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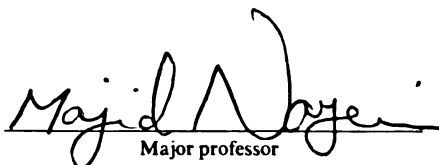
Development and Analysis of An Optimal
Bounding Ellipsoid Algorithm Using
Stochastic Approximation

presented by

Ming-Shou Liu

has been accepted towards fulfillment
of the requirements for

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**DEVELOPMENT AND ANALYSIS OF AN
OPTIMAL BOUNDING ELLIPSOID ALGORITHM
USING
STOCHASTIC APPROXIMATION**

By

Ming-Shou Liu

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ABSTRACT

DEVELOPMENT AND ANALYSIS OF AN OPTIMAL BOUNDING ELLIPSOID ALGORITHM USING STOCHASTIC APPROXIMATION

By

Ming-Shou Liu

Set-membership-based (SM) techniques provide an alternative to conventional system identification algorithms. This research is concerned with a class of SM algorithms known as the *optimal bounding ellipsoid* (OBE) algorithms which combines the weighted least square error minimization approach with a bounded error constraint. More specifically, OBE algorithms focus on finding an optimal hyper-ellipsoidal region for the feasible set when the additive model noise is bounded. Previous work has shown that OBE techniques have great potential in many applications involving parametric modelling.

The published OBE algorithms, however, either do not include a formal convergence proof or do not show interpretable geometrical meaning. In this research, a modified OBE algorithm, which is both interpretable and convergent, is introduced. This algorithm is referred to as *set-membership stochastic approximation* (SM-SA) algorithm because of its resemblance to the stochastic approximation (SA) method developed by Robbins and Monro in 1951.

Recently, a general class of OBE algorithms including all methods published to date, is unified into a single framework called the *unified OBE* (UOBE) algorithm. UOBE is based on generalized weighted recursive least squares in which very broad classes of “forgetting factors” and data weights may be employed. Different instances of UOBE are distinguished

by their weighting policies and the criteria for determining optimal weight values. SM-SA is a special case of UOBE, and, through the analysis of this research and the paper by Deller, Nayeri, and Krunz, the convergence properties of SM-SA, which will be introduced later, can be applied to the UOBE framework.

In this dissertation, the updating criteria and optimal weights for shrinking feasible sets for SM-SA are first analyzed. Then asymptotic properties and performance of SM-SA are discussed. Simulation results for different SM algorithms and comparisons to conventional recursive least squares are also examined. Finally, the UOBE formulation and its convergence properties are introduced.

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Ming-Shou Liu
August 3, 1993

*To my Mother and my wife
For their Love, Support, and Sacrifice*

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Acronyms and Abbreviations

Auto-regressive with exogenous input (ARX)

Auto-regressive-moving-average with exogenous input (ARMAX)

Multiple-input–multiple-output (MIMO)

Optimal bounding ellipsoid (OBE)

Optimal volume ellipsoid (OVE)

Recursive least squares (RLS)

Set-membership (SM)

Set-membership stochastic approximation (SM-SA)

Set-membership weighted recursive least squares (SM-WRLS)

Single-input–single-output (SISO)

Stochastic approximation (SA)

Unified optimal bounding ellipsoid (UOBE)

Weighted recursive least squares (WRLS)

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Chapter 1

Introduction and Background

“Man creates models of his natural and man-made environments to understand and explain them better and as prelude to action” [8]. Indeed, from social science to engineering and natural science, modeling plays an important roll to help humankind describe functions of processes and to predict future actions. For example, in economics, modeling is used to study inflation, to predict the trends of stock markets, and to maximize profits on investments. In engineering applications, modeling can be used to analyze the characteristics of a process, optimize the performance, and study the functions and dynamics before a large plant is actually built and large amounts of money and effort are invested. In many branches of science, modeling is used as an aid to describe and understand reality.

The term “system identification” generally refers to the determination of a mathematical model for a system or a process by observing its input-output relationships. Based on the model structures that are used, system identification methods are generally classified into two categories: Nonparametric and parametric. A nonparametric model is described by a curve, function, or table. Impulse responses and frequency diagrams are two examples of a nonparametric model. In this dissertation, we will consider the parametric model only. Such models are characterized by a parametric vector. Typical parametric models are *auto-regressive with exogenous input* (ARX), *auto-regressive-moving-average with exogenous input* (ARMAX) [16], and state-space structures.

For modeling engineering systems and industrial processes in practice, some basic characteristics of the system, such as linearity and stability, are assumed to be known. Further-

more, in many cases, we assume to have basic knowledge about the mathematical structure of the system such as the order of the system, the type of the model, and so on, so that it is possible to derive a specific mathematical model of the system dynamics. Consequently, only a set of parameters in the model equation are left to be determined. In this research, we derive a new algorithm for identifying model parameters which has well-analyzed convergence properties and an intuitively meaningful optimization criterion. This algorithm is applicable when the system is corrupted by additive and *bounded* noise.

1.1 Real-Time Identification and the Recursive Least Square Estimator

In classical system design, modeling is accomplished once during the design stage. This is called off-line identification. In most adaptive systems, on-line or recursive identification is desirable to allow optimum adaptation in the face of environmental uncertainty and changing environmental conditions.

From computational point of view, the memory requirement for recursive estimators is quite modest since not all the data are stored. In off-line identification, however, all the previous data have to be memorized.

A variety of well-known recursive techniques such as *least square error* (LSE), maximum likelihood, instrumental variables [16], and stochastic approximation [1] have been used to identify system parameters in real time. Among these techniques, the LSE method is most popular because of its simplicity and well-understood criteria.

To review the LSE method, consider a simple linear regression model

$$y(t) = \mathbf{x}^T(t)\boldsymbol{\theta}$$

where $\mathbf{x}(t)$ is an n -vector of known quantities and $\boldsymbol{\theta}$ is an n -vector of unknown parameters. The LSE estimate of $\boldsymbol{\theta}$ at time n is obtained by minimizing the cost function

$$V(\hat{\boldsymbol{\theta}}) = \frac{1}{2} \sum_{t=1}^n \left(y(t) - \mathbf{x}^T \hat{\boldsymbol{\theta}} \right)^2$$

Recursive least squares (RLS) [16] estimates are obtained through an on-line iterative computation of the off-line LSE estimates. *Weighted recursive least square* (WRLS) is a variation of RLS which minimizes

$$V(\hat{\theta}) = \frac{1}{2} \sum_{t=1}^n w_t \left(y(t) - \mathbf{x}^T \hat{\theta} \right)^2$$

where $\{w_n\}$ is a weighting sequence.

1.2 Introduction to Stochastic Approximation

Stochastic approximation algorithm is concerned with the problem of estimating the parameters of a linear system by using the noisy observations.

Let $f : \mathbb{R}^l \rightarrow \mathbb{R}^m$ be an unknown function with a unique zero $\mathbf{a}$, i.e.

$$f(\mathbf{a}) = 0 \tag{1.1}$$

We assume that $f(\cdot)$ can be observed at any $\mathbf{x} \in \mathbb{R}^l$. *Stochastic approximation* (SA) refers to a general class of recursive algorithms for estimating an equation of the form (1.1) when the observation $f(\cdot)$ is corrupted by random noise.

The origins of SA can be traced back to a paper by Robbins and Monro [1] in 1951. This paper is concerned with the problem of estimating the roots, say $\mathbf{a}$, of an unknown regression function $f(\mathbf{x})$, which is the mean value of an observable random variable $Z(\mathbf{x})$. Robbins and Monro prove that $\mathbf{a}$ can be estimated recursively by

$$\hat{\mathbf{a}}_{n+1} = \hat{\mathbf{a}}_n + l_n Z(\hat{\mathbf{a}}_n) \tag{1.2}$$

where l_n is a sequence of positive scalars that converges to zero. The reason why $\{l_n\}$ must converge to zero is because convergence of $\hat{\mathbf{a}}_n$ can take place only if the product $l_n Z(\hat{\mathbf{a}}_n) = 0$. Since $Z(\hat{\mathbf{a}}_n) \neq 0$ with probability one, l_n must be zero. This ensures that the noise is rejected by paying less and less attention to the noisy observations.

Robbins and Monro state the sufficient conditions for convergence of the estimate $\hat{\mathbf{a}}$ to

the true parameter in probability [1]:

$$\begin{cases} l_n \geq 0 \\ \sum_{n=1}^{\infty} l_n = \infty \\ \sum_{n=1}^{\infty} (l_n)^2 < \infty. \end{cases} \quad (1.3)$$

Any sequence $\{l_n\}_{n=0}^{\infty}$ which satisfies (1.3) is called a *sequence of type 1/n*. A common choice for the sequence $\{l_n\}$ is $l_n = 1/n$. Ljung [12] further weakened the second condition in (1.3) to

$$\sum_{n=1}^{\infty} (l_n)^p < \infty$$

for some $p > 1$. Blum [2] (1954) extended Robbins and Monro's theorem to the multidimensional case.

In the next two sections we will briefly review the development of “bounding ellipsoid” algorithms and investigate the problems associated with the previous work on these methods. We will then demonstrate that these problems can be overcome by an SA formulation.

1.3 Introduction to Set-Membership Theory

Set-membership (SM) identification refers to a class of recursive techniques for estimating parameters of linear systems or signal models under *a priori* information that constrains the solutions to certain sets [25]. In contrast to conventional system identification schemes (e.g. maximum likelihood, LSE, etc.) which yield point estimates of the parameters, an SM algorithm yields *sets* of parameter estimates which are compatible with noise constraints. These sets are generally called “membership sets.” Based on certain set-theoretic checking criteria, SM algorithms select and use only the useful data to update the parameter estimates and refine the membership set. When data do not help refine these membership sets, the effort of updating the parameters can be avoided.

SM theory was pioneered by Krasovski [3], Schweppe [4], Witsenhausen [5], and Bertsekas and Rhodes [7] in the late 1960s, in the control and system science domain for estimating the state of a linear dynamic system. The optimum estimate is the smallest time-varying

ellipsoid which contains the unknown system states.

A major appeal of parameter set estimation is that it does not rely on the availability of a probabilistic description of the uncertainty affecting the model. It simply delineates the range of the parameter space yielding model-output values within specified ranges about the observed output samples. On the other hand, since the estimates are updated only when the input data contain sufficient information, some SM algorithms are more efficient in some applications.

In recent years, SM-based signal processing is receiving considerable attention and is becoming increasingly popular around the world as is evident from the extensive literature on this topic [14, 20, 23, 26, 29, 31, 36, 39]. This research is restricted to the class of algorithms known as *optimal bounding ellipsoid* (OBE) algorithms which follow a *bounded error* constraint.

The original OBE algorithm is attributable to Fogel and Huang [14] in 1982. Let us refer to this algorithm as “F-H/OBE.” Deller [25] and Deller and Luk [26] have reformulated the F-H/OBE algorithm so that the center of the hyper-ellipsoid is exactly the conventional WRLS solution. This algorithm is referred to as *set-membership weighted recursive least square* (SM-WRLS) algorithm in order to distinguish it from the F-H/OBE algorithm. The main deficiency of these algorithms is the lack of well-understood convergence properties. This problem led to the development of the *modified OBE* algorithm by Dasgupta and Huang in 1987. We will refer to this algorithm by “D-H/OBE.”

Recently a general class of OBE algorithms, including all methods published to date as well as the algorithm presented in this work, is unified into a single framework called the *unified OBE (UOBE)* algorithm [38, 39]. Historically, this research on the “SM-SA” (presented in this thesis) algorithm, and that of Krunz on the “Dual-SM-WRLS” algorithm [33, 34], precedes and contributed to the development of UOBE. We shall later discuss the fact that SM-SA is an important special case of UOBE and show how the many of the results of this work apply to all algorithms in broader UOBE formulation in Chapter 5.

Another OBE-like technique called *optimal volume ellipsoid* (OVE) algorithm was proposed more recently [31, 32] by Cheung, Yurkovich, and Passino to estimate parameter sets.

It is proven that the algorithm is truly optimal in the sense that the volume of ellipsoid is smallest at each recursive computation. Simulation studies in [31, 32] compare the performance of OVE and F-H/OBE. They show that OVE gives tighter parameter bounds. The estimates given by OVE, however, do not show significant improvement over the estimates given by F-H/OBE.

1.4 Review of Previous Work on OBE Algorithms

In principle, F-H/OBE and SM-WRLS are the same in spirit. However, SM-WRLS has a structure which is simpler and more elegant than the F-H/OBE because of the choice of the weighting sequence (see [38, 39]). Thus we begin with a review of SM-WRLS^{*}; similar results on F-H/OBE and D-H/OBE are found in [38, 39]. Note that the theoretical development of SM-WRLS can handle a complex vector, *multiple-input-multiple-output* (MIMO) model, we consider only real scalars here for simplicity [28]. Let

$$y_n = \sum_{i=1}^p f_i y_{n-i} + \sum_{j=0}^q g_j u_{n-j} + v_n \quad (1.4)$$

$$= \boldsymbol{\theta}^{*T} \mathbf{x}_n + v_n \quad (1.5)$$

where $\boldsymbol{\theta}^{*T} = [f_1, \dots, f_p, g_0, \dots, g_q]$, $\mathbf{x}_n^T = [y_{n-1}, \dots, y_{n-p}, u_n, u_{n-1}, \dots, u_{n-q}]$, y_n and u_n are real measurable outputs and inputs, respectively, and v_n is an unknown noise process. Let $m = p + q + 1$. We assume that $m > 1$ and, for each time n , v_n is known to be bounded according to

$$v_n^2 \leq \gamma_n \quad (1.6)$$

where $\{\gamma_n\}$ is a known positive sequence.

Let $w(n)$ be a parameter set at time n such that all $\boldsymbol{\theta} \in w(n)$ are feasible parameter estimates consistent with the error bounding in (1.6). In conjunction with the model of

^{*}Generalized results for the more general weighting policies of UOBE are found in [38, 39] to which the present research contributed.

form (1.5), $w(n)$, which is a “hyper-strip” region, can be expressed as

$$w(n) = \{\boldsymbol{\theta} : \boldsymbol{\theta} \in \Re^m, (y_n - \boldsymbol{\theta}^T \mathbf{x}_n)^2 \leq \gamma_n\}$$

Intersection of these sets over a given time range $t \in [1, n]$ form convex polytopes of feasible parameters

$$\Omega(n) = \bigcap_{i=1}^n w(i). \quad (1.7)$$

In general, $\Omega(n)$ is an irregular convex set that is difficult to describe and track analytically. In conjunction with the WRLS processing, however, $\Omega(n)$ can be shown to be contained in a hyper-ellipsoid superset $\bar{\Omega}(n)$ [25]

$$\bar{\Omega}(n) = \left\{ \boldsymbol{\theta} : (\boldsymbol{\theta} - \boldsymbol{\theta}_n)^T \frac{\mathbf{C}(n)}{\kappa_n} (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \leq 1 \right\}$$

where $\mathbf{C}(n)$ is the weighted covariance matrix,

$$\mathbf{C}(n) = \mathbf{C}(n-1) + \lambda_n \mathbf{x}_n \mathbf{x}_n^T \quad (1.8)$$

κ_n is a scalar quantity,

$$\kappa_n = \boldsymbol{\theta}_n^T \mathbf{C}(n) \boldsymbol{\theta}_n + \sum_{i=1}^n \lambda_i \gamma_i \left(1 - \frac{y_i^2}{\gamma_i} \right)$$

and $\boldsymbol{\theta}_n$, the center of $\bar{\Omega}(n)$, is the WRLS estimate at time n using the weights $\{\lambda_i\}_{i=1}^n$ and can be computed recursively using

$$\begin{aligned} \mathbf{C}(n) &\triangleq \mathbf{P}^{-1}(n) \\ G_n &= \mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n \\ \varepsilon_n &= y_n - \boldsymbol{\theta}_{n-1}^T \mathbf{x}_n \\ \mathbf{P}(n) &= \mathbf{P}(n-1) - \frac{\lambda_n \mathbf{P}(n-1) \mathbf{x}_n \mathbf{x}_n^T \mathbf{P}(n-1)}{1 + \lambda_n G_n} \\ \boldsymbol{\theta}_n &= \boldsymbol{\theta}_{n-1} + \lambda_n \mathbf{P}(n) \mathbf{x}_n \varepsilon_n \end{aligned}$$

$$\kappa_n = \kappa_{n-1} + \lambda_n \gamma_n - \frac{\lambda_n \varepsilon_n^2}{1 + \lambda_n G_n}$$

The relationship between $\bar{\Omega}(n)$ and $\Omega(n)$ for two dimensional case is shown in Fig. 1.1.

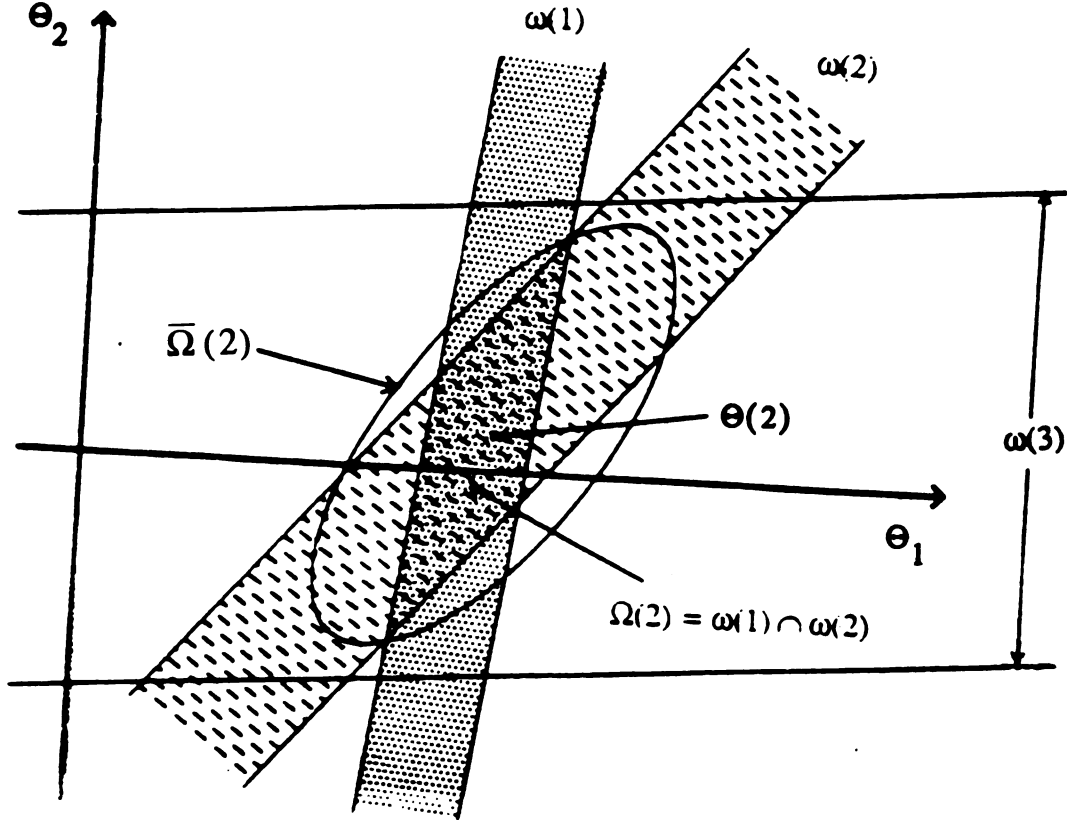


Figure 1.1. Relationship between $\bar{\Omega}(n)$ and $\Omega(n)$ for the two dimensional-case.

The SM-WRLS algorithm begins with a large ellipsoid, $\bar{\Omega}(0)$, which contains all admissible values of the model parameter vectors. The objective is to employ the weights λ_n to sequentially minimize the volume of the ellipsoid defined by

$$\mu_v\{\bar{\Omega}(n)\} \triangleq \det\{\kappa_n C^{-1}(n)\}$$

It has been shown that the optimal weight λ_n^* is the *unique* positive root of the quadratic equation [14, 38]

$$a_n \lambda_n^2 + b_n \lambda_n + c_n = 0 \quad (1.9)$$

where

$$\begin{aligned} a_n &= (m-1)G_n^2 \gamma_n \\ b_n &= 2mG_n \gamma_n - \kappa_{n-1}G_n^2 - G_n \gamma_n + \varepsilon_n^2 G_n \\ c_n &= m\gamma_n - m\varepsilon_n^2 - \kappa_{n-1}G_n. \end{aligned} \quad (1.10)$$

When the roots of equation (1.9) are both negative, the data do not contain useful information and hence updating can be avoided. Since a_n is always positive, if c_n is negative then there exists a unique positive root for (1.9). When $c_n \geq 0$, b_n is nonnegative because according to (1.10)

$$\begin{aligned} b_n &= (2m-1)G_n \gamma_n + \varepsilon_n^2 G_n - \kappa_{n-1}G_n^2 \\ &\geq mG_n \gamma_n + \varepsilon_n^2 G_n - \kappa_{n-1}G_n^2 \\ &= G_n(m\gamma_n + \varepsilon_n^2 - \kappa_{n-1}G_n) \\ &\geq G_n(m\gamma_n - m\varepsilon_n^2 - \kappa_{n-1}G_n) \\ &= G_n \underbrace{\alpha_3}_{c_n} \geq 0. \end{aligned}$$

Therefore both roots are negative. So there is no optimal weight at this time and updating can be avoided. Thus the estimates are updated if and only if $c_n < 0$. The sign of c_n serves as an indication for updating. Similar result for *trace* minimization in which optimization is applied to $\mu_t\{\bar{\Omega}(n)\} \triangleq \text{tr}\{\kappa_n \mathbf{C}^{-1}(n)\}$ can be found in [14, 38, 39].

1.5 Motivation for Developing the New Algorithm

Although the SM-WRLS algorithm and related algorithms have been applied in many areas, it is difficult to formally analyze the convergence properties. From simulation results, we have found that with F-H/OBE and SM-WRLS the optimal weight λ_n^* might drift to infinity as more data are used. The reason for this anomaly is as follows: The elements of the

weighted covariance matrix $\mathbf{C}(n)$ in (1.8) may become very large when more data are used. So, relatively speaking, the weight λ_n^* in (1.8) must be large enough in order for $\mathbf{x}_n$ to make a difference and to effectively shrink the volume of ellipsoid. To illustrate, we consider an AR(2) system given by

$$y_n = 1.6y_{n-1} - 0.68y_{n-2} + v_n. \quad (1.11)$$

v_n is a colored noise sequence generated by passing a white sequence through a Butterworth bandpass filter of order 14 with passband edge frequencies at $\omega = 1.75$ rad and 2.75 rad. The passband attenuation is -1dB and stopband attenuation is -60dB. This system is stable since the poles are located at $-0.8 \pm 0.2j$ in the z -plane. Figure 1.2 shows that the gradual growth of the weights as more data are used. In this situation, even if the estimate θ_n approaches the true parameter asymptotically, the large weights still make this algorithm difficult to implement on finite word-length machines. This is just an example in which the optimal weight may diverge. In order to deal with the convergence issues, Dasgupta and Huang introduced the D-H/OBE algorithm [20] in 1987. In this algorithm, the updated covariance matrix is a convex combination of $\mathbf{C}(n-1)$ and the new data outer product,

$$\mathbf{C}(n) = (1 - \lambda_n^*)\mathbf{C}(n-1) + \lambda_n^*\mathbf{x}_n\mathbf{x}_n^T.$$

The number $(1 - \lambda_n^*)$ is referred to as a “forgetting factor” by Dasgupta and Huang. In a sense, this D-H/OBE represents significant progress in OBE theory because it involves a formal convergence proof for the weights λ_n^* . However, the optimal weight λ_n^* in D-H/OBE is obtained by minimizing κ_n at each n . Since the magnitude of κ_n is not clearly related to the size of the membership sets (see [30, 38, 39]), this makes the D-H/OBE algorithm difficult to interpret geometrically.

Therefore, the objectives of this research are:

- To develop a new algorithm to remedy the trade-off between interpretability in weighting strategy and convergence issues.
- To provide a convergence proof for the new algorithm.

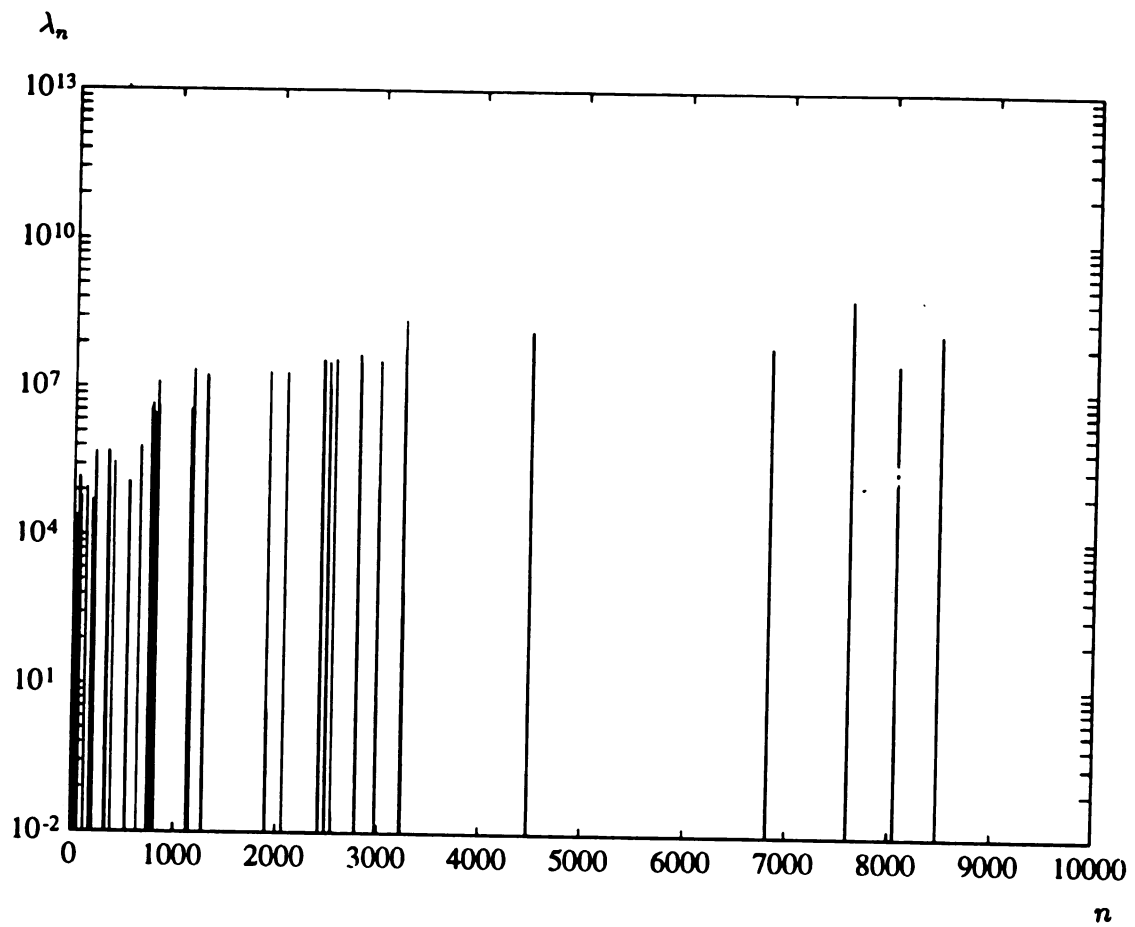


Figure 1.2. The weights become very large when more data are used.

- To perform simulations to study and compare the performance of the new algorithm with that of existing methods.

In the next chapter, the new algorithm “set-membership stochastic approximation” (SM-SA) is derived for an ARX model with bounded noise. SM-SA is developed in a systematic way to accommodate bounded weights and covariance matrix. The relationship of SM-SA to the stochastic approximation approach is investigated. If the weights of SM-SA are chosen to be $1/n$, then SM-SA is exactly the RLS estimator.

In Chapter 3, the convergence properties of SM-SA are discussed. First we show that the sequence of optimal weights $\{\lambda_n^*\}$ converges to zero as n approaches infinity. The convergence of the weights to zero is important because it is a good indication that the estimates converge. This is so since we pay less attention to noisy observations when λ_n^* is close to zero. The main point in this chapter is the convergence of the ellipsoid and its center, that serves as the estimate of the SM-SA. For RLS, the performance of the estimator is highly dependent on the statistics of the noise sequence. In SM-SA we will see that the dependency is much less than that of RLS.

Simulation studies are presented in Chapter 4. In light of the theory and analysis established in the previous two chapters, we will examine the performance of SM-SA excited by different noise sequences.

The UOBE framework is introduced in Chapter 5. We will show that the convergence properties of SM-SA can be applied to the more general weighting policies of UOBE.

Chapter 6 lists the conclusions of this research. We will also suggest some possible directions for future study.

Chapter 2

The Development of the SM-SA Algorithm

In the previous chapter, we have observed that in SM-WRLS, the weights λ_n^* may become very large, particularly for large n , leading to huge numbers in computations and numerical instabilities. In this work, we will show a feature that tends to promote convergence of the ellipsoid by preventing the “drift” of the covariance matrix toward infinity. In this chapter, we will start deriving the SM-SA algorithm by modifying SM-WRLS so that the covariance matrix is normalized to the sum of the weights. We will develop the optimality test based on the minimum volume criterion and find the similarities between SM-SA and SM-WRLS.

2.1 The Set-Membership Stochastic Approximation Algorithm

We begin with an ARX model given in (1.5) and let the noise bound be the same as in (1.6). For convenience, we rewrite the system here:

$$y_n = \sum_{i=1}^p f_i y_{n-i} + \sum_{j=0}^q g_j u_{n-j} + v_n = \boldsymbol{\theta}^{*T} \mathbf{x}_n + v_n \quad (2.1)$$

In an *unweighted* RLS algorithm, the problem of numerically unstable quantities can be eliminated by normalizing the covariance matrix to the time n , that is, by replacing $\mathbf{C}(n)$ by $(1/n)\mathbf{C}(n)$. In principle, if the sequence $\{\mathbf{x}_n\}$ represents a stationary stochastic process

with appropriate ergodicity properties, then $(1/n)\mathbf{C}(n)$ will tend to $E[\mathbf{x}_n\mathbf{x}_n^T]$. Clearly, however, this strategy may not work for weights determined by SM considerations as $\mathbf{C}(n)$ may grow much faster than n . In the SM-SA approach, SM-WRLS is modified so that the covariance matrix is normalized to the sum of the weights.

Let λ_n^{SM} be the optimal weight of SM-WRLS at time n . Define

$$\Sigma_n \triangleq \sum_{t=1}^n \lambda_t^{SM} = \Sigma_{n-1} + \lambda_n^{SM}, \quad \Sigma_0 \triangleq 0$$

and

$$\begin{aligned} \mathbf{R}(n) &\triangleq \frac{1}{\Sigma_n} \mathbf{C}(n) \\ &= \frac{1}{\Sigma_n} [\mathbf{C}(n-1) + \lambda_n^{SM} \mathbf{x}_n \mathbf{x}_n^T] \\ &= \frac{1}{\Sigma_n} [\Sigma_{n-1} \mathbf{R}(n-1) + \lambda_n^{SM} \mathbf{x}_n \mathbf{x}_n^T] \\ &= \frac{\Sigma_{n-1}}{\Sigma_n} \mathbf{R}(n-1) + \frac{\lambda_n^{SM}}{\Sigma_n} \mathbf{x}_n \mathbf{x}_n^T. \end{aligned}$$

If we define the normalized weight λ_n by

$$\lambda_n = \frac{\lambda_n^{SM}}{\Sigma_n}, \quad (2.2)$$

then

$$\frac{\Sigma_{n-1}}{\Sigma_n} = 1 - \frac{\lambda_n^{SM}}{\Sigma_n} = 1 - \lambda_n.$$

Thus the normalized covariance matrix $\mathbf{R}(n)$ can be written as

$$\mathbf{R}(n) = (1 - \lambda_n) \mathbf{R}(n-1) + \lambda_n \mathbf{x}_n \mathbf{x}_n^T \quad (2.3)$$

where $\lambda_n \in [0, 1]$. Remarkably, this approach in which $\mathbf{R}(n)$ is a convex combination of past normal matrix and incoming outer product, results in the same “covariance” matrix and recursive formula as those in D-H/OBE. However, SM-SA has been developed systematically from SM-WRLS expressly to connect the gap between the convergence issues and geomet-

rical meaning in previous work. This philosophy is quite different from the development of D-H/OBE in which geometrical interpretability is sacrificed.

Replacing λ_n for λ_n^{SM} leads to the modification of the SM-WRLS original recursions. The description of the corresponding ellipsoid is given in the following theorem.

by

Theorem 2.1 *The hyper-ellipsoid region corresponding to the SM-SA algorithm is given by*

$$\Gamma(n) = \left\{ \boldsymbol{\theta} \mid (\boldsymbol{\theta} - \boldsymbol{\theta}_n)^T \frac{\mathbf{R}(n)}{\kappa_n} (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \leq 1 \right\} \quad (2.4)$$

where κ_n is the quantity

$$\kappa_n = \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i (\gamma_i - y_i^2) + \boldsymbol{\theta}_n^T \mathbf{R}(n) \boldsymbol{\theta}_n. \quad (2.5)$$

$\boldsymbol{\theta}_n$, which is the weighted LSE estimate at time n using weights $\{\lambda_n\}$, can be computed recursively using the relations

$$\mathbf{P}(n) \triangleq \mathbf{R}^{-1}(n) \quad (2.6)$$

$$\mathbf{G}_n = \mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n \quad (2.7)$$

$$\varepsilon_n = y_n - \boldsymbol{\theta}_{n-1}^T \mathbf{x}_n \quad (2.8)$$

$$\mathbf{P}(n) = \frac{1}{1 - \lambda_n} \left[\mathbf{P}(n-1) - \frac{\lambda_n \mathbf{P}(n-1) \mathbf{x}_n \mathbf{x}_n^T \mathbf{P}(n-1)}{1 - \lambda_n + \lambda_n \mathbf{G}_n} \right] \quad (2.9)$$

$$\boldsymbol{\theta}_n = \boldsymbol{\theta}_{n-1} + \lambda_n \mathbf{P}(n) \mathbf{x}_n \varepsilon_n \quad (2.10)$$

$$\kappa_n = (1 - \lambda_n) \kappa_{n-1} + \lambda_n \gamma_n - \frac{\lambda_n (1 - \lambda_n) \varepsilon_n^2}{1 - \lambda_n + \lambda_n \mathbf{G}_n}. \quad (2.11)$$

Proof: From the bounding condition $v_n^2 \leq \gamma_n$, the accumulated inequality holds:

$$\sum_{i=1}^n v_i^2 \leq \sum_{i=1}^n \gamma_i.$$

That is,

$$\sum_{i=1}^n (y_i - \theta^{*T} \mathbf{x}_i)^2 \leq \sum_{i=1}^n \gamma_i.$$

The relationship remains true if the individual constraints are first multiplied by any non-negative quantity. Thus, we have

$$\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i (y_i - \theta^T \mathbf{x}_i)^2 \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i \quad (2.12)$$

where

$$\prod_{j=n+1}^n (\cdot) \triangleq 1.$$

Therefore, given a sequence of weights λ_n^* , (2.12) represents a global membership set, say $\Gamma(n)$, to which θ^* must belong. Since $\Omega(n)$ in (1.7) is the smallest known set, it must be true that $\Omega(n) \subset \Gamma(n)$. Let us denote any general element of $\Gamma(n)$ by θ . Upon expanding the square, we then have that

$$\begin{aligned} & \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i^2 - 2 \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i \theta^T \mathbf{x}_i \\ & + \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \theta^T \mathbf{x}_i \mathbf{x}_i^T \theta \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i. \end{aligned}$$

The following quadratic inequality in θ emerges after rearranging the terms

$$\begin{aligned} & \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i^2 - 2 \theta^T \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i \mathbf{x}_i \\ & + \theta^T \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T \right] \theta \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i \quad (2.13) \end{aligned}$$

For the WRLS solution, we have [21]

$$\theta_n = \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T \right]^{-1} \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i y_i \right]. \quad (2.14)$$

Thus

$$\left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T \right] \boldsymbol{\theta}_n = \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i y_i. \quad (2.15)$$

Substituting (2.15) into (2.13), we have

$$\begin{aligned} & \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i^2 - 2\boldsymbol{\theta}^T \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T \right] \boldsymbol{\theta}_n \\ & + \boldsymbol{\theta}^T \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T \right] \boldsymbol{\theta} \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i. \end{aligned} \quad (2.16)$$

For an off-line estimator, the covariance $\mathbf{R}(n)$ can be computed as

$$\mathbf{R}(n) = \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T.$$

Thus

$$\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i^2 - 2\boldsymbol{\theta}^T \mathbf{R}(n) \boldsymbol{\theta}_n + \boldsymbol{\theta}^T \mathbf{R}(n) \boldsymbol{\theta} \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i.$$

Since $\mathbf{R}(n)$ is symmetric, this can be written as

$$\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i y_i^2 + (\boldsymbol{\theta} - \boldsymbol{\theta}_n)^T \mathbf{R}(n) (\boldsymbol{\theta} - \boldsymbol{\theta}_n) - \boldsymbol{\theta}_n^T \mathbf{R}(n) \boldsymbol{\theta}_n \leq \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \gamma_i. \quad (2.17)$$

Since κ_n is given by

$$\kappa_n = \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i (\gamma_i - y_i^2) + \boldsymbol{\theta}_n^T \mathbf{R}(n) \boldsymbol{\theta}_n,$$

(2.17) is reduced to

$$(\boldsymbol{\theta} - \boldsymbol{\theta}_n)^T \mathbf{R}(n) (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \leq \kappa_n,$$

and (2.4) and (2.5) are established.

The matrix inversion lemma [16]:

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$$

in which $\mathbf{A}$ and $\mathbf{C}$ are nonsingular matrices, and $\mathbf{B}$ and $\mathbf{D}$ are matrices with appropriate dimensions, can be used in (2.3) to obtain

$$\mathbf{P}(n) = (1 - \lambda_n)^{-1}\mathbf{P}(n-1) - \frac{(1 - \lambda_n)^{-1}\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T(1 - \lambda_n)^{-1}\mathbf{P}(n-1)}{\lambda_n^{-1} + \mathbf{x}_n^T(1 - \lambda_n)^{-1}\mathbf{P}(n-1)\mathbf{x}_n}. \quad (2.18)$$

Replacing $\mathbf{x}_n^T\mathbf{P}(n-1)\mathbf{x}_n$ by $G(n)$ in (2.18) yields

$$\begin{aligned} \mathbf{P}(n) &= \frac{1}{1 - \lambda_n} \left[\mathbf{P}(n-1) - \frac{(1 - \lambda_n)^{-1}\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\mathbf{P}(n-1)}{\lambda_n^{-1} + (1 - \lambda_n)^{-1}G_n} \right] \\ &= \frac{1}{1 - \lambda_n} \left[\mathbf{P}(n-1) - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\mathbf{P}(n-1)}{1 - \lambda_n + \lambda_n G_n} \right]. \end{aligned}$$

For the WRLS estimate $\boldsymbol{\theta}_n$, we have

$$\begin{aligned} \boldsymbol{\theta}_n &= \mathbf{P}(n) \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i y_i \right] \\ &= \frac{1}{1 - \lambda_n} \left[\mathbf{P}(n-1) - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\mathbf{P}(n-1)}{1 - \lambda_n + \lambda_n G_n} \right] \left[\sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i y_i \right] \\ &= \frac{1}{1 - \lambda_n} \left[\mathbf{P}(n-1) - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\mathbf{P}(n-1)}{1 - \lambda_n + \lambda_n G_n} \right] \\ &\quad \left[(1 - \lambda_n) \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^{n-1} (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i y_i + \lambda_n \mathbf{x}_n y_n \right] \\ &= \boldsymbol{\theta}_{n-1} - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\boldsymbol{\theta}_{n-1}}{1 - \lambda_n + \lambda_n G_n} + \frac{1}{1 - \lambda_n} \left[\lambda_n\mathbf{P}(n-1)\mathbf{x}_n y_n - \frac{\lambda_n^2\mathbf{P}(n-1)\mathbf{x}_n G_n y_n}{1 - \lambda_n + \lambda_n G_n} \right] \\ &= \boldsymbol{\theta}_{n-1} - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\boldsymbol{\theta}_{n-1}}{1 - \lambda_n + \lambda_n G_n} + \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n y_n}{1 - \lambda_n} \left(1 - \frac{\lambda_n G_n}{1 - \lambda_n + \lambda_n G_n} \right) \\ &= \boldsymbol{\theta}_{n-1} - \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n\mathbf{x}_n^T\boldsymbol{\theta}_{n-1}}{1 - \lambda_n + \lambda_n G_n} + \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n y_n}{1 - \lambda_n + \lambda_n G_n} \\ &= \boldsymbol{\theta}_{n-1} + \frac{\lambda_n\mathbf{P}(n-1)\mathbf{x}_n \varepsilon_n}{1 - \lambda_n + \lambda_n G_n}. \end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{P}(n)\mathbf{x}_n &= \frac{1}{1-\lambda_n} \left[\mathbf{P}(n-1)\mathbf{x}_n - \frac{\lambda_n \mathbf{P}(n-1)\mathbf{x}_n G_n}{1-\lambda_n + \lambda_n G_n} \right] \\
&= \frac{\mathbf{P}(n-1)\mathbf{x}_n}{1-\lambda_n} \left(1 - \frac{\lambda_n G_n}{1-\lambda_n + \lambda_n G_n} \right) \\
&= \frac{\mathbf{P}(n-1)\mathbf{x}_n}{1-\lambda_n + \lambda_n G_n}.
\end{aligned}$$

Therefore

$$\boldsymbol{\theta}_n = \boldsymbol{\theta}_{n-1} + \lambda_n \mathbf{P}(n)\mathbf{x}_n \varepsilon_n.$$

To show (2.11), consider

$$\begin{aligned}
\kappa_n &= (1-\lambda_n) \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^{n-1} (1-\lambda_j) \right) \lambda_i (\gamma_i - y_i^2) + \lambda_n (\gamma_n - y_n^2) \\
&\quad + (\boldsymbol{\theta}_{n-1}^T + \lambda_n \mathbf{x}_n^T \mathbf{P}(n) \varepsilon_n) \mathbf{R}(n) (\boldsymbol{\theta}_{n-1} + \lambda_n \mathbf{P}(n)\mathbf{x}_n \varepsilon_n) \\
&= (1-\lambda_n) \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^{n-1} (1-\lambda_j) \right) \lambda_i (\gamma_i - y_i^2) + \lambda_n (\gamma_n - y_n^2) \\
&\quad + \boldsymbol{\theta}_{n-1}^T \mathbf{R}(n) \boldsymbol{\theta}_{n-1} + 2\lambda_n \mathbf{x}_n^T \boldsymbol{\theta}_{n-1} \varepsilon_n + \lambda_n^2 \mathbf{x}_n^T \mathbf{P}(n) \mathbf{x}_n \varepsilon_n^2 \\
&= (1-\lambda_n) \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^{n-1} (1-\lambda_j) \right) \lambda_i (\gamma_i - y_i^2) + \lambda_n (\gamma_n - y_n^2) + 2\lambda_n \boldsymbol{\theta}_{n-1}^T \mathbf{x}_n \varepsilon_n \\
&\quad + (1-\lambda_n) \boldsymbol{\theta}_{n-1}^T \mathbf{R}(n-1) \boldsymbol{\theta}_{n-1} + \lambda_n \boldsymbol{\theta}_{n-1}^T \mathbf{x}_n \mathbf{x}_n^T \boldsymbol{\theta}_{n-1} + \lambda_n^2 \mathbf{x}_n^T \frac{\mathbf{P}(n-1)\mathbf{x}_n \varepsilon_n^2}{1-\lambda_n + \lambda_n G_n} \\
&= (1-\lambda_n) \kappa_{n-1} + \lambda_n (\gamma_n - y_n^2) + \lambda_n (\mathbf{x}_n^T \boldsymbol{\theta}_{n-1})^2 + 2\lambda_n \mathbf{x}_n^T \boldsymbol{\theta}_{n-1} \varepsilon_n + \lambda_n^2 \frac{G_n \varepsilon_n^2}{1-\lambda_n + \lambda_n G_n} \\
&= (1-\lambda_n) \kappa_{n-1} + \lambda_n \gamma_n - \lambda_n \left[y_n^2 - 2(\mathbf{x}_n^T \boldsymbol{\theta}_{n-1}) y_n + (\mathbf{x}_n^T \boldsymbol{\theta}_{n-1})^2 \right] + \lambda_n^2 \frac{G_n \varepsilon_n^2}{1-\lambda_n + \lambda_n G_n} \\
&= (1-\lambda_n) \kappa_{n-1} + \lambda_n \gamma_n - \lambda_n \varepsilon_n^2 + \lambda_n^2 \frac{G_n \varepsilon_n^2}{1-\lambda_n + \lambda_n G_n} \\
&= (1-\lambda_n) \kappa_{n-1} + \lambda_n \gamma_n - \frac{\lambda_n (1-\lambda_n) \varepsilon_n^2}{1-\lambda_n + \lambda_n G_n}.
\end{aligned}$$

Thus the proof is complete. $\square$

2.2 Initial Values

The covariance matrix in SM-SA is

$$\mathbf{R}(n) = \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T.$$

In order to start the recursive algorithm, it is necessary to choose initial values for $\boldsymbol{\theta}_n$ and $\mathbf{P}(n)$. If we incorporate the initial condition into the covariance matrix, then it will become

$$\mathbf{R}(n) = \left[\prod_{i=1}^n (1 - \lambda_i) \right] \mathbf{R}_0 + \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 - \lambda_j) \right) \lambda_i \mathbf{x}_i \mathbf{x}_i^T. \quad (2.19)$$

Note that $[\prod_{i=1}^n (1 - \lambda_i)] \mathbf{R}_0$ will not be zero if $\mathbf{R}_0$ is chosen to be positive definitive.

Generally speaking, $\mathbf{P}(0) = \mathbf{R}_0^{-1}$ reflects the confidence in the initial estimates $\boldsymbol{\theta}_0$. If $\mathbf{P}(0)$ is large, from (2.10) we see that the next estimates, $\boldsymbol{\theta}_1$, will jump away from its initial value. Without any *a priori* information about $\boldsymbol{\theta}_0$, it is common to take

$$\boldsymbol{\theta}_0 = 0 \quad \text{and} \quad \mathbf{P}(0) = \sigma \mathbf{I}.$$

where σ is a “large” number, typically 10^6 so that the impact of the choices of initial values is small. From (2.19) we can also see that this choice will ensure the estimates to be close to the off-line LSE estimate.

Note that given any initial condition $\mathbf{P}(0)$, if

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n (1 - \lambda_i) = 0 \quad (2.20)$$

then the recursive estimates will approach to the off-line estimates asymptotically. Using material in Appendix A, (2.20) implies that

$$\sum_{n=1}^{\infty} \lambda_n^* = \infty$$

From theory of stochastic approximation, as discussed in Chapter 1, this is an essential

condition for the convergence of estimates to the true parameters.

2.3 The Optimal Weight

The optimal weight λ_n^* of SM-SA is found by using the minimization criterion 1 introduced in Chapter 1. That is, λ_n^* minimizes

$$\mu_v\{\bar{\Omega}(n)\} \triangleq \det\{\kappa_n \mathbf{R}^{-1}(n)\}.$$

Let $\mathbf{B}(n) \triangleq \kappa_n \mathbf{R}^{-1}(n)$. Then by definition,

$$\begin{aligned} \mathbf{B}(n) &= \kappa_n \mathbf{P}(n) \\ &= \mathbf{B}(n-1) \frac{\kappa_n}{(1-\lambda_n)\kappa_{n-1}} \left[\mathbf{I} - \frac{\lambda_n \mathbf{x}_n \mathbf{x}_n^T \mathbf{B}(n-1)}{\kappa_{n-1}(1-\lambda_n + \lambda_n G_n)} \right]. \end{aligned}$$

Define

$$h_n \triangleq 1 - \lambda_n + \lambda_n G_n, \quad (2.21)$$

and

$$d_n \triangleq \frac{1}{1-\lambda_n} \frac{\kappa_n}{\kappa_{n-1}} = 1 + \frac{\lambda_n \gamma_n}{(1-\lambda_n)\kappa_{n-1}} - \frac{\lambda_n \varepsilon_n^2}{\kappa_{n-1} h_n}. \quad (2.22)$$

Then

$$\mathbf{B}(n) = \mathbf{B}(n-1) \left\{ d_n \mathbf{I} - \frac{d_n \lambda_n \mathbf{x}_n}{\kappa_{n-1} h_n} [\mathbf{B}(n-1) \mathbf{x}_n]^T \right\}$$

and

$$\det \mathbf{B}(n) = \det \mathbf{B}(n-1) \det \left\{ d_n \mathbf{I} - \frac{d_n \lambda_n \mathbf{x}_n}{\kappa_{n-1} h_n} [\mathbf{B}(n-1) \mathbf{x}_n]^T \right\}.$$

Since $\mathbf{B}(n-1)$ is a known quantity at time n , it suffices to minimize the volume ratio $V(\lambda_n)$ given by

$$V(\lambda_n) \triangleq \frac{\det \mathbf{B}(n)}{\det \mathbf{B}(n-1)}.$$

Thus

$$V(\lambda_n) = \det \left\{ d_n \mathbf{I} - \frac{d_n \lambda_n \mathbf{x}_n}{\kappa_{n-1} h_n} [\mathbf{B}(n-1) \mathbf{x}_n]^T \right\}. \quad (2.23)$$

Using the matrix identity [26]

$$\det(c\mathbf{I} + \mathbf{y}\mathbf{z}^T) = c^{m-1}(c + \mathbf{y}^T\mathbf{z}) \quad (2.24)$$

and letting

$$c = d_n, \quad \mathbf{y} = -\frac{d_n \lambda_n \mathbf{x}_n}{\kappa_{n-1} h_n}, \quad \mathbf{z} = \mathbf{B}(n-1)\mathbf{x}_n,$$

then (2.23) becomes

$$\begin{aligned} V(\lambda_n) &= d_n^{m-1} \left(d_n - \frac{d_n \lambda_n \mathbf{x}_n^T \mathbf{B}(n-1)\mathbf{x}_n}{\kappa_{n-1} h_n} \right) \\ &= d_n^m \left(1 - \frac{\lambda_n \mathbf{x}_n^T \kappa_{n-1} \mathbf{P}(n-1)\mathbf{x}_n}{\kappa_{n-1} h_n} \right) \\ &= d_n^m \left(\frac{\kappa_{n-1} h_n - \lambda_n \kappa_{n-1} G_n}{\kappa_{n-1} h_n} \right) \\ &= d_n^m \left(\frac{h_n - \lambda_n G_n}{h_n} \right) \\ &= \frac{d_n^m (1 - \lambda_n)}{h_n}. \end{aligned} \quad (2.25)$$

To minimize the volume ratio with respect to λ_n , we must differentiate (2.25) and set the result to zero:

$$\begin{aligned} \frac{\partial}{\partial \lambda_n} V(\lambda_n) &= \frac{\partial}{\partial \lambda_n} \left[d_n^m (1 - \lambda_n) h_n^{-1} \right] \\ &= F(\lambda_n) \frac{d_n^{m-1}}{h_n^2} \end{aligned} \quad (2.26)$$

where

$$F(\lambda_n) = (1 - \lambda_n) m h_n \frac{\partial d_n}{\partial \lambda_n} - d_n G_n. \quad (2.27)$$

Since

$$\begin{aligned} \frac{\partial d_n}{\partial \lambda_n} &= \frac{1}{(1 - \lambda_n)^2} \frac{\kappa_n}{\kappa_{n-1}} + \frac{1}{(1 - \lambda_n)} \frac{\partial}{\partial \lambda_n} \frac{\kappa_n}{\kappa_{n-1}} \\ &= \frac{1}{(1 - \lambda_n)} d_n + \frac{1}{(1 - \lambda_n)} \left[-1 + \frac{\gamma_n}{\kappa_{n-1}} - \frac{\varepsilon_n^2 [1 - \lambda_n - \lambda_n h_n]}{h_n^2 \kappa_{n-1}} \right], \end{aligned} \quad (2.28)$$

(2.27) becomes

$$F(\lambda_n) = mh_n \left[d_n - 1 + \frac{\gamma_n}{\kappa_{n-1}} - \frac{\varepsilon_n^2(1 - \lambda_n - \lambda_n h_n)}{h_n^2 \kappa_{n-1}} \right] - d_n G_n. \quad (2.29)$$

The optimal weight λ_n^* is found by setting $F(\lambda_n) = 0$. Thus

$$mh_n \left[d_n - 1 + \frac{\gamma_n}{\kappa_{n-1}} - \frac{\varepsilon_n^2(1 - \lambda_n^* - \lambda_n^* h_n)}{h_n^2 \kappa_{n-1}} \right] = d_n G_n. \quad (2.30)$$

Expanding the h_n and d_n , we have a quadratic equation

$$a_n \lambda_n^2 + b_n \lambda_n + c_n = 0 \quad (2.31)$$

where

$$\begin{aligned} a_n &= m\gamma_n - m\varepsilon_n^2 + mG_n^2\gamma_n - 2mG_n\gamma_n - \kappa_{n-1}G_n + \kappa_{n-1}G_n^2 + G_n\gamma_n - G_n^2\gamma_n - \varepsilon_n^2G_n \\ b_n &= 2m\varepsilon_n^2 - 2m\gamma_n + 2mG_n\gamma_n + 2\kappa_{n-1}G_n - \kappa_{n-1}G_n^2 - G_n\gamma_n + \varepsilon_n^2G_n \\ c_n &= m\gamma_n - m\varepsilon_n^2 - \kappa_{n-1}G_n. \end{aligned}$$

This SM technique is called *set-membership stochastic approximation (SM-SA)* because of the resemblance to the stochastic approximation method. The central difference between D-H/OBE and SM-SA is that the λ_n^* in their algorithm is obtained by minimizing κ_n , while in SM-SA, it is obtained by minimizing the volume ratio. Indeed, there is an ellipsoid associated with Dasgupta-Huang's OBE at each step, but the meaningful measures of its size are ignored in the optimization process – sacrificing interpretability.

2.4 Data Selection

In this section, we will derive the criterion for updating. From (2.25) and the definitions of h_n and d_n in (2.21) and (2.22), it is easy to see that $V(0) = 1$ and $V(\lambda_n) > 0$ for all $0 < \lambda_n < 1$. Note that d_n and h_n are unity when $\lambda_n^* = 0$. Thus from (2.26) and (2.29), we

have

$$\left. \frac{\partial V(\lambda_n)}{\partial \lambda_n} \right|_{\lambda_n=0} = F(0) \quad (2.32)$$

$$= m \left(\frac{\gamma_n}{\kappa_{n-1}} - \frac{\varepsilon_n^2}{\kappa_{n-1}} \right) - G_n \quad (2.33)$$

$$= \frac{c_n}{\kappa_{n-1}} \quad (2.34)$$

where $\kappa_{n-1} > 0$ by definition (2.4) for $\lambda_{n-1} \in [0, 1)$. Equation (2.34) reveals some interesting relationships between the sign of c_n and the volume ratio, because if $c_n < 0$, there exists some λ_n such that the volume ratio is less than unity, which means that the volume shrinks. Before we develop the main theorem on this point, we need some preliminary lemmas:

Lemma 2.1 *If $G_n < 1$, then $h_n > 0$ for all $\lambda_n \in (1, \frac{1}{1-G_n})$ and*

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^-} h_n = 0^+$$

If $G_n > 1$, then $h_n > 0$ for all $\lambda_n \in (\frac{1}{1-G_n}, 1)$ and

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^+} h_n = 0^+$$

where h_n is defined in (2.21).

Proof: By definition,

$$\begin{aligned} h_n &= 1 - \lambda_n + \lambda_n G_n \\ &= 1 - \lambda_n(1 - G_n). \end{aligned}$$

If $G_n < 1$, then $1/(1 - G_n) > 1$. So the interval $(1, 1/(1 - G_n))$ is not empty. For all $\lambda_n \in (1, 1/(1 - G_n))$, we have

$$1 < \lambda_n < \frac{1}{1 - G_n}.$$

Multiplying each term by $(1 - G_n)$,

$$0 < (1 - G_n) < \lambda_n(1 - G_n) < 1.$$

Thus, for all $\lambda_n \in (1, 1/(1 - G_n))$, $h_n = 1 - \lambda_n(1 - G_n) > 0$ and

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^-} h_n = 0^+.$$

If $G_n > 1$, then $1/(1 - G_n) < 0$. Therefore the interval $(1/(1 - G_n), 0)$ is not empty and for all $\lambda_n \in (1/(1 - G_n), 0)$, we have

$$\frac{1}{1 - G_n} < \lambda_n < 0.$$

Multiplying each side by $(1 - G_n)$, we obtain

$$0 < \lambda_n(1 - G_n) < 1.$$

Thus $h_n > 0$ when $\lambda_n \in (1/(1 - G_n), 0)$ and

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^+} h_n = 0^+.$$

□

Lemma 2.2

$$\lim_{\lambda_n \rightarrow 1^-} V(\lambda_n) = +\infty$$

Proof: From (2.25), we have

$$V(\lambda_n) = \frac{d_n^m(1 - \lambda_n)}{h_n}.$$

Since $\lim_{\lambda_n \rightarrow 1^-} h_n \neq 0$, it suffices to show that

$$\lim_{\lambda_n \rightarrow 1^-} d_n^m(1 - \lambda_n) = +\infty.$$

From the definition of d_n ,

$$\begin{aligned} d_n^m(1 - \lambda_n) &= \frac{1}{(1 - \lambda_n)^m} \left(\frac{\kappa_n}{\kappa_{n-1}} \right)^m (1 - \lambda_n) \\ &= \frac{1}{(1 - \lambda_n)^{m-1}} \left(\frac{\kappa_n}{\kappa_{n-1}} \right)^m. \end{aligned}$$

The positiveness of κ_n can be immediately seen from (2.4). Thus at any time n , $\kappa_n/\kappa_{n-1} > 0$. So $d_n^m(1 - \lambda_n)$ approaches $+\infty$ as λ_n approaches 1^- . *if $0 < \gamma_n$ increases* $\square$

Theorem 2.2 *If $c_n \geq 0$, then there is no extremum of $V(\lambda_n)$ in the open interval $(0, 1)$.*

Proof: From (2.34) we know that

$$\left. \frac{\partial V(\lambda_n)}{\partial \lambda_n} \right|_{\lambda_n=0} \geq 0$$

when $c_n \geq 0$. First we consider the case when $c_n > 0$, for which there are three possible sub-cases:

1. Suppose $G_n < 1$. Then there are two asymptotes at $\lambda_n = 1$ and $\lambda_n = \frac{1}{1-G_n} > 1$.

Claim 2.1 *If $c_n > 0$ and $G_n < 1$, then*

$$\lim_{\lambda_n \rightarrow 1^+} d_n = -\infty$$

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^-} d_n = -\infty.$$

Proof of Claim 2.1: From (2.22),

$$d_n \triangleq \frac{1}{1 - \lambda_n} \frac{\kappa_n}{\kappa_{n-1}} = 1 + \frac{\lambda_n \gamma_n}{(1 - \lambda_n) \kappa_{n-1}} - \frac{\lambda_n \varepsilon_n^2}{\kappa_{n-1} h_n}$$

Note that since

$$\lim_{\lambda_n \rightarrow 1^+} h_n > 0,$$

and since $\kappa_{n-1} > 0$ at time n , the second term in the right hand side will dominate in the limit. So

$$\lim_{\lambda_n \rightarrow 1^+} d_n = -\infty.$$

Similar arguments also apply when λ_n approaches $1/(1 - G_n)^-$, because in this condition, $1 - \lambda_n$ does approach zero, but h_n approaches 0^+ from Claim 2.1. So the last term dominates ^{not} and ^{Lemma 2}

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^-} d_n = -\infty.$$

□

Note that for all $\lambda_n \in (1, 1/(1 - G_n))$, the sign of $1 - \lambda_n$ and h_n remain unchanged, so from (2.25) and Claim 2.1, the value of $V(\lambda_n)$ near the two asymptotes ($\lambda_n = 1$ and $\lambda_n = 1/(1 - G_n)$) will both approach $+\infty$ or $-\infty$, depending on whether m is even or odd. Note that in Lemma 2.2 we have shown that $V(\lambda_n)$ approaches $+\infty$ when λ_n^* approaches 1^- , so minima can not exist in $(0, 1)$ because otherwise more than two extrema will exist. Therefore there are only two possibilities in this case, as illustrated in Fig. 2.1.

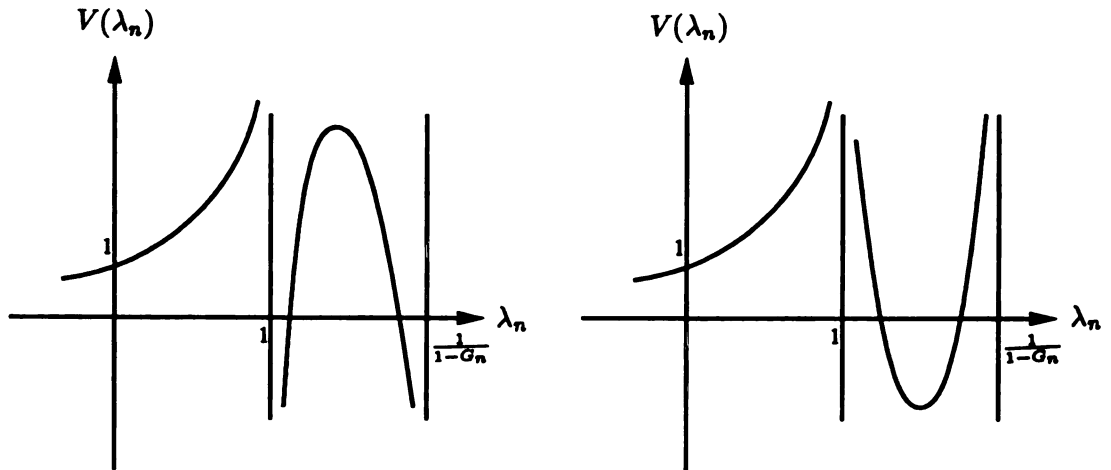


Figure 2.1. Plots of $V(\lambda_n)$ when $c_n > 0$ and $G_n < 1$.

2. Suppose $G_n = 1$. Then there is only one asymptote at $\lambda_n = 1$. Note that in this condition, $a_n = -m\varepsilon_n^2 - \varepsilon_n^2 < 0$. So $c_n/a_n < 0$. This means that the extrema have different signs. If there exists one extremum in the region of $(0, 1)$, then there must be another extremum in $(0, 1)$ since $\lim_{\lambda_n \rightarrow 1^-} V(\lambda_n) = +\infty$. This is contrary to the fact that the roots' signs must be different. Therefore we conclude that there is no extremum in $(0, 1)$.

3. Suppose $G_n > 1$. Then there are two asymptotes at $\lambda_n = 1$ and $\lambda_n = \frac{1}{1-G_n} < 0$. We claim that

$$\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^+} V(\lambda_n) = +\infty. \quad (2.35)$$

For convenience, we rewrite (2.25) as follows

$$V(\lambda_n) = \frac{d_n^m(1 - \lambda_n)}{h_n}.$$

Note that $\lim_{\lambda_n \rightarrow \frac{1}{1-G_n}^+} h_n = 0^+$ from Lemma 2.1, $1 - \lambda_n > 0$, and

$$d_n = \frac{1}{1 - \lambda_n} \frac{\kappa_n}{\kappa_{n-1}} = 1 + \frac{\lambda_n \gamma_n}{(1 - \lambda_n) \kappa_{n-1}} - \frac{\lambda_n \varepsilon_n^2}{\kappa_{n-1} h_n} \rightarrow +\infty. \quad (2.36)$$

Equation (2.36) holds because $h_n \rightarrow 0^+$ and λ_n is negative. Now (2.35) is satisfied which implies that there must be one local minimum in the interval $(1/(1-G_n), 0)$. If there is one extremum in the interval of $(0, 1)$, then there must be another extremum since $\lim_{\lambda_n \rightarrow 1^-} V(\lambda_n) = +\infty$. This situation can be seen in Fig. 2.2, which shows the existence of more than two roots in (2.31)—contradiction. Thus we conclude that there is no extremum for $\lambda_n \in (0, 1)$.

Suppose $c_n = 0$. Then

$$\left. \frac{\partial V(\lambda_n)}{\partial \lambda_n} \right|_{\lambda_n=0} = 0$$

and

$$\left. \frac{\partial^2 V(\lambda_n)}{\partial \lambda_n^2} \right|_{\lambda_n=0} > 0.$$

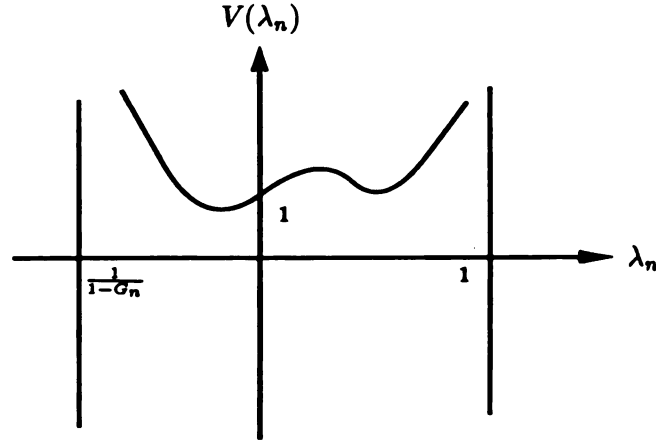


Figure 2.2. When $G_n > 1$ and $c_n > 0$, if there exists one extremum of $V(\lambda_n)$ in $(0,1)$, then there exist more than two extrema.

This means that $V(\lambda_n)$ is convex in the neighborhood of $\lambda_n = 0$ and there is one minimum when $\lambda_n = 0$. If there exists some extrema in $(0,1)$, there are at least three extrema which violates (2.31) since the number of extrema must be even. So we conclude that there is no extremum in $(0,1)$, and if there exist some λ_n in $(0,1)$ such that (2.31) holds, it must be a saddle point. The only possibilities with and without saddle points can be seen in Fig. 2.3.

□

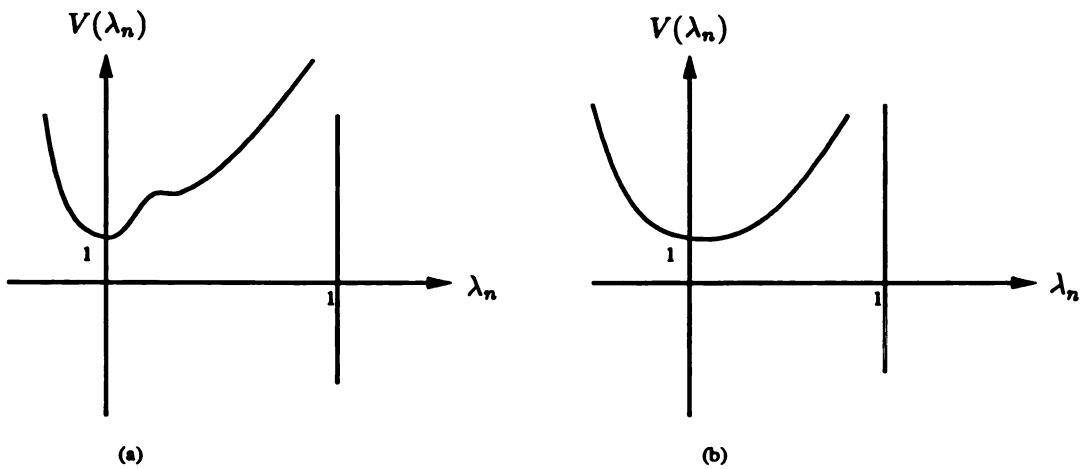


Figure 2.3. When $c_n = 0$, the the only two possibilities for $V(\lambda_n)$ in $(0,1)$ are: (a) There is a saddle point, or (b) no extrema and saddle points exist.

Theorem 2.3 Suppose $c_n < 0$. Then $V(\lambda_n)$ has a unique minimum in the open interval $(0, 1)$.

Proof: When $c_n < 0$, we have

$$\left. \frac{\partial V(\lambda_n)}{\partial \lambda_n} \right|_{\lambda_n=0} < 0.$$

By construction of the algorithm, there are two λ_n 's such that

$$\frac{\partial V(\lambda_n)}{\partial \lambda_n} = 0.$$

Also, from Lemma 2.2,

$$\lim_{\lambda_n \rightarrow 1^-} V(\lambda_n) = +\infty,$$

so there is at least one minimum in $(0, 1)$. If there exist more than one extremum in $(0, 1)$, then there must be at least three extrema as shown in Fig. 2.4, contrary to (2.31). Therefore, there must be exactly one minimum in the interval $[0, 1)$. $\square$

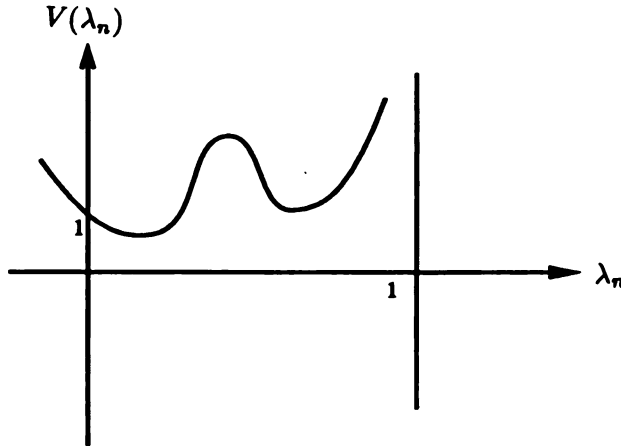


Figure 2.4. When $c_n < 0$, if more than one extremum appear in $(0, 1)$, then there must be at least three extrema.

From Theorems 2.2 and 2.3, we conclude that there exists an optimal weight if and only if $c_n < 0$.

Theorem 2.4 *There are always two real roots of the quadratic equation (2.31). Furthermore, if $c_n < 0$, then there is a unique root $\lambda_n^* \in (0, 1)$ with*

$$\lambda_n^* = \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}. \quad (2.37)$$

Theorem 2.4 implies that if $c_n < 0$, then the optimal weight λ_n^* , which belongs to $(0, 1)$, can be obtained from (2.37).

Proof: First we will show that the real roots of (2.37) exist. Let a_n , b_n , and c_n be coefficients of (2.31). For convenience, we rewrite them:

$$\begin{aligned} a_n &= m\gamma_n - m\varepsilon_n^2 + mG_n^2\gamma_n - 2mG_n\gamma_n - \kappa_{n-1}G_n + \kappa_{n-1}G_n^2 + G_n\gamma_n - G_n^2\gamma_n - \varepsilon_n^2G_n \\ b_n &= 2m\varepsilon_n^2 - 2m\gamma_n + 2mG_n\gamma_n + 2\kappa_{n-1}G_n - \kappa_{n-1}G_n^2 - G_n\gamma_n + \varepsilon_n^2G_n \\ c_n &= m\gamma_n - m\varepsilon_n^2 - \kappa_{n-1}G_n. \end{aligned}$$

Define $\bar{a}_n$ and $\bar{b}_n$ as:

$$\bar{a}_n \triangleq (m-1)G_n^2\gamma_n \quad (2.38)$$

$$\bar{b}_n \triangleq G_n(2m\gamma_n - \kappa_{n-1}G_n - \gamma_n + \varepsilon_n^2). \quad (2.39)$$

Then

$$a_n = c_n - \bar{b}_n + \bar{a}_n$$

and

$$b_n = -2c_n + \bar{b}_n.$$

Note that the expression of $\bar{a}_n$ and $\bar{b}_n$ are exactly the same as the coefficients of the quadratic equation in (1.9). Let the solutions of (2.31) be λ_n^1 and λ_n^2 . Then

$$\lambda_n^1 = \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}$$

$$\lambda_n^2 = \frac{-b_n - \sqrt{b_n^2 - 4a_n c_n}}{2a_n}$$

Note that

$$\begin{aligned} b_n^2 - 4a_n c_n &= (-2c_n + \bar{b}_n)^2 - 4c_n(c_n - \bar{b}_n + \bar{a}_n) \\ &= 4c_n^2 - 4c_n \bar{b}_n + \bar{b}_n^2 - 4c_n^2 + 4c_n \bar{b}_n - 4c_n \bar{a}_n \\ &= \bar{b}_n^2 - 4\bar{a}_n c_n \\ &= G_n^2(\gamma_n^2 + 4\varepsilon_n^2 m^2 \gamma_n - 2G_n \kappa_{n-1} \gamma_n - 2\varepsilon_n^2 \gamma_n + G_n^2 \kappa_{n-1}^2 - 2\varepsilon_n^2 G_n \kappa_{n-1} + \varepsilon_n^4) \\ &= G_n^2(G_n^2 \kappa_{n-1}^2 + \gamma_n^2 + \varepsilon_n^4 - 2G_n \kappa_{n-1} \gamma_n + 2\varepsilon_n^2 \gamma_n - 2\varepsilon_n^2 G_n \kappa_{n-1}) \\ &\quad + 4\varepsilon_n^2 m^2 \gamma_n G_n^2 - 4G_n^2 \varepsilon_n^2 \gamma_n \\ &= G_n^2(G_n \kappa_{n-1} - \gamma_n - \varepsilon_n^2)^2 + 4\varepsilon_n^2 G_n^2 m^2 \gamma_n - 4\varepsilon_n^2 G_n^2 \gamma_n \\ &= G_n^2(G_n \kappa_{n-1} - \gamma_n - \varepsilon_n^2)^2 + 4\varepsilon_n^2 G_n^2 \gamma_n (m^2 - 1) \geq 0. \end{aligned}$$

Thus two real roots always exist.

To prove that $\lambda_n^* = \lambda_n^1$, consider the following four cases:

1. Suppose $a_n < 0$ and $b_n < 0$. Then $\sqrt{b_n^2 - 4a_n c_n} < |b_n|$ since $a_n c_n > 0$. So $-b_n \pm \sqrt{b_n^2 - 4a_n c_n} > 0$. But since $a_n < 0$, both solutions are negative. This violates the existence of positive λ_n^* when $c_n < 0$. Therefore we can rule out this possibility.
2. Suppose $a_n > 0$ and $b_n > 0$. Then $\lambda_n^2 < 0$. The only possibility to obtain a positive root is to choose $\lambda_n^* = \lambda_n^1$.
3. Suppose $a_n > 0$ and $b_n < 0$. Then $\sqrt{b_n^2 - 4a_n c_n} > |b_n|$ since $a_n c_n < 0$. So $\lambda_n^2 < 0$ and $\lambda_n^1 > 0$. To obtain a positive root, we must have $\lambda_n^* = \lambda_n^1$.

4. Suppose $a_n < 0$ and $b_n > 0$. Then both roots are positive when $c_n < 0$. Since $a_n c_n > 0$, $\sqrt{b_n^2 - 4a_n c_n} < |b_n|$ and hence $\lambda_n^1 < \lambda_n^2$. To obtain the unique root in $(0, 1)$, we must have $\lambda_n^* = \lambda_n^1$.

We conclude that $\lambda_n^* = \lambda_n^1$ when $c_n < 0$. □

2.5 Relations Between SM-SA and SM-WRLS

As shown in Theorems 2.2 and 2.3, the ellipsoid will shrink if and only if c_n in equation (2.31) is negative. If we compare c_n in (2.31) and c_n in (1.9), we find that they are exactly the same quantity. This fact led us to explore whether relationships exist between SM-SA and SM-WRLS.

The following simulation shows some apparent equivalencies between these two algorithms. Consider the system

$$y_n = 1.6y_{n-1} - 0.68y_{n-2} + v_n \quad (2.40)$$

where $\{v_n\}$ is a sequence of pseudo-random numbers uniformly distributed in $[-0.5, 0.5]$. A set of 120,000 data points is generated using the model. Table 2.1 and Table 2.2 are the results of the simulations for SM-SA and SM-WRLS using the system in (2.40). It can be seen that both algorithms update the parameters at the same time and result in the same “volume measure” and estimates.

These empirical results, and others due to Krunz *et al.* [33, 34], led Deller *et al.* [38, 39] to the discovery that the volume measure defined by $\det \mathbf{B}(n)$, the data points selected, and the parameter estimates $\theta(n)$ is independent of the choice of weighting sequences when UOBE is considered. Thus the relationship between SM-SA and F-H/OBE is expected to be the same as that between SM-SA and SM-WRLS. This implies that the present research on SM-SA has much broader implications for the entire class of UOBE algorithms. Details will be discussed in Chapter 5.

On the other hand, SM-SA has some features which are not necessarily general. The

simulation results also reveal that the average weights satisfy Robbins-Monro's requirements in (1.3) under the condition that the noise input is a pseudo-random number with uniform distribution in $[-\sqrt{\gamma}, \sqrt{\gamma}]$. Figure 2.5 shows the relationship between the time and the "averaged optimal weight" from six experiments. The vertical and horizontal axes are logarithms of weight and time, respectively. The two dotted lines are logarithms of $1/n$ and $1/n^2$. The curve of the averaged weights is very close to the curve of $1/n$ and hence the estimates could approach to the true parameter. Therefore, experimental results show that SM-SA has a great potential in theoretical development and its theoretical analysis can help explain some features in SM-WRLS which cannot be obtained directly from the original algorithm.

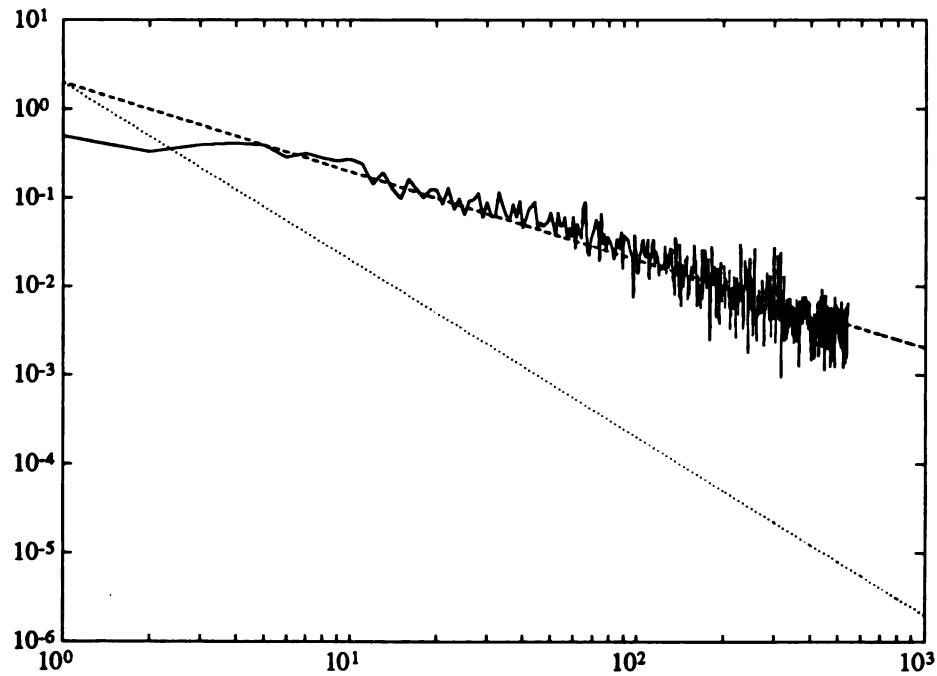


Figure 2.5. The relationship between logarithms of time and weight

SM-WRLS			SM-SA		
time	volume measure	ε_n	time	volume measure	ε_n
1	158803088.0	0.045102	1	625000000.0	0.045102
2	6109073.5	0.325878	2	12203776.0	0.325878
3	75912.859375	-1.841745	3	93237.984375	-1.853359
4	67.021507	6.571213	4	66.320511	8.059097
5	19.407402	0.208971	5	19.290403	0.212660
6	17.094551	-0.261806	6	17.005600	-0.262197
7	6.657953	-0.219103	7	6.645070	-0.222424
8	2.566225	0.210988	8	2.566096	0.208843
9	0.861677	-0.235759	9	0.861815	-0.236259
10	0.399760	-0.076875	10	0.399806	-0.076872
11	0.191688	-0.458952	11	0.191711	-0.458865
12	0.191604	0.157472	12	0.191627	0.157516
13	0.169054	-0.062481	13	0.169072	-0.062429
17	0.168957	-0.355722	17	0.168976	-0.355732
18	0.130553	-0.420661	18	0.130561	-0.420695
19	0.102843	0.197666	19	0.102846	0.197650
20	0.089835	0.242533	20	0.089836	0.242511
21	0.051043	-0.607100	21	0.051041	-0.607123
23	0.044455	-0.433844	23	0.044454	-0.433845
41	0.044317	-0.433763	41	0.044316	-0.433759
55	0.043492	0.340501	55	0.043491	0.340501
58	0.043200	0.330139	58	0.043199	0.330139
:	:	:	:	:	:
115034	0.000072	-0.498830	115034	0.000072	-0.498830
115258	0.000072	-0.500641	115258	0.000072	-0.500641
115280	0.000072	-0.496111	115280	0.000072	-0.496111
115463	0.000072	-0.491712	115463	0.000072	-0.491711
116302	0.000072	-0.483166	116302	0.000072	-0.483166
116350	0.000071	-0.499281	116350	0.000071	-0.499281
116643	0.000070	-0.503710	116643	0.000070	-0.503710
116738	0.000070	-0.501659	116738	0.000070	-0.501659
116934	0.000070	-0.500541	116934	0.000070	-0.500541
117060	0.000069	-0.499384	117060	0.000069	-0.499384
117354	0.000069	-0.491482	117354	0.000069	-0.491482
117706	0.000069	-0.502020	117706	0.000069	-0.502020
118516	0.000068	-0.503093	118516	0.000068	-0.503093
118709	0.000068	-0.498520	118709	0.000068	-0.498520

Table 2.1. The data-selection and the performance of SM-WRLS and SM-SA for the AR model (2.50).

SM-WRLS			SM-SA		
time	a_1	a_2	time	a_1	a_2
1	0.000000	0.000000	1	0.000000	0.000000
2	2.659994	0.000000	2	2.662042	0.000000
3	2.659676	-2.490193	3	2.661741	-2.496096
4	2.504596	-1.994164	4	2.505078	-1.995054
5	2.485668	-1.956294	5	2.485804	-1.956493
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
115034	1.602177	-0.678578	115034	1.602176	-0.678578
115258	1.600430	-0.676362	115258	1.600430	-0.676362
115280	1.601612	-0.677525	115280	1.601612	-0.677525
115463	1.601638	-0.677555	115463	1.601639	-0.677555
116302	1.596280	-0.673320	116302	1.596280	-0.673320
116350	1.594492	-0.670988	116350	1.594492	-0.670988
116643	1.591593	-0.668343	116643	1.591593	-0.668343
116738	1.602305	-0.678133	116738	1.602305	-0.678133
116934	1.604294	-0.680194	116934	1.604294	-0.680194
117060	1.602866	-0.678490	117060	1.602866	-0.678490
117354	1.603187	-0.678821	117354	1.603187	-0.678821
117706	1.597315	-0.678648	117706	1.597315	-0.678648
118516	1.600800	-0.681055	118516	1.600799	-0.681054
118709	1.605809	-0.682360	118709	1.605808	-0.682360

Table 2.2. Estimates of SM-WRLS and SM-SA for the AR model (2.50) where $a_1 = 1.6$ and $a_2 = -0.68$.

Chapter 3

Convergence Issues

In the previous chapter, we have developed the SM-SA algorithm with a selective updating scheme and non-increasing volume. In this chapter, the convergence properties of this algorithm will be discussed. However, for all the estimation problems, the parameters in model (2.1) can not be determined unless some conditions are imposed on the input signal. A commonly assumed condition is *persistence of excitation* (PE)^{*} in which the input signal is sufficiently rich in frequency components. PE has been defined in a variety of ways [15, 17, 19, 20, 21, 22]. The following definition has been used extensively in adaptive control and system identification.

Definition 3.1 *A signal $u(t)$ is said to be persistently exciting (PE) of order n if for all t there is an integer N such that*

$$\varrho_1 I < \sum_{k=t+1}^{t+N} \varphi(k) \varphi^T(k) < \varrho_2 I \quad (3.1)$$

where $\varrho_1, \varrho_2 > 0$ and the vector $\varphi(t)$ is given by

$$\varphi(t) = [u(t-1) \cdots u(t-n)]^T$$

^{*}Please read “PE” as “persistence of excitation” or “persistently exciting” as appropriate.

The underlying meanings of the PE condition have been discussed extensively in the literature [6, 15, 17, 10]. From a frequency-domain point of view, a stationary signal $u(t)$ is persistently exciting of order n if its frequency spectrum is non-zero at n points (or more). That is, its frequency components have to be rich enough so as to drive the estimates toward the true parameters. According to this definition, white noise is persistently exciting of all orders.

Research in SM-theory, however, is concerned with PE of $\mathbf{x}_n$, the regression vector. Anderson and Johnson [15], and Bitmead [17] proposed that the PE of the input signal implies the PE of $\mathbf{x}_n$ under some mild restrictions on the system function. The major result is given in the following theorem:

Theorem 3.1 [15, 17] *Consider a linear system with transfer function $H(z) = B(z)/A(z)$ where $A(z) = z^n - a_1 z^{n-1} - \dots - a_p z^{n-p}$ and $B(z) = b_0 z^n + b_1 z^{n-1} + \dots + b_q z^{n-q}$, which are co-prime, and with input sequence $\{u_k\}$ and bounded output sequence $\{y_k\}$. Then with $\mathbf{x}_n^T = [y_{n-1}, \dots, y_{n-p}, u_n, u_{n-1}, \dots, u_{n-q}]$, PE of $\{u_n\}$ implies PE of $\{\mathbf{x}_n\}$.*

Non-coprimeness of $A(z)$ and $B(z)$ usually renders the parameters non-identifiable. This discussion, however, is far beyond the scope of this research.

For LSE estimates of the parameters of a finite impulse response model with n parameters, the estimate is consistent and the variance of the estimate goes to zero if the input signal is persistently exciting of order n . In order to deal with the convergence issues in D-H/OBE, Dasgupta and Huang claimed that if for all t ,

$$\varrho_1 I < \sum_{k=t+1}^{t+N} \mathbf{x}_k \mathbf{x}_k^T < \varrho_2 I \quad (3.2)$$

for some $N \in \mathcal{N}$ and $\varrho_1, \varrho_2 > 0$, then the covariance matrix

$$\mathbf{C}(n) = (1 - \lambda_n^*) \mathbf{C}(n-1) + \lambda_n^* \mathbf{x}_n \mathbf{x}_n^T$$

in D-H/OBE, which has the same form as $\mathbf{R}(n)$ in SM-SA, is nonsingular. From Theorem 3.1, if the system function is co-prime, then the statement in (3.2) is validated. However, examining the proof of their argument, we find that the condition (3.2) is not sufficient because of the time index mismatch. In (3.2), k varies over *all* the natural numbers between $(t+1)$ to $(t+N)$. In their proof, however, the time index varies only over those instances at which updates take place. Consequently, we require a more strict definition for PE which would accommodate non-uniform time indexing.

Given that $\mathbf{x}_n$ is PE, the question of whether the subsequence of $\{\mathbf{x}_n\}$ in which the updates take place is PE or not is still an open issue and is difficult to prove formally. However, this problem can be simplified if we impose the *mixing* condition to the noise sequence. The mixing condition will be introduced in Section 3.4. For the moment, we assume that the subsequence of $\{\mathbf{x}_n\}$ used in updates is PE. For SM-SA with ARX model, we give the following theorem:

Theorem 3.2 *Given the update equation:*

$$\mathbf{R}(n) = (1 - \lambda_n)\mathbf{R}(n-1) + \lambda_n \mathbf{x}_n \mathbf{x}_n^T, \quad \mathbf{R}(0) = \mathbf{R}_0 > 0 \quad (3.3)$$

the eigenvalues of $\mathbf{R}(n)$, for all n , will remain bounded if the signal energy is bounded, i.e., there exists an $L \in \mathbb{R}$ such that $\mathbf{x}_n^T \mathbf{x}_n \leq L < \infty$ for all n , and $0 \leq \lambda_n \leq 1$.

Let $\{t_n\}$ be a subsequence such that $c_{t_n} < 0 \forall n$. If there exists an $N \in \mathcal{N}$ and $\varrho_1, \varrho_2 > 0$ such that for all k

$$\varrho_1 \mathbf{I} < \sum_{k=n+1}^{n+N} \mathbf{x}_{t_k} \mathbf{x}_{t_k}^T < \varrho_2 \mathbf{I}, \quad (3.4)$$

then $\mathbf{R}(n)$ is nonsingular.

Proof: Use induction. Note that from the above recursion, $\mathbf{R}(n) \geq 0$ for all n . So it suffices to show that $\text{trace}(\mathbf{R}(n)) < \infty$.

1. $n = 1$:

$$\mathbf{R}(1) = (1 - \lambda_1)\mathbf{R}(0) + \lambda_1 \mathbf{x}_n \mathbf{x}_n^T$$

$$\begin{aligned} \text{tr}[\mathbf{R}(1)] &= (1 - \lambda_1)\text{tr}[\mathbf{R}_0] + \lambda_1 \text{tr}[\mathbf{x}_1 \mathbf{x}_1^T] \\ &= (1 - \lambda_1)\text{tr}[\mathbf{R}_0] + \lambda_1 \mathbf{x}_1^T \mathbf{x}_1 \\ &\leq \max[\text{tr}(\mathbf{R}_0), L]. \end{aligned}$$

2. Let

$$\text{tr}[\mathbf{R}(n)] \leq \max[\text{tr}(\mathbf{R}_0), L] < \infty.$$

We have to show that

$$\text{tr}[\mathbf{R}(n + 1)] \leq \max[\text{tr}(\mathbf{R}_0), L] < \infty.$$

Since

$$\mathbf{R}(n + 1) = (1 - \lambda_{n+1})\mathbf{R}(n) + \lambda_{n+1} \mathbf{x}_{n+1} \mathbf{x}_{n+1}^T,$$

$$\begin{aligned} \text{tr}[\mathbf{R}(n + 1)] &= (1 - \lambda_{n+1})\text{tr}[\mathbf{R}(n)] + \lambda_{n+1} \mathbf{x}_{n+1}^T \mathbf{x}_{n+1} \\ &\leq \max[\text{tr}(\mathbf{R}(n)), \mathbf{x}_{n+1}^T \mathbf{x}_{n+1}] \\ &\leq \max[\text{tr}(\mathbf{R}(n)), L] \\ &\leq \max[\max(\text{tr}(\mathbf{R}_0), L), L] \\ &= \max[\text{tr}(\mathbf{R}_0), L]. \end{aligned}$$

Therefore, the sum of all eigenvalues is bounded for all n . So the eigenvalues have to remain bounded.

The non-singularity of $\mathbf{R}(n)$ under the assumptions in (3.4) that the subsequence of $\{\mathbf{x}_n\}$ in which updates take place is PE has been proven in the paper by Dasgupta and Huang [20]. □

Theorem 3.2 ensures that the covariance matrix $\mathbf{R}(n)$ in (3.3) is always bounded and the boundedness property holds for all the noise processes. Although boundedness of $\mathbf{R}(n)$ is sufficient for proofs of some fundamental properties, positive definiteness of $\mathbf{R}(n)$ is essential in order to prove the convergence of parameter sets as discussed in the next few sections.

3.1 Fundamental Convergence Properties of SM-SA

The most important feature of SM-SA, which differs from previous work, is the convergence of weighting sequence $\{\lambda_n^*\}$. In this section, the fundamental convergence properties of SM-SA are analyzed.

[Not necessary to converge to a point.] ok
Theorem 3.3 *Let $(\mathcal{R}^m, \|\cdot\|)$ be a normed space. Given the $\mathbf{R}(n)$ in (3.3) and the update equation (2.7)-(2.11). Let λ_n^* be defined as*

$$\lambda_n^* = \begin{cases} \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}, & \text{if } c_n < 0 \\ 0, & \text{otherwise} \end{cases}.$$

If infinitely many data points are selected, then the SM-SA algorithm developed in Theorem 2.1 converges in the sense that

$$\lim_{n \rightarrow \infty} \lambda_n^* = 0 \quad (3.5)$$

$$\lim_{n \rightarrow \infty} \varepsilon_n^2 \in [0, \gamma_n]. \quad (3.6)$$

Furthermore, if $\mathbf{R}(n)$ is positive definite, then

$$\lim_{n \rightarrow \infty} \|\theta_n - \theta_{n-1}\| = 0. \quad (3.7)$$

If there exists positive real numbers ϱ_1 and ϱ_2 and integer N_1 such that for all k

$$\varrho_1 I < \sum_{n=k+1}^{k+N_1} \mathbf{x}_n \mathbf{x}_n^T < \varrho_2 I \quad (3.8)$$

then [31]

$$\lim_{n \rightarrow \infty} \|\theta^* - \theta_n\|_2^2 \leq \frac{4}{\varrho_1} \lim_{n \rightarrow \infty} \sum_{i=n+1}^{n+N_1} \gamma_i \quad (3.9)$$

The proof of this theorem is very significant because, as a member of UOBE [39], SM-SA sets the stage for a formal proof of the converging behavior of this family. Before we prove this theorem, we need the following lemma:

Lemma 3.1 *Consider the coefficients of the quadratic equation (2.31) and the conditions stated in Theorem 3.3. Let c_n^- and c_n^+ be defined by*

$$c_n^- = \begin{cases} c_n, & \text{if } c_n < 0 \\ 0, & \text{otherwise} \end{cases} \quad (3.10)$$

$$c_n^+ = \begin{cases} c_n, & \text{if } c_n > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Then $\lim_{n \rightarrow \infty} c_n^- = 0$ if and only if $\lim_{n \rightarrow \infty} \lambda_n^* = 0$.

Proof of Lemma 3.1: From Theorem 2.4 and the definition of λ_n^* in Theorem 3.3, if $c_n < 0$, then

$$\lambda_n^* = \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}$$

Let $\bar{a}_n$ and $\bar{b}_n$ be the same as in (2.38) and (2.39), then from the proof of theorem 2.4, we have

$$b_n^2 - 4a_n c_n = \bar{b}_n^2 - 4\bar{a}_n c_n$$

If $c_n < 0$, then λ_n^* becomes

$$\lambda_n^* = \frac{2c_n^- - \bar{b}_n + \sqrt{\bar{b}_n^2 - 4(m-1)G_n^2 c_n^- \gamma}}{2a_n} \quad (3.11)$$

Suppose $\lim_{n \rightarrow \infty} c_n^- = 0$. If $\lim_{n \rightarrow \infty} a_n \neq 0$, then from (3.11) we know that $\lim_{n \rightarrow \infty} \lambda_n^* = 0$.

If $\lim_{n \rightarrow \infty} a_n = 0$, then we claim that $\lim_{n \rightarrow \infty} b_n \neq 0$ because from the definitions of $\bar{a}_n$ and

$\bar{b}_n$, we can write

$$a_n + b_n = -c_n + \bar{a}_n$$

and for $c_n < 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_n + b_n) &= \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \\ &= \lim_{n \rightarrow \infty} b_n \\ &= \lim_{n \rightarrow \infty} (-c_n^- + \bar{a}_n) \\ &= \lim_{n \rightarrow \infty} \bar{a}_n \\ &= (m-1)G_n^2 \gamma_n > 0. \end{aligned}$$

That is, $\lim_{n \rightarrow \infty} b_n > 0$. Thus from (2.31),

$$\lim_{n \rightarrow \infty} \lambda_n^* = - \lim_{n \rightarrow \infty} \frac{c_n}{b_n} = 0$$

Suppose $\lim_{n \rightarrow \infty} \lambda_n^* = 0$, it follows from (3.11) that when $c_n < 0$, then in the limit

$$2c_n^- - \bar{b}_n = -\sqrt{\bar{b}_n^2 - 4(m-1)G_n^2 c_n^- \gamma_n}$$

Thus

$$\lim_{n \rightarrow \infty} [4c_n^{2-} - 4c_n^- \bar{b}_n + \bar{b}_n^2] = \lim_{n \rightarrow \infty} [\bar{b}_n^2 - 4(m-1)G_n^2 c_n^- \gamma_n]$$

and

$$\lim_{n \rightarrow \infty} [4c_n^{2-} - 4c_n^- \bar{b}_n] = \lim_{n \rightarrow \infty} [-4(m-1)G_n^2 c_n^- \gamma_n]$$

Suppose $\lim_{n \rightarrow \infty} c_n^- \neq 0$. Then

$$\lim_{n \rightarrow \infty} c_n^- = - \lim_{n \rightarrow \infty} [(m-1)G_n^2 \gamma_n + \bar{b}_n].$$

The numerator of equation (3.11) becomes

$$2c_n^- - \bar{b}_n + \sqrt{\bar{b}_n^2 - 4(m-1)G_n^2\gamma\bar{b}_n + 4(m-1)^2G_n^4\gamma_n^2} = 2c_n^- - \bar{b}_n + \sqrt{(\bar{b}_n - 2(m-1)G_n^2\gamma_n)^2} \quad (3.12)$$

Comparing (3.12) and (3.11), we have in the limit

$$\sqrt{(\bar{b}_n - 2(m-1)G_n^2\gamma)^2} = \sqrt{\bar{b}_n^2 - 4(m-1)G_n^2c_n\gamma_n} \quad (3.13)$$

Note that in left hand side of (3.13), $\sqrt{(\bar{b}_n - 2(m-1)G_n^2\gamma)^2} < |\bar{b}_n|$; while in right hand side of (3.13), $\sqrt{\bar{b}_n^2 - 4(m-1)G_n^2c_n\gamma_n} > |\bar{b}_n|$ since $c_n < 0$. This is a contradiction. So the only possibility when $\lim_{n \rightarrow \infty} \lambda_n^* = 0$ is $\lim_{n \rightarrow \infty} c_n = 0$. $\square$

Proof of theorem 3.3: Suppose λ_n^* does not converge to zero. Then there exists a strictly increasing sequence $(m_n)_{n=1}^\infty$ such that $\lambda_{m_n}^* \geq a$ for some $a > 0$. From Lemma 3.1, we have $\limsup_{n \rightarrow \infty} c_n < 0$. From (2.34),

$$\left. \frac{\partial V(\lambda_{m_n})}{\partial \lambda_{m_n}} \right|_{\lambda_{m_n}=0} = F(0) = \frac{c_{m_n}}{\kappa_{m_n-1}}.$$

For the “volume measure” of $\Gamma(m_n)$, we have

$$\det \mathbf{B}(m_n) = \det \kappa_{m_n} \mathbf{P}(m_n) = \kappa_{m_n}^m \det \mathbf{P}(m_n)$$

Since $\mathbf{R}(m_n)$ is always bounded, $\det \mathbf{P}(m_n) = \det \mathbf{R}^{-1}(m_n) > 0$. Also note that $\{\det \mathbf{B}(n)\}_{n=1}^\infty$ is non-increasing, so κ_n is bounded. Since $\limsup_{n \rightarrow \infty} c_n \neq 0$, there exists infinitely many n such that

$$\left. \frac{\partial V(\lambda_{m_n})}{\partial \lambda_{m_n}} \right|_{\lambda_{m_n}=0} = \frac{c_{m_n}}{\kappa_{m_n-1}} < -b$$

where $b > 0$. So there are infinitely many n such that $V(\lambda_{m_n}^*) \leq c$ for some $0 < c < 1$. Let M_{m_n} be the volume at time m_n , then $M_{m_n} = V_0 V_{m_1} V_{m_2} \cdots V_{m_n}$. Thus $\lim_{n \rightarrow \infty} M_{m_n} = 0$.

Note that $\{M_n\}$ is non-increasing. Thus we have $\lim_{n \rightarrow \infty} M_n = 0$. By definition,

$$M_n = \det \kappa_n \mathbf{P}(n) = \kappa_n^m \det \mathbf{P}(n).$$

Since $\det \mathbf{P}(n)$ is positive definite by Theorem 3.2, $\lim_{n \rightarrow \infty} \kappa_n^m = 0$. This leads to the conclusion that $\lim_{n \rightarrow \infty} \kappa_n = 0$. Let $\tilde{\theta}_n = \theta_n - \theta^*$, then

$$\begin{aligned} \varepsilon_n &= y_n - \theta_{n-1}^T \mathbf{x}_n \\ &= \theta^{*T} \mathbf{x}_n + v_n - \theta_{n-1}^T \mathbf{x}_n \\ &= -\tilde{\theta}_{n-1}^T \mathbf{x}_n + v_n \end{aligned} \tag{3.14}$$

Since the volume of the ellipsoid converges to zero, the estimates converge to the true parameters. Thus $\lim_{n \rightarrow \infty} \|\tilde{\theta}_{n-1}\| = 0$ and hence $\lim_{n \rightarrow \infty} \tilde{\theta}_{n-1}^T \mathbf{x}_n = 0$. Thus, from (3.14) we have $\lim_{n \rightarrow \infty} |\varepsilon_n - v_n| = 0$. Since $\lim_{n \rightarrow \infty} \kappa_n = 0$ and since $c_n = m\gamma_n - m\varepsilon_n^2 - \kappa_{n-1}G_n$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} c_n &= \lim_{n \rightarrow \infty} m(\gamma_n - \varepsilon_n^2) - \kappa_n G_n \\ &= \lim_{n \rightarrow \infty} m(\gamma_n - \varepsilon_n^2) \\ &= \lim_{n \rightarrow \infty} m(\gamma_n - v_n^2) \geq 0 \end{aligned}$$

By the definition of c_n^- in Lemma 3.1, $\lim_{n \rightarrow \infty} c_n^- = 0$ and hence $\lim_{n \rightarrow \infty} \lambda_n^* = 0$, contrary to the assumption. Thus we conclude that the sequence $\{\lambda_n^*\}$ must converge to zero.

It follows from (2.10) that $\lim_{n \rightarrow \infty} \|\theta_n - \theta_{n-1}\| = 0$ when $\lim_{n \rightarrow \infty} \lambda_n^* = 0$. Thus (3.7) is satisfied.

Consider the coefficient c_n in (2.31). At any time n , we can write

$$c_n = c_n^+ + c_n^-$$

where c_n^+ and c_n^- were defined in Lemma 3.1. Since $\lim_{n \rightarrow \infty} c_n^- = 0$, we have

$$\liminf_{n \rightarrow \infty} c_n = \liminf_{n \rightarrow \infty} (c_n^+ + c_n^-) \geq 0.$$

Let $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$m(\gamma_n - \varepsilon_n^2) - \kappa_{n-1}G_n > -m\varepsilon.$$

Let $n > N$. Then

$$\gamma_n - \varepsilon_n^2 > -\varepsilon + \frac{\kappa_{n-1}G_n}{m} > -\varepsilon. \quad (3.15)$$

Let ε be arbitrarily small, then we have

$$\gamma - \varepsilon_n^2 \geq 0.$$

Thus $\varepsilon_n^2 \in [0, \gamma]$ and, hence, (3.6) is satisfied.

Due to the slow convergence of the ellipsoid for large n , one can argue that for sufficiently large n , there exists a positive integer M such that $M > N_1$ and

$$|y_{n+k} - \theta_n^T \mathbf{x}_{n+k}| < \sqrt{\gamma_{n+k}}, \quad 1 \leq k \leq M$$

In other words, no update takes place on $\{n+1, \dots, n+M\}$. Substituting for y_{n+k} by $\check{\theta}^T \mathbf{x}_{n+k} + v_{n+k}$ and noting that $|v_{n+k}| < \sqrt{\gamma_{n+k}}$, we can write

$$|(\check{\theta} - \theta_n)^T \mathbf{x}_{n+k}| = |(\check{\theta} - \theta_n)^T \mathbf{x}_{n+k} + v_{n+k} - v_{n+k}| \quad (3.16)$$

$$\leq |(\check{\theta} - \theta_n)^T \mathbf{x}_{n+k} + v_{n+k}| + |v_{n+k}| \quad (3.17)$$

$$\leq 2\sqrt{\gamma_{n+k}} \quad (3.18)$$

for $1 \leq k \leq N_1$. Therefore, squaring both side of (3.18) and, then, summing over k , $1 \leq k \leq N_1$, it is found that

$$(\check{\theta} - \theta_n)^T \left[\sum_{i=n+1}^{n+N_1} \mathbf{x}_i \mathbf{x}_i^T \right] (\check{\theta} - \theta_n) < 4 \sum_{i=n+1}^{n+N_1} \gamma_i$$

Hence, using (3.8), it follows that

$$\|\theta^* - \theta_n\|^2 < \frac{4}{\ell_1} \sum_{i=n+1}^{n+N} \gamma_i$$

□

Theorem 3.3 assumes that infinitely many data points are selected. The algorithm, however, may stop at some finite time N , that is, optimal weights no longer exist after N , when noise excitations are “too colored” or noise bounds are over-estimated. In this case the convergence properties (3.5) and (3.7) still hold since the volume of the ellipsoid and the estimates will remain constant, and the optimal weights λ_n^* become zero after time N . For c_n , recall that

$$c_n = m(\gamma_n - \varepsilon_n^2) - \kappa_{n-1} G_n.$$

Since $c_n \geq 0$ for $n > N$,

$$m(\gamma_n - \varepsilon_n^2) \geq \kappa_{n-1} G_n > 0 \quad (3.19)$$

Equation (3.19) ensures that (3.6) still holds even if the algorithm stops at finite time.

3.2 Convergence of Parameter Sets and Suboptimal Test

In this section, we will first investigate the condition under which the membership sets converge to a point. Then, a “suboptimal test” criterion is introduced. We will find that employing the suboptimal test not only improves computational complexity, but also results in excellent convergence properties.

3.2.1 Convergence of Parameter Sets

Since the volume of the ellipsoid in SM-SA is non-increasing, it converges for all the stationary signals and linear time-invariant models. The ideal performance for SM-SA is the convergence of membership sets to a point asymptotically. In this section, we will study the asymptotical properties of parameter sets.

Theorem 3.4 Let $\{t_n\}$ be a subsequence such that $\gamma_{t_n} - \varepsilon_{t_n}^2 < 0 \forall n$. If there exists an $N \in \mathcal{N}$ and $\rho_1, \rho_2 > 0$ such that (3.4) holds, then the volume of the ellipsoid converges to zero if $\{t_n\}$ is infinite.

Proof: In Theorem 3.3 we have shown that $\lim_{n \rightarrow \infty} \lambda_n^* = 0$. Let c_n^- be defined as in (3.10). Then from Lemma 3.1, $\lim_{n \rightarrow \infty} c_n^- = 0$. That is,

$$\lim_{n \rightarrow \infty} [m(\gamma_n - \varepsilon_n^2) - \kappa_{n-1} G_n] = 0. \quad (3.20)$$

Handwritten notes: $c_n^- \rightarrow 0$, $\kappa_{n-1} G_n$, $\gamma_n - \varepsilon_n^2$, G_n becomes ε_n by assumption

Note that the ellipsoid at time n is defined by

$$\Gamma(n) = \{\theta \mid (\theta - \theta_n)^T \mathbf{R}(n) (\theta - \theta_n) < \kappa_n\}.$$

Thus it is obvious that κ_n is positive if $\Gamma(n)$ is non-empty.

We claim that there exists a $\sigma > 0$ such that

$$\sigma < \frac{\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1)}. \quad (3.21)$$

Handwritten note: $\checkmark$

To show (3.21), consider

$$\frac{\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1)} = \frac{\mathbf{x}_n^T \kappa_{n-1} \mathbf{P}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \kappa_{n-1}^m \det \mathbf{P}(n-1)} \quad (3.22)$$

$$= \frac{\mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n}{\kappa_{n-1}^{m-1} \|\mathbf{x}_n\|^2 \det \mathbf{P}(n-1)} \quad (3.23)$$

Since the PE condition defined by (3.4) holds, $\mathbf{P}(n)$ is positive definite. Therefore

$$0 < \lambda_{\min} \|\mathbf{x}_n\|^2 \leq \mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n,$$

where $\lambda_{\min}$ is the minimum eigenvalue of $\mathbf{P}(n-1)$. Now

$$\frac{\lambda_{\min} \|\mathbf{x}_n\|^2}{\kappa_{n-1}^{m-1} \|\mathbf{x}_n\|^2 \det \mathbf{P}(n-1)} \leq \frac{\mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n}{\kappa_{n-1}^{m-1} \|\mathbf{x}_n\|^2 \det \mathbf{P}(n-1)} = \frac{\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1)}.$$

That is,

$$\frac{\lambda_{\min}}{\kappa_{n-1}^{m-1} \det \mathbf{P}(n-1)} \leq \frac{\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1)}.$$

By the positive definite property of $\mathbf{P}(n)$, we have $\det \mathbf{P}(n-1) < \infty$. Also $\kappa_{n-1}^{m-1} < \infty$.

Thus we conclude that there exists a $\sigma > 0$ such that

$$\sigma < \frac{\lambda_{\min}}{\kappa_{n-1}^{m-1} \det \mathbf{P}(n-1)} \leq \frac{\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n}{\|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1)},$$

and, hence, (3.21) is satisfied.

In the remaining part of this proof, we consider the subsequence $\{t_n\}$ only. In order to simplify the notation, we replace the time index t_n by n for the rest of this proof. Since $\mathbf{x}_n$ satisfies (3.4), there exists a δ such that there are infinitely many n for which $\|\mathbf{x}_n\|^2 > \delta$. Let $\epsilon > 0$. According to (3.20), there exists an N_1 such that if $n > N_1$ then

$$\left| m(\gamma_n - \varepsilon_n^2) - \kappa_n G_n \right| < \epsilon \delta \sigma. \quad (3.24)$$

Let $n > N_1$. Since $\gamma_n - \varepsilon_n^2 < 0$, we have from (3.24) that

$$m \left| \gamma_n - \varepsilon_n^2 \right| + \kappa_{n-1} G_n < \epsilon \delta \sigma.$$

Thus

$$0 < \kappa_{n-1} G_n = \kappa_{n-1} \mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n = \mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n < \epsilon \delta \sigma.$$

By (3.21),

$$0 < \sigma \|\mathbf{x}_n\|^2 \det \mathbf{B}(n-1) < \mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n < \epsilon \delta \sigma.$$

Thus

$$\det \mathbf{B}(n-1) < \frac{\delta}{\|\mathbf{x}_n\|^2} \epsilon.$$

Note that there exists an $N > N_1$ such that

$$\|\mathbf{x}_N\|^2 > \delta.$$

Therefore

$$\det \mathbf{B}(N-1) < \frac{\delta}{\|\mathbf{x}_N\|^2} \epsilon < \epsilon.$$

Since $\det \mathbf{B}(n)$ is non-increasing,

$$\det \mathbf{B}(n) < \epsilon \text{ for all } n > N. \quad (3.25)$$

By definition, the volume of the ellipsoid converges to zero. $\square$

Recall that $\varepsilon_n = -\tilde{\boldsymbol{\theta}}_{n-1}^T \mathbf{x}_n + v_n$. Therefore $\gamma_n - \varepsilon_n^2 < 0$ if and only if

$$|v_n - \tilde{\boldsymbol{\theta}}_{n-1}^T \mathbf{x}_n|^2 > \gamma_n \quad (3.26)$$

If the volume of the ellipsoid keeps shrinking, the value $\tilde{\boldsymbol{\theta}}_{n-1}^T \mathbf{x}_n$ will decrease to a small number as n becomes large. When $\tilde{\boldsymbol{\theta}}_{n-1}^T \mathbf{x}_n$ is small, from (3.26) we see that v_n must be sufficiently close to the bound so that $\gamma_n - \varepsilon_n^2 < 0$ for infinitely many n . This is very important for the convergence of the parameter set. In Section 3.3.2 we will show that if either the noise bound is over-estimated or the realization of the noise process does not visit an arbitrary neighborhood of the bound infinitely often, then the ellipsoid can not converge to a point asymptotically.

3.2.2 A Suboptimal Test for Innovation in the SM-SA Algorithm

Theorem 3.4 leads to the pursuit of a suboptimal test. Recall that one important feature for members of UOBE is “selective updating.” That is, no updating is taken when incoming data do not contain useful information. Sub optimal test is a small modification of the updating criterion associated with SM-SA. Instead of checking $c_n = m(\gamma_n - \varepsilon_n^2) - \kappa_{n-1}G_n$, the suboptimal test checks only $\bar{c}_n \triangleq \gamma_n - \varepsilon_n^2$ since the negativeness of $\bar{c}_n$ implies the

negativeness of c_n . Note that the unique root λ_n^* on $(0, 1)$ still exists for suboptimal test.

Suboptimal testing has been studied in previous work [28]. The computational benefits of employing a suboptimal test can be substantial since the updating criterion is reduced to $\mathcal{O}(m)$ computations where m is the model order. The weights obtained from suboptimal test, however, may not be optimal in the sense of volume ratio minimization. In order to obtain similar convergence properties as we do in SM-SA, define

$$\tilde{c}_n = \begin{cases} c_n & \text{if } \bar{c}_n < 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.2 *Given the coefficients of the quadratic equation (2.31) and the conditions stated in Theorem 3.3, we have $\lim_{n \rightarrow \infty} \tilde{c}_n = 0$ if and only if $\lim_{n \rightarrow \infty} \lambda_n^* = 0$, where λ_n^* is obtained by solving the quadratic equation*

$$a_n \lambda_n^2 + b_n \lambda_n + c_n = 0$$

when $\bar{c}_n < 0$ and is set to be zero when $\bar{c}_n \geq 0$.

We do not intend to give formal proof for this lemma because $\bar{c}_n < 0$ implies that $c_n < 0$ and hence the proof follows directly from that of Lemma 3.1. Only note that c_n in proof of lemma 3.1 must change to $\tilde{c}_n$. The importance of the suboptimal test in theoretical analysis can be seen in the next theorem.

Theorem 3.5 *Let $(\mathcal{H}^m, \|\cdot\|)$ be a normed space. Given the $\mathbf{R}(n)$ in (3.3) and the update equations (2.7)-(2.11), if $\mathbf{R}(n)$ is positive definite and the suboptimal test is employed with infinitely many data points selected, then the SM-SA algorithm developed in Theorem 2.1 converges in the sense that*

$$\lim_{n \rightarrow \infty} \lambda_n^* = 0 \tag{3.27}$$

$$\lim_{n \rightarrow \infty} \det \mathbf{B} = 0 \tag{3.28}$$

Proof: If we change c_n in the proof of Theorem 3.3 to $\bar{c}_n$, then by Lemma 3.2, the proof of (3.27) follows directly from the proof of Theorem 3.3. Since infinitely many data points are taken using the suboptimal test, there exists infinitely many points such that $\gamma_n - \varepsilon_n^2 < 0$. By Theorem 3.4, the volume of the ellipsoid converges to zero and hence (3.28) is satisfied. $\square$

Again, if the algorithm stops at finite time N , then $\lim_{n \rightarrow \infty} \lambda_n^* = 0$ for $n > N$. Thus (3.27) still holds. In this case, however, the volume of the ellipsoid cannot converge to zero. Simulation results will be presented in the next chapter in support of the above analysis.

3.3 Convergence of the Parameter Set with White Noise Case or Errors Disturbances

White noise has been studied frequently in the analysis of many estimation algorithms as well as model structures. A sequence $\{v_n\}$ is defined to be white noise if

$$Ev_n = 0 \tag{3.29}$$

and

$$Ev_i v_{i-j} = \begin{cases} 0 & j \neq 0 \\ \sigma^2 & j = 0 \end{cases} \tag{3.30}$$

where σ^2 is referred to as *variance* of the process. If (3.30) holds, $\{v_n\}$ is said to be *uncorrelated*. It is common to assume that $\{v_n\}$ is second moment ergodic so that the statistical properties in (3.29) and (3.30) can be realized by time averages, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n v_i = 0 \tag{3.31}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n v_i v_{i-j} = \begin{cases} 0 & j \neq 0 \\ \sigma^2 & j = 0 \end{cases} \tag{3.32}$$

For LSE algorithms, white properties have been shown to be sufficient for the convergence of the estimates for the structure in (2.1) [13]. Since the estimates in SM-SA belong to a class of LSE estimates, it is of interest to know whether the white property of noise excitation also applies to SM-SA.

In this section, we will first consider a special class of white noise which is called *i.i.d. white noise* in which the random variables v_i and v_j are not only uncorrelated but also *independent*, and identically distributed for all i, j . Following the analysis of the *i.i.d.* case, the general white noise processes defined in (3.29)-(3.30) or (3.31)-(3.32) are considered.

3.3.1 IID White Noise Case

Theorem 3.6 *Suppose the noise process $\{v_n\}$ is an i.i.d. white noise in which $\{v_n\}$ is zero mean and uniformly distributed on $[-\sqrt{\gamma}, \sqrt{\gamma}]$ for all n . If SM-SA is employed with noise bound given by*

$$v_n^2 < \gamma \text{ for all } n,$$

then the volume of the ellipsoid converges to zero.

Proof: Let $F \triangleq \{n \in \mathbb{N} : c_n < 0\} = \{n \in \mathbb{N} : c_n^- \neq 0\}$ and let $n \in F$. Since $\lim_{n \rightarrow \infty} c_n^- = 0$,

*(F is infinite set
if primary data is not deleted)*

$$\lim_{n \rightarrow \infty} m(\gamma - \varepsilon_n^2) - \kappa_{n-1} G_n = 0.$$

Note that

$$\kappa_{n-1} G_n = \kappa_{n-1} \mathbf{x}_n^T \mathbf{P}(n-1) \mathbf{x}_n = \mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n.$$

Thus

$$\lim_{n \rightarrow \infty} c_n^- = \lim_{n \rightarrow \infty} [m(\gamma - \varepsilon_n^2) - \mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n] = 0. \quad (3.33)$$

Equation (3.33) implies that in the limit

$$\gamma - \varepsilon_n^2 = \frac{\mathbf{x}_n^T \mathbf{B}_{n-1} \mathbf{x}_n}{m}.$$

We may assume that $\|\mathbf{x}_n\| \neq 0$ since the probability of $\|\mathbf{x}_n\| = 0$ is zero. Suppose the volume of the ellipsoid does not converge to zero, that is, $\lim_{n \rightarrow \infty} \det \mathbf{B}(n-1) \neq 0$. Since $\mathbf{B}(n-1)$ is positive definite, $\mathbf{x}_n^T \mathbf{B}(n-1) \mathbf{x}_n > 0$. Thus

$$\lim_{n \rightarrow \infty} \gamma - \varepsilon_n^2 > 0 \text{ for } n \in F.$$

For other $n \notin F$, the inequality $\gamma - \varepsilon_n^2 > 0$ still holds since $m(\gamma - \varepsilon_n^2) > \kappa_{n-1} G_n$. Thus we have

$$\lim_{n \rightarrow \infty} \gamma - \varepsilon_n^2 > 0 \text{ for all } n, \quad (3.34)$$

and there exists an $\epsilon > 0$ and a subsequence $\{m_n\}_{n=1}^\infty$ such that

$$\gamma - \varepsilon_{m_n}^2 > \epsilon \quad \forall n. \quad (3.35)$$

Since $\varepsilon_n = -\tilde{\theta}_{n-1}^T \mathbf{x}_n + v_n$, we have

$$\begin{aligned} \gamma - \varepsilon_{m_n}^2 &= (\sqrt{\gamma} + |\varepsilon_{m_n}|)(\sqrt{\gamma} - |\varepsilon_{m_n}|) \\ &= (\sqrt{\gamma} + |\varepsilon_{m_n}|)(\sqrt{\gamma} - |v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}|) \\ &> \epsilon. \end{aligned}$$

Thus

$$\begin{aligned} \sqrt{\gamma} - |v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| &> \frac{\epsilon}{\sqrt{\gamma} + |\varepsilon_{m_n}|} \\ &> \frac{\epsilon}{2\sqrt{\gamma}} \\ &= \epsilon'. \end{aligned}$$

Therefore (3.35) holds if and only if

$$|v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| < \sqrt{\gamma} - \epsilon' \quad \forall n. \quad (3.36)$$

Expanding (3.36), we have

$$-(\sqrt{\gamma} - \epsilon') < v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < \sqrt{\gamma} - \epsilon',$$

and thus

$$-\sqrt{\gamma} + (\epsilon' + \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}) < v_{m_n} < \sqrt{\gamma} - (\epsilon' - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n})$$

1. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0$. If $-\sqrt{\gamma} < v_{m_n} < -\sqrt{\gamma} + \epsilon'$ (see Fig. 3.1(a)), then $\sqrt{\gamma} - \epsilon' < |v_{m_n}| < \sqrt{\gamma}$ and thus

$$\begin{aligned} |v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| &= |v_{m_n}| + |\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| \\ &> \sqrt{\gamma} - \epsilon' + |\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| \\ &> \sqrt{\gamma} - \epsilon', \end{aligned}$$

violating (3.36). Consequently, v_{m_n} cannot be in the shaded area in Fig. 3.1 (a).

Since v_n is uniformly distributed, the probability that (3.36) holds is less than $\frac{2\sqrt{\gamma}-\epsilon'}{2\sqrt{\gamma}}$.

2. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0$. If $\sqrt{\gamma} - \epsilon' < v_{m_n} < \sqrt{\gamma}$, then

$$\begin{aligned} |v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| &= v_{m_n} + |\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| \\ &> \sqrt{\gamma} - \epsilon' + |\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| \\ &> \sqrt{\gamma} - \epsilon', \end{aligned}$$

violating (3.36). The graph interpretation is in Fig. 3.1 (b). Since v_n is uniformly distributed, the probability that (3.36) holds is less than $\frac{2\sqrt{\gamma}-\epsilon'}{2\sqrt{\gamma}}$.

Let the probability of $-\tilde{\theta}_{n-1}^T \mathbf{x}_n > 0$ be q_n . Then, the probability that (3.36) is satisfied is less than

$$q_{m_n} \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} + (1 - q_{m_n}) \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} = \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} < 1 \quad (3.37)$$

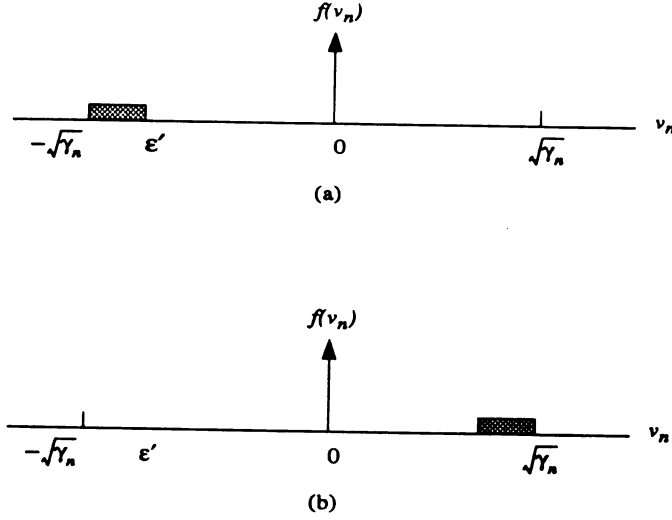


Figure 3.1. (a) If $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0$ and v_{m_n} in shaded area, then a contradiction results. (b) If $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0$ and v_{m_n} in shaded area, then a contradiction results.

Let A_n be the event that (3.36) holds at time m_n . Then from (3.37),

$$P(A_n) \leq \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} < 1.$$

Now, given (3.36) holds at time n , we claim that

$$P(A_{n+1}|A_n) \leq \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} < 1, \quad (3.38)$$

where A_{n+1} is the event that (3.36) holds at m_{n+1} . To show (3.38), consider

$$\begin{aligned} & P(A_{n+1}|A_n) \\ &= P(A_{n+1} \cap \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} | A_n) + P(A_{n+1} \cap \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} | A_n) \\ &= P(A_{n+1} | \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} \cap A_n) P(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} | A_n) \\ &\quad + P(A_{n+1} | \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} \cap A_n) P(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} | A_n) \\ &\leq P(-\sqrt{\gamma} + \epsilon' < v_{m_{n+1}} < \sqrt{\gamma} | \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} \cap A_n) P(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} | A_n) \\ &\quad + P(-\sqrt{\gamma} < v_{m_{n+1}} < \sqrt{\gamma} - \epsilon' | \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} \cap A_n) P(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} | A_n). \end{aligned}$$

Since $\{v_n\}$ is an *i.i.d.* white noise process,

$$P\left(-\sqrt{\gamma} + \epsilon' < v_{m_{n+1}} < \sqrt{\gamma} \mid \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} \cap A_n\right) = \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}$$

and

$$P\left(-\sqrt{\gamma} < v_{m_{n+1}} < \sqrt{\gamma} - \epsilon' \mid \{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} \cap A_n\right) = \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}.$$

Thus

$$P(A_{n+1} | A_n) \leq \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} P\left(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}}\} > 0 \mid A_n\right) + \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} P\left(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} < 0\} \mid A_n\right).$$

Let $q_{n+1} = P\left(\{\tilde{\theta}_{m_{n+1}-1}^T \mathbf{x}_{m_{n+1}} > 0\} \mid A_n\right)$. Then

$$P(A_{n+1} \mid A_n) = q_{n+1} \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} + (1 - q_{n+1}) \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}} = \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}.$$

Thus (3.38) is proved and

$$P(A_{n+1} \cap A_n) = P(A_{n+1} \mid A_n)P(A_n) \leq \left(\frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}\right)^2$$

A similar method can also be used to show that

$$P(A_{n+2} | A_{n+1} \cap A_n) \leq \frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}$$

and

$$P(A_{n+2} \cap A_{n+1} \cap A_n) = P(A_{n+2} | A_{n+1} \cap A_n)P(A_{n+1} \cap A_n) \leq \left(\frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}\right)^3.$$

By induction, we have

$$\lim_{n \rightarrow \infty} P(A_n \cap A_{n-1} \cap A_{n-2} \cap \dots \cap A_1) \leq \lim_{n \rightarrow \infty} \left(\frac{2\sqrt{\gamma} - \epsilon'}{2\sqrt{\gamma}}\right)^n = 0.$$

Thus we conclude that the ellipsoid converges to zero. $\square$

3.3.2 General White Noise Case

Set-membership-based algorithms are derived from noise bounds. In Theorem 3.4 we showed that if there exists infinitely many data points such that $\gamma_n - \varepsilon_n^2 < 0$, then the volume of the ellipsoid converges to zero. Since $\varepsilon_n = v_n - \tilde{\theta}_{n-1}^T \mathbf{x}_n$, the geometrical meaning of $\gamma_n - \varepsilon_n^2 < 0$ is that v_n must be very close to the boundaries if more data points are to be used. Before we study the general white noise case, consider the cases in which the noise bounds are over-estimated or when the noise processes are highly correlated such that

$$v_n^2 < \gamma_n - \epsilon \quad \forall n \geq N \quad (3.39)$$

for some N and ϵ .

Claim 3.1 *Suppose there exists N and ϵ such that (3.39) holds. Then the ellipsoid does not shrink to a point.*

Proof: We prove this claim by contradiction. Consider the N and ϵ in (3.39). Suppose the ellipsoid converges to a point. Let $n \geq N$. Then

$$(\sqrt{\gamma_n} - |v_n|)(\sqrt{\gamma_n} + |v_n|) > \epsilon.$$

Therefore

$$\begin{aligned} \sqrt{\gamma_n} - |v_n| &> \frac{\epsilon}{\sqrt{\gamma_n} + |v_n|} \\ &> \frac{\epsilon}{\sqrt{\gamma_n} + \sqrt{\gamma_n}} \\ &> \frac{\epsilon}{2\sqrt{\gamma_n}}. \end{aligned}$$

Since $\mathbf{x}_n$ is bounded, by the stability assumption on (1.5), and since the volume of the ellipsoid is assumed to converge to zero, there exists an $N_1 \geq N$ such that

$$|\tilde{\theta}_{n-1}^T \mathbf{x}_n| < \frac{\epsilon}{3\sqrt{\gamma}} < \frac{\epsilon}{3\sqrt{\gamma_n}},$$

where $\gamma = \max_{n \in N} \{\gamma_n\}$. Let $n \geq N_1$, then

$$\begin{aligned}
\gamma_n - \varepsilon_n^2 &= (\sqrt{\gamma_n} + |\varepsilon_n|)(\sqrt{\gamma_n} - |\varepsilon_n|) \\
&= (\sqrt{\gamma_n} + |\varepsilon_n|)(\sqrt{\gamma_n} - |v_n - \tilde{\theta}_{n-1}^T \mathbf{x}_n|) \\
&> (\sqrt{\gamma_n} + |\varepsilon_n|)(\sqrt{\gamma_n} - |v_n| - |\tilde{\theta}_{n-1}^T \mathbf{x}_n|) \\
&> (\sqrt{\gamma_n} + |\varepsilon_n|)(\frac{\epsilon}{2\sqrt{\gamma_n}} - |\tilde{\theta}_{n-1}^T \mathbf{x}_n|) \\
&> (\sqrt{\gamma_n} + |\varepsilon_n|)(\frac{\epsilon}{2\sqrt{\gamma_n}} - \frac{\epsilon}{3\sqrt{\gamma_n}}) \\
&= (\sqrt{\gamma_n} + |\varepsilon_n|)(\frac{\epsilon}{6\sqrt{\gamma_n}}) \\
&> \frac{\epsilon}{6}
\end{aligned}$$

Now the convergence of the ellipsoid to a point implies that κ_n converges to zero. So there exists an $N_2 \geq N_1$ such that

$$\kappa_{n-1} G_n < \frac{\epsilon}{6} \quad \forall n \geq N_2.$$

Let $n > N_2$. Then $n > N$ and

$$\begin{aligned}
c_n &= m(\gamma_n - \varepsilon_n^2) - \kappa_{n-1} G_n \\
&> \frac{m\epsilon}{6} - \frac{\epsilon}{6} \\
&> 0.
\end{aligned}$$

This result is contrary to the assumption since the algorithm stops taking points after time N_2 . Thus we conclude that the volume of the ellipsoid can not converge to zero. $\square$

Claim 3.1 points out the impact of the noise bounds on the volume of the ellipsoids and thus leads us to study the relationship between the white noise process and the noise bounds. Let $\{v_n\}$ be a white noise process defined in (3.29)-(3.30). If $\{v_n\}$ is *i.i.d.*, then from theorem 3.6 the volume of the ellipsoid converges to zero. The key point in the proof is that the magnitude of the noise can be arbitrarily close to the boundaries. If we eliminate

the condition of independence, then the noise may not be arbitrarily close to the boundaries.

Consider the following example.

Let $\{v_n\}$ be a noise process taking only discrete values at $-\sqrt{\gamma}$, $-\sqrt{2\gamma/3}$, 0 , $\sqrt{2\gamma/3}$, and $\sqrt{\gamma}$. Let A be an event defined in an appropriate probability space, and v_0 be the initial condition such that $P(v_0 \in A) = 2/5$.

If $v_0 \in A$, then $\{v_n\}$ is defined as

$$v_n = \begin{cases} \sqrt{2\gamma/3}, & w_n \geq 0 \\ -\sqrt{2\gamma/3}, & w_n < 0 \end{cases}.$$

Otherwise

$$v_n = \begin{cases} \sqrt{\gamma}, & 1 < w_n \leq 3 \\ 0, & -1 < w_n \leq 1 \\ -\sqrt{\gamma}, & -3 \leq w_n \leq -1 \end{cases}.$$

Sequence $\{w_n\}$ is an *i.i.d.* white noise with mean zero and uniform distribution on $[-3, 3]$.

The mass function of $\{v_n\}$ is

$$P(v_n = -\sqrt{2\gamma/3}) = P(v_n = \sqrt{2\gamma/3}) = \frac{1}{2}P(v_0 \in A) = \frac{1}{5}$$

$$P(v_n = -\sqrt{\gamma}) = P(v_n = 0) = P(v_n = \sqrt{\gamma}) = \frac{1}{3}P(v_0 \notin A) = \frac{1}{5}.$$

Thus, $(v_n, n \geq 1)$ is uniformly distributed over $\{-\sqrt{\gamma}, -\sqrt{2\gamma/3}, 0, \sqrt{2\gamma/3}, \sqrt{\gamma}\}$, and hence we can write $v_n^2 \leq \gamma$ for all n . The conditional expectations are given by

$$E(v_n | v_0 \in A) = \sqrt{2\gamma/3}P(w_n \geq 0) - \sqrt{2\gamma/3}P(w_n < 0) = 0$$

$$E(v_n | v_0 \notin A) = \sqrt{\gamma}P(1 < w_n \leq 3) - \sqrt{\gamma}P(-3 \leq w_n \leq -1) = 0.$$

Therefore,

$$E(v_n) = E(v_n | v_0 \in A)P(v_0 \in A) + E(v_n | v_0 \notin A)P(v_0 \notin A) = 0.$$

Also, we find that

$$E(v_s v_t \mid v_0 \in A) = \begin{cases} \frac{2\gamma}{3}, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases},$$

and

$$E(v_s v_t \mid v_0 \notin A) = \begin{cases} \frac{2\gamma}{3}, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}.$$

Consequently,

$$E(v_s v_t) = \begin{cases} \frac{2\gamma}{3}, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}.$$

As a result, $\{v_n\}$ is a white noise with mean zero and variance $\frac{2\gamma}{3}$. Furthermore, if second moment conditional ergodicity is assumed, as defined in (3.31)-(3.32), given v_0 , then v_n is also second moment ergodic. If RLS algorithm is used to identify the model parameters in (2.1) with this noise model, then the estimates converge to the true parameters since v_n is uncorrelated. This may not happen to SM-SA if the initial condition $v_0 \in A$, simply because v_n can not be arbitrarily close to the boundaries. By Claim 3.1, the ellipsoid cannot shrink to a point.

In this presented example v_n can not be close to the boundaries since the dependency between v_n and v_0 can not be eliminated, even for large n . In addition to this noise process, there are other white noise processes in which $\{v_n\}$ rarely hits the boundaries. In this case, the performance of SM-SA may not be favorable.

For the following example, we will assume $\{w_n\}$ to be a sequence of pseudo-random numbers with uniform distribution on $[-0.5, 0.5]$. First, let $\{v_n\}$ be a white noise process generated by passing $\{w_n\}$ through a first-order all-pass filter

$$H(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}.$$

Figure 3.2 and 3.3 show the sample correlation coefficients of $\{w_n\}$ and $\{v_n\}$.

Both sequences have identical sample mean and second moment properties. Consider the systems

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} + v_n$$

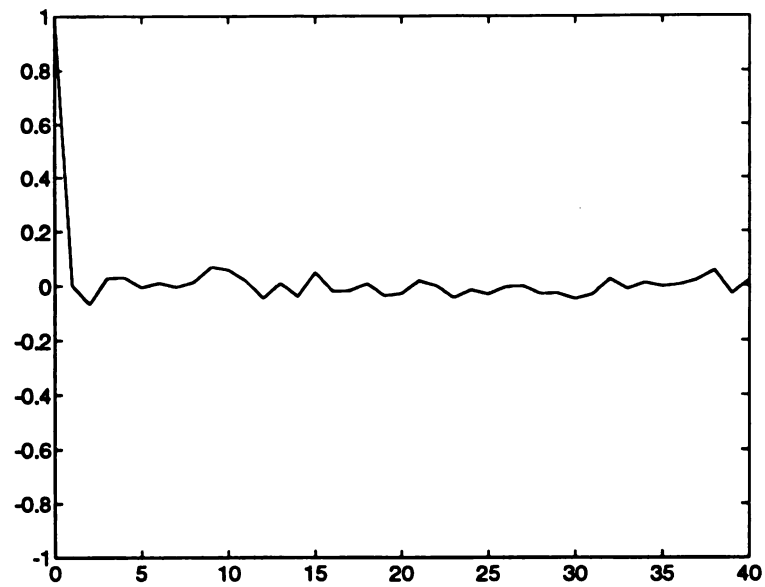


Figure 3.2. Sample correlation coefficients of $\{w_n\}$ in the example on page 63 up to time lag 40.

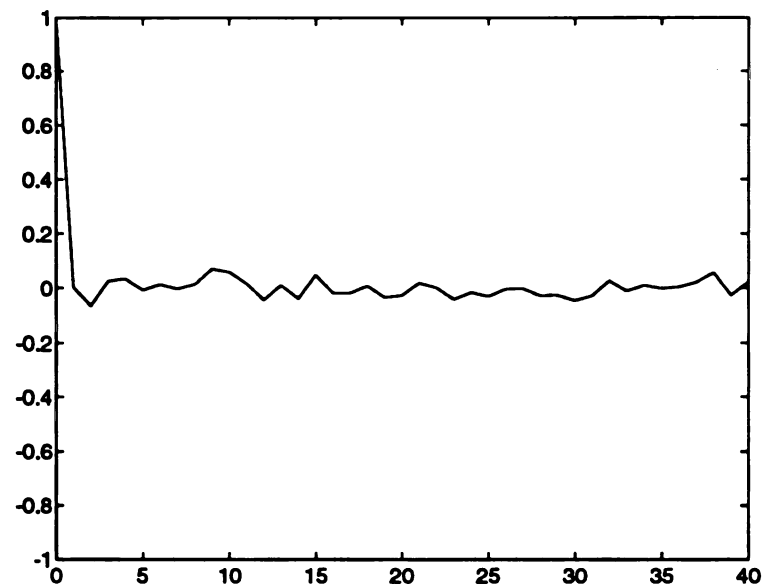


Figure 3.3. Sample correlation coefficients of $\{v_n\}$ in the example on page 63 up to time lag 40.

and

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} + w_n$$

where $a_1 = 2.1$, $a_2 = -1.48$, and $a_3 = 0.34$. Comparing the estimates obtained by RLS and SM-SA in Fig. 3.4 (a) and (b), we find that the behavior of SM-SA for these two noise processes is completely different while the estimates obtained by RLS are almost the same because both noise processes $\{v_n\}$ and $\{w_n\}$ are white.

The difference in SM-SA performance can be explained by analyzing the visitation frequency of the noise bounds. Fig. 3.5 compares the relative frequency of $\{w_n\}$ and $\{v_n\}$ at each amplitude interval. In these figures, the horizontal axis represents the amplitude and the vertical axis represents the number of times the noise amplitude falls in a specific interval. We see that the distribution of $\{w_n\}$ at each interval is close to be uniform on $[-0.5, 0.5]$, while the distribution of $\{v_n\}$ appears Gaussian, although it is not, within the interval $[-0.9, 0.9]$. Since $\{v_n\}$ rarely stays in the neighborhood of the noise bounds, the ellipsoid cannot shrink enough to provide good estimates as shown in Fig. 3.6.

However, if we pass $\{v_n\}$ through a soft-limiter such that v_n is set to be 0.5 or -0.5 if $v_n > 0.5$ and $v_n < -0.5$, respectively, and otherwise left unchanged, then the frequency of visitation to be a neighborhood of the bound, namely $\gamma = 0.25$, increases as depicted in Fig. 3.7. This new $\{v_n\}$ still seems to have white properties as shown in Fig. 3.8. However, from Fig. 3.9, we find that the performance of SM-SA in this case is much better than before.

If we multiply the original $\{v_n\}$ by two and pass it through the same soft-limiter, then the rate of visiting the neighborhood of the noise bounds by $\{v_n\}$ is much higher as shown in Fig. 3.10. Figure 3.11 shows that in this case the performance of SM-SA is extremely good.

If we examine the proof of Theorem 3.6, we find that the convergence of the ellipsoid does not depend on the first and the second moment properties of the noise input. That is, even if the conditions defined in (3.29) – (3.32) are not satisfied, the estimates may still converge to the true parameters if the “close-to-boundaries” condition is satisfied. For example, let $\{v_n\}$ be a noise process obtained by passing $\{w_n\}$ (i.i.d. noise, see page 61)

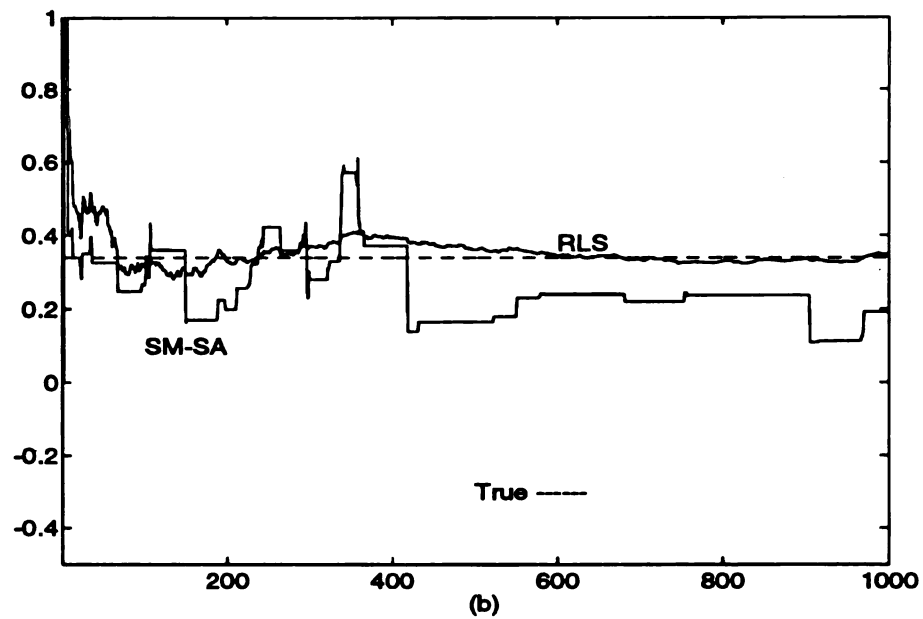
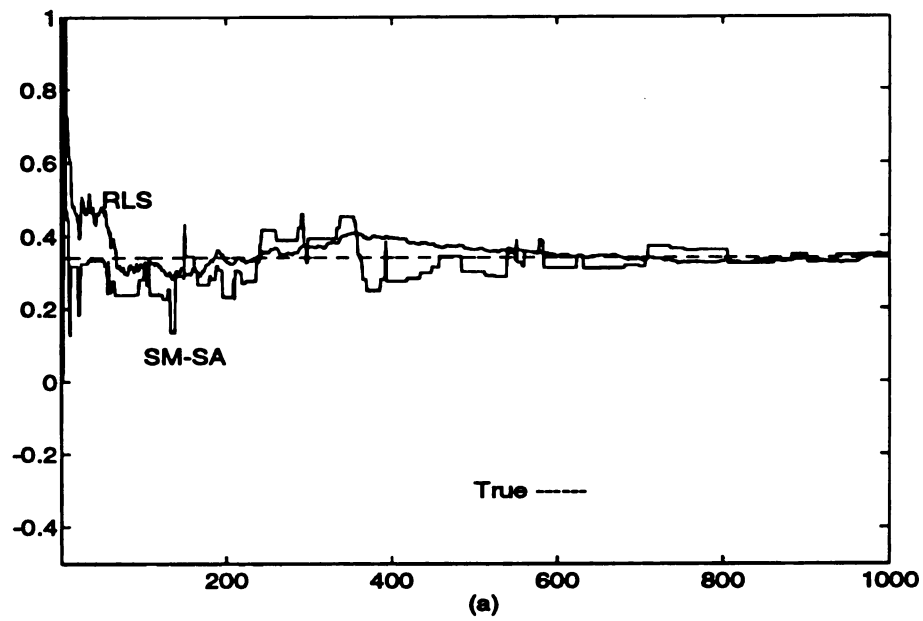


Figure 3.4. Estimates of the parameter a_3 when the system is driven by (a) an *i.i.d.* white noise (b) a white noise but not *i.i.d.*.

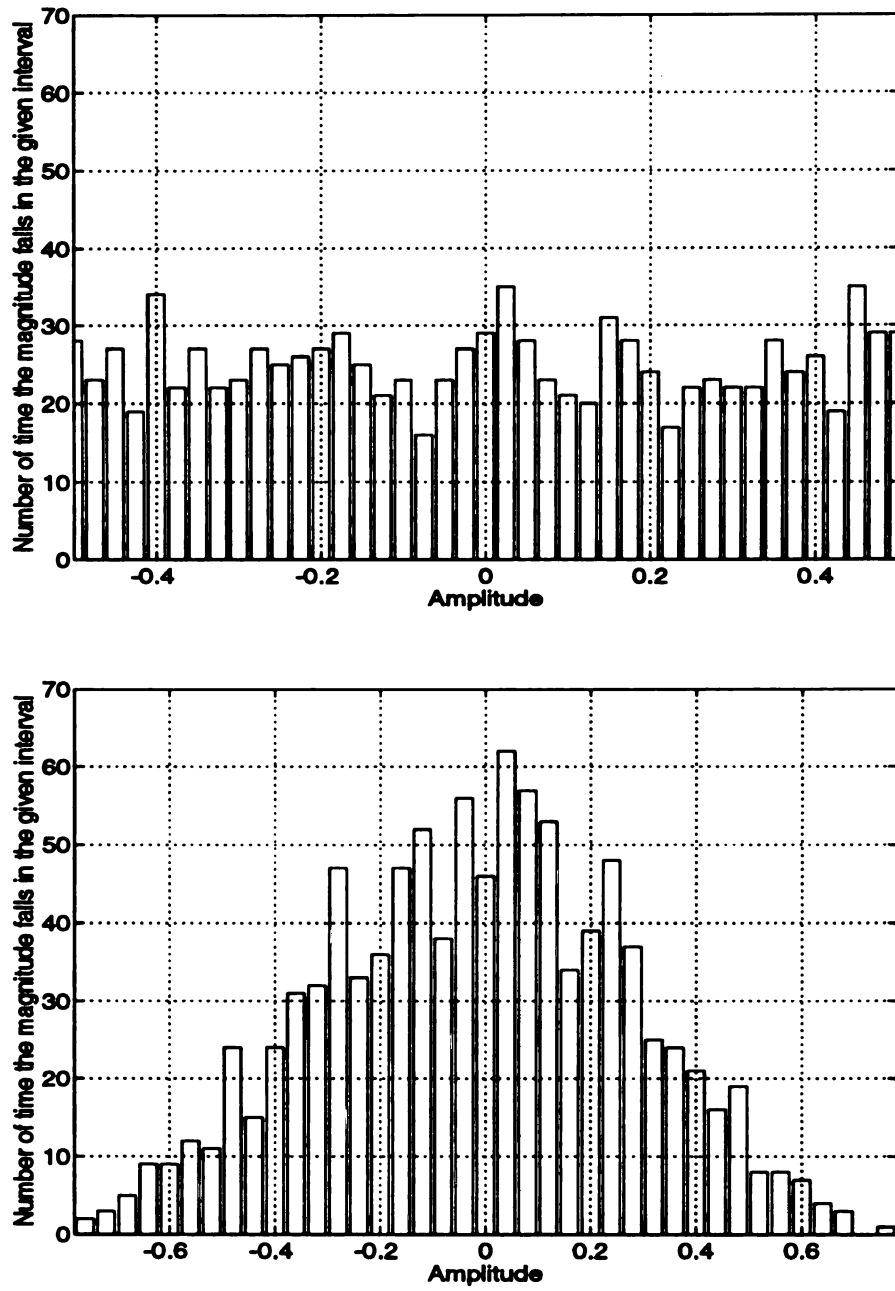


Figure 3.5. Distribution of $\{w_n\}$ and $\{v_n\}$ at each amplitude interval.

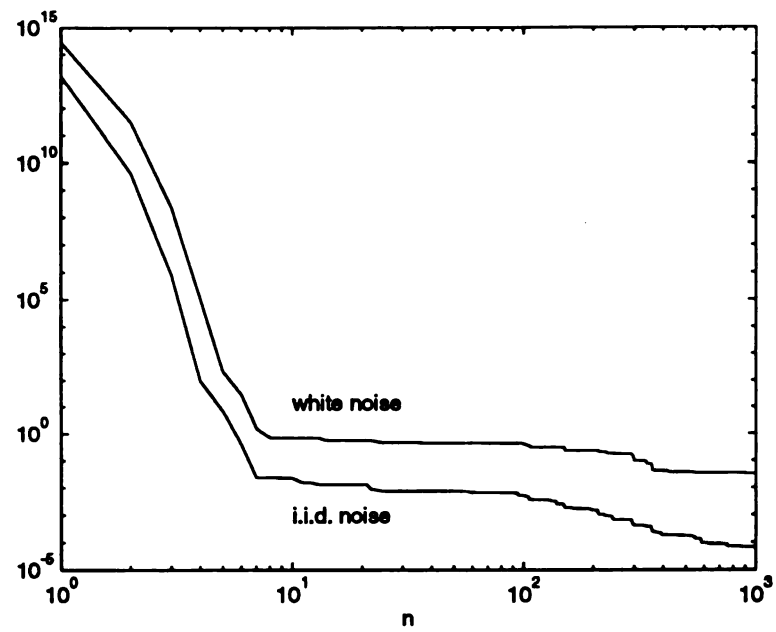


Figure 3.6. The logarithmic plot of volume versus the actual time n . We see that the volume of ellipsoid is much smaller when the noise is *i.i.d.*

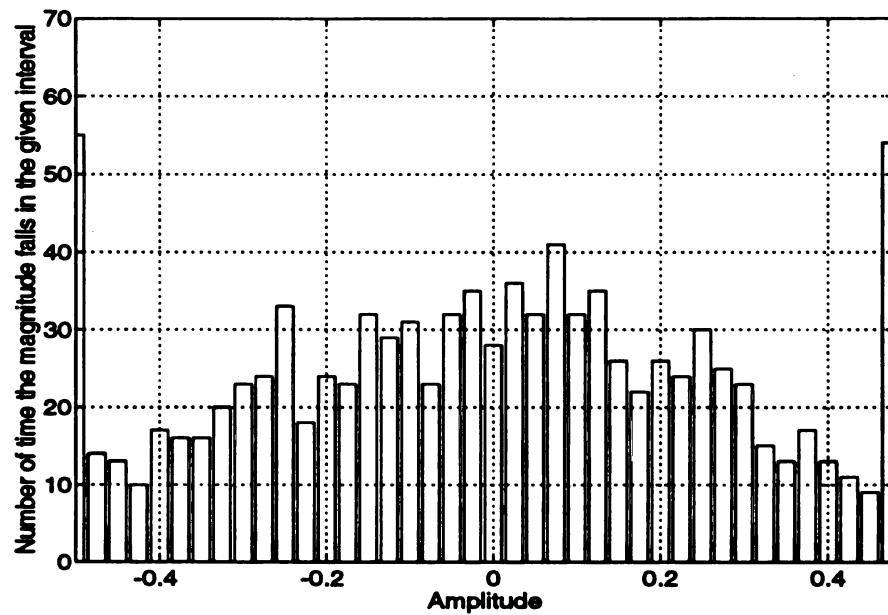


Figure 3.7. Distribution of a white noise at each amplitude interval after passing $\{v_n\}$ through a soft-limiter.

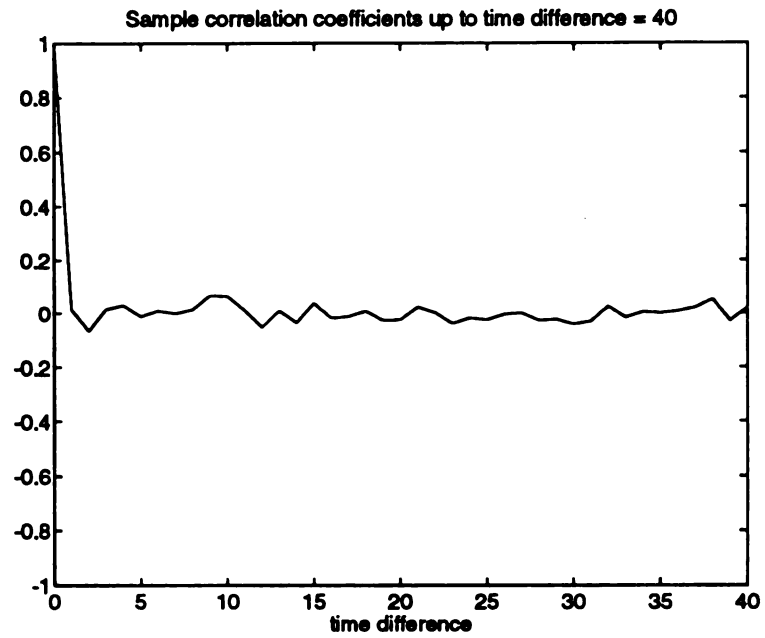


Figure 3.8. Sample correlation coefficients of $\{v_n\}$ after passing through a soft-limiter.

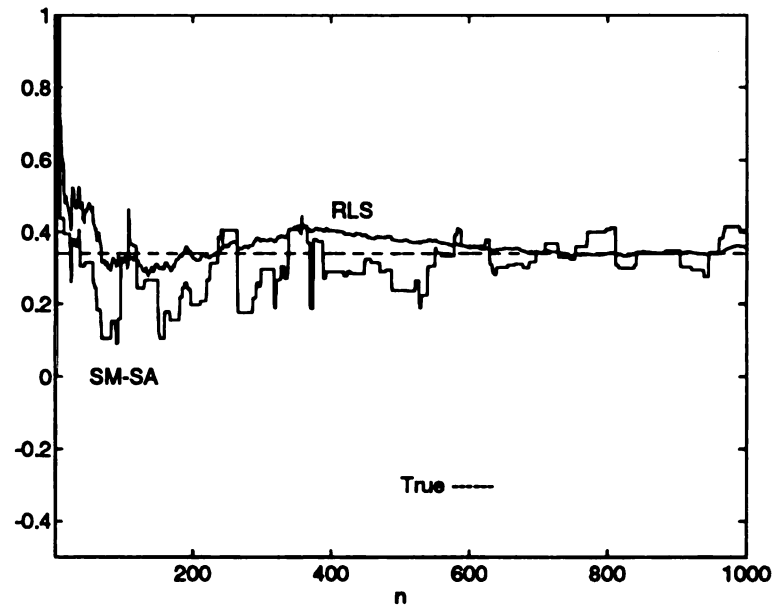


Figure 3.9. Estimates of the parameter a_3 when v_n is passed through a soft-limiter.

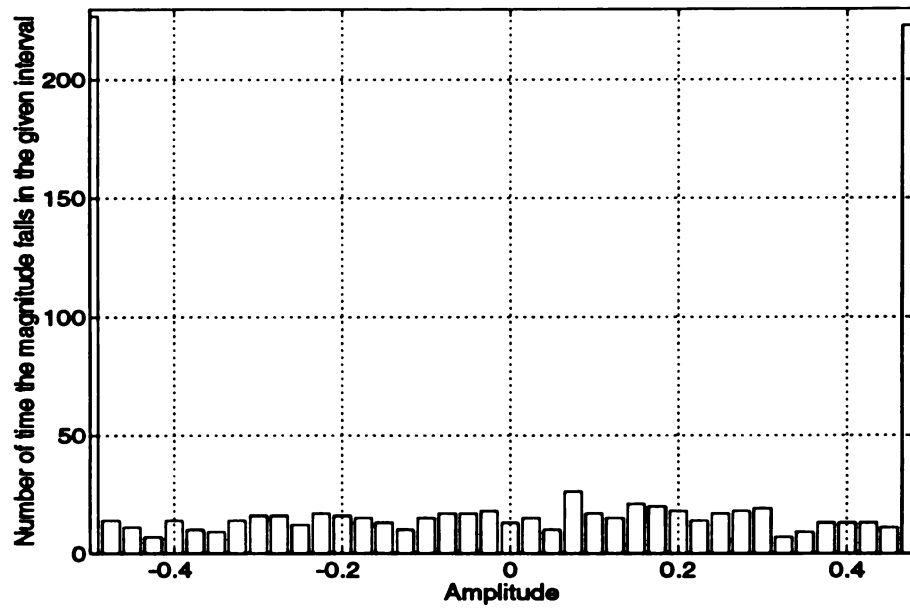


Figure 3.10. Distribution of a white noise at each amplitude interval after multiplying v_n by two and then passing it through a soft-limiter.

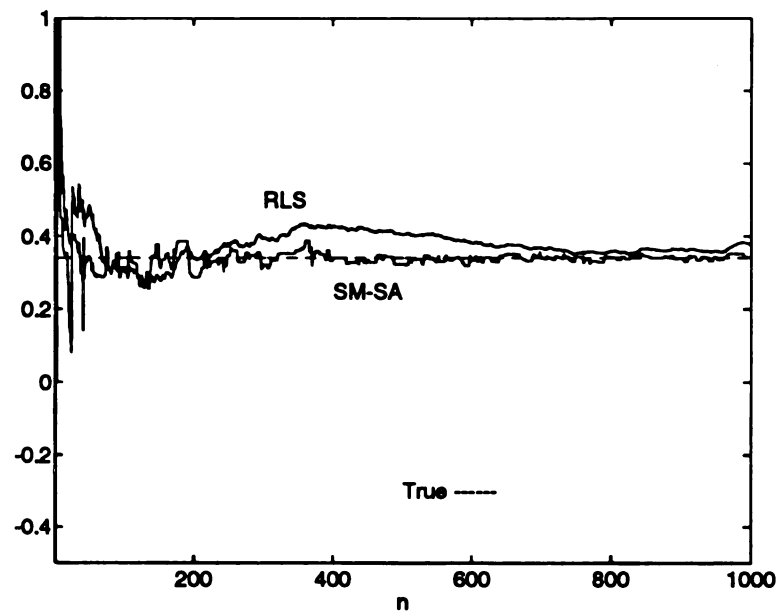


Figure 3.11. Estimates of the parameter a_3 when v_n is multiplied by two and then passed through a soft-limiter.

through an all-pole filter described by

$$H(z) = \frac{1}{1 + 0.5z^{-1}}$$

The output is bounded on $[-1, 1]$. From its sample correlation coefficients in Fig. 3.12, we know that $\{v_n\}$ is colored noise. Figure 3.13 shows that the performances of SM-SA and

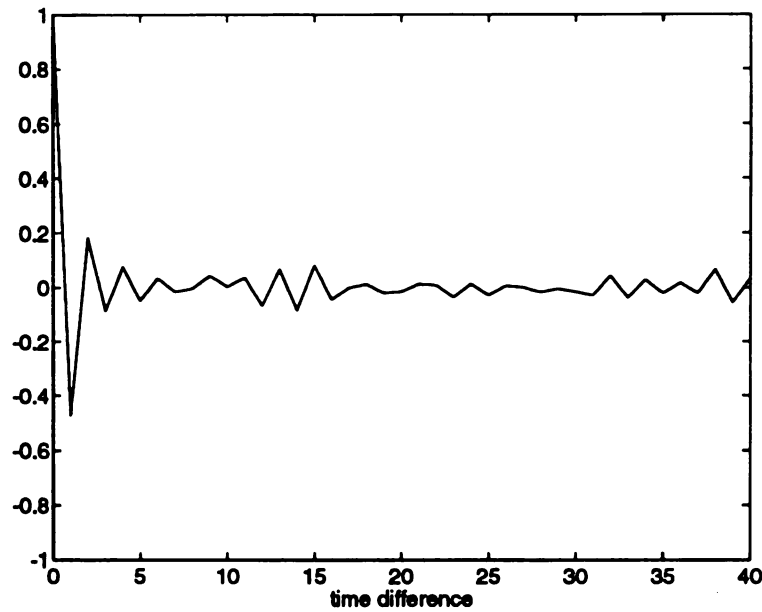


Figure 3.12. Sample correlation coefficients of $\{v_n\}$ up to time lag 40.

RLS are severely affected by the autocovariance of $\{v_n\}$.

Again, the distribution of $\{v_n\}$ is examined in Fig. 3.14 where we find that $\{v_n\}$ rarely appear in the neighborhood of the noise bounds. If we multiply $\{v_n\}$ by two and pass it through a soft-limiter, as we described before, then $\{v_n\}$ stays in the neighborhood of the noise bounds, ± 0.5 , most of the time. Note that in this case the new $\{v_n\}$ is still a color noise as shown by its sample correlation coefficients in Fig. 3.15. However, the performance of SM-SA is excellent as shown in Fig. 3.16, while RLS is still affected by the autocovariance.

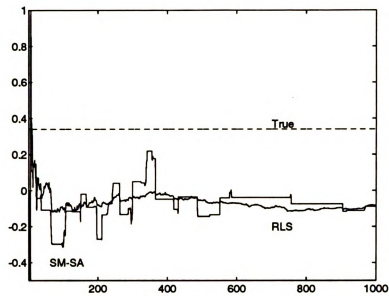


Figure 3.13. Estimates of a_3 by SM-SA and RLS when the noise is color.

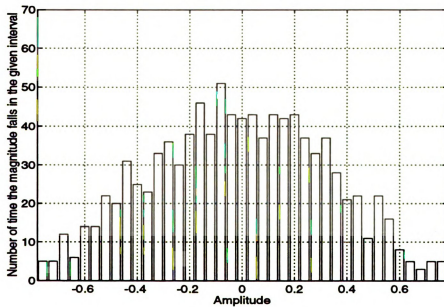


Figure 3.14. Distribution of a color noise at each amplitude interval.

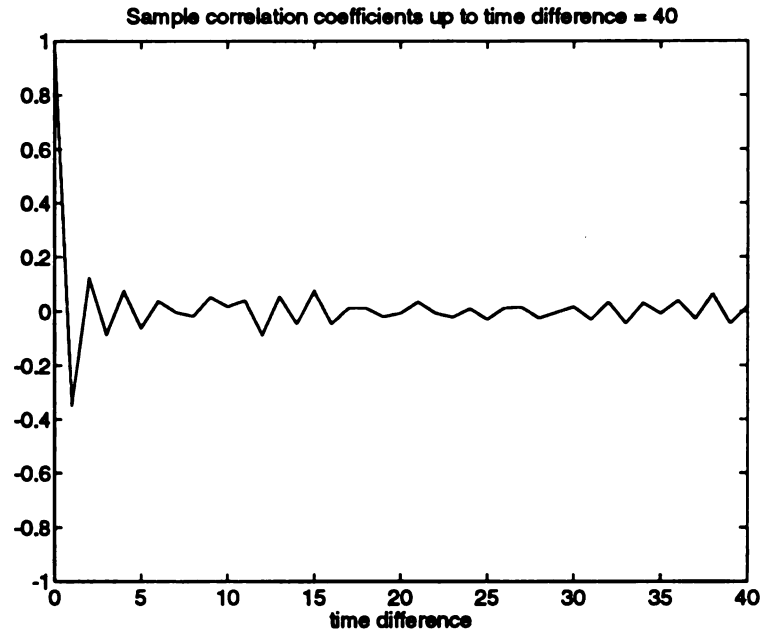


Figure 3.15. Sample correlation coefficients of a noise sequence obtained by passing $\{v_n\}$ through a soft-limiter.

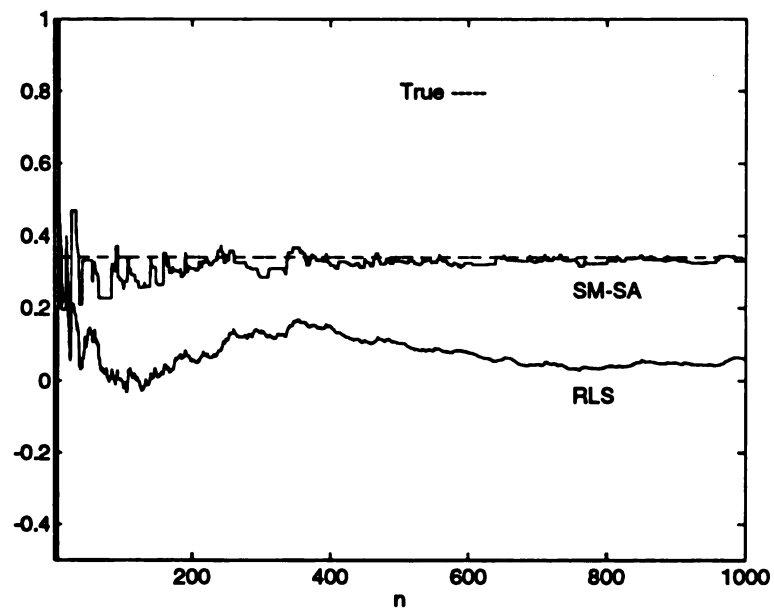


Figure 3.16. Estimates of the parameter a_3 when the color noise $\{v_n\}$ is passed through a soft-limiter.

The above examples show that the whiteness of the noise processes is not important in the performance of SM-SA, although it is important to the conventional RLS. The most important factor for the convergence of SM-SA is the probability for which the noise visits a neighborhood of the bounds. Thus, while SM-SA employs similar recursions to RLS, the two algorithms have totally different convergence properties. The convergence properties in RLS arise from the second moment properties; while the ones in SM-SA come from the noise bounding assumption.

3.4 Independence and The Mixing Conditions

To generalize the convergence of SM-SA to colored noise and give a more rigorous treatment, we first introduce the *measure-preserving transformation* T . Let $(S, \mathcal{F})$ be a measurable space consisting of a set S and σ -algebra $\mathcal{F}$ of subsets of S . Let P be a probability measure defined on $(S, \mathcal{F})$ so that $(S, \mathcal{F}, P)$ is a probability space. In our case, if X is a real set bounded by $\sqrt{\gamma}$,

$$X = [-\sqrt{\gamma}, \sqrt{\gamma}],$$

then $S = X^\infty$, the collection of all real sequences $(v_0, v_1, \dots)$. Let $T : (S, \mathcal{F}) \rightarrow (S, \mathcal{F})$ be a measurable transformation on $(S, \mathcal{F}, P)$, then T is said to be *measure-preserving* if and only if $P(T^{-1}A) = P(A)$ for all $A \in \mathcal{F}$ [10]. In order to simplify our discussion, we can take the measure-preserving transformation T as a *one-sided shifter*,

$$T(v_0, v_1, \dots) = (v_1, v_2, \dots).$$

Note that $P(T^{-1}A) = P(A)$ implies that $P(T^{-k}A) = P(A)$ for $k = 1, 2, \dots$. Thus $P\{(v_0, v_1, \dots, v_{n-1}) \in B_n\} = P\{(v_k, v_{k+1}, \dots, v_{n+k-1}) \in B_n\}$ for all n and k and n -dimensional Borel sets B_n .

Definition 3.2 Let T be measure-preserving on $(S, \mathcal{F}, P)$, T is said to be independent if for all $A, B \in \mathcal{F}$,

$$P(A \cap T^{-n}B) = P(A)P(B) \quad \forall n \in \mathcal{N}.$$

T is said to be mixing if for all $A, B \in \mathcal{F}$,

$$\lim_{n \rightarrow \infty} P(A \cap T^{-n}B) = P(A)P(B). \quad (3.40)$$

The mixing condition is a very reasonable assumption since it reflects the physical characteristics of many common random processes. In this research, let $\{v_n\}$ be the noise input of the model in (1.5), then $\{v_n\}$ satisfies the mixing condition if the dependency between v_i and v_j decreases to zero when $|i - j| \rightarrow \infty$.

For most system identification problems, we assume that the PE condition on $\{\mathbf{x}_n\}$ holds over all natural numbers. Since the SM-SA has a “selective updating” feature, we should confine the consideration of the PE condition to the subset of $\{\mathbf{x}_n\}$ in which updates take place. Since we do not know in advance which points will be used in the updates, this requires that all the infinite subsets of $\{\mathbf{x}_n\}$ are PE. However, this problem can be alleviated if the mixing condition is imposed on $\{v_n\}$. Consider the following corollary given by Bitmead [17]:

Corollary 3.1 If $\mathbf{x}_n$ is such that $E(\mathbf{x}_n \mathbf{x}_n^T)$ is positive definite and either $\mathbf{x}_n$ is ergodic or mixing, then there exists a constant $\beta > 0$ and a finite integer N such that

$$\sum_{i=n}^{n+N} \mathbf{x}_i \mathbf{x}_i^T \geq \beta \mathbf{I} > 0 \quad (3.41)$$

holds infinitely almost surely.

In the present work, the upper bound of (3.41) is guaranteed by the stability assumption. Note that all the subsequences of $\{\mathbf{x}_n\}$ are also mixing if $\{\mathbf{x}_n\}$ is mixing. Thus if $E(\mathbf{x}_n \mathbf{x}_n^T)$ is bounded below at the points which updates take place, then (3.4) is satisfied.

The next two corollaries serve as an extension of Theorem 3.6. First we consider the independent case in which the mean of the noise process does not have to be zero and the distribution does not have to be uniform.

Corollary 3.2 *Let $\{v_n\}$ be a noise sequence and let T be independent on $(S, \mathcal{F}, P)$ according to Definition 3.2. Given ϵ , let $D_{\epsilon+} = [\sqrt{\gamma} - \epsilon, \sqrt{\gamma}]$ and $D_{\epsilon-} = [-\sqrt{\gamma}, -\sqrt{\gamma} + \epsilon]$. If SM-SA is employed and there exists a δ such that for all n*

$$P(v_n \in D_{\epsilon+}) > \delta, \text{ and } P(v_n \in D_{\epsilon-}) > \delta,$$

then the volume of the ellipsoid converges to zero.

Proof: Consider the proof of Theorem 3.6. If the volume of the ellipsoid does not converge to zero, then (3.36) must hold. Given ϵ' , by assumption there exists a δ' such that for all n

$$P(v_n \in D_{\epsilon'+}) > \delta', \text{ and } P(v_n \in D_{\epsilon'-}) > \delta'.$$

1. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0$. Then the probability that (3.36) holds is less than $1 - \delta'$.
2. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0$. Then the probability that (3.36) holds is also less than $1 - \delta'$.

If we change the quantity $\frac{2\sqrt{\gamma}-\epsilon'}{2\sqrt{\gamma}}$ in the proof of Theorem 3.6 by $1 - \delta'$, then we have

$$\lim_{n \rightarrow \infty} P(A_n \cap A_{n-1} \cap A_{n-2} \cap \dots A_1) \leq \lim_{n \rightarrow \infty} (1 - \delta')^n = 0$$

and thus the ellipsoid converges to zero. □

The key point to the proof of the convergence of the parameter set is the independence between v_n and $\tilde{\theta}_{n-1}^T \mathbf{x}_n$. In order to extend the convergence property from independence to mixing condition, SM-SA has to be modified. The reason is that for the original SM-SA with the ARX model, $\mathbf{x}_n$ contains y_{n-1} , the output at time $n - 1$, thus we cannot separate the dependency between v_n and $\tilde{\theta}_{n-1}^T \mathbf{x}_n$ unless $\{v_n\}$ is independent. However, if we modify the SM-SA so that $\mathbf{x}_n$ is not changed unless an update takes place, provided that for large n the time difference between two updates is large, then we can apply the mixing condition to show convergence. This is possible since $\mathbf{x}_n$ depends only on the events up to the previous update. This modified version is identical to the SM-SA algorithm except for the regression vector $\mathbf{x}_n$. In SM-SA, $\mathbf{x}_n = [y_{n-1}, \dots, y_{n-p}, u_n, u_{n-1}, \dots, u_{n-q}]^T$, while in the modified algorithm, $\mathbf{x}_n = [y_{t_k}, \dots, y_{t_k-p}, u_n, u_{t_k}, \dots, u_{t_k-q}]^T$ where y_n, u_n are the same as in (1.5), and $\{t_i\}$ is a subsequence such that $c_{t_i} < 0$ and $t_k < n \leq t_{k+1}$. Note that in this situation, the algorithm is identical to RLS with a decimated input. The corresponding decimator has a time-varying sampling rate, say D_n , which is controlled by the data selection process. Generally, if y_n is wideband, then D_n is large and the data picked are usually sparse.

All the convergence proofs developed previously for SM-SA are still applicable to this modified algorithm, since those convergence properties are not affected by this modification. The following corollary applies to this modified algorithm.

Corollary 3.3 *Let $\{v_n\}$ be a noise sequence and let T be mixing on $(S, \mathcal{F}, P)$ according to Definition 3.2. Given ϵ and N , let $D_{\epsilon+} = [\sqrt{\gamma} - \epsilon, \sqrt{\gamma}]$ and $D_{\epsilon-} = [-\sqrt{\gamma}, -\sqrt{\gamma} + \epsilon]$. If the modified SM-SA is employed and there exists a δ such that for all n*

$$P(v_n \in D_{\epsilon+}) > \delta, \text{ and } P(v_n \in D_{\epsilon-}) > \delta,$$

and there exists an $M > N$ such that there are infinite number of interval $[n - 1, n - M]$ on which no update takes place, then the volume of the ellipsoid converges to zero.

Proof: Consider the proof of Theorem 3.6. Again let $\{m_n\}$ be the subsequence such that $\gamma - \varepsilon_{m_n}^2 > \epsilon$ for some $\epsilon > 0$. If the volume of the ellipsoid does not converge to zero, then (3.36) must hold. For convenience, we rewrite (3.36)

$$|v_{m_n} - \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n}| < \sqrt{\gamma} - \epsilon' \quad \text{where } \epsilon' = \frac{\epsilon}{2\sqrt{\gamma}}. \quad (3.42)$$

Given ϵ' , then, by assumption, there exists a δ' such that for all n

$$P(v_n \in D_{\epsilon'+}) > \delta', \quad \text{and} \quad P(v_n \in D_{\epsilon'-}) > \delta'.$$

1. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0$, then the probability that (3.42) holds is less than $1 - \delta'$.
2. Suppose $\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0$, then the probability that (3.42) holds is also less than $1 - \delta'$.

Again let A_n be the event that (3.42) holds at time m_n , then

$$\begin{aligned} P(A_n) &\leq P(-\sqrt{\gamma} + \epsilon' < v_{m_n} < \sqrt{\gamma} \mid \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0) P(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0) \\ &\quad + P(-\sqrt{\gamma} < v_{m_n} < \sqrt{\gamma} - \epsilon' \mid \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0) P(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0). \end{aligned}$$

Note that $\{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\}$ does not depend on the events which take place after the previous update. Let the time difference between m_n and the time index of the previous update be k . Define $W_{n-k} \triangleq \{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\}$ and $X_n \triangleq \{-\sqrt{\gamma} + \epsilon' < v_{m_n} < \sqrt{\gamma}\}$. Then $W_{n-k}, X_n \in \mathcal{F}$ and $P(X_n) \leq 1 - \delta'$. By the definition of mixing condition in (3.40),

$$\lim_{k \rightarrow \infty} P(X_n \cap W_{n-k}) = P(X_n)P(W_{n-k}). \quad (3.43)$$

Suppose $P(W_{n-k})$ is not zero. Then (3.43) can be written as

$$\lim_{k \rightarrow \infty} \frac{P(X_n \cap W_{n-k})}{P(W_{n-k})} = \lim_{k \rightarrow \infty} P(X_n \mid W_{n-k}) = P(X_n).$$

Let $0 < \xi < \delta'$, then there exists an N such that

$$|P(X_n \mid W_{n-N}) - P(X_n)| < \xi,$$

and, hence,

$$P(X_n) - \xi < P(X_n | W_{n-N}) < P(X_n) + \xi. \quad (3.44)$$

Note that (3.44) holds only if no update takes place on $\left[\frac{m_n - N - 1}{2}, \frac{m_n + 1}{2} \right]$ $[n - N, n]$.

Due to the slow convergence of the ellipsoid for large n , most incoming data are rejected. By assumption, there exists a positive integer $M \geq N$ such that no update takes place on $\{n - 1, n - 2, \dots, n - M\}$. Thus (3.44) is satisfied and

$$\begin{aligned} P(A_n) &\leq P\left(-\sqrt{\gamma} + \epsilon' < v_{m_n} < \sqrt{\gamma} \mid \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\right) P\left(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\right) \\ &\quad + P\left(-\sqrt{\gamma} < v_{m_n} < \sqrt{\gamma} - \epsilon' \mid \tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0\right) P\left(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0\right) \\ &\leq (P(-\sqrt{\gamma} + \epsilon' < v_{m_n} < \sqrt{\gamma}) + \xi) P\left(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\right) \\ &\quad + (P(-\sqrt{\gamma} < v_{m_n} < \sqrt{\gamma} - \epsilon') + \xi) P\left(\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0\right) \\ &\leq 1 - \delta' + \xi < 1, \end{aligned}$$

and

$$\begin{aligned} &P(A_n | A_{n-N}) \\ &\leq P\left(-\sqrt{\gamma} + \epsilon' < v_{m_n} < \sqrt{\gamma} \mid \{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\} \cap A_{n-N}\right) P\left(\{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} > 0\} \mid A_{n-N}\right) \\ &\quad + P\left(-\sqrt{\gamma} < v_{m_n} < \sqrt{\gamma} - \epsilon' \mid \{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0\} \cap A_{n-N}\right) P\left(\{\tilde{\theta}_{m_n-1}^T \mathbf{x}_{m_n} < 0\} \mid A_{n-N}\right) \\ &\leq 1 - \delta' + \xi < 1. \end{aligned}$$

Therefore,

$$P(A_n \cap A_{n-N}) = P(A_n | A_{n-N}) P(A_{n-N}) \leq (1 - \delta' + \xi)^2 < 1.$$

Note that

$$P(A_n \cap A_{n-1} \cap \dots \cap A_{n-N}) \leq P(A_n \cap A_{n-N}) \leq (1 - \delta' + \xi)^2 < 1,$$

and

$$\begin{aligned}
& \lim_{k \rightarrow \infty} P(A_{n+kN} \cap A_{n+kN-1} \cdots \cap A_n) \\
& \leq \lim_{k \rightarrow \infty} P(A_{n+kN} \cap A_{n+(k-1)N} \cap \cdots \cap A_n) \\
& \leq \lim_{k \rightarrow \infty} (1 - \delta' + \xi)^k = 0.
\end{aligned}$$

That is, the probability that the ellipsoid converges to zero is one.

□

Chapter 4

Simulation Studies

Set-membership based algorithms have shown great potential in applications for many signal processing and control systems problems involving parametric modeling [25, 26, 28], such as speech processing, image coding, neural networks, and adaptive control. This chapter is concerned with simulation studies using the SM-SA algorithm with optimal and suboptimal strategies for the properties discussed in Chapter 2 and 3.

The simulations performed in this chapter are designed for those noise processes which are commonly seen in engineering applications. These simulations not only serve as a test for theoretical analysis, but also provide heuristic descriptions for those phenomena which cannot be proven by rigorous mathematics even though they work for most engineering applications. We will see that the SM-SA performs better than conventional RLS in many problems if the noise bound can be correctly estimated.

In Section 4.1 we use an AR(3) time-invariant model to examine the general properties discussed in Chapter 3, including the behavior of the estimates, the volume of the ellipsoid, the estimation error, and the resemblance of SM-SA to stochastic approximation. Section 4.2 deals with a time-varying system derived from a speech signal. Comparisons to the conventional RLS are also made. Section 4.3 is concerned with the suboptimal test when the model is time-invariant. Finally, a more general case in which the noise bounds are over-estimated for the time-varying system is considered in Section 4.4.

For all the simulations in this chapter, the models are driven by the noise process $\{S_n\}$ which is the name representing all the noise processes summarized in Table 4.1. Note that

S_n	Characteristics of the noise processes
w_n	IID white noise with distribution $U[-0.5, 0.5]$
v_n	Obtained by $\frac{5}{6}(w_n + 0.1)$
z_n	Zero mean binary Bernoulli sequence obtained from $\{w_n\}$
t_n	Biased binary Bernoulli sequence obtained from $\{w_n\}$

Table 4.1. Noise processes used in the simulations in this chapter.

the sequence $\{z_n\}$ in Table 4.1 is obtained by

$$z_n = \begin{cases} 0.5 & w_n \geq 0 \\ -0.5 & w_n < 0 \end{cases},$$

and $\{t_n\}$ is obtained by

$$t_n = \begin{cases} 0.5 & w_n \geq -0.081 \\ -0.5 & w_n < -0.081 \end{cases}.$$

4.1 Performance of SM-SA for Time-Invariant Systems

In this section, an AR(3) model given by

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} + S_n \quad (4.1)$$

is considered. This model has three poles at $0.8 \pm 0.2j$ and 0.5 when $a_1 = 2.1$, $a_2 = -1.48$, and $a_3 = 0.34$. For these simulations, a set of 50,000 data points is generated to identify the model parameters using $\{S_n\}$ with

$$S_n^2 \leq 0.25 \quad (4.2)$$

Note that S_n should be replaced by the random sequences given in Table 4.1.

4.1.1 SM-SA with Pseudo-Random Excitation

In this section, the model (4.1) is driven by a pseudo random sequence $\{w_n\}$ with mean $\bar{\mu} = -8.9 \times 10^{-4}$, which is obtained using a C-language version of the machine-independent random number generator URAND describe in [11]. Only 1.58% of data points are selected by SM-SA. Figure 4.1 shows the non-increasing volume measure μ_n on a logarithmic scale for the model in (4.1). Figure 4.2 shows that the estimation errors $|\varepsilon_n|$ falls into the dead zone $[0, 0.5]$. The first 1000 SM-SA estimates can be seen in Fig. 4.3. We observe that the estimates are “almost” consistent and converge at a reasonable speed with only 1.58% of the total data points used.

Figure 4.4 illustrates the weighting sequence $\{\lambda_n^*\}$ averaged over six experiments. It shows that the averaged $\{\lambda_n^*\}$ is type $1/n$ sequence, which implies that it satisfies Robins-Monro’s conditions for convergence.

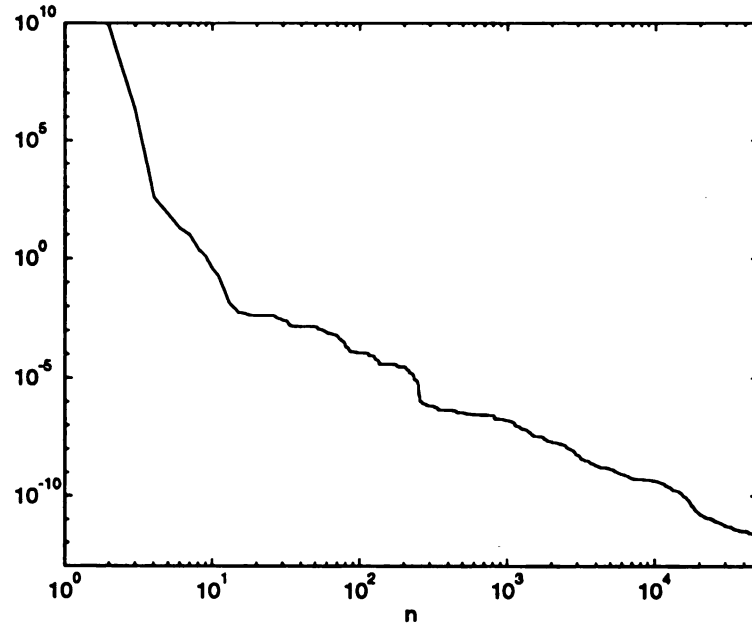


Figure 4.1. The logarithmic plot of μ_n versus time n for the linear time-invariant AR(3) model with pseudo-random excitation.

For LSE algorithms, the estimates are consistent when the noise excitation is white. If

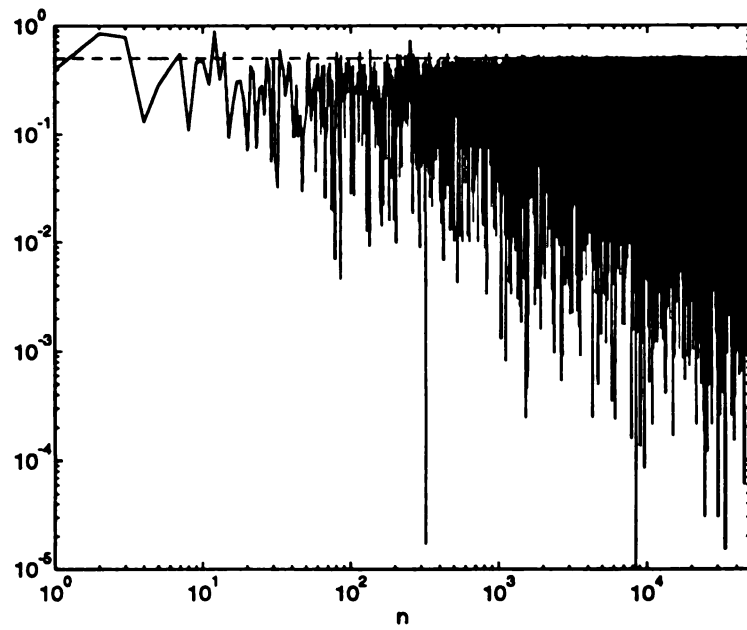


Figure 4.2. The logarithmic plot of $|\varepsilon_n|$ versus time n for the linear time-invariant AR(3) model with pseudo random excitation.

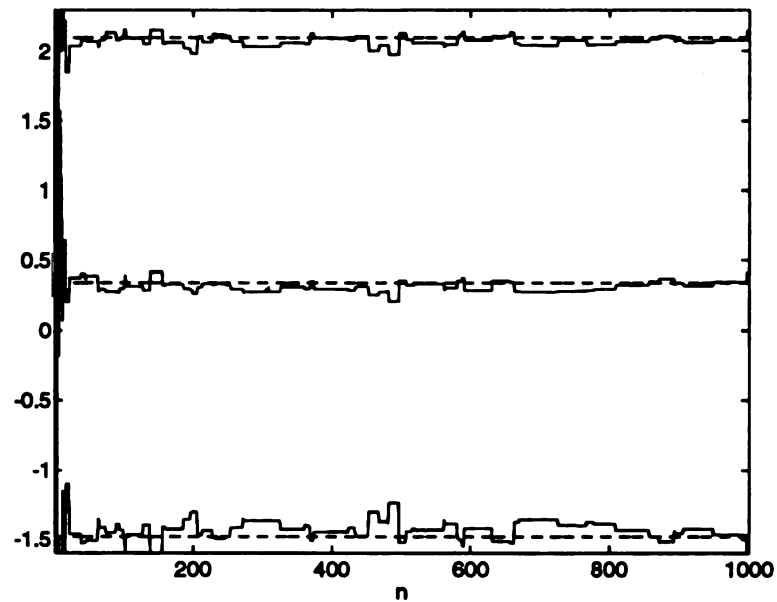


Figure 4.3. First 1000 estimates of the parameters in the AR(3) model obtained using SM-SA and a pseudo-random excitation, where $a_1 = 2.1$, $a_2 = -1.48$, and $a_3 = 0.34$.

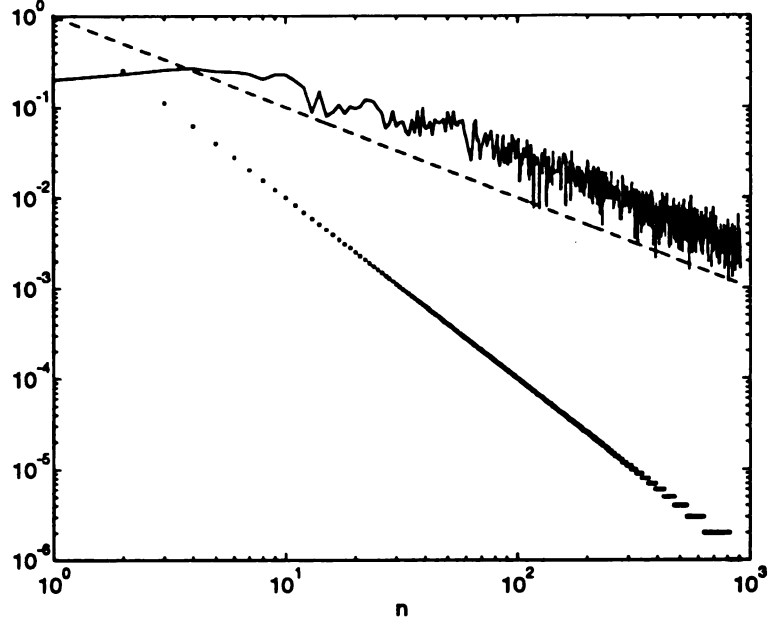


Figure 4.4. The relationship between logarithms of time and weight.

we let $v_n = \frac{5}{6}(w_n - 0.1)$, then the noise bound given in (4.2) still holds for $\{v_n\}$ but the sample mean is now $-\frac{1}{12}$. Figure 4.5 compares the performance of RLS and SM-SA. It is seen that RLS is severely affected by the bias of the noise excitation. The bias in SM-SA, however, is very small. Note that only 0.89% of the data points are used by SM-SA.

4.1.2 AR Models with Binary Bernoulli Excitation

SM-SA algorithm is derived from bounded noise constraints. The weight λ_n^* is optimal in the sense that the feasible set is minimized given that the previous information is known. We may view the size of the feasible sets as a “confidence interval.” If the optimal weight exists most of the time, it is expected that the performance of SM-SA is much better than RLS because of the optimal weights. From the analyses in the previous chapter, we know that optimal weight λ_n^* exists when the magnitude of the noise is close to the bound. In this simulation, a zero-mean Bernoulli process $\{z_n\}$ is employed to drive the system. Let $\gamma = 0.25$, then $\{z_n\}$ always hits the bounds. 80.11% of the data points are used. This is not surprising because $\{z_n\}$ is always at the boundaries and thus updating is carried out

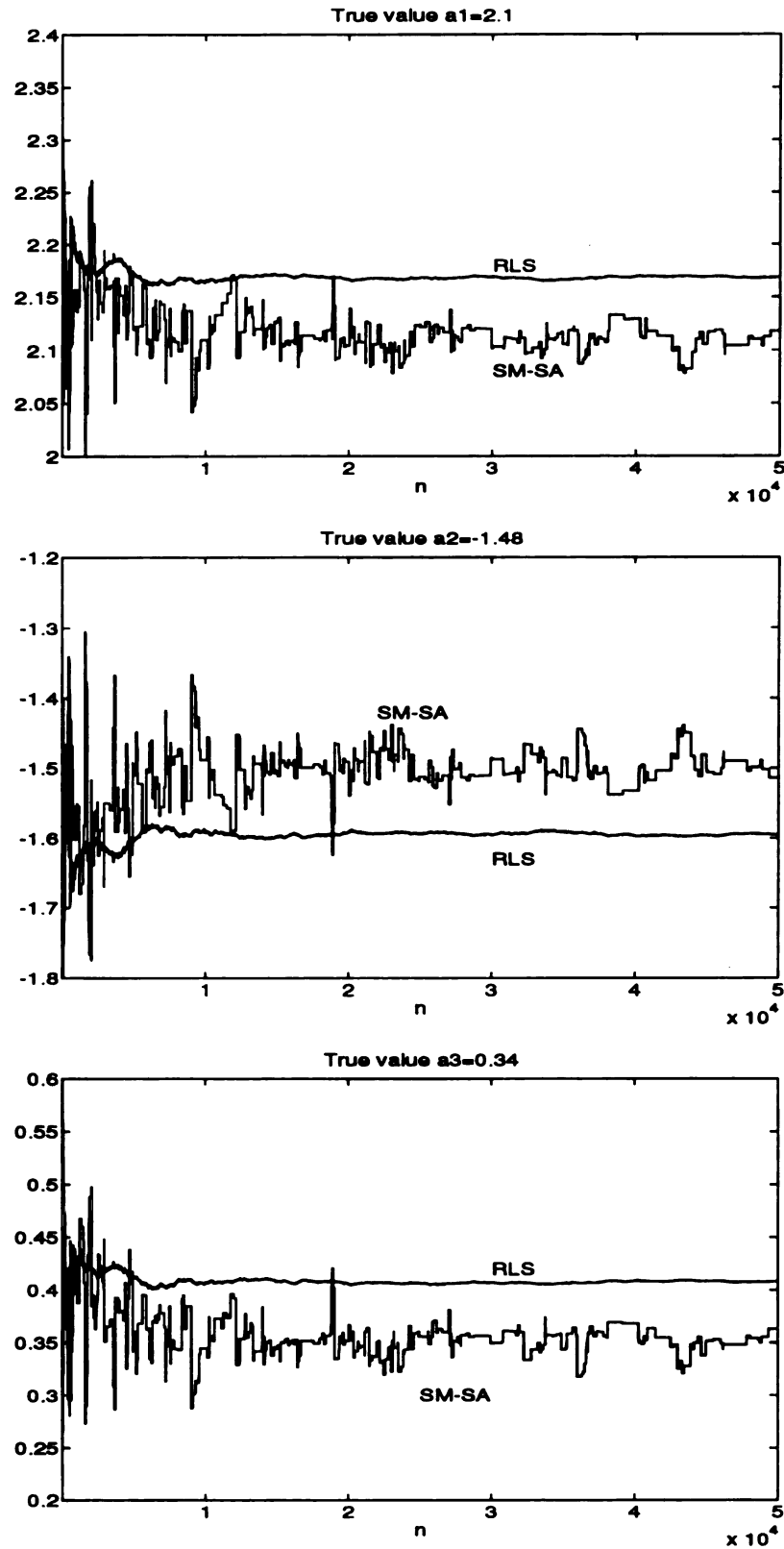


Figure 4.5. The estimates of model parameters by RLS and SM-SA for biased noise sequence.

most of the time. One of the advantages of this extensive updating is that the volume of the ellipsoid converges very fast to a very small number as indicated in Fig. 4.6.

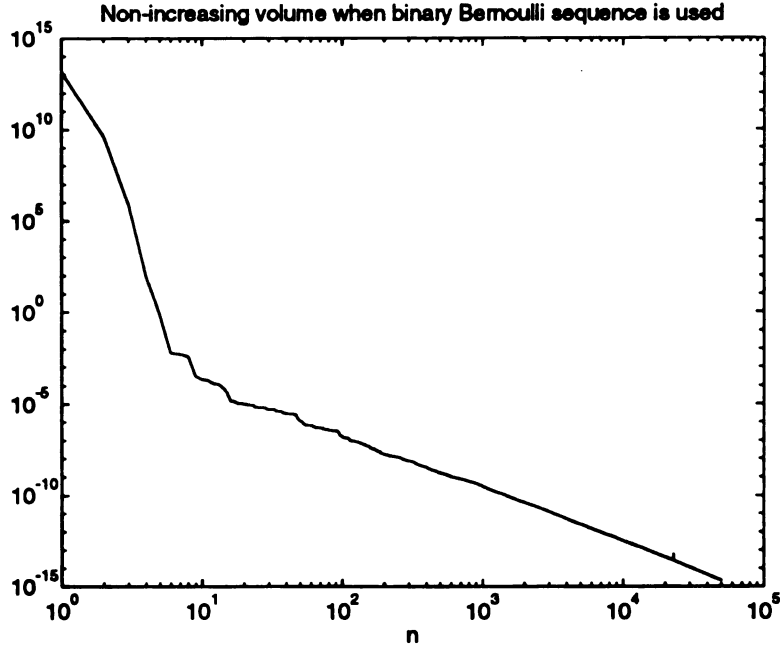


Figure 4.6. The logarithmic plot of μ_n versus time n when a binary Bernoulli sequence is used.

A comprehensive comparison of SM-SA and RLS is shown in Fig. 4.7.

If we replace $\{z_n\}$ by a non-zero mean Bernoulli sequence $\{t_n\}$, then, according to Fig. 4.8, the performance of SM-SA with 78.16% of the data points selected is not affected by the non-zero mean $\{t_n\}$. However, the RLS estimates are clearly influenced.

4.2 Performance of SM-SA for Time-Varying System

In this section, a speech model with time-varying coefficients

$$y_n = b_1(n)y_{n-1} + b_2(n)y_{n-2} + b_3(n)y_{n-3} + w_n \quad (4.3)$$

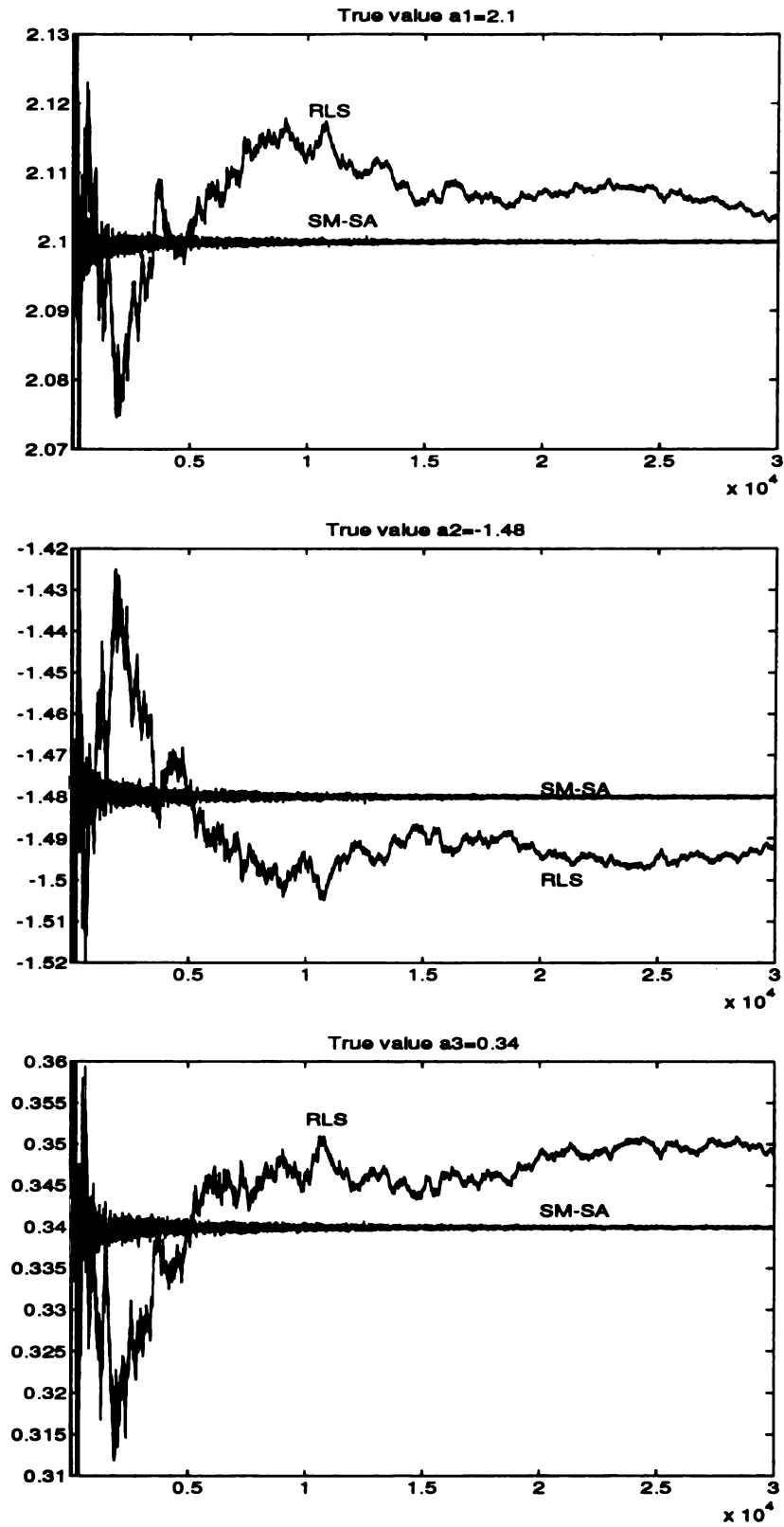


Figure 4.7. First 30,000 estimates obtained using SM-SA and RLS when a binary Bernoulli sequence is used.

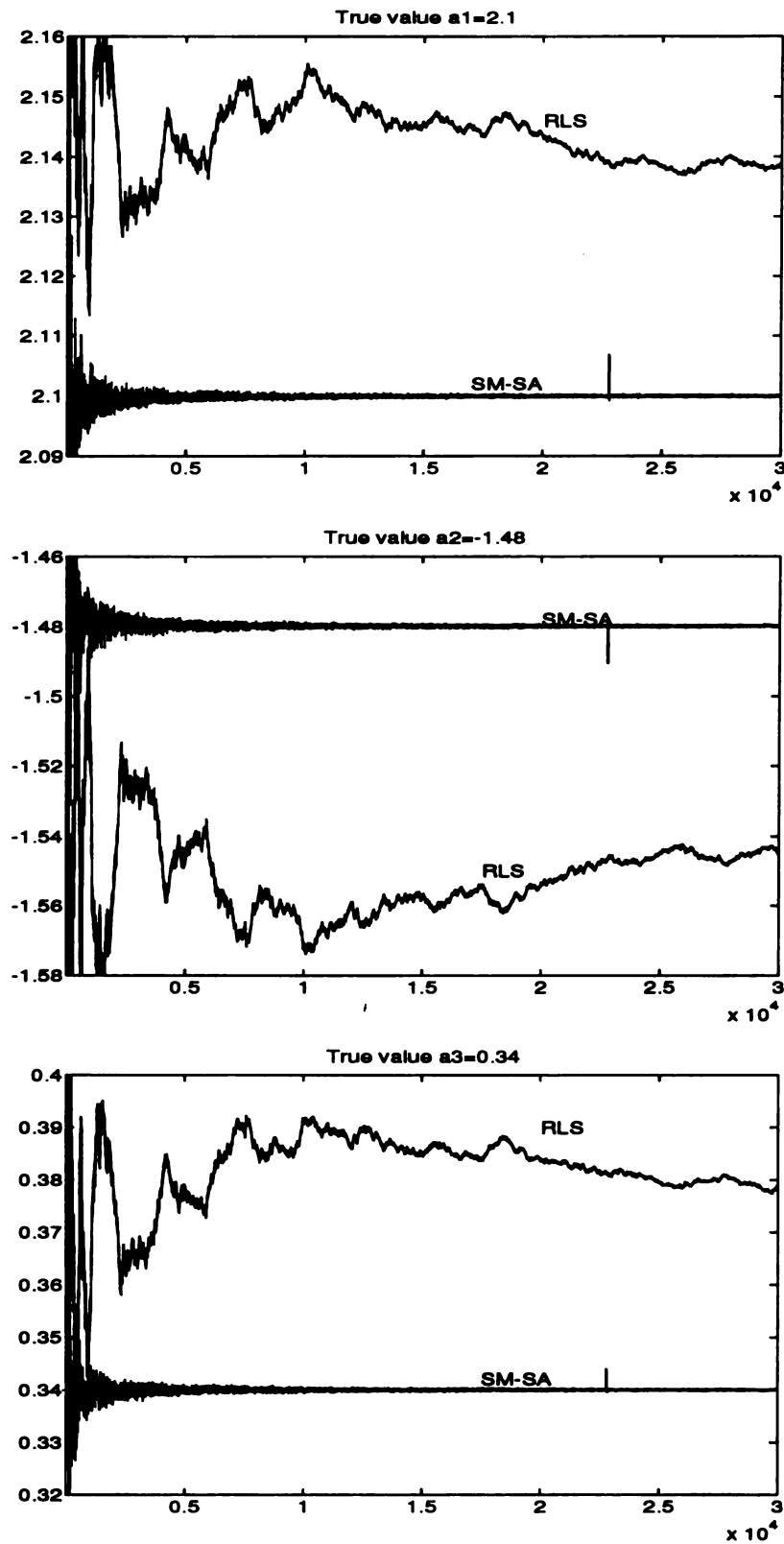


Figure 4.8. First 30,000 estimates obtained using SM-SA and RLS when a biased binary Bernoulli sequence is used.

is considered. Simulations for speech signals have been performed by Odeh [28]. Here we use a similar experiment just to examine the performance of SM-SA. True parameters in this model are derived using short-term linear prediction analysis [37] of order two with a 256-point Hamming window and Levinson-Durbin recursion (see [37]) on an utterance of the word “four” by an adult male speaker. The acoustic waveform is shown in Fig. 4.9. The data were sampled at 10kHz after 4.7kHz low-pass filtering. While more meaningful

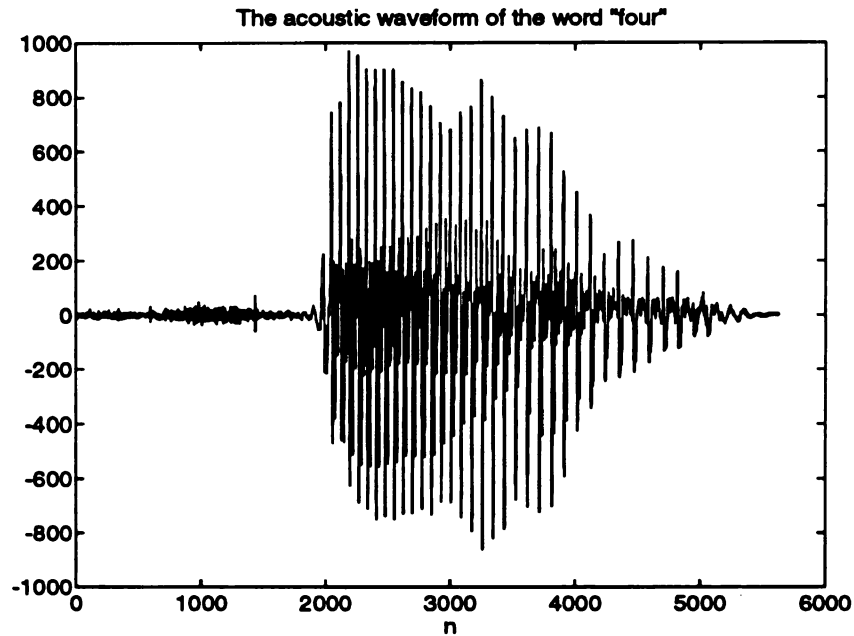


Figure 4.9. The acoustic waveform of the word “four”.

analysis of speech would involve model orders of 10-14 [37], this small number of parameters is used here so that the results are easily illustrated [28]. The true parameters for the word “four” are shown in Fig. 4.10. A set of 5376 data points were generated by driving this model by w_n . In this simulation, only 2.15% of data points are used by SM-SA with good performance. Simulation results by using SM-SA and RLS algorithms can be seen from Figs. 4.11 to 4.12. Similar results by using SM-WRLS can be seen in [28] in which 1.86% of data points are used. In each figure there are two curves, one for the true parameters as reference, the other for the estimates obtained by using SM-SA or RLS.

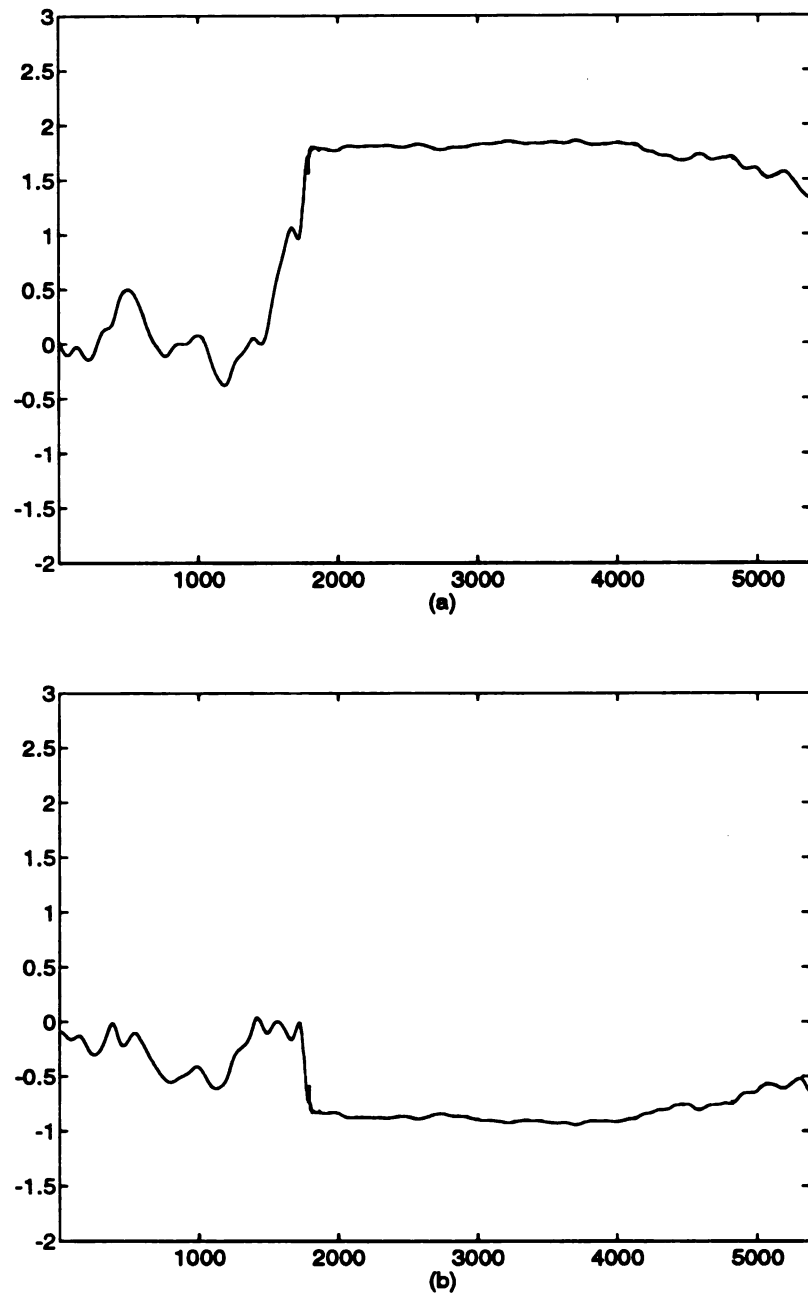


Figure 4.10. The "true" parameters for the word "four": (a) $b_1(n)$ (b) $b_2(n)$.

The power of the SM-SA algorithm is evident in this simulation. Upon comparing the estimates obtained by SM-SA and conventional RLS algorithms, we find that SM-SA shows better tracking capability than conventional RLS. For most algorithms, there is a trade-off between improved computational complexity and improved performance. For SM-SA, however, improved performance comes along with improved computational efficiency. More discussion about SM-SA in time-varying system is left in section 4.4.

4.3 Suboptimal Test

More computational saving can be achieved by employing the suboptimal test. In Section 3.2.2 we have shown that many convergence properties of the suboptimal SM-SA are similar to the “optimal” SM-SA. In this section, simulations and comparisons to the conventional RLS are performed to verify the performance of SM-SA and its convergence properties. Only the time-invariant AR(3) model given in (4.1), will be examined. The reason we do not apply suboptimal test to speech signal is explained in Section 4.4.

4.3.1 AR Models with Pseudo-Random Excitation

First the model (4.1) is driven by $\{w_n\}$. Only 0.46% of data points are used by the suboptimal strategy. Compared with 1.58% in “optimal” SM-SA, this is a significant saving. But more importantly, the checking procedure is $\mathcal{O}(m)$. Figure 4.13 shows the non-increasing volume measure μ_n in logarithmic scale. Fig. 4.14 shows that the estimation errors $|\varepsilon_n|$ also falls into the dead zone $[0, 0.5]$. The first 2000 estimates can be seen in Fig. 4.15. We see that the estimates are consistent and converge at a reasonable speed with only 0.46% of total data points used.

Figure 4.16 illustrates that $\{\lambda_n^*\}$ is a $1/n$ -type sequence, satisfying Robins and Monro’s conditions for convergence.

Only 0.19% of data are used if $\{v_n\}$ is the driving process, Note that the same experiment performed in Section 4.1.1 used 1.03% of the data points. Figure 4.17 compares the performance of RLS and SM-SA. We see that the effect of biased noise on suboptimal SM-SA is much less than it is on RLS.

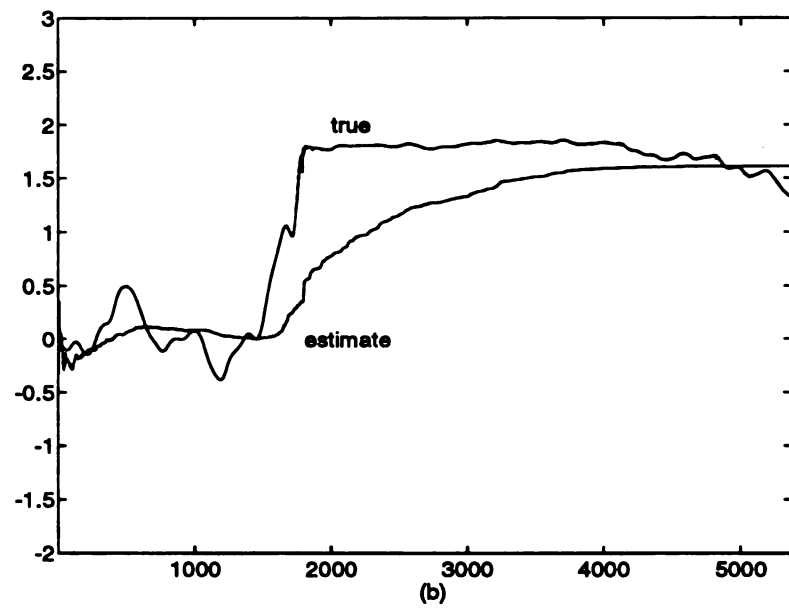
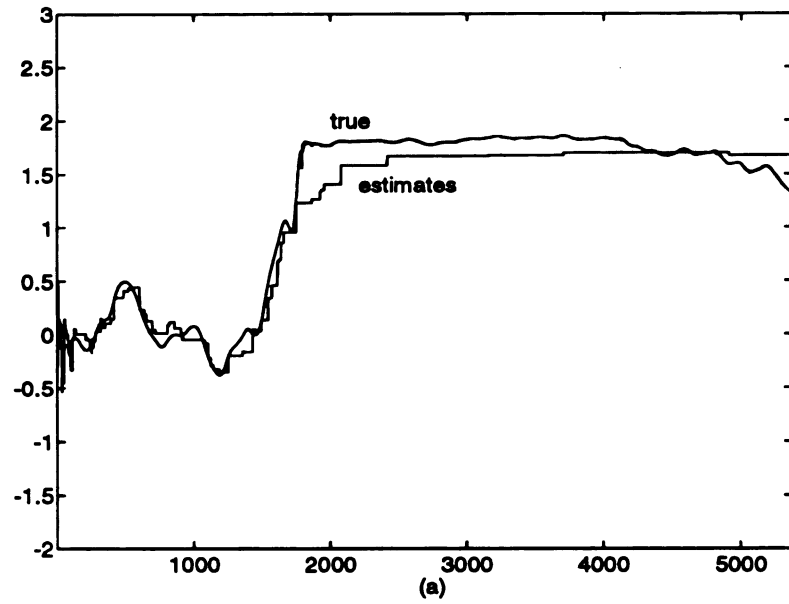


Figure 4.11. Estimates of $b_1(n)$ for the word "four" using (a) SM-SA (b) RLS.

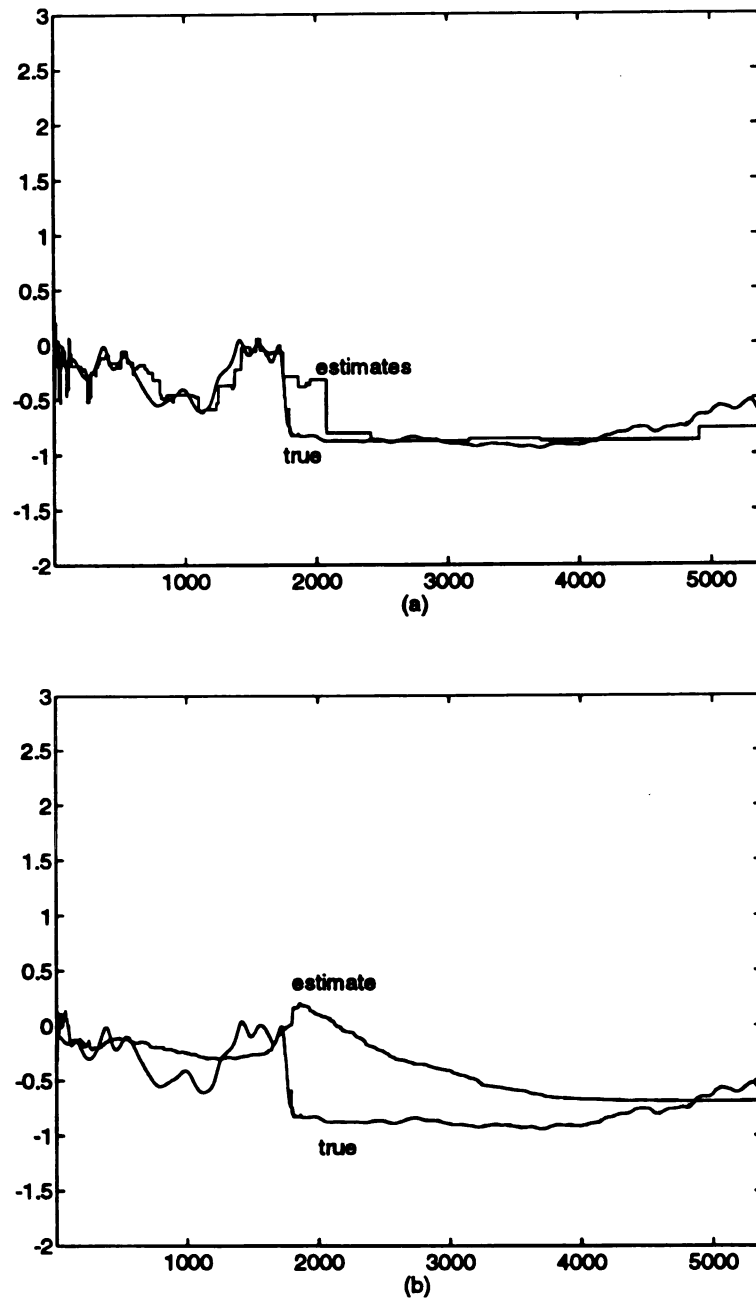


Figure 4.12. Estimates of $b_2(n)$ for the word "four" using (a) SM-SA (b) RLS.

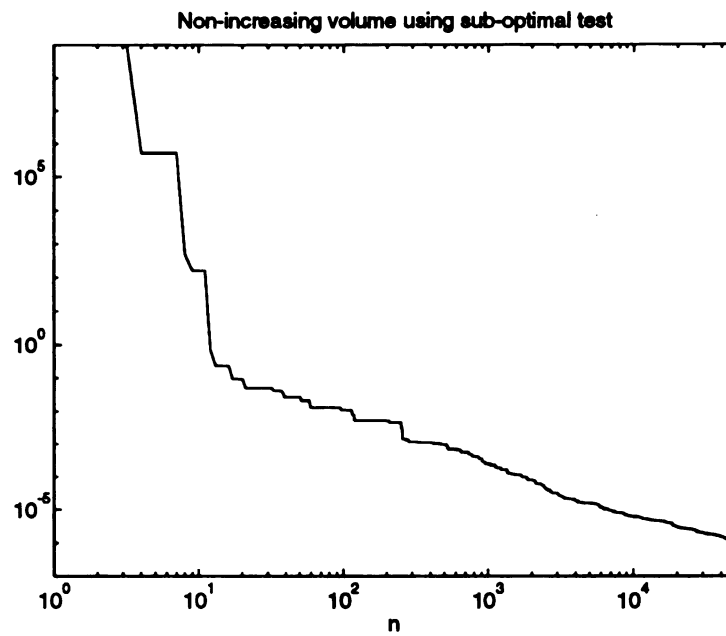


Figure 4.13. The logarithmic plot of μ_n versus time n for the AR(3) model when the suboptimal test is used.

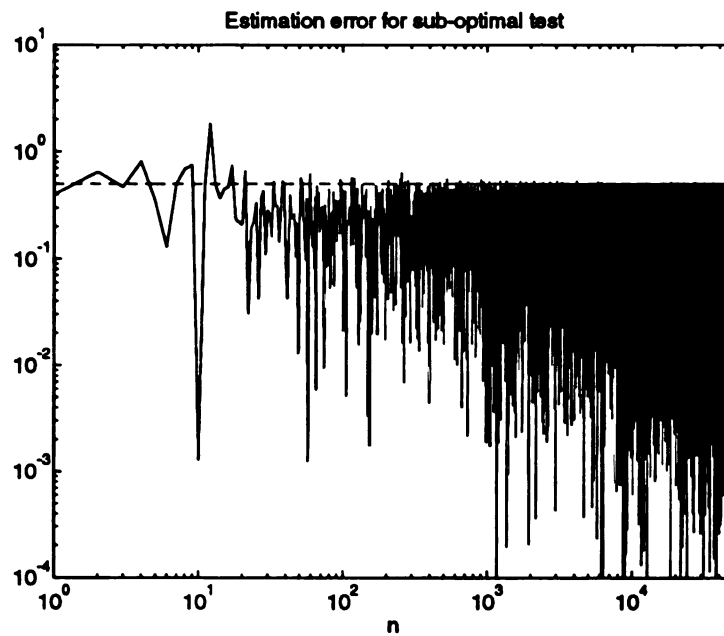


Figure 4.14. The logarithmic plot of $|\varepsilon_n|$ versus time n for the AR(3) model.

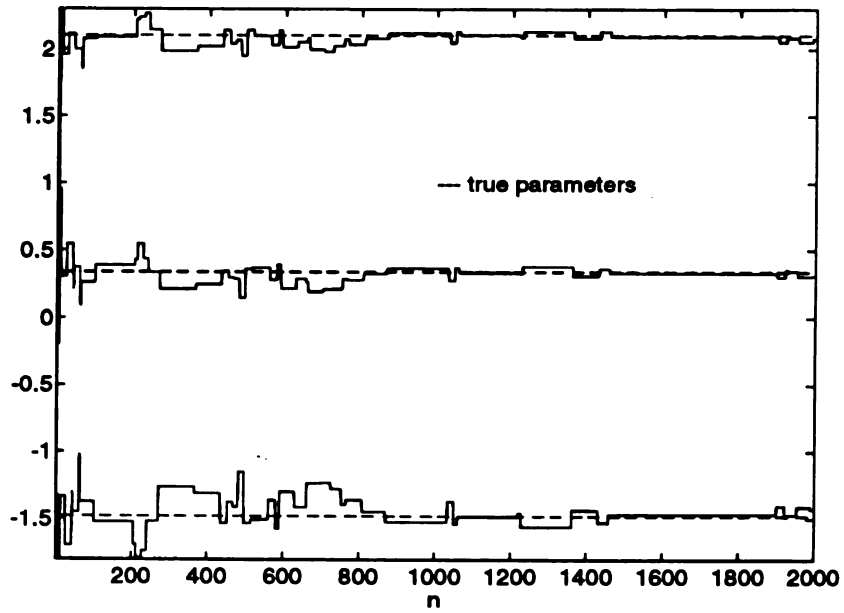


Figure 4.15. First 2000 estimates of the parameters using SM-SA, where $a_1 = 2.1$, $a_2 = -1.48$, and $a_3 = 0.34$.

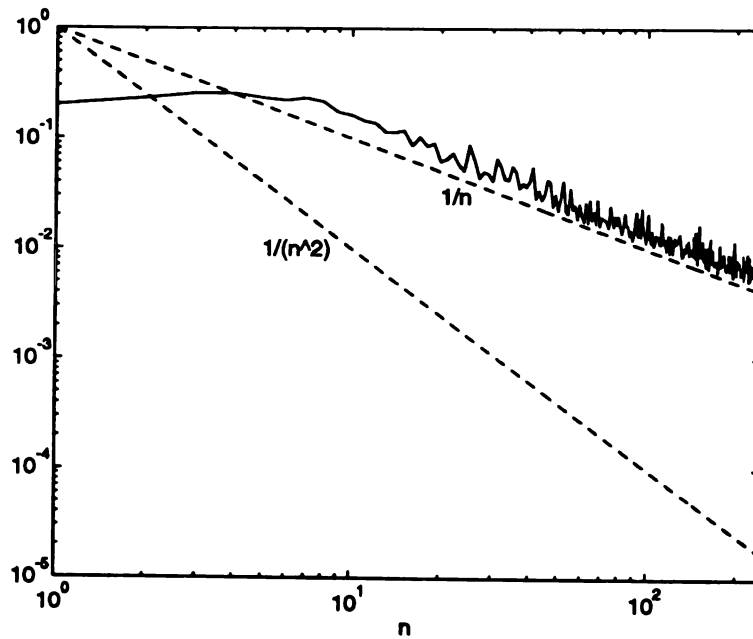


Figure 4.16. The relationship between logarithms of time and weight.

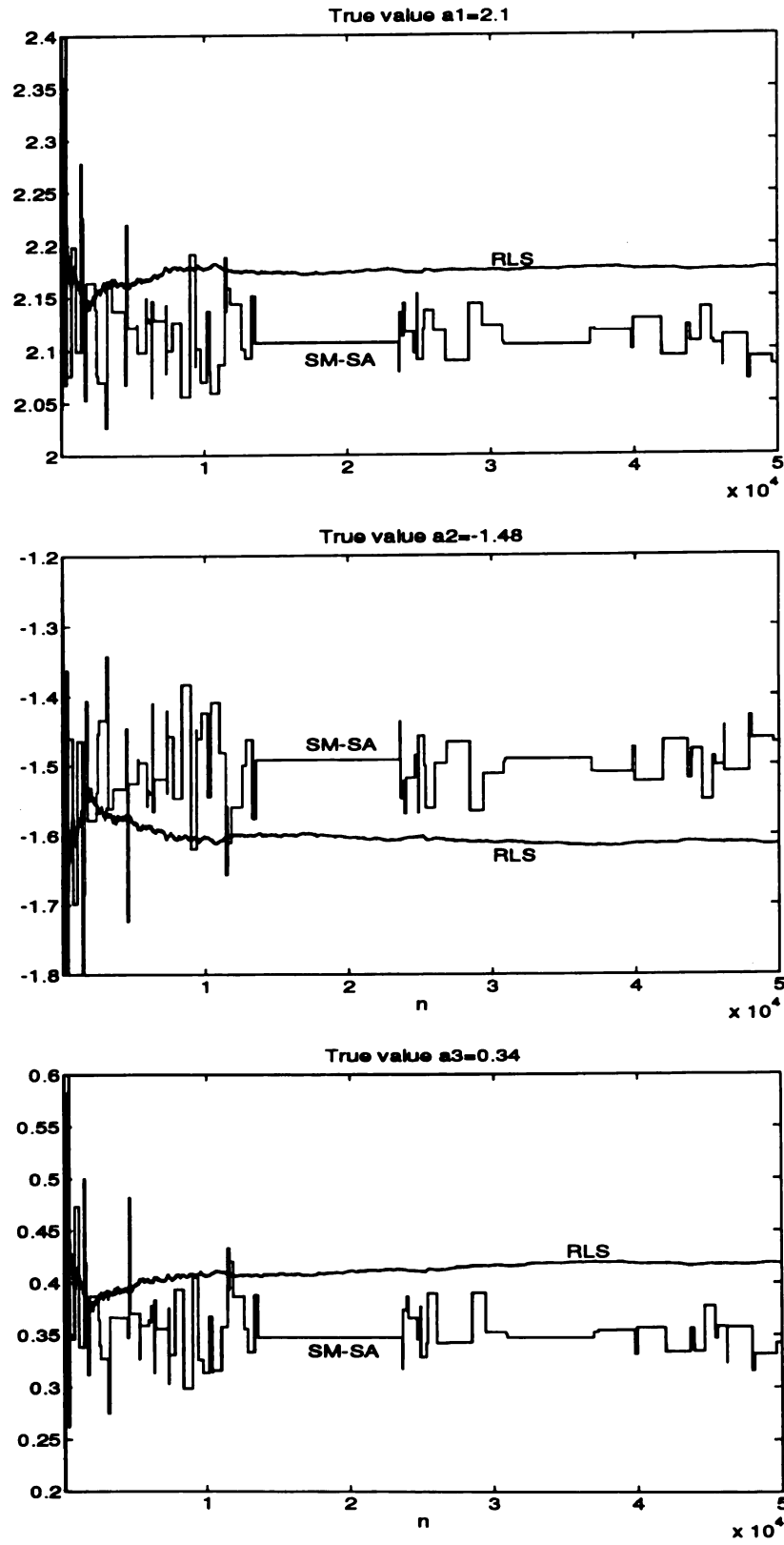


Figure 4.17. Estimates of parameter using RLS and SM-SA for the suboptimal test.

4.3.2 AR Models with Binary Bernoulli Excitation

Let $\gamma = 0.25$ and let the AR(3) model is driven by $\{z_n\}$. In the simulation for suboptimal test, 49.86% of data points are used. The corresponding experiment for optimal SM-SA used 80.11% of data points.

Figure 4.18 compares the performance between SM-SA and RLS. We find that SM-SA shows excellent performance in the convergence of estimates even for the suboptimal test.

If we change the noise process to $\{t_n\}$, Fig. 4.19 shows that the performance of SM-SA is not affected by the bias of excitation, which is similar to the result for the “optimal” SM-SA. In this case, 48.91% of data points are used, while 78.16% data points are used in the corresponding optimal SM-SA.

4.4 More Studies on Time-varying Systems

SM-SA algorithm is derived assuming a time-invariant system. In the previous simulation, we have seen that SM-SA has some tracking ability. The adaptability of SM-WRLS algorithm have been studied in previous work [28]. Although we do not intend to pursue theoretical analysis of SM-SA for adaptivity of SM-SA, we can make some heuristic descriptions. Consider the weighted covariance $\mathbf{R}(n)$

$$\mathbf{R}(n) = (1 - \lambda_n^*)\mathbf{R}(n-1) + \lambda_n^* \mathbf{x}_n \mathbf{x}_n^T$$

The $(1 - \lambda_n^*)$ term in this equation can be viewed as a *forgetting factor* for the algorithm. The smaller the quantity λ_n^* , the lesser the impact of the current point. By controlling the magnitude of the weights, the algorithm can decide the importance between the new and old data. This is why SM-SA can trace some time-varying systems. In this situation, the ellipsoid cannot shrink too fast, and since the parameters are always changing, the ellipsoid cannot converge to zero either. A better way to estimate system parameters is to “over-estimate” the noise bound because we can prevent the ellipsoid from shrinking to a very small set. For the speech signal we discussed in this chapter, if we “over-estimate” the noise

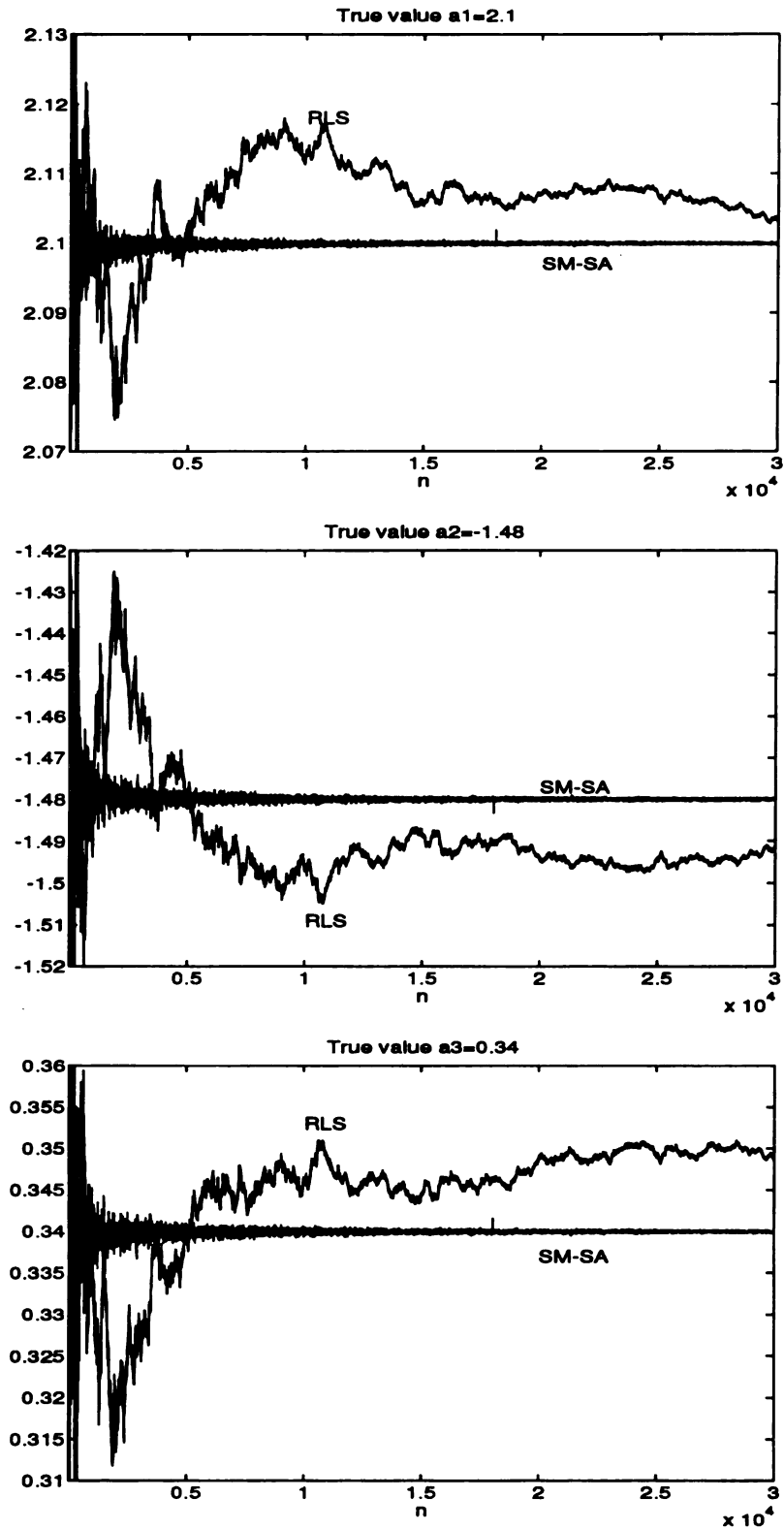


Figure 4.18. First 30,000 estimates obtained by SM-SA and RLS when a zero mean binary Bernoulli sequence is used and suboptimal test is employed.

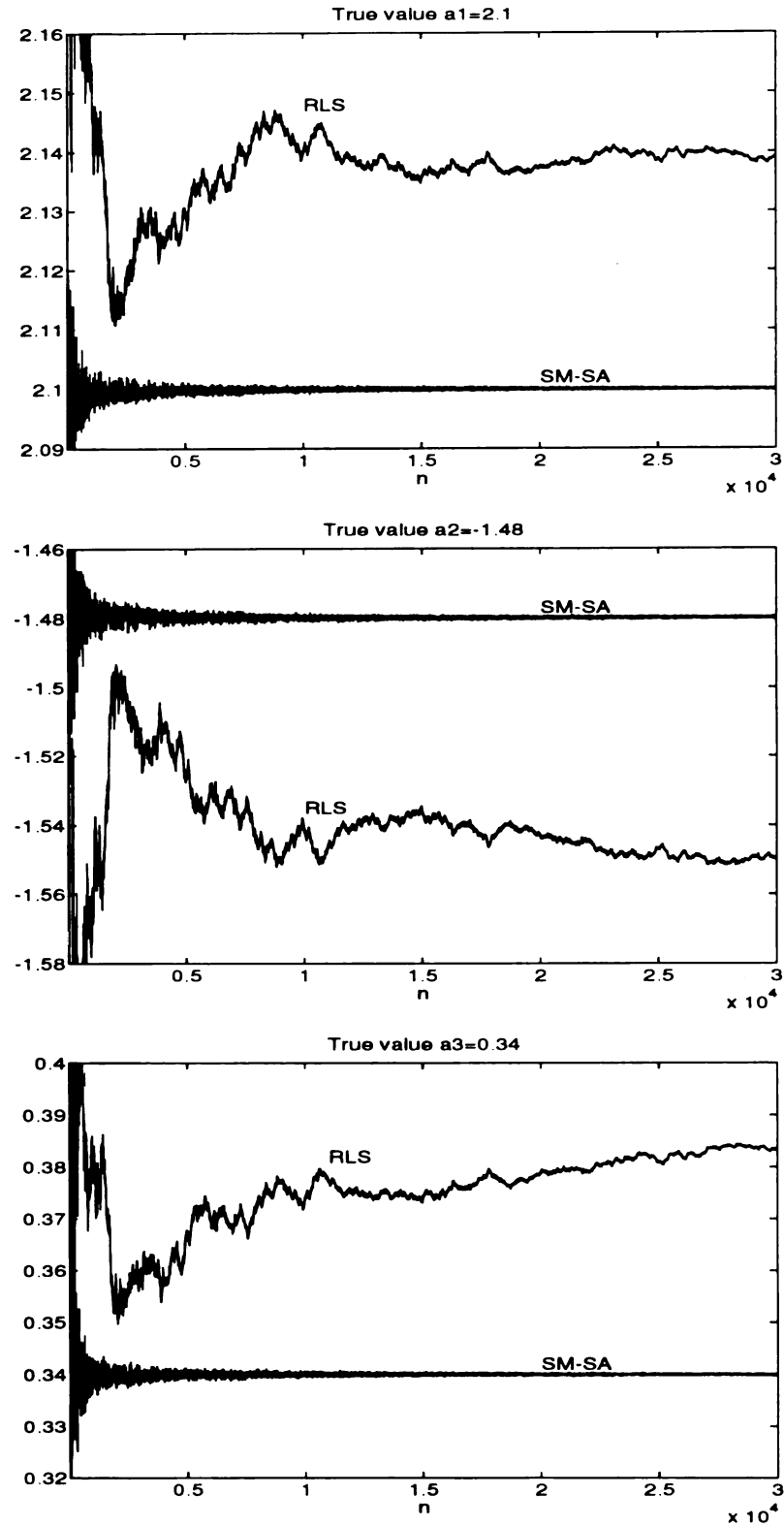


Figure 4.19. First 30,000 estimates obtained by SM-SA and RLS when a bias binary Bernoulli sequence is used and suboptimal test is employed.

bound so that the bound be chosen as

$$w_n^2 \leq 0.5$$

but the real bound is

$$w_n^2 \leq 0.25$$

we will find that the algorithm has better tracking ability as shown in Fig. 4.20. In this case 4.09% of data points are used, which is more than before.

Note that SM-SA is developed for time-invariant systems. If it works in a time-varying system, some assumptions may be violated, for example κ_n may become negative. Thus it is not appropriate to apply the suboptimal test to time-varying system since a negative $\tilde{c}_n$ may not imply a negative c_n . Some simulations show that if we apply the suboptimal test to time-varying systems, the results are often not encouraging.

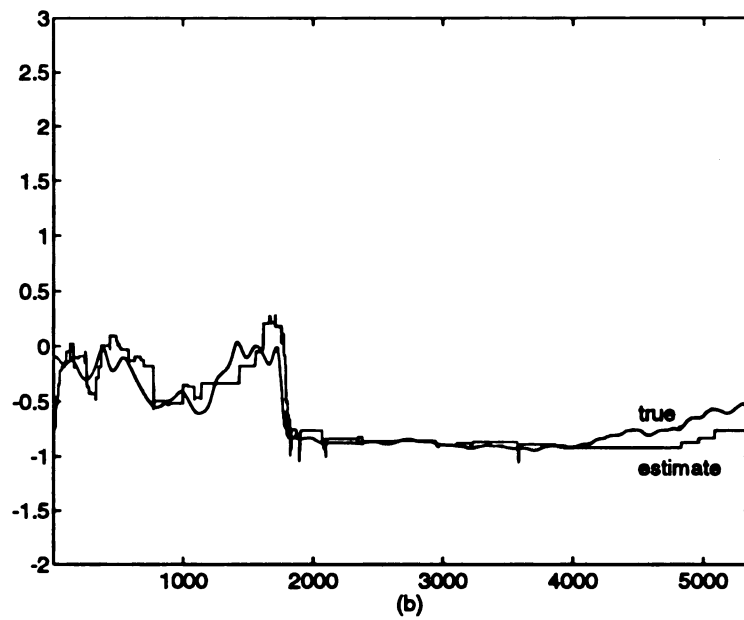
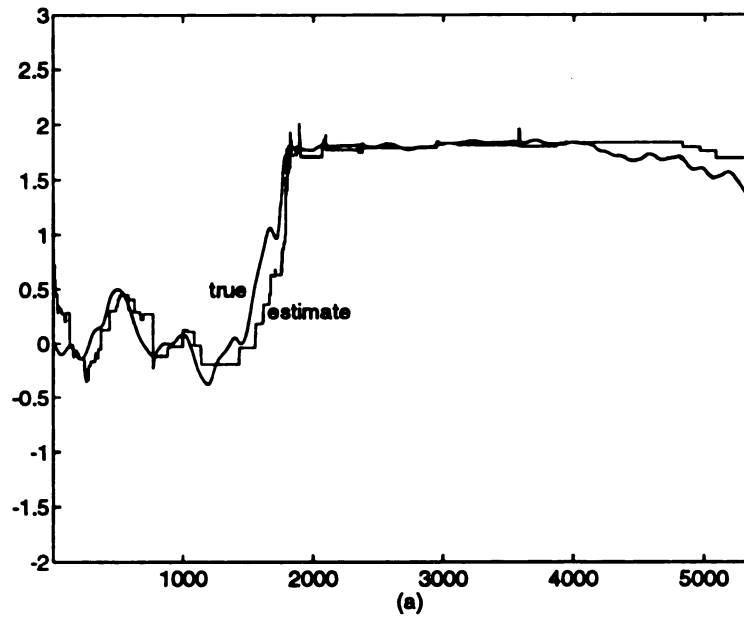


Figure 4.20. Estimates of (a) $b_1(n)$ (b) $b_2(n)$ by using SM-SA algorithm for the word “four” when noise bounds are over-estimated.

Chapter 5

The Convergence Properties of UOBE Algorithms

UOBE algorithm, a general class of OBE algorithms, is based on generalized WRLS in which very broad classes of “forgetting factors” and data weights may be employed. Historically, this research on the SM-SA algorithm, and that of Krunz on the “Dual-SM-WRLS” algorithm [33, 34], precedes and contributed to the development of UOBE. In this chapter, we will generalize the convergence properties of SM-SA to all UOBE algorithms and show the significant implication of this research to broader applications. A brief review of UOBE is given here. Details are found in [38, 39].

5.1 The UOBE Algorithm

Given the system in (1.5) and the noise bound in (1.6), the hyper-ellipsoid set $\bar{\Omega}(n)$ associated with UOBE is defined by

$$\bar{\Omega}(n) = \left\{ \boldsymbol{\theta} \mid (\boldsymbol{\theta} - \boldsymbol{\theta}_n)^T \frac{\mathbf{C}(n)}{\kappa_n} (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \leq 1 \right\}$$

where $\boldsymbol{\theta}_n$ is the WRLS estimates obtained by minimizing

$$V(\boldsymbol{\theta}) = \frac{1}{n} \sum_{\tau=1}^n w_{n,\tau} (y_{\tau} - \mathbf{x}_{\tau}^T \boldsymbol{\theta})^2$$

in which

$$w_{n,\tau} = \begin{cases} \alpha_n w_{n-1,\tau} & \tau \leq n-1 \\ \beta_n & \tau = n \end{cases}$$

and κ_n is a scalar quantity,

$$\kappa_n = \theta_n^T \mathbf{C}(n) \theta_n + \sum_{\tau=1}^n w_{n,\tau} (\gamma_{\tau} - y_{\tau}^2)$$

$\mathbf{C}(n)$ is fundamentally defined as the sum of the weighted outer products,

$$\mathbf{C}(n) = \alpha_n \mathbf{C}(n-1) + \beta_n \mathbf{x}(n) \mathbf{x}^H(n)$$

Different algorithms in UOBE are distinguished by the weighting sequences $\{\alpha_n\}$ and $\{\beta_n\}$, in conjunction with the optimization criterion employed in selecting them. Three criteria have been applied to the optimization process:

Optimization Criterion 1 Minimize the determinant of the inverse ellipsoid matrix,

$$\mu_v\{\bar{\Omega}(n)\} \triangleq \det\{\kappa_n \mathbf{C}^{-1}(n)\}.$$

Optimization Criterion 2 Minimize the trace of the inverse ellipsoid matrix,

$$\mu_t\{\bar{\Omega}(n)\} \triangleq \text{tr}\{\kappa_n \mathbf{C}^{-1}(n)\}.$$

Optimization Criterion 3 Minimize the parameter κ_n .

Criteria 1 and 2 were first suggested by Fogel and Huang [14], while criterion 3 was used by Dasgupta and Huang [20]. In the single-output case in which $\bar{\Omega}(n)$ is clearly interpretable as an ellipsoid in $\Re^m$, $\mu_v\{\bar{\Omega}(n)\}$ is proportional to the square of the volume of the ellipsoid, while $\mu_t\{\bar{\Omega}(n)\}$ is proportional to the sum of squares of its semi-axes. Criterion 3 has been used in conjunction with a specific weighting strategy to achieve a rigorous proof in certain sense [20].

Let us adopt the policy of writing the weights α_n and β_n as functions of a single param-

eter to be optimized at time n , say λ_n so that

$$\begin{aligned}\alpha_n &= \alpha_n(\lambda_n) \\ \beta_n &= \beta_n(\lambda_n)\end{aligned}\tag{5.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ should now be considered to be sequences of functions. If we choose $\alpha_n(\lambda_n) = 1$ and $\beta_n(\lambda_n) = \lambda_n$, then UOBE is exactly the SM-WRLS algorithm. SM-SA is obtained by choosing $\alpha_n(\lambda_n) = (1 - \lambda_n)$ and $\beta_n(\lambda_n) = \lambda_n$ and optimization criterion 1. The extension of the optimization process of SM-SA to criterion 2 is left as a future work.

5.2 Convergence Properties of UOBE

The most important theorem for the theoretical framework of UOBE is given as follows [38, 39]:

Theorem 5.1 *Consider a UOBE algorithm in which volume or trace is to be minimized. Then the following are independent of the choices of function sequences $\{\alpha_n(\lambda_n)\}$ and $\{\beta_n(\lambda_n)\}$:*

1. *the sequence of measures of optimality ($\{\mu_v(n)\}$ or $\{\mu_t(n)\}$);*
2. *the data points selected (times for which there exists $\lambda_n^* > 0$);*
3. *the parameter estimate, θ_n .*

However, in a UOBE algorithm with κ minimization, none of these items is independent of the sequence $\{\alpha_n(\lambda_n)\}$ and $\{\beta_n(\lambda_n)\}$.

Theorem 5.1 concludes that any UOBE algorithm with volume minimization, regardless of the weighting policy, will “behave” similarly in three senses. By using this theorem, we can extend the convergence properties from SM-SA to UOBE:

Corollary 5.1 *Consider the UOBE algorithms in which volume is to be minimized. Let $\{t_n\}$ be a subsequence such that update takes place at time t_n . If there exists an $N \in \mathcal{N}$ and $\varrho_1, \varrho_2 > 0$ such that for all n*

$$\varrho_1 \mathbf{I} < \sum_{k=n+1}^{n+N} \mathbf{x}_{t_k} \mathbf{x}_{t_k}^T < \varrho_2 \mathbf{I}, \quad (5.2)$$

then

$$\lim_{n \rightarrow \infty} \|\boldsymbol{\theta}_n - \boldsymbol{\theta}_{n-1}\| = 0.$$

In addition to (5.2), if the noise process $\{v_n\}$ is independent, and given ϵ there exists a δ such that for all n

$$P(v_n \in D_{\epsilon+}) > \delta, \text{ and } P(v_n \in D_{\epsilon-}) > \delta,$$

where $D_{\epsilon+} = [\sqrt{\gamma} - \epsilon, \sqrt{\gamma}]$ and $D_{\epsilon-} = [-\sqrt{\gamma}, -\sqrt{\gamma} + \epsilon]$, then the volume of the ellipsoid converges to zero and the estimates obtained using any UOBE algorithm converge to the true parameters.

This corollary is very important since, through the research on SM-SA, the convergence properties of all UOBE algorithms are established*. This implies that SM-SA has much broader implications for the entire class of UOBE algorithms.

*The original OBE paper by Fogel and Huang [14] is sometimes misunderstood to indicate the convergence of the bounding ellipsoid to a point under ordinary conditions on $\{v_n\}$ (see [38]). Now the convergence properties are proved.

Chapter 6

Conclusion

6.1 Concluding Remarks

6.1.1 Summary of Landmark Developments of SM-Based Techniques

Set-membership-based algorithms are developed to estimate the parameters of a linear system in which the noise perturbation is additive and bounded.

After the first publication of the OBE algorithm by Fogel and Huang [14] in 1982, SM-techniques attracted the attention of many research groups in the signal processing field. However, the lack of convergence proof has been a serious weakness for the OBE-like algorithms, although it shows good performance in many applications.

In order to remedy the convergence problem, Dasgupta and Huang proposed a modified OBE [20] in 1987. Instead of minimizing the volume of the ellipsoid, they minimize the upper bound of the set. This algorithm is convergent in the sense that the optimal weight and the difference of the estimates converge to zero. However, it has been criticized by some researchers [30, 38, 39] due to the lack of geometrical meaning.

The Adaptive Signal Processing and Speech Processing Laboratories at Michigan State University have been pioneers in the research on SM-based techniques. The problem of the trade-off between the interpretability and convergence led to the study of SM-SA. During the course of this research, it is found in parallel work that almost all OBE algorithms, can be combined into a unified frame work called UOBE, and, SM-SA is just a special case of UOBE obtained by choosing $\alpha_n(\lambda_n) = 1 - \lambda_n$ and $\beta_n(\lambda_n) = \lambda_n$, and volume minimization.

The name SM-SA comes from the structural resemblance to the stochastic approximation method.

6.1.2 Contributions of this Research

Selective updating so as to improve the computational load is the common feature among all the members in UOBE. In contrast to RLS estimation in which the parameter estimates are updated at every iteration, UOBE updates the estimates only if the incoming data contains sufficient information. In addition to this advantage, the major contribution of this work is the establishment of an algorithm which is both interpretable and convergent.

The interpretability of SM-SA comes from the optimization process in which the optimal weight is found by minimizing the volume of the ellipsoid defined by

$$\mu_v\{\bar{\mathbf{\Omega}}(n)\} \triangleq \det\{\kappa_n \mathbf{R}^{-1}(n)\}.$$

The convergence properties come from the boundedness of the covariance matrix. In this work, we have shown that:

1. The optimal weight λ_n^* converges to zero.
2. The squares of the estimation error sequence fall into a dead zone given by $[0, \gamma_n]$ asymptotically.
3. If the noise sequence $\{v_n\}$ is independent and the probabilities are non-zero in the neighborhoods of the noise bounds, then the volume of the ellipsoid converges to zero.
4. If the noise process is mixing, then under a slight modification of SM-SA, the volume of the ellipsoid converges to zero.

More importantly, through the analysis on SM-SA, the convergence of estimates presented in Chapter 3 can be extended to all UOBE algorithms.

The other contribution is the illustration of the central difference between SM-SA and RLS. While both algorithms have the same recursive formulation, they show completely different behaviors. The reason is because the weight of SM-SA at time n is a random

variable which depends on all the previous and present noise samples, while the weight of conventional RLS or WRLS is deterministic. We have shown the difference in Chapters 3 and 4 by theoretical analysis and simulation studies.

6.2 Future Work

This research is concerned with the theoretical analysis of a special case of UOBE, SM-SA. While we have seen in Chapter 4 that SM-SA performs well in tracking time-varying parameters, we should not rely on an algorithm, which is developed in a time-invariant setting, to perform well in a time-varying environment. It is of practical interest to develop the adaptive SM-SA since most engineering applications deal with time-varying models.

Secondly, we have shown the convergence of SM-SA only on *volume minimization*, while it is believed that the convergence properties under *trace minimization* are similar, it is still necessary to provide a rigorous proof.

Finally, it is often difficult to get an accurate estimate for the noise bounds. In order not to violate the bounded error assumption, it is natural to give conservative noise bounds. As we have discussed, this often leads to a bad estimates. One way to solve this problem is to use *soft* noise bounds in which the bounded noise assumption may be violated sometimes. Thus the author suggests the study of soft-bound SM-SA algorithm as a further work.

In conclusion, the following topics are proposed for future research:

1. To develop an adaptive SM-SA algorithm.
2. To establish the convergence proof for SM-SA under the trace minimization.
3. To extend the model from ARX to ARMAX.
4. To develop a soft-bound SM-SA algorithm.

APPENDICES

APPENDIX A

Infinite Product

Definition A.1 *Given an infinite product $\prod_{n=1}^{\infty} u_n$, let $p_n = \prod_{k=1}^n u_k$. If no factor u_n is zero, we say the product converges if there exists a number $p \neq 0$ such that $\{p_n\}$ converges to p . If $\{p_n\}$ converges to zero, we say the product diverges to zero.*

The following theorem is given without proof [9]:

Theorem A.1 *Assume that each $a_n \geq 0$. Then the product $\prod(1 - a_n)$ converges if, and only if, the series $\sum a_n$ converges.*

BIBLIOGRAPHY

BIBLIOGRAPHY

- [1] H. Robbins and S. Monro, "A stochastic approximation method," *Ann. Math. Stat.*, vol. 22, pp. 400-407, 1951.
- [2] J. Blum, "Multidimensional stochastic approximation method," *Ann. Math. Stat.*, vol. 25, pp. 737-744, 1954.
- [3] N.N. Krasnovski, "On the theory of controllability and observability of linear dynamic systems," *Prikl. Mat. Mech.*, vol. 28, no. 1, 1964.
- [4] F.C. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Trans. Automatic Control*, vol. AC-13, pp. 22-28, February 1968.
- [5] H.S. Witsenhausen, "Sets of possible states of linear systems given perturbed observation," *IEEE Trans. Automatic Control*, vol. AC-13, pp. 556-568, October 1968.
- [6] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.
- [7] D.P. Bertsekas and I.B. Rhodes, "Recursive state estimation for a set-membership description of uncertainty," *IEEE Trans. Automatic Control*, vol. AC-16, pp. 117-128, April 1971.
- [8] J.M. Mendel, *Discrete Techniques of Parameter Estimation: The equation Error Formulation*, Marcel Dekker, New York, 1973.
- [9] T.M. Apostol, *Mathematical Analysis*, Addison-Wesley, 1975.
- [10] R.B. Ash and M.F. Gardner, *Topics in Stochastic Process*, Academic Press, 1975.
- [11] G.E. Forsythe and M.A. Malcolm, *Compute Methods for Mathematical Computations*, Prentice-Hall, 1977.
- [12] L. Ljung, "Analysis of Recursive Stochastic Algorithms," *IEEE Trans. Automatic Control*, vol. AC-22, pp. 551-575, October 1977.
- [13] E. Fogel, "System identification via membership set constraints with energy constrained noise," *IEEE Trans. Automatic Control*, vol. AC-24, pp. 752-758, October 1979.
- [14] E. Fogel and Y.F. Huang, "On the value of information in system identification - bounded noise case," *Automatica*, vol. 18, pp. 229-238, 1982.
- [15] B.D.O. Anderson and C.R. Johnson, Jr., "Exponential convergence of adaptive identification and control algorithms," *Automatica*, vol. 18, pp. 1-13, January, 1982.

- [16] L.Ljung and T. Söderström, *Theory and Practice of Recursive Identification*, Cambridge, MA: MIT press, 1983.
- [17] R.R. Bitmead, "Persistence of excitation conditions and the convergence of adaptive schemes," *IEEE Trans. on Information Theory*, vol. IT-30, No. 2, pp. 183-191, March 1984.
- [18] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, (2nd edition), McGraw-Hill, 1984
- [19] G.C. Goodwin and K.S. Sin, *Adaptive Prediction, Filtering, and Control*, Prentice-Hall, 1984
- [20] S. Dasgupta and Y.F. Huang, "Asymptotically convergent modified recursive least-squares with data-dependent updating and forgetting factor for systems with bounded noise," *IEEE Trans. on Information Theory*, vol. IT-33, pp. 383-392, May 1987.
- [21] T. Soderstrom and P. Stoica, *System Identification*, Prentice-Hall, 1988
- [22] K.J. Åström and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989
- [23] J.P. Norton, "Recursive computation of inner bounds for the parameters of linear model," *Int. J. Control*, vol. 50, pp. 2423-2430, 1989.
- [24] J.R. Deller, Jr. and S.F. Odeh, "Implementing the optimal bounding ellipsoid algorithm on a fast processor," *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Process. '89*, Glasgow, vol. 2, pp. 1067-1070, May 1989.
- [25] J.R. Deller, Jr., "Set-membership identification in digital signal processing," *IEEE ASSP Magazine*, vol. 6, pp. 4-22, October 1989.
- [26] J.R. Deller, Jr. and T.C. Luk, "Linear prediction analysis of speech based on set-membership theory," *Computer Speech and Language*, vol. 3, pp. 301-327, October 1989.
- [27] R.A. Decarlo, *Linear Systems*, Prentice-Hall, 1989
- [28] S.F. Odeh *Algorithms and Architectures for Adaptive Set Membership-Based Signal Processing*, Ph.D dissertation, Michigan State University, 1990.
- [29] A.K. Rao, Y.F. Huang, and S. Dasgupta, "ARMA parameter estimation using a novel recursive estimation algorithm with selective updating," *IEEE Trans. on Acoust., Speech, and Signal Process.*, vol. 38, pp. 447-457, March 1990.
- [30] J.P. Norton and S.H. Mo, "Parameter bounding for time-varying systems," *Mathematics and Computers in Simulation*, vol. 32, pp. 527-534, 1990.
- [31] M.F. Cheung *On Optimal Algorithms for Parameter Set Estimation*, Ph.D dissertation, Ohio State University, 1991.
- [32] M.F. Cheung, S. Yurkovich, and K.M. Passino, "An optimal value ellipsoid algorithm for parameter set estimation" *Proc. of the 30th Conference on Decision and Control*, Brighton, pp. 969-974, December 1991.

- [33] M. Krunz *Numerical Stability and Convergence Issues in the SM-WRLS Algorithm*, M.S. Thesis, Michigan State University, 1992.
- ✓ [34] M. Nayeri, J.R. Deller, Jr., and M.M. Krunz, "Convergence and colored noise issues in bounding ellipsoid identification," *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Process.* '92, San Francisco, vol. 5, pp. 337-340, March 1992.
- [35] M. Nayeri, J.R. Deller, Jr., and M.S. Liu, "Optimal bounding error algorithms with adaptive tracking capability," *Proc. of the 26th Annual Asilomar Conference On Signals, Systems, and Computers*, Pacific Grove, October 1992.
- [36] M. Nayeri, M.S. Liu, and J.R. Deller, Jr., "An interpretable and converging set membership algorithm," *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Process.* '93, Minneapolis, April, 1993.
- [37] J.R. Deller, J.G. Proakis, and J.H.L. Hansen, *Discrete-Time Processing of Speech Signals*, New York, Macmillan, 1993.
- [38] J.R. Deller, Jr., M. Nayeri, S.F. Odeh "Least squares identification with error bounds for real-time signal processing and control," *Proceeding of the IEEE*, to appear.
- [39] J.R. Deller, Jr., M. Nayeri, M.S. Liu "Unifying the landmark developments in OBE processing," *Int. Journal Automatic Control and Signal Process.*, to appear.

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