This is to certify that the

dissertation entitled

Comparative Statics Under Uncertainty
for Decision Models
with More Than One Choice Variable

presented by

Gyemyung Choi

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Economics

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Major professor

Date May 11, 1992

COMPARATIVE STATICS UNDER UNCERTAINTY FOR DECISION MODELS WITH MORE THAN ONE CHOICE VARIABLE

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A DISSERTATION

Submitted to
Michigan State University
in partial fullfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1992

ABSTRACT

COMPARATIVE STATICS UNDER UNCERTAINTY FOR DECISION MODELS WITH MORE THAN ONE CHOICE VARIABLE

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Any economic decision model under uncertainty contains random and nonrandom parameters, the choice variables, an objective function, and a set of decision makers. Decision models can be divided into several types according to how many random parameters, choice or outcome variables they have. An important question in the study of economic decision models involving randomness is how does a particular type of a change in a random parameter affect the level of the choice variables selected by a decision maker.

Using a general one random-two choice-one outcome(1-2-1) model, we investigate the comparative static properties of three types of changes in randomness: simple increases in risk, relatively strong increases in risk, and first degree stochastic dominant(FSD) shifts. Several theorems are presented giving conditions on the economic model and risk taking characteristics of the decision maker that are sufficient to obtain unambiguous comparative static results for the three types of changes in randomness. A diagrammatical method is introduced, which can deal with corner solutions and may have a pedagogical value.

Specific two random-two choice-one outcome(2-2-1) models are also considered, but only modest results are obtained. The 2-2-1 model is quite difficult to analyze because of the problems of dealing with both the joint cumulative distribution function and two first-order conditions. Finally, we examine a specific two random-one choice-two

outcome(2-1-2) model including cases where the risks are not independent of one another, and present sufficient conditions for signing the effects on the choice variable of an arbitrary Rothschild and Stiglitz increase in risk and a FSD shift.

Dedicated to my parents

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest appreciation to my advisor, Professor Jack Meyer, for all the time, energy, and encouragement given to me throughout this dissertation. His openness in sharing his knowledge and insight has given me the tools to pursue the high professional standards and academic excellence that he has attained. Next, I would like to thank my committee members: Professors Carl Davidson and Larry Martin.

Finally, I am indebted to my father-in-law for his support during my years of study at Michigan State.

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CHAPTER ONE

INTRODUCTION

Almost every aspect of economic behavior is affected by uncertainty. It is widely believed that the underlying determinants of supply and demand have significant stochastic components, and the number and arrival times of customers at a store are stochastic. As would be expected, human behavior has adapted to uncertainty in a variety ways. Insurance, futures markets, and stock markets are three of the most important institutions that facilitate the reallocation of risk among individuals and firms.

A theory of choice under certainty does not provide an adequate explanation of an economic decision maker's response to a stochastic environment. For example, it cannot explain why some people buy a lottery ticket or insurance. Therefore, a theory of choice under uncertainty is naturally needed. Any economic decision model under uncertainty has several components: 1) the random and nonrandom parameters, 2) the choice(or control) variables, 3) the objective function, 4) the set of decision makers.

Decision models which include random parameters can be divided into two types. One type, called specific models, are constructed to explain a specific economic phenomenon. The model structure and variables have a specific interpretations. Another type, referred to as general models, are formulated to include many specific models as special cases. General models are usually more difficult to analyze than specific models.

There are two important comparative static questions in the area of risk analysis. The first one is concerned with determination of the effect of a particular type of change in randomness on the expected utility of a decision maker. The second one is concerned with determination of the direction of change for choice variables selected by the decision maker when a particular type of change in a random parameter occurs in an economic decision model. This paper deals with the latter question.

There are many types of changes in randomness whose effects can be analyzed. For example, an increase in the first two moments or variations in the maximum and minimum value of a random variable are types of changes in randomness whose effects may be examined. Also, increases in risk in the Rothschild and Stiglitz sense or first and second degree stochastically dominant shifts are types of changes in the random parameter that can be analyzed. Various groups of decision makers may prefer one type of change in randomness to the current situation, but another group of decision makers may not.

There are also several assumptions that can be made concerning the objective function. Under the expected utility hypothesis, the individual, when faced with alternative risky prospects or lotteries over a set of outcomes, will always choose that prospect which yields the highest mathematical expectation of some von Neumann-Morgenstern utility function $u(\cdot)$ defined over the set of outcomes. Assuming utility depends only on one outcome variable, denoted z, then the individual's problem is to choose the choice variable to maximize $Eu(z)$, where z usually depends on random parameters, choice variables, and a set of nonrandom parameters. Some conditions concerning the function z are often needed in order to prove comparative static results.

Finally, there are many types of decision makers. For instance, one group of decision makers prefers risk, but another group does not. Even if individuals in a group of decision makers are the same in the sense that they dislike risk, each individual's preference for risk can differ. Thus, decision makers are grouped according to their risk taking characteristics using such assumptions as decreasing absolute risk aversion.

An influential theoretical contribution to risk analysis is found in a pair of articles by Rothschild and Stiglitz[1970,1971]. R-S develop a general definition of an increase in risk by specifying conditions on the change in cumulative distribution function of a random parameter, and show that a R-S increase in risk reduces expected utility for all risk-averse decision makers. Also, they use a general decision model to analyze a risk-averse economic agent's choice's respond to a R-S increase in risk.

At about the same time, Sandmo[1971] develops a mean-preserving linear transformation of the random variable to represent a particular type of an increase in risk. Using a specific rather than a general decision model, he examines the effect of this increase in risk. Sandmo determines the effect of the risk increase on the output decision of a competitive firm exhibiting decreasing absolute risk aversion (DARA). Other specific models are presented at about this time by Arrow[1971], Fishburn and Porter[1976], and Mossin[1968].

Most papers dealing with comparative statics under uncertainty assume that the first-order and second-order conditions are satisfied to guarantee an interior or a bounded solution. Recently, Dionne, Eeckhoudt and Gollier[1991] address the issue of a corner or an unbounded solution. They emphasize that the assumption of an interior or a bounded solution is quite restrictive and may rule out some interesting cases. This is discussed further in chapter 2.

We use only general decision model formulations, beginning with Kraus'[1979] decision model. In this model, utility depends only on one outcome variable which in turn depends on one random parameter, a choice variable, and also a set of nonrandom parameters. In this model it is assumed that the agent chooses α to maximize $Eu(z(\tilde{x},\alpha,\lambda))$. The outcome variable, z, depends on a random variable, $\tilde{x}$, a choice variable, α, and a set of nonrandom parameters, λ. We call this model the one random-one choice-one outcome(1-1-1) general decision model.

Specific models which are special cases of this 1-1-1 model include the following. One is the Sandmo model of the competitive firm, in which z takes the form $z(\tilde{x},\alpha,\lambda) = \tilde{x}\cdot\alpha - c(\alpha) - \lambda$, where $\tilde{x}$ is the price of output, α is output level, λ is fixed cost, and $c(\cdot)$ is a variable cost function. Another is the standard portfolio model analyzed by Arrow[1971] and Fishburn and Porter[1976]. For this model, $z(\tilde{x},\alpha,\lambda) = \lambda_1\cdot(\alpha\cdot(\tilde{x} - \lambda_2) + \lambda_2)$, where $\tilde{x}$ is the return to a risky asset, α is the proportion invested in the risky asset, and λ_1 and λ_2 represent initial wealth, and the return of a riskless asset, respectively.

Many researchers have analyzed 1-1-1 models, but only a few have considered two choice variable models. Batra and Ullah[1974] examine a competitive firm's input decisions under output price uncertainty for the two input case. Feder[1977] considers a general decision model which is developed to include in it many economic decision models frequently encountered by economists. He does not, however, investigate the problem of determining the direction of impact on individual choice variables, except for the one choice variable case. Feder, Just and Schmitz(FJS)[1977] investigate an international trade model with one random and two choice variables, and Katz, Paroush and Kahana(KPK)[1982] explore the optimal policy of a price discriminating firm which operates under price uncertainty in one of two markets.

We use the following notation to represent the one random–two choice–one outcome(1–2–1) decision model. Assume the agent chooses α and δ to maximize $Eu(z(\tilde{x},\alpha,\delta,\lambda))$, where the outcome variable, z, depends on a random variable, $\tilde{x}$, two choice variables, α and δ, and a set of nonrandom parameters, λ. Batra and Ullah's model takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot f(\alpha,\delta) - \lambda_1\cdot\alpha - \lambda_2\cdot\delta$, where $\tilde{x}$ is the price of output, α and δ are two inputs, $\lambda_i(i=1,2)$ represent the prices of the inputs, and f is a production function. The price discrimination model of KPK has outcome variable given by $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot R_1(\alpha) + \lambda_1\cdot R_2(\delta) - c(\alpha + \delta) - \lambda_2$. In this case $\tilde{x}$ represents randomness in market 1, α and δ are the firm's sales in market 1 and market 2 respectively, λ_1 represents demand conditions in market 2, and λ_2 is fixed cost. In addition, $R_i(i=1,2)$ is the revenue function of market i, and $c(\cdot)$ is the variable cost function.

Researchers such as Kraus[1979], Meyer and Ormiston [1985,1989], and Black and Bulkley[1989] have exhaustively analyzed the general 1–1–1 model. The general 1-2-1 model, however, remains largely unexplored. All 1-2-1 models in the literature, except Feder's model[1977], are specific rather than general models. In its general form, the 1-2-1 model requires special assumptions to solve. Only by imposing a fairly rigid structure on the general 1-2-1 model can the comparative statics be determined.

The issue of a corner solution is more interesting in the two choice variable case than in the one choice variable case. Assuming a corner solution in a 1-1-1 model does not allow comparative statics to be conducted for small changes because the only choice variable is determined by the constraint. However, in a 1-2-1 model, because there are two choice variables and one of them can be an interior solution, comparative static analysis can still be carried out. The usual algebraic method does not work unless the first-order conditions are satisfied as equalities. That is the main reason we adopt a diagrammatical tool in order to handle corner solutions.

In addition to expanding the number of choice variables, we can expand the number of random variables. Hadar and Seo[1990] extend the standard portfolio model with only one risky asset by considering a portfolio model containing two risky assets, but no riskless assets. Thus, there is still only one choice variable. H-S avoid the subject of stochastic dependence among risky assets, by assuming independence. After examining the portfolio model with two risky assets and one choice variable, they then extend the model to include n risky assets and n-1 choice variables but make little progress. Recently, Meyer and Ormiston [1991] extend Hadar-Seo's portfolio model in an important direction by considering the cases where the two random variables are not independent of one another. Also, Meyer[1991] investigates an insurance demand model with two random variables and one choice variable.

Each of these decision models is a two random–one choice–one outcome(2–1–1) model. A general form for these models assumes the agent chooses α to maximize $Eu(z(\tilde{x},\tilde{y},\alpha,\lambda))$, where the outcome variable, z, depends on two random variables, $\tilde{x}$ and $\tilde{y}$, one choice variable, α, and a set of nonrandom parameters, λ. Hadar and Seo's portfolio model assumes z takes the form $z(\tilde{x},\tilde{y},\alpha) = \alpha\cdot\tilde{x} + (1-\alpha)\cdot\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are the returns to the risky assets, and α is the proportion invested in the risky asset $\tilde{x}$. Meyer's insurance demand model has $z(\tilde{x},\tilde{y},\alpha,\lambda) = (\lambda_1 - \tilde{x}) + \alpha\cdot(\tilde{x} - \lambda_2) + \tilde{y}$, where the support of $\tilde{x}$ is $[0,\lambda_1]$, α is the coinsurance rate, and λ_2 is the insurance premium.

Combining these extensions we have the two random–two choice–one outcome(2–2–1) model. In it the agent is assumed to choose α and δ to maximize $Eu(z(\tilde{x},\tilde{y},\alpha,\delta,\lambda))$, where the outcome variable, z, depends on two random variables, $\tilde{x}$ and $\tilde{y}$, two choice variables, α and δ, and a set of nonrandom parameters, λ. Specific examples of the 2–2–1 model include the following models. First is the portfolio model with one riskless asset, two risky assets, and two choice variables, where $z(\tilde{x},\tilde{y},\alpha,\delta,\lambda) = \alpha\cdot\tilde{x} + \delta\cdot\tilde{y} + (1 - \alpha - \delta)\cdot\lambda$. In ths model $\tilde{x}$ and $\tilde{y}$ are the returns to risky assets, α and δ are the proportion invested in the risky assets $\tilde{x}$ and $\tilde{y}$ respectively, and λ is the return to the riskless asset. Another example is Feder's general decision model where $z(\tilde{x},\tilde{y},\alpha,\delta,\lambda) = \tilde{x}\cdot\tilde{y}\cdot f(\alpha,\delta) + g(\alpha,\delta) + \lambda$. Again, $\tilde{x}$ and $\tilde{y}$ are random parameters, α and δ are choice variables, and λ is a nonrandom parameter. In this model f and g are real-valued functions of only the control variables.

Extension of either the 2-1-1 or 1-2-1 models to the 2-2-1 model is quite difficult because of the problems of stochastic dependence between random variables, and of two first-order conditions. Until now, the 2-2-1 model remains largely unexplored and only modest results are presented here.

Finally, Sandmo[1970], Block and Heineke[1973], and Dardanoni [1988] examine a decision model composed of one random, one choice, but two outcome variables. In these models, the utility function depends on two outcome variables, each of which in turn depends on one random variable, one choice variable, and a set of nonrandom parameters. In general form an agent is assumed to choose α to maximize $Eu(z_1(\tilde{x},\alpha,\lambda),z_2(\tilde{x},\alpha,\lambda))$, where the outcome variables, z_1 and z_2, depend on a random parameter, $\tilde{x}$, a choice variable, α, and a set of nonrandom parameters, λ. We define this model as the one random-one choice-two outcome(1-1-2) model. Sandmo's[1970] two period consumption-savings model, in which $z_1(\tilde{x},\alpha,\lambda) = \alpha$ and $z_2(\tilde{x},\alpha,\lambda) = \tilde{x}\cdot(\lambda_1 - \alpha) + \lambda_2$, is a special case where $\tilde{x}$ is the return to savings, α is the consumption in the first period, and the nonrandom parameters λ_1 and λ_2 represent initial wealth and the income of the second period, respectively.

The relationships among the models presented above are given as follows. First, the set of 1-1-1 models is a subset of 1-2-1 models because a 1-1-1 model can be obtained from a 1-2-1 model by simply holding a choice variable in the 1-2-1 model fixed. The set of 1-1-1 models is also a subset of 2-1-1 models because a 1-1-1 model can be obtained from a 2-1-1 model by assuming a degenerate random variable in the 2-1-1 model. In a similar manner, the sets of 1-2-1 models or 2-1-1 models are subsets of 2-2-1 models, and the set of 1-1-2 models is a subset of 2-1-2 models.

This research is organized as follows. Chapter 2 reviews the literature concerning the 1-1-1 model. Chapter 3 investigates a general 1-2-1 model and developes several theorems concerning the effects of changes in randomness. A diagrammatical method, which handles corner solutions and may have a pedagogical value, is introduced in that chapter. Chapter 4 considers two specific 2-2-1 models but presents only modest results. Chapter 5 extends a specific 1-1-2 model to a 2-1-2 model.

CHAPTER TWO

LITERATURE REVIEW

This chapter provides an opportunity to review the historical background in this area and to present various definitions and results which are necessary for chapter 3. We do this by reviewing the literature concerning the 1-1-1 model.

In the 1-1-1 model, utility depends on one outcome variable which in turn depends on one random parameter, one choice variable, and a set of nonrandom parameters. In this model it is assumed that the decision maker chooses α to maximize $Eu(z(\tilde{x},\alpha,\lambda))$. The outcome variable, z, depends on a random variable, $\tilde{x}$, a choice variable, α, and a set of nonrandom parameters, λ. The essential feature of this framework is that the contour map of the objective function $u(z(\tilde{x},\alpha,\lambda))$ in $\tilde{x}$, α space depends on the scalar-valued intermediate function z and is independent of u.

This particular formulation has several advantages. First, in this decision framework, utility depends only on one outcome variable, z. Thus, problems involving multidimensionality of utility, pointed out by Kihlstrom and Mirman[1974], are avoided. Second, the measures of absolute and relative risk aversion introduced by Pratt[1964] and Arrow[1971] and described below, can be used directly. Finally, in a world of certainty, all decision makers select the same level of α. To see this last point, note that if the random varible $\tilde{x}$ is fixed at x_0, then all decision makers select α so as to maximize $z(x_0,\alpha,\lambda)$ no matter what their risk taking preferences are. This allows the analysis to focus on the effects of randomness and risk aversion without worrying about differences in behavior which would arise even in a world of certainty.

Given the existence of a von Neumann–Morgenstern utility function, Pratt and Arrow propose the function $R_A(z) = - u''(z)/u'(z)$ as a measure of risk aversion. The measure has several positive features. First, it does not depend on which utility function is used to represent preferences. Second, it is related to risk when the risk is small. Finally, it completely and uniquely represents preferences. Pratt and Arrow also talk about another measure of risk aversion $R_R(z) = z \cdot R_A(z)$. This is known as the measure of relative risk aversion. While it is generally assumed that most individuals prefer more to less and are risk averse($R_A \geq 0$), further assumptions concerning preferences are <u>often</u> needed in order to prove comparative static results. Nonincreasing absolute and nondecreasing relative risk aversion are widely assumed to be exhibited by individuals. Recently, the conditions of $R_R(z) \leq 1$ or $R_R(z) \leq 2$ have been used.

An important question in the study of economic decision models involving randomness is how does a particular type of a change in random variable $\tilde{x}$ affect the level of the choice variable selected by a decision maker. This question has received much attention in the literature both in general as well as specific decision models. Examples of general theoretical analyses are articles by Feder[1977], Kraus[1979] and Katz[1981], Meyer and Ormiston [1983,1985,1989], Black and Bulkley[1989], and Ormiston [1992]. Examples of analyses of specific models are articles by Mossin[1968], Arrow[1971], Sandmo[1971], Fishburn and Porter[1976], and Cheng, Magil and Shafer[1987], to name just a few. We begin by reviewing the general model given by Kraus.

Kraus[1979] presents the general 1-1-1 model stated earlier and shows that it includes many interesting decision models as special cases. He first considers a special type of an increase in risk, termed a global increase in risk or an introduction of risk. This is an increase in risk from an initial nonrandom situation, where $x = \bar{x}$, to a situation when $\tilde{x}$ is random with mean $\bar{x}$. Kraus derives a sufficient condition for signing the effect on the α selected by a risk averse decision maker of this global increase in risk. The condition requires that z_α be convex or concave in $\tilde{x}$ and have an appropriate single crossing of the $\tilde{x}$ axis. Kraus presents the following theorem.

<u>Theorem 2.1</u>: Assuming that decision makers choose α in order to maximize $Eu(z(\tilde{x},\alpha,\lambda))$ where $u'(z) > 0$, $u''(z) < 0$, $z_x > 0$, and $z_{\alpha\alpha} < 0$, then all risk averse decision makers, when faced with a global increase in risk, will decrease the optimal value of α if $z_{\alpha x} \geq 0$ and $z_{\alpha xx} \leq 0$.

Katz[1981] provides a shorter proof of Kraus' result, and makes the result and its application more accessible. Theorem 2.1 can be adapted for other combinations of assumptions about $z(\tilde{x},\alpha,\lambda)$.
If $z_x < 0$, $z_{\alpha x} \leq 0$, and $z_{\alpha xx} \leq 0$, then the random variable can be transformed to $y = -x$ and the theorem re-expressed in terms of y.
If $z_x > 0$, $z_{\alpha x} \leq 0$, and $z_{\alpha xx} \geq 0$, then redefining the choice variable as $a = -\alpha$, the theorem can be applied. In a similar manner, if $z_x < 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \geq 0$, then both the random variable and the choice variable can be redefined. We shall restrict the discussion to the case

where $z_x \geq 0$.

Other more general changes in the random parameter have been considered. Rothschild and Stiglitz[1970] develop a characterization of $\tilde{x}$ becoming riskier for univariate probability distributions. They demonstrate that the following three conditions on a pair of cumulative distribution functions F and G with equal finite means are equivalent.

RS1: G can be obtained from F by the addition of noise; that is, there exists a pair of random variables $\tilde{x}$ and $\tilde{\xi}$ with $E(\xi|x) = 0$ such that F and G are the distributions of $\tilde{x}$ and $\tilde{x} + \tilde{\xi}$ respectively.

RS2: Every risk-averse expected utility maximizer prefers F to G; that is, $\int_0^B u(x)\cdot dF(x) \geq \int_0^B u(x)\cdot dG(x)$ for every concave function $u(\cdot)$.

RS3: G can be obtained from F by a sequence of one or more mean-preserving spreads, where informally speaking, a mean-preserving spread consists of a transfer of probability mass out of one region of the real line to both the left and the right of the convex hull of this region, in a manner that preserves the mean of the distribution.

These conditions are equivalent to and summarized by the following definition. The characterization in definition 2.1, which was offered as an alternative to the use of variance or standard deviation as a measure of comparative risk, has led to the development of several powerful analytical results in the economic theory of risk under uncertainty.

Definition 2.1: $G(x)$ is riskier than $F(x)$ if and only if $\int_0^s [G(x) - F(x)]dx \geq 0$ for all s and equals zero when $s = B$.

At the same time, Hadar and Russell[1969] address the question of when $F(\cdot)$ is prefered to $G(\cdot)$ by all agents in some well-defined class. The second definition of the R-S increase in risk asks exactly this question for the case of the well-defined class being risk averse agents. Stochastic dominance(SD) has been used to describe a particular set of rules for ranking random variables. These rules apply to pairs of random variables, and indicate when one is to be ranked higher than the other by specifying a condition which the difference between their CDFs must satisfy. First degree stochastic dominant(FSD) improvements and second degree stochastic dominant(SSD) improvements are defined as follows.

<u>Definition 2.2</u>: $F(x)$ FSD $G(x)$ if and only if $F(x) - G(x) \leq 0$ for all $x \in [0,B]$.

<u>Definition 2.3</u>: $F(x)$ SSD $G(x)$ if and only if $\int_0^s [F(x) - G(x)]dx \leq 0$ for all $s \in [0,B]$.

After developing a general definition of an increase in risk, Rothschild and Stiglitz[1971] use a general two argument decision model to analyze all risk-averse decision makers' response to a R-S increase in risk. R-S demonstrate that assuming the agent chooses α to maximize $Eu(\tilde{x},\alpha)$, then $u_{\alpha xx} < 0$ or $u_{\alpha xx} > 0$ is required to predict the direction of change in α selected by all risk-averse decision makers when $\tilde{x}$ undergoes an arbitrary R-S increase in risk. Most of their paper involves setting up several different problems in the literature and asking when the conditions are satisfied. Because their model is a two argument model, it is not reviewed further at this time.

After presenting Theorem 2.1, Kraus[1979] also addresses the question of how does a R–S increase in risk affect the level of the choice variable, α, selected by a risk averse decision maker, in the 1–1–1 model. Unfortunately, Kraus was unable to answer the question, seemingly because the R–S condition cannot be generally satisfied. The R–S sign condition for the 1–1–1 model requires that $u_z \cdot z_{\alpha xx} + 2u_{zz} \cdot z_{\alpha x} \cdot z_x + z_\alpha \cdot (u_{zz} \cdot z_{xx} + z_x^2 \cdot u_{zzz})$ be signed. As long as interior solutions are assumed, the first-order condition for the 1–1–1 model, $Eu'(z) \cdot z_\alpha = 0$, implies that z_α must change sign. Hence the last term of this expression is difficult to sign.

Sandmo[1971] represents an increase in risk in a different manner. He transforms $\tilde{x}$ according to $t(\tilde{x}) = \bar{x} + \gamma \cdot (\tilde{x} - \bar{x})$, where $\bar{x}$ is the mean of $\tilde{x}$ and a nonrandom parameter γ is greater than or equal to one. This represents a particular type of an increase in risk. Sandmo alters the outcomes of the random variable $\tilde{x}$ using a nondecreasing function $t(x)$ whose domain is all possible realizations of $\tilde{x}$. That is, each possible outcome of the original random variable is mapped into a new value thereby defining a new random variable. The transformation $t(x)$ is referred to as a deterministic transformation in order to distinguish it from the stochastic transformation introduced by Rothschild and Stiglitz [1970] in definition 1. Note that $Et(x) = E[\bar{x} + \gamma \cdot (\tilde{x} - \bar{x})] = \bar{x}$. Thus, Sandmo uses a mean-preserving linear deterministic transformation. Sandmo also linearly transforms $\tilde{x}$ according to $t(\tilde{x}) = \tilde{x} + \theta$, where θ is a positive nonrandom parameter, to represent a special type of a FSD improvement in the random variable $\tilde{x}$.

Using a specific 1–1–1 model, Sandmo analyzes the effects on α of a global increase in risk, an increase in risk represented by an increase in γ, and a FSD improvement represented by an increase in θ. The comparative static results are all local, in the sense that the results hold only at $\gamma = 1$ or $\theta = 0$ and for small changes in γ or θ. Because the results are for a specific rather than general model, details are not needed here.

In analyzing decision models involving risk, at least three methods have been used to represent a change in random variable $\tilde{x}$. These include changing the CDF for $\tilde{x}$, transforming $\tilde{x}$ deterministically, transforming $\tilde{x}$ stochastically. Under quite general conditions the three methods of representing a change in $\tilde{x}$ are equivalent, and it is primarily a matter of convenience as to which to use.

Meyer and Ormiston[1983] address the question of what conditions on $z(\tilde{x},\alpha,\lambda)$ are required to derive <u>determinate</u> comparative static results concerning the effect of an arbitrary R-S increase in risk on the choice variable selected by all risk averse decision makers. While Meyer and Ormiston answer the question, unfortunately, they conclude that an arbitrary R-S increase in risk, or an FSD shift, causes all risk averse decision makers to decrease the choice variable if and only if the optimal value of the choice variable does not depend on the random variable. Clearly, this is not an interesting case.

This negative result has one important implication for studying comparative statics under uncertainty. In order to obtain interesting comparative statics results, additional restrictions must be imposed on the risk taking characteristics of the decision maker, the objective function, or the type of change in randomness. This same conclusion is supported by examination of the literature. Invariably, restrictions are made before the researcher is able to obtain interesting comparative static results.

Although a global increase in risk imposes an additional restriction sufficiently strong to yield determinate comparative static results for all risk-averse decision makers, the added restriction is rather severe, and limits significantly the situations to which those results can be applied. Meyer and Ormiston[1985] extend the results of Kraus[1979] and Katz[1981] by developing a definition of an increase in risk which includes a global increase in risk as a special case, yet yields determinate comparative static analysis for arbitrary risk averse decision makers. This type of risk increase, termed a strong increase in risk, is defined as follows.

Definition 2.4: G(x) represents a strong increase in risk from F(x) if G(x) - F(x) satisfies

(a) $\int_a^s [G(x) - F(x)]dx \geq 0 \quad \forall\ s \in [a,f]$

(b) $\int_a^f [G(x) - F(x)]dx = 0$

(c) G(x) - F(x) is nonincreasing on (b,e), where the support of F is contained in [b,e], the support of G is contained in [a,f], and $a \leq b \leq e \leq f$.

The first two conditions simply require that a strong increase in risk be a R-S increase in risk. The third property is the added condition which identifies this particular type of risk increase, and which allows determinate statements to be made concerning the effect of a strong increase in risk on the choice variable selected by a risk averse decision maker. Strong increases in risk transfer probability mass from points in the interval (b,e) to [a,b] and [e,f], i.e. outcomes which were previously possible become less likely, and other outcomes either larger or smaller than previously possible now have a nonzero probability of occurring. Thus, strong increases in risk include risk increases from a risky position as well as from a no risk position. Meyer and Ormiston present the following theorem.

Theorem 2.2: Assuming that decision makers choose α in order to maximize $Eu(z(\tilde{x},\alpha,\lambda))$ where $u'(z) \geq 0$, $u''(z) \leq 0$, $z_x \geq 0$, and $z_{\alpha\alpha} < 0$, then all risk averse decision makers, when faced with a strong increase in risk, will decrease the optimal value of α if $z_{\alpha x} \geq 0$ and $z_{\alpha xx} \leq 0$.

Note that the restrictions on $u(z)$ and $z(\tilde{x},\alpha,\lambda)$ in this theorem are exactly those conditions determined by Katz[1981] to be sufficient for similar comparative static results for global increases in risk. Thus, Theorem 2.2 is a generalization of Theorem 2.1.

Recently, Black and Bulkley[1989] extend the result of Meyer and Ormiston[1985] by introducing a type of risk increase, termed a relatively strong increase in risk. The relatively strong increase in risk includes a strong increase in risk as a special case and is also sufficient to sign the effect on α given the same conditions on $u(z)$ and $z(\tilde{x},\alpha,\lambda)$. Black and Bulkley define the relatively strong increase in risk by using the ratio of probability densities. Their definition is:

Definition 2.5: $G(x)$ represents a relatively strong increase in risk compared with $F(x)$ if

(a) $\int_a^f [G(x) - F(x)]dx = 0$

(b) For all points in the interval[c,d], $f(x) \geq g(x)$ and for all points outside this interval $f(x) \leq g(x)$ where $a \leq b \leq c \leq d \leq e \leq f$, with [a,f] being the supports of x under G(x) and [b,e] being the supports under F(x)

(c) f(x)/g(x) is non-decreasing in the interval [b,c)

(d) f(x)/g(x) is non-increasing in the interval (d,e]

Conditions (a) and (b) are sufficient for G(x) to represent a R-S increase in risk, and also to represent a strong increase in risk. It is the case where b = c and d = e that is considered by Meyer and Ormiston[1985]. The main result of Black and Bulkley is summarized in the following theorem.

<u>Theorem 2.3</u>: Assuming that decision makers choose α in order to maximize $Eu(z(\tilde{x},\alpha,\lambda))$ where $u'(z) > 0$, $u''(z) \leq 0$, $z_x > 0$, and $z_{\alpha\alpha} < 0$, then all risk averse decision makers, when faced with a relatively strong increase in risk, will decrease α if $z_{\alpha x} \geq 0$ and $z_{\alpha xx} \leq 0$.

The result of B-B is exactly same as that of M-O, except that relatively strong increases replace strong increases in risk.

Meyer and Ormiston[1989] observe that the literature concerning the transformation approach to risk analysis has focused almost totally on the linear mean-preserving transformation proposed by Sandmo, and little has been done to explore the usefulness of more general <u>nonlinear</u> transformations. M-O introduce a type of risk increase, termed a simple increase in risk, which includes the Sandmo linear transformation as a special case. The definition is given as follows.

Definition 2.6: The transformation $t(x)$ represents a simple increase in risk for a random variable given by $F(x)$ if the function $k(x) = t(x) - x$ satisfies

(a) $\int_0^B k(x)dF(x) = 0$

(b) $\int_0^s k(x)dF(x) \leq 0 \quad \forall s \in [0,B]$

(c) $k'(x) \geq 0$

The transformation $t(x)$ is assumed to be nondecreasing, continuous and piecewise differentiable. The nondecreasing assumption combined with monotonic preferences for outcomes ensures that the transformation does not reverse the preference ordering over the outcomes of the original random variable. Meyer and Ormiston show that if the function $k(x)$ satisfies the first two conditions, then it reduces expected utility for all risk averse decision makers. Thus, the transformation can be interpreted as an increase in risk in the Rothschild and Stiglitz sense. The third property, $k'(x) \geq 0$, is the added condition which identifies this particular type of risk increases, and allows general statements to be made concerning the effect of a simple increase in risk on the choice variable selected by a group of decision makers. The simple increase in risk which is a subclass of a R-S increase in risk is carried out using a nonlinear deterministic transformation of the random variable. As stated earlier, Sandmo considers a special type of the simple increase in risk where $k(x) = (\gamma - 1)\cdot(\tilde{x} - \bar{x})$ and $k'(x) = (\gamma - 1) \geq 0$. We call an increase in risk represented by an increase in γ an increase in γ. Meyer and Ormiston present the following theorem.

Theorem 2.4: An economic decision maker choosing α to maximize

$Eu(z(\tilde{x},\alpha,\lambda))$ will decrease the optimal value of α selected when the random variable is transformed according to $t(x) = x + k(x)$ if

(a) $u(z)$ displays decreasing absolute risk aversion,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $t(x)$ is a simple increase in risk.

It is interesting to note that there is no relationship between a simple and a strong increase in risk. Also, a simple increase in risk requires more restrictive conditions on preferences and the objective function, such as DARA and $z_{xx} < 0$, than does a strong increase in risk, to derive unambiguous comparative static results.

Very recently, Ormiston[1992] observes that increases in risk have been frequently examined in the literature, but general FSD changes in $\tilde{x}$, using a deterministic transformation, have not been examined. The definition and main result of Ormiston are given as follows.

<u>Definition 2.7</u>: A deterministic transformation represents a FSD improvement in $\tilde{x}$ if and only if $k(x) \geq 0$ for all x in [0,B].

<u>Theorem 2.5</u>: Assuming that decision makers choose α in order to maximize $Eu(z(\tilde{x},\alpha,\lambda))$ where $u'(z) \geq 0$, $u''(z) \leq 0$, $z_x \geq 0$, and $z_{\alpha\alpha} < 0$, then all risk averse decision makers exhibiting DARA, when faced with a FSD improvement in $\tilde{x}$, will increase the optimal value of α if $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$ and $k'(x) \leq 0$.

Sandmo[1971] considers a special case of the FSD improvement in $\tilde{x}$ where $k(x) = \theta > 0$ and $k'(x) = 0$. That is, the FSD transformation used by Sandmo is linear in $\tilde{x}$. We call a FSD improvement represented by an increase in θ an increase in θ. Cheng, Magil and Shafer[1987] analyze the comparative static effects of a first degree stochastically dominant shift in the distribution of a random variable in several variants of the standard portfolio model.

All the above analysis assumes the first order condition is satisfied as an equality. Dionne, Eeckhoudt and Gollier[1991] note that the assumption of an interior or a bounded solution can be quite restrictive and may rule out some interesting cases. The following example is given to highlight the issue. Consider a specific 1-1-1 model, in which z takes the form $z(\tilde{x},\alpha,\lambda) = \alpha\cdot(\tilde{x} - \lambda_1) + \lambda_2$, where $\tilde{x}$ is a random variable, α is a choice variable, and λ_1 and λ_2 are nonrandom parameters. This formulation includes as special cases the standard portfolio problem, the problem of the competitive firm with constant marginal costs and the insurance problem; that is, λ_2 can be interpreted as an initial endowment or a fixed cost, while λ_1 may represent either a marginal cost of production, a sure interest rate or a marginal cost of insurance. Obviously here $z_x \geq 0$, when $\alpha \geq 0$, $z_{\alpha\alpha}$ and z_{xx} are equal to zero; which means that the outcome variable z is linear in α and x. The utility function u(z) is assumed to be three times differentiable with $u'(z) > 0$ and $u''(z) \leq 0$.

If we constrain the choice variable α to take values in the interval [0,1], then a corner solution can be defined as a solution determined by the constraint; that is, either $\alpha = 0$ or $\alpha = 1$ is a corner solution. Similarly, if we assume that α takes values in the interval $[0,\infty]$, then an unbounded solution, $\alpha = \infty$, may occur and $\alpha = 0$ is referred to a corner solution. Note that corner solutions are not unbounded but are <u>constrained</u> solutions in this discussion.

First, consider the case where $0 \le \alpha \le 1$ and $u''(z) = 0$. Assuming $\bar{x} > \lambda_1$, then a corner solution, $\alpha = 1$, occurs since the first order condition evaluated at $\alpha = 0$, $Eu'(\lambda_2)(\tilde{x} - \lambda_1)$, is positive <u>and</u> the second order condition is always <u>zero</u>. In a similar manner, when $\bar{x} < \lambda_1$, a corner solution, $\alpha = 0$, occurs. If $\bar{x} = \lambda_1$, then solutions are undetermined. Thus, the set of decision makers exhibiting $u''(z) = 0$ should be ruled out for a unique interior solution.

Next, consider the case where $0 \le \alpha \le 1$ and $u''(z) < 0$. If $\bar{x} < \lambda_1$, then a corner solution, $\alpha = 0$, occurs because $Eu'(\lambda_2)\cdot(\tilde{x} - \lambda_1) < 0$ and the second order condition is always negative. However, in this case, a unique interior solution may occur if $\bar{x} > \lambda_1$. That is, the condition $\bar{x} > \lambda_1$ is necessary for interior solutions, $0 < \alpha < 1$. Suppose that a decision maker is risk neutral. Then the decision maker will choose $\alpha = 0$, when $\bar{x} < \lambda_1$, while $\alpha = 1$, when $\bar{x} > \lambda_1$. Therefore, by continuity of preferences, some risk averse agents with a relatively low risk aversion ($R_A \to 0$) select the same when $\bar{x} > \lambda_1$ or $\bar{x} < \lambda_1$.

In this chapter we have reviewed the literature concerning the 1-1-1 model to see what types of changes in randomness have been used in either general or specific decision models. In the next chapter we use a general 1-2-1 decision model formulation and consider the effects of three types of changes in the random parameter: a simple increase in risk, a relatively strong increase in risk, and a FSD improvement in $\tilde{x}$. SSD shifts will not be explictly considered since they can be thought of as combinations of an FSD and an MPC shift.

CHAPTER THREE

The 1-2-1 MODEL

3.0 Introduction

An important question in the study of economic decision models involving randomness is how does a particular type of a change in random variable $\tilde{x}$ affect the level of the choice variable selected by a decision maker. In the 1-1-1 model, the problems of dealing with two first order conditions that occur in decision models with two choice variables do not arise. Because of this, the 1-1-1 model has received much attention in the literature both in general as well as specific decision models.

On the other hand, the 1-2-1 model has received much less attention in the literature. Several 1-2-1 models have been examined, but only in the context of specific decision models and only with less general changes in $\tilde{x}$. For example, Batra and Ullah[1974], Feder[1977], Feder, Just and Schmitz(FJS)[1977], and Katz, Paroush and Kahana(KPK) [1982] present such models and analysis.

These models analyze the effects of three types of changes in randomness: an increase in risk represented by an increase in γ, a global increase in risk, and a FSD improvement in $\tilde{x}$ represented by an increase in θ. The effects of more general types of changes in the random parameter, for the general 1-2-1 decision model, have not been analyzed. This will be done in this chapter.

We extend the 1-2-1 models in the literature by considering a general 1-2-1 model, and by analyzing the effects of three general types of changes in randomness: a simple increase in risk, a relatively strong increase in risk, and a more general FSD improvement. In addition, a diagram method is introduced, which handles corner solutions and may have a pedagogical value. The usual algebraic method does not work unless the first order conditions are satisfied as equalities.

The issue of a corner solution is more interesting in the two choice variable case than in the one choice variable case. Assuming a corner solution in a 1-1-1 model does not allow comparative statics to be conducted for small changes because the only choice variable is determined by the constraint. However, in a 1-2-1 model, because there are two choice variables and one of them can be an interior solution, comparative statics analysis can still be carried out. All 1-2-1 models in the literature so far assume interior solutions.

This chapter proceeds as follows. In the next section we first review the literature concerning the 1-2-1 model and then discuss two types of assumptions which make the 1-2-1 model tractable. Several theorems concerning the effects of changes in randomness are developed. Section 2 illustrates the use of our findings by extending a number of published results. Section 3 examines a specific 1-2-1 model which includes the Sandmo model of the competitive firm as a special case, and considers corner solutions.

3.1 A Generalization

3.1.1 Literature Review

Batra and Ullah[1974] examine a competitive firm's input decisions under output price uncertainty for the two input case. For Batra and

Ullah's model, the outcome variable z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot f(\alpha,\delta) - \lambda_1\cdot\alpha - \lambda_2\cdot\delta$, where $\tilde{x}$ is the price of output, α and δ are two inputs, $\lambda_i (i=1,2)$ represent the prices of the inputs, and f is a production function. They analyze the effects on α and δ of a global increase in risk, an increase in γ, and an increase in θ. Batra and Ullah find that if $z_{\alpha\delta} = \tilde{x}\cdot f_{\alpha\delta} > 0$, then determinate comparative static results concerning the effects on α and δ of changes in randomness can be derived.

Although the outcome variable z for Feder[1977]'s decision model has a specific form such that $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot f(\alpha,\delta) + g(\alpha,\delta) + \lambda$, the model can be classified as a general decision model because it does include many specific models as special cases. Feder transforms the random variable $\tilde{x}$ according to $t(\tilde{x}) = \tilde{x} + \gamma\cdot(\tilde{x} - \bar{x})$, where $\bar{x}$ is the mean of $\tilde{x}$ and γ is a positive nonrandom parameter. He then demonstrates that an increase in γ implies the R-S increase in risk. Feder analyzes the effect on $f(\alpha,\delta)$ of an increase in γ. He also determines the effect of an increase in θ, and a change in the nonrandom parameter, λ. It should be noted that even though Feder assumes a general form for the objective function which includes more than one choice variable, he did not derive comparative statics concerning the effects on <u>the choice variables</u>, α and δ, of changes in the random and nonrandom parameters, <u>except</u> for the one choice variable case. Interestingly, Feder does not consider a global increase in risk, a change whose effects are usually analyzed.

Feder, Just and Schmitz(FJS)[1977] investigate an international trade model with one random and two choice variables, in which z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot[f(\alpha) + \delta - \lambda_1] + g(\lambda_2 - \alpha - \lambda_3\cdot\delta) - \lambda_4\cdot\delta$.

Here λ_2 is the fixed amount of capital, which can be used to produce two goods, F and G, or to store F. $f(\cdot)$ and $g(\cdot)$ are production functions for F and G, respectively. It is assumed that a fixed amount of F has to be consumed. This is denoted λ_1. The amount of capital required for storings F is proportional to the amount stored and given by $\lambda_3 \cdot \delta$. The amount of capital used to produce F is denoted α. Then the available capital for producing G is $\lambda_2 - \alpha - \lambda_3 \cdot \delta$. λ_4 represents the ratio of F and G's prices. The random variable $\tilde{x}$ captures randomness in price of F. FJS transform the random variable $\tilde{x}$ according to $t(\tilde{x}) = \bar{x} + \gamma \cdot (\tilde{x} - \bar{x})$ to represent an increase in risk in the R-S sense. FJS examine the effects on α and δ of an increase in γ and an increase in θ. However, they also do not consider the effect of a global increase in risk. Note that $z_{\alpha\delta} = \lambda_3 \cdot g''(\cdot) < 0$ is a characteristic of their model.

Katz, Paroush and Kahana(KPK)[1982] examine the optimal policy of a price discriminating firm which operates under price uncertainty in one of two markets. The price discrimination model of KPK has outcome variable given by $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x} \cdot R_1(\alpha) + \lambda_1 \cdot R_2(\delta) - c(\alpha + \delta) - \lambda_2$. In this case $\tilde{x}$ represents randomness in market 1, α and δ are the firm's sales in market 1 and market 2 respectively, λ_1 represents demand conditions in market 2, and λ_2 is fixed cost. In addition, $R_i(i=1,2)$ is the revenue function of market i, and $c(\cdot)$ is the variable cost function. KPK assume that $\tilde{x}$ can be written as: $\tilde{x} = \bar{x} + \gamma \cdot \tilde{\xi}$ where $E(\xi) = 0$, $E(\tilde{\xi}^2) = 1$, and $\text{Prob}(\xi > -(\bar{x}/\gamma)) = 1$. They analyze the effects on α and δ of a global increase in risk, an increase in γ, and an increase in $\bar{x}$. Changing γ is a mean-preserving linear transfomation which is a special type of a R-S increase in risk. An increase in $\bar{x}$ represents a special case of a FSD improvement in $\tilde{x}$.

These models are all specific and only consider quite restrictive changes in the random parameter. This observation provides us an impetus for this chapter. Decision models with two choice variables are rather difficult to analyze because of the problems of handling two first order conditions. To formulate a general 1-2-1 model, we look at these four specific models to ask what general structure makes determinate comparative statics possible.

3.1.2 The Comparative Statics Problem

In the general 1-2-1 model, utility depends on one outcome variable which in turn depends on one random parameter, two choice variables, and a set of nonrandom parameters. In this model the decision maker is assumed to choose α and δ to maximize $Eu(z(\tilde{x},\alpha,\delta,\lambda)) = \int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$. The outcome variable, z, depends on a random variable, $\tilde{x}$, two choice variables, α and δ, and a set of nonrandom parameters, λ. It is assumed that the random variable of concern, $\tilde{x}$, is defined by a CDF, denoted F(x), with support in the interval [0,B]. This formulation includes all four specific 1-2-1 models which were just reviewed as special cases.

The utility function u(z) is assumed to be three times differentiable with $u'(\cdot) > 0$ and $u''(\cdot) < 0$; thus, the decision maker is a strict risk averter. The function $z(\tilde{x},\alpha,\delta,\lambda)$ is assumed three times differentiable with $z_{\alpha\alpha} < 0$, $z_{\delta\delta} < 0$, and $z_{\alpha\alpha}\cdot z_{\delta\delta} - z_{\alpha\delta}^2 > 0$. This condition on z, combined with $u''(\cdot) < 0$, ensures that the second-order condition for the maximization problem is satisfied. To simplify the discussion, we will focus on the case where $z_x \geq 0$. This assumption, combined with $u'(z) > 0$, indicates that higher values of the random variable are preferred to lower values. Initially, to focus on interior solutions to the maximization problem, it is assumed that both $z_\alpha = 0$ and $z_\delta = 0$ are satisfied for some finite α and δ for all $x \in [0,B]$.

Given these assumptions, the first and second order conditions of the 1-2-1 model can be written as:

$$H_1(\alpha,\delta,\lambda) = \partial Eu(z)/\partial\alpha = Eu'(z)\cdot z_\alpha = \int_0^B u'(z)\cdot z_\alpha\cdot dF(x) = 0$$

$$H_2(\alpha,\delta,\lambda) = \partial Eu(z)/\partial\delta = Eu'(z)\cdot z_\delta = \int_0^B u'(z)\cdot z_\delta\cdot dF(x) = 0$$

$$H_{11} = E[u'(z)\cdot z_{\alpha\alpha} + u''(z)\cdot z_\alpha^2] < 0$$

$$H_{22} = E[u'(z)\cdot z_{\delta\delta} + u''(z)\cdot z_\delta^2] < 0$$

$$H_{12} = E[u'(z)\cdot z_{\alpha\delta} + u''(z)\cdot z_\alpha\cdot z_\delta]$$

$$H = H_{11}\cdot H_{22} - H_{12}^2 > 0$$

The comparative static questions addressed here are how do the optimal values of α and δ change when random variable $\tilde{x}$ undergoes a simple increase in risk, a relatively strong increase in risk, or a FSD improvement in $\tilde{x}$. In its general form, a 1-2-1 model requires a fairly rigid structure for the comparative statics to be determined.

To see this, suppose that a relatively strong increase in risk occurs. Then, by using Cramer's rule, we have the following comparative statics: $\partial\alpha/\partial(r.s.) = (1/H)\cdot[\, - (\partial H_1/\partial(r.s.))\cdot H_{22} + (\partial H_2/\partial(r.s.))\cdot H_{12}\,]$, $\partial\delta/\partial(r.s.) = (1/H)\cdot[\, - (\partial H_2/\partial(r.s.))\cdot H_{11} + (\partial H_1/\partial(r.s.))\cdot H_{12}\,]$, where $\partial H_1/\partial(r.s.)$ and $\partial\alpha/\partial(r.s.)$ represent the effect of a relatively strong increase in risk on H_1 and α, respectively. Notice that even if the signs of $\partial H_1/\partial(r.s.)$ and $\partial H_2/\partial(r.s.)$ are known, this is not sufficient for deriving unambiguous comparative statics because H_{12} can be positive or negative. For example, suppose that $\partial H_1/\partial(\text{r.s.}) < 0$ and $\partial H_2/\partial(\text{r.s.}) < 0$. Then, we have unambiguous comparative statics if $H_{12} > 0$.

Assumptions such as $\partial H_2/\partial(\text{r.s.}) = 0$ allow the comparative statics to be simplified. Given $\partial H_2/\partial(\text{r.s}) = 0$, $\partial\alpha/\partial(\text{r.s.})$ has the same sign as $\partial H_1/\partial(\text{r.s.})$ since $H_{22} < 0$. Similarly, $\partial\delta/\partial(\text{r.s.})$ and $\partial H_1/\partial(\text{r.s.})\cdot H_{12}$ have the same sign. This is one of the conditions which are present in the specific models reviewed earlier that makes determinate comparative statics possible. The condition $z_{\delta x} = 0$, which is satisfied in KPK, simplifies the analysis. It implies that z_δ does not depend on the random variable; thus, the condition $H_2 = 0$ is equivalent to $z_\delta = 0$. That is, a change in $\tilde{x}$ does not affect the condition $H_2 = 0$.

Another condition, $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$, is met in several of specific 1–2–1 models in the literature. This is the case for the Batra and Ullah[1974] model, in which $H_1 = Eu'(z)\cdot z_\alpha = Eu'(\cdot)\cdot(\tilde{x}\cdot f_\alpha - \lambda_1) = 0$ and $H_2 = Eu'(z)\cdot z_\delta = Eu'(\cdot)\cdot(\tilde{x}\cdot f_\delta - \lambda_2) = 0$. Because $f_\alpha/f_\delta = \lambda_1/\lambda_2$, this implies that $(z_{\alpha x}/z_{\delta x})\cdot z_\delta = (f_\alpha/f_\delta)\cdot(\tilde{x}\cdot f_\delta - \lambda_2) = \tilde{x}\cdot f_\alpha - \lambda_1 = z_\alpha$. The condition is also satisfied in Feder[1977] and FJS[1977]. In order to make a 1–2–1 model tractable, we shall assume either that $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ where $(z_{\alpha x}/z_{\delta x})$ is nonrandom, or that $z_{\delta x} = 0$.

3.1.3 The Comparative Static Results for the Case: $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$

Even though there are two first order conditions, $H_1 = 0$ and $H_2 = 0$, in the 1–2–1 model, notice that looking at either of them individually is the same as looking at a 1–1–1 model. Therefore, Theorem 2.3 – 2.5 derived for the 1–1–1 model by Black and Bulkley [1989], Meyer and Ormiston[1989], and Ormiston[1992] can be used. First, we deal with simple increases in risk.

Let $H_1(\alpha,\delta,\lambda,\theta)$ denote the derivative with respect to the choice variable α of expected utility when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$, where $0 \le \theta \le 1$; that is,

$$H_1(\alpha,\delta,\lambda,\theta) = \int_0^B u'(z(x + \theta k(x),\alpha,\delta,\lambda))\cdot z_\alpha(x + \theta k(x),\alpha,\delta,\lambda)\cdot dF(x).$$

Note that $H_1(\alpha,\delta,\lambda,0) = 0$ for the initial optimal values of α and δ.

<u>Corollary 3.1</u>: $\partial H_1/\partial\theta|_{\theta=0} < 0$ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA

(b) $z_x \ge 0$, $z_{xx} \le 0$, $z_{\alpha x} \ge 0$, and $z_{\alpha xx} \le 0$

(c) $t(x)$ represents a simple increase in risk

<u>Proof</u>: The proof of corollary 3.1 is given in Meyer and Ormiston[1989] and is simply sketched here. For the initial optimal value of α and δ,

$$\partial H_1/\partial\theta|_{\theta=0} = \int_0^B [u'(z)\cdot z_{\alpha x} + u''(z)\cdot z_\alpha\cdot z_x]\cdot k(x)\cdot dF(x)$$
$$= \int_0^B u'(z)\cdot z_{\alpha x}\cdot k(x)\cdot dF(x) + \int_0^B u''(z)\cdot z_\alpha\cdot z_x\cdot k(x)\cdot dF(x)$$

M–O show that under the conditions of the corollary, $\partial H_1/\partial\theta|_{\theta=0} < 0$.

<u>Corollary 3.1'</u>: $\partial H_2/\partial\theta|_{\theta=0} < 0$ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA

(b) $z_x \ge 0$, $z_{xx} \le 0$, $z_{\delta x} \ge 0$, and $z_{\delta xx} \le 0$

(c) $t(x)$ represents a simple increase in risk

<u>Theorem 3.1</u>: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} = 0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ at $H_1 = 0$ and $H_2 = 0$ and $z_{\alpha\delta} > 0$,

(d) t(x) represents a simple increase in risk.

<u>Proof</u>: If $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ is a characteristic of the model, and $z_{\alpha x}$ and $z_{\delta x}$ do not depend on the random variable, then $H_1 = (z_{\alpha x}/z_{\delta x})\cdot H_2$ at $H_i = 0$. Therefore, $\partial H_2/\partial\theta = (z_{\delta x}/z_{\alpha x})\cdot\partial H_1/\partial\theta$. By using Cramer's rule,

$$\partial\alpha/\partial\theta|_{\theta=0} = (1/H)\cdot[-(\partial H_1/\partial\theta)\cdot H_{22} + (\partial H_2/\partial\theta)\cdot H_{12}]$$

$$= -(1/H)\cdot(\partial H_1/\partial\theta)\cdot[H_{22} - (z_{\delta x}/z_{\alpha x})\cdot H_{12}].$$

Simplifying the term $[H_{22} - (z_{\delta x}/z_{\alpha x})\cdot H_{12}]$ using $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$, $[H_{22} - (z_{\delta x}/z_{\alpha x})\cdot H_{12}] = E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\alpha x}\cdot z_{\delta\delta} - z_{\delta x}\cdot z_{\alpha\delta})$. Substituting this into $\partial\alpha/\partial\theta|_{\theta=0}$, $\partial\alpha/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\alpha x}z_{\delta\delta} - z_{\delta x}z_{\alpha\delta})$. By corollary 3.1, $\partial H_1/\partial\theta|_{\theta=0} < 0$. Thus, under the conditions of the theorem, $\partial\alpha/\partial\theta|_{\theta=0} < 0$. By using a similar procedure, we can show that $\partial\delta/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\delta x}z_{\alpha\alpha} - z_{\alpha x}z_{\alpha\delta}) < 0$.

Theorem 3.1 gives conditions sufficient to yield unambiguous comparative static results concerning the effect on α and δ of a simple increase in risk. Condition (a) restricts the set of decision makers to those exhibiting DARA. DARA is generally thought to be a reasonable assumption concerning preferences. Condition (b) restricts the model. Note that $z_{xx} < 0$ is allowed in the 1-1-1 model, but in the 1-2-1 model, $z_{xx} = 0$ is required to make $(z_{\alpha x}/z_{\delta x})$ nonrandom. Also, $z_{\alpha xx} = 0$ since $z_{xx} = 0$. Condition (c) further restricts the model. The condition that $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ is assumed to make the 1-2-1 model tractable. The condition $z_{\alpha\delta} > 0$ is added to allow determinate statements to be made concerning the effect on α and δ of a change in the random parameter.

Condition (d) requires that an increase in risk be simple. Even though a simple increase in risk is a subclass of the R-S increase in risk, it includes the Sandmo linear transformation represented as an increase in γ as a special case. While these conditions are restrictive, they do include the models presented by Batra and Ullah, Feder, and FJS as special cases.

Now, we deal with relatively strong increases in risk. It is interesting to observe that since Kraus[1979], the CDF approach has been used to analyze the effect of this type of increases in risk. Let $\partial H_1/\partial(r.s.)$ denote the effect on H_1 of a relatively strong increase in risk; that is, $\partial H_1/\partial(r.s.) = \int_0^B u'(z)\cdot z_\alpha\cdot d[G(x) - F(x)]$, where G(x) represents a relatively strong increase in risk from F(x). For the optimal value of α and δ under F(x), $H_1(\alpha,\delta,\lambda) = \int_0^B u'(z)\cdot z_\alpha\cdot dF(x) = 0$.

Corollary 3.2: Assume that a relatively strong increase in risk occurs. Then, $\partial H_1/\partial(r.s.) < 0$ if

(a) $u'(z) > 0$ and $u''(z) < 0$

(b) $z_x \geq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$

Proof: The proof of corollary 3.2 is given in Black and Bulkley[1989] and is simply sketched here. B-B demonstrate that given the assumptions about $z(\tilde{x},\alpha,\delta,\lambda)$ and u(z), $\partial H_1/\partial(r.s.) = \int_0^B u'(z)\cdot z_\alpha\cdot d[G(x) - F(x)] < 0$ for the optimal value of α and δ under F(x), where G(x) represents a relatively strong increase in risk from F(x).

Corollary 3.2': Assume that a relatively strong increase in risk occurs. Then, $\partial H_2/\partial(r.s.) < 0$ if

(a) $u'(z) > 0$ and $u''(z) < 0$

(b) $z_x \geq 0$, $z_{\delta x} \geq 0$, and $z_{\delta xx} \leq 0$

<u>Theorem 3.2</u>: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ if

(a) $u'(z) > 0$ and $u''(z) < 0$,

(b) $z_x \geq 0$, $z_{xx} = 0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ at $H_i = 0$ and $z_{\alpha\delta} > 0$,

(d) $G(x)$ represents a relatively strong increase in risk from $F(x)$.

<u>Proof</u>: By using a similar procedure which is provided for theorem 3.1,

$$\partial\alpha/\partial(r.s.) = -(1/H)\cdot(\partial H_1/\partial(r.s.))\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\alpha x}\cdot z_{\delta\delta} - z_{\delta x}\cdot z_{\alpha\delta}),$$

$$\partial\delta/\partial(r.s.) = -(1/H)\cdot(\partial H_1/\partial(r.s.))\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\delta x}\cdot z_{\alpha\alpha} - z_{\alpha x}\cdot z_{\alpha\delta}).$$

By corollary 3.2, $\partial H_1/\partial(r.s.) < 0$. Thus, one can conclude that under the conditions of the theorem, the optimal value of α and δ are decreased.

Theorem 3.2 gives conditions sufficient to yield unambiguous comparative static results concerning the effect on α and δ of a relatively strong increase in risk. Condition (a) only requires that the decision maker be risk averse. Conditions (b) and (c) are the same as in theorem 3.1. Condition (d) requires that an increase in risk be relatively strong. While a relatively strong increase in risk is a special type of the R-S increase in risk, it includes a global increase in risk as a special case. Note that even though there is no relationship between simple and relatively strong increases in risk, more restrictive condition on preferences, such as DARA, is required in theorem 3.1 than in theorem 3.2.

Finally, we deal with a FSD improvement in the random variable.

The transformation approach is used here as in corollary 3.1.

Corollary 3.3: $\partial H_1/\partial\theta|_{\theta=0} > 0$ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) u(z) displays DARA

(b) $z_x \geq 0$, $z_{xx} \leq 0$, and $z_{\alpha x} \geq 0$

(c) t(x) represents a FSD improvement in $\tilde{x}$ and $k'(x) \leq 0$

Proof: The proof of corollary 3.3 is given in Ormiston[1992] and is simply sketched here. For the initial optimal value of α and δ,

$$\partial H_1/\partial\theta|_{\theta=0} = \int_0^B (u'(z)\cdot z_{\alpha x} + u''(z)\cdot z_\alpha\cdot z_x)\cdot k(x)\cdot dF(x)$$
$$= \int_0^B u'(z)\cdot z_{\alpha x}\cdot k(x)\cdot dF(x) + \int_0^B [-R_A(z)\cdot z_x\cdot k(x)]\cdot u'(z)\cdot z_\alpha\cdot dF(x).$$

Ormiston demonstrates that under the conditions of the corollary, $\partial H_1/\partial\theta|_{\theta=0} > 0$. This is a local result in the sense that the result holds only at $\theta = 0$ and for small changes in θ.

Corollary 3.3': $\partial H_2/\partial\theta|_{\theta=0} > 0$ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) u(z) displays DARA

(b) $z_x \geq 0$, $z_{xx} \leq 0$, and $z_{\delta x} \geq 0$

(c) t(x) represents a FSD improvement in $\tilde{x}$ and $k'(x) \leq 0$

Theorem 3.3: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will increase the optimal values of α and δ when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) u(z) displays DARA,

(b) $z_x \geq 0$, $z_{xx} = 0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$ at $H_i = 0$ and $z_{\alpha\delta} > 0$,

(d) $k(x) \geq 0$ and $k'(x) \leq 0$.

Proof: By using a similar procedure which is provided for theorem 3.1,

$$\partial\alpha/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\alpha x}\cdot z_{\delta\delta} - z_{\delta x}\cdot z_{\alpha\delta})$$

$$\partial\delta/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot E(1/z_{\alpha x})\cdot u'(z)\cdot(z_{\delta x}\cdot z_{\alpha\alpha} - z_{\alpha x}\cdot z_{\alpha\delta}).$$

By corollary 3.3, $\partial H_1/\partial\theta|_{\theta=0} > 0$; that is, given the assumptions about $z(\tilde{x},\alpha,\delta,\lambda)$, $u(z)$, and $k'(x)$, the sign of $H_1(\alpha,\delta,\lambda,\theta)$, evaluated at the initial optimal values of α and δ and $\theta = 0$, changes from zero to positive as a result of a FSD improvement.

Theorem 3.3 gives conditions sufficient to yield unambiguous comparative static results concerning the effect on α and δ of a FSD improvement in $\tilde{x}$. As in theorem 3.1, condition (a) requires that preferences exhibit DARA. Conditions (b) and (c) are the same as in Theorem 3.1 and 3.2. Condition (d) implies that not all FSD improvements in $\tilde{x}$ permit the effect on α and δ to be signed; that is, the condition $k'(x) \leq 0$ is also required. The specific 1-2-1 models reviewed earlier consider a special case of these FSD improvements; that is, $k(x) = \theta > 0$ and $k'(x) = 0$ where θ is a positive nonrandom parameter.

3.1.4 The Comparative Static Results for the Case: $z_{\delta x} = 0$

In this section, we consider a 1-2-1 model in which $z_{\delta x} = 0$, and address the same comparative static questions as in 3.1.3. The condition $z_{\delta x} = 0$ allows the comparative static analysis to be simplified. We first write down the following three theorems, and then demonstrate that they can be proved by using either the algebraic or graphical approach. The algebraic method will be only used for proving

Theorem 3.6. This is because Theorem 3.4 – 3.6 will be also proved by using a diagram method.

Theorem 3.4: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ (decrease α and increase δ) when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $t(x)$ represents a simple increase in risk.

Theorem 3.5: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ(decrease α and increase δ) if

(a) $u'(z) > 0$ and $u''(z) < 0$,

(b) $z_x \geq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $G(x)$ represents a relatively strong increase in risk from $F(x)$.

Theorem 3.6: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will increase the optimal values of α and δ (increase α and decrease δ) when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $k(x) \geq 0$ and $k'(x) \leq 0$.

Proof: $z_{\delta x} = 0$ implies that the condition $H_2 = 0$ is equivalent to $z_\delta = 0$. Thus, $\partial H_2/\partial\theta = 0$. Then we have the following comparative statics: $\partial\alpha/\partial\theta = -(1/H)\cdot[(\partial H_1/\partial\theta)\cdot H_{22}]$, $\partial\delta/\partial\theta = (1/H)\cdot[(\partial H_1/\partial\theta)\cdot H_{12}]$. Note that $H_{12} = Eu'(z)\cdot z_{\alpha\delta}$ since $z_\delta = 0$ from $H_2 = 0$, and $\partial H_1/\partial\theta > 0$ by corollary 3.3. Thus, under the conditions of the theorem, $\partial\alpha/\partial\theta > 0$ and $\partial\delta/\partial\theta > 0$ when $z_{\alpha\delta} > 0$, while $\partial\alpha/\partial\theta > 0$ and $\partial\delta/\partial\theta < 0$ when $z_{\alpha\delta} < 0$.

Now, we introduce a diagram method to solve the same problem and to be available for handling corner solutions. While the diagram approach is a supplement to the algebraic approach, all of our results derived for the 1-2-1 model in which $z_{\delta x} = 0$ can also be proved by the use of a simple diagram in the (α,δ) plane. Furthermore, if a corner solution exists, the key assumption of the comparative statics, that the first-order condition is satisfied with equality, does not hold, yet the diagram methods still can be used.

Define m_1 and m_2 as the set of points in the (α,δ) plane which yield $H_1(\alpha,\delta,\lambda) = 0$ and $H_2(\alpha,\delta,\lambda) = 0$ respectively. To ensure the existence of the m_1 and m_2 curves, it is assumed that $H_1 = 0$ and $H_2 = 0$ are satisfied for some finite α and δ. It is interesting to note that the m_1 and m_2 curves are similar to the reaction curves which are used to examine a Cournot duopoly model.

Lemma 3.1: If $H_{12} >(<) 0$ for all points in (α,δ) plane, then the m_1 and m_2 curves have positive(negative) slopes and at the intersection point, |slope of m_1| > |slope of m_2|, where $|\cdot|$ means absolute value.

Proof: Totally differentiating $H_1(\alpha,\delta,\lambda) = 0$ yields $H_{11}\cdot d\alpha + H_{12}\cdot d\delta = 0$. For a movement along m_1, $d\delta/d\alpha = -H_{11}/H_{12}$. Notice that H_{12} is evaluated at (α,δ) satisfying $H_1 = 0$. Thus, if $H_{12} >(<) 0$ for all points in (α,δ) plane, then the m_1 curve has positive(negative) slope since $H_{11} < 0$ from the second order condition. In a similar manner, $d\delta/d\alpha = - H_{12}/H_{22}$ where H_{12} is evaluated at (α,δ) satisfying $H_2(\alpha,\delta,\lambda) = 0$. Let η donote $(-H_{11}/H_{12}) - (-H_{12}/H_{22})$, where H_{11}, H_{12}, and H_{22} are evaluated at (α,δ) satisfying <u>both</u> $H_1 = 0$ and $H_2 = 0$. Then, because $H = H_{11}\cdot H_{22} - H_{12}^2 > 0$, $\eta = (-H_{11}/H_{12}) - (-H_{12}/H_{22}) = [-(H_{11}\cdot H_{22} - H_{12}^2)]/[H_{12}\cdot H_{22}]$ will be positive or negative according as H_{12} is positive or negative.

The diagram method introduced in Lemma 3.1 has several positive features. First, it can be used to prove Theorem 3.4 – 3.6. That is, it is supplement to the algebraic method. Second, ambiguous results under the algebraic approach may be clear under the diagram approach. One example is given in section 3.2.1. Finally, it can handle corner solutions, which will be discussed in section 3.1.5.

<u>Theorem 3.4</u>: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ (decrease α and increase δ) when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $t(x)$ represents a simple increase in risk.

<u>Proof</u>: $z_{\delta x} = 0$ implies that the condition $H_2 = 0$ is not affected by changes in the random parameter, and the sign of H_{12} is the same as that of $z_{\alpha\delta}$. Notice that $H_{12} = E[u'(z)\cdot z_{\alpha\delta} + u''(z)\cdot z_\alpha\cdot z_\delta] = Eu'(z)\cdot z_{\alpha\delta}$ since

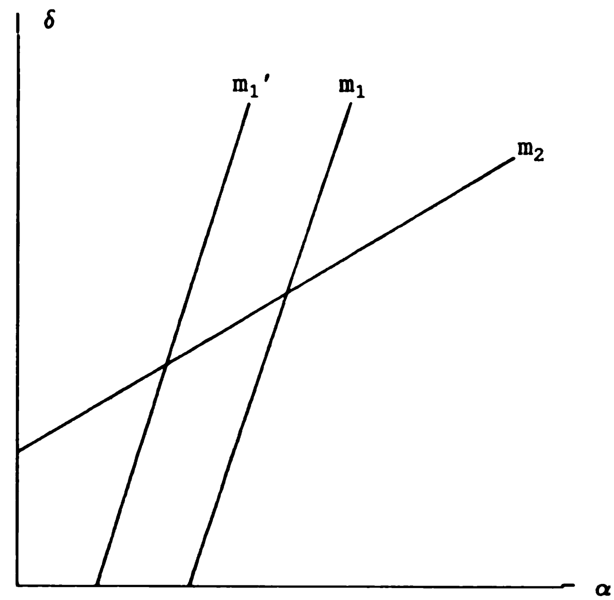

Figure 3.1.1

$z_{\alpha\delta} > 0$

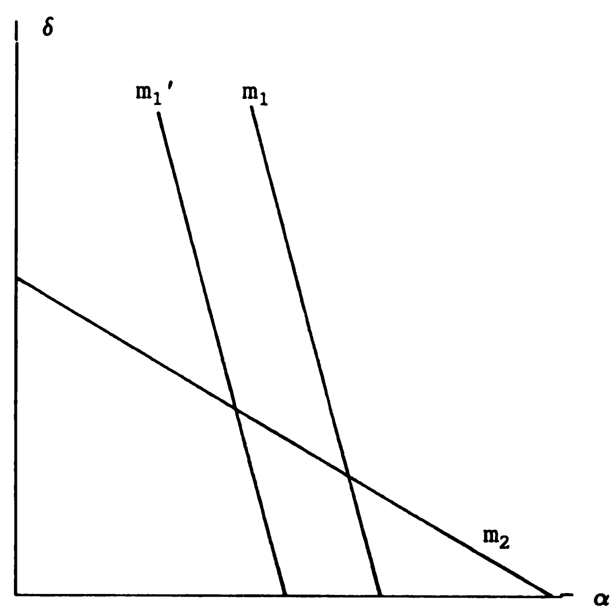

Figure 3.1.2

$z_{\alpha\delta} < 0$

$z_\delta = 0$ from $H_2(\alpha,\delta,\lambda) = 0$. Thus, the simple increase in risk does not change the m_2 curve. However, it will shift the m_1 curve to the left to position m_1' because $\partial H_1/\partial\theta|_{\theta=0} < 0$ from corollary 3.1. This is because $H_{11} < 0$ implies that α should be decreased when holding δ fixed. From lemma 3.1, the m_1 and m_2 curves have the relative positions shown in figure 3.1.1 and 3.1.2, according to the sign of $z_{\alpha\delta}$. The conclusions of the corollary follow immediately from the figures.

Theorem 3.4 gives conditions sufficient to yield unambiguous comparative static results concerning the effect on α and δ of a simple increase in risk. Condition (a) requires that preferences exhibit DARA. Condition (b) restricts the model. However, it does not require that z_{xx} must be equal to zero. Condition (c) further restricts the model. Note that unambiguous comparative static results can be derived even if $z_{\alpha\delta} < 0$. The condition that $z_{\delta x} = 0$ is assumed to make the 1-2-1 model tractable, but it is a severe restriction on the model. Condition (d) is the same as in theorem 3.1.

Theorem 3.5: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will decrease the optimal values of α and δ (decrease α and increase δ) if

(a) $u'(z) > 0$ and $u''(z) < 0$,

(b) $z_x \geq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $G(x)$ represents a relatively strong increase in risk from $F(x)$.

Proof: The proof is similar to that which is provided for theorem 3.4 and is simply sketched here. Given $z_{\delta x} = 0$, the condition $H_2 = 0$ is equivalent to $z_\delta = 0$. Note that z_δ does not include the random parameter. Thus, by corollary 3.2, a relatively strong increase in risk shifts the m_1 curve to the left, but does not affect the m_2 curve.

Theorem 3.6: An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x},\alpha,\delta,\lambda))\cdot dF(x)$ will increase the optimal values of α and δ (increase α and decrease δ) when the random variable is transformed according to $t(x) = x + \theta\cdot k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$,

(c) $z_{\delta x} = 0$ and $z_{\alpha\delta} >(<) 0$,

(d) $k(x) \geq 0$ and $k'(x) \leq 0$.

Proof: The proof is similar to that which is provided for theorem 3.4 and is simply sketched here. Given the conditions of the theorem, a FSD improvement in the random parameter shifts the m_1 curve to the right, but does not affect the m_2 curve. The conclusions of the theorem follow immediately from figures 3.1.1 and 3.1.2.

3.1.5 The Graphical Approach and Corner Solutions

The main purpose of this section is to show that the graphical approach can handle corner solutions. Another example occurs in section 3.3. Let α_c and δ_c denote initial optimal values of α and δ, and α_a and δ_a new optimal values of α and δ resulted from a change in $\tilde{x}$. Assuming that the choice variables, α and δ, take values in the interval $[0,\infty]$, then a corner solution which was defined as a constrained solution occurs when $\alpha_c = 0$ or $\delta_c = 0$. There are two interesting possible corner solutions under initial situations: 1) $\alpha_c > 0$ and $\delta_c = 0$, 2) $\alpha_c = 0$ and

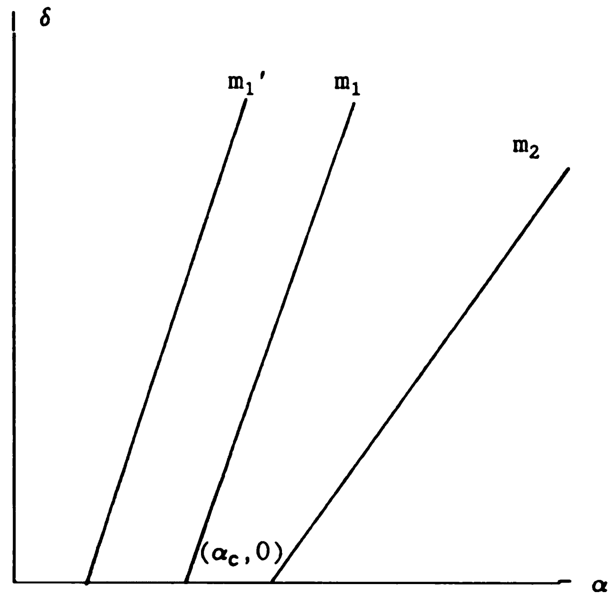

Figure 3.2.1

$z_{\alpha\delta} > 0$

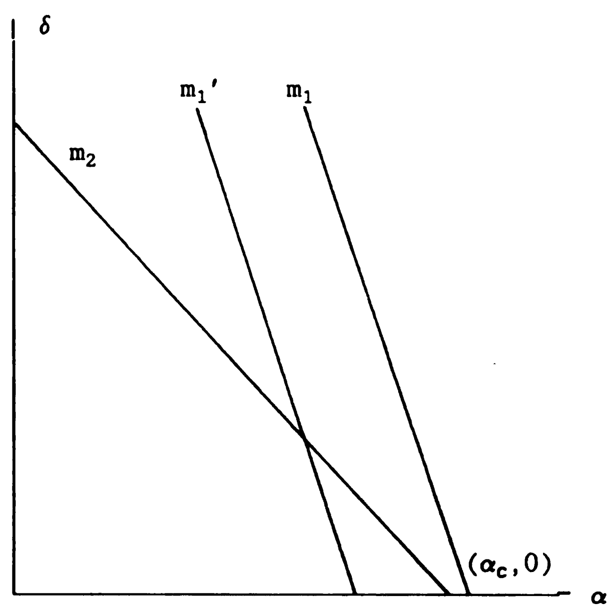

Figure 3.2.2

$z_{\alpha\delta} < 0$

$\delta_c > 0$. The first case is discussed here. To simplify the discussion, it is assumed that $z_{\delta x} = 0$, and $z_x \geq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$.

We only consider the first case since corner solutions will be discussed further in section 3.3. If $\alpha_c > 0$ and $\delta_c = 0$, then $\alpha_c > 0$ and $\delta_c = 0$ must satisfy both $Eu'(\cdot)\cdot z_\alpha(\tilde{x},\alpha_c,0,\lambda) = 0$ and $Eu'(\cdot)\cdot z_\delta(\tilde{x},\alpha_c,0,\lambda) \leq 0$. By lemma 3.1, the m_1 and m_2 curves have positive(negative) slopes when $z_{\alpha\delta} >(<) 0$. Because the set of points above the m_2 curve represents $Eu'(\cdot)\cdot z_\delta(\tilde{x},\alpha_c,0,\lambda) < 0$, the m_2 curve should lie to the right of the m_1 curve if $z_{\alpha\delta} > 0$, while the m_2 curve should rest below the m_1 curve if $z_{\alpha\delta} < 0$. Lemma 3.1 implies that the m_1 and m_2 curves have the relative positions shown in figures 3.2.1 and 3.2.2. Suppose that a relatively strong increase in risk occurs. Then, by corollary 3.2, it shifts the m_1 curve to the left to position m_1', but does not affect the m_2 curve. From figure 3.2.1, one can conclude that if $z_{\alpha\delta} > 0$, then $\alpha_a < \alpha_c$ and $\delta_a = \delta_c = 0$. In a similar manner, from figure 3.2.2, one can conclude that if $z_{\alpha\delta} < 0$, then $\alpha_a < \alpha_c$ but in this case δ_a may(need not) be positive. That is, the decision maker may increase δ from $\delta_c = 0$, when faced with a relatively strong increase in risk, only if $z_{\alpha\delta} < 0$.

3.2 Examples of Applications

3.2.1 The Model of Batra and Ullah[1974]

For Batra and Ullah's model, the outcome variable z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot f(\alpha,\delta) - \lambda_1\cdot\alpha - \lambda_2\cdot\delta$. It is assumed that $\tilde{x}$ takes values in the interval [0,B], and f_α and f_δ, the marginal products of the inputs, are positive. Hence, $z_x(= f(\alpha,\delta)) \geq 0$, $z_{xx} = 0$, and $z_{\alpha x}(= f_\alpha)$, $z_{\delta x}(= f_\delta) > 0$. The condition $z_\alpha = (z_{\delta x}/z_{\alpha x})\cdot z_\delta$ is also satisfied.

Therefore, Theorems 3.1 - 3.3 can be applied to this model in order to examine the effects of three general types of changes in randomness as long as $z_{\alpha\delta}(= \tilde{x}\cdot f_{\alpha\delta}) > 0$.

Hartman[1975] shows that the partial approach implicit in Batra and Ullah's analysis(p.542) is incorrect and then argues that the effects on input decisions of a global increase in risk and an increase in γ are the same as each other if the firm exhibits DARA. Note that Batra and Ullah's mistake is to consider the shifts in the m_1 and m_2 curves holding the other fixed when investigating the effect of a global increase in risk on the input decisions. Corollary 3.2 and Lemma 3.1 imply that a strong increase in risk shifts the m_1 curve leftward and the m_2 curve downward. A two stage decision process is assumed in Hartman's analysis; that is, the firm faced with a global increase in risk first chooses the appropriate level of output and then, given the level of output, it chooses the inputs to minimize cost.

Using theorem 3.1 and 3.2, we show that the two stage decision process need not be assumed. The effects of a relatively strong and a simple increase in risk on the choice variables can be rewritten as:

$$\partial\alpha/\partial(r.s.) = -(1/H)\cdot(\partial H_1/\partial(r.s.)\cdot[(1/f_\alpha)\cdot Eu'(z)\cdot(\tilde{x}\cdot f_\alpha f_{\delta\delta} - \tilde{x}\cdot f_\delta f_{\alpha\delta})]$$

$$\partial\delta/\partial(r.s.) = -(1/H)\cdot(\partial H_1/\partial(r.s.)\cdot[(1/f_\alpha)\cdot Eu'(z)\cdot(\tilde{x}\cdot f_\delta f_{\alpha\alpha} - \tilde{x}\cdot f_\alpha f_{\alpha\delta})]$$

$$\partial\alpha/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot[(1/f_\alpha)\cdot Eu'(z)\cdot(\tilde{x}\cdot f_\alpha f_{\delta\delta} - \tilde{x}\cdot f_\delta f_{\alpha\delta})]$$

$$\partial\delta/\partial\theta|_{\theta=0} = -(1/H)\cdot(\partial H_1/\partial\theta)\cdot[(1/f_\alpha)\cdot Eu'(z)\cdot(\tilde{x}\cdot f_\delta f_{\alpha\alpha} - \tilde{x}\cdot f_\alpha f_{\alpha\delta})].$$

By corollary 3.1 and 3.2, $\partial H_1/\partial(r.s.)$ and $\partial H_1/\partial\theta$ are same in sign under DARA. Thus, the effects on α and δ of a relatively strong and a simple increase in risk are the same as each other if the firm exhibits DARA.

To further extend the analysis, consider the case where $z_{\alpha\delta}(=\tilde{x}\cdot f_{\alpha\delta})$

< 0. For this case, it is not obvious how to employ the algebraic method to derive determinate comparative statics. However, the use of the diagram approach does allow us to derive results which might not be obvious under the algebraic approach. Given that $z_\alpha = (z_{\alpha x}/z_{\delta x})\cdot z_\delta$, $H_{12} = E[u'(z)\cdot z_{\alpha\delta} + (z_{\delta x}/z_{\alpha x})\cdot u''(z)\cdot z_\alpha^2]$. Thus, $z_{\alpha\delta}(=x\cdot f_{\alpha\delta}) < 0$ implies that the m_1 and m_2 curves have negative slopes. Lemma 3.1 indicates that both curves have the relative positions shown in figure 3.1.2. The strong or simple increase in risk shifts the m_1 curve leftward and the m_2 curve downward. Hence, one can conclude that all risk averse decision makers (exhibiting DARA), when faced with a relatively strong or a simple increase in risk, will not increase both α and δ even if $f_{\alpha\delta} < 0$.

3.2.2 The Model of Feder[1977]

The outcome variable z for Feder's decision model has a specific form such that $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot f(\alpha,\delta) + g(\alpha,\delta) + \lambda$. The condition $z_\alpha = (z_{\delta x}/z_{\alpha x})\cdot z_\delta$ holds in this model. To see this, look at the first order conditions: $H_1(\alpha,\delta,\lambda) = Eu'(z)\cdot(\tilde{x}\cdot f_\alpha + g_\alpha) = 0$ and $H_2(\alpha,\delta,\lambda) = Eu'(z)\cdot(\tilde{x}\cdot f_\delta + g_\delta) = 0$. Because $f_\alpha/f_\delta = g_\alpha/g_\delta$, this implies that $(z_{\alpha x}/z_{\delta x})\cdot z_\delta = (f_\alpha/f_\delta)\cdot(\tilde{x}\cdot f_\delta + g_\delta) = \tilde{x}\cdot f_\alpha + g_\alpha = z_\alpha$. Obviously here $z_{xx} = 0$.

Feder writes(p. 509) that "it is not possible to determine the direction of impact on the different control variables ···. Definite results, however, can be obtained for the function f." Here we present simple conditions about the function z which are sufficient for determining the direction of changes in choice variables when the random parameter $\tilde{x}$ undergoes the three types of changes in randomness discussed in section 3.1.

Applying Theorem 3.1, all risk-averse decision makers exhibiting

DARA, when faced with a simple increase in risk, will decrease the optimal values of α and δ if $z_x(= f(\alpha,\delta)) \geq 0$, $z_{\alpha x}(= f_\alpha(\alpha,\delta))$, $z_{\delta x}(= f_\delta(\alpha,\delta)) \geq 0$, and $z_{\alpha\delta}(= \tilde{x}\cdot f_{\alpha\delta} + g_{\alpha\delta}) > 0$. Assuming that $g(\alpha,\delta)$ is additively separable, then the condition about $z_{\alpha\delta}$ can be simplified because $g_{\alpha\delta} = 0$. Feder does not consider the effect of a global increase in risk on $f(\alpha,\delta)$. Theorem 3.2, however, can be applied to examine the effect of a relatively strong increase in risk <u>on</u> α and δ. That is, all risk-averse decision makers, when faced with a relatively strong increase in risk, will decrease the optimal values of α and δ if $z_x(= f(\alpha,\delta)) \geq 0$, $z_{\alpha x}(= f_\alpha(\alpha,\delta))$, $z_{\delta x}(= f_\delta(\alpha,\delta)) \geq 0$, and $z_{\alpha\delta}(= \tilde{x}\cdot f_{\alpha\delta} + g_{\alpha\delta}) > 0$. Similarly, using theorem 3.3, we can also investigate the effect on α and δ of a FSD improvement in $\tilde{x}$. Note that other combinations of assumptions concerning the function z can also be considered.

3.2.3 The Model of Feder, Just and Schmitz(FJS)[1977]

For FJS's model, the function z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot[f(\alpha) + \delta - \lambda_1] + g(\lambda_2 - \alpha - \lambda_3\cdot\delta) - \lambda_4\cdot\delta$. FJS assume that $\tilde{x}$ takes values in the interval [0,B], the production functions, $f(\cdot)$ and $g(\cdot)$, are increasing and concave, and the nonrandom parameters, $\lambda_i(i=1,4)$, are all nonnegative. Let Δ denote $f(\alpha) + \delta - \lambda_1$. Notice that $z_x(= \Delta)$ can be positive or negative. Also, $z_{xx} = 0$, and $z_{\alpha x}(= f'(\alpha))$, $z_{\delta x}(= 1) > 0$. To see whether the condition $z_\alpha = (z_{\delta x}/z_{\alpha x})\cdot z_\delta$ at $H_i = 0$ is satisfied in this model, look at the first order conditions: $H_1(\alpha,\delta,\lambda) = Eu'(\cdot)\cdot(\tilde{x}\cdot f'(\cdot) - g'(\cdot)) = 0$ and $H_2(\alpha,\delta,\lambda) = Eu'(\cdot)\cdot(\tilde{x} - g'(\cdot)\cdot\lambda_3 - \lambda_4) = 0$. Because $f' = g'/(g'\cdot\lambda_3 + \lambda_4)$, this implies that $(z_{\alpha x}/z_{\delta x})\cdot z_\delta = f'\cdot(\tilde{x} - g'\cdot\lambda_3 - \lambda_4) = \tilde{x}\cdot f' - g' = z_\alpha$. The condition $f' = g'/(g'\cdot\lambda_3 + \lambda_4)$ also implies $1 - \lambda_3 \cdot f' > 0$ since $g' \cdot (1 - \lambda_3 \cdot f') = f' \cdot \lambda_4 > 0$.

Even though the condition $z_\alpha = (z_{\alpha x}/z_{\delta x}) \cdot z_\delta$ holds, Theorem 3.1 – 3.3 cannot be applied since $z_{\alpha\delta}(= \lambda_3 \cdot g''(\cdot)) < 0$. Whenever $z_{\alpha\delta} < 0$, the signs of $z_{\alpha x} \cdot z_{\delta\delta} - z_{\delta x} \cdot z_{\alpha\delta}$ and $z_{\delta x} \cdot z_{\alpha\alpha} - z_{\alpha x} \cdot z_{\alpha\delta}$ are not usually determined. However, in FJS's model, $z_{\alpha x} \cdot z_{\delta\delta} - z_{\delta x} \cdot z_{\alpha\delta} = \lambda_3 \cdot g'' \cdot (\lambda_3 \cdot f' - 1) > 0$ and $z_{\delta x} \cdot z_{\alpha\alpha} - z_{\alpha x} \cdot z_{\alpha\delta} = \tilde{x} \cdot f'' + g'' \cdot (1 - \lambda_3 \cdot f') < 0$ because $1 - \lambda_3 \cdot f'(\cdot) > 0$.

First, we demonstrate that with DARA, a simple increase in risk leads to an increase(a decrease) in α and a decrease(an increase) in δ if $\Delta >(<) 0$. From the proof of theorem 3.1, we know that

$$\partial\alpha/\partial\theta|_{\theta=0} = - (1/H) \cdot (\partial H_1/\partial\theta) \cdot (1/z_{\alpha x}) \cdot Eu'(z) \cdot (z_{\alpha x} \cdot z_{\delta\delta} - z_{\delta x} \cdot z_{\alpha\delta})$$

$$\partial\delta/\partial\theta|_{\theta=0} = - (1/H) \cdot (\partial H_1/\partial\theta) \cdot (1/z_{\alpha x}) \cdot Eu'(z) \cdot (z_{\delta x} \cdot z_{\alpha\alpha} - z_{\alpha x} \cdot z_{\alpha\delta}).$$

Corollary 3.1 implies that if $\Delta >(<) 0$, then $\partial H_1/\partial\theta <(>) 0$. Therefore, $\partial\alpha/\partial\theta|_{\theta=0} >(<) 0$ and $\partial\delta/\partial\theta|_{\theta=0} <(>) 0$ if $\Delta >(<) 0$. Second, by using a similar procedure, we can show that with $u'(\cdot) > 0$ and $u''(\cdot) < 0$, a relatively strong increase in risk leads to an increase(a decrease) in α and a decrease(an increase) in δ if $\Delta >(<) 0$. Finally, it can be shown that with DARA, a FSD improvement in $\tilde{x}$ leads to a decrease in α and an increase in δ if $\Delta >(<) 0$ and $k'(x) \le(\ge) 0$.

3.2.4 The model of Katz, Paroush and Kahana(KPK)[1982]

The price discrimination model of KPK has outcome variable given by $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x} \cdot R_1(\alpha) + \lambda_1 \cdot R_2(\delta) - c(\alpha + \delta) - \lambda_2$. In KPK model, $R_i' > 0$, $R_i'' \le 0$, and c', $c'' > 0$. A characteristic of the model is $z_{\delta x} = 0$. Also, $z_x = R_1 \ge 0$, $z_{xx} = 0$, $z_{\alpha x} = R_1' > 0$, and $z_{\alpha xx} = 0$. Thus, <u>if</u> interior solutions are assumed, then theorem 3.4 – 3.6 can be applied to this model to examine the effects of three types of changes in $\tilde{x}$. Applying theorem 3.4, the risk averse firm exhibiting DARA, when faced with a simple increase in risk, will decrease α and increase δ. Note that $z_{\alpha\delta} = - c''(\cdot) < 0$. Using theorem 3.5 and 3.6, we can also analyze the effects on the choice variables of a relatively strong increase in risk and a FSD improvement in $\tilde{x}$.

3.3 A Two Output Competitive Firm Model

Typical competitive firms produce more than one good. Many oil companies produce natural gas and other related chemical products. Numerous competitive farmers produce two or more products. When the output price is random for one output, the competitive multiproduct firm is able to spread its risks by ouptut diversification.

Here we investigate a model of a competitive multiproduct firm, in which z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = \tilde{x}\cdot\alpha + \lambda_1\cdot\delta - c(\alpha,\delta) - \lambda_2$, where $\tilde{x}$ is the price of product α, α and δ are output levels, λ_1 is the price of product δ, λ_2 is fixed cost, and $c(\alpha,\delta)$ is a variable cost function. We assume that the variable cost function is monotone increasing and convex: c_α, $c_\delta > 0$, $c_{\alpha\alpha}$, $c_{\delta\delta} > 0$, and $c_{\alpha\alpha}\cdot c_{\delta\delta} - c_{\alpha\delta}^2 > 0$.

Sandmo[1971] shows that price uncertainty has a negative output effect on the competitive firm producing a single product. We demonstrate that price uncertainty can have a positive output effect on a competitive multiproduct firm. Since our focus is on the effects of price uncertainty on product choice as well as on output decisions, we will also consider corner solutions. The primary purpose of this section is to illustrate the diagram method introduced in 3.1.5. This section proceeds as follows. We first analyze the effects of the three types of changes in randomness discussed in 3.1, and then examine the possible impacts of three kinds of taxation on the firm's output decisions.

Let α_c and δ_c denote initial optimal values of α and δ, and α_a and δ_a new optimal values of α and δ resulted from a change in $\tilde{x}$. We assume that the choice variables, α and δ, take values in the interval $[0,\infty]$. Under the initial situation, there are three possible outcomes. The firm may specialize in the risky output, α, or the riskless output, δ, or produce both α and δ. First, consider the effects of a relatively strong increase in risk on α and δ. Because a relatively strong increase in risk includes a global increase in risk as a special case, the following corollaries can be applied to examine the impacts of price uncertainty on a multiproduct firm's output decisions <u>when</u> uncertainty is introduced from a nonrandom initial situation.

<u>Corollary 3.4</u>: Assume $\alpha_c > 0$ and $\delta_c = 0$. That firm is producing risky but not riskless output. Suppose that the firm is risk averse and a relatively strong increase in risk occurs. If $c_{\alpha\delta} < 0$, then $\alpha_a < \alpha_c$ and $\delta_a = 0$. If $c_{\alpha\delta} > 0$, then $\alpha_a < \alpha_c$ and δ_a may be positive.

<u>Proof</u>: We can use the diagram method introduced in 3.1.5 because $z_{\delta x} = 0$. If $\alpha_c > 0$ and $\delta_c = 0$, then $\alpha_c > 0$ and $\delta_c = 0$ must satisfy both $Eu'(\cdot)\cdot(\tilde{x} - c_\alpha(\alpha_c,0)) = 0$ and $Eu'(\cdot)\cdot(\lambda_1 - c_\delta(\alpha_c,0)) \le 0$. Because the set of points above the m_2 curve represents $\lambda_1 - c_\delta(\alpha,\delta) < 0$, the m_2 curve should lie to the right of the m_1 curve if $c_{\alpha\delta} < 0$, while the m_2 curve should rest in below the m_1 curve if $c_{\alpha\delta} > 0$. Note that H_{12} and $c_{\alpha\delta}$ are opposite in sign since $z_{\alpha\delta} = -\, c_{\alpha\delta}$. Lemma 3.1 implies that the m_1 and m_2 curves have the relative positions shown in figures 3.2.1 and 3.2.2. By corollary 3.2, a relatively strong increase in risk shifts the m_1 curve to the left to position m_1'. The case of $c_{\alpha\delta} < 0$ is obvious

from figure 3.2.1. In the case of $c_{\alpha\delta} > 0$, if the leftward shift in the m_1 curve resulted from a relatively strong increase in risk is enough, then δ_a may(need not) be positive.

Corollary 3.4 demonstrates positive output effect on riskless good of price uncertainty. If $c_{\alpha\delta} < 0$, then the risk-averse firm continues to specialize in risky product α. Thus, $c_{\alpha\delta} > 0$ is a necessary condition for the risk increase to cause diversification or switching into the riskless good. Only if the marginal cost of δ also declines as α is reduced, the risk-averse firm may start producing product δ.

<u>Corollary 3.5</u>: Assume $\alpha_c > 0$ and $\delta_c > 0$. Suppose that the firm is risk averse and a relatively strong increase in risk occurs. Then, if $c_{\alpha\delta} < 0$, $\alpha_a < \alpha_c$ and $\delta_a < \delta_c$. If $c_{\alpha\delta} > 0$, $\alpha_a < \alpha_c$ and $\delta_a > \delta_c$.

<u>Proof</u>: Suppose that $c_\alpha(0,0) = 0$ and $c_\delta(0,0) = 0$, which are sufficient conditions for product diversification. In this case, the m_1 and m_2 curves have a positive α-intercept and a positive δ-intercept, respectively. By corollary 3.2, a relatively strong increase in risk shifts the m_1 curve to the left. See figures 3.1.1 and 3.1.2 for the rest of the proof. Also, applying theorem 3.5 gives rise to the corollary. This is because the corollary assumes an interior solution.

<u>Corollary 3.6</u>: Assume $\alpha_c = 0$ and $\delta_c > 0$. Suppose that the firm is risk averse and a relatively strong increase in risk occurs. Then, $\alpha_a = 0$ and $\delta_a = \delta_c$.

<u>Proof</u>: If $\alpha_c = 0$ and $\delta_c > 0$, then $\alpha_c = 0$ and $\delta_c > 0$ must satisfy both $Eu'(\cdot)\cdot(\tilde{x} - c_\alpha(0,\delta_c)) \le 0$ and $Eu'(\cdot)\cdot(\lambda_1 - c_\delta(0,\delta_c)) = 0$. Because

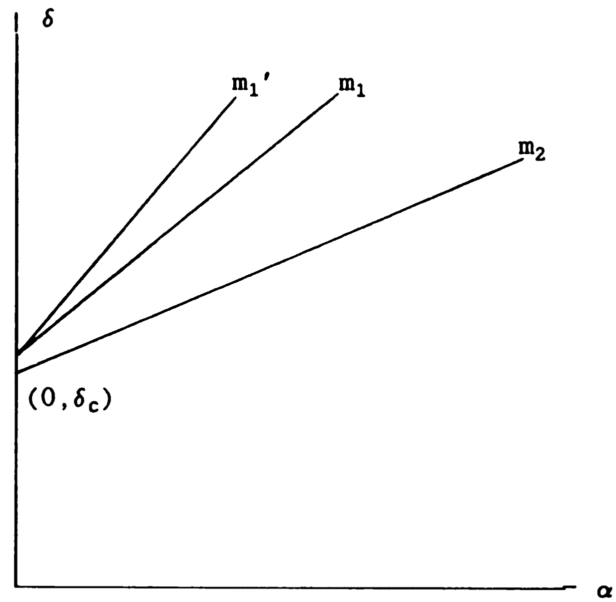

Figure 3.3.1

$c_{\alpha\delta} < 0$

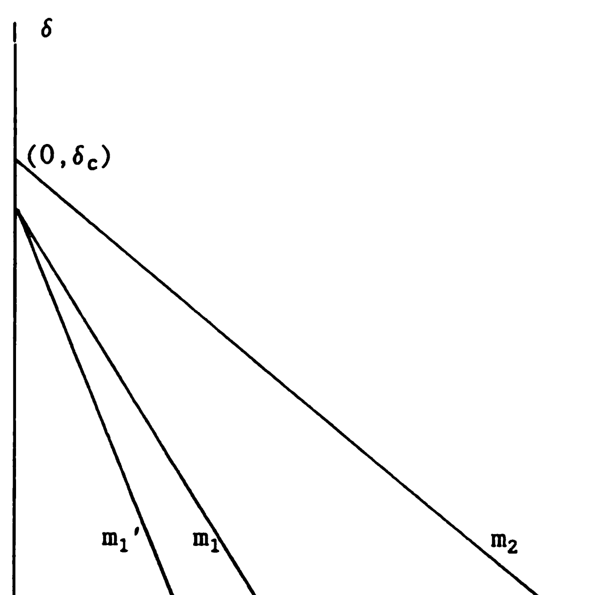

Figure 3.3.2

$c_{\alpha\delta} > 0$

the set of points to the right of the m_1 curve represents $Eu'(\cdot)\cdot(\tilde{x} - c_\alpha(\alpha,\delta)) < 0$, the m_2 curve should be below the m_1 curve if $c_{\alpha\delta} < 0$, while the m_2 curve should be above the m_1 curve if $c_{\alpha\delta} > 0$. Lemma 3.1 implies that the m_1 and m_2 curves have the relative positions shown in figures 3.3.1 and 3.3.2. By corollary 3.2, a relatively strong increase in risk shifts the m_1 curve to the left to position m_1'. The conclusions immediately follow from the figures. It is interesting to note that in this case product diversification never occurs and the optimal value of δ remains unchanged, regardless of the sign of $c_{\alpha\delta}$.

The effects on α and δ of a simple increase in risk and a FSD change in $\tilde{x}$ can also be examined by using a similar procedure, and the details are omitted here.

Finally, to illustrate other uses of the graphical approach the possible impacts of three kinds of taxation on α and δ are determined. These involve shifts in nonrandom parameters. To simplify the analysis, it is assumed that $\alpha_c > 0$ and $\delta_c > 0$. First, consider the effect of an increase in a profits tax on α and δ. Katz[1983] points out that the relative risk aversion measure based on profits, used by Sandmo[1971], leads to two problems; the measure associated with the negative profits can be negative, and risk aversion measures are not properly defined on profits but on wealth. Following Katz, the model should be slightly changed. Let λ_0 be the firm's initial wealth, and λ_3 the profit tax rate. Then, the firm's final wealth which is assumed to be nonnegative is given by $z(\tilde{x},\alpha,\delta,\lambda) = \lambda_0 + [\tilde{x}\cdot\alpha + \lambda_1\cdot\delta - c(\alpha,\delta) - \lambda_2]\cdot(1-\lambda_3)$. Given DARA <u>and</u> IRRA, an increase in λ_3 shifts the m_1 curve to the right, since

$\partial H_1/\partial\lambda_3 = - Eu''(z)\cdot(\tilde{x} - c_\alpha)\cdot[\tilde{x}\cdot\alpha + \lambda_1\cdot\delta - c(\alpha,\delta) - \lambda_2] = [1/(1-\lambda_3)]\cdot E[R_R(z)\cdot u'(z)\cdot(\tilde{x} - c_\alpha)] - [\lambda_0/(1-\lambda_3)]\cdot E[R_A(z)\cdot u'(z)\cdot(\tilde{x} - c_\alpha)]$. Obviously here $\partial H_2/\partial\lambda_3 = 0$. Thus, the risk averse firm exhibiting DARA and IRRA, when faced with an increase in λ_3, will increase α and δ if $c_{\alpha\delta} < 0$, while increase α and decrease δ if $c_{\alpha\delta} > 0$.

Second, consider the effect of an increase in a specific tax on α and δ. Assuming that a specific tax is imposed on good α, then the outcome variable z takes the form $z(\tilde{x},\alpha,\delta,\lambda) = (\tilde{x}-\lambda_3)\cdot\alpha + \lambda_1\cdot\delta - c(\alpha,\delta) - \lambda_2$. In this case, λ_3 represents a specific tax. It is clear to see that $\partial H_1/\partial\lambda_3 < 0$ under DARA, and $\partial H_2/\partial\lambda_3 = 0$. Thus, the risk averse firm exhibiting DARA, when faced with an increase in a specific tax imposed on good α, will decrease α and δ if $c_{\alpha\delta} < 0$, while decrease α and increase δ if $c_{\alpha\delta} > 0$.

Finally, consider the effect of an increase in an ad valorem sales tax on α and δ. Let λ_3 be an ad valorem sales tax. Assuming that an ad valorem sales tax is imposed on the firm, the firm's problem is to choose α and δ to maximize $Eu[(1-\lambda_3)\cdot(\tilde{x}\cdot\alpha + \lambda_1\cdot\delta) - c(\alpha,\delta) - \lambda_2]$. Then, $\partial H_1/\partial\lambda_3 = - Eu'(z)\cdot\tilde{x} - \lambda_1\cdot\delta\cdot Eu''(z)\cdot((1-\lambda_3)\cdot\tilde{x} - c_\alpha) - \alpha\cdot Eu''(z)\cdot((1-\lambda_3)\cdot\tilde{x} - c_\alpha)\tilde{x}$. The first term is negative, and the second term is also nonpositive under DARA. However, the last term may be positive or negative under DARA, and will be positive under IARA. Thus, the whole expression may be negative or positive under DARA or IARA. Clearly, $\partial H_2/\partial\lambda_3 = - \lambda_1 < 0$. It is very difficult to determine the effect of a change in the sales tax rate on the firm's output decision. The increase in λ_3 reduces not only the mean of $\tilde{z}$ but also leads to a decrease in risk. Therefore, the ultimate impact of a change in the tax rate on the m_1 curve depends on two forces which operate in opposite directions.

CHAPTER FOUR

THE 2-2-1 MODEL

4.0 Introduction

Determining the consequences of a change in a random parameter is an important and frequently studied comparative statics problem. This problem has been examined, but most often for decision models including only one source of randomness. Papers which include multiple sources of risk usually assume independent risks. Thus, an important direction in which this comparative static analysis can and should be extended is to examine decision models with multiple sources of randomness including cases where the risks are not independent of one another.

Recently, Hadar and Seo[1990] extend the standard portfolio model with only one risky asset by considering a portfolio model with more than one risky asset. However, they avoid the issues involving stochastic dependence by imposing the strong and simplifying restriction that the random parameters are independently distributed. Meyer and Ormiston [1992] extend Hadar and Seo's portfolio model with only two risky assets by considering the case where the returns to risky assets are not independently distributed.

In addition to expanding the number of random variables, we can expand the number of choice variables. Combining these extensions we have the two random-two choice-one outcome(2-2-1) model. In it the agent is assumed to choose α and δ to maximize $Eu(z(\tilde{x},\tilde{y},\alpha,\delta,\lambda))$, where the outcome variable, z, depends on two random parameters, $\tilde{x}$ and $\tilde{y}$, two choice variables, α and δ, and a set of nonrandom parameters, λ. Hadar and Seo[1990] investigate a specific 2-2-1 model, but make limited progress. Extension of either the 2-1-1 or 1-2-1 models to the 2-2-1 model is quite difficult because of the problems of dealing with the joint CDF and two first-order conditions. Until now, the 2-2-1 model remains largely unexplored and only modest results are presented here.

Two specific 2-2-1 models are examined in this chapter. One is the decision model presented by Feder[1977], and another is the model of a competitive multiproduct firm. When investigating the first model, we impose a strong assumption which allows the use of the mean-standard deviation(MS) approach. However, under that assumption, the 2-2-1 model is transformed into a 1-2-1 model, and the problems of dealing with the joint CDF do not arise. On the other hand, the second model is analyzed using the expected utility(EU) framework. To avoid the subject of stochastic dependence, we assume that the random parameters are independently distributed.

This chapter is organized as follows. Section 1 reviews the literature concerning the 2-1-1 and 2-2-1 models. Section 2 first reviews the literature concerning moment based decision models, and then examines Feder's decision model under the MS framework. Section 3 investigates a two output competitive firm model under the EU framework, and considers corner solutions.

4.1 Literature Review

A general form for the 2-1-1 model assumes the agent chooses α to maximize $Eu(z(\tilde{x},\tilde{y},\alpha,\lambda)) = \int_0^B\int_0^B u(z(\tilde{x},\tilde{y},\alpha,\lambda))\cdot d^2H(x,y)$, where the outcome variable, z, depends on two random parameters, $\tilde{x}$ and $\tilde{y}$, one choice variable, α, and a set of nonrandom parameters, λ. In addition, the joint cumulative distribution function(CDF) for $\tilde{x}$ and $\tilde{y}$ are denoted $H(x,y)$. The conditional and marginal CDFs for $\tilde{x}$ are denoted $F(x|y)$ and $F(x)$, respectively, and for $\tilde{y}$ they are $G(y|x)$ and $G(y)$. If $\tilde{x}$ and $\tilde{y}$ are independently distributed then $F(x|y) = F(x)$ for all y, $G(y|x) = G(y)$ for all x, and $H(x,y) = F(x)\cdot G(y)$. To simplify notation, the symbol $d^2H(x,y)$ is used to denote $[\partial^2 H(x,y)/\partial x\cdot\partial y]\cdot dx\cdot dy$. Finally, the random parameters $\tilde{x}$ and $\tilde{y}$ are assumed to take values in the interval $[0,B]$.

An important question is how does a decision maker adjust the optimal value of α when the random parameter $\tilde{x}$ undergoes some types of changes in randomness. In working out the details, two conceptual questions should be addressed, as a preliminary matter. The first asks which changes in a random parameter are most usefully analyzed. This question arises because the various definitions of risk increases or stochastically dominant shifts formulated for single random parameter models do not necessarily have the same meaning in the multiple random parameter case. The second question deals with which assumption to make concerning the other random parameters, as the parameter of interest is shifted. This is an especially important question when the random parameters are stochastically dependent.

Random variable $\tilde{y}$ is stochastically dependent on $\tilde{x}$ if the conditional distribution function for $\tilde{y}$ given x is not degenerate for each x, nor is it the same for each x. For example, if $\tilde{y}$ is equal in distribution to $\tilde{x} + \tilde{\xi}$ where $E(\xi|x) = 0$, which is the first definition of the Rothschild and Stiglitz increase in risk, then $\tilde{y}$ is stochastically dependent on $\tilde{x}$ and is said to be a stochastic transformation of $\tilde{x}$ because the conditional distribution function for $\tilde{y}$ given x is not degenerate for each x. When the conditional distribution functions are the same for each x then $\tilde{y}$ is independent of $\tilde{x}$. If the conditional distribution functions are degenerate for each x, then nonstochastic dependence occurs and $\tilde{y} = t(\tilde{x})$ for some function $t(\cdot)$. In this case, $\tilde{y}$ is said to be a deterministic transformation of $\tilde{x}$.

The literature concerning the 2-1-1 model includes the analysis of incomplete insurance markets in Doherty and Schlesinger[1983,1985,1986], and of the portfolio models in Kira and Ziemba[1980], Hadar and Seo [1990] and Meyer and Ormiston[1992]. Eeckhoudt and Kimball[1991] consider insurance demand with background risk. Note that most 2-1-1 models are formulated within a specific rather than a general decision model. We only review the portfolio models, beginning with the model given by Hadar and Seo[1990].

The two risky asset portfolio model presented by Hadar and Seo assumes the outcome variable z takes the form $z(\tilde{x},\tilde{y},\alpha) = \alpha\cdot\tilde{x} + (1-\alpha)\cdot\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are the returns to the risky assets, and α is the proportion invested in the risky asset $\tilde{x}$. The utility function is assumed to be three times continuously differentiable, nondecreasing and concave in z. Hadar and Seo assume that given the initial joint distribution of returns, the agent attains a unique, regular, interior maximum at α_0 satisfying $0 < \alpha_0 < 1$. However, this assumption can be quite restrictive and may rule out some interesting cases. For instance, if $\tilde{x}$ and $\tilde{y}$ are independently distributed and $u''(z) < 0$, then a sufficient condition for interior solutions is $\bar{x} = \bar{y}$.

The structure of the model is quite simple. To see this, let $\psi(x;y,\alpha)$ denote the derivative with respect to α of utility; that is,

$\psi(x;y,\alpha) = \partial u(z)/\partial\alpha$ where z is the outcome variable. Then

$\partial\psi/\partial x = [u'(z) + u''(z)\alpha(x-y)] = [u'(z) + u''(z)\cdot(\alpha x + (1-\alpha)y) - y\cdot u''(z)]$

$= u'(z)[1 - R_R(z) + y\cdot R_A(z)].$

$\partial^2\psi/\partial x^2 = u''(z)\cdot\alpha\cdot[1 - R_R(z) + y\cdot R_A(z)] + u'(z)\cdot[-R_R'(z)\cdot\alpha + y\cdot R_A'(z)\cdot\alpha].$

Thus, $\psi_x \geq 0$ for all $y \geq 0$ and $0 < \alpha < 1$ if $u' \geq 0$, $u'' \leq 0$, and $R_R \leq 1$. Also, $\psi_{xx} \leq 0$ for all $y \geq 0$ and $0 < \alpha < 1$ if $u' \geq 0$, $u'' \leq 0$, $R_R \leq 1$, $R_R' \geq 0$, and $R_A' \leq 0$.

Hadar and Seo avoid the issues involving stochastic dependence by imposing the strong and simplifying restriction that the random parameters are independently distributed. This assumption allows them to alter $\tilde{x}$ without changing $\tilde{y}$, and without changing how $\tilde{y}$ depends stochastically on $\tilde{x}$. That is, $\tilde{x}$ can be changed keeping the marginal and the conditional CDFs for $\tilde{y}$ unchanged. They then determine conditions on the decision maker's preferences that are necessary and sufficient for a first degree stochastic dominant(FSD) shift or a mean-preserving contraction(MPC) in $\tilde{x}$ to cause an increase in the optimal value of α. The conditions on utility are the same as those found in the single risky asset case by Fishburn and Porter[1976] for FSD changes, and Rothschild and Stiglitz[1971] for MPCs. Thus, with independence, an extension to two risky assets preserves the findings from the one risky asset case. It is interesting to note that the simple conditions on the utility function do not come from the independence assumption, but from the simple structure of the model.

When $\tilde{x}$ and $\tilde{y}$ are stochastically dependent, either the marginal or at least one of the conditional CDFs for $\tilde{y}$ <u>must</u> change as $\tilde{x}$ is changed. The question of which of these, if any, to hold fixed as the parameter of interest $\tilde{x}$ is altered is one which requires careful attention. Kira and Ziemba[1980] address this issue when investigating the effect of a FSD change in $\tilde{x}$ on portfolio composition when $\tilde{x}$ and $\tilde{y}$ are not independently distributed. They allow the marginal CDF for $\tilde{y}$ to be changed, thus choosing to hold the conditional CDF for $\tilde{y}$ fixed, as $\tilde{x}$ is changed. Given the new marginal CDF for $\tilde{x}$ and the unchanged conditional CDFs for $\tilde{y}$, the new joint distribution of $\tilde{x}$ and $\tilde{y}$ resulting from the change in $\tilde{x}$ is uniquely determined. Depending on the initial joint distribution of $\tilde{x}$ and $\tilde{y}$, this assumption of Kira and Ziemba can lead to unusual reactions to a change in $\tilde{x}$. This is because the change in $\tilde{x}$ causes a simultaneous change in the marginal CDF for $\tilde{y}$. Changes which appear to be improvements in $\tilde{x}$ can lead to even greater improvements in $\tilde{y}$. As a result, Kira and Ziemba's conditions for determining the effect of a FSD change in $\tilde{x}$ are rather complex and have not been very useful in economic analysis.

Unlike Kira and Ziemba[1980], Meyer and Ormiston[1992] allow the conditional CDF for $\tilde{y}$ to be changed, thus choosing to hold the marginal CDF for $\tilde{y}$ fixed, as $\tilde{x}$ is changed. Meyer and Ormiston present two counter examples to show that an arbitrary R-S decrease in risk or an FSD dominant shift in the marginal CDF for $\tilde{x}$ without changing the marginal CDF for $\tilde{y}$ may alter a stochastic relationship measured by, for instance, correlation or conditional expectation, and therefore, may lead to unusual comparative static results.

In presenting theorems concerning changes in $\tilde{x}$, M-O use stochastic and deterministic transformations to represent a change in the random parameter $\tilde{x}$. This is because these transformations of $\tilde{x}$ do not affect

$\tilde{y}$, keeping its marginal CDF fixed. In addition, the stochastic dependence of $\tilde{y}$ on $\tilde{x}$ is kept similar to that which prevailed initially. Furthermore, the stochastic transformation allows the characterization of comparative risk for univariate CDFs introduced by Rothschild and Stiglitz[1970] and Hadar and Russell[1969] to be used directly. Meyer and Ormiston demonstrate that even if the independence is not assumed, the same conditions on the utility function as those in Hadar and Seo [1990] are still necessary and sufficient for the usual comparative static results to follow. They also emphasize that the important assumption of Hadar and Seo is not the independence assumption itself, but their ceteris paribus assumption concerning how a random variable is changed and what is not changed.

Now, turn to the 2-2-1 model. After examining the portfolio model with two risky assets and one choice variable, Hadar and Seo[1990] then extend the model to include n risky assets and n-1 choice variables but make little progress. H-S show that the conditions which result in an increase in asset i when the portfolio contains only two risky assets are still necessary when there are n risky assets. On the other hand, they were unable to show that these conditions are also sufficient except in the case of an exponential utility function. Only by imposing a severe restriction on the risk taking characteristics of the decision maker and on the joint cumulative distribution function, they can analyze the portfolio model which has a simple model structure.

4.2 Feder's Decision Model

4.2.1 Literature Review

Two different approaches to representing an agent's preferences over strategies yielding random payoffs are in wide use. Under the mean-standard deviation(MS) approach, the agent is assumed to rank the alternatives according to the value of some function defined over the first two moments of the random payoff, while the expected utility(EU) criterion assumes that the expected value of some utility function defined over payoffs is used instead. It is well known that some restrictions must be placed on either the agents' preferences or the distribution of random terms in order to guarantee consistency between the EU and MS approaches. All such restrictions presented in the literature, such as requiring that the agent's utility function be quadratic or that the random alternatives be normally distributed, have serious theoretical defects and/or have no empirical support.

Meyer[1987] identifies a restriction which is sufficient to ensure the consistency between the MS and EU approaches and confirms that it holds in many economic decision models. The results of Meyer can be viewed as ones which improve moment based decision models in at least two ways. First, the location and scale(LS) condition described below is more acceptable, in the sense that the condition does not require any special assumptions about the form of the utility function or the distribution of random terms, and second, the various hypotheses and assumptions that make EU analysis so powerful are translated into equivalent conditions in the MS framework.

Definition 4.1: Two cumulative distribution functions $G_1(\cdot)$ and $G_2(\cdot)$ are said to differ only by location and scale parameters λ_1 and λ_2 if $G_1(x) = G_2(\lambda_1 + \lambda_2 \cdot x)$ with $\lambda_2 > 0$.

In order to more formally specify the LS condition, and at the same time establish the relationship it implies concerning the preference representation under the MS and EU approaches, consider the following. Assume a choice set in which all random variables Z_i differ from one another only by location and scale parameters. Let X be the random variable obtained from one of the Z_i using the normalizing transformation $X = (Z_i - \mu_i)/\sigma_i$ where μ_i and σ_i are the mean and standard deviation of Z_i. All Z_i, no matter which was selected to define X, are equal in distribution to $\mu_i + \sigma_i \cdot X$. Thus, the expected utility from Z_i for any agent with utility function $u(\cdot)$ can be written as:

$Eu(Z_i) = \int_0^B u(\mu_i + \sigma_i \cdot x) \cdot dF(x) = V(\sigma_i, \mu_i)$, where the normalized random variable X takes values in the interval [0,B].

Notice that the MS framework can be used only if the outcome variable z is <u>linear</u> in the random parameter; that is, $z_{xx} = 0$. In this case the choice set differs from one another only by location and scale parameters. While the effect of a change in the random parameter, which differs only by location and scale parameters, can be analyzed under the MS analysis, it should be noted that the effect of a completely arbitrary change in the random parameter can not be analyzed. The risk-increasing <u>linear</u> transformation $\tilde{x} + \gamma \cdot (\tilde{x} - \bar{x})$, which is often used in the EU framework, satisfies the location and scale condition, since $\tilde{x} + \gamma \cdot (\tilde{x} - \bar{x}) = -\gamma \cdot \bar{x} + (1 + \gamma) \cdot \tilde{x} = \lambda_1 + \lambda_2 \cdot \tilde{x}$ where $\lambda_1 = -\gamma \cdot \bar{x}$ and $\lambda_2 = 1 + \gamma > 0$. Thus, the effect of such a change can be analyzed under the MS framework.

The following results of Meyer[1987] prove useful:

a) $V_\mu > 0$ if and only if $u' > 0$,

b) $V_\sigma < 0$ if and only if $u'' < 0$,

c) $S(\sigma,\mu) = -V_\sigma/V_\mu > 0$ if $u' > 0$ and $u'' < 0$,

d) $V(\sigma,\mu)$ is a concave function for σ and μ if and only if $u'' < 0$,

e) S_μ <(=) 0 if and only if $u(\cdot)$ displays DARA(CARA),

f) S_σ >(=) 0 if and only if $u(\cdot)$ displays DARA(CARA)(see Appendix).

4.2.2 The Comparative Statics Problem

We consider an extension of Feder's decision model, in which z takes the form $z(\tilde{x},\tilde{y},\alpha,\delta,\lambda) = \tilde{x}\cdot\tilde{y}\cdot f(\alpha,\delta) + g(\alpha,\delta) + \lambda$, where $\tilde{x}$ and $\tilde{y}$ are random parameters, α and δ are choice variables, and λ is a nonrandom parameter. In this model f and g are real-valued functions of only the control variables. The 2-2-1 model includes the 1-2-1 model presented by Feder[1977] as well as specific 1-1-1 models such as the standard portfolio model and the competitive firm model as special cases.

We begin by assuming that the random variables take values in the interval [0,B]. The utility function u(z) is assumed to be three times differentiable with $u'(z) > 0$ and $u''(z) < 0$. The function z is assumed three times differentiable with $z_{\alpha\alpha} < 0$, $z_{\delta\delta} < 0$, and $z_{\alpha\alpha}\cdot z_{\delta\delta} - z_{\alpha\delta}^2 > 0$. This condition on z, combined with $u'' < 0$, ensures that the second order condition for the maximization problem is satisfied. To simplify the discussion, we focus on interior solutions to the maximization problem.

Even though the LS condition is satisfied in this model, to ensure that changes in the random parameters of the model will change only location and scale of members of the choice set, it is assumed that $\tilde{x}\cdot\tilde{y}$ can be written as $\tilde{x}\cdot\tilde{y} = \phi_1 + \phi_2\cdot\tilde{\xi}$, where $E(\tilde{\xi}) = 0$ and $Var(\tilde{\xi}) = 1$.
Let ϕ_1 and ϕ_2 be the expected value and standard deviation of $\tilde{x}\cdot\tilde{y}$,

respectively; that is, $\phi_1 = E(x\cdot y) = \bar{x}\cdot\bar{y} + Cov(x,y)$, and $\phi_2 = [Var(x\cdot y)]^{1/2}$. Then, $\tilde{x}\cdot\tilde{y} = \phi_1 + \phi_2\cdot\tilde{\xi}$ is a generalization of the risk increasing linear transformation introduced by Sandmo. However, under this assumption, the problems of dealing with the joint CDF for $\tilde{x}$ and $\tilde{y}$ do not arise, and the 2-2-1 model is transformed into a 1-2-1 model. Note that $Var(x\cdot y)$ itself will be a parameter of interest. More commonly, comparative static properties of changes in parameters such as $Var(x)$ or $Var(y)$ will be of interest. Thus, $Var(x\cdot y)$ can be written in terms of these more interesting parameters (see Appendix).

Given these assumptions, the agent's problem is to choose α and δ to maximize $V(\sigma,\mu)$, where $\sigma = \phi_2\cdot f(\alpha,\delta)$, $\mu = \phi_1\cdot f(\alpha,\delta) + g(\alpha,\delta) + \lambda$. Thus, the first-order and second-order conditions can be written as:

$$H_1 = \partial V(\sigma,\mu)/\partial\alpha = V_\sigma\cdot(\partial\sigma/\partial\alpha) + V_\mu\cdot(\partial\mu/\partial\alpha) = 0$$

$$H_2 = \partial V(\sigma,\mu)/\partial\delta = V_\sigma\cdot(\partial\sigma/\partial\delta) + V_\mu\cdot(\partial\mu/\partial\delta) = 0$$

$$H_{11} = [V_{\sigma\sigma}\cdot(\partial\sigma/\partial\alpha) + V_{\sigma\mu}\cdot(\partial\mu/\partial\alpha)]\cdot(\partial\sigma/\partial\alpha) + V_\sigma\cdot(\partial^2\sigma/\partial\alpha^2) + [V_{\mu\mu}\cdot(\partial\mu/\partial\alpha) + V_{\sigma\mu}\cdot(\partial\sigma/\partial\alpha)]\cdot(\partial\mu/\partial\alpha) + V_\mu\cdot(\partial^2\mu/\partial\alpha^2) < 0$$

$$H_{22} = [V_{\sigma\sigma}\cdot(\partial\sigma/\partial\delta) + V_{\sigma\mu}\cdot(\partial\mu/\partial\delta)]\cdot(\partial\sigma/\partial\delta) + V_\sigma\cdot(\partial^2\sigma/\partial\delta^2) + [V_{\mu\mu}\cdot(\partial\mu/\partial\delta) + V_{\sigma\mu}\cdot(\partial\sigma/\partial\delta)]\cdot(\partial\mu/\partial\delta) + V_\sigma\cdot(\partial^2\mu/\partial\delta^2) < 0$$

$$H_{12} = [V_{\sigma\sigma}\cdot(\partial\sigma/\partial\delta) + V_{\sigma\mu}\cdot(\partial\mu/\partial\delta)]\cdot(\partial\sigma/\partial\alpha) + V_\sigma\cdot(\partial^2\sigma/\partial\alpha\partial\delta) + [V_{\mu\mu}\cdot(\partial\mu/\partial\delta) + V_{\sigma\mu}\cdot(\partial\sigma/\partial\delta)]\cdot(\partial\mu/\partial\alpha) + V_\mu\cdot(\partial^2\mu/\partial\alpha\partial\delta)$$

$$H = H_{11}\cdot H_{22} - H_{12}^2 > 0.$$

The comparative static questions addressed here are how do the optimal values of α and δ change when a parameter which can affect the agent's decision is changed. Parameters which can influence the agent's choice are: $\bar{x}$, $\bar{y}$, $Cov(x,y)$, $Var(x)$, $Var(y)$, $Cov(x^2,y^2)$, and λ.

4.2.3 The Comparative Static Results

Notice that the condition $f_\alpha/f_\delta = g_\alpha/g_\delta$ is a characteristic of the model. Thus, at $H_1 = 0$ and $H_2 = 0$, $H_1 = (f_\alpha/f_\delta)\cdot H_2$. To simplify the analysis, it is assumed that $g(\alpha,\delta)$ is additively separable and linear in α and δ; thus, $g_{\alpha\delta} = 0$, $g_{\alpha\alpha} = 0$, and $g_{\delta\delta} = 0$. The case where $f_\alpha > 0$ and $f_\delta > 0$ is considered here. Also, it is assumed that g_α and g_δ have negative signs. Assuming a parameter, $\bar{x}$, is increased, then we have the following comparative statics concerning the effect of a change in $\bar{x}$: $\partial\alpha/\partial\bar{x} = -(1/H)\cdot(\partial H_1/\partial\bar{x})\cdot[H_{22} - (f_\delta/f_\alpha)\cdot H_{12}]$. Simplifying the last term, $H_{22} - (f_\delta/f_\alpha)\cdot H_{12} = V_\sigma\cdot\phi_2\cdot f_{\delta\delta} + V_\mu\cdot\phi_2\cdot f_{\delta\delta} - (f_\delta/f_\alpha)\cdot(V_\sigma\cdot\phi_2\cdot f_{\alpha\delta} + V_\mu\cdot\phi_1\cdot f_{\alpha\delta}) = (1/f_\alpha)\cdot(V_\mu\cdot\phi_1 + V_\sigma\cdot\phi_2)(f_\alpha\cdot f_{\delta\delta} - f_\delta\cdot f_{\alpha\delta})$. Substituting this into $\partial\alpha/\partial\bar{x}$, $\partial\alpha/\partial\bar{x} = -(1/H)\cdot(\partial H_1/\partial\bar{x})\cdot(1/f_\alpha)\cdot(V_\mu\cdot\phi_1 + V_\sigma\cdot\phi_2)(f_\alpha\cdot f_{\delta\delta} - f_\delta\cdot f_{\alpha\delta})$.
By using a similar procedure, we can demonstrate that
$\partial\delta/\partial\bar{x} = -(1/H)\cdot(\partial H_1/\partial\bar{x})\cdot(1/f_\alpha)\cdot(V_\mu\cdot\phi_1 + V_\sigma\cdot\phi_2)\cdot(f_\delta\cdot f_{\alpha\alpha} - f_\alpha\cdot f_{\alpha\delta})$.

The first order conditions imply $V_\mu\cdot\phi_1 + V_\sigma\cdot\phi_2 > 0$, since interior solutions are assumed here. Now, we assume that $f_{\alpha\delta} > 0$. This condition is required to make the last term in $\partial\alpha/\partial\bar{x}$ or $\partial\delta/\partial\bar{x}$ have determinate sign. Under these assumptions, the comparative static results will depend <u>only</u> on $\partial H_1/\partial\bar{x}$. That is, if $\partial H_1/\partial\bar{x} > 0$, then $\partial\alpha/\partial\bar{x} > 0$ and $\partial\delta/\partial\bar{x} > 0$.

The following comparative statics are straightforward, and some parameters such as f_α and ϕ_2 are omitted.

$$\partial H_1/\partial\bar{x} = -[S_\sigma\cdot(1/2)\cdot(\mathrm{Var}(xy))^{-1/2}\cdot(2\bar{x}\cdot\mathrm{Var}(y) - 2\bar{y}\cdot\mathrm{Cov}(x,y)) + S_\mu\cdot\bar{y}]$$
$$- [S(\sigma,\mu)\cdot(1/2)\cdot(\mathrm{Var}(xy))^{-1/2}\cdot(2\bar{x}\cdot\mathrm{Var}(y) - 2\bar{y}\cdot\mathrm{Cov}(x,y))] + \bar{y},$$
$$\partial H_1/\partial\bar{y} = -[S_\sigma\cdot(1/2)\cdot(\mathrm{Var}(xy))^{-1/2}\cdot(2\bar{y}\cdot\mathrm{Var}(x) - 2\bar{x}\cdot\mathrm{Cov}(x,y)) + S_\mu\cdot\bar{x}]$$
$$- [S(\sigma,\mu)\cdot(1/2)(\mathrm{Var}(xy))^{-1/2}\cdot(2\bar{y}\cdot\mathrm{Var}(x) - 2\bar{x}\cdot\mathrm{Cov}(x,y))] + \bar{x},$$

$\partial H_1/\partial Cov(x,y) = -[S_\sigma \cdot (1/2) \cdot (Var(xy))^{-1/2} \cdot (-2Cov(x,y) - 2\bar{x} \cdot \bar{y}) + S_\mu]$

$- [S(\sigma,\mu) \cdot (1/2) \cdot (Var(xy))^{-1/2} \cdot (-2Cov(x,y) - 2\bar{x} \cdot \bar{y})] + 1 > 0$

under DARA,

$\partial H_1/\partial Var(x) = -[S_\sigma \cdot (1/2) \cdot (Var(xy))^{-1/2} \cdot (Var(y) + \bar{y}^2)] - [S(\sigma,\mu) \cdot (1/2) \cdot$

$(Var(xy))^{-1/2} \cdot (Var(y) + \bar{y}^2)] < 0$ under DARA,

$\partial H_1/\partial Var(y) = -[S_\sigma \cdot (1/2) \cdot (Var(xy))^{-1/2} \cdot (Var(x) + \bar{x}^2)] - [S(\sigma,\mu) \cdot (1/2) \cdot$

$(Var(xy))^{-1/2} \cdot (Var(x) + \bar{x}^2)] < 0$ under DARA,

$\partial H_1/\partial Cov(x^2,y^2) = - S_\sigma \cdot (1/2) \cdot (Var(xy))^{-1/2} - S(\sigma,\mu) \cdot (1/2) \cdot (Var(xy))^{-1/2} < 0$

under DARA,

$\partial H_1/\partial \lambda = - S_\mu \cdot \phi_2 > 0$ under DARA.

Many of the comparative static results of the 1-2-1 model carry through to the case in which there are two sources of risk. Under DARA, a reduction in variance of $\tilde{x}$ or $\tilde{y}$ increases the optimal values of α and δ. Also, under DARA, the effect of a change in λ is the same as that in the 1-2-1 model. However, other intuitively appealing results from the 1-2-1 model do not hold unambiguously in the case of two sources of randomness.

An increase in $\bar{x}$ or $\bar{y}$ does not unambiguously result in an increase in α and δ. This is because it increases the expected value of z, μ, but <u>may</u> also increase the standard deviation, σ. The effect of an increase in $\bar{x}$ on the standard deviation of z can be positive or negative according as the term $\bar{x} \cdot Var(y) - \bar{y} \cdot Cov(x,y)$ is positive or negative. If the effect is negative, then the increase in $\bar{x}$ unambiguously results in an increase in α and δ under DARA. However, if the effect is positive, the increase in $\bar{x}$ does not unambiguously result in an increase in α and δ even under DARA.

The term Cov(x,y) has two effects. First, a high Cov(x,y) increases the expected value of the outcome variable, z. Second, a higher Cov(x,y) reduces the standard deviation of z. This second effect runs counter to intuition, based on the notion that low(negative) covariance will smooth out z. That intuition is not wrong, but the mathematical expression formalizing that intuition involves the covariance between x^2 and y^2 and the covariance between x and y. The variance of $\tilde{x}\cdot\tilde{y}$ is $E(x^2\cdot y^2) - [E(x\cdot y)]^2$. Cov(x,y) increases the second term and thus reduces the variance, while $Cov(x^2,y^2)$ increases the first term and thus increases the variance(see Appendix). The decomposition presented here points out the importance of a distinction between Cov(x,y) and $Cov(x^2,y^2)$. Thus, under DARA, an increase in the term Cov(x,y) unambiguously leads to an increase in the optimal value of α and δ, while an increase in the term $Cov(x^2,y^2)$ results in a decrease in α and δ.

4.3 A Two Output Competitive Firm Model

In this section, we examine a specific 2-2-1 model, in which the outcome variable z takes the form $z(\tilde{x},\tilde{y},\alpha,\delta,\lambda) = \tilde{x}\cdot\alpha + \tilde{y}\cdot\delta - c(\alpha,\delta) - \lambda$, where $\tilde{x}$ and $\tilde{y}$ are the prices of products α and δ, respectively, α and δ are two products, λ is fixed cost, and $c(\alpha,\delta)$ is a variable cost function. Note that the model is an extension to the 1-2-1 model investigated in chapter 3.3. Assuming $\tilde{x}$ and $\tilde{y}$ are independently distributed, the agent's problem is to choose α and δ to maximize $Eu(z) = \int_0^B\int_0^B u(z)dF(x)dG(y)$. It is assumed that the decision maker is a risk averter; that is, $u' > 0$ and $u'' < 0$, and the variable cost function has the same properties as in chapter 3.3.

Under these assumptions, the first-order conditions can be written as: $H_1 = Eu'(\tilde{x} - c_\alpha(\alpha,\delta)) = Eu'(\bar{x} - c_\alpha(\alpha,\delta)) + Cov(u',\tilde{x}) \le 0$,

$H_2 = Eu'(\tilde{y} - c_\delta(\alpha,\delta)) = Eu'(\bar{y} - c_\delta(\alpha,\delta)) + Cov(u',\tilde{y}) \le 0$.

The question addressed here is how do the optimal values of α and δ change when an increase in risk occurs from an initial nonrandom situation, where $x = \bar{x}$ and $y = \bar{y}$, to a situation when both $\tilde{x}$ and $\tilde{y}$ are randoms with means $\bar{x}$ and $\bar{y}$, respectively. Let α_c and δ_c denote the optimal values of α and δ under certainty and α_a and δ_a when both product's prices become random. We assume that α and δ take values in the interval $[0,\infty]$. Then a corner solution occurs when $\alpha_c = 0$ or $\delta_c = 0$. The case where $\alpha_c = 0$ and $\delta_c = 0$ is not interesting. The diagram approach is used to deal with corner solutions.

Lemma 4.1: If $\tilde{x}$ and $\tilde{y}$ are independent, then $Cov(u',x) \le 0$.

Proof: If $\tilde{x}$ and $\tilde{y}$ are independently distributed, then $Cov(u',x) = Eu'(x - \bar{x}) = \int_0^B\int_0^B u'(\cdot)(x - \bar{x})dF(x)dG(y)$. It is convenient to write $u' = u'(x,y)$ as it depends on both random parameters. Because $\int_0^B\int_0^B u'(\bar{x},y)(x - \bar{x})dF(x)dG(y) = 0$, we have

$Eu'(x - \bar{x}) = \int_0^B\int_0^B [u'(x,y) - u'(\bar{x},y)](x - \bar{x})dF(x)dG(y)$.

Note that under $u'' < 0$, the integrand is nonpositive since $u'(x,y) - u'(\bar{x},y)$ and $(x - \bar{x})$ are opposite in sign for all x in $[0,B]$. Thus, $Cov(u',x) \le 0$. The strict equality holds when $\alpha = 0$.

Corollary 4.1: Assume that the risk averse firm specializes in product α under certainty; that is, $\alpha_c > 0$ and $\delta_c = 0$.

If $c_{\alpha\delta} < 0$, then $\alpha_a < \alpha_c$ and $\delta_a = \delta_c = 0$.

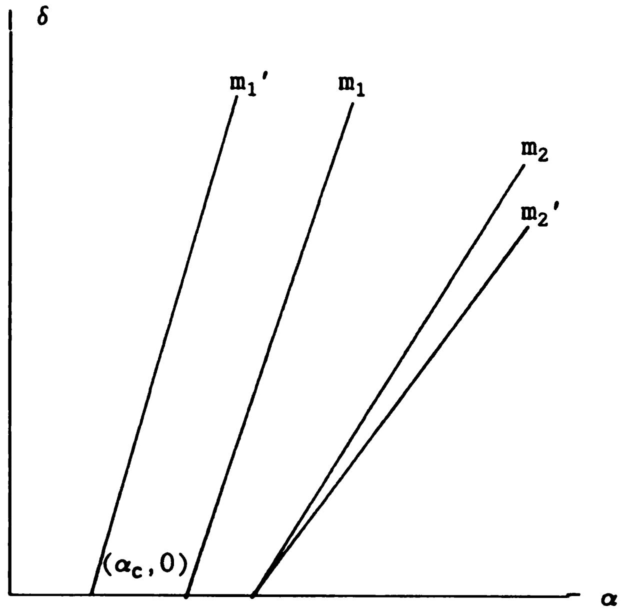

Figure 4.1.1

$c_{\alpha\delta} < 0$

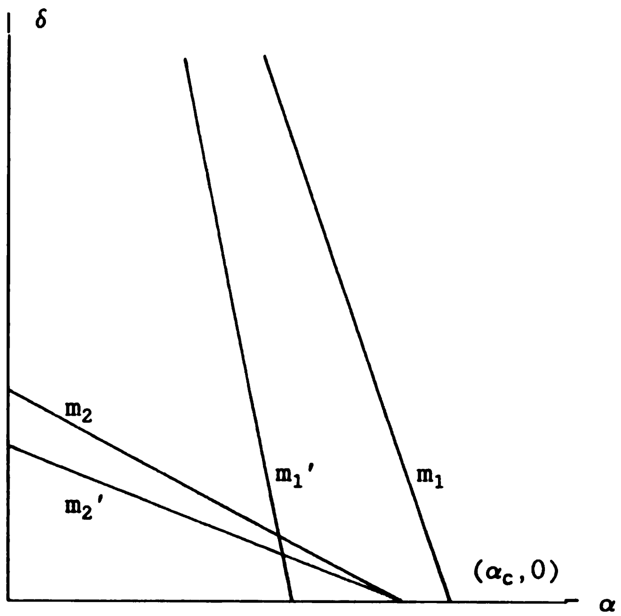

Figure 4.1.2

$c_{\alpha\delta} > 0$

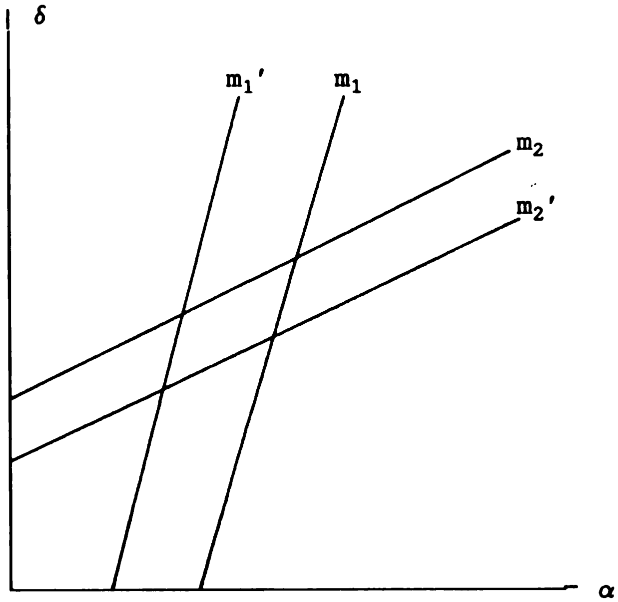

Figure 4.2.1

$c_{\alpha\delta} < 0$

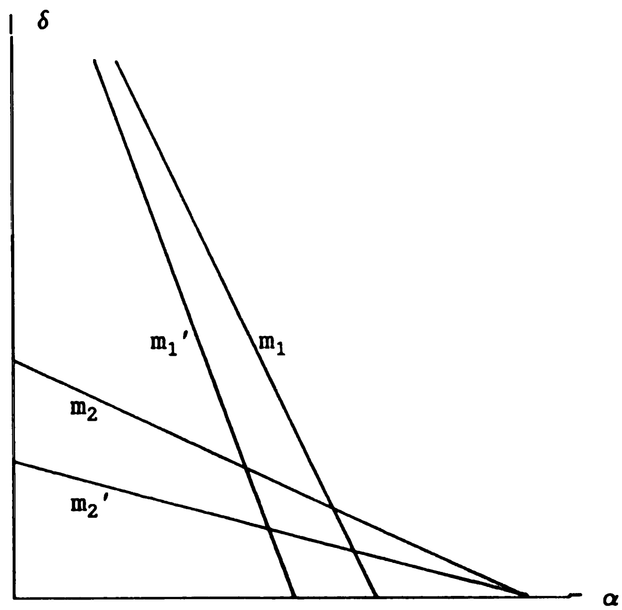

Figure 4.2.2

$c_{\alpha\delta} > 0$

If $c_{\alpha\delta} > 0$, then $\alpha_a < \alpha c$ and δ_a may(need not) be positive.

Proof: If $\alpha_c > 0$ and $\delta_c = 0$, then $\alpha_c > 0$ and $\delta_c = 0$ must satisfy $\bar{x} - c_\alpha(\alpha_c,0) = 0$ and $\bar{y} - c_\delta(\alpha_c,0) \leq 0$. Because the set of points above the m_2 curve represents $\bar{y} - c_\delta < 0$, the m_2 curve should lie to the right of the m_1 when $c_{\alpha\delta} < 0$, while the m_2 should rest in below the m_1 when $c_{\alpha\delta} > 0$. Lemma 3.1 implies that the m_1 and m_2 curves have the relative positions shown in figures 4.1.1 and 4.1.2. By lemma 4.1, the introduction of uncertainty shifts the m_1 curve to the left, and the m_2 curve downward. This is because $H_{11} < 0$ and $H_{22} < 0$. The case of $c_{\alpha\delta} < 0$ is obvious from the figure. Also, it is clear to see that $c_{\alpha\delta} > 0$ is a necessary condition for product diversification.

The case where $\alpha_c = 0$ and $\delta_c > 0$ is not considered since it is analytically symmetric to the case investigated in corollary 4.1. This corollary only requires that the decision maker be risk averse, and indicates that if $c_{\alpha\delta} > 0$, then price uncertainty may result in product diversification. The diversification occurs if a product which is not produced under certainty is produced when its price is random.

Corollary 4.2: Assume that the risk averse firm produces both goods under certainty; that is, $\alpha_c > 0$ and $\delta_c > 0$.
If $c_{\alpha\delta} < 0$, then $\alpha_a < \alpha_c$ and $\delta_a < \delta_c$.
If $c_{\alpha\delta} > 0$, then the firm may increase the output of one product, but will not increase the output of both products.

Proof: The proof is similar to that which is provided for corollary 4.1 and is simply sketched here. By lemma 4.1, the introduction of uncertainty from an initial nonrandom situation shifts the m_1 curve to the left, and the m_2 curve downward. The conclusion of the corollary follows immediately from figures 4.2.1 and 4.2.2.

APPENDIX

<u>Proof of $S_\sigma \geq 0$</u>: (Leathers)

$S(\sigma,\mu) = -V_\sigma/V_\mu$. $S_\mu = (1/V_\mu^2)\cdot[-V_{\sigma\mu}V_\mu + V_\sigma V_{\mu\mu}] = (1/V_\mu)\cdot[-V_{\sigma\mu} - S(\sigma,\mu)V_{\mu\mu}]$ <(=) 0 under DARA(CARA)[Meyer, 1987, p.425]. $S_\mu \leq 0$ implies that $V_{\sigma\mu} \geq 0$ and $V_{\sigma\mu}/V_{\mu\mu} \leq V_\sigma/V_\mu$. By property (d), $V_{\sigma\sigma}/V_{\sigma\mu} \leq V_{\sigma\mu}/V_{\mu\mu}$. Combining these we have $V_{\sigma\sigma}/V_{\sigma\mu} \leq V_{\sigma\mu}/V_{\mu\mu} \leq V_\sigma/V_\mu$. $S_\sigma(\sigma,\mu) = (V_{\sigma\mu}/V_\mu)\cdot[-V_{\sigma\sigma}/V_{\sigma\mu} + V_\sigma/V_\mu] \geq 0$ because $V_{\sigma\mu} \geq 0$ and $(V_\sigma/V_\mu - V_{\sigma\sigma}/V_{\sigma\mu}) \geq 0$.

<u>Proof of $\phi_2^2 = Var(x\cdot y)$</u>:

$Var(x\cdot y) = E((x\cdot y)^2) - [E(x\cdot y)]^2$. Expanding the first term, $E(x^2\cdot y^2) = E(x^2)\cdot E(y^2) + Cov(x^2,y^2) = [Var(x) + \bar{x}^2]\cdot[Var(y) + \bar{y}^2] + Cov(x^2,y^2)$. Also, $[E(x\cdot y)]^2 = [\bar{x}\cdot\bar{y} + Cov(x,y)]^2 = \bar{x}^2\cdot\bar{y}^2 + 2\bar{x}\cdot\bar{y}Cov(x,y) + (Cov(x,y))^2$. Combining these we have $Var(x\cdot y) = Var(x)\cdot Var(y) + \bar{x}^2\cdot Var(y) + \bar{y}^2\cdot Var(x) + Cov(x^2,y^2) - (Cov(x,y))^2 - 2\bar{x}\cdot\bar{y}\cdot Cov(x,y)$.

CHAPTER FIVE

THE 2-1-2 MODEL

5.0 Introduction

Many theoretically interesting and policy relevant economic problems can be formulated by means of a two outcome(or argument) model, where a decision maker's utility function has two outcomes related to each other through a linear budget constraint. This framework which captures the essence of decision making(the notion of tradeoffs within an economic constraint) is commonly used to investigate the effects of taxes on labor supply or of varying interest rates on savings. Some parameters in the two outcome model are generally assumed to include randomness. Nominal wage rate uncertainty in the labor supply model is one example. It is most common in markets where earnings include commissions or when present work yields an unknown future income.

Sandmo[1970], Rothschild and Stiglitz[1971], Dardanoni[1988], and Ormiston and Schlee[1992] examine a decision model composed of one random, one choice, but two outcome variables. In this model, the utility function depends on two outcome variables, each of which in turn depends on one random variable, one choice variable, and a set of nonrandom parameters. They address the following question: how does a decision maker adjust the choice variable when the random parameter undergoes an increase in risk or first and second degree stochastically dominant shifts? We extend these earlier works by considering two sources of randomness including cases where the risks are not independent of one another, and we analyze the effects of two types of changes in randomness: FSD shifts and R-S increases in risk.

In analyzing decision models involving risk, at least three methods have been used to represent a change in random variable $\tilde{x}$. These include changing the CDF for $\tilde{x}$, transforming $\tilde{x}$ stochastically, and transforming $\tilde{x}$ deterministically. The three methods are employed in this chapter, with special emphasis on the stochastic transformation which is used to examine the case where $\tilde{x}$ and $\tilde{y}$ are stochastically dependent. This is because it allows the characterization of comparative risk for univariate CDFs, introduced by Rothschild and Stiglitz[1970] and Hadar and Russell[1969], to be used directly.

This chapter proceeds as follows. In the next section the literature concerning the 1-1-2 model is reviewed first. In section 2, the effects of FSD and MPS changes in $\tilde{x}$ within a specific 2-1-2 model are examined using the CDF and transformation approaches. The analysis shows that the conditions sufficient to make determinate comparative static statements concerning the effects of FSD shifts and MPSs on the choice variable are exactly those conditions determined in the 1-1-2 models.

5.1 Literature Review

In the 1-1-2 model, the utility function depends on two outcome variables, each of which in turn depends on one random variable, one choice variable, and a set of nonrandom parameters. In general form an agent is assumed to choose α to maximize $EU(z_1(\tilde{x},\alpha,\lambda),z_2(\tilde{x},\alpha,\lambda))$, where the outcome variables, z_1 and z_2, depend on a random parameter, $\tilde{x}$, a choice variable, α, and a set of nonrandom parameters, λ.

Although the 1-1-2 model can represent wider ranges of economic decisions, it raises several difficult problems. First, in this decision framework, utility depends on two outcomes, z_1 and z_2. Thus, problems involving multidimensionality of utility arise. Next, the risk aversion measures derived for univariate utility function cannot be used directly. Finally, in a world of certainty, each decision maker may select a different level of α. To see this last point, note that if the random variable $\tilde{x}$ is fixed at x_0, then a decision maker whose preference is represented by utility function U(,) selects α so as to maximize $U(z_1(x_0,\alpha,\lambda),z_2(x_0,\alpha,\lambda))$. Thus, another decision maker who has different utility function V(,) may select a different level of α. This causes us to worry about differences in behavior which would arise even in a world of certainty.

There are two interesting and frequently studied comparative static problems. The first asks what conditions are needed to predict the direction of change in the optimal value of α when an increase in risk occurs from a nonrandom initial situation, where $x = \bar{x}$, to a situation when $\tilde{x}$ is random with mean $\bar{x}$. This question arises because the comparative static analysis in single outcome variable models should be changed in the two outcome variable case. The second question is concerned with determination of the direction of change for the choice variable selected by a decision maker when the random parameter $\tilde{x}$ undergoes an increase in risk or first and second degree stochastically dominant shifts. This is an especially important question when the utility function is not additively separable in outcome variables. Mirman[1971] addresses the first question, and Sandmo[1970], Rothschild and Stiglitz[1971], Block and Heineke[1973], Dardanoni[1988], and

Ormiston and Schlee[1992] address the second. This chapter deals with the latter question. We begin by reviewing a specific 1-1-2 model presented by Mirman.

Mirman[1971] considers a two-period consumption and saving model, in which $z_1(\tilde{x},\alpha,\lambda) = \lambda - \alpha \geq 0$ and $z_2(\tilde{x},\alpha,\lambda) = \tilde{x}\cdot f(\alpha) \geq 0$. Here $\tilde{x}$ represents the uncertainty of the second period technology, α is investment for the second period, λ is initial endowment which can be used for consumption in the first period or invested for consumption in the second period, and $f(\cdot)$ is a production function with $f'(\cdot) > 0$ and $f''(\cdot) < 0$. Mirman assumes the utility function is additively separable in outcome variables; that is, $U(z_1,z_2) = u(z_1) + v(z_2)$ where u', $v' > 0$ and u'', $v'' < 0$. It is the assumption concerning the utility function that allows him to investigate the effect on α of a global increase in risk. Let α_c denote the optimal value of α under a nonrandom initial situation where $x = \bar{x}$. Mirman's main result is:

<u>Theorem 5.1</u>: Assuming that a decision maker chooses α to maximize $E[u(\lambda-\alpha) + v(\tilde{x}\cdot f(\alpha))]$ where u', $v' > 0$, u'', $v'' < 0$, and $f' > 0$, $f'' < 0$, and that $0 < \alpha_c < \lambda$, then the decision maker, when faced with a global increase in risk, will decrease the optimal value of α if $R_R(z_2) \leq 1$ and $R_R'(z_2) \geq 0$, where $R_R(z_2) = - z_2\cdot v''(z_2)/v'(z_2)$.

The general 1-1-2 model of Rothschild and Stiglitz[1971] has outcome variables given by $z_1(\tilde{x},\alpha,\lambda) = \alpha$ and $z_2(\tilde{x},\alpha,\lambda) = \tilde{x}$. They show that the conditions of $U_{\alpha xx} < 0$ or $U_{\alpha xx} > 0$ are sufficient for signing the effect of a R-S increase in risk on the choice variable selected by a risk-averse decision maker. Unfortunately, for many specific 1-1-2 models the conditions specified by R-S are too restrictive to allow one to obtain interesting comparative static results. Additional restrictions must be imposed on the risk taking characteristics of the decision maker and/or the decision maker's behavior under certainty.

The specific 1-1-2 model analyzed by Sandmo[1970] takes the form $z_1(\tilde{x},\alpha,\lambda) = \lambda_1 - \alpha$ and $z_2(\tilde{x},\alpha,\lambda) = \lambda_2 \cdot \alpha + \tilde{x}$, where $\tilde{x}$ is future income (denoted the income risk case), α is saving, and λ_1 and λ_2 are the first period income and the return to saving, respectively. Obviously here z_1 does not include randomness, and z_2 is linear in α and $\tilde{x}$. The utility function $U(z_1,z_2)$ defined over z_1, $z_2 > 0$, which denote respectively present and future consumption, is assumed to be continuous, increasing, concave and at least three times differentiable. In order to derive determinate comparative static results concerning the effect of an increase in risk represented by an increase in γ, Sandmo proposes the <u>temporal</u> risk aversion function: $R_A(z_1,z_2) = - U_{22}(z_1,z_2)/U_2(z_1,z_2)$ where $U_2 = \partial U(z_1,z_2)/\partial z_2$ and $U_{22} = \partial^2 U(z_1,z_2)/\partial z_2^2$. Sandmo's main result is:

<u>Theorem 5.2</u>: Assuming that a decision maker chooses α to maximize $EU(\lambda_1 - \alpha, \lambda_2 \cdot \alpha + \tilde{x})$, then the decision maker, when faced with an increase in γ, will increase the optimal value of α if (a) $\partial\alpha/\partial\bar{x} < 0$ for all $\bar{x}$ and λ_2 under a nonrandom situation where $x = \bar{x}$ and (b) $\partial R_A(z_1,z_2)/\partial\alpha < 0$.

This theorem gives conditions sufficient to yield determinate comparative static results concerning the effect on saving of a special type of a R-S increase in risk. Condition (a) requires that the decision maker's behavior under certainty should be known, and condition (b) restricts the set of decision makers to those exhibiting decreasing temporal risk aversion with respect to the choice variable α. Dardanoni [1988] extends the result by showing that the same conditions are still sufficient for an arbitrary R-S increase in risk.

Sandmo considers another case, in which z_1 is the same as before and $z_2(\tilde{x},\alpha,\lambda) = \tilde{x}\cdot\alpha + \lambda_2$. In this case $\tilde{x}$ is the return to saving (denoted the capital risk case), α is saving, and λ_2 is the second period income. Note that $\tilde{x}$ represents a multiplicative risk. He addresses the same question as does in the additive risk case, but is unable to derive unambiguous comparative static results. That is, Sandmo makes the distinction between the additive and multiplicative risks in the study of saving decisions in a two-period model with a general(nonseparable) utility function. Block and Heineke[1973] consider a specific 1-1-2 model, whose structure is the same as Sandmo's[1970], to examine the labor supply decision of a single economic agent. B-H are <u>also</u> unable to derive determinate comparative static results concerning the effect of an increase in γ when $\tilde{x}$ represents a multiplicative risk.

The specific 1-1-2 model of Dardanoni[1988] assumes the outcome variables take the form $z_1(\tilde{x},\alpha,\lambda) = \lambda_1 - \alpha$ and $z_2(\tilde{x},\alpha,\lambda) = \tilde{x}\cdot\alpha + \lambda_2$, where $\tilde{x}$ is a random variable, α is a choice variable, and $\lambda_i(i=1,2)$ are nonrandom parameters. This model includes Sandmo's model of optimal savings and Block and Heineke's model of labor supply as special cases. Notice that the random parameter $\tilde{x}$ represents a multiplicative risk. The assumptions concerning utility function are the same as those in Sandmo[1970]. In order to derive unambiguous comparative static results concerning the effect of an arbitrary Rothschild and Stiglitz increase in risk, Dardanoni uses the proportional risk aversion function introduced by Zeckhauser and Keeler[1970] and Menezes and Hanson[1971]: $R_p(z_1,z_2) = - (z_2-\lambda_2)\cdot U_{22}(z_1,z_2)/U_2(z_1,z_2)$. His main result is:

Theorem 5.3: Assuming that a decision maker chooses α to maximize $EU(\lambda_1-\alpha,\tilde{x}\cdot\alpha+\lambda_2)$, then the decision maker, when faced with an arbitrary Rothschild and Stiglitz increase in risk, will increase the optimal value of α if (a) $\partial\alpha/\partial\bar{x} < 0$ for all $\bar{x}$ and λ_2 under a nonrandom situation where $x = \bar{x}$ and (b) $\partial R_p(z_1,z_2)/\partial\alpha < 0$.

Again, theorem 5.3 implies that meaningful and intuitive predictions on comparative statics in the 1-1-2 model may be obtained using both the results from individual's behavior under certainty and some restrictions on the utility function in terms of risk preferences. It is interesting to observe that if $\partial\alpha/\partial\bar{x}$ and $\partial R_p(z_1,z_2)/\partial\alpha$ are opposite in sign, then we may not unambiguously predict the decision maker's response to a R-S increase in risk. However, there is no legitimate assumption as to whether $\partial\alpha/\partial\bar{x}$ is in general likely to be positive or negative; nor is it clear whether R_p is in general likely to be increasing or decreasing in α.

In the general 1-1-2 model given by Ormiston and Schlee[1992] it is assumed that the agent chooses α to maximize $EU(\alpha,z(\tilde{x},\alpha))$. O-S provide conditions on preferences that are necessary and sufficient for a dominating shift in the initial CDF, representing FSD improvements in $\tilde{x}$ or MPCs, to cause an unambiguous change in the optimal level of the choice variable α. The identification of necessary conditions allows one to see precisely the tradeoffs between restrictions on preferences and CDF changes in obtaining interesting comparative static results. They then investigate the implications of their results for a linear model which includes the labor supply and consumption-savings model.

5.2 A Specific 2-1-2 Model

5.2.1 The Comparative Statics Problem

A specific two random-one choice-two outcome(2-1-2) model is considered in this section. In it the agent is assumed to choose α to maximize $EU(\lambda-\alpha,\tilde{x}\cdot\alpha+\tilde{y}) = \int_0^B\int_0^B U(\lambda-\alpha,\tilde{x}\cdot\alpha+\tilde{y})\cdot d^2H(x,y)$, where $\tilde{x}$ and $\tilde{y}$ are random parameters which take values in the interval $[0,B]$, α is a choice variable, and λ is a nonrandom parameter. In this case $z_1 = \lambda - \alpha$ and $z_2 = \tilde{x}\cdot\alpha + \tilde{y}$. The utility function $U(z_1,z_2)$ defined over z_1 and z_2 is assumed to be continuous, increasing, concave and at least three times differentiable.

The joint cumulative distribution function(CDF) for $\tilde{x}$ and $\tilde{y}$ are denoted $H(x,y)$. The conditional and marginal CDFs for $\tilde{x}$ are denoted $F(x|y)$ and $F(x)$, respectively, and for $\tilde{y}$ they are $G(y|x)$ and $G(y)$. If $\tilde{x}$ and $\tilde{y}$ are independently distributed then $F(x|y) = F(x)$ for all y, $G(y|x) = G(y)$ for all x, and $H(x,y) = F(x)\cdot G(y)$. To simplify notation, the symbol $d^2H(x,y)$ is used to denote $[\partial^2H(x,y)/\partial x\cdot\partial y]\cdot dx\cdot dy$.

To simplify the analysis, the initial joint distribution $H^o(x,y)$ is assumed to be such that the agent attains a unique, regular, interior maximum at α_o satisfying $0 < \alpha_o < \lambda$. This is defined by the first order condition: $\eta(\alpha_o) = \int_0^B\int_0^B \psi(x;y,\alpha_o)\cdot d^2H^o(x,y) = 0$ where $\psi(x;y,\alpha_o) = - U_1(\lambda-\alpha_o,x\cdot\alpha_o+y) + x\cdot U_2(\lambda-\alpha_o,x\cdot\alpha_o+y)$. It is also assumed that the second order condition is always satisfied at $\alpha = \alpha_o$; that is,

$\eta'(\alpha_0) = \int_0^B\int_0^B \partial\psi(x;y,\alpha_0)/\partial\alpha \cdot d^2H^o(x,y) < 0$. As usual, determining how the optimal value for α changes as the random variable $\tilde{x}$ is altered involves determining the sign of $\eta(\alpha_0)$ after $\tilde{x}$ has been changed.

The proofs of our theorems will be simplified if we utilize the following corollary.

Corollary 5.1: (a) ψ_x >(<) 0 if and only if $\partial\alpha/\partial\bar{x}$ >(<) 0 for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$. (b) ψ_{xx} >(<) 0 if $\partial\alpha/\partial\bar{x}$ <(>) 0 for all $\bar{x}$ and $\partial R_p(z_1,z_2)/\partial\alpha$ <(>) 0 where $R_p(z_1,z_2) = -(z_2-y)\cdot U_{22}(z_1,z_2)/U_2(z_1,z_2)$.

Proof: (a). Under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$, $\partial\alpha/\partial\bar{x} = -(1/H)\cdot(-U_{12}\cdot\alpha + \bar{x}\cdot\alpha\cdot U_{22} + U_2)$ where $H = U_{11} - 2\bar{x}\cdot U_{12} + \bar{x}^2\cdot U_{22} < 0$. The statement follows from $\partial\psi(x;y,\alpha)/\partial x = -U_{12}\cdot\alpha + x\cdot\alpha\cdot U_{22} + U_2$.
(b). The proof is given in Dardanoni[1988] and is simply sketched here. Dardanoni shows that the given conditions are sufficient for determining the sign of $\psi_{xx} = \alpha^2\cdot(-U_{122} + x\cdot U_{222}) + 2x\cdot U_{22}$.

5.2.2 The Comparative Static Results

The comparative static analysis concerning the effect of a change in a random parameter for decision models with two sources of randomness should resolve the problems of handling the joint CDF for $\tilde{x}$ and $\tilde{y}$. Hadar and Seo[1990] avoid the problems by assuming independence, while Meyer and Ormiston[1991] propose the stochastic and deterministic transformations as a method to solve them. For the case of independent $\tilde{x}$ and $\tilde{y}$, the comparative static analysis can be carried out by replacing $H^o(x,y) = F^o(x)\cdot G(y)$ with $H^1(x,y) = F^1(x)\cdot G(y)$. FSD improvements in the

CDF for $\tilde{x}$ are represented by $F^1(x) \le F^0(x)$ for all x in [0,B]. For MPSs the condition is: $\int_0^s [F^1(x)-F^0(x)]\cdot dx \ge 0$ for all s in [0,B], with equality for s = B.

Theorem 5.4: Assume a) $\tilde{x}^i$ and $\tilde{y}$ are stochastically independent, i = 0,1; b) $\tilde{x}^1$ FSD $\tilde{x}^0$; c) $EU(\lambda-\alpha_i,\tilde{x}^i\cdot\alpha_i+\tilde{y})$ is maximized at $0 < \alpha_i < \lambda$. Then α_1 >(<) α_0 if and only if $\partial\alpha/\partial\bar{x}$ >(<) 0 for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$.

Proof: The FOC is: $\eta(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot d^2H^0(x,y) = 0$. Subtracting this from the similar expression with $H^1(x,y)$ replacing $H^0(x,y)$ yields:

$$\eta^*(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot d^2[H^1(x,y) - H^0(x,y)]$$
$$= \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot d[F^1(x) - F^0(x)]\cdot dG(y).$$

By part (a) of corollary 5.1, $\eta^*(\alpha_0)$ >(<) 0 if and only if $\partial\alpha/\partial\bar{x}$ >(<) 0 for all $\bar{x}$. Then, the second order condition implies α_1 >(<) α_0.

Theorem 5.5: Assume a) $\tilde{x}^i$ and $\tilde{y}$ are stochastically independent, i = 0,1; b) $\tilde{x}^1$ MPS $\tilde{x}^0$; c) $EU(\lambda-\alpha_i,\tilde{x}^i\cdot\alpha_i+\tilde{y})$ is maximized at $0 < \alpha_i < \lambda$. Then α_1 >(<) α_0 if $\partial\alpha/\partial\bar{x}$ <(>) 0 for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$ and $\partial R_P(z_1,z_2)/\partial\alpha$ <(>) 0.

Proof: Using the same procedure as in the proof of theorem 5.4, we have $\eta^*(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot d[F^1(x) - F^0(x)]\cdot dG(y)$. By part (b) of corollary 5.1, $\eta^*(\alpha_0)$ >(<) 0 <u>if</u> $\partial\alpha/\partial\bar{x}$ <(>) 0 for all $\bar{x}$ and $\partial R_P(z_1,z_2)/\partial\alpha$ <(>) 0. Then, from the second order condition α_1 >(<) α_0.

The conditions in theorem 5.4 and 5.5 are the same as those found in the 1-1-2 model by Block and Heineke[1973] for FSD shifts, and

Dardanoni[1988] for MPSs. Thus, with independence, extension to the 2-1-2 model preserves the findings from the 1-1-2 model.

Now, consider the case where $\tilde{x}$ and $\tilde{y}$ are stochastically dependent. Rothschild and Stiglitz[1970] use a stochastic transformation of $\tilde{x}$ in one of their three definitions of an increase in the riskiness of $\tilde{x}$. According to the first definition(RS1), $\tilde{x}^1$ is riskier than $\tilde{x}^0$ if $\tilde{x}^1 =^d [\tilde{x}^0 + \tilde{\xi}]$, where $\tilde{\xi}$ satisfies $E(\tilde{\xi}|x^0) = 0$ and " $=^d$ " represents "is equal in distribution to". Theorem 5.6 uses this to extend theorem 5.5 to the case of dependent random variables.

<u>Theorem 5.6</u>: Assume a) $\tilde{x}^i$ and $\tilde{y}$ are stochastically dependent; b) $\tilde{x}^1$ is obtained from $\tilde{x}^0$ by adding $\tilde{\xi}$ which satisfies $E(\tilde{\xi}|x^0) = 0$, and $\tilde{\xi}$ is independent of $\tilde{y}$; c) $EU(\lambda-\alpha_i, \tilde{x}^i \cdot \alpha_i + \tilde{y})$ is maximized at $0 < \alpha_i < \lambda$. Then $\alpha_1 >(<) \alpha_0$ if $\partial\alpha/\partial\bar{x} <(>) 0$ for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$ and $\partial R_P(z_1, z_2)/\partial\alpha <(>) 0$.

<u>Proof</u>: Because $\tilde{\xi}$ is independent of $\tilde{y}$, $(\tilde{x}^1|y) = (\tilde{x}^0|y) + \tilde{\xi}$ for each y, and therefore, the conditional random variable $(\tilde{x}^1|y)$ is riskier than $(\tilde{x}^0|y)$ for each y. Note that these are univariate random variables. Rothschild and Stiglitz have shown that $\int_0^s [F^1(x|y) - F^0(x|y)]dx \geq 0$ for all s in [0,B] with equality at s=B. This is used in determining the effect of changing $\tilde{x}$. The first order condition can be written as: $\eta(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot dF^0(x|y)\cdot dG(y) = 0$. Subtracting this from the similar expression with $F^1(x|y)$ replacing $F^0(x|y)$ yields: $\eta^*(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot[dF^1(x|y) - dF^0(x|y)]\cdot dG(y)$. It is interesting to observe that the <u>independent</u> stochastic transformation of $\tilde{x}$ allows the characterization of comparative risk for univariate CDFs to be used

directly. Now, note that if $\partial\alpha/\partial\bar{x}$ <(>) 0 for all $\bar{x}$ under a nonrandom situation and $\partial R_p(z_1,z_2)/\partial\alpha$ <(>) 0, then $\psi(x;y,\alpha)$ is convex(concave) in x for all $0 < \alpha < \lambda$. Thus, the result of R-S concerning convex(concave) functions indicates that $\eta^*(\alpha_0)$ >(<) 0.

Theorem 5.6 contains Theorem 5.5 as a special case. To see this, recall that Theorem 5.5 assumes that $\tilde{x}^1$ is a MPS of $\tilde{x}^0$, and that each $\tilde{x}^i$ is independent of $\tilde{y}$. As mentioned earlier, R-S have shown that when $\tilde{x}^1$ is a MPS from $\tilde{x}^0$, then there exists a ξ such that $\tilde{x}^1 =^d \tilde{x}^0 + \xi$, where ξ satisfies $E(\xi|x^0) = 0$. Furthermore, even though ξ may depend on $\tilde{x}^0$, because $\tilde{x}^0$ is independent of $\tilde{y}$, ξ can be selected to be independent of $\tilde{y}$. Define $\tilde{x}^{1'} = \tilde{x}^0 + \xi$. Notice that $\tilde{x}^{1'}$ is equal to $\tilde{x}^0 + \xi$, and that $\tilde{x}^1$ and $\tilde{x}^{1'}$ are equal in distribution to one another. Furthermore, the pair $(\tilde{x}^{1'},\tilde{y})$ has the same joint CDF as does $(\tilde{x}^1,\tilde{y})$. Note that for expected utility maximizing decision maker, random variables are completely described by their joint distribution function. Thus, if α increases when ξ is added to $\tilde{x}^0$, it also increases when $\tilde{x}^0$ is replaced by $\tilde{x}^1$.

The procedure of adding independent noise to $\tilde{x}$ can be used to obtain any R-S increase in risk for the case of independent $\tilde{x}$ and $\tilde{y}$, but can only yield a subset of all R-S increases in risk for the case of dependent $\tilde{x}$ and $\tilde{y}$. When $\tilde{x}^0$ is not independent of $\tilde{y}$, it may not be possible to represent the R-S increase in risk by adding a ξ which is independent of $\tilde{y}$. This is because ξ must be allowed to depend on $\tilde{x}^0$ in order to represent an arbitrary R-S increase in risk, and this may also require that ξ depend on $\tilde{y}$ when $\tilde{x}^0$ and $\tilde{y}$ are not independent of one another.

Theorem 5.6 indicates that when $\tilde{x}$ is made riskier by means of an independent stochastic transformation, the independence assumption is not necessary. The independent stochastic transformation can make a random variable $\tilde{x}$, for any y, be riskier <u>in the sense of</u> Rothschild and Stiglitz, without changing the stochastic relationship measured by, for instance, conditional expectation. To remove the independence we have to sufficiently limit changes in the stochastic relationship.

Having shown how stochastic transformations can be used to extend theorem 5.5 concerning MPSs, it is quite straightforward to similarly extend theorem 5.4. Random variable $\tilde{x}^1$ dominates $\tilde{x}^0$ in the first degree if $\tilde{x}^1 = \tilde{x}^0 + \tilde{\xi}$, where the support of $\tilde{\xi}$ is nonnegative.

<u>Theorem 5.7</u>: Assume a) $\tilde{x}^i$ and $\tilde{y}$ are stochastically dependent; b) $\tilde{x}^1$ is obtained from $\tilde{x}^0$ by adding $\tilde{\xi}$, where $\tilde{\xi}$ has nonnegative support; c) $EU(\lambda-\alpha_i,\tilde{x}^i\cdot\alpha_i+\tilde{y})$ is maximized at $0 < \alpha_i < \lambda$. Then $\alpha_1 >(<) \alpha_0$ if and only if $\partial\alpha/\partial\bar{x} >(<) 0$ for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$.

<u>Proof</u>: Note that even though $\tilde{\xi}$ need not be independent of $\tilde{y}$, $(\tilde{x}^1|y) = (\tilde{x}^0|y) + (\tilde{\xi}|y)$ for each y, and although $(\tilde{\xi}|y)$ may depend on y, each is nonnegative. Thus, the conditional random variable $(\tilde{x}^1|y)$ dominates $(\tilde{x}^0|y)$ in the first degree. That is, their CDFs satisfy $[F^1(x|y) - F^0(x|y)]dx \leq 0$ for all x in [0,B] and all y. As in the proof of theorem 5.6, $\eta^*(\alpha_0) = \int_0^B\int_0^B \psi(x;y,\alpha_0)\cdot[dF^1(x|y) - dF^0(x|y)]\cdot dG(y)$. By part (a) of corollary 5.1, $\psi(x;y,\alpha)$ is increasing(decreasing) in x for all $0 < \alpha < \lambda$ if and only if $\partial\alpha/\partial\bar{x} >(<) 0$ for all $\bar{x}$. Thus, the FSD result of Hadar

and Russell[1969] indicates that $\eta^{*}(\alpha_0)$ >(<) 0.

Independence of $\tilde{\xi}$ and $\tilde{y}$ is not required in Theorem 5.7 because if $\tilde{\xi}$ is nonnegative, it is nonnegative for all $\tilde{y}$. Thus, when $\tilde{\xi}$ is added to $\tilde{x}$, $\tilde{x}$ improves in a FSD sense for all realizations of $\tilde{y}$. This same thing does not happen when $\tilde{\xi}$ increases the riskiness of $\tilde{x}$. Since independence is not required of $\tilde{\xi}$, a corollary to Theorem 5.7 can be presented in terms of deterministic transformations.

A deterministic transformation of random variable $\tilde{x}$ replaces every realization of $\tilde{x}$ by a new value determined according to a function defined over the support of $\tilde{x}$. Formally, a deterministic transformation is any nondecreasing function t(x) whose domain is all possible realizations of $\tilde{x}$. Note that deterministic transformations provide a sufficiently strong ceteris paribus restriction for the case of FSD improvements, but not for the case of MPSs.

Corollary 5.2: Assume a) $\tilde{x}^i$ and y are stochastically dependent, $i = 0,1$; b) $\tilde{x}^0$ is transformed in $\tilde{x}^1$ using deterministic transformation t(x) which satisfies $t(x) - x = k(x) \geq 0$ for all x in [0,B]; c) $EU(\lambda-\alpha_i,\ \tilde{x}^i\cdot\alpha_i + \tilde{y})$ is maximized at $0 < \alpha_i < \lambda$. Then α_1 >(<) α_0 if and only if $\partial\alpha/\partial\bar{x}$ >(<) 0 for all $\bar{x}$ under a nonrandom situation where $x = \bar{x}$ and $y = \bar{y}$.

To show that Theorem 5.7 generalizes Theorem 5.4, Corollary 5.2 is used. Assume that $\tilde{x}^1$ is a FSD improvement over $\tilde{x}^0$. This implies that $F^1(x) \leq F^0(x)$ for all x. For continuously distributed random variables, this same shift from F^0 to F^1 can be accomplished using the deterministic transformation: $t(x) = \inf\{\xi: F^1(\xi) \geq F^0(x)\}$. Define $\tilde{x}^{1\prime}$ =

$t(\tilde{x}^0)$. While $\tilde{x}^{1'}$ is not necessarily the same as $\tilde{x}^1$, it dose have the same marginal distribution and joint distribution with $\tilde{y}$ and hence leads to the same selection of α.

5.3 Summary and Conclusions

This chapter extended a specific 1-1-2 model to a 2-1-2 model. Especially, using stochastic transformations, the case where $\tilde{x}$ and $\tilde{y}$ are not independently distributed was investigated. However, the analysis shows that the conditions for signing the effect on the choice variable α selected by a decision maker of FSD shifts and R-S increases in risk are exactly those conditions determined in the 1-1-2 models.

Extension of the 2-1-2 model to a 2-2-2 model remains for further research. An interesting 2-2-2 model is the integration of both the labor and savings decisions into a single model of household decision making in which the supply of labor and savings are simultaneously determined and both factor returns are random.

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