This is to certify that the

thesis entitled

The Stress Field of a Spheroidal Sliding Inclusion Under Shear

presented by

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has been accepted towards fulfillment of the requirements for

Master's degree in Mechanics

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Major professor

Date 2-28-92

THE STRESS FIELD
OF A SPHEROIDAL SLIDING INCLUSION
UNDER SHEAR

By

Sheng Peiying

A THESIS

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

MASTER OF SCIENCE

Department of Materials Science and Mechanics

1992

ABSTRACT

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By

Sheng Peiying

The elastic fields due to a spheroidal inhomogeneity under a uniform shear stress applied at infinity and a spheroidal inclusion which undergoes a uniform shear eigenstrain are investigated. The interface between the matrix and the inclusion cannot sustain shear tractions and is free to slip. Initially, Boussinesq displacement potentials in forms of infinite series are used to solve these two problems. However, we discover that only finite number terms of potentials will give the solution. As a result, it is found that for the inhomogeneity problem the stresses in the spheroidal subdomain vanish and the spheroid rotates without deformation, and that for the eigenstrain problem the stresses in both the inclusion and the matrix vanish, and there is no deformation in the matrix, while the inclusion does deform but keeps its original shape and orientation.

ACKNOWLEDGEMENTS

I wish to thank my advisor, professor Iwona Maria Jasiuk, for her valuable guidance, the financial support she offered and her genuine concern of my sick daughter. Such understanding and support gave me strength to overcome difficulties to accomplish my career goals.

I would like to thank professor Dahsin Liu, professor Martin Ostoja Starzewski and professor Gary L. Cloud for their support.

I would also like to thank my wife for all her love and support during my studies, especially, when she is loaded with her Ph.D. dissertation writing and the care of our daughter.

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1. INTRODUCTION

1.1 PROBLEM STATEMENT

If a region Ω in a matrix D has an eigenstrain, which is a nonelastic strain such as thermal expansion, initial strain, plastic strain, etc., it is called an inclusion. If a subdomain Ω in a matrix D has moduli different from those of the matrix, then it is called an inhomogeneity. Both the inclusion problem and inhomogeneity problem can be solved basically in the same way in view of mathematics, so that sometimes they are called the inclusion problems for convenience (Fig. 1)[1].

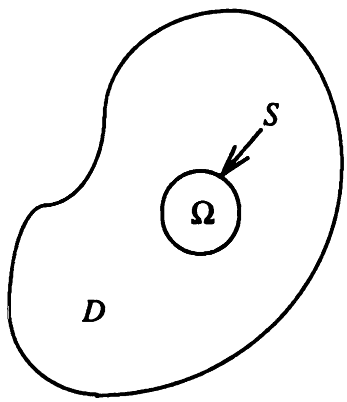

Fig. 1. Inclusion in a matrix

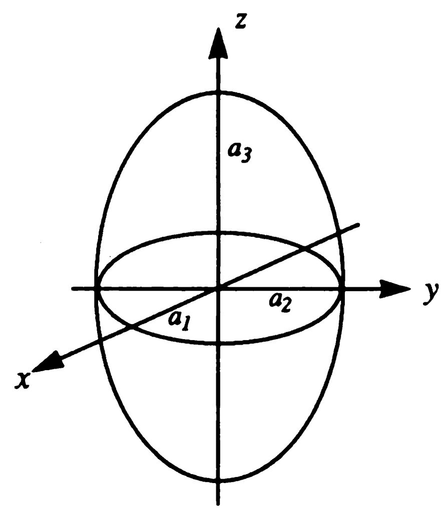

Fig. 2. Prolate spheroid

Fibers in composite materials, aggregates in concrete, voids in porous material, and others can be represented by inclusions, in particular, by the inclusions with the ellipsoidal shape in geometry. Therefore, the solution of an ellipsoidal inclusion problem is important for the engineering applications. The stress field near the inclusion depends on its interface condition, elastic constants, types of loading, shape of materials, and others. At the interface, the inclusion could be perfectly bonded to the matrix, free to slip along the interface, debonded from the matrix, or a combination of these.

In this thesis we consider an infinite, elastic, homogeneous and isotropic material having a prolate spheroid which is also elastic, homogeneous and isotropic , but has elastic properties different from those of the matrix. The prolate spheroid is the special ellipsoid in which the three semi-axes have the relation $a_1 = a_2 < a_3$. We want to determine stress and displacement fields when the infinite body containing a spheroidal inclusion is under shear loading at infinity

$$\sigma_{zx} = S_0 \tag{1.1}$$

where S_0 is a constant shear stress, and the coordinates are chosen in such a way that the z-axis coincides with the major semi-axis of the spheroid, and the x-axis with the minor axis (Fig. 2). The equation (1.1) is the boundary condition at infinity. We assume that the sliding is allowed at the interface between the inhomogeneity and the matrix, and shear tractions are specified to vanish, but no debonding takes place. In the other words, the stresses and normal displacements are continuous, and shear stresses are zero on the interface. These boundary conditions at the interface S (see Fig.1) can be expressed as follows

$$[\sigma_{ij}]\, n_j = 0 \qquad \text{on } S \tag{1.2}$$

$$[u_i]\, n_i = 0 \qquad \text{on } S \tag{1.3}$$

$$\sigma_{ij} n_j - \sigma_{jk} n_j n_k n_i = 0 \qquad \text{on } S \tag{1.4}$$

where σ_{ij}.........stress

u_i.........displacement

n_i.........normal vector on S

[ ]........the difference of quantities out of and in inclusion,(out)-(in)

i, j, k.....x, y, z

1.2 BACKGROUND

The inclusion problems have been investigated by many researchers. Mura[1] gave a comprehensive review of this subject. Jasiuk[2] investigated the sliding inclusions in details. The papers that are valuable to us are due to Goodier[3], Sadowsky and Sternberg[4], Edwards[5], Eshelby[6,7], Ghahremani[8], Tsuchida, Mura, Jasiuk, Furuhashi and Dundurs[9]-[15], and others.

The earlier papers deal with the case in which the bonding between the inclusion and the matrix is assumed to be perfect, i.e., all stresses and displacements are continuous at the interface. These boundary conditions are described with the equations

$$[\sigma_{ij}]\, n_j = 0 \qquad \text{on } S \tag{1.5}$$

$$[u_i] = 0 \qquad \text{on } S \tag{1.6}$$

Goodier[3] considered the axisymmetric elasticity solutions to investigate the disturbance effect of inclusions with simple spherical and cylindrical geometry and discussed the concentration of stress around these two types of inclusions.

Sadowsky and Sternberg[4] obtained an exact closed form solution for the stress around a spheroidal cavity in an infinite elastic body, which is subjected to arbitrary uniform stresses at infinity in the plane perpendicular to the axis of revolution of the cavity. They used the displacement potential approach.

Edwards[5] also used Boussinesq's approach in the theory of elasticity and chose Legendre functions as the potentials. He obtained exact closed form solutions for the distribution of stress around a spheroidal inclusion in an infinite elastic body under all four types of uniform loading conditions at infinity, i.e.,

case 1: xy plane state of hydrostatic stress

$$\sigma_x = \sigma_y = 1 \tag{1.7}$$

case 2: xy plane state of pure shearing stress

$$\sigma_x = -\sigma_y = 1 \tag{1.8}$$

case 3: state of uniaxial tension

$$\sigma_z = 1 \tag{1.9}$$

case4: zx plane state of pure shearing stress

$$\sigma_{zx} = 1 \tag{1.10}$$

where the coordinates are designated as in Fig.2. The interface was assumed to be perfectly bonded. These solutions for the basic loadings are useful because, for the problem of the prolate spheroidal inclusion in an infinite body subjected to an arbitrary uniform loading, the stress distribution throughout the body could be obtained as a linear combination of the solution for the basic loading conditions.

Eshelby[6,7] solved the inclusion problems by using Green's function method and expressed a solution in a form of elliptic integral. He pointed out a remarkable feature of the stress and strain field of a particular type of inclusion, that is, if the region is an ellipsoid, the strain and stress are uniform everywhere inside the inclusion. It should be noted that the interface needs to be perfectly bonded for this conclusion to be held.

The condition of perfect bond at the interface between the matrix and the inclusion is not true for some real materials. There could be a discontinuity of displacements, such as debonding, sliding and the combination of them at the interface. For example, grains in polycrystal, particles in soils, and fibers in composite materials may be subjected to sliding along their interfaces. Thus it is valuable to consider more complex boundary conditions, among which is a sliding situation which is designated in equations (1.2)-(1.4).

A number of recent papers considered sliding inclusions. Ghahremani[8] investigated the case of a sliding spherical inhomogeneity under tension at infinity.

Mura and Furuhashi[9] pointed out that when sliding is allowed along the inclusion surface, the well-known solution of Eshelby[6] must be substantially modified. They solved the problem of inclusion with a sliding interface by superposing the Eshelby's solution (perfect bonding) and Volterra's solution (sliding). This way they relaxed the stresses at the interface of perfectly bonded inclusion by distributing dislocations along the surface of inclusion. They reported a striking result that when an ellipsoidal inclusion undergoes a shear eigenstrain and the inclusion is free to slip along the interface, the stress field vanishes everywhere in the inclusion and in the matrix, and there exists a shear deformation that transforms an ellipsoid into the identical one without changing its orientation. It should be noted that their solution is restricted to such a case that $a_1 \neq a_2 \neq a_3$, where a_1, a_2, and a_3 are the semi-axes of the ellipsoid.

Tsuchida, et al.[10] used Papkovich-Neuber displacement potential and solved the elastic field of an elliptical inclusion in an infinite body. The boundary conditions at the interface are the same as those in [9], i.e., the interface between the inclusion and the matrix can not sustain shear tractions and the inclusion is free to slip along the interface. But the eigenstrain is of the different type. The inclusion undergoes a uniform expansion instead of the shear strain in [9]. It is concluded that in contrast to the perfectly bonded case, the solution for the sliding case involves infinite series. Also the stress field is nonzero. The infinite series of potentials were used before by Tsuchida and Mura[11] to evaluate the stress field in an elastic half space having a spheroidal inhomogeneity under an all-around tension parallel to the plane boundary.

Kouris, et al.[12] employed Papkovich-Neuber displacement potentials, the same method as that in [10] but different from that in [9], and reexamed the stress field when an elliptic subdomain in a free body undergoes a uniform shear type of eigenstrain or the body is subjected to a uniform shear loading at infinity. No resistance was allowed along the interface of the ellipse. In this solution, although the displacement potentials were taken in a form of infinite series, the results from the calculations show that only a finite number of terms of the infinite series contributes to the stress field. For the case of the shear eigenstrain the only nonzero term of the displacement potential corresponds to the rigid body motion, so that the whole stress field vanishes. For the case of the uniform shear loading at infinity the finite terms of the potentials give zero stress field in the inhomogeneity, so the inhomogeneity acts like an elliptic cavity of the same shape in an infinite plate. The result is consistent to that reported in [9], because $a_1 \neq a_2$, while $a_3 = 0$. As the limit situation of the ellipse, a circular inclusion was also studied. The stress field, however, is not zero, when an inhomogeneity has the shape of the circle. That means the solution for the circular inhomogeneity can not be obtained from the limit of the ellipse. There exists the discontinuity of the stress when the ellipse approaches the circle, which is called an anomaly of solution. This is not true for the case of perfect bonding at the interface in which the stress of the circular inclusion is the limit of that of the elliptic inclusion.

Jasiuk, et al.[13-15] considered the inclusion in the shape of prolate spheroid having the sliding boundary condition, i.e., the inclusion was free to slip along its interface with the matrix and no shear traction existed on the interface. Both types of inclusion problems, in which the inclusion was undergoing a uniform eigenstrain and the inhomogeneity was subjected to a uniform stress at infinity, were investigated. All the possible uniform loadings, all-around tension, uniaxial tension parallel to or shear stress in the plane perpendicular to the rotational axis of the spheroid(corresponding to the previous case1-case3, or equation(1.7)-(1.9)) at infinity were considered except the second kind of shear loading ,$\sigma_{zx} = S_0$ (corresponding to the above-mentioned case 4 or equation(1.10)). The method used to solve the above problems was similar to the one of Edwards'[5] and Tsuchida and Mura's[11]. The prolate harmonic functions in the form of infinite series were taken as the displacement potentials. It was discovered that stresses in the sliding spheroidal inclusion are not uniform, unlike in the perfectly bonded inclusion, where the stress field is uniform[6]. Similar to the paper[10], the conclusion was made that infinite series representation of displacement potentials is required for the sliding inclusion, contrary to perfectly bonded inclusion, where finite series give the exact solusion[5]. When the sliding inclusion is of spherical shape, only finite series are required[8].

In summary, a lot of research has been done on the stress and strain fields of an infinite body containing an ellipsoidal, in particular spheroidal inclusion which undergoes uniform eigenstrains or is subjected to uniform stresses at infinity. The Boussinesq's or the Papkovich's displacement potentials methods of elasticity are suitable in solving the inclusion problems, as well as the Green's function approach. The displacement potentials and the stress fields are obviously different when the interface between the inclusion and the matrix behaves differently. For the case of perfect bonding interface the finite number of terms of potential functions gives the exact solution and the stress field in the spheroidal inclusion is uniform. For the case of sliding boundary conditions at the interface, however, the infinite series of potential functions are required and the stress field is not uniform any more. The stress field vanishes if an inclusion of ellipsoidal shape undergoes a shear eigenstrain or an elliptical inhomogeneity in two dimensions is subjected to a uniform shear stresses at infinity. This is not true when the ellipsoidal inclusion goes to spherical one, or when a spheroidal inclusion has an shear eigenstrain or a spheroidal inhomogeneity is subjected to a uniform shear stress in the plane perpendicular to its rotational axis at infinity, or an elliptic inclusion becomes a circular one.

As a continuity and supplement of Jasiuk's work on the prolate spheroidal inclusion with a sliding interface condition(corresponding to the cases 1, 2 and 3), this thesis mainly investigates the case of an infinite body with a sliding inhomogeneity inside, which is subjected to a uniform shear stress $\sigma_{zx} = S_0$ at infinity(corresponding to the case 4).

2. METHOD OF SOLUTION

We assume that inclusion is of prolate spheroidal shape which ranges from needle-like shape to spherical shape. Therefore, the prolate spheroidal coordinates are adopted. Based on the Boussinesq's approach of elasticity in three dimensions, the stress and strain fields are expressed by prolate spheroidal harmonic potential functions in a form of infinite series. Satisfying the boundary condition at the interface and at infinity, the algebraic equation system containing the potential functions and their coefficients is obtained. Finally the linear equation system is solved for the coefficients so that the displacement and stress fields are found according to the basic equations of elasticity. The calculations are carried out numerically.

2.1 COORDINATE SYSTEM

It is convenient to choose a prolate spheroidal coordinate system for the problem with the inclusion of spheroidal geometry. As a result, the boundary conditions at the interface can be easily expressed.

Let the origin of a Cartesian coordinate system, O, be at the center of the prolate spheroidal inhomogeneity and the z axis coincide with inclusion's rotational axis, i.e., its major axis, so that all the three axes in Cartesian coordinates are the symmetric axes of the spheroid (Fig.3). Then an orthogonal curvilinear coordinate system-a prolate spheroidal coordinate system is used. The prolate spheroidal coordinate system(α, β, γ) is related to the Cartesian coordinate system(x,y,z) by the transformation equations[2],

$$x = c\sinh\alpha\sin\beta\cos\gamma = c\bar{q}\bar{p}\cos\gamma \tag{2.1}$$

$$y = c\sinh\alpha\sin\beta\sin\gamma = c\bar{q}\bar{p}\sin\gamma \tag{2.2}$$

$$z = c\cosh\alpha\cos\beta = cqp \tag{2.3}$$

where c is the focal length of the prolate spheroid, and the range of variables are

$$0 \le \alpha < \infty, \quad 0 \le \beta \le \pi, \quad 0 \le \gamma \le 2\pi. \tag{2.4}$$

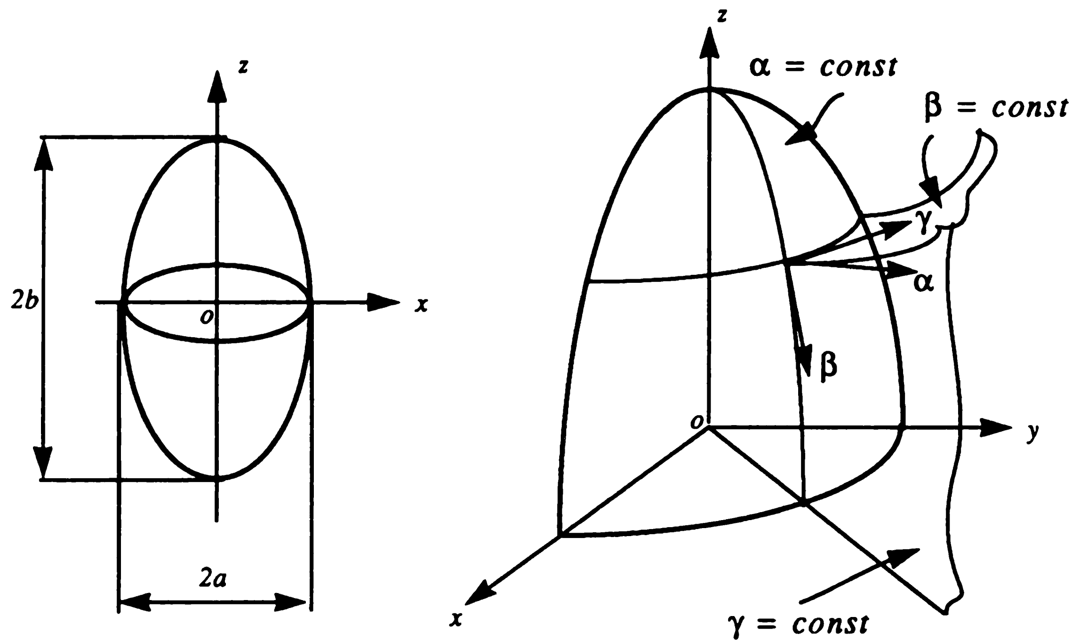

Fig. 3. Coordinate system

The surfaces α=const, β=const and γ=const consist of an orthogonal family of prolate spheroids, hyperboloids of two sheets and half planes passing the meridian curves.

For simplicity the following new variables are introduced

$$q = \cosh\alpha \tag{2.5}$$

$$\bar{q} = \sinh\alpha \tag{2.6}$$

$$p = \cos\beta \tag{2.7}$$

$$\bar{p} = \sin\beta, \tag{2.8}$$

with the relationship

$$q^2 - \bar{q}^2 = 1 \tag{2.9}$$

$$p^2 + \bar{p}^2 = 1 \tag{2.10}$$

and their ranges are

$$1 \le q < \infty \tag{2.11}$$

$$0 \le \bar{q} < \infty \tag{2.12}$$

$$-1 \le p \le 1 \tag{2.13}$$

$$0 \le \bar{p} \le 1 \tag{2.14}$$

The differential of arc length is

$$(ds)^2 = (d\alpha/h)^2 + (d\beta/h)^2 + (d\gamma/h_3)^2 \tag{2.15}$$

where

$$h = \frac{1}{c\sqrt{q^2 - p^2}}, \qquad h_3 = \frac{1}{c\bar{q}\bar{p}} \tag{2.16}$$

The surface of the spheroid is defined by $\alpha = \alpha_0$ or the corresponding variable $q_0 = \cosh\alpha_0$. The minor and major semi-axes of the spheroid are

$$a = c\bar{q}_0, \qquad b = cq_0 \tag{2.17}$$

with the relationship

$$b^2 - a^2 = c^2 \tag{2.18}$$

2.2 METHOD OF SOLUTION

The problem considered in this thesis is solved by using the displacement approach. If no body forces are accounted for, the displacement equations of equilibrium for elasticity in three-dimensions are

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \times \nabla e = 0 \tag{2.19}$$

where $\underset{\sim}{u}$........displacement vector

ν........Poisson's ratio

e........dilatation, $\nabla \bullet \underset{\sim}{u}$

∇^2......$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, Laplacian operator

∇........$\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors in x, y, z direction respectively.

The general solution of the displacement equation of equilibrium is given as[14]

$$2G\underset{\sim}{u} = \nabla (\phi_0 + \underset{\sim}{R} \cdot \underset{\sim}{\beta}) + 4(1-\nu)\underset{\sim}{\beta} + 2\nabla \times \underset{\sim}{\Lambda} + \nabla \times (\underset{\sim}{R}\lambda_0) \tag{2.20}$$

where $\underset{\sim}{R}$........position vector

G.......shear modulus

$\underset{\sim}{\beta}(\phi_1, \phi_2, \phi_3)$, $\underset{\sim}{\Lambda}(\lambda_1, \lambda_2, \lambda_3)$, ϕ_0, λ_0 ....harmonic functions.

The displacement components in the Cartesian coordinate system are

$$\begin{aligned} 2Gu_x = {} & \frac{\partial}{\partial x}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) - 4(1-\nu)\phi_1 \\ & + 2\left(\frac{\partial \lambda_3}{\partial y} - \frac{\partial \lambda_2}{\partial z}\right) + \left(y\frac{\partial \lambda_0}{\partial z} - z\frac{\partial \lambda_0}{\partial y}\right) \end{aligned} \tag{2.21}$$

$$\begin{aligned} 2Gu_y = {} & \frac{\partial}{\partial y}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) - 4(1-\nu)\phi_2 \\ & + 2\left(\frac{\partial \lambda_1}{\partial z} - \frac{\partial \lambda_3}{\partial x}\right) + \left(z\frac{\partial \lambda_0}{\partial x} - x\frac{\partial \lambda_0}{\partial z}\right) \end{aligned} \tag{2.22}$$

$$\begin{aligned} 2Gu_z = {} & \frac{\partial}{\partial z}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) - 4(1-\nu)\phi_3 \\ & + 2\left(\frac{\partial \lambda_2}{\partial x} - \frac{\partial \lambda_1}{\partial y}\right) + \left(x\frac{\partial \lambda_0}{\partial y} - y\frac{\partial \lambda_0}{\partial x}\right) \end{aligned} \tag{2.23}$$

But not all the harmonic functions in equations(2.21)-(2.23) are required to be present in order to satisfy the displacement equations of equilibrium (2.19). The different combinations of these harmonic functions were proposed by different scholars. For example, the formulation in terms of ϕ_0, ϕ_1, ϕ_2 and ϕ_3 is due to Papkovich and Neuber, and ϕ_0, ϕ_3, and λ_3 due to Boussinesq. In this thesis Boussinesq's appoach is used as follows

$$2Gu_x = \frac{\partial\phi_0}{\partial x} + z\frac{\partial\phi_3}{\partial x} + 2\frac{\partial\lambda_3}{\partial y} \tag{2.24}$$

$$2Gu_y = \frac{\partial\phi_0}{\partial y} + z\frac{\partial\phi_3}{\partial y} - 2\frac{\partial\lambda_3}{\partial x} \tag{2.25}$$

$$2Gu_z = \frac{\partial\phi_0}{\partial z} + z\frac{\partial\phi_3}{\partial z} - (3-4\nu)\,\phi_3 \tag{2.26}$$

The displacements in Cartesian coordinate system can be transformed into the prolate spheroidal coordinate system by using the transformation tensor or matrix as follows

$$u_i = A_{ij}u_j \tag{2.27}$$

or

$$\begin{bmatrix} u_\alpha \\ u_\beta \\ u_\gamma \end{bmatrix} = [A] \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{2.28}$$

where i....................α, β, γ

j...................x, y, z

$u_\alpha, u_\beta, u_\gamma$......displacements in the spheroidal coordinate system

u_x, u_y, u_z.......displacements in the Cartesian coordinate system

$[A]$.................transformation matrix from Cartesian coordinate system to spheroidal coordinate system, which is derived from the direction cosines between the two systems as follows

$$[A] = \begin{bmatrix} ch q\bar{p}\cos\gamma & ch q\bar{p}\sin\gamma & ch\bar{q}p \\ ch\bar{q}p\cos\gamma & ch\bar{q}p\sin\gamma & -chq\bar{p} \\ -\sin\gamma & \cos\gamma & 0 \end{bmatrix} \tag{2.29}$$

Using the equations (2.28) and (2.29), the transformed displacements in the prolate spheroidal coordinate system are(appendix A)

$$2Gu_\alpha = h\bar{q}\frac{\partial\phi_0}{\partial q} + ch\bar{q}p\left[q\frac{\partial\phi_3}{\partial q} - (3-4\nu)\phi_3\right] + 2h\frac{q}{\bar{\bar{q}}}\frac{\partial\lambda_3}{\partial\gamma} \tag{2.30}$$

$$2Gu_\beta = -h\bar{p}\frac{\partial\phi_0}{\partial p} - chq\bar{p}\left[p\frac{\partial\phi_3}{\partial p} - (3-4\nu)\phi_3\right] + 2h\frac{p}{\bar{\bar{p}}}\frac{\partial\lambda_3}{\partial\gamma} \tag{2.31}$$

$$2Gu_\gamma = \frac{1}{c\bar{q}\bar{p}}\frac{\partial\phi_0}{\partial\gamma} + \frac{qp}{\bar{\bar{q}}\bar{p}}\frac{\partial\phi_3}{\partial\gamma} + 2ch^2\bar{q}\bar{p}\,(p\frac{\partial\lambda_3}{\partial p} - q\frac{\partial\lambda_3}{\partial q}) \tag{2.32}$$

The corresponding stresses in the prolate spheroidal coordinate system can be derived from the displacements (2.30)-(2.32) by using the strain-displacement relations and the Hooke's law. These stresses are

$$\sigma_\alpha = h^2\left[\bar{q}^2\frac{\partial^2\phi_0}{\partial q^2} + c^2h^2\bar{p}^2\,(q\frac{\partial\phi_0}{\partial q} - p\frac{\partial\phi_0}{\partial p})\right]$$

$$+ \; ch^2\left[q\bar{q}^2p\frac{\partial^2\phi_3}{\partial q^2} + \{c^2h^2q^2\bar{p}^2 - 2(1-\nu)\bar{q}^2\}\,p\frac{\partial\phi_3}{\partial q} - (c^2h^2p^2 + 2\nu)\,q\bar{p}^2\frac{\partial\phi_3}{\partial p}\right] \tag{2.33}$$

$$+ 2h^2q\left[\frac{\partial^2\lambda_3}{\partial\gamma\partial q} - \frac{q}{\bar{q}^2}\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_\beta = h^2\left[\bar{p}^2\frac{\partial^2\phi_0}{\partial p^2} + c^2h^2\bar{q}^2\left(q\frac{\partial\phi_0}{\partial q} - p\frac{\partial\phi_0}{\partial p}\right)\right]$$

$$+ \quad ch^2\left[qp\bar{p}^2\frac{\partial^2\phi_3}{\partial p^2} - \{c^2h^2\bar{q}^2p^2 + 2\,(1-\nu)\,\bar{p}^2\}\,q\frac{\partial\phi_3}{\partial p} + (c^2h^2q^2 - 2\nu)\,\bar{q}^2p\frac{\partial\phi_3}{\partial q}\right] \quad (2.34)$$

$$- \; 2h^2p\left[\frac{\partial^2\lambda_3}{\partial p\partial\gamma} + \frac{p}{\bar{p}^2}\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_\gamma = \frac{1}{c^2\bar{q}^2\bar{p}^2}\frac{\partial^2\phi_0}{\partial\gamma^2} + h^2\left[q\frac{\partial\phi_0}{\partial q} - p\frac{\partial\phi_0}{\partial p}\right]$$

$$+ \quad \frac{qp}{c\bar{q}^2\bar{p}^2}\frac{\partial^2\phi_3}{\partial\gamma^2} + ch^2\left[(q^2 - 2\nu\bar{q}^2)\,p\frac{\partial\phi_3}{\partial q} - (p^2 + 2\nu\bar{p}^2)\,q\frac{\partial\phi_3}{\partial p}\right] \quad (2.35)$$

$$+ \; 2h^2\left[p\frac{\partial^2\lambda_3}{\partial p\partial\gamma} - q\frac{\partial^2\lambda_3}{\partial\gamma\partial q}\right] + \frac{2}{c^2\bar{q}^2\bar{p}^2}\frac{\partial\lambda_3}{\partial\gamma}$$

$$\sigma_{\alpha\beta} = h^2\bar{q}\bar{p}\left[-\frac{\partial^2\phi_0}{\partial q\partial p} + c^2h^2\left(q\frac{\partial\phi_0}{\partial p} - p\frac{\partial\phi_0}{\partial q}\right)\right]$$

$$- \; ch^2\bar{q}\bar{p}\left[qp\frac{\partial^2\phi_3}{\partial q\partial p} + \{c^2h^2p^2 - (1-2\nu)\}\,q\frac{\partial\phi_3}{\partial q} - \{c^2h^2q^2 + (1-2\nu)\}\,p\frac{\partial\phi_3}{\partial p}\right] \quad (2.36)$$

$$+ \quad \frac{h^2}{\bar{q}\bar{p}}\left[\bar{q}^2p\frac{\partial^2\lambda_3}{\partial\gamma\partial q} - q\bar{p}^2\frac{\partial^2\lambda_3}{\partial p\partial\gamma} - 2qp\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_{\beta\gamma} = -\frac{h}{c\bar{q}}\left[\frac{\partial^2\phi_0}{\partial p\partial\gamma} + \frac{p}{\bar{p}^2}\frac{\partial\phi_0}{\partial\gamma}\right]$$

$$- \; h\frac{q}{\bar{q}}\left[p\frac{\partial^2\phi_3}{\partial p\partial\gamma} + \{\frac{1}{\bar{p}^2} - 2(1-\nu)\}\frac{\partial\phi_3}{\partial\gamma}\right] \tag{2.37}$$

$$+ \; \frac{hp}{c\bar{q}\bar{p}^2}\frac{\partial^2\lambda_3}{\partial\gamma^2} + ch^3\bar{q}\bar{p}^2\left[q\frac{\partial^2\lambda_3}{\partial q\partial p} - p\frac{\partial^2\lambda_3}{\partial p^2} + c^2h^2\{2qp\frac{\partial\lambda_3}{\partial q} - (q^2+p^2)\frac{\partial\lambda_3}{\partial p}\}\right]$$

$$\sigma_{\gamma\alpha} = \frac{h}{c\bar{p}}\left[\frac{\partial^2\phi_0}{\partial\gamma\partial q} - \frac{q}{\bar{q}^2}\frac{\partial\varphi_0}{\partial\gamma}\right]$$

$$+ \; h\frac{p}{\bar{p}}\left[q\frac{\partial^2\phi_3}{\partial\gamma\partial q} - \{\frac{1}{\bar{q}^2} + 2(1-\nu)\}\frac{\partial\varphi_3}{\partial\gamma}\right] \tag{2.38}$$

$$+ \; \frac{hq}{c\bar{q}^2\bar{p}}\frac{\partial^2\lambda_3}{\partial\gamma^2} + ch^3\bar{q}^2\bar{p}\left[p\frac{\partial^2\lambda_3}{\partial q\partial p} - q\frac{\partial^2\lambda_3}{\partial q^2} + c^2h^2\{(q^2+p^2)\frac{\partial\lambda_3}{\partial q} - 2qp\frac{\partial\lambda_3}{\partial p}\}\right]$$

2.3 BOUNDARY CONDITIONS

The boundary conditions for the spheroidal inclusion having freely sliding surface and subjected to the uniform shear stress at infinity have been given in Cartesian coordinate system by the equations (1.1)-(1.4). In the spheroidal coordinate system which we use in this thesis they are in the following forms.

The boundary conditions at the interface $\alpha = \alpha_0$ (or $q=q_0$) become

$$u_\alpha\big|_{\alpha=\alpha_0} = \bar{u}_\alpha\big|_{\alpha=\alpha_0} \tag{2.39}$$

$$\sigma_\alpha\big|_{\alpha=\alpha_0} = \bar{\sigma}_\alpha\big|_{\alpha=\alpha_0} \tag{2.40}$$

$$\sigma_{\alpha\beta}\big|_{\alpha=\alpha_0} = 0 \tag{2.41}$$

$$\bar{\sigma}_{\alpha\beta}\big|_{\alpha=\alpha_0} = 0 \tag{2.42}$$

$$\sigma_{\gamma\alpha}\big|_{\alpha=\alpha_0} = 0 \tag{2.43}$$

$$\bar{\sigma}_{\gamma\alpha}\big|_{\alpha=\alpha_0} = 0 \tag{2.44}$$

The stress boundary conditions at infinity,when pure shear loading is applied, are

$$\sigma_{xz}\big|_{r\to\infty} = S_0 \tag{2.45}$$

$$\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}\big|_{r\to\infty} = 0 \tag{2.46}$$

where r=$\sqrt{x^2+y^2}$, and S_0 is a constant. The bar above a letter refers to the inclusion.

2.4 HARMONIC POTENTIALS

Having the formulas for stresses and displacements and the boundary conditions set up, we now need to find appropriate potential functions so that the stresses and displacements obtained from the potential functions by the equations(2.30)-(2.38) can satisfy the boundary conditions(2.39)-(2.46).

In view of the superposition principle of elasticity, we consider the stress and displacement field in the matrix as a sum of two parts,

$$u_i = \tilde{u}_i + u_i^0 \tag{2.47}$$

$$\sigma_{ij} = \tilde{\sigma}_{ij} + \sigma_{ij}^0 \tag{2.48}$$

where the term with the superscript *0* is the undisturbed field in absence of the inhomogeneity caused by only the applied loading at infinity, and the other term is due to the disturbance caused by the spheroidal inhomogeneity.

The main concern in this investigation is to choose the proper harmonic potential function in the equations (2.30)-(2.38) for the stress and displacement field due to the disturbance of the prolate spheroidal inhomogeneity. Generally, the harmonic functions which satisfy the displacement equations of equilibrium in the form of prolate spheroidal coordinates are [2]

$$\begin{bmatrix} P_n(q) \\ Q_n(q) \end{bmatrix} \begin{bmatrix} P_n(p) \\ Q_n(p) \end{bmatrix} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \tag{2.49}$$

or

$$\begin{bmatrix} P_n^m(q) \\ Q_n^m(q) \end{bmatrix} \begin{bmatrix} P_n^m(p) \\ Q_n^m(p) \end{bmatrix} \begin{bmatrix} \cos m\gamma \\ \sin m\gamma \end{bmatrix} \tag{2.50}$$

where $P_n(p)$, $P_n(q)$ and $Q_n(p)$, $Q_n(q)$ are Legendre's functions of the first and the second kind, and $P_n^m(p), P_n^m(q)$ and $Q_n^m(p), Q_n^m(q)$ are the associated Legendre's functions of the first and the second kind of order *m*. They are defined as follows

$$P_n(p) = \frac{1}{2^n n!} \frac{d^n}{dp^n} (p^2 - 1)^n \tag{2.51}$$

$$P_n(q) = \frac{1}{2^n n!} \frac{d^n}{dp^n} (q^2 - 1)^n \tag{2.52}$$

$$Q_n(p) = \frac{1}{2}P_n(p)\log\frac{1+p}{1-p} - W_{n-1}(p) \tag{2.53}$$

$$Q_n(q) = \frac{1}{2}P_n(q)\log\frac{q+1}{q-1} - W_{n-1}(q) \tag{2.54}$$

$$P_n^m(p) = (1-p^2)^{m/2}\frac{d^m}{dp^m}P_n(p) \tag{2.55}$$

$$P_n^m(q) = (q^2-1)^{m/2}\frac{d^m}{dq^m}P_n(q) \tag{2.56}$$

$$Q_n^m(p) = (1-p^2)^{m/2}\frac{d^m}{dp^m}Q_n(p) \tag{2.57}$$

$$Q_n^m(q) = (q^2-1)^{m/2}\frac{d^m}{dq^m}Q_n(q) \tag{2.58}$$

$$W_{n-1}(z) = \sum_{r=1}^{n}\frac{1}{r}P_{r-1}(z)P_{n-r}(z) \qquad z=p,q \tag{2.59}$$

The definition (2.55) has the other form

$$P_n^m(p) = (1-p^2)^{-m/2}\int_1^p\int_1^p\cdots\cdots\int_1^p P_n(p)\,(dp)^m \tag{2.60}$$

These two definitions have a certain relation each other. One of them equals to a constant times the other.

Considering the feature of symmetry and antisymmetry of the geometry and loading of the problem and examining the stress and displacement resulting from the formulas (2.30)-(2.38), it is suitable to choose m=1 and to choose $\cos\gamma$ for the potentials ϕ_0 and ϕ_3, and $\sin\gamma$ for λ_3, respectively. Furthermore, those Legendre functions which will lead to the singularity of stresses at the origin of the coordinate system (corresponding to the center of the inhomogeneity, q=1) should be excluded from the candidates of potentials for the inclusion, as well as the Legendre functions which will give the infinite stress at infinity (the corresponding $q \to \infty$) must be omitted from the possible potentials for the matrix. As a result, harmonic functions $Q_n^1(q)$ are sustained for the matrix, $P_n^1(q)$ for the inclusion, and $P_n^1(p)$ for both of them. Finally, the following linear combination of selected harmonic functions is chosen as the potential function for the matrix and inclusion respectively,

a) for the matrix region ($\alpha \geq \alpha_0$), or in D-Ω

$$\phi_0 = S_0 \sum_{n=0}^{\infty} A_n Q_n^1(q) P_n^1(p) \cos\gamma \tag{2.61}$$

$$\phi_3 = S_0 \sum_{n=1}^{\infty} B_n Q_n^1(q) P_n^1(p) \cos\gamma \tag{2.62}$$

$$\lambda_3 = S_0 \sum_{n=0}^{\infty} C_n Q_n^1(q) P_n^1(p) \sin\gamma \tag{2.63}$$

b) for the inclusion region ($\alpha \leq \alpha_0$), or in Ω

$$\phi_0 = S_0 \sum_{n=1}^{\infty} \bar{A}_n P_n^1(q) P_n^1(p) \cos\gamma \tag{2.64}$$

$$\phi_3 = S_0 \sum_{n=1}^{\infty} \bar{B}_n P_n^1(q) P_n^1(p) \cos\gamma \tag{2.65}$$

$$\lambda_3 = S_0 \sum_{n=1}^{\infty} \bar{C}_n P_n^1(q) P_n^1(p) \sin\gamma \tag{2.66}$$

where $P_n^1(p)$ is defined by expression (2.55) for any value of n except 0 when the expression(2.60) is used, because $P_0^1(p)$ with the former definition vanishes, and $A_n, B_n, C_n, \bar{A}_n, \bar{B}_n$ and $\bar{C}_n$ are unknown constants which will be found by substituting the potential function into the boundary conditions.

2.5 UNDISTURBED FIELD

The undisturbed stress and displacement field is the field in the body without the inclusion inside. The undisturbed field of the infinite body under the pure shear loading $\sigma_{zx} = S_0$ at infinity in prolate spheroidal coordinate system is

$$2Gu_\alpha^0 = S_0 c^2 h (2q^2 - 1) p\bar{p}\cos\gamma \tag{2.67}$$

$$2Gu_\beta^0 = S_0 c^2 h q\bar{q} (2p^2 - 1) \cos\gamma \tag{2.68}$$

$$2Gu_\gamma^0 = -S_0 cqp \sin\gamma \tag{2.69}$$

$$\sigma_\alpha^0 = S_0 2c^2 h^2 q\bar{q} p\bar{p} \cos\gamma \tag{2.70}$$

$$\sigma_\beta^0 = -S_0 2c^2 h^2 q\bar{q} p\bar{p} \cos\gamma \tag{2.71}$$

$$\sigma_\gamma^0 = 0 \tag{2.72}$$

$$\sigma_{\alpha\beta}^0 = S_0 (2c^2 h^2 q^2 \bar{q}^2 - 2q^2 + 1) \cos\gamma \tag{2.73}$$

$$\sigma_{\beta\gamma}^0 = S_0 chq\bar{p} \sin\gamma \tag{2.74}$$

$$\sigma_{\gamma\alpha}^0 = -S_0 chp\bar{q} \sin\gamma \tag{2.75}$$

where the superscript 0 refers to the undisturbed field.

In order to use these expressions in the boundary conditions, we express equations(2.67), (2.70), (2.73) and (2.75) in terms of Legendre's functions as follows

$$\frac{2Gu_\alpha^0}{S_0 h\cos\gamma} = \frac{1}{3}c^2(2q^2-1)P_2^1(p) \tag{2.76}$$

$$\frac{\sigma_\alpha^0}{S_0 c^2 h^4\cos\gamma} = 2c^2 q\bar{q}\left(\frac{1}{3}q^2-\frac{1}{7}\right)P_2^1(p) - \frac{4}{35}c^2 q\bar{q}P_4^1(p) \tag{2.77}$$

$$\frac{\sigma_{\alpha\beta}^0\bar{p}}{S_0 c^2 h^4\cos\gamma} = \frac{3}{35}c^2(-7q^4-2q^2+1)P_1^1(p) + \frac{4}{45}c^2(3q^4-2q^2+1)P_3^1(p)$$

$$- \frac{8}{315}c^2(2q^2-1)P_5^1(p) \tag{2.78}$$

$$\frac{\sigma_{\gamma\alpha}^0\bar{p}}{S_0 c h^5\sin\gamma} = -c^4\bar{q}\left(\frac{1}{3}q^4-\frac{2}{7}q^2+\frac{5}{63}\right)P_2^1(p) - c^4\bar{q}\left(-\frac{4}{35}q^2+\frac{4}{77}\right)P_4^1(p)$$

$$- \frac{8}{693}c^4\bar{q}P_6^1(p) \tag{2.79}$$

2.6 SOLUTION

Substituting the harmonic potentials(2.61)-(2.66) in section 2.4 into the expressions for stresses and displacements(2.30)-(2.38) in section 2.3, we obtain the stress and displacement field due to the disturbance of the inhomogeneity for the matrix and the inhomogeneity. Then these fields are combined with the undisturbed ones, given by equations (2.76)-(2.79), to form the total stress and displacement fields which satisfy the equations of equilibrium and the boundary conditions.

The undisturbed stresses (2.70)-(2.75) in the section 2.5, caused by the applied loading at infinity, of course, satisfy the boundary conditions at infinity. The stresses due to the inhomogeneity disturbance derived from potentials (2.61)-(2.63) vanish at infinity. So the sum of the two parts of stresses satisfies the boundary conditions at infinity.

In order to satisfy the sliding boundary conditions at the interface, we substitute the above-mentioned total stress and displacement field for the matrix and the field for the inhomogeneity into the the boundary condition equations (2.39)-(2.44). Before the equation system is solved, the following has to be done:

1. For solving the boundary condition equation system at the interface, all terms containing the variable p should be transformed in the forms of Legendre's function $P_n^1(p)$ by using the following recursion formulas[16],

$$pP_n^1(p) = \frac{n}{2n+1}P_{n+1}^1(p) + \frac{n+1}{2n+1}P_{n-1}^1(p) \tag{2.80}$$

$$(1-p^2)\frac{d}{dp}P_n^1(p) = (n+1)pP_n^1(p) - nP_{n+1}^1(p)\,. \tag{2.81}$$

2. Of the potentials for the matrix, the two terms with A_0, C_0 cause much more complicated operation when they are transformed in the forms of Legedre's function. To simplify the calculation, the stress and displacement field due to the two terms is worked out directly. Furthermore, it is noticed that the fields brought out by the two terms respectively have singularity as $\bar{p} = 0$, however, the singularity in the combination of the two fields vanishes if the ratio of A_0 to C_0 is chosen as 2 . The combination of the two fields(appendix B) is substituted into the boundary condition equations to find the coefficient C_0 , and then A_0 is taken as $2C_0$.

3. The resulting displacement includes the rigid body motion when the displacement equation of equilibrium is used for a problem of elasticity. If two terms of the potentials represent the same rigid body motion, the system of equations will be singular. Examining the displacements resulting from the chosen potentials(2.61)-(2.66), it is found

that $\bar{A}_1$ and $\bar{C}_1$correspond to the same rigid body translation of the inclusion in the x direction. So one of them,$\bar{C}_1$ is chosen to be zero.

Finally, the boundary condition equations at the interface are obtained in the following way.

The condition of continuity of normal displacement(2.39),

$u_\alpha|_{\alpha=\alpha_0} = \bar{u}_\alpha|_{\alpha=\alpha_0}$, becomes

$$\sum_{N=1}^{\infty} [\ \{K^A_{n,n}A_n + K^B_{n,n-1}B_{n-1} + K^B_{n,n+1}B_{n+1} + K^C_{n,n}C_n\ \}$$

$$- \frac{1}{\Gamma}\{\bar{K}^A_{n;n}\bar{A}_n + \bar{K}^B_{n,n-1}\bar{B}_{n-1} + \bar{K}^B_{n,n+1}\bar{B}_{n+1} + \bar{K}^C_{n,n}\bar{C}_n\}\ ]\ P^1_n(p) \qquad (2.82)$$

$$= -\frac{1}{3}c^2(2q_0^2 - 1)\,P^1_2(p)\,.$$

The condition of continuity of normal tractions (2.40) which can be expressed as

$\sigma_\alpha|_{\alpha=\alpha_0} = \bar{\sigma}_\alpha|_{\alpha=\alpha_0}$ becomes

$$\sum_{n=1}^{\infty} [\ \{L^A_{n,n-2}A_{n-2} + L^A_{n,n}A_n + L^A_{n,n+2}A_{n+2}$$

$$+\ L^B_{n,n-3}B_{n-3} + L^B_{n,n-1}B_{n-1} + L^B_{n,n+1}B_{n+1} + L^B_{n,n+3}B_{n+3}$$

$$+\ L^C_{n,n-2}C_{n-2} + L^C_{n,n}C_n + L^C_{n,n+2}C_{n+2}\ \}$$

$$-\{\bar{L}^A_{n,n-2}\bar{A}_{n-2} + \bar{L}^A_{n,n}\bar{A}_n + \bar{L}^A_{n,n+2}\bar{A}_{n+2}$$

$$+\ \bar{L}^B_{n,n-3}\bar{B}_{n-3} + \bar{L}^B_{n,n-l}\bar{B}_{n-1} + \bar{L}^B_{n,n+1}\bar{B}_{n+1} + \bar{L}^B_{n,n+3}\bar{B}_{n+3} \qquad (2.83)$$

$$+\bar{L}^C_{n,n-2}\bar{C}_{n-2} + \bar{L}^C_{n,n}\bar{C}_n + \bar{L}^C_{n,n+2}\bar{C}_{n+2}\ \}\ ]\ P^1_n(p)$$

$$+ C_0 \frac{2}{3\bar{q}_0} P_2^1(p)$$

$$= -2c^2 q_0 \bar{q}_0 \left(\frac{1}{3} q_0^2 - \frac{1}{7}\right) P_2^1(p) + \frac{4}{35} c^2 q_0 \bar{q}_0 P_4^1(P)$$

The conditions of vanishing shear traction(2.41)-(2.44) which can be expressed as

$$\sigma_{\alpha\beta}\big|_{\alpha=\alpha_0} = 0, \quad \bar{\sigma}_{\alpha\beta}\big|_{\alpha=\alpha_0} = 0, \quad \sigma_{\gamma\alpha}\big|_{\alpha=\alpha_0} = 0 \text{ and } \bar{\sigma}_{\gamma\alpha}\big|_{\alpha=\alpha_0} = 0$$

become

$$\sum_{n=1}^{\infty} [S^A_{n,n-3} A_{n-3} + S^A_{n,n-1} A_{n-1} + S^A_{n,n+1} A_{n+1} + S^A_{n,n+3} A_{n+3}$$

$$+ \quad S^B_{n,n-4} B_{n-4} + S^B_{n,n-2} B_{n-2} + S^B_{n,n} B_n + S^B_{n,n+2} B_{n+2} + S^B_{n,n+4} B_{n+4}$$

$$+ \quad S^C_{n,n-3} C_{n-3} + S^C_{n,n-1} C_{n-1} + S^C_{n,n+1} C_{n+1} + S^C_{n,n+3} C_{n+3}] P^1_n(p) \tag{2.84}$$

$$+ C_0 \left[\{-3q_0^2 + 2\tfrac{1}{5}\} \frac{q_0}{\bar{q}_0^2} P_1^1(p) + \frac{2}{15} \frac{q_0}{\bar{q}_0^2} P_3^1(p) \right]$$

$$= -\frac{3}{35} c^2 (-7q_0^4 - 2q_0^2 + 1) P_1^1(p) - \frac{4}{45} c^2 (3q_0{}^4 - 2q_0^2 + 1) P_3^1(p)$$

$$+ \frac{8}{315} c^2 (2q_0^2 - 1) P_5^1(p)$$

$$\sum_{n=1}^{\infty} [\bar{S}^A_{n,n-3} \bar{A}_{n-3} + \bar{S}^A_{n,n-1} \bar{A}_{n-1} + \bar{S}^A_{n,n+1} \bar{A}_{n+1} + \bar{S}^A_{n,n+3} \bar{A}_{n+3}$$

$$+ \quad \bar{S}^B_{n,n-4} \bar{B}_{n-4} + \bar{S}^B_{n,n-2} \bar{B}_{n-2} + \bar{S}^B_{n,n} \bar{B}_n + \bar{S}^B_{n,n+2} \bar{B}_{n+2} + \bar{S}^B_{n,n+4} \bar{B}_{n+4} \tag{2.85}$$

$$+ \quad \bar{S}^C_{n,n-3} \bar{C}_{n-3} + \bar{S}^C_{n,n-1} \bar{C}_{n-1} + \bar{S}^C_{n,n+1} \bar{C}_{n+1} + \bar{S}^C_{n,n+3} \bar{C}_{n+3}] P^1_n(p) \quad = 0$$

$$\sum_{n=1}^{\infty} [T^A_{n,n-4}A_{n-4} + T^A_{n,n-2}A_{n-2} + T^A_{n,n}A_n + T^A_{n,n+2}A_{n+2} + T^A_{n,n+4}A_{n+4}$$

$$+ \quad T^B_{n,n-5}B_{n-5} + T^B_{n,n-3}B_{n-3} + T^B_{n,n-1}B_{n-1}$$

$$+ \quad T^B_{n,n+1}B_{n+1} + T^B_{n,n+3}B_{n+3} + T^B_{n,n+5}B_{n+5}$$

$$+ \quad T^C_{n,n-4}C_{n-4} + T^C_{n,n-2}C_{n-2} + T^C_{n,n}C_n + T^C_{n,n+2}C_{n+2} + T^C_{n,n+4}C_{n+4}]\, P^1_n(p)$$

$$+C_0\left[\{\frac{1}{3}\frac{q_0^3}{\bar{q}_0} + \frac{2}{3}\bar{q}_0 q_0 - \frac{1}{7}\frac{q_0}{\bar{q}_0}\}\, c^2 P^1_2(p) - \frac{2}{35}c^2\frac{q_0}{\bar{q}_0}P^1_4(p) \right]$$

$$= (\frac{1}{3}q_0^4 - \frac{2}{7}q_0^2 + \frac{5}{63})\, c^4\bar{q}_0 P^1_2(p) + (-\frac{4}{35}q_0^2 + \frac{4}{77})\, c^4\bar{q}_0 P^1_4(p) + \frac{8}{693}c^4\bar{q}_0 P^1_6(P) \quad (2.86)$$

and

$$\sum_{n=1}^{\infty} [\bar{T}^A_{n,n-4}\bar{A}_{n-4} + \bar{T}^A_{n,n-2}\bar{A}_{n-2} + \bar{T}^A_{n,n}\bar{A}_n + \bar{T}^A_{n,n+2}\bar{A}_{n+2} + \bar{T}^A_{n,n+4}\bar{A}_{n+4}$$

$$+ \quad \bar{T}^B_{n,n-5}\bar{B}_{n-5} + \bar{T}^B_{n,n-3}\bar{B}_{n-3} + \bar{T}^B_{n,n-1}\bar{B}_{n-1}$$

$$+ \quad \bar{T}^B_{n,n+1}\bar{B}_{n+1} + \bar{T}^B_{n,n+3}\bar{B}_{n+3} + \bar{T}^B_{n,n+5}\bar{B}_{n+5}$$

$$+ \quad \bar{T}^C_{n,n-4}\bar{C}_{n-4} + \bar{T}^C_{n,n-2}\bar{C}_{n-2} + \bar{T}^C_{n,n}\bar{C}_n + \bar{T}^C_{n,n+2}\bar{C}_{n+2} + \bar{T}^C_{n,n+4}\bar{C}_{n+4}]\, P^1_n(p)$$

$$= 0 \quad (2.87)$$

where $\Gamma = \bar{G}/G$, $q_0 = \cosh\alpha_0$, $\bar{q}_0 = \sinh\alpha_0$. All the coefficients,

$K^A_{i,j}$, $K^B_{i,j}$, $K^C_{i,j}$, $L^A_{i,j}$, $L^B_{i,j}$, $L^C_{i,j}$, $S^A_{i,j}$, $S^B_{i,j}$, $S^C_{i,j}$, $T^A_{i,j}$, $T^B_{i,j}$, $T^C_{i,j}$

are the functions of $Q_n^1(q_0)$ and ν as shown in the appendix C. All the coefficients with a bar can be obtained from the corresponding coefficients without the bar by replacing $Q_n^1(q_0)$ and ν by $P_n^1(q_0)$ and $\bar{\nu}$, respectively.

Equating the coefficients of $P_n^1(p)$, for the positive integer n, in both sides of each of the boundary condition equations (2.82)-(2.87), we get an infinite system of linear algebraic equations for the unknowns, $A_n, B_n, C_n, \bar{A}_n, \bar{B}_n$ and $\bar{C}_n$.

For solving the system of equations, it is convenient to use the following recursion formulas which are derived from [16],

$$Q_{n+1}^1(q) = \frac{1}{n}[(2n+1)qQ_n^1(q) - (n+1)Q_{n-1}^1(q)] \tag{2.88}$$

$$Q_n^{1\prime}(q) = \frac{1}{q^2-1}[nQ_{n+1}^1(q) - (n+1)qQ_n^1(q)] \tag{2.89}$$

$$Q_n^{1\prime\prime}(q) = \frac{1}{1-q^2}\left[2qQ_n^{1\prime}(q) - \{n(n+1) - \frac{1}{1-q^2}\}Q_n^1(q)\right]. \tag{2.90}$$

The recursion formulas for $P_{n+1}^1(q), P_n^{1\prime}(q), P_n^{1\prime\prime}(q)$ and $P_{n+1}^1(p), P_n^{1\prime}(p), P_n^{1\prime\prime}(p)$ have the same forms as those for $Q_{n+1}^1(q), Q_n^{1\prime}(q), Q_n^{1\prime\prime}(q)$, equations (2.88)-(2.90).

In the numerical calculations, the infinite system of equations is truncated with n=N in such a way that the number of equations equals to the number of unknowns. By using the exclusive program , the coefficients $A_n, B_n, C_n, \bar{A}_n, \bar{B}_n$ and $\bar{C}_n$ are evaluated, so that the complete potentials are obtained. Then it is easy to find the stress and displacement fields anywhere in the inclusion and in the matrix by substituting the potentials into the expressions for the stress and displacement (2.30)-(2.38).

3. RESULTS AND DISCUSSION

1. The unknowns in the boundary conditions

Solving the system of equations (2.82)-(2.87) for the boundary conditions at the interface, we obtained all the unknown coefficients A_n , B_n , C_n , $\overline{A}_n$, $\overline{B}_n$ and $\overline{C}_n$. We confirm that A_n , $\overline{A}_n$, C_n and $\overline{C}_n$ are zero when n is an odd number, and B_n and $\overline{B}_n$ are zero when n is an even number, which can be inferred from the symmetry of the problem.

The second very important observation is that the non-zero terms of the coefficients are A_0, A_2 , B_1 , C_0 , C_2 , $\overline{A}_2$ and $\overline{B}_1$ only. In view of this, we conclude that it is possible to get a closed form analytical solution by setting up a finite series of potentials, although it is known that the infinite series of potentials should be used in order to obtain the stress field of a prolate spheroidal sliding inclusion under other kinds of loading [13-14].

2. The stress field

A very interesting conclusion of our work is that the stress field in the prolate spheroidal inhomogeneity, with the interface free to slip under shear loading σ_{zx} at infinity, vanishes so that the inhomogeneity behaves just like a cavity. As to the distribution of stress field, it does not matter whether the inclusion is perfectly bonded at its interface or is allowed to slip freely along the interface, if we have the special case of a cavity. In order to confirm our solution, we used an analytical method for the prolate spheroidal inclusion perfectly bonded at the interface and took the shear modulus, $\overline{G} = 0$. We obtained the same stress and displacement field in the matrix as that for the sliding inclusion and $\overline{G} \neq 0$. Furthermore, decreasing the shear modulus $\overline{G}$ continuously towards the limit, $\overline{G} = 0$, for the prolate spheroidal sliding inhomogeneity, the same stress and displacement fields were obtained once again.

Since the condition of the prolate spheroidal sliding inclusion under shear loading σ_{zx} is equivalent to that of the perfectly bonded inclusion with shear modulus $\overline{G} = 0$ and the field of elasticity of the latter is obtained by finite series of potentials, it is reasonable to attempt to derive the field of the former also by finite series of potentials.

Mura and Furuhashi[9] reported the zero stress field in the inclusion for the similar case. They found that when an ellipsoidal inclusion, which is free to slip along the interface, undergoes a shear eigenstrain and the interface can not sustain any shear traction , the stress field vanishes everywhere in the inclusion. Kouris, et al.[12] considered a problem of elliptic inhomogeneity under a uniform shear stress applied at infinity and obtained the solution which is the same as that of elliptic hole in the matrix under shear. As the extension of this 2-dimensional problem to the 3-dimensions, our solution of zero stress field in the inhomogeneity is expected.

It should be noted that the stress field in the prolate spheroidal sliding inclusion does not vanish if the shear stress σ_{xy} instead of σ_{zx} is applied at infinity. The field is given by Jasiuk[14].

3. The anomaly of solution

The solution for the stress field can not be obtained from our formulas in this thesis when the spheroidal inclusion goes to a spherical one, because the two semi-axes have the same lengths, a=b, which leads to the focal length c=0 and the related singularity. However, Ghahremani[8] solved for the stress field of the sliding spherical inhomogeneity under the uniform tension applied at infinity by using the Papkovich's approach. By means of the superposition principle for elasticity, the stress field in the sliding inhomogeneity subjected the shear stress at infinity could be obtained easily. This way, we find that the stress field does not vanish when the inhomogeneity is in the shape of sphere. In conclusion, the solution for a spherical inhomogeneity can not be obtained from the limit of spheroidal inhomogeneity for sliding case. This is called the anomaly of solution.

The anomaly was reexamined by Kouris et al.[12] for the two dimensional case. As a result, they showed that the plate with a freely sliding circular inclusion under shear has nonzero stress in the inclusion which can be obtained from the solution given by Muskhelishvili[17]. When investigating the ellipsoidal sliding inclusion which undergoes a uniform shear eigenstrain , Mura and Furuhashi[9] pointed that in the case of degeneracy of ellipsoids to the spheroid and sphere, non-vanishing stress field is left in the material after slight relaxation by the interface sliding.

It should be noted that the anomaly does not occur for the case of the inclusion perfectly bonded in which the solution for sphere can be obtained from the limit of the ellipsoid(same for the sliding inclusion in two dimension)[9]. Once more, the anomaly does not come for the case of the spheroidal sliding inclusion under shear in the plane parallel to the xy plane as shown in the paper by Jasiuk[14].

4. The deformation

The prolate spheroidal inhomogeneity which is free to slip along its interface has only the displacements corresponding to the rigid body motion, the rotation of the inhomogeneity around the axis y, when the uniform shear stress σ_{zx} is applied at infinity. This means the spheroid keeps its shape under the shear loading considered in our paper. It can be observed from the harmonic potentials used in the paper. In particular, it is more convenient to transform the potential functions into Cartesian coordinate system. In that case, the displacements are

$$u_x = \omega z \tag{3.1}$$

$$u_z = -\omega x. \tag{3.2}$$

where ω is a constant. The displacements represent the rigid body rotation around the y-axis.

The similar situation was found by Mura and Furuhashi[9] for the ellipsoidal sliding inclusion undergoing a uniform shear eigenstrain. They pointed that there exists a shear deformation which transforms an ellipsoid into the identical ellipsoid without changing its orientation. It is not, however, the case of sphere that is always deformed to an ellipsoid by shear. Jasiuk[15] showed a figure for the configuration before and after deformation of the elliptical sliding inclusion undergoing a uniform eigenstrain. Although above mentioned inclusions sustain their shapes under the loading, the difference is that there is no deformation in our inhomogeneity.

5. The potential

The Boussinesq potentials approach was used and the spheroidal harmonic functions in terms of Legendre function were taken as the potentials. It is a suitable way to investigate the spheroidal inclusion problems. Although we got the undisturbed stress field by the harmonic function $\phi_0 = S_0 xy$ expressed in Cartesian system for convenience, it can be transformed to the Legendre function in the spheroidal system,

$$\phi_0 = S_0 c^2 P_2^1(q) P_2^1(p) . \tag{3.3}$$

The same method was successfully used by Edwards[5] for the spheroidal inclusion bonded perfectly and by Jasiuk[14] for the spheroidal sliding inclusion(Papkovich approach of potentials was also used).

6. The eigenstrain problem

As the mentioned at the beginning of the thesis, there are two types of the inclusion problems, the inhomogeneity problem and the eigenstrain problem which can be solved mathematically in the same way.

Here we discuss the eigenstrain problem.

We consider an isotropic and infinitely extended elastic body, containing a prolate spheroidal subdomain that may have elastic constants different from those of the matrix. The configuration is the same as that shown in Fig.2. The problem is to solve the stress fields caused by the uniform shear eigenstrain, ε_{zx}^0 in the inclusion.

The potentials are chosen as the same as those for the inhomogeneity in formulas (2.61)-(2.66) except that the constant S_0 is taken as $2G\varepsilon_{zx}^0$

The boundary conditions at infinity are that all stresses vanish

$$\sigma_{ij} = 0. \tag{3.4}$$

The boundary condition at the interface are the same as those represented in equations(2.39)-(2.44). But, for the matrix there is no more undisturbed stress field caused by the applied shear stress at infinity, equations(2.67)-(2.75), and the displacements in the inclusion will be the sum of two parts. If the inclusion is allowed to deform freely without the matrix constrain, then the displacement is $\bar{u}^0$ which can be derived from the given eigenstrain and the strain-displacement relation as follows

$$\frac{2G\bar{u}_\alpha^0}{S_0 h\cos\gamma} = \frac{1}{3}c^2(2q^2-1)P_2^1(p)\,. \tag{3.5}$$

The corresponding displacements of elasticity, $\bar{u}^e$ in the inclusion can be found by substituting the potentials into equation(2.30)-(2.32). The total displacement is the sum of $\bar{u}^0$ and $\bar{u}^e$,

$$\bar{u}_i = \bar{u}_i^0 + \bar{u}_i^e \tag{3.6}$$

By using strain-displacement equations and Hooke's law, the stresses can be obtained in the forms of the equations(2.33)-(2.38).

Following the same route as for the inhomogeneity problem, we can solve the eigenstrain problem. The boundary conditions at infinity are satisfied automatically. By substituting the displacements and stresses into the boundary conditions at the interface, the system of equations similar to (2.82)-(2.87) is obtained.

The condition of continuity of normal displacement,

$u_\alpha\big|_{\alpha=\alpha_0} = \bar{u}_\alpha\big|_{\alpha=\alpha_0}$, becomes

$$\sum_{N=1}^{\infty}[\{K_{n,n}^A A_n + K_{n,n-1}^B B_{n-1} + K_{n,n+1}^B B_{n+1} + K_{n,n}^C C_n\}$$

$$- \frac{1}{\Gamma} \{ \bar{K}^A_{n,n} \bar{A}_n + \bar{K}^B_{n,n-1} \bar{B}_{n-1} + \bar{K}^B_{n,n+1} \bar{B}_{n+1} + \bar{K}^C_{n,n} \bar{C}_n \} \;] P^1_n(p) \tag{3.7}$$

$$= \frac{1}{3} c^2 (2q_0^2 - 1) P^1_2(p) .$$

The condition of continuity of normal tractions which can be expressed as

$\sigma_\alpha\big|_{\alpha = \alpha_0} = \bar{\sigma}_\alpha\big|_{\alpha = \alpha_0}$ becomes

$$\begin{aligned}
&\sum_{n=1}^{\infty} [\; \{ L^A_{n,n-2} A_{n-2} + L^A_{n,n} A_n + L^A_{n,n+2} A_{n+2} \\
&+ \; L^B_{n,n-3} B_{n-3} + L^B_{n,n-1} B_{n-1} + L^B_{n,n+1} B_{n+1} + L^B_{n,n+3} B_{n+3} \\
&+ \; L^C_{n,n-2} C_{n-2} + L^C_{n,n} C_n + L^C_{n,n+2} C_{n+2} \; \} \\
&- \{ \bar{L}^A_{n,n-2} \bar{A}_{n-2} + \bar{L}^A_{n,n} \bar{A}_n + \bar{L}^A_{n,n+2} \bar{A}_{n+2} \\
&+ \; \bar{L}^B_{n,n-3} \bar{B}_{n-3} + \bar{L}^B_{n,n-1} \bar{B}_{n-1} + \bar{L}^B_{n,n+1} \bar{B}_{n+1} + \bar{L}^B_{n,n+3} \bar{B}_{n+3} \\
&+ \; \bar{L}^C_{n,n-2} \bar{C}_{n-2} + \bar{L}^C_{n,n} \bar{C}_n + \bar{L}^C_{n,n+2} \bar{C}_{n+2} \; \} \;] P^1_n(p) \\
&+ C_0 \frac{2}{3\bar{q}_0} P^1_2(p) \\
&= 0
\end{aligned} \tag{3.8}$$

The conditions of vanishing shear traction which can be expressed as

$\sigma_{\alpha\beta}\big|_{\alpha = \alpha_0} = 0 , \quad \bar{\sigma}_{\alpha\beta}\big|_{\alpha = \alpha_0} = 0 , \quad \sigma_{\gamma\alpha}\big|_{\alpha = \alpha_0} = 0$ and $\bar{\sigma}_{\gamma\alpha}\big|_{\alpha = \alpha_0} = 0$

become

$$\sum_{n=1}^{\infty} [S^A_{n,n-3}A_{n-3} + S^A_{n,n-1}A_{n-1} + S^A_{n,n+1}A_{n+1} + S^A_{n,n+3}A_{n+3}$$

$$+ \quad S^B_{n,n-4}B_{n-4} + S^B_{n,n-2}B_{n-2} + S^B_{n,n}B_n + S^B_{n,n+2}B_{n+2} + S^B_{n,n+4}B_{n+4}$$

$$+ \quad S^C_{n,n-3}C_{n-3} + S^C_{n,n-1}C_{n-1} + S^C_{n,n+1}C_{n+1} + S^C_{n,n+3}C_{n+3}\,]\,P^1_n(p) \tag{3.9}$$

$$+C_0 \quad \left[\{-3q_0^2 + 2\frac{1}{5}\} \frac{q_0}{\bar{q}_0^2} P^1_1(p) + \frac{2}{15}\frac{q_0}{\bar{q}_0^2} P^1_3(p) \right]$$

$$= 0$$

$$\sum_{n=1}^{\infty} [\bar{S}^A_{n,n-3}\bar{A}_{n-3} + \bar{S}^A_{n,n-1}\bar{A}_{n-1} + \bar{S}^A_{n,n+1}\bar{A}_{n+1} + \bar{S}^A_{n,n+3}\bar{A}_{n+3}$$

$$+ \quad \bar{S}^B_{n,n-4}\bar{B}_{n-4} + \bar{S}^B_{n,n-2}\bar{B}_{n-2} + \bar{S}^B_{n,n}\bar{B}_n + \bar{S}^B_{n,n+2}\bar{B}_{n+2} + \bar{S}^B_{n,n+4}\bar{B}_{n+4} \tag{3.10}$$

$$+ \quad \bar{S}^C_{n,n-3}\bar{C}_{n-3} + \bar{S}^C_{n,n-1}\bar{C}_{n-1} + \bar{S}^C_{n,n+1}\bar{C}_{n+1} + \bar{S}^C_{n,n+3}\bar{C}_{n+3}\,]\,P^1_n(p)$$

$$=0$$

$$\sum_{n=1}^{\infty} [T^A_{n,n-4}A_{n-4} + T^A_{n,n-2}A_{n-2} + T^A_{n,n}A_n + T^A_{n,n+2}A_{n+2} + T^A_{n,n+4}A_{n+4}$$

$$+ \quad T^B_{n,n-5}B_{n-5} + T^B_{n,n-3}B_{n-3} + T^B_{n,n-1}B_{n-1}$$

$$+ \quad T^B{}_{n,n+1}B_{n+1} + T^B_{n,n+3}B_{n+3} + T^B_{n,n+5}B_{n+5}$$

$$+ \quad T^C_{n,n-4}C_{n-4} + T^C_{n,n-2}C_{n-2} + T^C_{n,n}C_n + T^C_{n,n+2}C_{n+2} + T^C_{n,n+4}C_{n+4}\,]\,P^1_n(p)$$

$$+C_0 \left[\{ \frac{1}{3}\frac{q_0^3}{\bar{q}_0} + \frac{2}{3}\bar{q}_0 q_0 - \frac{1}{7}\frac{q_0}{\bar{q}_0} \} c^2 P_2^1(p) - \frac{2}{35} c^2 \frac{q_0}{\bar{q}_0} P_4^1(p) \right] \tag{3.11}$$

$=0$

and

$$\sum_{n=1}^{\infty} [\bar{T}^A_{n,n-4}\bar{A}_{n-4} + \bar{T}^A_{n,n-2}\bar{A}_{n-2} + \bar{T}^A_{n,n}\bar{A}_n + \bar{T}^A_{n,n+2}\bar{A}_{n+2} + \bar{T}^A_{n,n+4}\bar{A}_{n+4}$$

$$+ \bar{T}^B_{n,n-5}\bar{B}_{n-5} + \bar{T}^B_{n,n-3}\bar{B}_{n-3} + \bar{T}^B_{n,n-1}\bar{B}_{n-1}$$

$$+ \bar{T}^B_{n,n+1}\bar{B}_{n+1} + \bar{T}^B_{n,n+3}\bar{B}_{n+3} + \bar{T}^B_{n,n+5}\bar{B}_{n+5}$$

$$+ \bar{T}^C_{n,n-4}\bar{C}_{n-4} + \bar{T}^C_{n,n-2}\bar{C}_{n-2} + \bar{T}^C_{n,n}\bar{C}_n + \bar{T}^C_{n,n+2}\bar{C}_{n+2} + \bar{T}^C_{n,n+4}\bar{C}_{n+4}]\, P^1_n(p)$$

$$= 0 \tag{3.12}$$

where $\Gamma = \bar{G}/G$, $q_0 = \cosh\alpha_0$, $\bar{q}_0 = \sinh\alpha_0$. All the coefficients,

$K^A_{i,j}$, $K^B_{i,j}$, $K^C_{i,j}$, $L^A_{i,j}$, $L^B_{i,j}$, $L^C_{i,j}$, $S^A_{i,j}$, $S^B_{i,j}$, $S^C_{i,j}$, $T^A_{i,j}$, $T^B_{i,j}$, $T^C_{i,j}$

are the functions of $Q^1_n(q_0)$ and ν as shown in the appendix C. All the coefficients with a bar can be obtained from the corresponding coefficients without the bar by replacing $Q^1_n(q_0)$ and ν by $P^1_n(q_0)$ and $\bar{\nu}$, respectively.

Equating the coefficients of $P^1_n(p)$, for the positive integer n, in both sides of each of the boundary condition equations (3.7)-(3.12), we get an infinite system of linear algebraic equations for the unknowns, A_n, B_n, C_n, $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_n$.

In the numerical calculation, the infinite system of equations is truncated with n=N in such a way that the number of equations equals to the number of unknowns. By us ing the exclusive program , the coefficients $A_n, B_n, C_n, \bar{A}_n, \bar{B}_n$ and $\bar{C}_n$ are evaluated, so that the complete potentials are obtained. Then it is easy to find the stress and displacement fields anywhere in the inclusion and in the matrix by substituting the potentials into the expressions for the stress and displacement (2.30)-(2.38). Again we find that our solution can be expressed in terms of finite series, or more specifically, only the coefficients $\bar{A}_2$ and $\bar{B}_1$ are non-zero, and the stress field in both the matrix and the inclusion vanishes. This stress field in the inclusion is consistent with the case of the inhomogeneity under remote shear loading as discussed in the section 2. After deformation, both the matrix and inclusion keep their original configuration. Furthermore, there is no strain in the matrix. Although there is a strain, the eigenstrain, in the inclusion, the inclusion keeps its shape and orientation after the deformation.

7. The closed form solution

In view of the above-mentioned reason we can find a closed form solution for both the inhomogeneity problem and the eigenstrain problem. The potential functions used for the closed form solution have the same form as those for the solution by the infinite series, but only a finite number of terms of the series is kept. Referring to the coefficients $A_n, B_n, C_n, \bar{A}_n, \bar{B}_n$ and $\bar{C}_n$ found for the infinite series approach, we chose the potentials for the closed form solution for the inhomogeneity problem as follows

a) for the matrix

$$\phi_0 = S_0 A_0 Q_0^1(q) P_0^1(p) \cos\gamma + S_0 A_2 Q_2^1(q) P_2^1(p) \cos\gamma \tag{3.13}$$

$$\phi_3 = S_0 B_1 Q_1^1(q) P_1^1(p) \cos\gamma \tag{3.14}$$

$$\lambda_3 = S_0 C_0 Q_0^1(q) P_0^1(p) \sin\gamma + S_0 C_2 Q_2^1(q) P_2^1(p) \sin\gamma \tag{3.15}$$

b) for the inclusion

$$\phi_0 = S_0 \bar{A}_2 P_2^1(q) P_2^1(p) \cos\gamma \tag{3.16}$$

$$\phi_3 = S_0 \bar{B}_1 P_1^1(q) P_1^1(p) \cos\gamma \tag{3.17}$$

where $A_0 = 2C_0$ in order to avoid the singularity of the stress.

In the same way for the solution from the infinite series of potentials(2.61)-(2.66), we get the stresses and the displacements at the interface due to these finite number of terms of potentials(3.13)-(3.17). Substituting these stresses and displacements and those of the undisturbed field into the boundary conditions(2.39)-(2.44) and solving the system of algebraic equations, we get the closed form solution for the inhomogeneity problem as follows,

$$A_0 = \frac{4}{3}c(1-\nu)B_1 \tag{3.18}$$

$$A_2 = \frac{1}{9}c(-3q^2+2-2\nu)B_1 \tag{3.19}$$

$$B_1 = \frac{cq_0}{(3q_0^2-2+\nu)Q_1(q_0) + \frac{1}{\bar{q}_0^2}(-q_0^2+2-\nu)} \tag{3.20}$$

$$C_0 = \frac{2}{3}c(1-\nu)B_1 \tag{3.21}$$

$$C_2 = \frac{1}{9}c(1-\nu)B_1 \tag{3.22}$$

$$\bar{A}_2 = \frac{1}{9}c(1-2\bar{\nu})\bar{B}_1 \tag{3.23}$$

$$\begin{aligned}\bar{B}_1 = \frac{\Gamma}{2(1-\bar{\nu})q_0}[cq_0(2q_0^2-1) \\ + \{(-6q_0^4+7q_0^2-2\nu q_0^2+1-2\nu)Q_1(q_0) \\ + (2\bar{q}_0^2+3-2\nu)\}B_1]\end{aligned} \tag{3.24}$$

This solution is consistent to that from the infinite series of potentials.

The same method is also suitable for the eigenstrain case. In that case only the potentials for the inclusion are needed in the simpler forms as follows

$$\phi_0 = S_0\bar{A}_2 P_2^1(q) P_2^1(p) \cos\gamma \tag{3.25}$$

$$\phi_3 = S_0\bar{B}_1 P_1^1(q) P_1^1(p) \cos\gamma. \tag{3.26}$$

In the same way, we get the closed form solution for the eigenstrain problem as follows,

$$\bar{A}_2 = \frac{1}{9}c(1-2\bar{\nu})\bar{B}_1 \tag{3.27}$$

$$\bar{B}_1 = \frac{c\Gamma(1-2q_0^2)}{2(1-\bar{\nu})} \tag{3.28}$$

This solution is also consistent to that from the infinite series of potentials.

Although the papers[10][13] and [14] showed that the closed form solution can not be obtained and the infinite series potentials should be used for the elliptic and prolate spheroidal inclusion problem in which the inclusion may slip along the interface and is subjected to the other kinds of loading, it is found that when the prolate spheroidal sliding inclusion undergoes the uniform shear eigenstrain ε_{zx}^0 or is subjected to the uniform shear stress σ_{zx} at infinity, a finite number of terms of potentials will give the solution. As a result, it is found that for the inhomogeneity problem the stresses in the spheroidal subdomain vanish and the spheroid rotates without deformation, and that for the eigenstrain problem the stresses in both the inclusion and the matrix vanish, and there is no deformation in the matrix, while the inclusion does deform but keeps its original shape and orientation.

APPENDIX A

The displacement and stress field

For ϕ_0

$$2Gu_\alpha = h\bar{q}\frac{\partial \phi_0}{\partial q}$$

$$2Gu_\beta = -h\bar{p}\frac{\partial \phi_0}{\partial p}$$

$$2Gu_\gamma = \frac{1}{c\bar{q}\bar{p}}\frac{\partial \phi_0}{\partial \gamma}$$

$$\sigma_\alpha = h^2\left[\bar{q}^2\frac{\partial^2 \phi_0}{\partial q^2} + c^2h^2\bar{p}^2\left(q\frac{\partial \phi_0}{\partial q} - p\frac{\partial \phi_0}{\partial p}\right)\right]$$

$$\sigma_\beta = h^2\left[\bar{p}^2\frac{\partial^2 \phi_0}{\partial p^2} + c^2h^2\bar{q}^2\left(q\frac{\partial \phi_0}{\partial q} - p\frac{\partial \phi_0}{\partial p}\right)\right]$$

$$\sigma_\gamma = \frac{1}{c^2\bar{q}^2\bar{p}^2}\frac{\partial^2 \varphi_0}{\partial \gamma^2} + h^2\left[q\frac{\partial \phi_0}{\partial q} - p\frac{\partial \phi_0}{\partial p}\right]$$

$$\sigma_{\alpha\beta} = h^2\bar{q}\bar{p}\left[-\frac{\partial^2 \phi_0}{\partial q \partial p} + c^2h^2\left(q\frac{\partial \phi_0}{\partial p} - p\frac{\partial \phi_0}{\partial q}\right)\right]$$

$$\sigma_{\beta\gamma} = -\frac{h}{c\bar{q}}\left[\frac{\partial^2 \phi_0}{\partial p \partial \gamma} + \frac{p}{\bar{p}^2}\frac{\partial \phi_0}{\partial \gamma}\right]$$

$$\sigma_{\gamma\alpha} = \frac{h}{c\bar{p}}\left[\frac{\partial^2 \phi_0}{\partial \gamma \partial q} - \frac{q}{\bar{q}^2}\frac{\partial \varphi_0}{\partial \gamma}\right]$$

For ϕ_3

$$2Gu_\alpha = ch\bar{q}p\left[q\frac{\partial\phi_3}{\partial q} - (3-4\nu)\,\phi_3\right]$$

$$2Gu_\beta = -chq\bar{p}\left[p\frac{\partial\phi_3}{\partial p} - (3-4\nu)\,\phi_3\right]$$

$$2Gu_\gamma = \frac{qp}{\bar{q}\bar{p}}\frac{\partial\phi_3}{\partial\gamma}$$

$$\sigma_\alpha = ch^2\left[q\bar{q}^2p\frac{\partial^2\phi_3}{\partial q^2} + \{c^2h^2q^2\bar{p}^2 - 2\,(1-\nu)\,\bar{q}^2\}\,p\frac{\partial\phi_3}{\partial q} - (c^2h^2p^2 + 2\nu)\,q\bar{p}^2\frac{\partial\phi_3}{\partial p}\right]$$

$$\sigma_\beta = ch^2\left[qp\bar{p}^2\frac{\partial^2\phi_3}{\partial p^2} - \{c^2h^2\bar{q}^2p^2 + 2\,(1-\nu)\,\bar{p}^2\}\,q\frac{\partial\phi_3}{\partial p} + (c^2h^2q^2 - 2\nu)\,\bar{q}^2p\frac{\partial\phi_3}{\partial q}\right]$$

$$\sigma_\gamma = \frac{qp}{c\bar{q}^2\bar{p}^2}\frac{\partial^2\phi_3}{\partial\gamma^2} + ch^2\left[(q^2 - 2\nu\bar{q}^2)\,p\frac{\partial\phi_3}{\partial q} - (p^2 + 2\nu\bar{p}^2)\,q\frac{\partial\phi_3}{\partial p}\right]$$

$$\sigma_{\alpha\beta} = -\ ch^2\bar{q}\bar{p}\left[qp\frac{\partial^2\phi_3}{\partial q\partial p} + \{c^2h^2p^2 - (1-2\nu)\}\,q\frac{\partial\phi_3}{\partial q} - \{c^2h^2q^2 + (1-2\nu)\}\,p\frac{\partial\phi_3}{\partial p}\right]$$

$$\sigma_{\beta\gamma} = -\ h\frac{q}{\bar{q}}\left[p\frac{\partial^2\phi_3}{\partial p\partial\gamma} + \{\frac{1}{\bar{p}^2} - 2\,(1-\nu)\}\frac{\partial\phi_3}{\partial\gamma}\right]$$

$$\sigma_{\gamma\alpha} = h\frac{p}{\bar{p}}\left[q\frac{\partial^2\phi_3}{\partial\gamma\partial q} - \{\frac{1}{\bar{q}^2} + 2\,(1-\nu)\}\frac{\partial\varphi_3}{\partial\gamma}\right]$$

For λ_3

$$2Gu_\alpha = 2h\frac{q}{\bar{q}}\frac{\partial\lambda_3}{\partial\gamma}$$

$$2Gu_\beta = 2h\frac{p}{\bar{p}}\frac{\partial\lambda_3}{\partial\gamma}$$

$$2Gu_\gamma = 2ch^2\bar{q}\bar{p}\,(p\frac{\partial\lambda_3}{\partial p} - q\frac{\partial\lambda_3}{\partial q})$$

$$\sigma_\alpha = 2h^2 q\left[\frac{\partial^2\lambda_3}{\partial\gamma\partial q} - \frac{q}{\bar{q}^2}\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_\beta = -\ 2h^2 p\left[\frac{\partial^2\lambda_3}{\partial p\partial\gamma} + \frac{p}{\bar{p}^2}\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_\gamma = 2h^2\left[p\frac{\partial^2\lambda_3}{\partial p\partial\gamma} - q\frac{\partial^2\lambda_3}{\partial\gamma\partial q}\right] + \frac{2}{c^2\bar{q}^2\bar{p}^2}\frac{\partial\lambda_3}{\partial\gamma}$$

$$\sigma_{\alpha\beta} = \frac{h^2}{\bar{q}\bar{p}}\left[\bar{q}^2 p\frac{\partial^2\lambda_3}{\partial\gamma\partial q} - q\bar{p}^2\frac{\partial^2\lambda_3}{\partial p\partial\gamma} - 2qp\frac{\partial\lambda_3}{\partial\gamma}\right]$$

$$\sigma_{\beta\gamma} = \frac{hp}{c\bar{q}\bar{p}^2}\frac{\partial^2\lambda_3}{\partial\gamma^2} + ch^3\bar{q}\bar{p}^2\left[q\frac{\partial^2\lambda_3}{\partial q\partial p} - p\frac{\partial^2\lambda_3}{\partial p^2} + c^2h^2\,\{2qp\frac{\partial\lambda_3}{\partial q} - (q^2+p^2)\frac{\partial\lambda_3}{\partial p}\}\right]$$

$$\sigma_{\gamma\alpha} = \frac{hq}{c\bar{q}^2\bar{p}}\frac{\partial^2\lambda_3}{\partial\gamma^2} + ch^3\bar{q}^2\bar{p}\left[p\frac{\partial^2\lambda_3}{\partial q\partial p} - q\frac{\partial^2\lambda_3}{\partial q^2} + c^2h^2\,\{(q^2+p^2)\frac{\partial\lambda_3}{\partial q} - 2qp\frac{\partial\lambda_3}{\partial p}\}\right]$$

APPENDIX B

The stress and displacement field of

$$\phi_0 = 2S_0 Q_0^1(q) P_0^1(p) \quad \text{and} \quad \lambda_3 = S_0 Q_0^1(q) P_0^1(p)$$

at the interface

$$u_\alpha = 0$$

$$\bar{p}\frac{2Gu_\beta}{S_0 h\cos\gamma} = \frac{2}{\bar{q}_0} P_1^1(p)$$

$$\bar{p}\frac{2Gu_\gamma}{S_0 c^2 h^2 \sin\gamma} = -\frac{2}{3c} P_2^1(p)$$

$$\frac{\sigma_\alpha}{S_0 c^2 h^4 \cos\gamma} = \frac{2}{3\bar{q}_0} P_2^1(p)$$

$$\bar{p}\frac{\sigma_{\alpha\beta}}{S_0 c^2 h^4 \cos\gamma} = \frac{q_0}{\bar{q}_0^2}(-3q_0^2 + 2\frac{1}{5}) P_1^1(p) + \frac{2}{15}\frac{q_0}{\bar{q}_0^2} P_3^1(p)$$

$$\bar{p}\frac{\sigma_{\gamma\alpha}}{S_0 c h^5 \sin\gamma} = \left(\frac{q_0^3}{3\bar{q}_0} + \frac{2}{3}\bar{q}_0 q_0 - \frac{q_0}{7\bar{q}_0}\right) c^2 P_2^1(p) - \frac{2}{35}\frac{q_0}{\bar{q}_0} c^2 P_4^1(p)$$

APPENDIX C

$C1 \quad K^{A}_{i,j}\ ,K^{B}_{i,j}\ ,K^{C}_{i,j}$

$$K^{A}_{n,n} = F_1(n)$$

$$K^{B}_{n,n-1} = F_2(n-1)\,W_3(n-1)$$

$$K^{B}_{n,n+1} = F_2(n+1)\,W_4(n+1)$$

$$K^{C}_{n,n} = F_3(n)$$

where

$$F_1(n) = E_1(n)$$

$$F_2(n) = E_2(n) + E_3(n)$$

$$F_3(n) = E_4(n)$$

where

$$E_1(n) = \bar{q}_0 Q^{1\prime}_n(q_0)$$

$$E_2(n) = c\bar{q}_0 q_0 Q^{1\prime}_n(q_0)$$

$$E_4(n) = 2\frac{q_0}{\bar{q}_0} Q^1_n(q_0)$$

$$E_3(n) = -c\bar{q}_0(3-4\nu)\,Q^1_n(q_0)$$

$W_i(n)$ is shown in the appendix D

C2 $L^{A}_{i,j}$, $L^{B}_{i,j}$, $L^{C}_{i,j}$

$$L^{A}_{n,n} = F_1(n) + F_2(n)\,W_6(n) + F_3(n)\,W_{33}(n)$$

$$L^{A}_{n,n-2} = F_2(n-2)\,W_5(n-2) + F_3(n-2)\,W_{32}(n-2)$$

$$L^{A}_{n,n+2} = F_2(n+2)\,W_7(n+2) + F_3(n+2)\,W_{34}(n+2)$$

$$L^{B}_{n,n-1} = F_4(n-1)\,W_3(n-1) + F_5(n-1)\,W_9(n-1)$$

$$F_6(n-1)\,W_{36}(n-1) + F_7(n-1)\,W_{30}(n-1)$$

$$L^{B}_{n,n+1} = F_4(n+1)\,W_4(n+1) + F_5(n+1)\,W_{10}(n+1)$$

$$+ + F_6(n+1)\,W_{37}(n+1) + F_7(n+1)\,W_{31}(n+1)$$

$$L^{B}_{n,n-3} = F_5(n-3)\,W_8(n-3) + F_6(n-3)\,W_{35}(n-3)$$

$$L^{B}_{n,n+3} = F_5(n+3)\,W_{11}(n+3) + F_6(n+3)\,W_{38}(n+3)$$

$$L^C_{n,n} = F_8(n) + F_9(n) W_6(n)$$

$$L^C_{n,n-2} = F_9(n-2) W_5(n-2)$$

$$L^C_{n,n+2} = F_9(n+2) W_7(n+2)$$

where

$$F_1(n) = q_0^2 E_1(n) + E_2(n)$$

$$F_2(n) = -E_1(n) - E_2(n)$$

$$F_3(n) = E_3(n)$$

$$F_4(n) = q_0^2 E_4(n) + E_5(n) + q_0^2 E_6(n)$$

$$F_5(n) = -E_4(n) - E_5(n) - E_6(n)$$

$$F_6(n) = E_7(n) - E_8(n)$$

$$F_7(n) = q_0^2 E_8(n)$$

$$F_8(n) = q_0^2 E_9(n) + q_0^2 E_{10}(n)$$

$$E_9(n) = -E_9(n) - E_{10}(n)$$

where

$$E_1(n) = \bar{q}_0^2 Q_n^{1\prime\prime}(q_0)$$

$$E_2(n) = q_0 Q_n^{1\prime}(q_0)$$

$$E_3(n) = -Q_n^1(q_0)$$

$$E_4(n) = c q_0 \bar{q}_0^2 Q_n^{1\prime\prime}(q_0)$$

$$E_5(n) = c q_0^2 Q_n^{1\prime}(q_0)$$

$$E_6(n) = -2c(1-\nu)\bar{q}_0^2 Q_n^{1\prime}(q_0)$$

$$E_7(n) = -c q_0 Q_n^1(q_0)$$

$$E_8(n) = -2c\nu q_0 Q_n^1(q_0)$$

$$E_9(n) = 2 q_0 Q_n^{1\prime}(q_0)$$

$$E_{10}(n) = -\frac{2q_0^2}{\bar{q}_0^2} Q_n^1(q_0)$$

$W_i(n)$ is shown in the appendix D.

$C3 \quad S^A_{i,j}\ , S^B_{i,j}\ , S^C_{i,j}$

$$S^A_{n,\,n-1} = F_1\,(n-1)\,W_{30}\,(n-1) + F_2\,(n-1)\,W_{36}\,(n-1)$$

$$+ F_3\,(n-1)\,W_3\,(n-1) + F_4\,(n-1)\,W_9\,(n-1)$$

$$S^A_{n,\,n+1} = F_1\,(n+1)\,W_{31}\,(n+1) + F_2\,(n+1)\,W_{37}\,(n+1)$$

$$+ F_3\,(n+1)\,W_4\,(n+1) + F_4\,(n+1)\,W_{10}\,(n+1)$$

$$S^A_{n,\,n-3} = F_2\,(n-3)\,W_{35}\,(n-3) + F_4\,(n-3)\,W_8\,(n-3)$$

$$S^A_{n,\,n+3} = F_2\,(n+3)\,W_{38}\,(n+3) + F_4\,(n+3)\,W_{11}\,(n+3)$$

$$S^B_{n,\,n} = F_5\,(n)\,W_{33}\,(n) + F_6\,(n)\,W_{41}\,(n)$$

$$+ F_7\,(n) + F_8\,(n)\,W_6\,(n) + F_9\,(n)\,W_{14}\,(n)$$

$$S^B_{n,\,n-2} = F_5\,(n-2)\,W_{32}\,(n-2) + F_6\,(n-2)\,w_{40}\,(n-2)$$

$$+ F_8\,(n-2)\,W_5\,(n-2) + F_9\,(n-2)\,W_{13}\,(n-2)$$

$$S^B_{n,n+2} = F_5(n+2)\,W_{34}(n+2) + F_6(n+2)\,W_{42}(n+2)$$

$$+ F_8(n+2)\,W_7(n+2) + F_9(n+2)\,W_{15}(n+2)$$

$$S^B_{n,n-4} = F_6(n-4)\,W_{39}(n-4) + F_9(n-4)\,W_{12}(n-4)$$

$$S^B_{n,n+4} = F_6(n+4)\,W_{43}(n+4) + F_9(n+4)\,W_{16}(n+4)$$

$$S^C_{n,n-1} = F_{10}(n-1)\,W_3(n-1) + F_{11}(n-1)\,W_9(n-1)$$

$$+ F_{12}(n-1)\,W_{30}(n-1) + F_{13}(n-1)\,W_{36}(n-1)$$

$$S^C_{n,n+1} = F_{10}(n+1)\,W_4(n+1) + F_{11}(n+1)\,W_{10}(n+1)$$

$$+ F_{12}(n+1)\,W_{31}(n+1) + F_{13}(n+1)\,W_{37}(n+1)$$

$$S^C_{n,n-3} = F_{11}(n-3)\,W_8(n-3) + F_{13}(n-3)\,W_{35}(n-3)$$

$$S^C_{n,n+3} = F_{11}(n+3)\,W_{11}(n+3) + F_{13}(n+3)\,W_{38}(n+3)$$

where

$$F_1(n) = q_0^2 E_1(n) + E_2(n)$$

$$F_2(n) = -E_1(n)$$

$$F_3(n) = E_3(n)$$

$$F_4(n) = -E_3(n)$$

$$F_5(n) = q_0^2 E_4(n) + E_7(n) + q_0^2 E_8(n)$$

$$F_6(n) = -E_4(n) - E_8(n)$$

$$F_7(n) = q_0^2 E_6(n)$$

$$F_8(n) = E_5(n) - (q_0^2 + 1) E_6(n)$$

$$F_9(n) = -E_5(n) + E_6(n)$$

$$F_{10}(n) = q_0^2 E_9(n) + q_0^2 E_{11}(n)$$

$$F_{11}(n) = -E_9(n) - E_{11}(n)$$

$$F_{12}(n) = q_0^2 E_{10}(n)$$

$$F_{13}(n) = -E_{10}(n)$$

where

$$E_1(n) = -\bar{q}_0 Q_n^{1\prime}(q_0)$$

$$E_2(n) = \bar{q}_0 q_0 Q_n^{1}(q_0)$$

$$E_3(n) = -\bar{q}_0 Q_n^{1\prime}(q_0)$$

$$E_4(n) = -c\bar{q}_0 q_0 Q_n^{1\prime}(q_0)$$

$$E_5(n) = -c\bar{q}_0 q_0 Q_n^{1\prime}(q_0)$$

$$E_6(n) = c(1-2\nu)\,\bar{q}_0 q_0 Q_n^{1\prime}(q_0)$$

$$E_7(n) = c\bar{q}_0 q_0^2 Q_n^{1}(q_0)$$

$$E_8(n) = c(1-2\nu)\,\bar{q}_0 Q_n^{1}(q_0)$$

$$E_9(n) = \bar{q}_0 Q_n^{1\prime}(q_0)$$

$$E_{10}(n) = -\frac{q_0}{\bar{q}_0} Q_n^{1}(q_0)$$

$$E_{11}(n) = -\frac{2q_0}{\bar{q}_0} Q_n^{1}(q_0)$$

$W_i(n)$ is shown in the appendix D.

$C4 \quad T^A_{i,j}\ , T^B_{i,j}\ , T^C_{i,j}$

$$T^A_{n,n} = F_1(n) + F_2(n)\,W_6(n) + F_3(n)\,W_{14}(n)$$

$$T^A_{n,n-2} = F_2(n-2)\,W_5(n-2) + F_3(n-2)\,W_{13}(n-2)$$

$$T^A_{n,n+2} = F_2(n+2)\,W_7(n+2) + F_3(n+2)\,W_{15}(n+2)$$

$$T^A_{n,n-4} = F_3(n-4)\,W_{12}(n-4)$$

$$T^A_{n,n+4} = F_3(n+4)\,W_{16}(n+4)$$

$$T^B_{n,n-1} = F_4(n-1)\,W_3(n-1) + F_5(n-1)\,W_9(n-1) + F_6(n-1)\,W_{19}(n-1)$$

$$T^B_{n,n+1} = F_4(n+1)\,W_4(n+1) + F_5(n+1)\,W_{10}(n+1) + F_6(n+1)\,W_{20}(n+1)$$

$$T^B_{n,n-3} = F_5(n-3)\,W_8(n-3) + F_6(n-3)\,W_{18}(n-3)$$

$$T^B_{n,n+3} = F_5(n+3)\,W_{11}(n+3) + F_6(n+3)\,W_{21}(n+3)$$

$$T^{B}_{n,\,n-5} = F_6\,(n-5)\,W_{17}\,(n-5)$$

$$T^{B}_{n,\,n+5} = F_6\,(n+5)\,W_{22}\,(n+5)$$

$$T^{C}_{n,\,n} = F_7\,(n) + F_8\,(n)\,W_6\,(n) + F_9\,(n)\,W_{14}\,(n)$$
$$+ F_{10}\,(n)\,W_{33}\,(n) + F_{11}\,(n)\,W_{41}\,(n)$$

$$T^{C}_{n,\,n-2} = F_8\,(n-2)\,W_5\,(n-2) + F_9\,(n-2)\,W_{13}\,(n-2)$$
$$+ F_{10}\,(n-2)\,W_{32}\,(n-2) + F_{11}\,(n-2)\,W_{40}\,(n-2)$$

$$T^{C}_{n,\,n+2} = F_8\,(n+2)\,W_7\,(n+2) + F_9\,(n+2)\,W_{15}\,(n+2)$$
$$+ F_{10}\,(n+2)\,W_{34}\,(n+2) + F_{11}\,(n+2)\,W_{42}\,(n+2)$$

$$T^{C}_{n,\,n-4} = F_9\,(n-4)\,W_{12}\,(n-4) + F_{11}\,(n-4)\,W_{39}\,(n-4)$$

$$T^{C}_{n,\,n+4} = F_9\,(n+4)\,W_{16}\,(n+4) + F_{11}\,(n+4)\,W_{43}\,(n+4)$$

where

$$F_1(n) = q_0^4 E_1(n)$$

$$F_2(n) = -2q_0^2 E_1(n)$$

$$F_3(n) = E_1(n)$$

$$F_4(n) = q_0^4 E_2(n)$$

$$F_5(n) = -2q_0^2 E_2(n)$$

$$F_6(n) = E_2(n)$$

$$F_7(n) = q_0^4 E_3(n) + q_0^2 E_4(n) + q_0^2 E_5(n)$$

$$F_8(n) = -2q_0^2 E_3(n) - (q_0^2 + 1) E_4(n) - (q_0^2 - 1) E_5(n)$$

$$F_9(n) = E_3(n) + E_4(n) - E_5(n)$$

$$F_{10}(n) = E_6(n) + q_0^2 E_7(n)$$

$$F_{11}(n) = -E_7(n)$$

where

$$E_1(n) = -c^2 Q_n^{1\prime}(q_0) + c^2 \frac{q_0}{\bar{q}_0^2} Q_n^1(q_0)$$

$$E_2(n) = -c^3 q_0 Q_n^{1\prime}(q_0) + c^3 \{ \frac{1}{\bar{q}_0^2} + 2(1-\nu) \} Q_n^1(q_0)$$

$$E_3(n) = -c^2 \frac{q_0}{\bar{q}_0^2} Q_n^1(q)$$

$$E_4(n) = -c^2 \bar{q}_0^2 q_0 Q_n^{1\prime\prime}(q_0)$$

$$E_5(n) = c^2 \bar{q}_0^2 Q_n^{1\prime}(q_0)$$

$$E_6(n) = -2c^2 \bar{q}_0^2 q_0 Q_n^1(q_0)$$

$$E_7(n) = c^2 \bar{q}_0^2 Q_n^{1\prime}(q_0)$$

$W_i(n)$ is shown in the appendix D.

APPENDIX D

$W_{i,j}$

$$W_1(n) = \frac{n}{2n+1}$$

$$W_2(n) = \frac{n+1}{2n+1}$$

$$W_3(n) = W_1(n)$$

$$W_4(n) = W_2(n)$$

$$W_5(n) = W_3(n)\,W_1(n+1)$$

$$W_6(n) = W_3(n)\,W_2(n+1) + W_4(n)\,W_1(n-1)$$

$$W_7(n) = W_4(n)\,W_2(n-1)$$

$$W_8(n) = W_5(n)\,W_1(n+2)$$

$$W_9(n) = W_5(n)\,W_2(n+2) + W_6(n)\,W_1(n)$$

$$W_{10}(n) = W_6(n)\,W_2(n) + W_7(n)\,W_1(n-2)$$

$$W_{11}(n) = W_7(n)\,W_2(n-2)$$

$$W_{12}(n) = W_8(n)\,W_1(n+3)$$

$$W_{13}(n) = W_8(n) W_2(n+3) + W_9(n) W_1(n+1)$$

$$W_{14}(n) = W_9(n) W_2(n+1) + W_{10}(n) W_1(n-1)$$

$$W_{15}(n) = W_{10}(n) W_2(n-1) + W_{11}(n) W_1(n-3)$$

$$W_{16} = W_{11}(n) W_2(n-3)$$

$$W_{17} = W_{12}(n) W_1(n+4)$$

$$W_{18}(n) = W_{12}(n) W_2(n+4) + W_{13}(n) W_1(n+2)$$

$$W_{19}(n) = W_{13}(n) W_2(n+2) + W_{14}(n) W_1(n)$$

$$W_{20}(n) = W_{14}(n) W_2(n) + W_{15}(n) W_1(n-2)$$

$$W_{21}(n) = W_{15}(n) W_2(n-2) + W_{16}(n) W_1(n-4)$$

$$W_{22}(n) = W_{16}(n) W_2(n-4)$$

$$W_{30}(n) = -\frac{n^2}{2n+1}$$

$$W_{31}(n) = \frac{(n+1)^2}{2n+1}$$

$$W_{32}(n) = W_{30}(n) W_1(n+1)$$

$$W_{33}(n) = W_{30}(n) W_2(n+1) + W_{31}(n) W_1(n-1)$$

$$W_{34}(n) = W_{31}(n)\,W_2(n-1)$$

$$W_{35}(n) = W_{32}(n+2)\,W_1(n+2)$$

$$W_{36}(n) = W_{32}(n)\,W_2(n+2) + W_{33}(n)\,W_1(n)$$

$$W_{37}(n) = W_{33}(n)\,W_2(n) + W_{34}(n)\,W_1(n-2)$$

$$W_{38}(n) = W_{34}(n)\,W_2(n-2)$$

$$W_{39}(n) = W_{35}(n)\,W_1(n+3)$$

$$W_{40}(n) = W_{35}(n)\,W_2(n+3) + W_{36}(n)\,W_1(n+1)$$

$$W_{41}(n) = W_{36}(n)\,W_2(n+1) + W_{37}(n)\,W_1(n-1)$$

$$W_{42}(n) = W_{37}(n)\,W_2(n-1) + W_{38}(n)\,W_1(n-3)$$

$$W_{43}(n) = W_{38}(n)\,W_2(n-3)$$

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