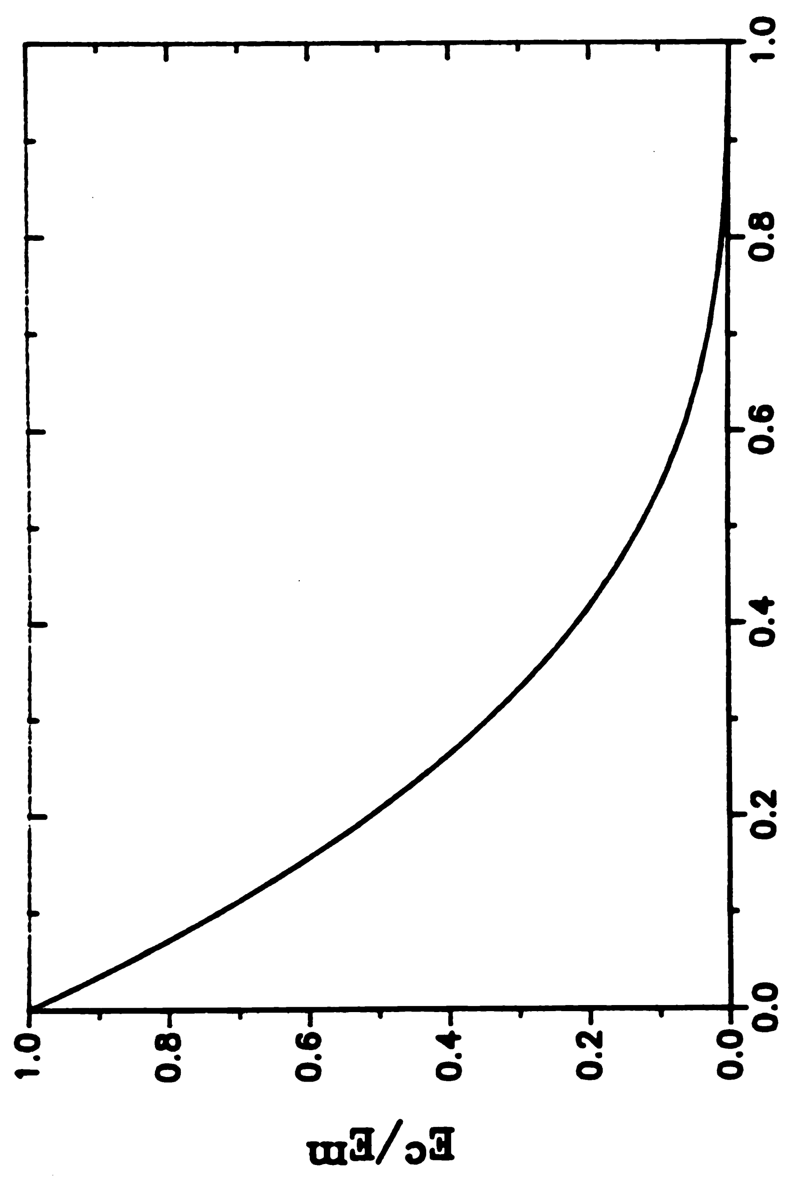

FIGURE 3.14 E^c/E^m vs. c for hole inclusions by differential scheme

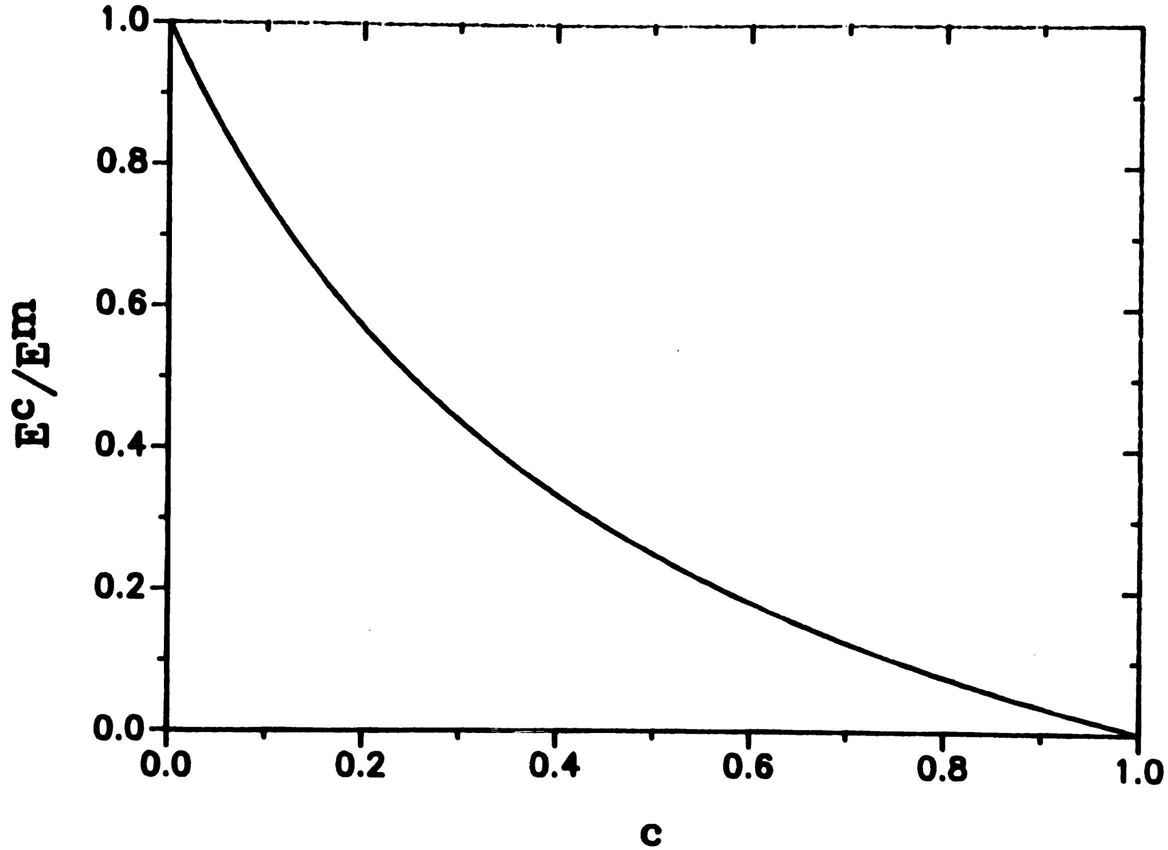

FIGURE 3.15 E^c/E^m vs. c for hole inclusions by Mori-Tanaka method

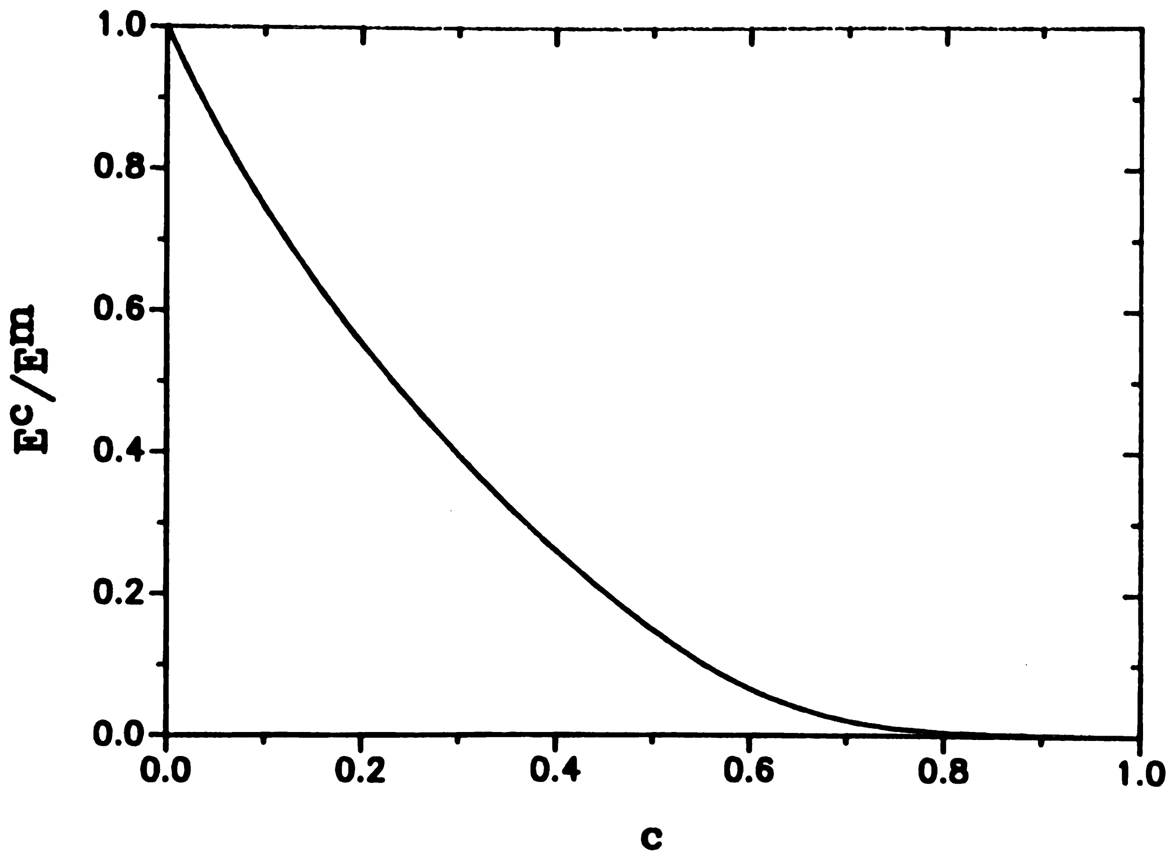

FIGURE 3.16 E^c/E^m vs. c for hole inclusions by generalized self-consistent method

3.2 RIGID INCLUSIONS

In this section the inclusion is regarded as the rigid material. The results are computed by taking $K^f \to \infty$ and $\mu^f \to \infty$ from the results of the elastic inclusions. Jasiuk et al.(1992) have already discussed the two theories, the self consistent method and the differential scheme, for this limit. However their results are referred here to be compared with those of the other theories. The final forms of the remaining theories are as below

(1) Mori - Tanaka method

$$\frac{1}{\bar{K}} = \frac{1}{K^m} - \frac{c}{K^m}\left(\frac{\frac{1}{\mu^m}+\frac{1}{K^m}}{\frac{1}{\mu^m}+c\frac{1}{K^m}}\right) \tag{3.12}$$

$$\frac{1}{\bar{\mu}} = \frac{1}{\mu^m} - 2A'\frac{c}{\mu^m}\left(\frac{\frac{1}{\mu^m}+\frac{1}{K^m}}{1+A'\frac{c}{\mu^m}}\right) \tag{3.13}$$

where

$$A' = \frac{1+\tilde{k}\frac{1}{\mu^m}}{\frac{2}{\mu^m}+\frac{3}{K^m}+\tilde{k}\frac{1}{\mu^m}\left(\frac{1}{\mu^m}+\frac{2}{K^m}\right)} \tag{3.14}$$

(2) The generalized self consistent method

$$\frac{1}{\bar{K}} = \frac{1}{K^m} - \frac{c}{K^m}\left(\frac{\frac{1}{\mu^m}+\frac{1}{K^m}}{\frac{1}{\mu^m}+c\frac{1}{K^m}}\right) \tag{3.15}$$

$$C_1(\frac{\mu^m}{\mu}) + C_2(\frac{\mu^m}{\mu}) + C_3 = 0 \tag{3.16}$$

where the coefficient C_1,C_2 and C_3 can be easily obtained from Appendix. These yield the two interesing limits. One is perfect bonding ($\tilde{k} \to \infty$) and the other is pure sliding ($\tilde{k} \to 0$).

3.2.1 PERFECT BONDING

Fig. (3.17) - (3.22) are for the effective elastic constants of the perfect bonding case which are obtained by using the Mori - Tanaka method and the generalized self consistent method. In perfect bonding case, the distinctly different results between the four effective medium theories come from the Poisson ratio. The self consistent method and the differential scheme allow the existence of the fixed value of the Poisson ratio. Both give the same fixed value $\nu^0 = 1/3$. However the Mori-Tanaka method and the generalized self consistent method do not show the fixed value of Poisson ratio as shown in Fig. (3.21) and (3.22).

3.2.2 PURE SLIDING

If there exist only sliding effect at the interface between matrix and inclusion, the values of the effective shear modulus and the effective Young's modulus are lower than those of the perfect bonding case. All theories agree to this reduction of μ and E. In contrast to them, the theories split when the area bulk modulus is concerned. The self consistent method and the differential scheme allow this reduction of the value of K while the Mori - Tanaka method and the generalized self consistent method give the same value of K as that of perfect bonding case.

A remarkable difference also happen when the effective Poisson ratio ν is taken into account. All theories except differential scheme predict the existence of the fixed value of the Poisson ratio when the volume fraction goes to one. By the way, the fixed value of the generalized self consistent method is $\nu^0 = 1/2$ while the fixed value of the other twomethods is $\nu^0 = 1$ (Fig. 3.23 - 3.26).

Another difference is founded in the effective shear modulus. Only the Mori - Tanaka method shows that the value of μ is not an infinite value but a finite value ($\mu = 3\mu^m$) which is independent of the elastic constant of the matrix at the point where the volume fraction is c = 1. However the other methods give an infinite value of μ at c = 1, which implies that the inclusion is rigid material.

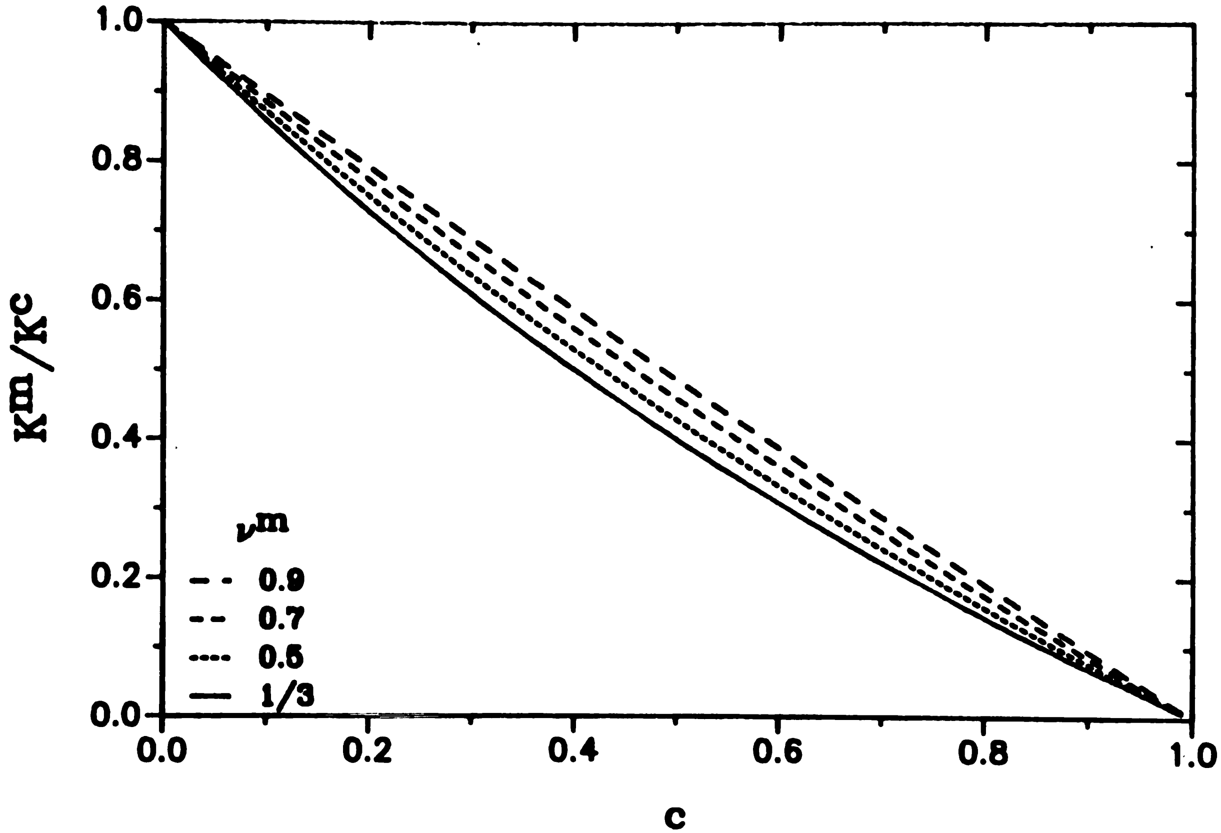

FIGURE 3.17 K^m/K^c vs. c for rigid inclusions by Mori-Tanaka method

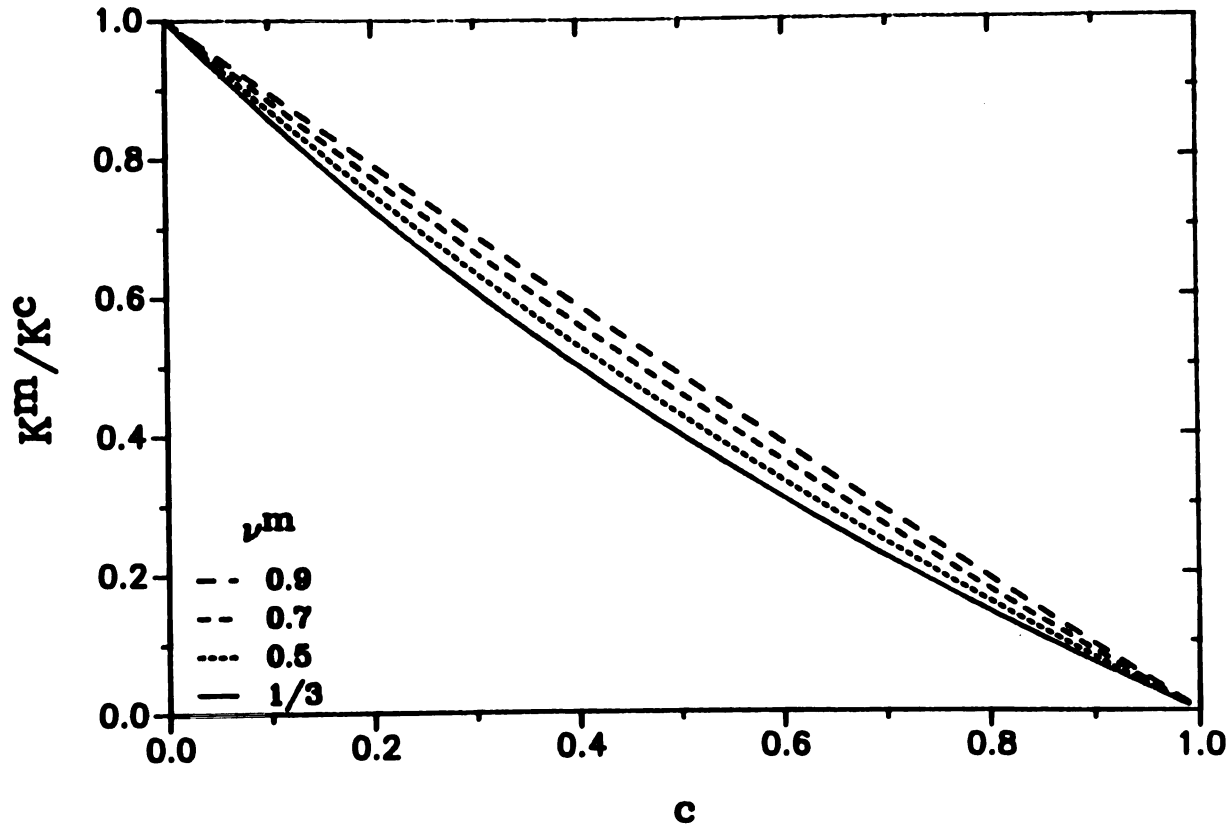

FIGURE 3.18 K^m/K^c vs. c for rigid inclusions by generalized self-consistent method

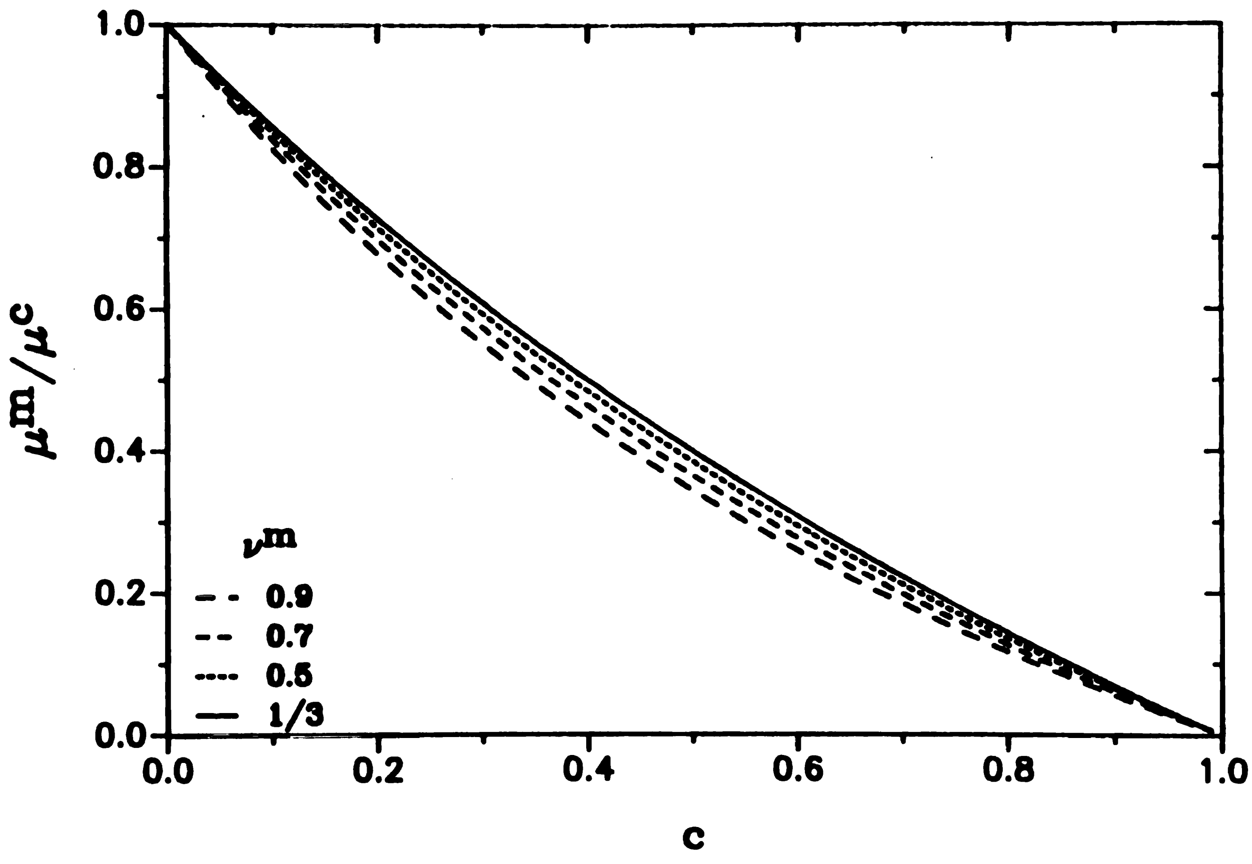

FIGURE 3.19 μ^m/μ^c vs. c for rigid inclusions and perfect bonding by Mori-Tanaka method

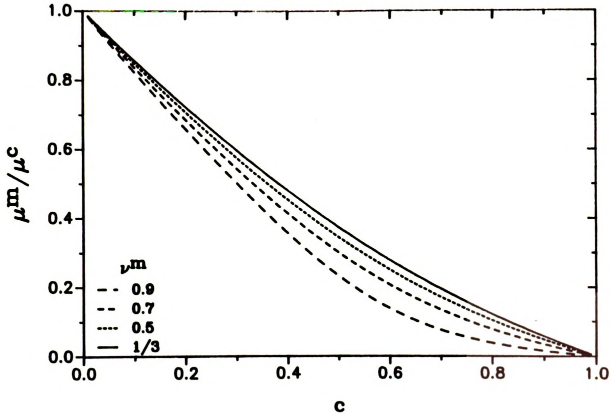

FIGURE 3.20 μ^m/μ^c vs. c for rigid inclusions and perfect bonding by generalized self-consistent method

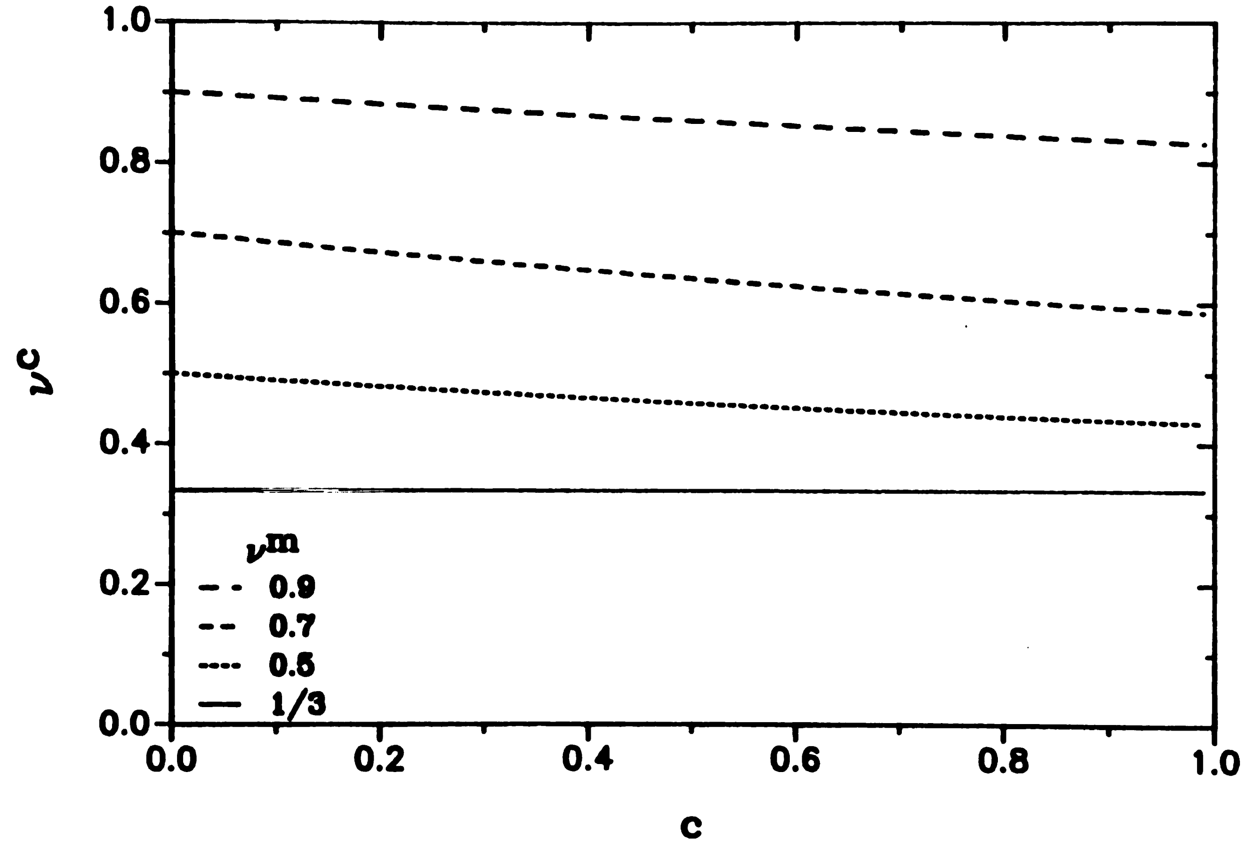

FIGURE 3.21 v^c vs. c for rigid inclusions and perfect bonding by Mori -Tanaka method

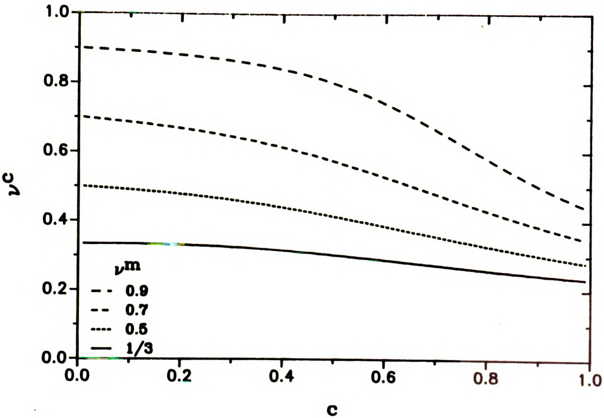

FIGURE 3.22 ν^c vs. c for rigid inclusions and perfect bonding by generalized self-consistent method

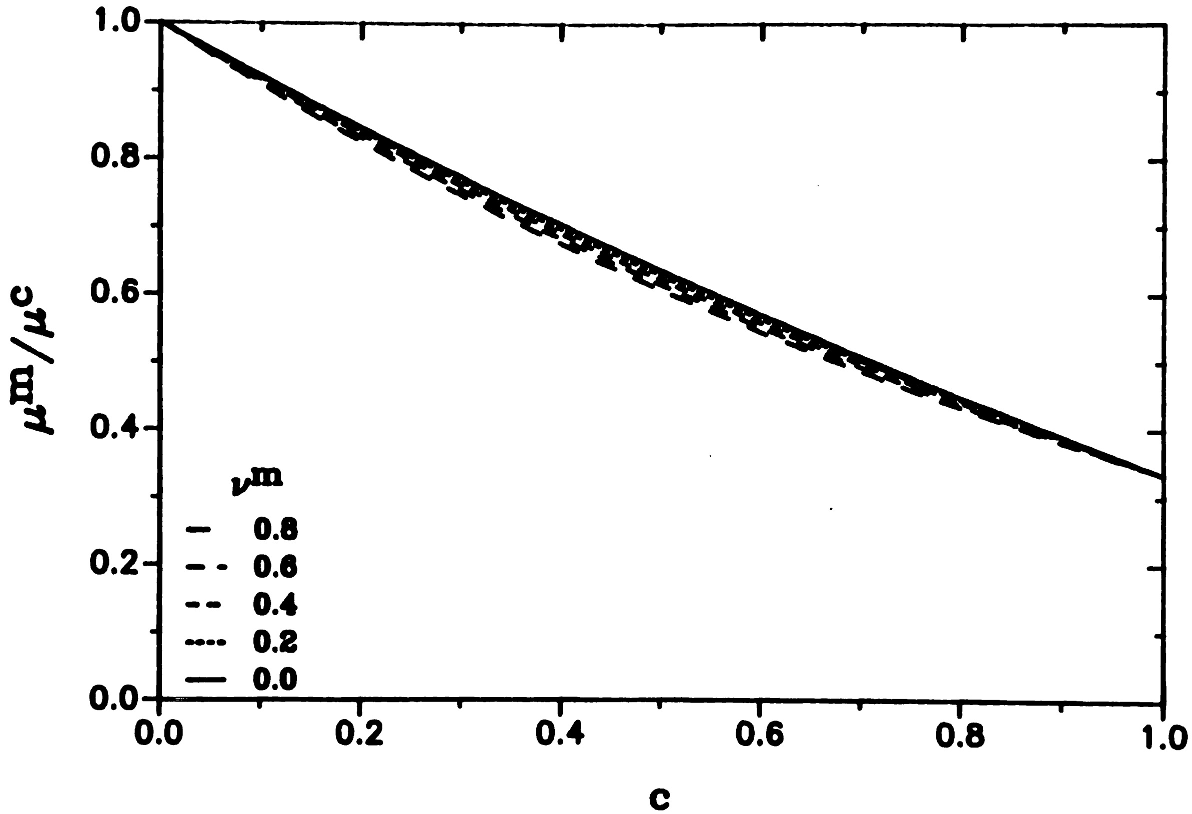

FIGURE 3.23 μ^m/μ^c vs. c for rigid inclusions and pure sliding by Mori -Tanaka method

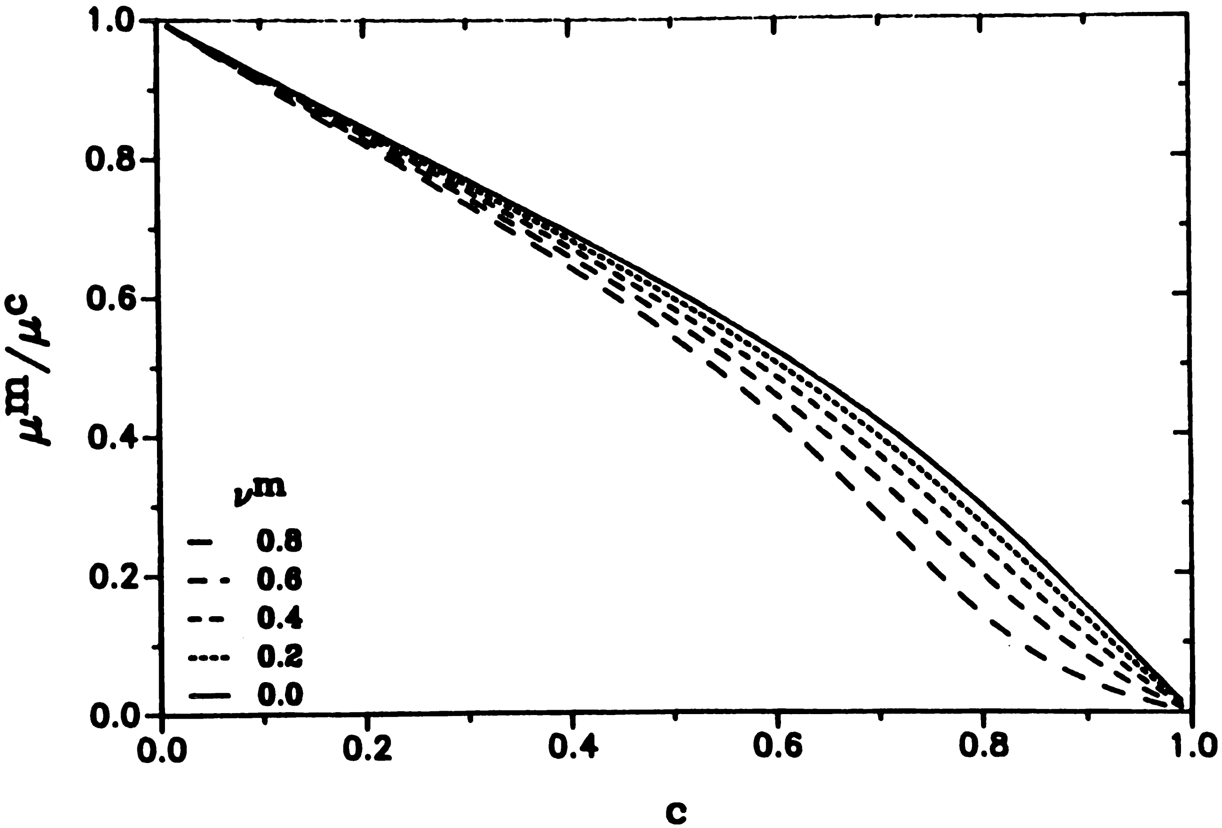

FIGURE 3.24 μ^m/μ^c vs. c for rigid inclusions and pure sliding by generalized self-consistent method

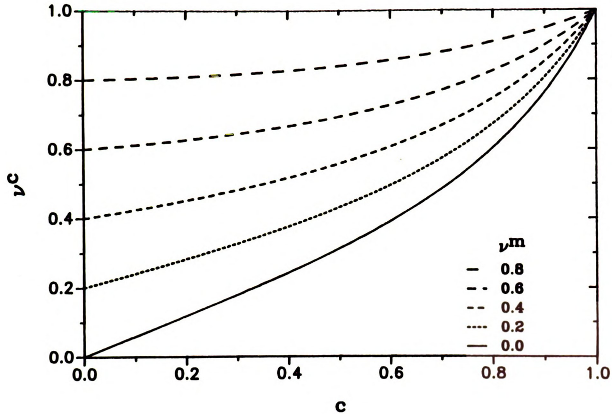

FIGURE 3.25 ν^c vs. c for rigid inclusions and pure sliding by Mori--Tanaka method

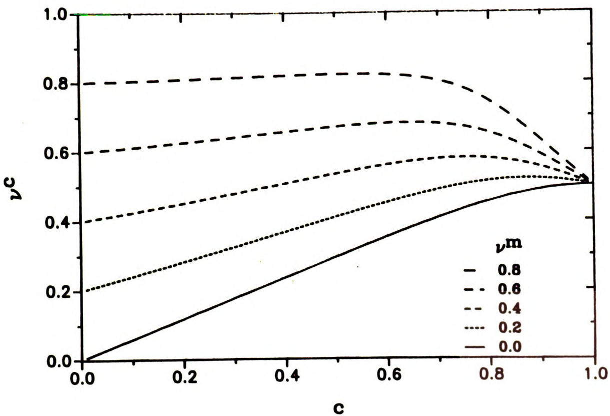

FIGURE 3.26 ν^c vs. c for rigid inclusions and pure sliding by generalized self-consistent method

3.3 EQUAL SHEAR MODULI

If the shear modulus of matrix is same as that of inclusion, the elastic constants of the composite materials are determined exactly (Hashin, 1965; Christensen, 1979). This can be applied to 2D or 3D composites but only to perfect bonding case. All effective medium theories used here give the exact values of the elastic constants. The area bulk modulus is given as

$$K = K^m + c\,(K^m + \mu^m)\left[\frac{K^f - K^m}{K^f + \mu^m + c\,(K^m - K^f)}\right] \tag{3.17}$$

It is well known that the upper and lower bounds of elastic constants coincide to each other for this limit case. It can be easily shown that the above bulk modulus is the same as the exact bound. The Poisson ratio and the Young's modulus which are obtained by the four theories also coincide to the law of mixtures as expected.

For the two dimensional composite materials, this results of exact elastic constants were also derived from the CLM theorem by Thorpe and Jasiuk (1992).

3.4 SAME MATERIALS WITH PURE SLIDING

It has been known that, for pure sliding case, the stress field of the composite materials is independent of the elastic constants of the materials if matrix and inclusions are made of the same material (Dundurs and Stippes, 1970). When the volume fraction of inclusion is dilute, the Young's modulus of this kind of composite is proved to be independent of the Poisson ratio (Thorpe and Jasiuk, 1991). Here it is founded that, at the whole range of volume fraction, the Young's modulus is independent of the Poisson ratio of the material as well as the dilute case, no matter which theories are used.

Fig.(3.35) - (3.38) are for the Young's modulus calculated by each theory and Fig.(3.27) - (3.34) for the Poisson ratio and the shear modulus. Only the diffential scheme shows that, as the volume fraction approach $c = 1$, the values of Poisson ratio and shear modulus go to a fixed value which is independent of the elastic constants of materials. More discussion with the CLM theorem is given in the next chapter.

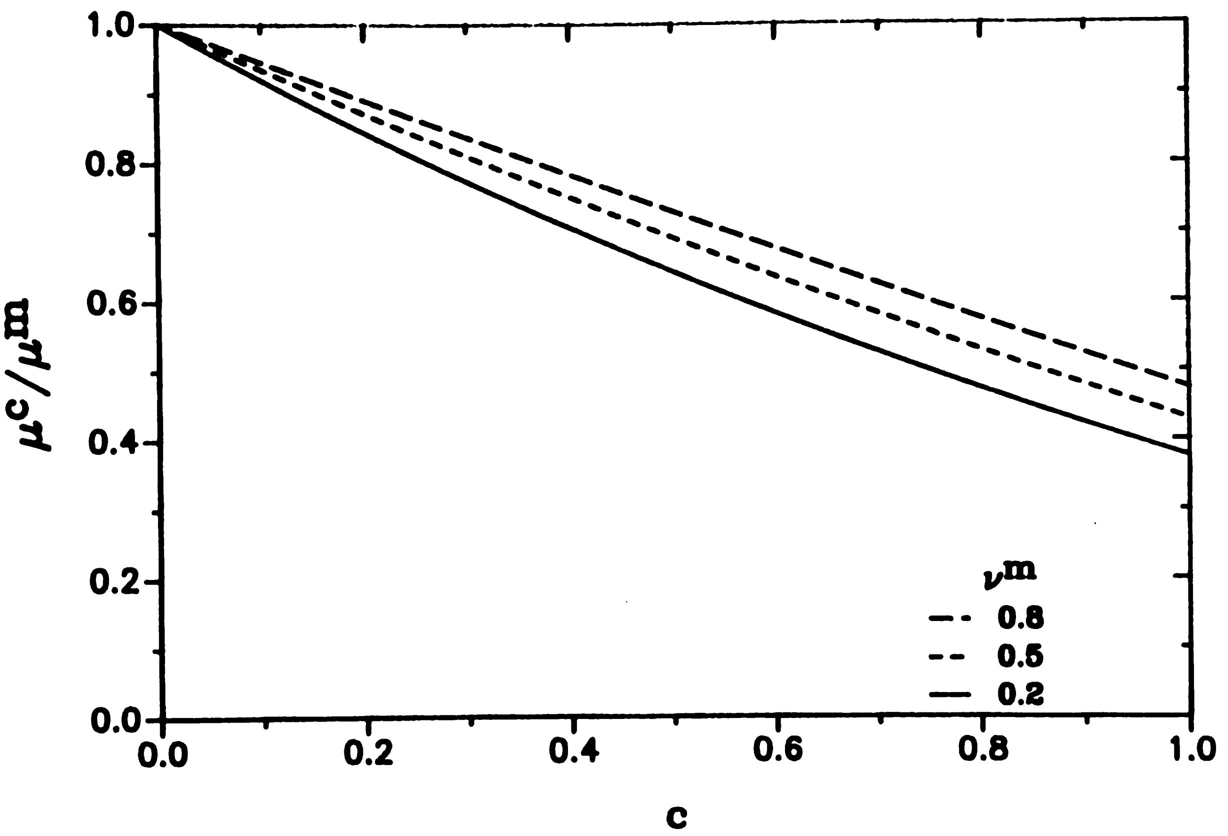

FIGURE 3.27 μ^c/μ^m vs. c for same materials with pure sliding by self-consistent method

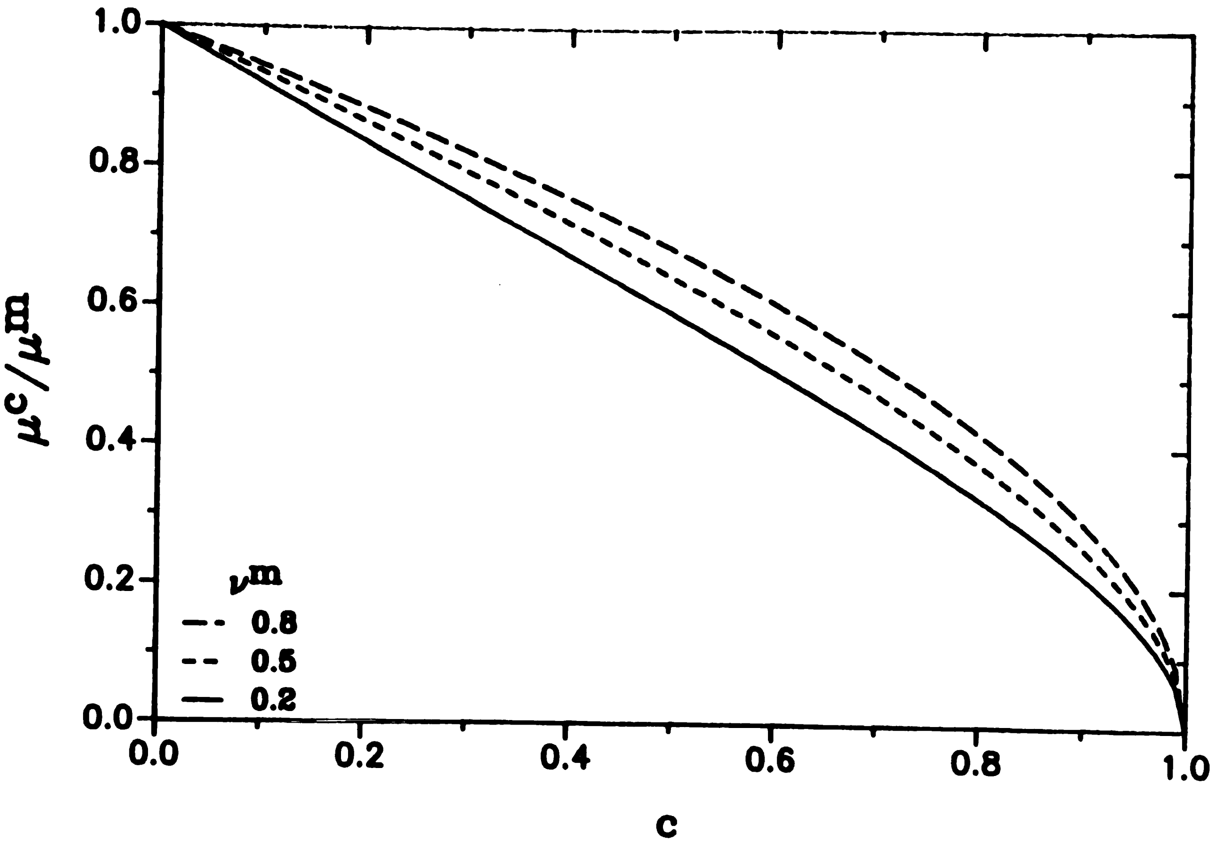

FIGURE 3.28 μ^c/μ^m vs. c for same materials with pure sliding by differential scheme

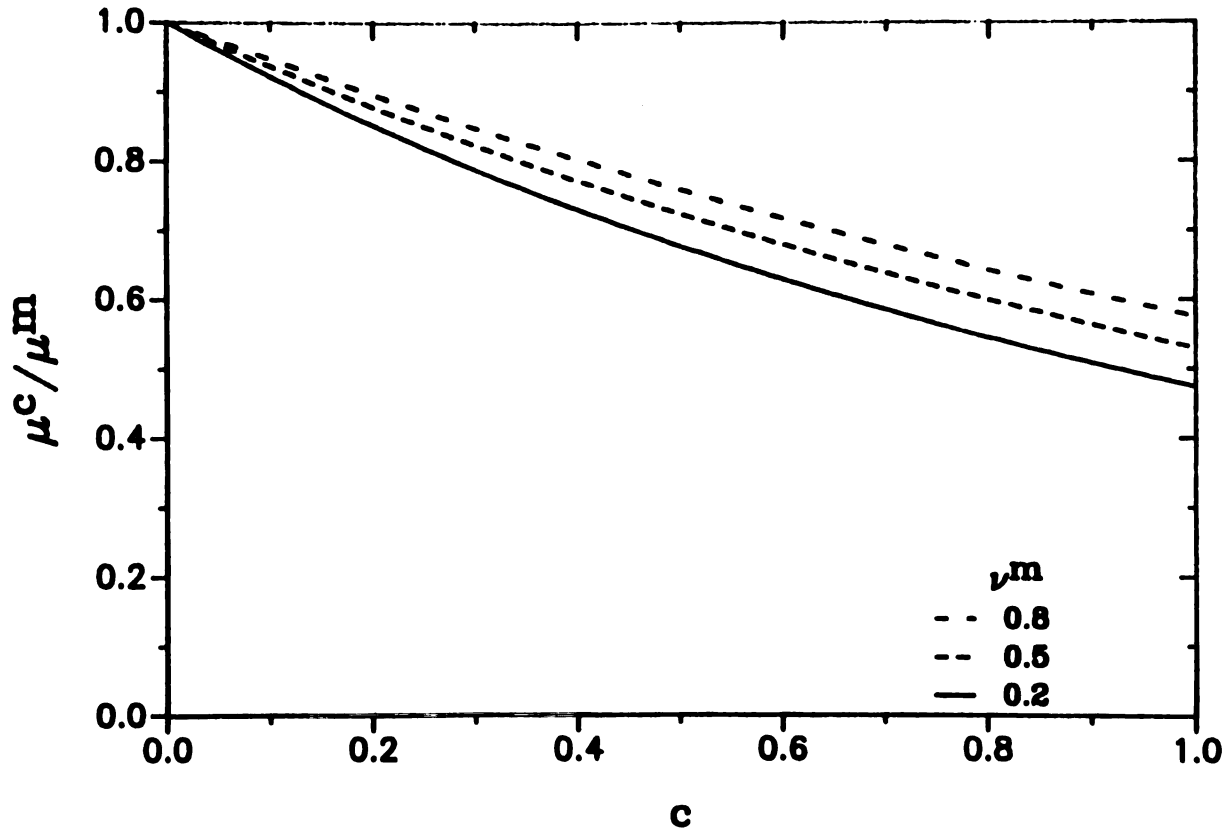

FIGURE 3.29 μ^c/μ^m vs. c for same materials with pure sliding by Mori-Tanaka method

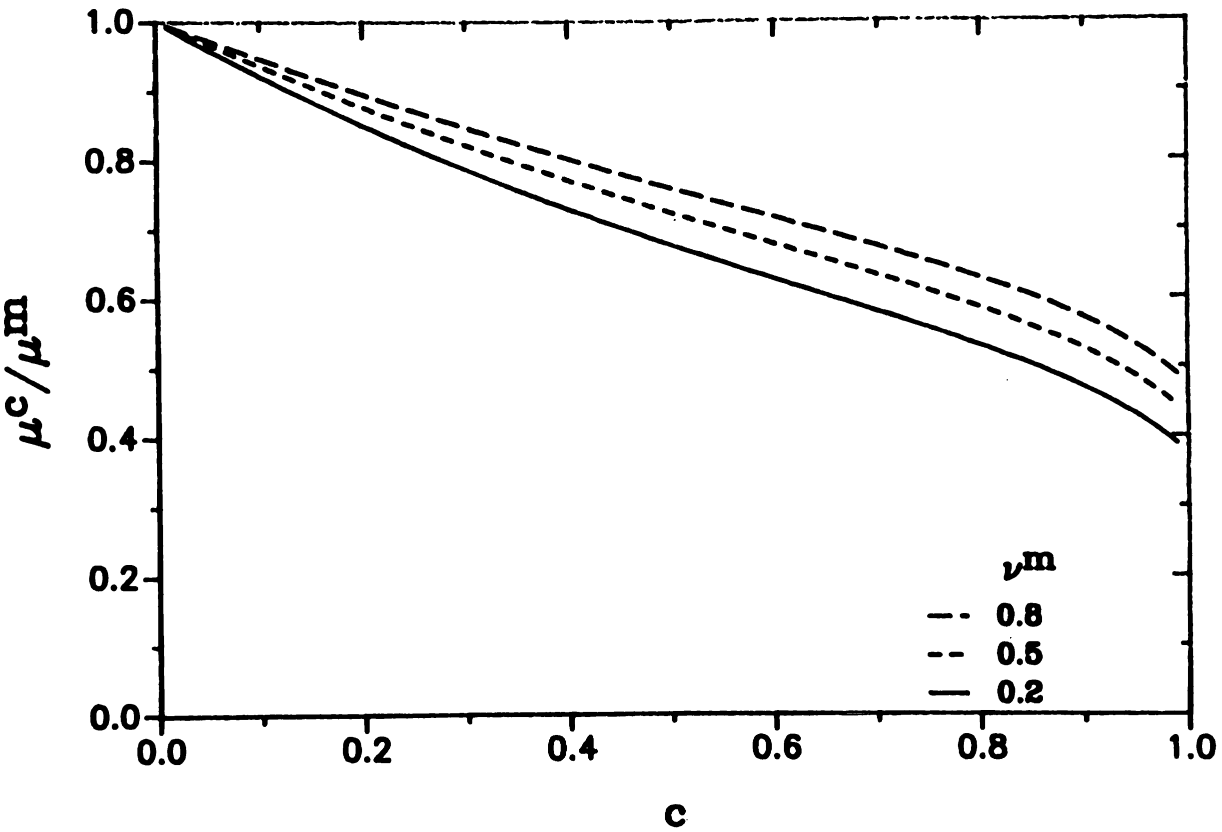

FIGURE 3.30 μ^c/μ^m vs. c for same materials with pure sliding by generalized self-consistent method

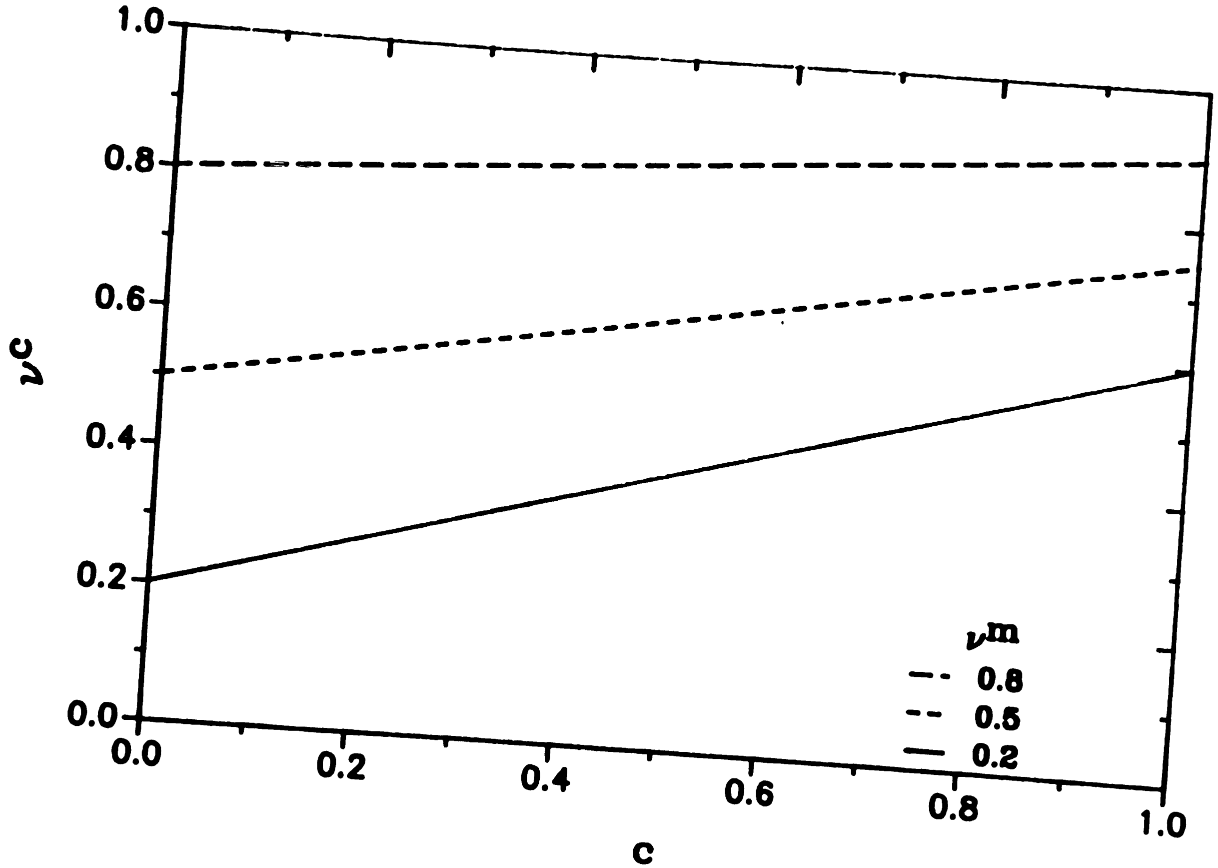

FIGURE 3.31 ν^c vs. c for same materials with pure sliding by self-consistent method

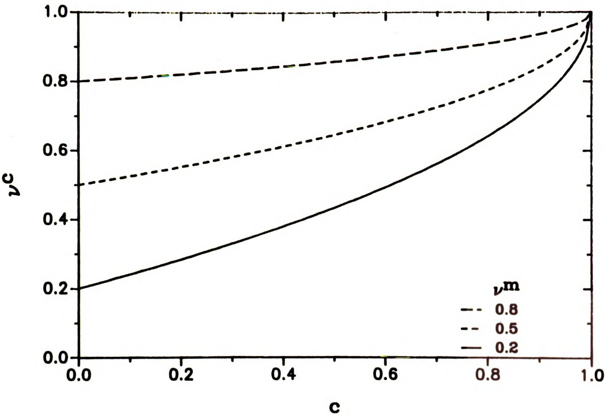

FIGURE 3.32 ν^c vs. c for same materials with pure sliding by differential scheme

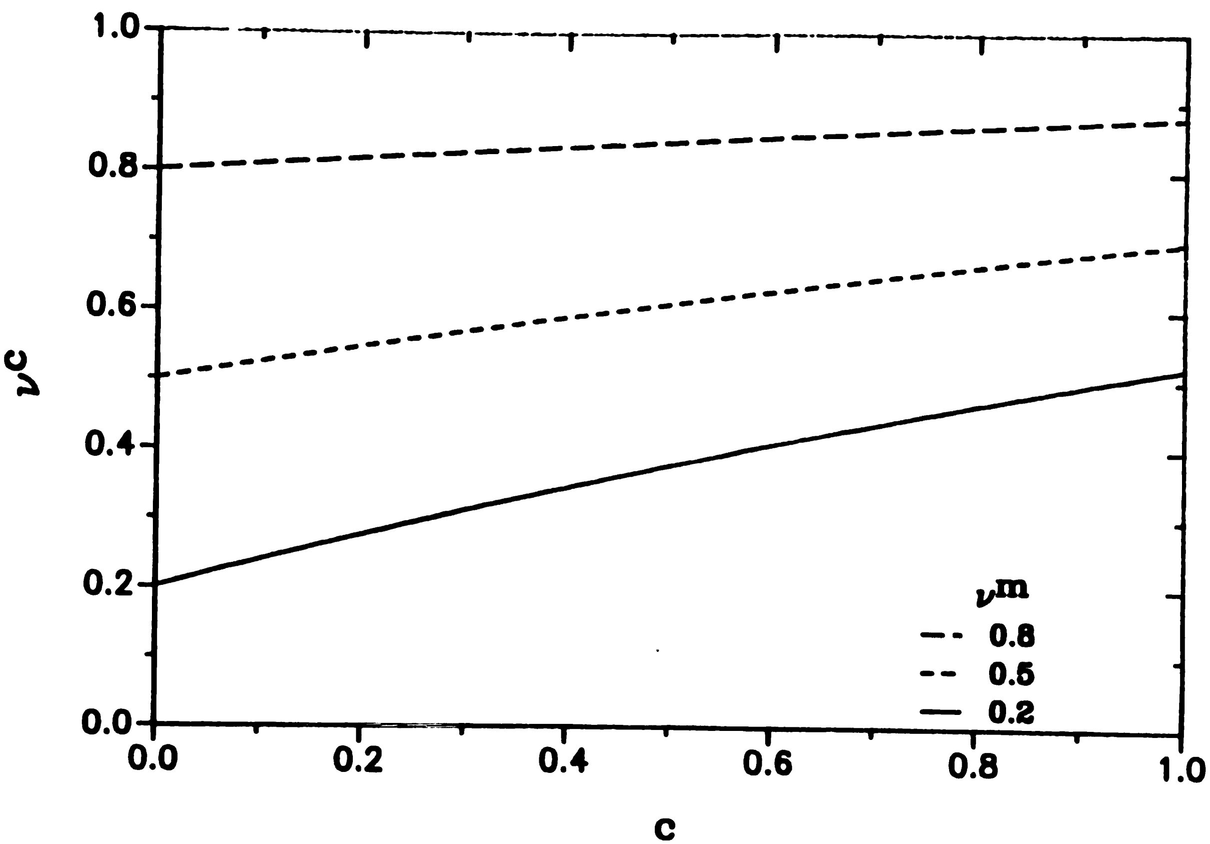

FIGURE 3.33 ν^c vs. c for same materials with pure sliding by Mori -Tanaka method

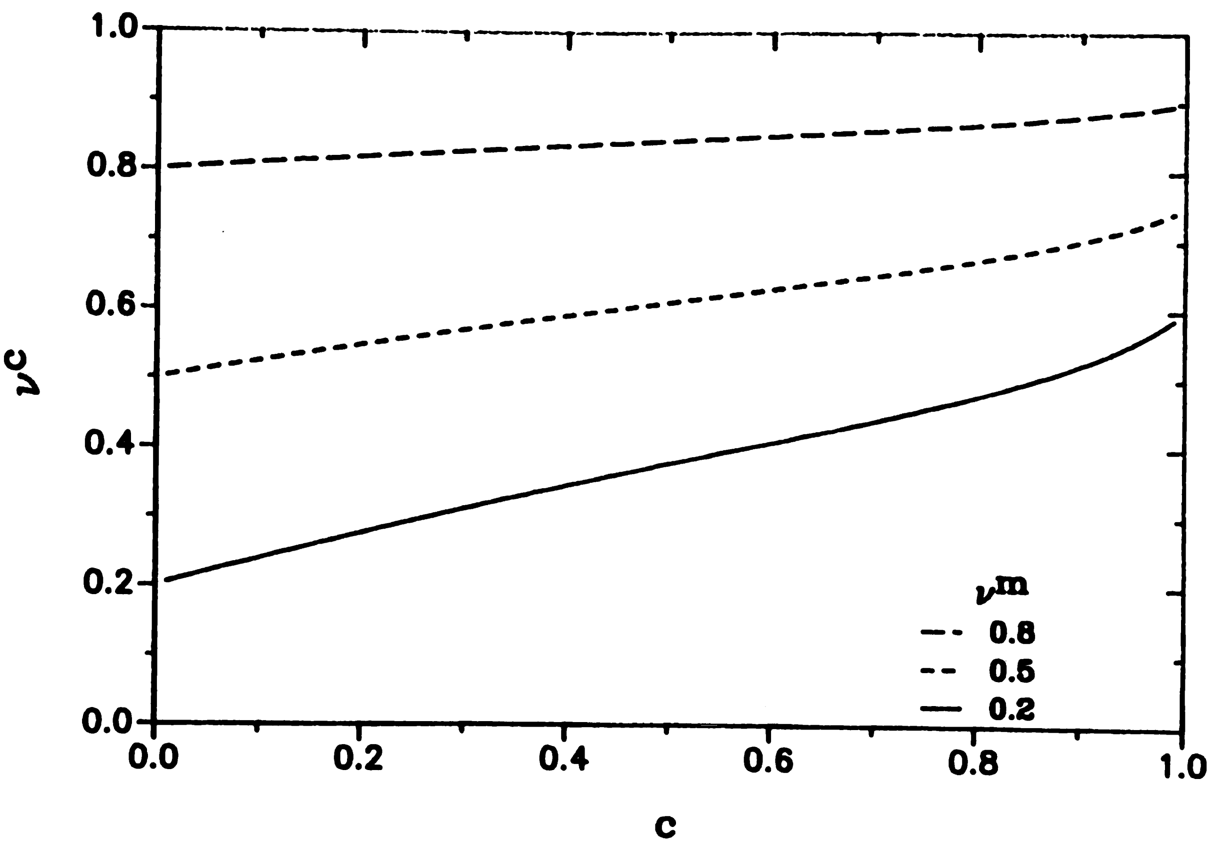

FIGURE 3.34 ν^c vs. c for same materials with pure sliding by generalized self-consistent method

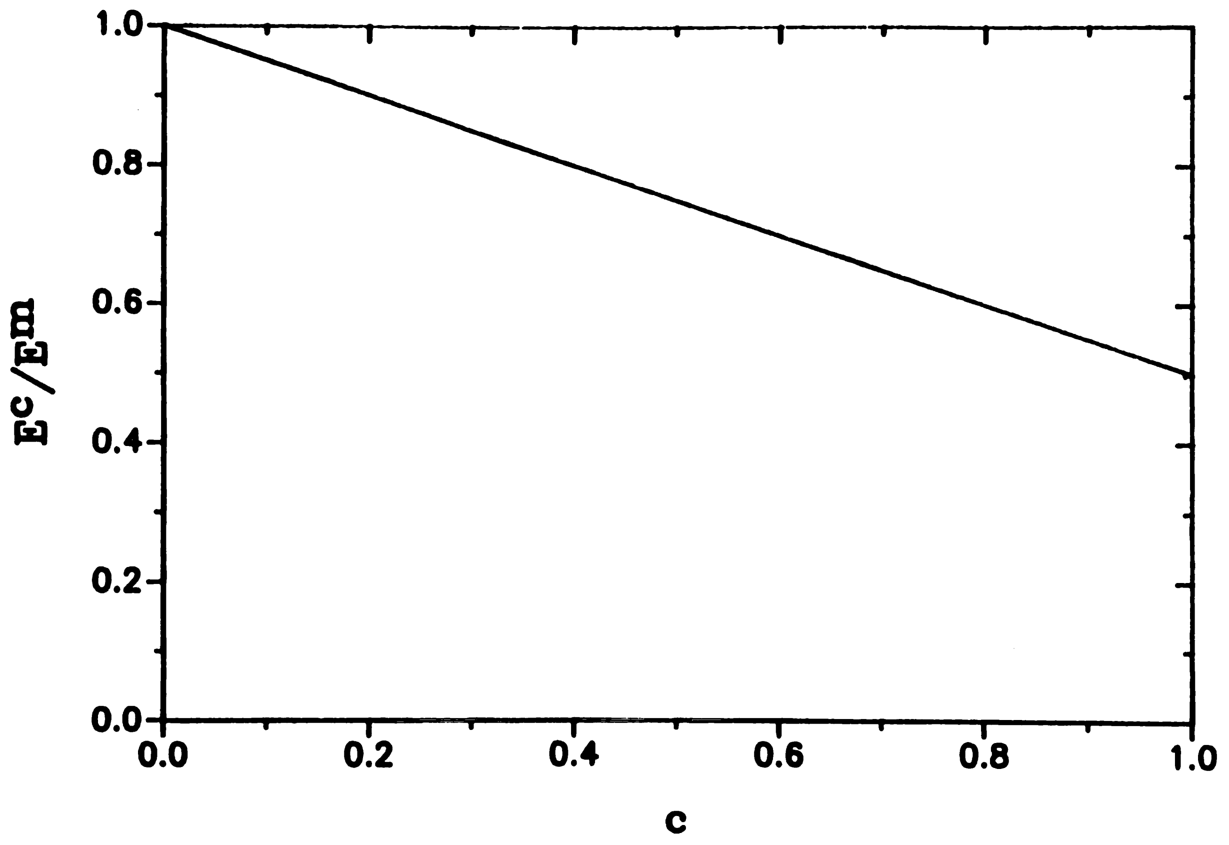

FIGURE 3.35 E^c/E^m vs. c for same materials with pure sliding by self-consistent method

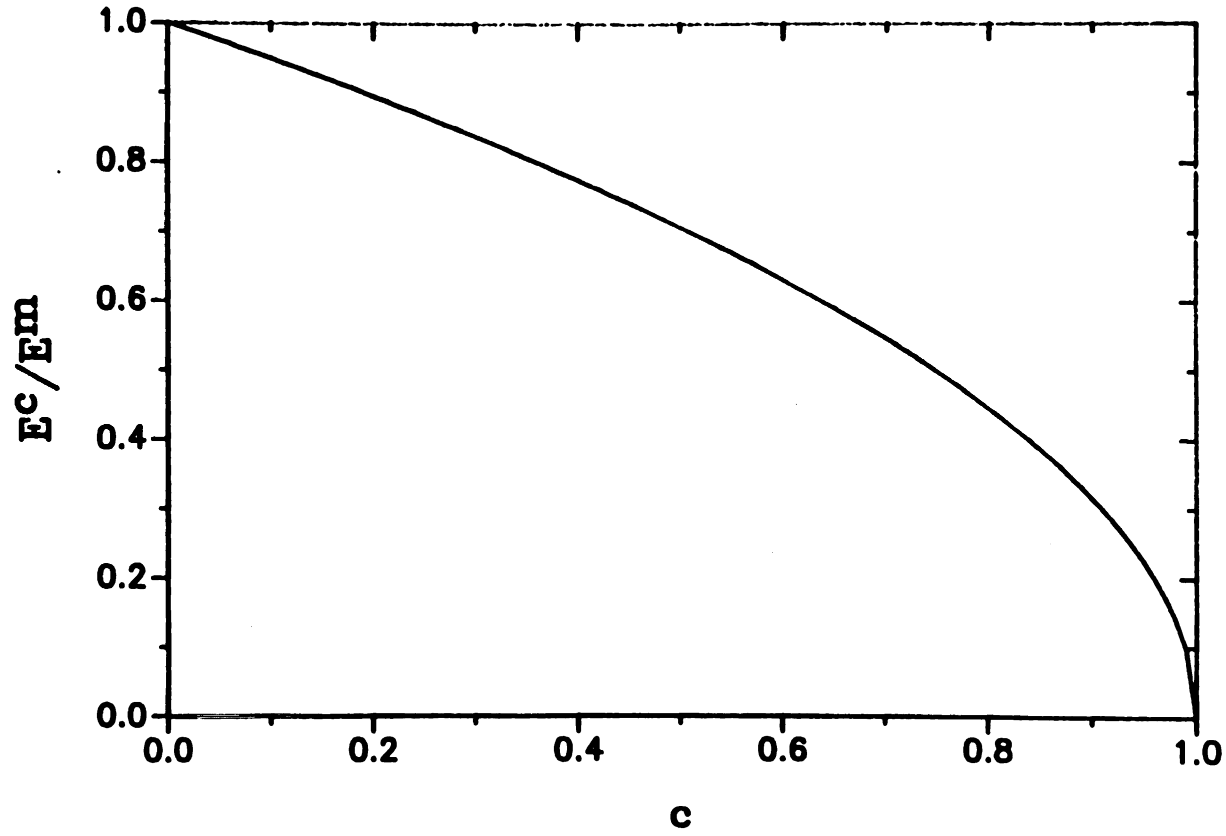

FIGURE 3.36 E^c/E^m vs. c for same materials with pure sliding by differential scheme

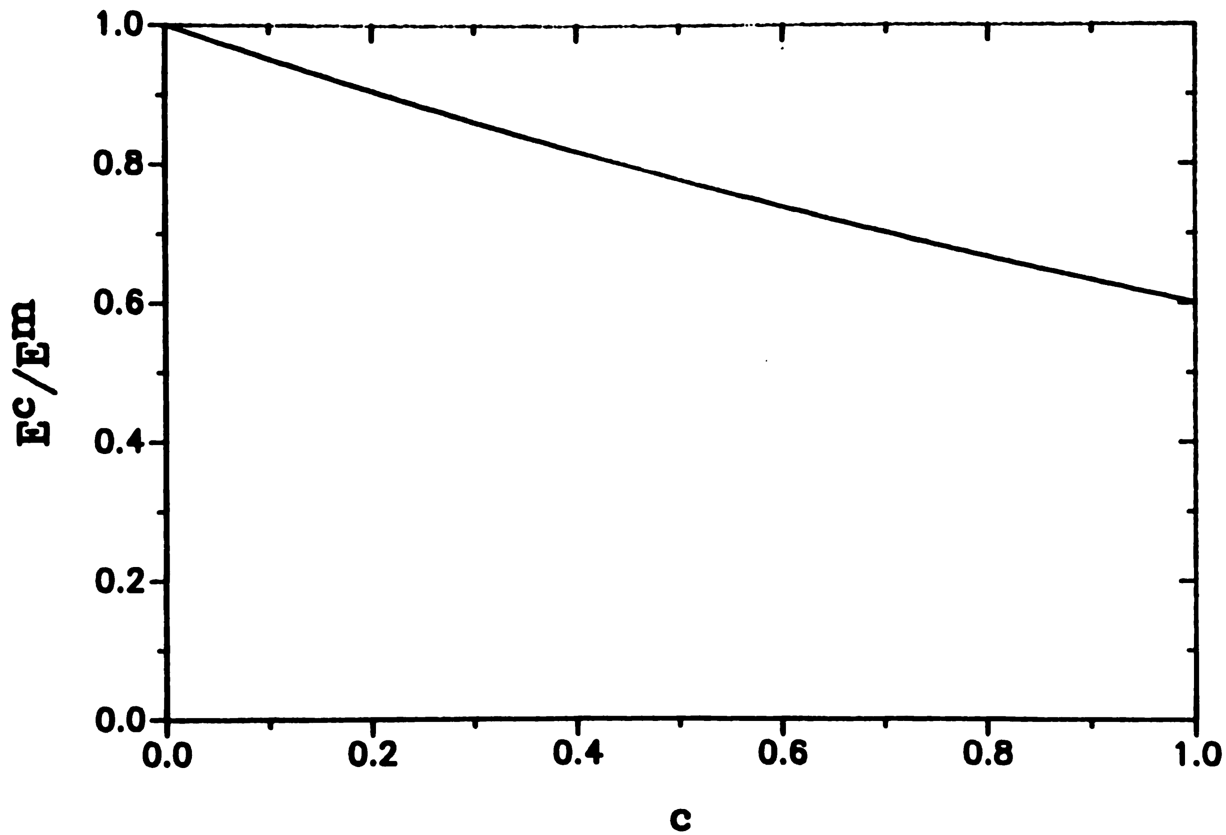

FIGURE 3.37 E^c/E^m vs. c for same materials with pure sliding by Mori-Tanaka method

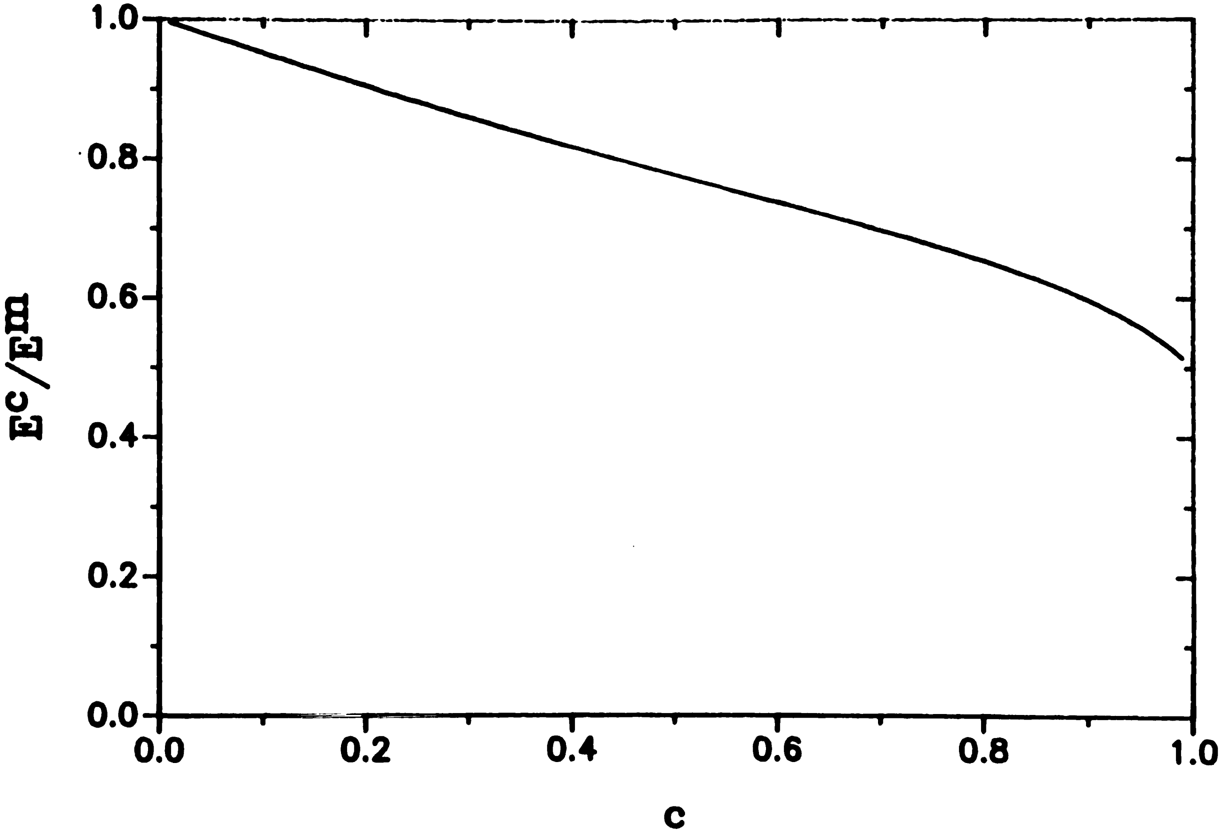

FIGURE 3.38 E^c/E^m vs. c for same materials with pure sliding by generalized self-consistent method

CHAPTER 4

CLM THEOREM AND THE RESULTS OF EFFECTIVE MEDIUM THEORIES

In this chapter the recently proven theorem for two dimensional composites is discussed comparing with the results of the effective medium theories. This theorem is stated as follow.

If the area bulk modulus and shear modulus are varied by the transformation (CLM transformation) as (Cherkaev et al., 1992; Thorpe and Jasiuk, 1992)

$$\frac{1}{K_t^m} = \frac{1}{K^m} + C \qquad \frac{1}{\mu_t^m} = \frac{1}{\mu^m} - C \tag{4.1}$$

$$\frac{1}{K_t^f} = \frac{1}{K^f} + C \qquad \frac{1}{\mu_t^f} = \frac{1}{\mu^f} - C \tag{4.2}$$

then

$$\frac{1}{K_t} = \frac{1}{K} + C \qquad \frac{1}{\mu_t} = \frac{1}{\mu} - C \tag{4.3}$$

where the subscript t denotes the transformed modulus and C is constant. This theorem is true only for two dimensional composites. It is also true that the Young's modulus is invariant under this CLM transformation.

The effective medium theories used in this thesis show that the Young's modulus of two dimensional composites containing holes is independent of the Poisson ratio of matrix. This agrees to the results of numerical simulations given by Day et al.(1991). Thorpe and Jasiuk (1992) have proved that the invariance of Young's modulus of the materials containing holes is derived from the CLM theorem. The CLM transformation leaves the inclusion as holes, which means that change of the Poisson ratio of matrix is basically identical to the CLM transformation. Therefore, for the composites of which inclusions are holes, the effective medium theories do not violate the CLM theorem.

For the same material with pure sliding discussed in section 3.3, the stress field is independent of the elastic constants of matrix as said before. In this limit the Young's modulus calculated by using the four effective medium theories is given to be independent of the Poisson ratio at whole range of volume fraction, which provides the possibility that the CLM theorem can be applied to the composite materials having imperfect bonding at interfaces.

The area bulk modulus and shear modulus obtained by the Mori - Tanaka method for the case of general inclusion are easily verified to follow the CLM theorem because the quantities in the parenthesis of the solutions of section 2.4 are invariant.

CHAPTER 5

CONCLUSIONS

The effective moduli of two dimensional composites are calculated by the effective medium theories ; the self consistent method, the differential scheme, the Mori - Tanaka method and the generalized self consistent method. The composites have the sliding interfaces between matrix and inclusions. The results of the self consistent method and the differential scheme are highly coupled forms while those of the Mori - Tanaka method and the generalized self consistent method are easy to be applied directly to an actual composite materials.

Several interesting limits are investigated to observe the differences between the effective medium theories and the effects of sliding interfaces on the effective moduli of composites. All the results of the effective medium theories used in this thesis reduce to the exact Eshelby's solution of a single inclusion problem when the volume fraction of inclusions is dilute. In some limit the big differences between the values of the effective moduli predicted by these theories are observed especially at the high volume fraction. The effect of the sliding interfaces make the value of the effective shear modulus and the effective Young's modulus lower than the values of perfect bonding case.

The above mentioned four effective medium theories do not violate the results of the CLM theorem of 2 D elasticity when the inclusion is hole. Furthermore they show the possibility that the CLM theorem can be applied to the case of sliding interfaces.

APPENDIX

The coefficients of equation for shear modulus of the generalized self consistent method are

$$C_1 = (B_1-2B_2)\,(A_2+A_4\eta^m) + 2\,(B_3+B_4\eta^m)\,(A_3-A_4) - A_0\,(B_3+B_4\eta^m) \tag{A.1}$$

$$C_2 = B_1A_4+2B_2\,[A_2+A_4\,(\eta^m-1)\,] - 2\,(A_3-A_4)\,[B_3+B_4\,(\eta^m-1)\,] - B_4A_0 \tag{A.2}$$

$$C_3 = 2\,[B_2A_4 - B_4\,(A_3-A_4)\,] \tag{A.3}$$

where

$$A_2 = [\,(\eta^f-3)\,\mu^m - (\eta^m-3)\,\mu^f]\,c^3 - 3\,(\mu^f-\mu^m)\,c^2 \tag{A.4}$$

$$A_3 = \frac{1}{(\eta^m+1)}\,\{\,[\,(\eta^m-3)\,\mu^f - (\eta^f-3)\,\mu^m]\,c^3 + 4\,(\mu^f-\mu^m)\,c^2$$

$$+ [\,(\eta^m+1)\,\mu^f + (\eta^f+1)\,\mu^m]\,c\,\} \tag{A.5}$$

$$A_4 = \frac{1}{(\eta^m+1)}\,\{\,[\,(\eta^m-3)\,\mu^f - (\eta^f-3)\,\mu^m]\,c^3 + 3\,(\mu^f-\mu^m)\,c^2$$

$$+ (\mu^f+\eta^f\mu^m)\,\} \tag{A.6}$$

$$A_0 = 2\,(\mu^f-\mu^m)\,c^2 \tag{A.7}$$

$$B_1 = 2c^2\,[\tilde{k}\,(\mu^f - \mu^m) - \mu^f\mu^m] \tag{A.8}$$

$$B_2 = \frac{1}{(\eta^m + 1)}\,\{\mu^f\mu^m\,(3 - 2c) + \tilde{k}\,[\mu^f\,(\eta^m - 1) - \mu^m\,(\eta^f - 1)\,c]$$

$$+ \tilde{k}\,(\mu^f + \eta^f\mu^m) - \mu^f\mu^m c^2 + \tilde{k}\,(\mu^f - \mu^m)\,c^2\,\} \tag{A.9}$$

$$B_3 = \tilde{k}\,[\mu^f\,(\eta^m + 3) - \mu^m\,(\eta^f + 3)\,]\,c^3 - 6\mu^f\mu^m c^3 + 3\mu^f\mu^m c^2 - 3\tilde{k}\,(\mu^f - \mu^m)\,c^2 \tag{A.10}$$

$$B_4 = \frac{1}{(\eta^m + 1)}\,\{3\mu^f\mu^m\,(2c^3 - 1) - \tilde{k}\,[\mu^f\,(\eta^m + 3) - \mu^m\,(\eta^f + 3)\,]\,c^3$$

$$- \tilde{k}\,(\mu^f + \eta^f\mu^m) - 3\mu^f\mu^m c^2 + 3\tilde{k}\,(\mu^f - \mu^m)\,c^2\,\} \tag{A.11}$$

where c is the volume fraction of inclusions and $\eta^{f,m} = 1 + 2\dfrac{\mu^{f,m}}{K^{f,m}}$.

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