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ABSTRACT

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In the vein of recent work of Sagan, Yeh and Ziegler, we study extremal problems connected with the Möbius function μ of certain families of subsets from O_n, the lattice of faces of the n-dimensional octahedron. In particular, we find that for lower order ideals $\mathcal{F}$ in O_n, $|\mu(\mathcal{F})|$ attains a maximum by taking roughly the lower two-thirds of the poset. For the case when $\mathcal{F}$ varies over all intervals of rank-selections, we give a proof which finds the extremal configuration when $n \equiv 4(5)$. When $n \not\equiv 4(5)$, our proof narrows down the extremal configuration to one of two possibilities. We include data which supports our conjecture that the maximum occurs by taking the ranks from approximately $\frac{2}{5}n$ to $\frac{4}{5}n$.

Purtill showed in a recent thesis that the coefficients of the *cd*-index Φ are non-negative for certain posets, including the Boolean algebra B_n and the n-octahedron. This work has been generalized by Stanley, who found the coefficients of Φ are non-negative for face-lattices of convex polytopes. Stanley has observed that the non-negativity of the coefficients of Φ immediately implies the arbitrary rank-selection results for face lattices of convex polytopes. The Möbius function is maximized by taking every other rank of the corresponding face lattice. For the record, we verify the details of Stanley's observation. In fact, Purtill's recurrence for the *cd*-index of B_n and O_n allow us to conclude that the odd and even alternating ranks are the

only extremal configurations for these two posets. We include a conjecture of Stanley which, if true, implies uniqueness of the extremal configuration for face lattices of convex polytopes.

Sagan, Yeh, and Ziegler also studied $L_n(q)$, the lattice of subspaces of an n-dimensional vector space over GF_q. For this poset they found all of $L_n(q)$ is the extremal configuration for the lower order ideal case. Using the inversion statistic, we show the interval of ranks and arbitrary rank-selection cases also have the same extremal configuration, i.e. the entire poset.

DEDICATION

To my parents Mary and Arthur,
and to the women in my family:
past, present and future.

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I would like to take this opportunity to thank my advisor Bruce Sagan for his encouragement and enthusiasm. Perhaps the best way to express my thanks is to paraphrase Roe Goodman, my Rutgers undergraduate mentor, who said the only way we can begin to repay our professors for their time and spirit is, in turn, to treat our own students with the same kindness and respect.

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0.1 Basic Notation

We follow [17, chapter 3] for most of the terminology and notation we will use in this dissertation. A partially ordered set (*poset*) P is a set P together with a binary relation $\leq$ (sometimes denoted by $\leq_P$ to indicate the poset P) which is reflexive, antisymmetric and transitive, i.e.

(i)	(reflexivity)	For all $x \in P$, $x \leq x$.
(ii)	(antisymmetry)	If $x \leq y$ and $y \leq x$, then $x = y$.
(iii)	(transitivity)	If $x \leq y$ and $y \leq z$, then $x \leq z$.

We use the notation $x < y$ to indicate $x \leq y$ and $x \neq y$. In a similar fashion, the notation $y > x$ means $x < y$. We say $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$; otherwise they are *incomparable*. The element y *covers* x (notation: $x \prec y$) if $x < z \leq y$ implies $z = y$.

Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $\eta : P \rightarrow Q$ whose inverse is also order-preserving. The *direct product* or *Cartesian product* of P and Q is the poset $P \times Q$ defined on the set

$$\{(x, y) : \ x \in P, \ y \in Q\}$$

such that

$$(x, y) \leq_{P \times Q} (x', y') \quad \text{if } x \leq_P x' \text{ and } y \leq_Q y'.$$

Given a finite poset P its *Hasse diagram* is a graph $G = G(V, E)$. Its vertices V consist of the elements of P. The edges E are determined by the cover relations of P. In other words, we draw an edge between x and y if $x \prec y$. To clearly show the "hierarchy" of P, we draw y above x.

A *chain* c in P is a subset of P so that every two elements are comparable. Thus, if the elements of c are $\{x_0, x_1, \ldots, x_k\}$, with $x_i < x_j$ when $i < j$, we can write c as

$$c: \quad x_0 < x_1 < \ldots < x_k.$$

We say this chain c in P is *unrefinable* (or *saturated*) if we can write c as

$$c: \quad x_0 \prec x_1 \prec \ldots \prec x_k.$$

A *maximal chain* in a poset P is an unrefinable chain from a minimal element in P to a maximal element in P.

Let P be a poset that has unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ (i.e., P is *bounded*). P is a *graded poset* of rank n if every maximal chain in P have the same length n. All the posets we study in this dissertation will be finite and graded. Thus, we have an associated rank function ρ defined by $\rho: \quad P \rightarrow \{0, 1, \ldots, n\}$, where

$$\rho(y) = \begin{cases} 0 & \text{if } y = \hat{0} \\ \rho(x) + 1 & \text{if } x \prec y. \end{cases}$$

The n-dimensional octahedron $\mathcal{O}_n$ is the convex hull of the $2n$ points

$$\{\pm e_1, \ldots, \pm e_n\}$$

in n-dimensional Euclidean space, where

$$e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i}).$$

For brevity, we will label these points by writing $+i$ for $+e_i$ and $-i$ for $-e_i$. Using this notation, we see that any proper face of $\mathcal{O}_n$ can be labeled using a *signed subset of* $\{1, \ldots, n\}$, i.e. a subset of the set $\{\pm 1, \ldots, \pm n\}$, say

$$S = \{a_1, \ldots, a_k\},$$

where S cannot contain both the elements $+i$ and $-i$.

We can associate a poset O_n with this geometrical object, called the *lattice of faces of the n-dimensional octahedron.* We construct O_n by taking all of the faces of $\mathcal{O}_n$ and ordering them by inclusion. With respect to the shorthand we have adopted, we can represent O_n as the poset of all signed subsets of $\{1, \ldots, n\}$ ordered by inclusion with the element $\hat{1}$ adjoined. (Here we need to have an element which represents the maximal face of $\mathcal{O}_n$, i.e., $\mathcal{O}_n$ itself, since the notation does not lend itself naturally to representing this element. Thus $\hat{1}$ is this maximal face.) The *rank* of any element $S \in O_n \setminus \hat{1}$ is given by $|S|$, where $|\cdot|$ denotes cardinality. Observe that $\hat{1}$ has rank $n+1$ in O_n.

For a family $\mathcal{F} \subseteq P$ we define its *completion* by

$$\hat{\mathcal{F}} = \mathcal{F} \cup \{\hat{0}, \hat{1}\}$$

and its *proper part* by

$$\overline{\mathcal{F}} = \mathcal{F} \setminus \{\hat{0}, \hat{1}\}.$$

An *interval* $[x, y]$ in P is defined by

$$[x, y] = \{z \in P : \ x \leq z \leq y\}, \text{ for } x \leq y.$$

Notice that $[x, x]$ consists of the element x. We do not allow the empty set to be an interval. Given a poset P, the *Möbius function* μ is defined recursively on intervals $[x, y]$ in P by

$$\begin{aligned} \mu(x, x) &= 1 \text{ for all } x \text{ in } P, \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z). \end{aligned}$$

For brevity, we let $\mu(P)$ denote the value of $\mu_P(\hat{0}, \hat{1})$. For x an element of P we let $\mu(x)$ denote $\mu_P(\hat{0}, x)$. Additionally, for any family $\mathcal{F}$ of elements of P we let $\mu(\mathcal{F})$

equal $\mu_{\mathcal{F}}(\hat{0},\hat{1})$.

Given a non-negative integers i and j, we let

$$[i] = \{1, 2, \ldots, i\}$$

and

$$[i,j] = \{i, i+1, \ldots, j\},$$

with the conventions that $[0] = \emptyset$ and $[i,j] = \emptyset$ for $j < i$. If $\text{rank}(\hat{1}) = r$, then for a family $\mathcal{F} \subseteq P$ and $S \subseteq [r-1]$, we define the ***rank-selected subposet***

$$\mathcal{F}(S) = \{x \in \mathcal{F} : \ \text{rank}(x) \in S\}.$$

In particular the i^{th} *rank level* of $\mathcal{F}$ is given by

$$\mathcal{F}(i) = \mathcal{F}(\{i\}) = \{x \in \mathcal{F} : \ \text{rank}(x) = i\},$$

and the ***interval*** $[i,j]$ ***of ranks of*** $\mathcal{F}$ (not to be confused with the closed interval $[x,y]$ of elements of P) is

$$\mathcal{F}[i,j] = \mathcal{F}([i,j]) = \{x \in \mathcal{F} : \ \text{rank}(x) \in [i,j]\}.$$

Also, we use the shorthand

$$\mathcal{F}[k] = \mathcal{F}([k]).$$

Finally, a *lower order ideal* $\mathcal{A}$ is a subset $\mathcal{A}$ of P such that if $x \in \mathcal{A}$ and $y \leq x$ then $y \in \mathcal{A}$.

0.2 Introduction

The questions studied in this thesis belong to a branch of mathematics called extremal combinatorics. Extremal combinatorics is concerned with finding the "best" configuration (according to some given criteria) among a set of possible arrangements.

As an example, in [17, Exercise 3.41a] Stanley posed the following extremal question: given a bounded poset P with a fixed number of elements, what is the maximum value of the Mōbius function of P? Ziegler answered this question for both bounded posets and graded posets. He also determined the extremal configuration in each situation [21].

Recent work of Sagan, Yeh and Ziegler [15] approached extremal problems involving the Mōbius function from a slightly different angle. They fixed the poset under consideration (in their case, the Boolean algebra B_n) and studied the maximum value attained by the Mōbius function μ over certain subsets of B_n. More specifically, if $\mathcal{F}$ is a family of subsets contained in the Boolean algebra B_n, then the $\max_{\mathcal{F}} |\mu(\mathcal{F})|$ has been found for three categories of families:

i. lower order ideals

ii. intervals of ranks

iii. arbitrary rank-selections.

The maxima are obtained by taking roughly the lower half, middle third, and every other rank of B_n, respectively. The lower order ideal case was first solved by Eckhoff [12] and Scheid [1], and viewed in the context of the reduced Euler characteristic by Björner and Kalai [8]. Niven [13] and de Bruijn [11] had previously solved the arbitrary rank-selection case, while the interval of ranks case was a new result.

In this dissertation, we will address analogous extremal problems for O_n, the lattice of faces of the n-dimensional octahedron. More specifically, by extending the techniques developed for B_n to O_n, we find the extremal configuration for lower order ideals is the lower two-thirds of the poset O_n. In chapter 2, we state our conjecture for the interval of ranks question. Our proof gives a complete answer when $n \equiv 4(5)$ and narrows down the answer to one of two possibilities for $n \not\equiv 4(5)$. Also, we include data supporting our conjecture. In chapter 3, we see that for arbitrary rank-selections the extremal configuration is to take every other rank of O_n. This is actually a simple observation made by Stanley that the non-negativity of the coefficients of the *cd*-index of B_n and O_n (proved by Purtill in his thesis) immediately imply the arbitrary rank-selection result. In fact, Stanley's observation applied to his new work on the *cd*-index of face lattices of convex polytopes L_P also shows the Möbius function is maximized for L_P by taking every other rank of the corresponding face lattice. In the first half of chapter 4 we answer the interval of rank and arbitrary rank-selection cases which were left undone in [15] for $L_n(q)$, the lattice of subspaces of an n-dimensional vector space over GF_q. In both cases, the extremal configuration is to take the entire poset. Finally, we complete chapter 4 by indicating our future research.

Chapter 1

Lower Order Ideals

In this chapter we will be concerned with maximizing $|\mu(\mathcal{F})|$ as $\mathcal{F}$ ranges over all lower order ideals in O_n. We first state the main result of this chapter:

Theorem 1.0.1 *If $\mathcal{F}$ is a lower order ideal in O_n, then*

$$|\mu(\mathcal{F})| \leq |\sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} \binom{n}{k}(-2)^k|,$$

with equality occurring if and only if

$$\mathcal{F} = O_n[k] \text{ with } k = \lfloor \tfrac{2n}{3} \rfloor.$$

(Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.)

Before proving Theorem 1.0.1, we will first specialize $\mathcal{F}$ to be a rank-selected lower order ideal, i.e., a lower order ideal of the form $O_n[k] \cup \{\hat{0}\}$. We show in Lemma 1.2.1 that $|\mu(O_n[k])|$ is maximized if we take k to be $\lfloor \frac{2n}{3} \rfloor$, i.e. the lower two-thirds of O_n. Once we generalize $\mathcal{F}$ to be any lower order ideal in O_n, we will see the ideal $O_n[\lfloor \frac{2n}{3} \rfloor] \cup \{\hat{0}\}$ is also the maximal configuration for Theorem 1.0.1.

1.1 Elementary Properties for O_n

We begin by determining the Möbius value μ of elements from O_n and rank-selected lower order ideals from O_n. For the most part, these results follow from known properties of the lattice O_n, the definition of the Möbius function, and the product theorem for the Möbius function.

The most important result we establish in this section is a recurrence for the $\mu(O_n[k])$'s (Corollary 1.1.5). We do this via a "reduced Euler characteristic" interpretation of the Möbius function. This recurrence, in its absolute value form (Corollary 1.1.7), allows us to quickly narrow down the possibilities for the maximum of the Möbius function for rank-selected lower order ideals. The summation formula for the $\mu(O_n[k])$'s allows us to sharpen and complete the argument for Theorem 1.0.1.

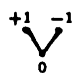
Let us first determine the Möbius value of elements from O_n.

Proposition 1.1.1 *Let x be an element of $O_n \setminus \hat{1}$ with rank k. Then $\mu(x) = (-1)^k$.*

Proof. First note that

$$O_n \setminus \hat{1} \cong \underbrace{V \times \cdots \times V}_{n},$$

where V is the poset with Hasse diagram

$V =$ 

.

This holds because of the following order-preserving bijection. Define $\eta : O_n - \hat{1} \rightarrow \underbrace{V \times \cdots \times V}_{n}$ by

$$x \longmapsto \vec{v} = (v_1, \ldots, v_n)$$

where

$$v_i = \begin{cases} +1, & \text{if } i \in x \\ -1, & \text{if } -i \in x \\ 0, & \text{if } i, -i \notin x. \end{cases}$$

Let $x = x_1, \ldots, x_k$ be an element of rank k in $O_n - \hat{1}$. Using the bijection η between $O_n - \hat{1}$ and $\underbrace{V \times \cdots \times V}_{n}$ and the product theorem for the Möbius function [17, Prop. 3.8.2], we obtain

$$\begin{aligned} \mu_{O_n - \hat{1}}(x) &= \mu_{V\times\cdots\times V}(\eta(x)) \\ &= \mu_V(\mathbf{0})^{n-k}\mu_V(\pm\mathbf{1})^k, \end{aligned}$$

where the elements $\mathbf{0}$, $+\mathbf{1}$ and $-\mathbf{1}$ correspond to the labeling of V. Noting $\mu_V(\mathbf{0}) = +1$ and $\mu_V(+\mathbf{1}) = \mu_V(-\mathbf{1}) = -1$ gives the result. □

We are able to express $\mu(O_n[k])$ in two ways: in terms of a summation formula (Corollary 1.1.2) or a recurrence (Corollary 1.1.5).

Corollary 1.1.2 *The following summation formula holds for $\mu(O_n[k])$:*

$$\mu(O_n[k]) = -\left(\sum_{j=0}^{k} \binom{n}{j}(-2)^j\right), \quad 0 \le k \le n.$$

Proof. Use Proposition 1.1.1 and the fact there are $\binom{n}{j}2^j$ elements of rank j in O_n. □

Corollary 1.1.3 $\mu(O_n) = (-1)^{n+1}$, $n \geq 1$.

Proof. It is enough to observe $\mu(O_n) = \mu(O_n[n])$ (recall $\hat{1}$ has rank $n+1$ in O_n). The result then immediately follows once we apply the binomial theorem to the summation expression for $\mu(O_n[n])$ given in Corollary 1.1.2. □

Combining Proposition 1.1.1 and Corollary 1.1.3 yields the following:

Proposition 1.1.4 *Let x be an element of O_n with rank k. Then $\mu(x) = (-1)^k$.*

Corollary 1.1.5 *The $\mu(O_n[k])$'s satisfy the recurrence*

$$\mu(O_n[k]) = -2\mu(O_{n-1}[k-1]) + \mu(O_{n-1}[k]),$$

where $n \geq 2$, $0 < k < n$, *with boundary conditions*

$$\mu(O_n[0]) = -1 \quad for\ n \geq 0$$

and

$$\mu(O_n[n]) = (-1)^{n+1} \quad for\ n \geq 1.$$

We have two different proofs of this recurrence. The first applies the interpretation of the Möbius function as counting certain chains in a poset. The second is a direct application of the summation formula for $\mu(O_n[k])$ and induction.

Before beginning our first proof, we briefly review the aforementioned interpretation of the the Möbius function. For P a bounded poset, we let $c_i = c_i(P)$ denote the number of chains of length i in $P \setminus \{\hat{0}, \hat{1}\}$. Here a chain of length i is a totally ordered subset containing $i+1$ elements.

A well-known result [17, Prop. 3.8.5] which relates chains with the Möbius function is

$$\mu(P) = -1 + c_0 - c_1 + c_2 - c_3 + \dots \tag{1.1}$$

We are now ready to use this formulation of the Möbius function.

Proof 1. Let

$$a_j = \#\text{ of chains of length j in } O_{n-1}[k-1]$$
$$b_j = \#\text{ of chains of length j in } O_{n-1}[k]$$
$$c_j = \#\text{ of chains of length j in } O_n[k].$$

We will refer to chains of "type c_j" to mean those chains of length j in $O_n[k]$. Similar terminology will be used for a_j and b_j. Using equation (1.1), we can translate the recurrence in terms of chain notation, i.e.,

$$-1+c_0-c_1+c_2-\ldots = -2(-1+a_0-a_1+a_2-\ldots) + (-1+b_0-b_1+b_2-\ldots) \quad (1.2)$$

Regrouping the terms on the right side of equation (1.2) above gives

$$-1+c_0-c_1+c_2-\ldots = -1+(2(1)+b_0)-(2a_0+b_1)+(2a_1+b_2)-(2a_2+b_3)+\ldots \quad (1.3)$$

The chains of type c_j : $x_1 < x_2 < \ldots < x_{j+1}$ consist of those chains of length j in $O_n[k]$. They belong to one of the following four classes:

- (i) neither $\{+n\}$ nor $\{-n\} \in x_i$ for any i
- (ii) $x_1 = \{+n\}$ ("the chain starts with $\{+n\}$")
- (iii) $x_1 = \{-n\}$ ("the chain starts with $\{-n\}$")
- (iv) all other remaining chains not accounted for in (i), (ii) or (iii).

If we let d_j denote the cardinality of the last category (iv) of chains, then we claim

$$\begin{aligned} c_j &= b_j + a_{j-1} + a_{j-1} + d_j \\ &= b_j + 2a_{j-1} + d_j, \end{aligned} \quad (1.4)$$

with the convention that $a_{-1} = 1$ counts the number of empty chains in $O_{n-1}[k-1]$. Clearly the chains of type b_j and category (i) have the same cardinality via the identity map. The collection of chains starting with $\{+n\}$ in $O_n[k]$ has the same cardinality as the type a_{j-1} chains by defining a natural bijection ν between these two sets. Let

$$c: \ \{+n\} = x_1 < x_2 < \ldots < x_{j+1}$$

be such a category (ii) chain. We have $n \in x_i$ for all i, so we can map c to the chain

$$\nu(c): \ x_2 \setminus \{+n\} < \ldots < x_{j+1} \setminus \{+n\},$$

which is a type a_{j-1} chain. Conversely, if

$$\tilde{c}: \ y_1 < \ldots < y_j$$

is a chain of type a_{j-1}, map it to the chain

$$\nu^{-1}(\tilde{c}): \ \{+n\} < y_1 \cup \{+n\} < \ldots < y_j \cup \{+n\}.$$

Clearly, $\nu^{-1} \circ \nu$ and $\nu \circ \nu^{-1}$ are the identity maps on category (ii) and type a_{j-1} chains, respectively, so these two collections have the same cardinality. We can similarly show the cardinality of category (iii) chains equals a_{j-1} by uniformly replacing "$+n$" by "$-n$" in this argument.

Substituting equation (1.4) for each of the c_j's appearing in the left side of equation (1.3), we see that all the terms on the right side of equation (1.3) cancel out. The remaining terms consist of an alternating sum of the d_j's. Hence the corollary will follow once we have shown this alternating sum equals zero, i.e.

$$d_0 - d_1 + d_2 - d_3 + \ldots + (-1)^{k-1} d_{k-1} = 0. \tag{1.5}$$

To complete this argument, we will construct a bijection λ between certain chains of type d_j and type d_{j+1}, according to where $\{\pm n\}$ first occurs in each type of chain. Here $\pm n$ stands for either $+n$ or $-n$ (but not both), depending upon which occurs in the chain. An important observation to make is that if $x_1 < \ldots < x_{j+1}$ is a chain of length j in $O_n[k]$, then rank $x_i \geq i$.

Let $d_{j,0}$ denote the cardinality of those type d_j chains $x_1 < \ldots < x_{j+1}$ in $O_n[k]$ with the property that if x_h is the least element in the chain containing $\pm n$ then $x_h = x_{h-1} \cup \{\pm n\}$. Let $d_{j,1}$ be defined similarly, except that $x_h \supset x_{h-1} \cup \{\pm n\}$ (strict containment). For $h = 1$, we let $x_0 = \emptyset$. Note that $d_j = d_{j,0} + d_{j,1}$ for all j. From the definitions of $d_{j,0}$ and $d_{j,1}$, we easily derive the following two properties:

$$(i) \quad d_{0,0} = 0$$

$$(ii) \quad d_{k-1,1} = 0.$$

Property (i) holds by virtue of the fact that type d_0 chains do not include the chains $+n$ or $-n$. Suppose property (ii) does not hold, i.e. $d_{k-1,1} > 0$. A typical type $d_{k-1,1}$ chain looks like

$$y_1 < \ldots < y_{h-1} < y_h \cup \{\pm n\} < y_{h+1} \cup \{\pm n\} < \ldots < y_k \cup \{\pm n\}$$

with $y_h \neq y_{h-1}$. But then

$$y_1 < \ldots < y_{h-1} < y_h < y_{h+1} < \ldots < y_k$$

is a chain of length $k-1$ (all the y_i's are distinct). Thus rank $y_k \geq k$ implies rank $(y_k \cup \{\pm n\}) \geq k+1$, contradicting the fact that $y_k \cup \{\pm n\}$ is an element from $O_n[k]$. Hence property (ii) holds.

Next we show that the identity

$$d_{j,1} = d_{j+1,0} \quad (0 \leq j \leq k-2) \tag{1.6}$$

holds. We define the map λ : *type $d_{j,1}$ chains* $\longrightarrow$ *type $d_{j+1,0}$ chains* by

$$x_1 < \ldots < x_{h-1} < x_h < \ldots < x_{j+1} \longmapsto x_1 < \ldots < x_{h-1} < x_h \backslash \{\pm n\} < x_h < \ldots < x_{j+1}$$

where h is the first position where $\pm n$ occurs in a type $d_{j,1}$ chain. We define the inverse map λ^{-1} : *type $d_{j+1,0}$ chains* $\longrightarrow$ *type $d_{j,1}$ chains* by

$$y_1 < \ldots < y_{h-1} < y_h < y_{h+1} < \ldots < y_{j+2} \longmapsto y_1 < \ldots < y_{h-1} < y_{h+1} < \ldots < y_{j+2}$$

where $h+1$ is the first position where $\pm n$ occurs in a type $d_{j+1,0}$ chain. It is easy to check $\lambda^{-1} \circ \lambda$ is the identity on type $d_{j,1}$ chains and $\lambda \circ \lambda^{-1}$ is the identity on type $d_{j+1,0}$ chains, so λ is a bijection.

Since $d_j = d_{j,0} + d_{j,1}$, the alternating sum of d_j's in equation (1.5) simplifies to verifying

$$d_{0,0} + (-1)^{k-1} d_{k-1,1} = 0,$$

which is indeed true by properties (i) and (ii).

Finally, the first boundary condition holds since

$$\begin{aligned} \mu(O_n[0]) &= -(\mu_{O_n}(\hat{0})) \\ &= -1. \end{aligned}$$

The second boundary condition holds by virtue of the fact $\mu(O_n[n]) = \mu(O_n)$, which equals $(-1)^{n+1}$ by Corollary 1.1.3. □

Proof 2. We first substitute the summation formula for $\mu(O_n[k])$ given in Corollary 1.1.2 into the expression $-2\mu(O_{n-1}[k-1]) + \mu(O_{n-1}[k])$. After shifting the index of summation in the sum corresponding to $-2\mu(O_{n-1}[k-1])$ and breaking off the $j = 0$th term from the sum corresponding to $\mu(O_{n-1}[k])$, we obtain

$$\begin{aligned} -2\mu(O_{n-1}[k-1]) + \mu(O_{n-1}[k]) &= 2\sum_{j=0}^{k-1} \binom{n-1}{j} (-2)^j - \sum_{j=0}^{k} \binom{n-1}{j} (-2)^j \\ &= -\sum_{j=1}^{k} \binom{n-1}{j-1} (-2)^j - \sum_{j=1}^{k} \binom{n-1}{j} (-2)^j - 1. \end{aligned} \tag{1.7}$$

Using the facts $\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$ and $\binom{n}{0}(-2)^0 = 1$, we combine the three terms on the right side of equation (1.7) to get

$$\begin{aligned} -2\mu(O_{n-1}[k-1]) + \mu(O_{n-1}[k]) &= -\sum_{j=1}^{k} (\binom{n-1}{j-1} + \binom{n-1}{j})(-2)^j - \binom{n}{0}(-2)^0 \\ &= -\sum_{j=0}^{k} \binom{n}{j} (-2)^j, \end{aligned}$$

which equals $\mu(O_n[k])$ by Corollary 1.1.2. □

An easy result that we will need in order to complete the proof of Theorem 1.0.1 is the following corollary:

Corollary 1.1.6 *When k is odd $\mu(O_n[k])$ is positive and when k is even $\mu(O_n[k])$ is negative* $(n \geq 1,\ 0 \leq k \leq n)$.

Proof. The proof proceeds via induction on n and applying the recurrence for $\mu(O_n[k])$ described in Corollary 1.1.5. By the boundary conditions from Corollary 1.1.5, we have $\mu(O_n[0]) = -1$ and $\mu(O_n[n]) = (-1)^{n+1}$, the latter being positive for n odd and negative for n even. In particular, the result holds for $n = 1$. Now assume $n > 1$. We have handled the case for $k = 0$ or n, so now we will consider $1 \leq k \leq n-1$. If k is odd, then by the induction hypothesis $\mu(O_{n-1}[k-1])$ is negative and $\mu(O_{n-1}[k])$ is positive. Hence the recurrence given in Corollary 1.1.5 implies $\mu(O_n[k])$ is positive. A similar argument demonstrates $\mu(O_n[k])$ is negative for k even. □

Putting Corollaries 1.1.5 and 1.1.6 together, we see that the $|\mu(O_n[k])|$'s also satisfy a recurrence.

Corollary 1.1.7 *The $|\mu(O_n[k])|$ satisfy the recurrence*

$$|\mu(O_n[k])| = 2|\mu(O_{n-1}[k-1])| + |\mu(O_{n-1}[k])|$$

with boundary conditions

$$|\mu(O_n[0])| = 1 \quad for\ n \geq 0$$

and

$$|\mu(O_n[n])| = 1 \quad for\ n \geq 1.$$

1.2 Rank-Selected Lower Order Ideals

Before we determine the extremal configuration for $|\mu(\mathcal{F})|$, $\mathcal{F}$ an arbitrary lower order ideal, it is natural for us to first study "simpler" ideals. In particular, we study ideals $\mathcal{F}$ which are rank-selected lower order ideals, i.e. lower order ideals of the form $\mathcal{F} = O_n[k] \cup \hat{0}$.

In order to state a special case of the main theorem of this chapter, we make a few more definitions. We say a sequence $a_0, a_1, \ldots, a_n$ of real numbers is *unimodal* if for some k, $0 \leq k \leq n$, we have

$$a_0 \leq \ldots \leq a_k \geq \ldots \geq a_n.$$

Similarly, a sequence is *strictly unimodal* if we replace the inequalities by strict inequalities in the definition of unimodal. Finally, a sequence $a_1, \ldots, a_n$ is *almost strictly unimodal* if the sequence is strictly unimodal or is of the form

$$a_1 < a_2 < \cdots < a_k = a_{k+1} > a_{k+2} > \cdots > a_n.$$

We are now ready to state a rank-selection lemma:

Lemma 1.2.1 *For fixed $n \geq 2$, $|\mu(O_n[k])|$ is strictly unimodal with unique maximum occurring when $k = \lfloor \frac{2n}{3} \rfloor$.*

Proof. The proof for rank-selected lower order ideals proceeds by induction on n. By the recurrence for $|\mu(O_n[k])|$, we quickly conclude the sequence $\{|\mu(O_{n+1}[k])|\}_{k=0}^{n+1}$ is almost unimodal and narrow down its maximum to one of two possibilities. We will complete the argument by considering the equivalence class of n modulo 3 and apply the summation formulas and recurrence relation for $\mu(O_n[k])$ determined in the previous section.

Table 1.1: $|\mu(O_n[k])|$ for $n = 2,3,4$

	k=1	2	3	4
n =2	3	1	0	0
3	5	7	1	0
4	7	17	15	1

For $n = 2,3$, and 4, the lemma is easily checked to be true. Refer to Table 1.1. Fix n and let k^* be the index k for which $|\mu(O_n[k])|$ is a maximum. By Corollary 1.1.7 $|\mu(O_n[k])|$ satisfies the recurrence

$$|\mu(O_n[k])| = 2|\mu(O_{n-1}[k-1])| + |\mu(O_{n-1}[k])|. \tag{1.8}$$

By the induction hypothesis the sequences $\{\mu(O_n[k])\}_{k=1}^{k*}$ and $\{\mu(O_n[k])\}_{k=k*}^{n}$ are strictly increasing and decreasing, respectively. Applying the recurrence to these monotone sequences, we conclude the sequences

$$\{\ |\mu(O_{n+1}[k])|\ \}_{k=1}^{k^*} \tag{1.9}$$

and

$$\{\ |\mu(O_{n+1}[k])|\ \}_{k=k^*+1}^{n+1} \tag{1.10}$$

are strictly increasing and strictly decreasing, respectively. Thus, equations (1.9) and (1.10) imply we have pinned down the index k corresponding to the maximum $|\mu(O_{n+1}[k])|$ as one of two possibilities: k^* or $k^* + 1$. A case-by-case argument using the equivalence class modulo 3 of n and applying the summation formula for $\mu(O_n[k])$ will give the result.

For ease in notation, let

$$\begin{aligned} a_k &= |\mu(O_{n-1}[k])| \\ b_k &= |\mu(O_n[k])| \end{aligned}$$

$$c_k = |\mu(O_{n+1}[k])|.$$

We first suppose $n \equiv 1(3)$. Since $n+1 \equiv 2(3)$, we wish to show

$$c_{k^*+1} > c_{k^*}.$$

Applying the recurrence (1.8), it suffices to show

$$2b_{k^*} + b_{k^*+1} > 2b_{k^*-1} + b_{k^*},$$

i.e.

$$b_{k^*} > 2b_{k^*-1} - b_{k^*+1}.$$

The summation formulas given in Corollary 1.1.2 and the fact $n \equiv 1(3)$ enable us to rewrite this as

$$\begin{aligned} b_{k^*} &> b_{k^*-1} + (b_{k^*-1} - b_{k^*+1}) \\ &= b_{k^*-1} + \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j, \end{aligned}$$

since k^* is even. By the induction hypothesis we have $b_{k^*} > b_{k^*-1}$. If we can show $\sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j$ is negative, then we will be finished. Write n as $3m+1$. Then $k^* = \lfloor \frac{2n}{3} \rfloor = 2m$. Thus

$$\begin{aligned} \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j &= \binom{3m+1}{2m}(-2)^{2m} + \binom{3m+1}{2m+1}(-2)^{2m+1} \\ &= \frac{(3m+1)!}{(2m)!\,m!}(-2)^{2m}\left[\frac{-1}{(m+1)(2m+1)}\right], \end{aligned}$$

which is negative, as desired.

Next we consider $n \equiv 2(3)$. Since $n+1 \equiv 0(3)$, we wish to show

$$c_{k^*+1} > c_{k^*}. \tag{1.11}$$

Using the same method as in the case for $n \equiv 1(3)$ (and noting k^* is now odd), we can rewrite equation (1.11) as

$$b_{k^*} > b_{k^*-1} - \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j.$$

By the induction hypothesis we have $b_{k^*} > b_{k^*-1}$, so it remains to show $\sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j$ is non-negative. If we write n as $3m+2$, then $k^* = \lfloor \frac{2n}{3} \rfloor = 2m+1$. Thus

$$\begin{aligned} \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j &= \frac{(3m+2)!}{(2m+1)!\, m!}(-2)^{(2m+1)} \left[\frac{1}{m+1} - \frac{2}{2m+2}\right] \\ &= 0. \end{aligned}$$

Note the above calculation implies

$$b_{k^*-1} = b_{k^*+1} \quad \text{for } n \equiv 2(3). \tag{1.12}$$

We will need this result for the next case.

For $n \equiv 0(3)$ and $n+1 \equiv 1(3)$, we wish to show

$$c_{k^*+1} < c_{k^*},$$

or

$$b_{k^*} < 2b_{k^*-1} - b_{k^*+1}. \tag{1.13}$$

Rewriting equation (1.13) in terms of elements from the $n-1$st row, we have

$$2a_{k^*-1} + a_{k^*} \;<\; 2\,(2a_{k^*-2} + a_{k^*-1}) - (2a_{k^*} + a_{k^*+1}),$$

i.e.

$$a_{k^*+1} < 4a_{k^*-2} - 3a_{k^*}. \tag{1.14}$$

Observe $n-1 \equiv 2(3)$, so by equation (1.12) we conclude $a_{k^*-2} = a_{k^*}$. (Recall here that the index k^* is the index of the maximum in the nth, not the $n-1$st, row. The index corresponding to the maximum in the $n-1$st row is k^*-1.) Hence equation (1.14) becomes

$$a_{k^*+1} < a_{k^*},$$

which holds by the induction hypothesis on the $n-1$st row. □

1.3 Arbitrary Lower Order Ideals

In this section we give the proof of Theorem 1.0.1. To do so, we develop the notions of the shadow $\Delta(S)$ and the dual shadow $\nabla(S)$ of elements S of a fixed rank from O_n. Using bipartite graph arguments, we obtain estimates for $\Delta(S)$ and $\nabla(S)$ reminiscent of Bollobás' work on shadows for hypergraphs [10].

For the proof of Theorem 1.0.1 we suppose we have a lower order ideal $\mathcal{F} \subseteq O_n$ which maximizes $|\mu(\mathcal{F})|$. By the shadow lemmas we find bounds for the "shape" of $\mathcal{F}$. Next we find rank-selected lower order ideals contained in $\mathcal{F}$ and containing $\mathcal{F}$. A lemma due to Baclawski [2, Lemma 4.6] and independently to Stečkin [20] enable us to "peel off" elements from these ideals. Hence, we are able to compare the Möbius values of these configurations with $\mathcal{F}$ and complete the proof.

Suppose we are given elements S all of the same rank in a poset. Since we are working with lower order ideals, we would naturally like to be able to estimate the number of elements in the poset covered by S. More formally, we define the *shadow* of a subset S of rank r in O_n by

$$\Delta(S) = \{B \in O_n(r-1) : \ B \subsetneq A \text{ for some } A \in S\}.$$

We then have an inequality involving $|\Delta(S)|$ and $|S|$.

Lemma 1.3.1 *(Shadow Lemma for O_n) If $S \subseteq O_n(r)$, where $r \geq \frac{2n+2}{3}$, then $|\Delta(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = O_n(\frac{2n+2}{3})$.*

Proof. For the inequality we utilize an edge-counting argument. Consider the bipartite graph G formed in the Hasse diagram of O_n by S and $\Delta(S)$. Each vertex $A \in S$ has degree r, so the graph G has exactly $r|S|$ edges. Also, every vertex $B \in \Delta(S)$ has degree at most $2(n-r+1)$, so the number of edges in G is at most $2(n-r+1)|\Delta(S)|$. Thus

$$r|S| \leq 2(n-r+1)|\Delta(S)|,$$

i.e.

$$|S| \leq \frac{2(n-r+1)}{r}|\Delta(S)|.$$

Thus when $r > \frac{2n+2}{3}$, the first part of the lemma follows.

If $n \equiv 2(3)$ and $r = \frac{2n+2}{3}$, then the above argument works as long as some vertex in $\Delta(S)$ does not have degree $\frac{2n+2}{3}$. If every vertex $B \in \Delta(S)$ has this degree, then in O_n the vertices of $\Delta(S)$ are only adjacent to vertices of S (and vice-versa). Hence if $S \subset O_n(r)$ (strict containment), this would contradict shellability of the chain complex of O_n ([5], [7]). (See Section 2.1 for information about shellability and the chain complex.) $\square$

In an analogous manner, we define the *dual shadow* of S, where S is a subset of rank r in O_n, by

$$\nabla(S) = \{B \in O_n(r+1) : \ B \supseteq A \text{ for some } A \in S\}.$$

We also have a dual shadow lemma for O_n. Since its proof is virtually identical to that of Lemma 1.3.1, we shall simply state the result.

Lemma 1.3.2 *(Dual Shadow Lemma for O_n) If $S \subseteq O_n(r)$ where $r \leq \frac{2n-1}{3}$, then $|\nabla(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = O_n(\frac{2n-1}{3})$.* $\square$

Now we are ready to give a proof of Theorem 1.0.1. Let $\mathcal{F} \subseteq O_n$ be a lower order ideal with maximum $|\mu(\mathcal{F})|$ and let k be the maximum rank of an element in $\mathcal{F}$. We will first derive some expressions that will enable us to compare $\mu(\mathcal{F})$ with $\mu(\mathcal{F}[k-1])$ and $\mu(\mathcal{F}[k-2])$, yielding an upper bound for k. The Shadow Lemma for O_n and the following proposition enable us to do this. Here $\max P$ denotes the set of maximal elements of the poset P. Recall that for P a bounded poset, $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$.

Proposition 1.3.3 *[2, Lemma 4.6] [20] Let P be a bounded poset. If $T \subseteq \max \bar{P}$ then*

$$\mu(P) = \mu(P \setminus T) - \sum_{x \in T} \mu(\hat{0}, x). \tag{1.15}$$

This result follows by counting the chains in P not containing elements of T and the chains in P containing elements of T. Proposition 1.3.3 is useful in the sense that it enables us to see how the Möbius function of a poset changes if we "peel off" some (or all) of its top elements.

Applying equation (1.15) to $P = \mathcal{F}$, $T = \mathcal{F}(k)$, and recalling $\mu(\hat{0}, x) = (-1)^k$ for $x \in O_n$ of rank k gives

$$\mu(\mathcal{F}) = \mu(\mathcal{F}[k-1]) - (-1)^k |\mathcal{F}(k)|,$$

or

$$\mu(\mathcal{F}[k-1]) = \mu(\mathcal{F}) + (-1)^k |\mathcal{F}(k)|. \tag{1.16}$$

Similarly applying equation (1.15) to $P = \mathcal{F}[k-1]$, $T = \mathcal{F}(k-1)$, substituting for $\mu(\mathcal{F}[k-1])$ in equation (1.16), and solving for $\mu(\mathcal{F}[k-2])$ gives

$$\mu(\mathcal{F}[k-2]) = \mu(\mathcal{F}) - (-1)^k (|\mathcal{F}(k-1)| - |\mathcal{F}(k)|). \tag{1.17}$$

Since $\mathcal{F}$ is a lower order ideal, we have $\Delta\mathcal{F}(k) \subseteq \mathcal{F}(k-1)$, so

$$|\mathcal{F}(k-1)| - |\Delta\mathcal{F}(k)| \geq 0.$$

Suppose $k > \lceil \frac{2n}{3} \rceil$. By the Shadow Lemma 1.3.1 we know

$$|\Delta\mathcal{F}(k)| > |\mathcal{F}(k)|.$$

Hence

$$|\mathcal{F}(k-1)| - |\mathcal{F}(k)| > 0.$$

After considering all the possibilities for the sign of $\mu(\mathcal{F})$ and the parity of k, we see one of equations (1.16), (1.17) implies $|\mu(\mathcal{F})|$ is not a maximum, contradicting the fact that $\mathcal{F} \subseteq O_n$ is an ideal with maximum $|\mu(\mathcal{F})|$. Hence $k \leq \lceil \frac{2n}{3} \rceil$.

We will now work with the Dual Shadow Lemma to extract further information about the structure of $\mathcal{F}$. Let

$$\mathcal{F}^c = \{B : \ B \in O_n \setminus \hat{\mathcal{F}}\}$$

and define

$$\ell = \min\{\text{rank}(B) : \ B \in \mathcal{F}^c\}.$$

Form $\mathcal{G} = \mathcal{F} \cup \mathcal{F}^c(\ell)$ and $\mathcal{H} = \mathcal{G} \cup \mathcal{F}^c(\ell+1)$. As before, we apply equation (1.15) first to $P = \mathcal{G}$, $T = \mathcal{F}^c(\ell)$ and then to $P = \mathcal{H}$, $T = \mathcal{F}^c(\ell+1)$ to obtain equations resembling equations (1.16) and (1.17):

$$\mu(\mathcal{G}) = \mu(\mathcal{F}) - (-1)^{\ell}|\mathcal{F}^c(\ell)| \tag{1.18}$$

$$\mu(\mathcal{H}) = \mu(\mathcal{F}) + (-1)^{\ell}(|\mathcal{F}^c(\ell+1)| - |\mathcal{F}^c(\ell)|). \tag{1.19}$$

Now $\nabla\mathcal{F}^c(\ell) \subseteq \mathcal{F}^c(\ell+1)$, implying $|\mathcal{F}^c(\ell+1)| - |\nabla\mathcal{F}^c(\ell)| \geq 0$. Suppose $\ell < \frac{2n-1}{3}$. We apply the Dual Shadow Lemma 1.3.2 to conclude

$$|\mathcal{F}^c(\ell+1)| - |\mathcal{F}^c(\ell)| > 0.$$

Once we consider all the possibilities for the sign of $\mu(\mathcal{F})$ and the parity of ℓ, we see that one of equations (1.18), (1.19) implies $|\mu(\mathcal{F})|$ is not a maximum. Hence we must have $\ell \geq \frac{2n-1}{3}$.

To finish this argument, we reason in the following manner: for each equivalence class of n modulo 3, the bounds for k and ℓ will enable us to find rank-selected lower order ideal configurations containing $\mathcal{F}$ and contained in $\mathcal{F}$, respectively. As before, we will apply equation (1.15) to obtain expressions for the Möbius function of these two lower order ideal configurations, apply a parity and sign argument, and then the rank-selection Lemma 1.2.1 to derive the required result.

By definition of k and ℓ, we have $k \geq \ell - 1$. We first consider the case $n \equiv 0(3)$. We have

$$k \geq \ell - 1 \geq \tfrac{2n}{3} - 1,$$

and from before

$$k \leq \tfrac{2n}{3},$$

implying

$$k = \tfrac{2n}{3} \text{ or } \tfrac{2n}{3} - 1.$$

For convenience, let $r = \frac{2n}{3}$. Then we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]),$$

where the first containment follows from the definition of ℓ and its bounds, while the second from the definition of k and its bounds. Note that r is even in this case, so by equation (1.16) we have

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) + |\mathcal{F}(r)|. \tag{1.20}$$

By the same token, equation (1.18) becomes

$$\mu(O_n[r]) = \mu(\mathcal{F}) - |O_n(r) \setminus \mathcal{F}(r)|. \tag{1.21}$$

If $\mu(\mathcal{F}) > 0$, then equation (1.20) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Hence $\overline{\mathcal{F}} = O_n[r-1] = O_n[\lfloor \frac{2n}{3} \rfloor - 1]$, contradicting the rank-selection Lemma 1.2.1. Otherwise $\mu(\mathcal{F}) < 0$, so equation (1.21) plus the maximality of $|\mu(\mathcal{F})|$ imply $O_n(r) = \mathcal{F}(r)$. Thus $\overline{\mathcal{F}} = O_n[r] = O_n[\lfloor \frac{2n}{3} \rfloor]$, as desired.

Suppose $n \equiv 1(3)$. Again, by the inequalities derived for k and ℓ and their definitions, we obtain

$$k \geq \ell - 1 \geq \frac{2n+1}{3} - 1 \text{ and } k \leq \frac{2n+1}{3},$$

implying

$$k = \frac{2n+1}{3} \text{ or } \frac{2n+1}{3} - 1.$$

Letting $r = \frac{2n+1}{3}$, we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]).$$

Notice r is odd in this case, so again equations (1.16) and (1.18) reduce to

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) - |\mathcal{F}(r)| \tag{1.22}$$

and

$$\mu(O_n[r]) = \mu(\mathcal{F}) + |O_n(r) \setminus \mathcal{F}(r)|. \tag{1.23}$$

If $\mu(\mathcal{F}) > 0$, then equation (1.23) and the maximality of $|\mu(\mathcal{F})|$ imply $O_n(r) = \mathcal{F}(r)$, so $\overline{\mathcal{F}} = O_n[\lfloor \frac{2n}{3} \rfloor + 1]$, contradicting Lemma 1.2.1. If $\mu(\mathcal{F}) < 0$, then equation (1.22) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Thus $k = r - 1$ and $\overline{\mathcal{F}} = O_n[\lfloor \frac{2n}{3} \rfloor]$, as desired.

Finally, suppose $n \equiv 2(3)$. We have the bounds

$$k \geq \frac{2n-1}{3} - 1 \text{ and } k \leq \frac{2n+2}{3}, \tag{1.24}$$

implying

$$k = \frac{2n-1}{3} - 1, \ \frac{2n-1}{3} \text{ or } \frac{2n-1}{3} + 1.$$

Letting $r = \frac{2n-1}{3}$ we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r+1]).$$

Suppose $k = r + 1$. We will see that this will lead to a contradiction. The choices for k will then reduce to $r - 1$ or r, and the remainder of the argument will proceed as in the $n \equiv 0(3)$ and $n \equiv 1(3)$ cases.

Since r is odd in this case, equations (1.16) and (1.17) become

$$\mu(\mathcal{F}[r]) = \mu(\mathcal{F}) + |\mathcal{F}(r+1)| \tag{1.25}$$

and

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) + |\mathcal{F}(r+1)| - |\mathcal{F}(r)|. \tag{1.26}$$

If $\mu(\mathcal{F}) > 0$ then equation (1.25) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r+1) = \emptyset$, contrary to our assumption on k. Thus $\mu(\mathcal{F}) < 0$. However, recall that the Shadow Lemma applied to $S = \mathcal{F}(r+1)$ has

$$|\mathcal{F}(r+1)| \leq |\Delta\mathcal{F}(r+1)| \leq |\mathcal{F}(r)|,$$

implying the difference $|\mathcal{F}(r+1)| - |\mathcal{F}(r)| \leq 0$. If the difference is negative, this contradicts the maximality of $|\mu(\mathcal{F})|$. If the difference is zero, then (by the Shadow Lemma again) $\mathcal{F}(r+1) = O_n(\frac{2n+2}{3})$. So $\overline{\mathcal{F}} = O_n[\frac{2n+2}{3}]$, contradicting Lemma 1.2.1.

The present situation is k is $r-1$ or r, so we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]).$$

Since r is odd, the same reasoning as in the $n \equiv 1(3)$ case shows that equations (1.22) and (1.23) continue to hold. Now if $\mu(\mathcal{F}) < 0$, equation (1.22) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Thus $\overline{\mathcal{F}} = O_n[r-1] = O_n[\lfloor \frac{2n}{3} \rfloor - 1]$, contradicting the rank-selection Lemma 1.2.1. Therefore, $\mu(\mathcal{F}) > 0$, so the maximality of $|\mu(\mathcal{F})|$ implies $O_n(r) \setminus \mathcal{F}(r) = \emptyset$, i.e. $\overline{\mathcal{F}} = O_n[r] = O_n[\lfloor \frac{2n}{3} \rfloor]$.

We thus conclude the extremal configuration occurring in Lemma 1.2.1 coincides with the extremal configuration for Theorem 1. □

Table 1.2 displays the extremal configuration and the Mõbius values of the lower order ideal case for O_n, $n = 1, \ldots, 25$.

Table 1.2: Lower Order Ideal Extremal Configuration for O_n, $n = 1, \ldots, 25$

n	Extremal Configuration $\mathcal{F}$	$\lvert\mu(O_n(\mathcal{F}))\rvert$
1	$O_1[1]$	1
2	$O_2[1]$	3
3	$O_3[2]$	7
4	$O_4[2]$	17
5	$O_5[3]$	49
6	$O_6[4]$	129
7	$O_7[4]$	351
8	$O_8[5]$	1023
9	$O_9[6]$	2815
10	$O_{10}[6]$	7937
11	$O_{11}[7]$	23297
12	$O_{12}[8]$	65537
13	$O_{13}[8]$	187903
14	$O_{14}[9]$	553983
15	$O_{15}[10]$	1579007
16	$O_{16}[10]$	4571137
17	$O_{17}[11]$	13516801
18	$O_{18}[12]$	38862849
19	$O_{19}[12]$	113213439
20	$O_{20}[13]$	335478783
21	$O_{21}[14]$	970522623
22	$O_{22}[14]$	2839740417
23	$O_{23}[15]$	8428126209
24	$O_{24}[16]$	24494735361
25	$O_{25}[16]$	71904919551

Chapter 2

Interval of Rank-Selections

In this chapter we are interested in maximizing $|\mu(O_n[i,j])|$ where $[i,j] \subseteq [n]$ is an interval of ranks from O_n. To do this, we will translate this problem to one of enumerating a certain class of permutations. A special edge-labeling of the poset O_n, called an R-labeling, enables us to perform this conversion.

The main result we will prove in this chapter is the following:

Theorem 2.0.4 *For intervals of rank-selections $[i,j] \subseteq [n]$ of O_n, where $n > 0$ is fixed and $n \neq 2$, $|\mu(O_n(S))|$ achieves a maximum at one (or possibly both) of*

$$S = \begin{cases} \left[\frac{2n}{5}, \frac{4n-5}{5}\right], \left[\frac{2n+5}{5}, \frac{4n}{5}\right] & \textit{when } n \equiv 0(5) \\ \left[\frac{2n+3}{5}, \frac{4n-4}{5}\right], \left[\frac{2n+3}{5}, \frac{4n+1}{5}\right] & \textit{when } n \equiv 1(5) \\ \left[\frac{2n+1}{5}, \frac{4n-3}{5}\right], \left[\frac{2n+6}{5}, \frac{4n+2}{5}\right] & \textit{when } n \equiv 2(5) \\ \left[\frac{2n+4}{5}, \frac{4n-2}{5}\right], \left[\frac{2n+4}{5}, \frac{4n+3}{5}\right] & \textit{when } n \equiv 3(5) \\ \left[\frac{2n+2}{5}, \frac{4n-1}{5}\right] & \textit{when } n \equiv 4(5). \end{cases}$$

For $n = 2$, the maxima occur when

$$S = [1,1] \textit{ or } [2,2].$$

We conjecture that the following result is true.

Conjecture 2.0.5 *For intervals of rank-selections $[i,j] \subseteq [n]$ of O_n, where $n > 0$ is fixed and $n \neq 2$, $|\mu(O_n(S))|$ achieves a unique maximum when*

$$S = [\lfloor \tfrac{2n+5}{5} \rfloor, \lfloor \tfrac{4n+1}{5} \rfloor].$$

For $n = 2$, the maxima occur when

$$S = [1,1] \text{ or } [2,2].$$

In section 3 we provide numerical evidence to support this conjecture.

We will return to the concept of edge-labelings, specifically EL- and CL-labelings, in the next chapter.

2.1 Edge and Chain Labelings

In order to motivate the importance of edge and chain labelings, we first briefly develop the notion of shellability of complexes. This topological condition implies a certain result about simplicial homology. We will see that CL-shellings and EL-shellings of order complexes are each combinatorial conditions which imply shellability of the order complex.

We follow [6] for all notation and terminology related to shellability and homology. A *simplicial complex* Δ is a collection of subsets of a finite set V (called the *vertex set*) such that

(i) if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$

(ii) if $v \in V$ then $\{v\} \in \Delta$.

All the complexes we will work with will be nonvoid, so $\emptyset \in \Delta$. The members of Δ are called *simplices* or *faces*, while the maximal faces of Δ are called *facets*. The *dimension* of a face $F \in \Delta$ is $|F| - 1$, while the dimension of a complex is

$$\dim \Delta = \max \{\dim F : \quad F \in \Delta\}.$$

A complex is *pure* is all of its facets are equicardinal.

Let Δ be a pure simplicial complex. A *shelling* of Δ is a linear order $F_1, F_2, \ldots, F_t$ of the facets of Δ satisfying the following condition:

> Given facets F_i and F_j with $i < j$, there exists a facet F_k with $k < j$ such that $F_i \cap F_j \subseteq F_k \cap F_j$, where $|F_k \cap F_j| = \dim \Delta - 1$.

We call a complex *shellable* if it admits a shelling. If $F_1, \ldots, F_t$ is a listing of the facets of Δ in a shelling order, let

$$\Delta_i = \{G \in \Delta : \quad G \subseteq F_k \text{ for some } k \leq i\}$$

and let

$$\mathcal{R}(F_i) = \{x \in F_i : \quad F_i \setminus x \in \Delta_{i-1}\}$$

be the *restriction* of F_i.

We next review simplicial homology and homology of a shellable complex. Let Δ be a simplicial complex of dimension d on the vertex set V. Assume that V has been given some linear order. Hence, if $F = \{v_0, v_1, \ldots, v_k\}$ is a nonempty face of dimension k in Δ, we will write it as $[v_0, v_1, \ldots, v_k]$ if $v_0 < v_1 < \ldots < v_k$ with respect to the linear ordering of V. Let $C_k(\Delta)$ denote the free abelian group generated by the set of k-dimensional faces of Δ written in canonical form. Notice that $C_{-1} \cong \mathcal{Z}$ and $C_0(\Delta) \cong \mathcal{Z}^{|V|}$, where $\mathcal{Z}$ denotes the integers. Define the *boundary operator*

$$\partial_k : \quad C_k(\Delta) \rightarrow C_{k-1}(\Delta)$$

by

$$\partial_k[v_0, v_1, \ldots, v_k] = \sum_{i=0}^{k}(-1)^k[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k].$$

(Here $\hat{x}_i$ means delete the element x_i.) One can verify that $\partial_k \circ \partial_{k+1}$ for all $k \in \mathcal{Z}$.

The kernel of the boundary operator ∂_k forms a group $Z_k(\Delta)$ called the *group of k-cycles*, i.e.,

$$Z_k(\Delta) = \langle\ \rho \in C_k(\Delta) : \ \ \partial_k(\rho) = 0\ \rangle.$$

Let $B_k(\Delta)$ be the group generated by the image of δ_{k+1},

$$B_k(\Delta) = \langle\ \sigma \in C_k(\Delta) : \ \ \sigma = \delta_{k+1}(\tau),\ \tau \in C_{k+1}(\Delta)\ \rangle,$$

called the *group of k-dimensional boundaries.* Notice that $B_k(\Delta) \subseteq Z_k(\Delta)$ for $k \in \mathcal{Z}$. The *reduced homology groups* are the quotient groups defined by

$$\tilde{H}_k(\Delta) = Z_k(\Delta)/B_k(\Delta).$$

We call a complex Δ *acyclic* over $\mathcal{Z}$ if $\tilde{H}_k(\Delta) = 0$ for all $k \in \mathcal{Z}$.

An important result about the reduced homology of a shellable complex is the following:

Theorem 2.1.1 *[6, Theorem 7.7.2] Let Δ be a shellable d-dimensional complex with facets $\mathcal{F}$. Furthermore, suppose*

$$\{F \in \mathcal{F} : \ \ \mathcal{R}(F) = F\} = \{F_1, \ldots, F_t\},$$

where $\mathcal{R}(F)$ is the restriction operator induced by some shelling. Then

$$\tilde{H}_i(\Delta) \cong \begin{cases} \mathcal{Z}^t & i = d \\ 0 & i \neq d. \end{cases}$$

Also, there are cycles $\rho_1, \ldots, \rho_t \in \tilde{H}_d(\Delta)$ uniquely determined by

$$\rho_k(F_j) = \begin{cases} 1 & j = k \\ 0 & otherwise. \end{cases}$$

Finally, $\{\rho_1, \ldots, \rho_t\}$ is a basis of the free group $\tilde{H}_d(\Delta)$.

For P a poset, we can define a simplicial complex $\Delta(P)$ called the *order complex.* We take the vertices of $\Delta(P)$ to be the elements of P and the faces of $\Delta(P)$ to be the chains of P. Notice that the facets of $\Delta(P)$ are simply the maximal chains of P.

If P is a finite graded poset, we say it is *shellable* if its order complex $\Delta(P)$ is shellable. It would be useful to have a combinatorial criterion to show that the order complex of a poset is shellable. The concepts of EL-labelings and CL-labelings each imply shellability of the order complex.

Let P be a finite graded poset. An *edge-labeling* of P is a map $\lambda : \{(x, y) \in P \times P : x \prec y\} \longrightarrow \Lambda$, where Λ is some poset (usually the integers). We say an unrefinable chain

$$x_0 \prec x_1 \prec \ldots \prec x_k$$

in a poset with edge-labeling λ is *rising* if

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \ldots \leq \lambda(x_{k-1}, x_k).$$

An edge labeling λ of a poset P is called an R-*labeling* if for every interval $[x, y]$ in P there is a unique rising maximal chain c in $[x, y]$. Furthermore, we call an R-labeling λ an *EL-labeling* (edge lexicographical labeling) if this unique rising maximal chain c is lexicographically least among all other maximal chains in $[x, y]$. A graded poset that admits an EL-labeling is said to be *EL-shellable* (edge lexicographically shellable).

Using the labeling of the elements of O_n in section 0.1, we see that O_n has a very natural R-labeling:

Proposition 2.1.2 *Let O_n be the lattice of signed subsets of the elements $[n]$ ordered by inclusion with $\hat{1}$ adjoined. If we label each edge $(x, y) \in O_n$ by*

$$\lambda(x, y) = \begin{cases} y \setminus x, & \textit{if } y \neq \hat{1} \\ 0, & \textit{if } y = \hat{1}, \end{cases}$$

then λ is an R-labeling of O_n. In fact, this λ is also an EL-labeling of O_n, so O_n is EL-shellable.

Proof. We first observe the edge-labeling λ of O_n is well-defined, for if $x \prec y$ with $y \neq \hat{1}$, then $|x| + 1 = |y|$. Hence $y \setminus x$ is a single integer. For $x \prec y = \hat{1}$ in O_n (i.e. x is a *coatom* of O_n), the edge (x, y) is labeled by the integer 0.

Notice that any maximal chain in the interval $[x, y]$, $y \neq \hat{1}$, is labeled by a sequence of integers in the set $\{y \setminus x\}$. Hence, the unique rising maximal chain c in $[x, y]$ consists of the elements of $y \setminus x$ ordered with respect to the natural order of the integers. (Here we are interchangeably thinking of a maximal chain in $[x, y]$ as a set of labeled edges from x to y.)

For $y = \hat{1}$ and $x = \{x_1, \ldots, x_k\}$ of rank $k \leq n$, any maximal chain in $[x, y] = [x, \hat{1}]$ is labeled by some signing of the $n - k + 1$ elements

$$\{0\} \cup [n] \setminus \{|x_1|, \ldots, |x_k|\} = \{0\} \cup \{z_1, \ldots, z_{n-k}\}$$

with $z_1 > z_2 > \ldots > z_{n-k}$. Hence, the unique rising chain c is

$$c: \quad x < x \cup \{-z_1\} < x \cup \{-z_1, -z_2\} < \ldots < x \cup \{-z_1, \ldots, -z_{n-k}\} < 0.$$

By construction, this chain is lexicographically least among all other maximal chains in the interval $[x, \hat{1}]$, so λ is in fact an EL-labeling of O_n. $\square$

The Boolean algebra has a well-known R-labeling.

Proposition 2.1.3 *Let B_n be the lattice of subsets of the elements $[n]$ ordered by inclusion. If we label each edge $(x, y) \in B_n$ by*

$$\lambda(x, y) = y \setminus x,$$

then λ is an R-labeling of B_n. In fact, this λ is also an EL-labeling of B_n, so B_n is EL-shellable.

Let P be a graded poset of rank n. Let

$$\mathcal{E}^*(P) = \{(c, x, y) : \ c \text{ is a maximal chain of } P, \ \ x, y \in c \text{ and } \ x \prec y\}.$$

A *chain-edge labeling* of P is a map $\lambda : \mathcal{E}^*(P) \to \Lambda$, where Λ is some poset (usually the integers), satisfying the following condition: if c and c' are two maximal chains in P

$$c: \ \ \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$$

$$c': \ \ \hat{0} = x'_0 \prec x'_1 \prec \ldots \prec x'_n = \hat{1}$$

whose first d edges coincide, then the corresponding labels must also coincide along these d edges, i.e.

$$\lambda(c, x_{i-1}, x_i) = \lambda(c', x'_{i-1}, x'_i) \quad \text{for } i = 1, \ldots, d.$$

Suppose we have assigned a chain-edge labeling λ to P. Let $[x, y]$ be an interval in P with $k = \rho(x, y)$ and $r : \hat{0} = r_0 \prec r_1 \prec \cdots \prec r_{\rho(x)} = x$ be a saturated chain from $\hat{0}$ to x. We call the pair $([x, y], r)$ a *rooted interval with root* r, denoted by $[x, y]_r$. If $c : x = x_0 \prec x_1 \prec \ldots \prec x_k = y$ is a maximal chain in $[x, y]_r$, then it has a well-defined induced labeling given by

$$\lambda(c, x_{i-1}, x_i) = \lambda(m, x_{\rho(x)+i-1}, x_{\rho(x)+i}), \ \ i = 1, \ldots, k,$$

where m is any maximal chain in P containing r and c. We say the maximal chain c in a rooted interval $[x, y]_r$ is *increasing* if

$$\lambda(c, x_0, x_1) < \lambda(c, x_1, x_2) < \ldots < \lambda(c, x_{k-1}, x_k).$$

A chain-edge labeling λ is a *CL-labeling* (chain lexicographical labeling) if for every rooted interval $[x, y]_r$ in P: (i) there is a unique increasing maximal chain c

in $[x,y]_r$, and (*ii*) for any other maximal chain c' in $[x,y]_r$, c is lexicographically least. Any graded poset which admits a CL-labeling is called *CL-shellable* (chain lexicographically shellable).

From the definitions it is easy to see that if P is EL-shellable then it is CL-shellable. A theorem of Björner links the ideas of EL-shellability, CL-shellability and shellability.

Proposition 2.1.4 *[5, Theorem 2.3] For P a graded poset we have the following implications: P is EL-shellable $\Rightarrow$ P is CL-shellable $\Rightarrow$ P is shellable.*

Using recursive atom orderings (a concept equivalent to CL-shellability), Björner and Wachs proved the following theorem:

Theorem 2.1.5 *[9, Theorem 4.5] Let P be a convex polytope and L_P be its lattice of faces. Then L_P is CL-shellable.*

We will need this result for the proof of Theorem 3.0.8.

2.2 Augmented Signed Permutations and Rank-selections

Given the R-labeling of O_n described in section 1, we can use this to convert the statement of Conjecture 2.0.5 to one of enumerating augmented signed permutations of the integers $[n]$. Before doing this, we first recall some known results about R-labelings and the Möbius function of rank-selections.

Let $\alpha_P(S) = \alpha(S)$ be the number of maximal chains in $P(S) \cup \{\hat{0}, \hat{1}\}$. We define the ***beta invariant*** by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha(T).$$

By the Principle of Inclusion and Exclusion, we have

$$\alpha(S) = \sum_{T \subseteq S} \beta(T).$$

We have the following proposition, due to Stanley.

Proposition 2.2.1 *[16, Proposition 14.1]* $\mu(P(S)) = (-1)^{|S|+1}\beta(S)$.

Let π be a permutation of the n letters $\{a_1, \ldots, a_n\}$. We will write all permutations using one-line notation:

$$\begin{aligned} \pi &= \pi(a_1)\pi(a_2)\cdots\pi(a_n) \\ &= \pi_1\pi_2\cdots\pi_n. \end{aligned}$$

We define the *descent set* of $\pi = \pi_1\pi_2\cdots\pi_n$ by

$$D(\pi) = \{i \mid \ \pi_i > \pi_{i+1}\}.$$

As an example, if $\pi = 4\,3\,1\,5\,2$, then $D(\pi) = \{1,2,4\}$. We let the set of all *augmented signed permutations* of $[n]$ be the following:

$$\begin{aligned} S_n^{\pm} = \{\sigma = s_1 \ldots s_n s_{n+1} : \ & \{s_1, \ldots, s_n\} \subset \{\bar{1}, \bar{2}, \ldots, \bar{n}\} \cup \{1, 2, \ldots, n\}, \\ & |s_i| \neq |s_j| \text{ for } i \neq j, \text{ and } s_{n+1} = 0\}. \end{aligned}$$

(Here we are using the bar notation $\bar{a}$ to denote $-a$.) For example,

$$S_2^{\pm} = \{120, 1\bar{2}0, \bar{1}20, \bar{1}\bar{2}0, 210, 2\bar{1}0, \bar{2}10, \bar{2}\bar{1}0\}.$$

Note that the maximal chains in O_n with respect to the labeling λ described in Proposition 2.1.2 correspond with the augmented signed permutations $S_n^{\pm}$. As an example, there are seven maximal chains in O_3 with descent set $\{1,2\}$:

$$\{32\bar{1}0, 3\bar{1}\bar{2}0, 31\bar{2}0, 21\bar{3}0, 2\bar{1}\bar{3}0, 1\bar{2}\bar{3}0, \bar{1}\bar{2}\bar{3}0\}$$

Observe that $|\mu(O_3[2])| = 7$. The fact that these two numbers coincide is explained by Corollary 2.2.3.

Theorem 2.2.2 *Let P be a poset of rank $n+1$ and $S \subseteq [n]$. If P admits an R-labeling, then*

$$\begin{aligned}\beta(S) &= \textit{the number of maximal chains in } P \textit{ with descent set } S\\ &\quad\ \textit{(with respect to the given R-labeling } \lambda).\end{aligned}$$

By Proposition 2.2.1, we have an important corollary.

Corollary 2.2.3 *Under the same hypotheses as Theorem 2.2.2 we have*

$$\begin{aligned}|\mu(P(S))| &= \textit{the number of maximal chains in } P \textit{ with descent set } S\\ &\quad\ \textit{(with respect to the given R-labeling } \lambda\ ).\end{aligned}$$

As a remark, Theorem 2.2.2 and Corollary 2.2.3 still hold if "R-labeling λ" is replaced by "EL-labeling λ" or "CL-labeling λ".

2.3 Proof of the Interval Case for O_n

We have established that O_n has an R-labeling in Proposition 2.1.2. In light of Corollary 2.2.3 we will now use the notation

$$\beta_n(S) = |\mu(O_n(S))|,$$

where $S \subseteq [n]$. We will also use the notation

$$\mathcal{B}_n(S) = \{\pi \in S_n^{\pm} : \ D(\pi) = S\}.$$

(Thus $\beta_n(S) = |\mathcal{B}_n(S)|$.)

The problem of maximizing $|\mu(O_n(S))|$, where S runs over all interval rank-selections $[i,j] \subseteq [n]$, is equivalent to maximizing the number of permutations in $\mathcal{B}_n(S)$, where S runs over all intervals of descents $[i,j] \subseteq [n]$. To tackle this problem, we fix $n > 0$ and form a triangular array of the $\beta_n[i,j]$'s. The jth *row* consists of the values

$$\beta_n[1, n-j+1],\ \beta_n[2, n-j+2],\ \ldots, \beta_n[j,n] \quad (j = 1, \ldots, n).$$

For example,

$$O_1: \quad \begin{array}{ccccc} & & 1 & & \end{array}$$

$$O_2: \quad \begin{array}{ccccc} & & 1 & & \\ & 3 & & 3 & \end{array}$$

$$O_3: \quad \begin{array}{ccccc} & & 1 & & \\ & 7 & & 5 & \\ 5 & & 11 & & 7 \end{array}$$

To prove Theorem 2.0.4, we look at various configurations in the triangles. In other words, we try to determine the maximum value in a given row, a given diagonal, or a given antidiagonal. Using bipartite graph arguments we have been able to determine the maximum value of the diagonal case completely. (Here a ***diagonal*** means the sequence of $\beta_n[i,j]$'s with j fixed, $1 \leq i \leq j$, and an ***antidiagonal*** means the sequence of $\beta_n[i,j]$'s with i fixed, $i \leq j \leq n$.) Applying the same techniques to the row and antidiagonal cases narrows down the maximum to one or two possibilities. Once we have found the maximum behavior in the row, diagonal and antidiagonal sequences, we will intersect this information to yield Theorem 2.0.4.

To get the reader accustomed to working with these augmented permutations, we now give a "faster" proof of Corollary 1.1.7. In fact, Proposition 2.3.1 gives a recurrence for any $\beta_n[i,j]$, not just the $\beta_n[1,j]$'s.

Proposition 2.3.1 *For the triangular array of $\beta_n[i,j]$'s, with $n > 1$, the following recurrences hold:*

$$(i) \quad \beta_n[i,j] = 2\beta_{n-1}[i-1,j-1] + 2\beta_{n-1}[i,j-1] + \beta_{n-1}[i,j], \quad 1 \le i \le j < n$$
$$(ii) \quad \beta_n[i,n] = 2\beta_{n-1}[i-1,n-1] + \beta_{n-1}[i,n-1], \quad 1 \le i \le n$$

with boundary conditions

$$\beta_1[1,1] = 1 \text{ and } \beta_n(\emptyset) = 1 \quad (n \ge 1).$$

Proof. Let

$$\begin{aligned} \mathcal{B}_n[i,j] &= \{\pi \in S_n^{\pm} : \; D(\pi) = [i,j]\} \\ \mathcal{B}_n^+[i,j] &= \{\pi \in \mathcal{B}_n[i,j] : \; \pi \text{ contains the element } n\} \\ \mathcal{B}_n^-[i,j] &= \{\pi \in \mathcal{B}_n[i,j] : \; \pi \text{ contains the element } \bar{n}\}. \end{aligned}$$

For shorthand we will sometimes use $\mathcal{B}$, $\mathcal{B}^+$ and $\mathcal{B}^-$, when context makes the value of the other parameters clear.

Each of the recurrences stated in this proposition follows easily after observing what happens if we remove the element n or $\bar{n}$ from a permutation in $S_n^{\pm}$. For (i), let $\pi = a_1 \cdots a_n a_{n+1} = a_1 \cdots a_n 0 \in \mathcal{B}_n[i,j]$. If $\pi \in \mathcal{B}^+$, then $a_i = n$. Deleting the element n from π gives the bijection

$$\mathcal{B}_n^+[i,j] \longleftrightarrow \mathcal{B}_{n-1}[i-1,j-1] \cup \mathcal{B}_{n-1}[i,j-1] \tag{2.1}$$

where the first term on the right side of equation (2.1) corresponds to $a_{i-1} > a_{i+1}$ and the second term to $a_{i-1} < a_{i+1}$. If $\pi \in \mathcal{B}^-$ then either $a_1 = \bar{n}$ or $a_{j+1} = \bar{n}$. Deleting the element $\bar{n}$ from π gives the bijection

$$\mathcal{B}_n^-[i,j] \longleftrightarrow \mathcal{B}_{n-1}[i-1,j-1] \cup \mathcal{B}_{n-1}[i,j] \cup \mathcal{B}_{n-1}[i,j-1]. \tag{2.2}$$

(Again, the right side of equation (2.2) corresponds to the three cases $a_1 = \bar{n}$, $a_{j+1} = \bar{n}$ with $a_j > a_{j+2}$, and $a_{j+1} = \bar{n}$ with $a_j < a_{j+2}$). After taking cardinalities, the correspondences in (2.1) and (2.2) imply recurrence (i).

Part (ii) follows from the bijections

$$\mathcal{B}_n^+[i,n] \longleftrightarrow \mathcal{B}_{n-1}[i-1,n-1] \cup \mathcal{B}_{n-1}[i,n-1] \tag{2.3}$$

and

$$\mathcal{B}_n^-[i,n] \longleftrightarrow \mathcal{B}_{n-1}[i-1,n-1]. \tag{2.4}$$

The boundary conditions follow from the facts

$$\mathcal{B}_1[1,1] = (1,0)$$

and

$$\mathcal{B}_n(\emptyset) = (\bar{n}, \overline{n-1}, \ldots, \bar{1}, 0) \quad \square$$

We will need some notation before starting the various proofs to find the maximum value in a given row, diagonal or antidiagonal from the triangular array of $\beta_n[i,j]$'s. Let $\pi = a_1 a_2 \cdots a_n$ be any strictly increasing (respectively, strictly decreasing) sequence of integers. For $a \notin \pi$ an integer, we let $\check{\pi}$ denote the sequence obtained by inserting $|a|$ in the sequence π so that it remains strictly increasing (respectively, strictly decreasing). Similarly, we let $\tilde{\pi}$ denote the sequence obtained by inserting $-|a|$ in the sequence π, and π' denote the sequence obtained by inserting a into the sequence π. In each case, we add the element $-|a|$ or a into the sequence π so that the resulting sequence remains strictly increasing (respectively, strictly decreasing). For $a \in \pi$ let $\hat{\pi}$ denote the sequence obtained by removing the element a from π. (Note that since π was monotone, the sequence $\hat{\pi}$ will still be monotone.)

Recall that a *strictly unimodal sequence* is a sequence $a_1, \ldots, a_n$ of the form

$$a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_n.$$

and a sequence $a_1, \ldots, a_n$ is *almost strictly unimodal* if the sequence is strictly unimodal or is of the form

$$a_1 < a_2 < \cdots < a_k = a_{k+1} > a_{k+2} > \cdots > a_n.$$

We are now ready to begin the proofs.

Proposition 2.3.2 *(Row Sequences) For $r \geq 0$ the sequence*

$$\beta_n[1, 1+r], \beta_n[2, 2+r], \ldots, \beta_n[n-r, n]$$

is almost strictly unimodal with maximum occurring at one (or both) of

$$\beta_n[\lfloor \tfrac{2n-2r}{3} \rfloor, \lfloor \tfrac{2n-2r}{3} \rfloor + r], \ \beta_n[\lceil \tfrac{2n-2r}{3} \rceil, \lceil \tfrac{2n-2r}{3} \rceil + r].$$

For $r = 0$ and $n \equiv 2(3)$, the sequence

$$\beta_n[1, 1], \beta_n[2, 2], \ldots, \beta_n[n, n]$$

is almost strictly unimodal with maximum occurring at

$$\beta_n[\lfloor \tfrac{2n-1}{3} \rfloor, \lfloor \tfrac{2n-1}{3} \rfloor] = \beta_n[\lfloor \tfrac{2n+4}{3} \rfloor, \lfloor \tfrac{2n+4}{3} \rfloor].$$

Proof. We first show $\beta_n[i-1, i+r-1] < \beta_n[i, i+r]$ for $i \leq \lfloor \frac{2n-2r}{3} \rfloor$. To do this we construct a bipartite graph G with vertex bipartition $V_1 = \mathcal{B}_n[i-1, i+r-1]$ and $V_2 = \mathcal{B}_n[i, i+r]$ and show that $|V_1| < |V_2|$.

Given $\pi = a_1 \cdots a_n 0 \in V_1$, we can write $\pi = \pi_1 \pi_2 \pi_3 0$ with

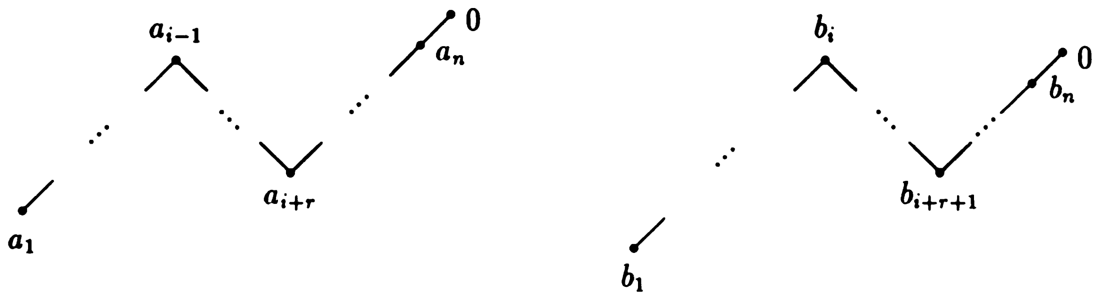
$$\begin{aligned}
\pi_1 &= a_1 \cdots a_{i-1} \\
\pi_2 &= a_i \cdots a_{i+r} \\
\pi_3 &= a_{i+r+1} \cdots a_n.
\end{aligned}$$

Note that

$$\pi = a_1 < \ldots < a_{i-1} > a_i > \ldots > a_{i+r} < a_{i+r+1} < \ldots < a_n < 0,$$

so π_1, π_2 and π_3 are monotone increasing, decreasing and increasing, respectively.

$\mathcal{B}_n[i-1, i+r-1]$ $\mathcal{B}_n[i, i+r]$



Let $\pi \in V_1$. For each $a \in \pi_3$ form the permutations

$$\sigma = \check{\pi}_1 \pi_2 \hat{\pi}_3 0,$$

i.e., remove the element a from π_3 and adjoin $|a|$ to π_1 with respect to the usual ordering, and

$$\tilde{\sigma} = \tilde{\pi}_1 \pi_2 \hat{\pi}_3 0,$$

i.e., remove the element a from π_3 and adjoin $-|a|$ to π_1 with respect to the usual ordering. Notice that σ and $\tilde{\sigma}$ are elements of $\mathcal{B}_n[i, i+r]$. Now given $\pi \in V_1$, draw an edge of G to every σ and $\tilde{\sigma}$ that can be obtained from π in the manner just described.

In the bipartite graph constructed, every vertex in V_1 has degree equal to $2|\pi_3| = 2(n-i-r)$. Conversely, each vertex in V_2 has degree at most $|\check{\pi}_1| = |\tilde{\pi}_1| = i$ since the element $a \in \pi_3$ is negative. Notice there are some vertices $\tau = b_1 \cdots b_n 0 \in V_2$ with degree $< i$. For instance, if $b_1 = \bar{n}$ then $\tilde{\sigma}$ cannot be obtained from a π with $\bar{n} \in \pi_3$ since we always have $\min \pi_3 > \min \pi_2$. Hence

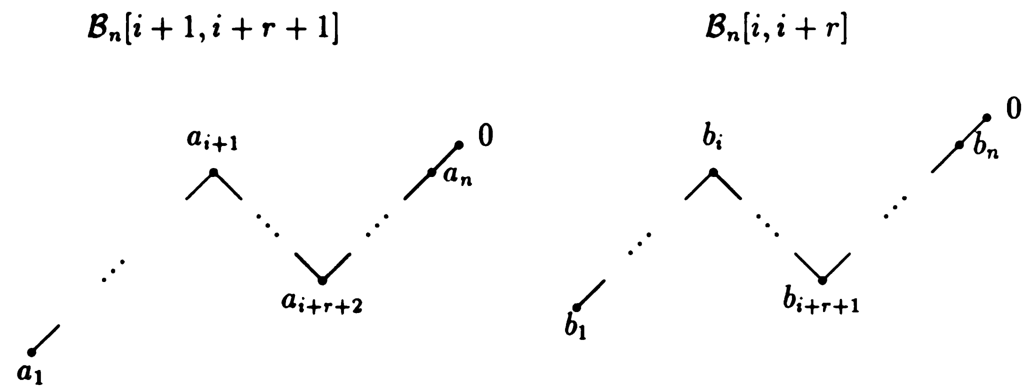
$$2(n-i-r)|V_1| < i|V_2|,$$

or

$$|V_1| < \frac{i}{2(n-i-r)}|V_2|.$$

Therefore, for $i \leq \frac{2n-2r}{3}$ we have $|V_1| < |V_2|$.

We next show $\beta_n[i, i+r] > \beta_n[i+1, i+r+1]$ for $\lceil\frac{2n-2r}{3}\rceil \leq i \leq n-r-1$. In a similar fashion, we construct a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i+1, i+r+1]$ and $V_2 = \mathcal{B}_n[i, i+r]$.



Given $\pi = \pi_1\pi_2\pi_3 0 \in V_1$, where

$$\begin{aligned}
\pi_1 &= a_1 < \ldots < a_i, \\
\pi_2 &= a_{i+1} > \ldots > a_{i+r+1}, \\
\pi_3 &= a_{i+r+2} < \ldots < a_n,
\end{aligned}$$

remove any $a \in \pi_1$ and draw an edge to every permutation in V_2 of the form

$$\tilde{\sigma} = \hat{\pi}_1\pi_2\tilde{\pi}_3 0.$$

Here we have added $-|a| < 0$ to π_3 with respect to the usual ordering because all the elements of $\tilde{\pi}_3$ must be negative. Observe $\tilde{\sigma} \in \mathcal{B}_n[i, i+r]$ and every vertex in V_1 has degree $|\pi_1| = i$. Every vertex in V_2 has degree at most $2(n-i-r)$ because

$|\tilde{\pi}_3| = n - i - r$ and there are two choices for the signing of an element $a \in \tilde{\pi}_3$ when it is put back into $\hat{\pi}_1$. The degree of a vertex could be less than $2(n-i-r)$. If $\tau = b_1 \cdots b_n 0 \in \mathcal{B}_n[i, i+r]$ with $b_{i+r+1} = \bar{n}$, then moving the element n into $b_1 \cdots b_{i-1}$ would give a permutation with i a member of its descent set. So we have

$$i|V_1| < 2(n-i-r)|V_2|.$$

Thus, we obtain $|V_1| < |V_2|$ for $i \geq \frac{2n-2r}{3}$.

The previous argument shows $\beta_n[n-r-1, n-1] > \beta_n[n-r, n]$ for $r \leq n-3$. If $r = n-1$ there is no argument since $\beta_n[0, n-1](= \beta_n(\emptyset))$ is not an element of the row sequence. For $r = n-2$ the result follows by using the actual summation formulas for $\beta_n[1, j]$. We wish to show

$$\begin{aligned} \beta_n[1, n-1] &> \beta_n[2, n] \\ &= \beta_n[1, 1] \end{aligned} \tag{2.5}$$

To obtain the equality in equation (2.5), we need a lemma.

Lemma 2.3.3 *For $S \subseteq [n]$ the following equality holds*

$$\beta_n(S) = \beta_n([n] \setminus S). \ \square$$

This lemma is proved by noting that there is a bijection $\mathcal{B}_n(S) \leftrightarrow \mathcal{B}_n([n] \setminus S)$ via the bar operator which sends $\pi = a_1 \cdots a_n 0 \in \mathcal{B}_n(S)$ to $\bar{\pi} = \bar{a}_1 \cdots \bar{a}_n 0 \in \mathcal{B}_n([n] \setminus S)$.

Now substitute the summation formula from Corollary 1.1.2 in equation (2.5). We see it is enough to show

$$|-\sum_{j=0}^{n-1} \binom{n}{j} (-2)^j| > |-\sum_{j=0}^{1} \binom{n}{j} (-2)^j|. \tag{2.6}$$

Applying the the binomial theorem the left side of equation (2.6) and simplifying the right side of equation (2.6) we obtain

$$|-(-1)^n + (-2)^n| = 2^n - 1 > |-1+2n| = 2n-1.$$

Here we are using the fact $n \geq 3$ to simplify the quantities appearing in the absolute value signs. (Recall $r = n-2 > 0$.) But $2^n > 2n$ when $n \geq 3$, so equation (2.6) holds for $n \geq 3$.

It remains to show

$$\beta_n[\lfloor \tfrac{2n}{3} \rfloor, \lfloor \tfrac{2n}{3} \rfloor] = \beta_n[\lceil \tfrac{2n}{3} \rceil, \lceil \tfrac{2n}{3} \rceil]$$

for $n \equiv 2(3)$. (This is the $r = 0$ case.) Write $n = 3m+2$. Then

$$\lfloor \tfrac{2n}{3} \rfloor = \lfloor \tfrac{6m+4}{3} \rfloor = 2m+1$$

and

$$\lceil \tfrac{2n}{3} \rceil = 2m+2$$

Note that

$$\begin{aligned} \beta_n[i,i] &= |\mu(O_n[i,i])| \\ &= (\text{number of elements in } O_n \text{ of rank } i) \ -1. \end{aligned}$$

Thus,

$$\begin{aligned} \beta_n[\lfloor \tfrac{2n}{3} \rfloor, \lfloor \tfrac{2n}{3} \rfloor] - \beta_n[\lceil \tfrac{2n}{3} \rceil, \lceil \tfrac{2n}{3} \rceil] &= \binom{3m+2}{2m+1} 2^{2m+1} - \binom{3m+2}{2m+2} 2^{2m+2} \\ &= 2^{2m+1} \tfrac{(3m+2)!}{(2m+2)!(m+1)!}[2m+2-2(m+1)] \\ &= 0. \ \square \end{aligned}$$

We conjecture that the row sequences obtain their maxima as follows.

Conjecture 2.3.4 *For fixed $0 \le r \le n-1$ the sequence*

$$\beta_n[1, 1+r], \beta_n[2, 2+r], \ldots, \beta_n[n-r, n]$$

is almost strictly unimodal. Its maxima only occur at

$$\begin{array}{ll} \beta_n\left[\lfloor \frac{2n-2r+1}{3} \rfloor, \lfloor \frac{2n+r+1}{3} \rfloor\right] & \text{for } n \not\equiv 2(3) \text{ or } r \neq 0 \\ \beta_n\left[\lfloor \frac{2n+1}{3} \rfloor, \lfloor \frac{2n+1}{3} \rfloor\right] = \beta_n\left[\lfloor \frac{2n+4}{3} \rfloor, \lfloor \frac{2n+4}{3} \rfloor\right] & \text{for } n \equiv 2(3) \text{ and } r = 0. \end{array}$$

We next fix j and consider the diagonal of elements $\beta_n[i, j]$, $i = 1, \ldots, j$. This time we are able to determine the complete maximum behavior of this sequence.

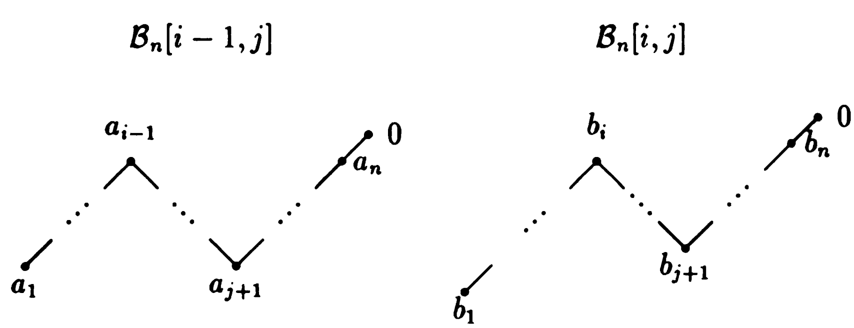
Proposition 2.3.5 *(Diagonal Sequences) The sequence*

$$\beta_n[1, j], \beta_n[2, j], \ldots, \beta_n[j, j]$$

is strictly unimodal. It attains its unique maximum at

$$\begin{array}{ll} \beta_n\left[\lfloor \frac{j+2}{2} \rfloor, j\right] & \text{for } j \neq n \\ \beta_n\left[\lfloor \frac{2n+3}{3} \rfloor, n\right] & \text{for } j = n. \end{array}$$

Proof. We first show $\beta_n[i-1, j] < \beta_n[i, j]$ for $i \le \frac{j+2}{2}$ $(j \neq n)$. As in Proposition 2.3.2, we form a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i-1, j]$ and $V_2 = \mathcal{B}_n[i, j]$.



Given $\pi = a_1 \cdots a_n 0 = \pi_1 a_{i-1} \pi_2 \pi_3 0 \in V_1$, where

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_{i-2} \\ \pi_2 &= a_i > \ldots > a_j \\ \pi_3 &= a_{j+1} < \ldots < a_n \end{aligned}$$

and $a \in \pi_2$, form the permutation

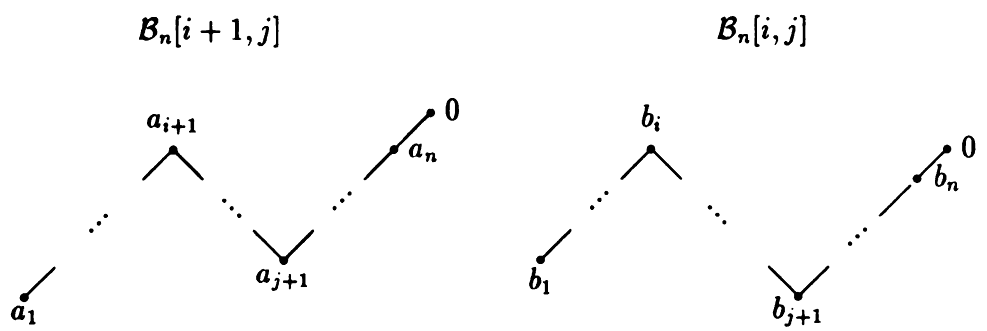
$$\sigma' = \pi_1' a_{i-1} \hat{\pi}_2 \pi_3 0.$$

Since $a \in \pi_2$, we know $a < a_{i-1}$, so $\sigma \in \mathcal{B}_n[i,j]$. For every $\pi \in V_1$, draw an edge to every σ that is constructed in the manner described. The degree of every vertex in V_1 is $|\pi_2| = j - i + 1$, while the degree of every vertex in V_2 is at most $|\pi_1'| = i - 1$. We obtain the edge count inequality

$$(j - i + 1)|V_1| < (i - 1)|V_2|$$

This inequality is strict. For instance, if $\tau = b_1 \cdots b_n 0 \in \mathcal{B}_n[i,j]$ with $b_1 = \bar{n}$, then we cannot put $\bar{n}$ into $b_{i+1} \cdots b_j$. If we did, then we would violate the property that for permutations $\pi \in \mathcal{B}_n[i-1,j]$ we have $\min \pi_2 > \min \pi_3$. So $|V_1| < |V_2|$ for $i \leq \frac{j+2}{2}$.

We next show $\beta_n[i,j] > \beta_n[i+1,j]$ for $i \geq \frac{j+1}{2}$ $(j \neq n)$. Construct a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i+1,j]$ and $V_2 = \mathcal{B}_n[i,j]$.



Given $\pi = a_1 \cdots a_n 0 = \pi_1 a_{i+1} \pi_2 \pi_3 0 \in V_1$, where

$$\pi_1 = a_1 < \ldots < a_i,$$

$$\pi_2 = a_{i+2} > \ldots > a_{j+1},$$

$$\pi_3 = a_{j+2} < \ldots < a_n,$$

and $a \in \pi_1$, draw an edge to the permutation

$$\sigma' = \hat{\pi}_1 a_{i+1} \pi_2' \pi_3 0.$$

Since $a \in \pi_1$, we know $a < a_{i+1}$, so $\sigma' \in \mathcal{B}_n[i,j]$. The degree of every vertex in V_1 is $|\pi_1| = i$, while the degree of each vertex in V_2 is at most $|\pi_2'| = (j - i + 1)$. We obtain the edge count inequality

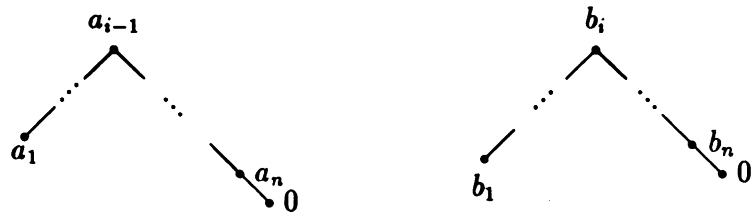
$$i|V_1| < (j - i + 1)|V_2|.$$

This inequality is strict. For instance, if $\tau = b_1 \cdots b_n 0 \in \mathcal{B}_n[i,j]$ with $b_j > b_{j+2}$, then if we were to move the element b_{j+1} into the sequence $b_1 \cdots b_{i-1}$, the resulting permutation would have the descent set $[i+1, j+1]$ rather than $[i+1, j]$. Therefore $|V_1| < |V_2|$ for $i \geq \frac{j+1}{2}$.

We now know $\{\beta_n[i,j]\}$, $1 \leq i \leq \lfloor \frac{j+2}{2} \rfloor$, is a strictly increasing sequence and $\{\beta_n[i,j]\}$, $\lceil \frac{j+1}{2} \rceil \leq i \leq j$, is a strictly decreasing sequence. Observe that $\lfloor \frac{j+2}{2} \rfloor = \lceil \frac{j+1}{2} \rceil$. Therefore, we may conclude the diagonal sequence $\{\beta_n[i,j]\}$, $1 \leq i \leq j$, $j \neq n$, is strictly unimodal with unique maximum $\beta_n[\lfloor \frac{j+2}{2} \rfloor, j]$.

We have characterized the monotonic behavior of all the diagonal sequences except the one with $j = n$. We now consider that sequence. To show $\beta_n[i-1,n] < \beta_n[i,n]$ for $i \leq \frac{2n+3}{3}$, we consider a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i-1,n]$ and $V_2 = \mathcal{B}_n[i,n]$.

$\mathcal{B}_n[i-1,n]$ $\qquad\qquad$ $\mathcal{B}_n[i,n]$



Given $\pi = a_1 \cdots a_n 0 = \pi_1 a_{i-1} \pi_2 0 \in V_1$, where

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_{i-2} \\ \pi_2 &= a_i > \ldots > a_n \end{aligned}$$

and $a \in \pi_2$, draw an edge to each of the permutations

$$\sigma = \check{\pi}_1 a_{i-1} \hat{\pi}_2 0$$

and

$$\tilde{\sigma} = \tilde{\pi}_1 a_{i-1} \hat{\pi}_2 0.$$

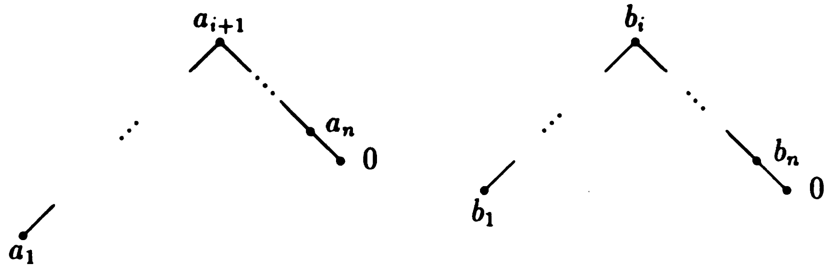
Since $D(\pi) = [i-1,n]$, we see that the elements $a_{i-1}, a_i, \ldots a_n$ are positive. So $a_{i-1} \geq a > 0$ implies $a_{i-1} > -a$. Each vertex in V_1 has degree equal to $2|\pi_2| = 2(n-i+1)$, while each vertex in V_2 has degree at most $|\check{\pi}_1| = |\tilde{\pi}_1| = (i-1)$. For instance, if $\tau = b_1 \cdots b_n 0 \in \mathcal{B}_n[i,n]$ with $b_1 = \bar{n}$, then we cannot put n into $b_{i+1} \cdots b_n$ since then $b_i < \max \pi_2 = n$ and the result would not be a permutation in $\mathcal{B}_n[i-1,n]$. We thus have the edge count inequality

$$2(n-i+1)|V_1| < (i-1)|V_2|,$$

so $|V_1| < |V_2|$ for $i \leq \frac{2n+3}{3}$.

To show $\beta_n[i,n] > \beta_n[i+1,n]$ for $i \geq \frac{2n+2}{3}$, we construct a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i+1,n]$, $V_2 = \mathcal{B}_n[i,n]$.

$\mathcal{B}_n[i+1,n]$ $\qquad\qquad$ $\mathcal{B}_n[i,n]$



Given $\pi = a_1 \cdots a_n 0 = \pi_1 \pi_2 0 \in V_1$, with

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_i \\ \pi_2 &= a_{i+1} > \ldots > a_n \end{aligned}$$

and $a \in \pi_1$, draw an edge to the permutation

$$\sigma = \hat{\pi}_1 \check{\pi}_2 0.$$

All the elements in π_2 are positive, so we must place $|a|$ in π_2 so that the resulting permutation $\sigma \in \mathcal{B}_n[i,n]$. Every vertex in V_1 has degree $|\pi_1| = i$, while each vertex in V_2 has degree at most $2|\check{\pi}_2| = 2(n-i+1)$. For example, if $\tau = b_1 \cdots b_n 0 \in \mathcal{B}_n[i,n]$ with $b_i = n$, then we cannot put n back into $\hat{\pi}_1 = b_1 \cdots b_{i-1}$. If we did, this would violate the property that max $\pi_1 <$ max π_2 for any permutation $\pi \in \mathcal{B}_n[i+1,n]$. We have the edge count inequality

$$i|V_1| < 2(n-i+1)|V_2|,$$

so $|V_1| < |V_2|$ for $i \geq \frac{2n+2}{3}$.

We have shown the sequence $\{\beta_n[i,n]\}$, $1 \leq i \leq \lfloor \frac{2n+3}{3} \rfloor$, is strictly increasing and the sequence $\{\beta_n[i,n]\}$, $\lceil \frac{2n+2}{3} \rceil \leq i \leq n$, is strictly decreasing. Hence, for $n \not\equiv 1(3)$ the sequence $\{\beta_n[i,n]\}$, $1 \leq i \leq n$, is strictly unimodal with unique maximum of $\beta_n[\lfloor \frac{2n+3}{3} \rfloor, n]$.

For $n \equiv 1(3)$ we apply Lemma 2.3.3 to the sequence sequence of $\beta_n[i,n]$'s. We obtain $\{\beta_n[1,i]\}$, $0 \le i \le \lfloor \frac{2n}{3} \rfloor$, is strictly increasing and $\{\beta_n[1,i]\}$, $\lceil \frac{2n-1}{3} \rceil \le i \le n-1$ is strictly decreasing. The permutations represented by these two sequences correspond to rank selections of the form $[1,i]$ from O_n, i.e., rank-selected lower order ideals. By Theorem 1.0.1 $\beta_n[1,\lfloor \frac{2n}{3} \rfloor] > \beta_n[1,\lceil \frac{2n-1}{3} \rfloor]$. Hence, the sequence $\{\beta_n[1,i]\}$, $0 \le i \le n-1$ is strictly unimodal with maximum $\beta_n[1,\lfloor \frac{2n}{3} \rfloor]$. Applying Lemma 2.3.3 again gives the result we claimed. □

The last sequence we consider is the one consisting of antidiagonal elements $\beta_n[i,j]$, $j = i, \ldots, n$.

Proposition 2.3.6 *(Antidiagonal Sequences) The sequence*

$$\beta_n[i,i], \beta_n[i,i+1], \ldots, \beta_n[i,n]$$

is almost strictly unimodal with maximum occurring at one (or both) of

$$\beta_n[i,\lfloor \tfrac{2n+i-1}{3} \rfloor],\ \beta_n[i,\lceil \tfrac{2n+i-1}{3} \rceil].$$

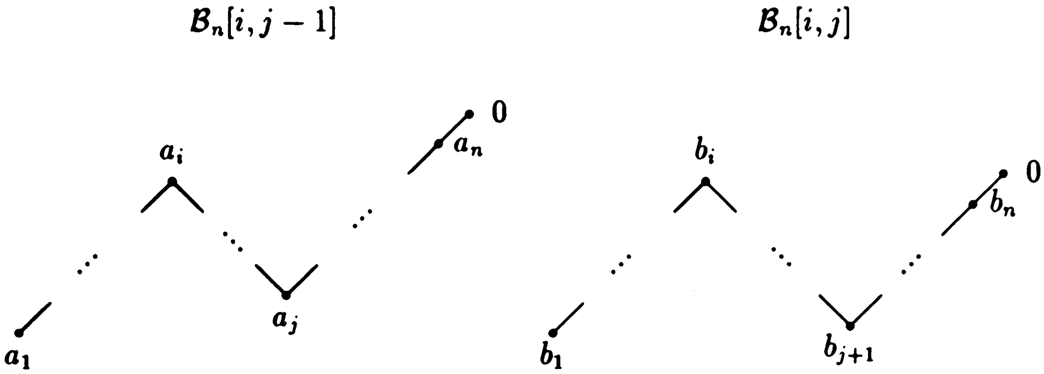
When $i = 1$ the antidiagonal sequence

$$\beta_n[i,i], \beta_n[i,i+1], \ldots, \beta_n[i,n]$$

is strictly unimodal with unique maximum of

$$\beta_n[i,\lfloor \tfrac{2n+i-1}{3} \rfloor].$$

Proof. We begin by forming a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i,j-1]$ and $V_2 = \mathcal{B}_n[i,j]$ $(j \ne n)$.



Given $\pi = \pi_1\pi_2 a_j \pi_3 0 \in V_1$, where

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_{i-1} \\ \pi_2 &= a_i > \ldots > a_{j-1} \\ \pi_3 &= a_{j+1} < \ldots < a_n \end{aligned}$$

and $a \in \pi_3$, form the permutations

$$\sigma = \pi_1 \check{\pi}_2 a_j \hat{\pi}_3 0$$

and

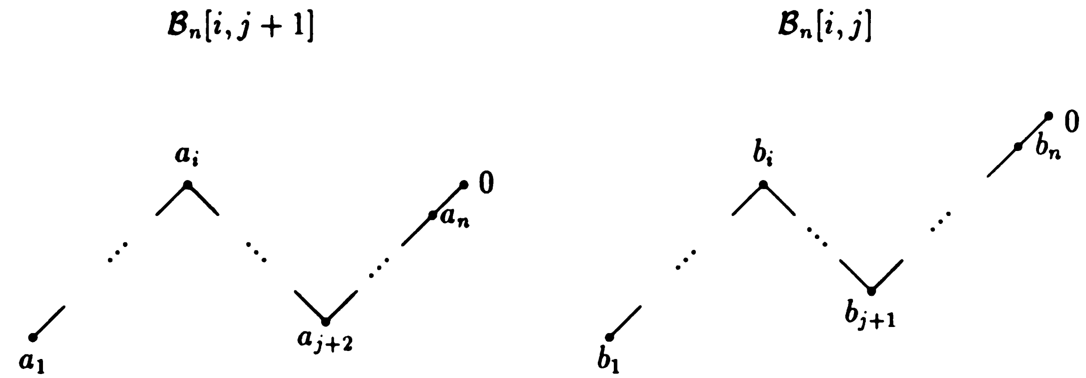
$$\tilde{\sigma} = \pi_1 \tilde{\pi}_2 a_j \hat{\pi}_3 0.$$

Since $a_j < a < 0$ for all $a \in \pi_3$, $|a| > a_j$. Hence $\sigma, \tilde{\sigma} \in V_2$. From the vertex π, draw an edge to each of the $|\pi_3| = 2(n-j)$ permutations in V_2 constructed in the manner described. The degree of each vertex in V_2 is at most $|\check{\pi}_2| = |\tilde{\pi}_2| = j - i + 1$. If $\tau = b_1 \ldots b_n 0 \in V_2$ with $b_i = n$, then we cannot place $\bar{n}$ into $b_{j+2} \cdots b_n$ to form a permutation in V_1 because the permutation $b_1 \cdots b_{i-1} b_{i+1} \cdots b_{j+1} \bar{n} b_{j+2} \cdots b_n 0$ has its jth element $b_{j+1} > \bar{n}$, violating $a_j < \min \pi_3$ for permutations $\pi = a_1 \cdots a_n 0 \in V_1$. We have the edge count inequality

$$2(n-j)|V_1| < (j-i+1)|V_2|,$$

so $|V_1| < |V_2|$ for $j \leq \frac{2n+i-1}{3}$.

Next we form a bipartite graph with vertex bipartition $V_1 = \mathcal{B}_n[i, j+1]$ and $V_2 = \mathcal{B}_n[i,j]$ $(j+1 \neq n)$.



Given $\pi = \pi_1\pi_2\pi_3 0 \in V_1$ and $a \in \pi_2$ with

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_i \\ \pi_2 &= a_{i+1} > \ldots > a_{j+1} \\ \pi_3 &= a_{j+2} < \ldots < a_n, \end{aligned}$$

form the permutation

$$\tilde{\sigma} = \pi_1 \hat{\pi}_2 \tilde{\pi}_3 0.$$

Observe that $\tilde{\sigma} \in V_2$ since all the elements of $\tilde{\pi}_3$ must be negative. For each $a \in \pi_3$, draw an edge from π to every $\tilde{\sigma}$ formed in this manner. We obtain the edge count inequality

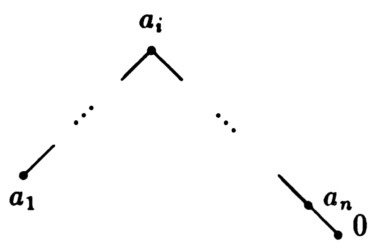
$$(j - i + 1)|V_1| < 2(n-j)|V_2|.$$

The degree of every vertex in V_1 is $|\pi_2| = j - i + 1$ and each vertex in V_2 has degree at most $2|\tilde{\pi}_3| = 2(n-j)$. If $\tau = b_1 \ldots b_n 0 \in V_2$ with $b_{j+1} = \bar{n}$, we cannot move $\bar{n}$ into

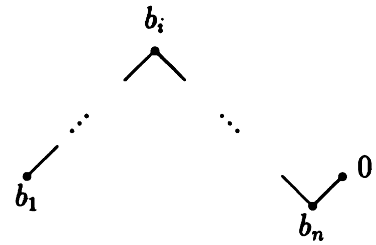
$b_{i+1}\cdots b_j$. If we did, then we would violate the property min $\pi_2 >$ min π_3 for any $\pi \in V_1$. Thus for $j \geq \frac{2n+i-1}{3}$ we have $|V_1| < |V_2|$.

Finally, we consider the terms of the form $\beta_n[i,n]$ and $\beta_n[i,n-1]$. Let $V_1 = \mathcal{B}_n[i,n]$ and $V_2 = \mathcal{B}_n[i,n-1]$ be a vertex partition. (Note $i \leq n-1$).

$\mathcal{B}_n[i,n]$



$\mathcal{B}_n[i,n-1]$



For $\pi = \pi_1\pi_2 0 \in V_1$, with

$$\begin{aligned} \pi_1 &= a_1 < \ldots < a_i \\ \pi_2 &= a_{i+1} > \ldots > a_n \end{aligned}$$

and $a \in \pi_2$, form the permutation

$$\tilde{\sigma} = \pi_1\hat{\pi}_2\tilde{a}0$$

Draw an edge from π to each of the $|\pi_2| = n-i$ permutations in V_2 formed in the described manner. Each vertex in V_2 has degree at most one. For instance, if $\tau = b_1\cdots b_n 0 \in V_2$ and any one of the elements $b_i,\ldots,b_{n-1}$ is negative, replacing b_n by $|b_n|$ in $b_i\cdots b_n$ will not give us back a permutation in V_1. We obtain the edge count inequality

$$(n-i)|V_1| < |V_2|,$$

so $|V_1| < |V_2|$ when $i \leq n-1$.

All parts of the proposition have now been proved except the case $i = 1$ which follows by appealing to the lower order ideal Theorem 1.0.1. □

Now we are ready to give the proof of Theorem 2.0.4. The theorem easily holds for $n = 1$ and $n = 2$. (Refer back to the pictures of the triangular arrays for O_1 and O_2 that were previously given.) Now fix $n \geq 3$. We first look at the candidates for the overall maximum for $\beta_n[i,j]$ that the row and diagonal propositions give us. From Proposition 2.3.5 the candidates for the maxima are those $\beta_n[i,j]$ with $i = \lfloor \frac{j+2}{2} \rfloor$, i.e.

$$\beta_n[r+1, 2r+1] \quad \text{for } r = 0, \ldots, 2r+1 \leq n-1 \tag{2.7}$$

$$\beta_n[r+2, 2r+2] \quad \text{for } r = 0, \ldots, 2r+2 \leq n-1. \tag{2.8}$$

Note that we do not need to consider the nth diagonal $\{\beta_n[i,n]\}$, $1 \leq i \leq n$, since by Lemma 2.3.3 this sequence coincides with the sequence $\{\beta_n[1,j]\}$, $1 \leq j \leq n$ (and the common entry $\beta_n[1,n] = 1$, which is not a maximum).

The Row Sequence Proposition 2.3.2 predicts the maximum occurs at

$$\beta_n\left[\lfloor \tfrac{2n-2r}{3} \rfloor, \lfloor \tfrac{2n+r}{3} \rfloor\right] \text{ or } \beta_n\left[\lceil \tfrac{2n-2r}{3} \rceil, \lceil \tfrac{2n+r}{3} \rceil\right].$$

Restated, these maximum candidates are

$$\beta_n\left[\tfrac{2n-2r}{3}, \tfrac{2n+r}{3}\right] \quad \text{when } 2n-2r \equiv 0(3) \tag{2.9}$$

$$\beta_n\left[\tfrac{2n-2r-1}{3}, \tfrac{2n+r-1}{3}\right], \beta_n\left[\tfrac{2n-2r+2}{3}, \tfrac{2n+r+2}{3}\right] \quad \text{when } 2n-2r \equiv 1(3) \tag{2.10}$$

$$\beta_n\left[\tfrac{2n-2r-2}{3}, \tfrac{2n+r-2}{3}\right], \beta_n\left[\tfrac{2n-2r+1}{3}, \tfrac{2n+r+1}{3}\right] \quad \text{when } 2n-2r \equiv 2(3) \tag{2.11}$$

Next we determine which row and diagonal candidates coincide. In other words, we look for those integer values of r in which a maximum candidate from the diagonal argument (equations (2.7) and (2.8)) is also a maximum candidate from the row argument (equations (2.9), (2.10) and (2.11)). Note that in both cases $r = j - i$ for the entry $\beta_n[i,j]$, so we are using the same parameter in the rows and diagonals. For

Table 2.1: Common Value of r for Row and Diagonal Results

	Diagonal equation	Row equation	Common value of r	Forcing
$2n-2r \equiv 1(3)$	(2.7)	(2.10)	$\frac{2n-4}{5}$	$n \equiv 2(5)$
	(2.8)	(2.10)	$\frac{2n-7}{5}$	$n \equiv 1(5)$
	(2.7)	(2.10)	$\frac{2n-1}{5}$	$n \equiv 3(5)$
	(2.8)	(2.10)	$\frac{2n-4}{5}$	$n \equiv 2(5)$
$2n-2r \equiv 2(3)$	(2.7)	(2.11)	$\frac{2n-5}{5}$	$n \equiv 0(5)$
	(2.8)	(2.11)	$\frac{2n-8}{5}$	$n \equiv 4(5)$
	(2.7)	(2.11)	$\frac{2n-2}{5}$	$n \equiv 1(5)$
	(2.8)	(2.11)	$\frac{2n-5}{5}$	$n \equiv 0(5)$

example, when $2n-2r \equiv 0(3)$, $\frac{2n-2r}{3} = r+1$ if and only if $r = \frac{2n-3}{5}$. Since r is an integer, we must have $n \equiv 4(5)$. Similarly, $\frac{2n-2r}{3} = r+2$ when $r = \frac{2n-6}{5}$, forcing $n \equiv 3(5)$.

The remaining cases when $2n-2r \equiv 1(3)$ and $2n-2r \equiv 2(3)$ follow in a similar manner. Table 2.1 displays the information needed to complete these cases.

We now rearrange our results according to the equivalence class of n modulo 5. In each case we decide if we can employ the antidiagonal sequence result to narrow down the two max candidates to one overall maximum. We will see that the antidiagonal Proposition 2.3.6 is only helpful when $n \equiv 4(5)$.

First suppose $n \equiv 0(5)$. Substituting $r = \frac{2n-5}{5}$ into the row maximum candidates for $2n-2r \equiv 2(3)$, we obtain the candidates

$$\beta_n\left[\frac{2n-2(\frac{2n-5}{5})-2}{3}, \frac{2n+(\frac{2n-5}{5})-2}{3}\right] = \beta_n\left[\frac{2n}{5}, \frac{4n-5}{5}\right]$$
$$\beta_n\left[\frac{2n-2(\frac{2n-5}{5})+1}{3}, \frac{2n+(\frac{2n-5}{5})+1}{3}\right] = \beta_n\left[\frac{2n+5}{5}, \frac{4n}{5}\right].$$

These two possibilities lie in the same row, so we cannot apply the antidiagonal result.

For $n \equiv 1(5)$ the row and diagonal arguments give

$$\beta_n\left[\frac{2n-2(\frac{2n-7}{5})-1}{3}, \frac{2n+(\frac{2n-7}{5})-1}{3}\right] = \beta_n\left[\frac{2n+3}{5}, \frac{4n-4}{5}\right]$$

(here $2n - 2r \equiv 1(3)$ and $r = \frac{2n-7}{5}$), and

$$\beta_n\left[\frac{2n-2(\frac{2n-2}{5})+1}{3}, \frac{2n+(\frac{2n-2}{5})+1}{3}\right] = \beta_n\left[\frac{2n+3}{5}, \frac{4n+1}{5}\right]$$

(here $2n - 2r \equiv 2(3)$ and $r = \frac{2n-2}{5}$). Both of these elements lie in the same antidiagonal. Recall that the antidiagonal proposition asserts

$$\begin{aligned} \beta_n[i, j-1] &< \beta_n[i, j] \quad \text{for } j \leq \lfloor \tfrac{2n+i-1}{3} \rfloor \\ \beta_n[i, j] &> \beta_n[i, j+1] \quad \text{for } j \geq \lceil \tfrac{2n+i-1}{3} \rceil. \end{aligned}$$

For $n \equiv 1(5)$ this says

$$\begin{aligned} \beta_n\left[\tfrac{2n+3}{5}, j-1\right] &< \beta_n\left[\tfrac{2n+3}{5}, j\right] \quad \text{for } j \leq \tfrac{4n-4}{5} \\ \beta_n\left[\tfrac{2n+3}{5}, j\right] &> \beta_n\left[\tfrac{2n+3}{5}, j+1\right] \quad \text{for } j \geq \tfrac{4n+1}{5}. \end{aligned}$$

Hence, Proposition 2.3.6 gives no additional information about the two candidates for the maximum.

For $n \equiv 2(5)$ the two candidates for the maximum are

$$\beta_n\left[\tfrac{2n+1}{5}, \tfrac{4n-3}{5}\right] \text{ and } \beta_n\left[\tfrac{2n+6}{5}, \tfrac{4n+2}{5}\right].$$

(Here $2n - 2r \equiv 1(3)$ with $r = \frac{2n-4}{5}$.) As when $n \equiv 0(5)$, these two possibilities lie in different antidiagonals, so the antidiagonal result cannot give any new information.

When $n \equiv 3(5)$ the candidates are

$$\beta_n\left[\tfrac{2n+4}{5}, \tfrac{4n-2}{5}\right] \text{ and } \beta_n\left[\tfrac{2n+4}{5}, \tfrac{4n+3}{5}\right].$$

(Here $2n - 2r \equiv 0(3)$ with $r = \frac{2n-6}{5}$ and $2n - 2r \equiv 1(3)$ with $r = \frac{2n-1}{5}$, respectively.) These two values lie in the same antidiagonal. However, for $n \equiv 3(5)$ Proposition 2.3.6 says

$$\begin{array}{ll} \beta_n\left[\frac{2n+4}{5}, j-1\right] < \beta_n\left[\frac{2n+4}{5}, j\right] & \text{for } j \le \frac{4n-2}{5} \\ \beta_n\left[\frac{2n+4}{5}, , j\right] > \beta_n\left[\frac{2n+4}{5}, j+1\right] & \text{for } j \ge \frac{4n+3}{5}, \end{array}$$

which yields no new information about the two candidates for the maximum.

The last case is when $n \equiv 4(5)$. We obtain the candidate maxima

$$\beta_n\left[\frac{2n+2}{5}, \frac{4n-1}{5}\right] \text{ and } \beta_n\left[\frac{2n+2}{5}, \frac{4n-6}{5}\right].$$

which lie in the same antidiagonal. (Here $2n - 2r \equiv 0(3)$ with $r = \frac{2n-3}{5}$ and $2n - 2r \equiv 2(3)$ with $r = \frac{2n-8}{5}$, respectively.) By Proposition 2.3.6 we have

$$\begin{array}{ll} \beta_n\left[\frac{2n+2}{5}, j-1\right] < \beta_n\left[\frac{2n+2}{5}, j\right] & \text{for } j \le \frac{4n-1}{5} \\ \beta_n\left[\frac{2n+2}{5}, , j\right] > \beta_n\left[\frac{2n+2}{5}, j+1\right] & \text{for } j \ge \frac{4n-1}{5}. \end{array}$$

Hence we can conclude in this case that the overall maximum is $\beta_n[\frac{2n+2}{5}, \frac{4n-1}{5}]$. $\square$

The result of Theorem 2.0.4 is unsatisfactory in the sense that, unlike the Boolean algebra, it does not determine the overall maximum for the interval rank-selection case (except when $n \equiv 4(5)$). Instead, for $n \not\equiv 4(5)$ it narrows down the extremal configuration to one of two possibilities. Part of the difficulty with the n-octahedron is that its triangle of $\beta_n[i, j]$'s is not symmetric along its rows like the Boolean algebra's triangle. More specifically, each row of the Boolean algebra's triangle is symmetric about its middle-most element (or elements, if the length of its row sequence is even.)

We have tried various methods to "sharpen" the bipartite arguments for O_n by redistributing the edges created between the different permutation types. Unfortunately, none were successful. However, the data for the interval of ranks case behaves quite beautifully, as you can see in Table 2.2. The data strongly suggests that the maximum occurs by taking the ranks $\lfloor \frac{2n+5}{5} \rfloor$ through $\lfloor \frac{4n+1}{5} \rfloor$.

Table 2.2: Extremal Configuration for Interval of Rank-selections from O_n

n	Extremal Configuration $\mathcal{F}$	$\lvert\mu(O_n(\mathcal{F}))\rvert$
1	$O_1[1,1]$	1
2	$O_2[1,1] = O_n[2,2]$	3
3	$O_3[2,2]$	11
4	$O_4[2,3]$	41
5	$O_5[3,4]$	161
6	$O_6[3,5]$	591
7	$O_7[3,5]$	2631
8	$O_8[4,6]$	11871
9	$O_9[4,7]$	52513
10	$O_{10}[5,8]$	231937
11	$O_{11}[5,9]$	993343
12	$O_{12}[5,9]$	4699903
13	$O_{13}[6,10]$	22111231
14	$O_{14}[6,11]$	102406721
15	$O_{15}[7,12]$	471223169
16	$O_{16}[7,13]$	2126966271
17	$O_{17}[7,13]$	10262581759
18	$O_{18}[8,14]$	49138667007
19	$O_{19}[8,15]$	232427864577
20	$O_{20}[9,16]$	1091852042241
21	$O_{21}[9,17]$	5058126423039
22	$O_{22}[9,17]$	24635153735679
23	$O_{23}[10,18]$	119106560870399
24	$O_{24}[10,19]$	570189596794881
25	$O_{25}[11,20]$	2712059387740161

Chapter 3

Arbitrary Rank-selections

Given a permutation $a_1 \cdots a_n$, we say it is *alternating* if either

$$a_1 < a_2 > a_3 < a_4 > \dots \tag{3.1}$$

or

$$a_1 > a_2 < a_3 > a_4 < \dots . \tag{3.2}$$

In the first case (equation (3.1)) we call the permutation an *alternating permutation with initial ascent* and in the second case (equation (3.2)) we call the permutation an *alternating permutation with initial descent.* We say a permutation $a_1 \cdots a_n$ has a *double ascent* if there exists an index k so that $a_k < a_{k+1} < a_{k+2}$.

Let P be a poset of rank n, $S \subseteq [n-1]$, and λ an R-labeling of P. Recall that $|\mu(P(S))|$ equals the number of maximal chains in P with descent set S with respect to an R-labeling λ. Proposition 2.1.3 gives an R-labeling λ of the Boolean algebra B_n. Note that the maximal chains in B_n with respect to this λ are simply permutations in the symmetric group S_n on n letters.

By Corollary 2.2.3, the question of maximizing the Möbius function over arbitrary rank-selections from the Boolean algebra is equivalent to finding an $S \subseteq [n-1]$ which maximizes the number of permutations of descent set S in the symmetric group S_n.

Sagan, Yeh and Ziegler [15, Theorem 1.2, part 2] used this interpretation of the arbitrary rank-selection problem to solve the B_n case. They constructed injective but not surjective maps from permutations in S_n with a double ascent to permutations in S_n with one less double ascent. They obtained the following result.

Theorem 3.0.7 *For arbitrary rank-selections $S \subseteq [n-1]$ of B_n, $|\mu(B_n(S))|$ attains a unique maximum when we take S to be*

$$S = \{1,3,5,\ldots\} \cap [n-1] \text{ or } S = \{2,4,6,\ldots\} \cap [n-1].$$

In this case $|\mu(B_n(S)| = E_n$, the nth Euler number.

In other words, for arbitrary rank-selections $S \subseteq [n-1]$ the Möbius function is maximized by taking every other rank from B_n. In terms of permutations from the symmetric group, this result says the alternating permutations with initial ascent (or the alternating permutations with initial descent) are the largest class with given descent set.

The arbitrary rank-selection question, viewed in the context of permutations in the symmetric group, was studied by Niven and de Bruijn. In [13] Niven also used an injection-but-not-surjection argument, whereas in [11] de Bruijn developed an algorithmic technique.

In this chapter we are interested in addressing the arbitrary rank-selection case for the n-octahedron. Rather than studying this question in terms of augmented signed permutations of $[n]$ which arise from the R-labeling of O_n in Proposition 2.1.2, we present an approach based upon a non-commutative polynomial Φ called the *cd*-index. In a recent thesis, Purtill [14] showed that for certain lattices L, $\Phi(L)$ has non-negative coefficients. (See Theorem 3.1.2.) Stanley [18] generalized this to face lattices of convex polytopes, which permitted him to conclude

Theorem 3.0.8 *(Stanley) Let L_P be lattice of faces of a convex polytope P, where the rank of L_P is $n+1$. For arbitrary rank-selections $S \subseteq [n]$ of L_P, $|\mu(L_P(S))|$ attains a maximum when we take S to be*

$$S = \{1,3,5,\ldots\} \cap [n] \text{ or } S = \{2,4,6,\ldots\} \cap [n].$$

In this chapter we reconstruct Stanley's proof in the case of the n-octahedron and show that these are the only S which maximize μ for O_n.

Theorem 3.0.9 *For arbitrary rank-selections $S \subseteq [n]$ of O_n, $|\mu(O_n(S))|$ attains a unique maximum when we take S to be*

$$S = \{1,3,5,\ldots\} \cap [n] \text{ or } S = \{2,4,6,\ldots\} \cap [n].$$

In this case $|\mu(O_n(S)| = E_n^{\pm}$, the nth signed Euler number.

3.1 The *ab*-index and the *cd*-index

This section serves as a brief introduction to the *ab*-index and the *cd*-index. We will follow [18] for all notation and terminology related to the *cd*-index. For those interested in studying the *cd*-index's origins, we refer to Bayer and Klapper's paper [4].

We will begin by describing Bayer and Billera's work on flag h-vectors of Eulerian posets. A key observation due to Fine links the *ab*-index with the *cd*-index. Once we review some algebraic properties of the *cd*-index, we summarize Purtill's dissertation work. For us, his most important results are the non-negativity of the coefficients of the *cd*-index of B_n and O_n, and a recurrence for each of their *cd*-indexes.

Let P be a finite, graded poset of rank $n+1$ that is bounded. Recall in chapter 2 that for $S \subseteq [n]$ we defined $\alpha(S) = \alpha_P(S)$, the number of maximal chains of

$P(S) \cup \{\hat{0}, \hat{1}\}$, and $\beta(S) = \beta_P(S)$, the beta invariant. This defines $\alpha : 2^n \to Z$, the ***flag f-vector*** of P, by

$$S \mapsto \alpha(S),$$

and $\beta : 2^n \to Z$, the ***flag h-vector*** of P, by

$$S \mapsto \beta(S).$$

We will now encode the flag h-vector (equivalently, the flag f-vector) of the poset P. First, define a monomial in the noncommutative variables a and b by

$$u_S = u_1 \cdots u_n,$$

where

$$u_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

(For our purposes, it will be helpful to think of a as "ascent" and b as "descent".) As an example, if $n = 5$ and $S = \{1, 4, 5\}$, then $u_S = baabb$. We form a non-commutative polynomial, called the ab-index, by

$$\Psi_P(a, b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

We define the degree of both a and b be 1 so that $\Psi_P(a, b)$ is homogeneous of degree n.

When P is an Eulerian poset (refer to [17, Chapter 3] for terminology), Bayer and Billera [3] showed the flag h-vector β_P satisfies certain linear relations called the generalized Dehn-Sommerville equations. In the literature these equations are also referred to as the Bayer-Billera relations. Fine observed that having β_P satisfy the Bayer-Billera relations is equivalent to having the ab-index contained in the algebra generated by the two elements $c = a + b$ and $d = ab + ba$.

Proposition 3.1.1 *[4, Theorem 4] (Fine) Let P be a finite graded poset that is also bounded. Then the flag h-vector β_P satisfies the Bayer-Billera relations if and only if $\Psi_P(a,b)$ can be written as a polynomial $\Phi_P(c,d)$ in the noncommutative variables $c = a+b$ and $d = ab+ba$.*

We call the polynomial $\Phi_P(c,d)$ the *cd-index* of P.

As noted in [18] the *cd*-index has some very nice algebraic properties. If $\Phi_P(c,d)$ exists (which it does for Eulerian posets such as B_n and O_n), then it is unique. (As noncommutative polynomials over any field K, $c = a+b$ and $d = ab+ba$ are algebraically independent.) If we define the degree of c to be 1 and the degree of d to be 2, then $\Phi_P(c,d)$ is homogeneous of degree n with integer coefficients. By the very nature of the variables $c = a+b$ and $d = ab+ba$, if $\Phi_P(c,d)$ exists then its associated *ab*-index $\Psi_P(a,b)$ is symmetric in a and b.

In his dissertation, Purtill proved the following result about the *cd*-index. For shorthand we will write $\Phi(P)$ for $\Phi_P(c,d)$.

Theorem 3.1.2 *[14, Theorem 6.1] For B_n and O_n, its cd-index $\Phi(B_n)$, respectively $\Phi(O_n)$, has non-negative coefficients.*

He also established recurrences for $\Phi(B_n)$ and $\Phi(O_n)$.

Proposition 3.1.3 *[14, Corollary 5.8] The following recurrence holds for the cd-index of the Boolean algebra:*

$$\Phi(B_{n+2}) = c\Phi(B_{n+1}) + \sum_{j=1}^{n} \binom{n}{j} \Phi(B_j) d \Phi(B_{n+1-j}), \quad n \geq 0,$$

with $\Phi(B_1) = 1$.

Proposition 3.1.4 *[14, Corollary 5.12] The following recurrence holds for the cd-index of the $n+1$-octahedron:*

$$\Phi(O_{n+1}) = c\Phi(O_n) + \sum_{j=1}^{n} 2^j \binom{n}{j} \Phi(B_j) d\Phi(O_{n-j}), \quad n \geq 0,$$

with $\Phi(O_0) = 1$.

Some values for $\Phi(B_n)$ and $\Phi(O_n)$ are given below:

$$\begin{aligned}
\Phi(B_1) &= 1 \\
\Phi(B_2) &= c \\
\Phi(B_3) &= c^2 + d \\
\Phi(B_4) &= c^3 + 2cd + 2dc \\
\Phi(B_5) &= c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2
\end{aligned}$$

$$\begin{aligned}
\Phi(O_0) &= 1 \\
\Phi(O_1) &= c \\
\Phi(O_2) &= c^2 + 2d \\
\Phi(O_3) &= c^3 + 6cd + 4dc \\
\Phi(O_4) &= c^4 + 14c^2d + 16cdc + 6dc^2 + 20d^2 \\
\Phi(O_5) &= c^5 + 30cdc^2 + 48c^2dc + 30c^3d + 100cd^2 + 64d^2c + 80dcd + 8dc^3
\end{aligned}$$

3.2 Arbitrary Rank-selections: L_P and O_n

In this section we will prove the arbitrary rank-selection result for face lattices of convex polytopes L_P. The results we will have to appeal to are very strong. Two such examples are Stanley's non-negativity of the coefficients of the cd-index for L_P and Björner's and Wachs' result that these face lattices are CL-shellable.

Before proving Theorem 3.0.8, we first verify some properties of ab-expansions of monomials in the variables c and d. When we formally substitute $c = a + b$ and $d = ab + ba$ into $\Phi(L_P)$ to recapture $\Psi(L_P)$, we will see the ab-words which receive the highest contribution from the coefficients of $\Phi(L_P)$ are the alternating ones: the *alternating initial ascent word* of length n, $\underbrace{ababab\ldots}_{n}$ and the *alternating initial descent word* of length n, $\underbrace{bababa\ldots}_{n}$. This conclusion requires Stanley's result about the non-negativity of the coefficients of $\Phi(L_P)$.

Since L_P is CL-shellable (Theorem 2.1.5), the coefficient β_S of a given word u_S in $\Psi_{L_P}(a,b)$ equals $|\mu(L_P(S))|$. Hence, we will have shown the rank-selections

$$S = \{1,3,5,\ldots\} \cap [n] \text{ and } S = \{2,4,6,\ldots\} \cap [n]$$

maximize $|\mu(L_P)|$.

We are able to conclude that the odd and even alternating rank-selections from L_P each maximize the Möbius function for the arbitrary rank-selection question. What we are not able to conclude at this time is a uniqueness result, i.e., that the alternating rank-selections are the only extremal configuration. We can, however, show uniqueness when $L_P = O_n$ via the recurrence for $\Phi(O_n)$ in Proposition 3.1.4. We also include a conjecture of Stanley which, if true, implies the uniqueness of the extremal configuration in Theorem 3.0.8 for any L_P.

We begin by defining a *convex polytope* P (or *polytope*) to be the convex hull of a non-empty finite set of points in Euclidean space. For instance, the n-dimensional octahedron is a convex polytope since it is the convex hull of the $2n$ points

$$\{\pm e_1, \ldots, \pm e_n\}$$

in n-dimensional Euclidean space, where

$$e_i = (\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{n-i}).$$

A *polyhedral complex* Δ (not to be confused with the shadow $\Delta(S)$ defined in chapter 1) is finite set of polytopes $\{\mathcal{P}_1, \ldots, \mathcal{P}_j\}$ in Euclidean space such that

(*i*) any face belonging to some $\mathcal{P}_i$ is a member of Δ

(*ii*) the intersection of any two members of Δ is again a member of Δ.

The maximal faces of a polyhedral complex are called *facets.* We say Δ is an *n-complex* if the dimension of all the facets of Δ is n. Finally, for an n-complex Δ we form its *face lattice* L_Δ by ordering the faces of Δ by inclusion and adjoining a maximal element $\hat{1}$. For example, the dimension of all the facets of the n-dimensional octahedron $\mathcal{O}_n$ is $n-1$. Thus, $L_{\mathcal{O}_n} = O_n$, the n-octahedron.

We next make some simple observations about monomials in the variables c and d. If $w = w_1 \cdots w_n$ is an ab-word of degree n, then we say w has a *double ascent at position* i if $w_i = w_{i+1} = a$ and has a *double descent at position* i if $w_i = w_{i+1} = b$.

Lemma 3.2.1 *Given any monomial w of degree n in the noncommutative variables $c = a + b$ and $d = ab + ba$, its ab-expansion satisfies*

(i) the alternating initial ascent word of length n occurs exactly once

(ii) the alternating initial descent word of length n occurs exactly once

(iii) any given ab-word of length n occurs at most once.

(iv) If $w = w'dw''$ with deg $w' = i - 1$ then its ab-expansion contains no ab-word with a double ascent or double descent at position i.

Proof. For properties (i) through (iii) we proceed by induction on the degree n of a monomial in c and d. By convention for $n = 0$ the only monomial in c and d is 1.

Now let w be a monomial in the variables c and d of degree $n \geq 1$. We have two cases to consider, depending upon the first element of w. If w begins with c, then remove this initial c. This leaves a monomial w' in the variables c and d of degree $n - 1$. By the induction hypothesis the ab-expansion of w' yields exactly one alternating initial ascent word of length $n-1$ and one alternating initial descent word of length $n - 1$. All the other possible ab-words of length $n - 1$ occur at most once.

Premultiply all of these monomials by $c = a+b$. This yields exactly one alternating initial ascent word of length n and one alternating initial descent word of length n. All the other possible length n monomials in a and b occur at most once. If not, then deleting the initial a or b from two duplicate monomials in a and b of length n would give two duplicate length $n-1$ monomials in a and b arising from the same cd-word of degree $n-1$. This would contradict the induction hypothesis.

In a similar fashion, we show the same result for w a monomial in the variables c and d of degree $n > 1$ whose first element is d. The difference here is we will have to apply the induction hypothesis for words in the variables c and d of degree $n-2$.

Remove the initial d from w. This leaves a monomial w' in the variables c and d of degree $n-2$. By the induction hypothesis for $n-2$ the ab-expansion of w' yields exactly one alternating initial ascent word of length $n-2$ and one alternating initial descent word of length $n-2$. All other possible ab-words of length $n-2$ occur at most once.

Now, premultiply all of these monomials by $d = ab + ba$. This gives exactly one alternating initial ascent word of length n and one alternating initial descent word of length n. All other possible length n ab-monomials occur at most once. If not, then delete the initial ab or ba from two of the duplicate ab-monomials of length n. This gives two duplicate length $n-2$ monomials in a and b arising from some word in c and d of degree $n-2$, contradicting the induction hypothesis.

For (iv), suppose $w = w'dw''$ with $deg\ w' = i-1$. Since the degree of w' is $i-1$, the d (of degree 2) which follows w' will contribute to the ith and $i+1$st positions of the ab-words formed from the ab-expansion of w. This element d can only contribute ab or ba. Hence, the ab-expansion of w contains no ab-word with a double ascent or double descent at the ith position. $\square$

The uniqueness result in Theorem 3.0.9 requires two key observations about $\Phi(O_n)$.

Lemma 3.2.2 *The following two properties hold for $\Phi(O_n)$:*

(i) all possible monomials of degree n in the variables c and d occur at least once in $\Phi(O_n)$

(ii) for $n \geq 0$ if m is some monomial of length n in the variables a and b that is not alternating, then there exists a monomial w in $\Phi(O_n)$ whose ab-expansion makes no contribution to m.

Proof. We first show (i) holds by induction on n. Property (i) is vacuously true for $n = 0$ since $\Phi(O_0) = 1$.

Now assume $n \geq 1$. By the recurrence of Proposition 3.1.4

$$\Phi(O_n) = c\Phi(O_{n-1}) + \sum_{j=1}^{n-1} 2^j \binom{n-1}{j} \Phi(B_j) d \Phi(O_{n-j-1}), \ n \geq 1, \tag{3.3}$$

with $\Phi(O_0) = 1$. Suppose w is a monomial of degree n in the variables c and d. If w begins with the variable c, remove it to form the monomial w' of degree $n-1$. By the induction hypothesis, w' appears at least once in $\Phi(O_{n-1})$. Hence the term $c\Phi(O_{n-1})$ appearing in equation (3.3) contains the monomial $w = cw'$. We do not have to worry about any possible cancellation since the coefficients of $\Phi(O_n)$ are clearly non-negative from equation (3.3).

Next, let w be a monomial in c and d of degree n beginning with the letter d. We similarly use the first term of the second summand appearing in equation (3.3) to verify property (i) for w. Remove d from w to form w', a monomial of degree $n-2$. By the induction hypothesis w' appears at least once in $\Phi(O_{n-2})$. Also $\Phi(B_1) = 1$. Hence the term in the summand corresponding to $j = 1$ in equation (3.3) creates a $w = 1dw'$ for $\Phi(O_n)$.

Property (ii) is vacuously true when $n = 0$ since there are no non-alternating words of length 1 in the variables a and b. For $n \geq 1$, (ii) will follow immediately from part (i) of this lemma and from (iv) of Lemma 3.2.1. Let m be a monomial of length n in the variables a and b that is not alternating. Then m has a double ascent or double descent. Without loss of generality we may assume m has a double ascent aa at the ith position. (The same argument works when m has a double descent bb at the ith position.) By property (i) we know every monomial of degree n in the variables c and d occurs at least once in $\Phi(O_n)$. In particular, monomials of the form $w = w'dw''$ with $deg\ w' = i-1$ occur at least once. By Lemma 3.2.1 the ab-expansion of any such w makes no contribution to m. □

We have an analogous version of Lemma 3.2.2 for B_n. Its proof is omitted since it is virtually identical to that of Lemma 3.2.2.

Lemma 3.2.3 *The following two properties hold for $\Phi(B_n)$:*

(i) *all possible monomials of degree $n-1$ in the variables c and d occur at least once in $\Phi(B_n)$*

(ii) *for $n \geq 1$ if m is some monomial of length $n-1$ in the variables a and b that is not alternating, then there exists a monomial w in $\Phi(B_n)$ whose ab-expansion makes no contribution to m.* □

It is well-known that the *nth Euler number* E_n counts the total number of alternating permutations in the symmetric group with odd (or even) descent set, i.e.

$$
\begin{aligned}
E_n &= |\{\pi \in S_n : \ D(\pi) = \{1,3,5,\ldots\} \cap [n-1]\}| && (3.4)\\
&= |\{\pi \in S_n : \ D(\pi) = \{2,4,6,\ldots\} \cap [n-1]\}|. && (3.5)
\end{aligned}
$$

A recurrence for the nth Euler numbers is

$$E_{n+2} = \sum_{j=0}^{n} \binom{n}{j} E_j E_{n-j+1}, \quad n \geq 1,$$

with $E_0 = E_1 = 1$. In a similar manner, Purtill defines the *nth signed Euler number* $E_n^{\pm}$ which counts the total number of augmented signed permutations that are alternating with odd (or even) descent set:

$$\begin{aligned} E_n^{\pm} &= |\{\pi \in S_n^{\pm} : \ D(\pi) = \{1,3,5,\ldots\} \cap [n]\}| && (3.6) \\ &= |\{\pi \in S_n^{\pm} : \ D(\pi) = \{2,4,6,\ldots\} \cap [n]\}|. && (3.7) \end{aligned}$$

A recurrence for the nth signed Euler numbers is

$$E_{n+1}^{\pm} = \sum_{j=0}^{n} 2^j \binom{n}{j} E_j E_{n-j}^{\pm}, \quad n \geq 1,$$

with $E_0^{\pm} = E_1^{\pm} = 1$. (See [14, section 6].)

From Lemmas 3.2.1 and 3.2.2, we can now very easily give the proofs of the arbitrary rank-selection theorems for L_P and O_n. We first consider the arbitrary rank-selection question for L_P, and then prove the uniqueness result for O_n. It is a well-known fact that L_P is Eulerian. Thus, $\Phi(L_P)$ exists by Bayer and Billera's work in [3] and Fine's equivalence in Proposition 3.1.1. By Stanley [18, Theorem 3.1.2], the coefficients of $\Phi(L_P)$ are non-negative.

Consider the ab-expansion of $\Phi(L_P)$ into $\Psi(L_P)$. By Lemma 3.2.1 each monomial in $\Phi(L_P)$ contributes exactly one term to the alternating initial ascent word of length n, exactly one term to the alternating initial descent word of length n, and at most one term to all other words of length n in the variables a and b. Thus, the coefficient of the alternating permutations of length n with initial ascent (equivalently, with initial descent) equals the number of monomials appearing in $\Phi(L_P)$. In turn, this number is just the sum of all the coefficients of $\Phi(L_P)$, i.e., $\Phi_{L_P}(1,1)$. (Here we are formally substituting $c = 1$ and $d = 1$ in $\Phi_{L_P}(c,d)$.)

For an *ab*-word u_S, $S \subseteq [n]$, we define

$$D(u_S) = S.$$

(Recall that when we introduced u_S, we said it would be helpful to think of the a's as ascents and the b's as descents.) Since L_P is CL-shellable (Theorem 2.1.5), the coefficient of any *ab*-word u_S in $\Psi(L_P)$ is $|\mu(L_P(S))|$.

We have shown the alternating *ab*-words of length n receive the greatest contribution from the coefficients of $\Phi(L_P)$. In fact, they receive the maximum possible contribution, since the expansion of each monomial in $\Phi(L_P)$ contributes at most one to each *ab*-word. Thus, the extremal configuration for rank-selections $S \subseteq [n]$ from L_P is

$$S = \{1,3,5,\ldots\} \cap [n] \text{ or } S = \{2,4,6,\ldots\} \cap [n].$$

This is Theorem 3.0.8.

We next show that for O_n the odd and even alternating rank-selections are the only extremal configuration. Suppose m is not an alternating word in the variables a and b. By property (ii) of Lemma 3.2.2 there exists a monomial w in $\Phi(O_n)$ whose *ab*-expansion does not make any contribution to m. Thus the coefficient of m in $\Psi(O_n)$ is less than that of the alternating initial ascent (or descent) *ab*-word of length n. $\square$

As a comment, we could use Lemma 3.2.3, the B_n counterpart of Lemma 3.2.2, in the argument just given to reprove Theorem 3.0.7. Also, for the proof of Theorem 3.0.9 we did not have to appeal to Stanley's non-negativity of the coefficients of $\Phi(L_p)$ and the CL-shellability of L_p. Instead, we could have used Purtill's non-negativity of the coefficients of $\Phi(O_n)$ and the fact O_n has an R-labeling.

In a personal communication with Stanley we brought to his attention that we cannot conclude uniqueness of the extremal configuration for L_P other than the cases when L_P is B_n or O_n. Consequently, Stanley has made the following conjecture:

Conjecture 3.2.4 *[19] Among all n-polytopes (or more generally Gorenstein* lattices of rank $n+1$), the simplex (i.e., the Boolean algebra) has the least cd-index coefficient-wise.*

If Stanley's conjecture is true, it would immediately imply the uniqueness result for Theorem 3.0.8: we would know that every *cd*-word occurs at least once in $\Phi(L_P)$, since every *cd*-word occurs at least once in $\Phi(B_n)$ with positive coefficients. This would give an L_P counterpart of part (i) of Lemma 3.2.2. Moreover, since part (ii) of Lemma 3.2.2 follows from part (i) of the same lemma and from Lemma 3.2.1, Lemma 3.2.2 would hold for any L_P, not just O_n and B_n.

In the table that follows, we include the Mōbius values of the extremal configuration for Theorem 3.0.9.

Table 3.1: Arbitrary Rank-selection from O_n: Maximum Möbius Value for $n = 1, \ldots, 10$

n	$E_n^{\pm}$
1	1
2	3
3	11
4	57
5	361
6	2763
7	24611
8	250737
9	2873041
10	36581523

Chapter 4

$L_n(q)$; Related Extremal Questions

In the first section of this chapter we address extremal problems for $L_n(q)$, the lattice of subspaces of an n-dimensional vector space over the finite field GF_q. For the lower order ideal case, Sagan, Yeh and Ziegler found the entire poset $L_n(q)$ is the extremal configuration. Their argument uses techniques analogous to the ones they developed for B_n. We determine that all of the ranks of the poset $L_n(q)$ is also the extremal configuration for its the interval of ranks and arbitrary rank-selection cases. As usual in such cases, we use the inversion statistic.

The second section of this chapter describes work-in-progress related to questions addressed in this dissertation.

4.1 $L_n(q)$: Interval of Ranks and Arbitrary Rank-selections

For $S \subseteq [n-1]$, we use the notation

$$(L_n(q), S) = \{W \in L_n(q) : \ \dim W \in S\}$$

to indicate the S rank-selection from $L_n(q)$. For $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define

$$\text{inv } \pi = |\ \{(i,j) : \ \pi_i > \pi_j \text{ and } i < j\}\ |.$$

For example, if $\pi = 31524 \in S_5$, then inv $\pi = 5$.

In chapter 2 we saw that we could translate questions involving the Möbius function of rank-selections S from B_n and O_n into questions of studying certain permutations with descent set S. For S rank-selections from $L_n(q)$ there is a way to compute $|\mu(L_n(q), S)|$ in terms of the inversion statistic.

Proposition 4.1.1 *[17, Theorem 3.12.3] Let $L_n(q)$ be the lattice of subspaces of an n-dimensional vector space over GF_q and let $S \subseteq [n-1]$. Then*

$$|\mu(L_n(q), S)| = \sum_{\pi \in S_n, D(\pi)=S} q^{inv\,\pi}.$$

We will use this interpretation of S rank-selections from $L_n(q)$ to establish its interval of rank and arbitrary rank-selection results.

Theorem 4.1.2 *For sufficiently large q and arbitrary rank-selections $S \subseteq [n-1]$ from $L_n(q)$, $|\mu(L_n(q), S)|$ attains a maximum when we take $S = [n-1]$. In other words, for large enough values of q,*

$$|\mu(L_n(q), S)| \leq q^{\binom{n}{2}}$$

with equality holding if and only if $S = [n-1]$.

Since the extremal configuration in Theorem 4.1.2 is an instance of an interval rank-selection, we have the immediate corollary:

Corollary 4.1.3 *For interval of rank-selections $S \subseteq [n-1]$ from $L_n(q)$, the extremal configuration coincides with the one in Theorem 4.1.2.*

The proof of Theorem 4.1.2 will follow once we establish a simple result about inversions of permutations in S_n.

Proposition 4.1.4 *For $\pi \in S_n$, we have the following bound*

$$inv\ \pi \leq \binom{n}{2}.$$

Furthermore, $\pi = n\ n-1 \cdots 2\ 1$ is the unique permutation in S_n with inv $\pi = \binom{n}{2}$.

Proof. For any $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, we have the following bound on the maximum number of inversions that the ith element $\sigma_i \in \sigma$ can contribute to inv σ:

$$|\{j: \ \sigma_i > \sigma_j \text{ and } i < j\}| \leq n - i. \tag{4.1}$$

Hence, the greatest number of possible inversions any $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$ could have is

$$\sum_{i=1}^{n} n - i = \binom{n}{2}.$$

Suppose inv $\pi = \binom{n}{2}$, where $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$. Thus for $1 \leq i \leq n$, π_i must attain the maximum possible number of inversions described in equation (4.1). When $i = 1$ equation (4.1) forces $\pi_1 = n$. The letters $[n-1]$ are now left, so when $i = 2$ equation (4.1) forces $\pi_2 = n - 1$. Continuing in this fashion, we see that

$$\pi_i = n - i + 1, \quad 1 \leq i \leq n. \ \square$$

Theorem 4.1.2 now follows immediately from Proposition 4.1.4. $|\mu(L_n(q), S)|$ is a polynomial in q with degree at most $\binom{n}{2}$. The degree equals $\binom{n}{2}$ only when $S = [n-1]$. There is only one such permutation $\pi \in S_n$ with $D(\pi) = \binom{n}{2}$, so we are done.

A more precise statement of what is meant by the phrase "large-enough q" in Theorem 4.1.2 is given in the next proposition.

Proposition 4.1.5 *The maximum in Theorem 4.1.2 is guaranteed to hold if $q \geq E_n$, where E_n is the nth Euler number.*

Proof. We have for $S \neq [n-1]$

$$|\mu(L_n(q), S)| = \sum_{i=0}^{\binom{n}{2}-1} c_i q^i, \tag{4.2}$$

where

$$c_i = |\ \{\pi \in S_n : \ D(\pi) = S, \text{ inv } \pi = i\}\ |.$$

Taking the most naive upper bound for each q^i appearing in equation (4.2) gives

$$\begin{aligned}
|\mu(L_n(q), S)| &\leq \sum_{i=0}^{\binom{n}{2}-1} c_i q^{\binom{n}{2}-1} \\
&= q^{\binom{n}{2}-1} \sum_{i=0}^{\binom{n}{2}-1} c_i \\
&= q^{\binom{n}{2}-1} |\ \{\pi \in S_n : \ D(\pi) = S\}\ | \\
&\leq q^{\binom{n}{2}-1}\, E_n \qquad (4.3) \\
&\leq q^{\binom{n}{2}}, \qquad (4.4)
\end{aligned}$$

where the inequality in equation (4.4) holds if $q \geq E_n$. The bound in equation (4.3) arises since the greatest number of permutations in S_n with the same descent set is E_n [11].

To see how good the estimate is in Proposition 4.1.5, we will need the following proposition:

Proposition 4.1.6 *For $S \subseteq [n-1]$ and $|S| = n-2$, $|\mu(L_n(q), S)|$ is of the form*

$$|\mu(L_n(q), S)| = q^{\binom{n}{2}-1} + \mathcal{O}(q^{\binom{n}{2}-2}) \quad (as\ q \to \infty). \tag{4.5}$$

Conversely, if for $S \subseteq [n-1]$ equation (4.5) holds, then $|S| = n-2$.

Proof. For the forward direction, it suffices to show there exists exactly one permutation $\pi \in S_n$ with inv $\pi = \binom{n}{2} - 1$ and with $D(\pi) = S$, where $S \subseteq [n-1], |S| = n-2$ is given. First suppose $S = [n-1] \setminus \{i\}$ with $i \neq 1$. Let $\pi = \pi_1 \cdots \pi_n \in S_n$ with $D(\pi) = S$. Thus

$$\pi_1 > \cdots > \pi_i < \pi_{i+1} > \pi_{i+2} > \cdots > \pi_n,$$

so either $\pi_1 = n$ or $\pi_{i+1} = n$. If $\pi_{i+1} = n$, then π_1 can contribute at most $n-2$ to the total number of inversions. Since $\pi_i < \pi_{i+1}$, π_i can contribute at most $n-i-1$ to the total number of inversions of π. By equation (4.1), we cannot achieve inv $\pi = \binom{n}{2} - 1$, so we must have $\pi_1 = n$.

Under the new assumption that $\pi_1 = n$, π_i can still contribute at most $n - i - 1$ to the total number of inversions of π. In order to have inv $\pi = \binom{n}{2} - 1$, π_i must contribute exactly $n - i - 1$ to the total number of inversions, implying

$$\pi_i > \pi_{i+2} > \pi_{i+3} > \cdots > \pi_n.$$

Also, the other π_j, $j \neq i$, must contribute the maximum number of inversions to π, i.e., inv $\pi_j = n - j$, $j \neq i$. Hence

$$\pi_1 > \pi_2 > \cdots > \pi_{i-1} > \pi_{i+1}.$$

But $\pi_i < \pi_{i+1}$, so we have

$$\pi_1 > \cdots > \pi_{i-1} > \pi_{i+1} > \pi_i > \pi_{i+2} > \pi_{i+3} > \cdots > \pi_n,$$

implying

$$\pi_j = \begin{cases} n - j + 1 & \text{for } j \neq i, i+1 \\ n - i & \text{for } j = i \\ n - i + 1 & \text{for } j = i + 1. \end{cases}$$

Next suppose $S = [n-1] \setminus \{1\} = [2, n-1]$. For $\pi = \pi_1 \cdots \pi_n \in S_n$ with $D(\pi) = S$ we have

$$\pi_1 < \pi_2 > \pi_3 > \cdots > \pi_n,$$

implying $\pi_2 = n$. Thus π_2 contributes $n-2$ to the total number of inversions of π. Furthermore, suppose inv $\pi = \binom{n}{2} - 1$. Since $\pi_1 < \pi_2$, π_1 can contribute at most $n-2$ to the total number of inversions of π. In fact, by equation (4.1) π_1 must contribute exactly $n-2$ to the total number of inversions of π. Similarly, the remaining terms must contribute $n-j$ to the total number of inversions ($j \neq 1$). Thus we have

$$\pi_1 > \pi_3 > \cdots \pi_n,$$

implying

$$\pi_j = \begin{cases} n-1 & \text{for } j = 1 \\ n & \text{for } j = 2 \\ n-j+1, & \text{for } j = 3, \ldots, n. \end{cases}$$

The converse is clearly true for $n = 3$. Now suppose $n > 3$ and $S \subseteq [n-1]$ with $|\mu(L_n(q), S)|$ satisfying equation (4.5). Then there exists exactly one $\pi = \pi_1 \cdots \pi_n \in S_n$ with $D(\pi) = S$ and inv $\pi = \binom{n}{2} - 1$. Suppose on the contrary that $|S| < n-2$. The permutation π has at least two ascents, i.e. $S \subseteq [n-1] \backslash \{k, l\}$ for some $k \neq l \in [n-1]$. This means $\pi_k < \pi_{k+1}$ and $\pi_l < \pi_{l+1}$. But then

$$|\{j : \ \pi_k > \pi_j \text{ and } k < j\}| \leq n-k-1$$

and

$$|\{j : \ \pi_l > \pi_j \text{ and } l < j\}| \leq n-l-1.$$

Thus inv $\pi \leq \binom{n}{2} - 2$, contradicting the fact inv $\pi = \binom{n}{2} - 1$. Thus $|S| = n-2$. □

We also have a recurrence for the $|\mu(L_n(q), S)|$ which appear in Proposition 4.1.6. This recurrence enables us to easily evaluate $|\mu(L_n(q), S)|$, and hence compare the value of q which Theorem 4.1.2 is guaranteed to hold (Proposition 4.1.5) with the actual smallest value of q for which it holds.

Proposition 4.1.7 *For $S \subseteq [n-1]$, with $S = [n-1] \setminus \{i\}$ and $n \geq 3$, $|\mu(L_n(q), S)|$ satisfies the following recurrence:*

For $i \neq 1, n-1$,

$$\begin{aligned} |\mu(L_n(q), S)| &= q^{n-1}|\mu(L_{n-1}(q), [n-2] \setminus \{i-1\})| + \\ &\quad q^{n-i-1}(|\mu(L_{n-1}(q), [n-2])| + |\mu(L_{n-1}(q), [n-2] \setminus \{i\})|). \end{aligned} \tag{4.6}$$

For $i = 1$

$$|\mu(L_n(q), S)| = q^{n-2}(|\mu(L_{n-1}(q), [n-2])| + |\mu(L_{n-1}(q), [2, n-2])|), \tag{4.7}$$

and for $i = n-1$

$$|\mu(L_n(q), S)| = q^{n-1}|\mu(L_{n-1}(q), [n-3])| + |\mu(L_{n-1}(q), [n-2])|. \tag{4.8}$$

The boundary conditions are:

$$\begin{aligned} &|\mu(L_n(q), [n-1])| = q^{\binom{n}{2}} \\ &|\mu(L_n(q), \emptyset)| = 1. \end{aligned}$$

Proof. Suppose $n \geq 3$ and $S = [n-1] \setminus \{i\}$ with $i \neq 1, n-1$. For $\pi = \pi_1 \cdots \pi_n \in S_n$ with $D(\pi) = S$, either $\pi_1 = n$ or $\pi_{i+1} = n$. Thus

$$|\mu(L_n(q), S)| = \sum_{\substack{\pi \in S_n, D(\pi) = S \\ \pi_1 = n}} q^{inv\,\pi} + \sum_{\substack{\pi \in S_n, D(\pi) = S \\ \pi_{i+1} = n}} q^{inv\,\pi}. \tag{4.9}$$

Deleting n from π gives

$$\begin{aligned} |\mu(L_n(q), S)| &= q^{n-1}|\mu(L_{n-1}(q), [n-2] \setminus \{i-1\})| + \\ &\quad q^{n-i-1}(|\mu(L_{n-1}(q), [n-2])| + |\mu(L_{n-1}(q), [n-2] \setminus \{i\})|). \end{aligned}$$

For $i = 1$ the first summand on the right side of equation (4.9) is zero, so by virtually the same argument we obtain the recurrence in equation (4.7). Similarly, for $i = n-1$

Table 4.1: Comparison of the Predicted q with the Actual q, $n = 1, \ldots, 10$

n	Predicted $q \geq E_n$	Actual q
3	2	2
4	5	2
5	16	3
6	61	3
7	272	3
8	1385	3
9	7936	3
10	50521	3

we note the second summand on the right side of equation (4.9) will only contribute one term when we delete n from π, so we also obtain equation (4.8). □

In Table 4.1 we compare the predicted q so that Theorem 4.1.2 is guaranteed to hold with the actual q. As one can see, there is room to improve the estimate in Proposition 4.1.5.

4.2 Related Extremal Questions

We are currently focusing upon four main extremal problems in our research. What follows is a brief description of each.

I. Classification of Posets with Rank-selected Extremal Configuration

For the Boolean algebra and the n-octahedron, the Mōbius function attains a maximum over lower order ideals by taking a certain rank-selected lower order ideal. (See [15] and Theorem 1.0.1). Given the results for these two posets, we would like to address a much more difficult question: identify those posets whose extremal configuration maximizing the Mōbius function over lower order ideals corresponds to a rank-selected lower order ideal. Since the Boolean algebra and the n-octahedron are examples of face lattices of convex polytopes, we expect the face lattices of convex polytopes to naturally belong to this class of posets.

II. New Applications of the *cd*-index

The *cd*-index is a relatively new object of study. As we saw in chapter 3, for certain posets the *cd*-index encodes information about the Möbius function of rank-selections. The answer to the question of maximizing the Möbius function over arbitrary rank-selections from the n-octahedron (and more generally, from any face lattice of a convex polytope) was an easy consequence of recent work of Purtill and Stanley on the coefficients of the *cd*-index. We are looking for other results for O_n and B_n which the *cd*-index answers quite naturally. Also, we would like to prove Stanley's Conjecture 3.2.4 or find a counterexample.

III. Extremal Questions for Other Posets

The q-analog of the Boolean algebra B_n is $L_n(q)$, so it would make sense for us to study the q-analog of O_n. This poset is $I_n(q)$, the poset of isotropic subspaces. For $I_n(q)$ we plan to address extremal questions analogous to the ones studied in this dissertation.

IV. Permutations and Their Inverses

One can study the absolute value of the Möbius function of rank-selected subposets from the Boolean algebra in terms of permutations in the symmetric group. A related question posed by Ira Gessel is: for fixed n let $f_n(A, B)$ be the number of permutations in the symmetric group on n elements with descent set A such that the descent set of the inverse is B. For what A and B is $f_n(A, B)$ the greatest? The data we have looked at suggests that A and B are "alternating". We hope to prove this result by applying an algorithmic technique similar to one developed by de Bruijn [11].

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