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AN INVESTIGATION OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF THE MATHEMATICAL LIMIT CONCEPT

By

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ABSTRACT

AN INVESTIGATION OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF THE MATHEMATICAL LIMIT CONCEPT

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The purpose of this is study is to describe prospective secondary mathematics teachers' understanding about the mathematical concept of limits in terms of the following types of knowledge: subject matter knowledge, curriculum knowledge, and pedagogical content knowledge.

The following research questions were addressed:

1. How well do prospective teachers understand the concept of limits?
2. What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the concept of limits?
3. What are prospective teachers' opinions about the involvement of the concept of limits in k-12 mathematics curriculum?
4. What are the possible misconceptions, difficulties, and errors the prospective teachers anticipate in teaching the concept of limits?

To provide structure in addressing these first two questions, a five-category model describing prospective secondary mathematics teachers' understanding of the concept of limits of sequences was employed and a questionnaire was developed. The keywords characterizing these categories are basic, computational, transitional, rigorous, and abstract. The test items in the questionnaire were designed to measure prospective teachers' understanding of the limit concept in terms of these five categories of the model. For the last two research questions, open-ended questions were embedded in the questionnaire. The open-ended questions were followed up by four in-depth interviews. Because of a

low response rate to the last two research questions, no conclusions were made but the data was presented and discussed.

Forty-two prospective secondary mathematics teachers participated in this study. The results indicate that this group of prospective teachers' understanding of the limit concept is more procedural oriented; there exist discrepancies between the participants' concept definitions and their concept images of limits; and the misconceptions, difficulties and errors produced by this group are similar to those found in research studies on students. Since misconceptions, difficulties, and errors prospective teachers possess might be passed to their students, implications for classroom teachers, mathematicians, mathematics educators, and teacher training institutions are presented.

To the Memory of my Parents

Lee Kuao Tung

and

Pen Wen Hung

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CHAPTER ONE

INTRODUCTION

This Changing Society Needs Mathematical Power

In today's world, the security and wealth of nations depend on their human resources. Mathematics has come to play a remarkably important role in this. Travers & Westbury (1989) pointed out the role of mathematics in the society:

> The importance of mathematics in the school curriculum reflects the vital role it plays in contemporary society. At the most basic level, a knowledge of mathematical concepts and techniques is indispensable in commerce, engineering, and the sciences. From the individual pupil's point of view, the mastery of school mathematics provides both a basic preparation for adult life and a broad entree into a vast array of career choices. From a societal perspective, mathematical competence is an essential component in the preparation of a numerate citizenry and it is needed to ensure the continued production of the highly-skilled personnel required by industry, technology and science (p.1).

Mathematics also opens the doors to careers for students and helps citizens make informed decisions. Mathematics provides knowledge to compete in a world of technology in which jobs demand workers to work smarter rather than harder. Working smarter means being able to absorb new ideas, to adapt to change, to cope with ambiguity, to perceive patterns, and to solve unconventional problems. All these abilities are enhanced by having mathematical power. *Everybody Counts: A report to the nation on the future of mathematics education* (1989) stated the reasons for requiring students to study mathematics in order to achieve that mathematical power so the students will be able "to learn practical skills for daily lives, to understand quantitative aspects of public policy, to develop problem-solving skills, and to prepare for careers (p.6)." *The National Council of Teachers of Mathematics' Professional Standards for Teaching Mathematics* (NCTM, 1991) emphasized what should be included in mathematical power:

> Mathematical power includes the ability to explore, conjecture, and reason logically; to solve non routine problems; to communicate about and through mathematics; and to connect ideas within mathematics and between mathematics and other intellectual activity. Mathematical power also involves the development of personal self-information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power (p.1).

Shortage of People Well-trained in Mathematics

The United States currently experiences a shortage of well-trained young people in this world of technology. *Everybody Counts* (1989) warned us,

> Not only do we face a shortage of personnel with mathematical preparation suitable to scientific and technological jobs, but also the level of mathematical literacy of the general public is completely inadequate to reach either our personal or national aspirations (p.6).

Lacking mathematical power, many of today's students are neither prepared for tomorrow's jobs nor even for today's occupations. Three quarters of Americans stop studying mathematics before completing career or job prerequisites. Most students leave school with mathematical knowledge neither sufficient to cope with on-the-job demands for problem-solving nor sufficient to meet the college requirements for mathematical literacy (Curriculum and Evaluation Standard for School Mathematics, 1989; Everybody Counts, 1989). More than any other subject, mathematics keeps students out of programs leading to scientific and professional careers (Douglas, 1986; Steen, 1987; Everybody Counts, 1989).

Calculus as a Gate-Keeper in the Mathematics Pipeline

The introduction of calculus is a critical stage in the mathematical education of students. Not only is calculus an important component of mathematical subject matter, but also students who take calculus are forced eventually to begin to think in new and different ways about mathematics (Orton, 1986). Fey (1984) argued that calculus plays an important role in the K-12 mathematics curriculum for the following reasons:

1. Calculus has a broad applicability to modeling change in the physical world.
2. Its study has a striking potential for revealing much about the history of mathematical ideas.
3. The subject matter naturally synthesizes and strengthens algebra and geometry skills and understandings acquired earlier.
4. It stimulates the development of geometric intuition (p.54).

Hence, calculus serves as a gate-keeper and bars many students from study in science, mathematics, physics (Redish, 1987), or engineering (Lathrop, 1987). Many other departments, such as biological science (Levin, 1987), and business (Prichett, 1987), require at least one term or even more than a year of study in calculus. As Lax (1986) claimed, "A calculus-deficient education would shunt students into a small corner of mathematics, instead of opening up its whole panorama (p.2)." In his introduction to the MAA report *Toward a Lean and Lively Calculus*, Douglas (1986) pointed this out clearly by saying:

> The United States is currently experiencing a shortage of young people studying mathematics, science, and engineering, and this shortage is expected to worsen. Calculus is the gateway and is fundamental to all such study. Hence every student who does not complete calculus is lost to further study in science, mathematics or engineering (p.iv).

Douglas (1987) continued:

> Calculus is taught to over three quarters of a million students a term; about a half billion dollars a year is spent on tuition for teaching calculus; and

> calculus is a prerequisite for more than half of the majors at colleges and universities. Almost everyone has a stake in calculus (p.5).

The role of calculus as a driving force for secondary school mathematics is a recent phenomenon (Fey, 1984). Calculus was at one time a subject to be learned only by an elite in the final stage of their college careers (Steen, 1987; Young, 1987). Today nearly one million students study calculus each year in the United States, usually completing their study of calculus by the end of their second year of college. Large numbers of students take calculus, because of an increased tendency on the part of other disciplines to require calculus. More recently, students in the biological and social sciences have been required to take a semester or even a year of calculus (Douglas, 1986). There are 100,000 calculus enrollments in two-year colleges, and another 600,000 in the four-year colleges and universities (Steen, 1987). Due to the fact that a considerable knowledge of calculus is regarded as a pre-requisite to further study in many disciplines, there is a great deal of pressure to learn concepts of calculus as early as possible. There are approximately 300,000 students taking some calculus in high schools and the number of students taking the Advanced Placement Calculus exams has been increasing 10% each year since 1960 (Tucker, 1987). We see that there is a trend for calculus to move down in the curriculum, from the last year of college to the first year of college, and now into the secondary schools.

Calculus Instruction in Crisis

A large segment of the mathematical community has recently come to believe that calculus instruction is in a state of crisis (Douglas, 1986; Orton, 1985, 1986; Steen, 1987, Tall, 1985). Douglas (1986) pointed this out in the MAA report by saying:

> Many students who start calculus do not complete it successfully. The country cannot afford this now, if it ever could. Further, many of those who do finish the course have taken a watered down, cookbook course in

> which all they learn are recipes, without even being taught what it is that they are cooking (p.iv).

Steen (1987) stated in the introduction of *Calculus for a New Century: A pump, not a filter*, that "in many universities, fewer than half of the students who begin calculus finish the term with a passing grade (p.xi)." Too many of these students fail their calculus courses or just barely pass, indicating they certainly have little understanding of calculus. Even among those who manage to survive with fair or good grades, knowledge of calculus is often superficial (Compton, 1987) and restricted to mere skill in solving routine computational exercises (Douglas, 1986; Steen, 1987). Students frequently have no idea what understanding of the concepts of calculus is nor do they realize the limit as the central idea for the calculus.

The Important Role of Limit Concept in Calculus

Many mathematicians and mathematics educators (Allendoerfer, 1963, Buchanan, 1965; Chaney, 1968; Confrey, 1980; Fless, 1988; Hight, 1963; Williams, 1989) have commented on the importance of the limit concept in learning the calculus. Allendoerfer (1963) stated that "many people assume that calculus is chiefly concerned with differentiation and integration, but this is a superficial point of view. The essential idea of calculus is that of a limit and without a clear exposition of limits any calculus course is a failure (p.484)." The mathematical concept of limit is one of the basic and most important concepts in calculus. Without an understanding of the notion of limit, the student's understanding of calculus will be at best superficial. There is a new trend of qualitative research on the learning of calculus (Confrey, 1980; Dreyfus, 1990; Dreyfus and Eisenberg, 1983; Davis and Vinner, 1986; Even, Lappan, and Fitzgerald, 1988; Orton, 1983a, 1983b; Tall and Schwarzenberger, 1978; Tall and Vinner, 1981; Vinner, 1983,

1987) suggesting that college students' understanding of fundamental calculus concepts, such as function, limit, derivative, and definite integral are undeveloped.

The theory of limits is important for learning mathematics both in the secondary schools, and in colleges. Hight (1963) argued that "the greatest gap between secondary school mathematics programs and colleges seems to be due to the present treatment of limits (p.205)." Hence a good understanding of the limit concept might narrow the gap. Chaney (1968) found a unit on limits to be very valuable to students preparing for college calculus. However, does the limit concept only enhance the learning of calculus specifically?

The Limit Concept Is Important

Many of the topics taught in the secondary schools and high schools cannot be adequately understood without an understanding of limit. Some of these topics include the circumference and area of a circle, finding the value of π, finding the area and volume of a sphere, finding the sums of geometric series, graphs of some given functions and asymptotes (Smith, 1959). An understanding of limits is central to a mature understanding of the real numbers (Confrey, 1980; Tall & Schwarzenberger, 1978), to the development of formulas for areas as well as volumes of geometrical figures (Buchanan, 1965), and an understanding of infinity (Fischbein et al., 1979). The limit concept is not only critical for the learning of calculus, but also for the following reasons:

1. The limit concept shows up in one of the earliest problems of geometry--the area of a circle. The area of a circle of unit radius cannot not be calculated explicitly in terms of rational numbers. The ancient Greeks approximated it by calculating the area of inscribed polygons with increasingly many sides. So, if A_n is the area of the inscribed regular n-gons, then $\pi = \lim_{n \to \infty} A_n$ by the formal definition of the limit of a sequence.

2. Various methods for approximate solution of equations, such as Newton's method, lead to sequences which converge to the (unknown) true solution of the equation.

3. The fundamental concepts of calculus are defined using limits. Thus without limits, calculus and much of higher mathematics would not exist.

4. Some mastery of the limit concept is also necessary for an understanding of the real number system. For example, what do we actually mean by an infinite decimal expansion? Why is it wrong to say $\sqrt{3} = 1.7320508$ (although this is what my calculator says). What do we mean by numbers like π and ε?

How the Limit Was Taught in School

Even though the mathematical concept of limit is important, students' notions about limits are frequently vague and incorrect. Many of the students have never encountered limits before they take calculus (Orton, 1987). Nearly every calculus course begins its new material with some consideration of limits either in terms of limit of functions or in terms of limit of sequences. Many of the textbooks start with the limit of functions' treatment and the limit of sequences has been treated as a special case of the limit of functions. In varying degrees of completeness and formality, the limit concept is treated only as the groundwork which is laid for later development of the concepts of derivative and integral. How is the concept of limit usually developed? First, class discussions of limits generally begin with some dynamic informal discussion of "f(x) getting close to L" as "x gets close to a." This informal introduction is quickly followed by a formal ε-δ definition of limit. Assignments and test questions rely on either algebraic manipulation and evaluation of limits or on some application of the formal definition (Fey, 1984). Typical assignments in the textbooks are of two varieties. First, students are asked to compute the limits of continuous functions; the task is merely one of substitution into a formula. In the second type of exercise, students are asked to find limits at the points of discontinuity; the expected solution requires an algebraic substitution and reduces the question to a case of numerical substitution or some memorized techniques for finding limits. Students' intuitive understanding of limits makes no connection with the formal treatment on which they have spent most of their time

(Fey, 1984). At the same time the formal treatment of the limit concept is far beyond their comprehension with the symbols, notations, and apparently unrelated subconcepts. In addition to these complications mentioned above, students have to deal with a combination of limits for the sequence of dependent variables and the sequence of independent variables.

The prevalence of misconceptions about limits (Confrey, 1980; Davis & Vinner, 1986; Davis, 1984, 1985; Dreyfus, 1990; Fischbein et al., 1979; Fless, 1988; Orton, 1983a, 1983b, 1987; Orton & Reynold, 1986; Sierpinska, 1987; Tall, 1981, 1985; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1989, 1991) is not surprising. The limit concept is not easy to understand because it involves an infinite process in some sense. Indeed, limits confused the best minds for centuries until finally, toward the end of the 19th-century, they were placed on what most mathematicians regard as a satisfactory foundation. (A more detailed description of the development of the limit concept will appear in Chapter Two.)

Limits of Sequences

Sequences and their limits occupy a position in analysis that is more basic and foundational than functions and their limits, derivatives, or integrals. Consider, for instance, the sequence of inscribed polygons used by the ancients to calculate the area of a circle. As another example, sequences formed by various iteration procedures, such as Newton's method or the method of successive substitution, are often used to solve equations. Sequences can be used to give an axiomatic development of the real number system; in fact, a development of the real number system by sequences leads into the concepts of fields, isomorphisms, and equivalence relations and equivalence classes, all of which are important concepts of an algebraic nature.

Sequences are simpler than functions because of their discrete nature, and yet they represent an infinite process of some sort; the domain of a sequence is an infinite set. Thus it is in studying sequences that the student encounters the limit concept in the simplest possible context, free of external complications such as needing to understand the nature of real intervals. Thus not only are sequences themselves of immense importance in mathematics in general, but they are the avenue mathematics educators most often recommend to introduce the limit concept in particular (Churchman, 1972; Curriculum and Evaluation Standards for School Mathematics, 1989; Goals for School Mathematics, 1963; Isaac, 1967; Macey, 1970; Shelton, 1965; Taylor, 1969,). Fort (1951) stated, "The easiest way to master limits is by the use of sequences and progressions (p.v)." In a guideline for teachers, Randolph (1957) put forth the idea that a treatment of limits should begin with a study of sequences (p.200).

More recently, *The National Council of Teachers of Mathematics' Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) provides a clear picture of what the K-12 mathematics curriculum ought to emphasize. Whether the goals are for K-4 or 5-8 mathematics, the NCTM strongly recommends development of pattern recognition. Patterns could be represented through a list of numbers, figures, graphs or objects. Pattern recognition exercises naturally suggest the question "what will happen if this pattern continues?" For example, one pattern exercise in Curriculum and Evaluation for School Mathematics for K-4 mathematics asks "given a list of numbers such as, 1, 1, 2, 3, 5, 8, 13 ,... tell what number comes next? What is your reason?" In addition to that, teachers could ask "what will happen if this pattern goes on forever? What is your reason?" The sequence introduced here not only enhances students' ability of pattern recognition, but also enhances the ability to grasp the notion of infinitely large, the infinitely small, the infinite process, and what is the product of this unending process. In Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), for 9-12 grades, the mathematics curriculum should include:

1. Investigating limiting processes by examining infinite sequences and series and area under curves; and
2. Understanding the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of a tangent line, and their applications in other disciplines.

The National Council of Teachers of Mathematics' Professional Standards for Teaching Mathematics (NCTM, 1991) recommended that teaching mathematics should focus on helping students learn to conjecture, invent, and solve problems which all relate to sequence recognition which in turn helps the learning of limit concept. Teachers should ask and stimulate their students by asking questions like the following:

1. "What would happen if ...? What if not?"
2. "Do you see a pattern?"
3. "What are some possibilities here?"
4. "Can you predict the next one? What about the last one? Would there be a last one?"

The concept of limit is an important one in mathematics. Yet this aspect of mathematical understanding is often neglected in the classroom (Orton, 1986; Orton & Reynolds, 1986). It may not be appropriate to teach limits formally in school before calculus, but the idea of a limit can easily be injected in a practical and intuitive way on many occasions. If the concept of limits were developed whenever appropriate throughout a mathematical education there would be more chance that sufficient time and experience had been made available for real meaning to have become integrated within the knowledge of structure. Mathematics learning consists very largely of building understanding of new concepts onto previous understood concepts (Orton, 1987). If students were taught to connect mathematical ideas and application from early on by exposing, inventing, conjecturing, and seeing patterns through the notion of sequences to the limit concept, then its introduction in calculus would be less painful for the students (Orton, 1987).

Research on Prospective Secondary Mathematics Teachers' Knowledge about Mathematical Limit Is Needed

Results from a report of the Secondary International Mathematics Study indicated that students of calculus in the United State were inferior to their counterparts in other countries. Students' poor performance in calculus showed a lack of understanding about some basic calculus concepts such as the notion of limits. Several studies have pointed to common misconceptions about limits experienced by students (Confrey, 1980; Davis, 1984, 1985; Davis & Vinner, 1986; Dreyfus, 1990; Fischbein et al., 1979; Fless, 1988; Orton, 1983a, 1983b, 1986, 1987; Orton & Reynold, 1986; Sierpinska, 1987; Tall, 1981, 1985; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1989, 1991). But there seems to be no study done on the teachers who teach calculus. What are their understandings about the limit concept? What are their misconceptions about the limit concept? Because a large and increasing number of students will probably start learning their calculus during formative stage in their development in secondary schools, the knowledge and understanding of calculus on the part of their teachers will be very important (MAA Note #6, 1986). What are their interpretations of the limit concept while in a teaching situation ? What kinds of misconceptions, difficulties and errors do they make? Are their misconceptions, difficulties, and errors similar to the students' as shown by the research? Understanding prospective teachers' misconceptions, difficulties, and errors help them become better prepared in their subject matter for teaching students. After all, today's prospective teachers are tomorrow's teachers.

Research on Teachers' Knowledge in General

Most of the early studies of teachers' knowledge emphasized relatively quantitative measures of teachers' knowledge, such as number of courses completed or performance on a standardized test (Ball, 1990; Carpenter, 1989; Even, 1989; Shulman, 1986). Carpenter (1989) stated that research on teacher thinking in the past has focused on generic processes. These generic processes are usually described as: whether teachers use a rational planning model; whether they plan in terms of lessons, units, or some other segment of time; to what factors they attribute students' successes and failures in a quantitative version. For the most part, past research has not seriously examined the subject matter taught as a variable to be investigated. Carpenter (1989) stated that teachers need a paradigm that blends the concern for the realities of classroom instruction with the rich analysis of the structure of knowledge and problem solving.

Peterson et al. (1987) argued that few researchers have attempted to take subject matter into account in analyzing teachers' beliefs within a specific topic area. Romberg & Carpenter (1986) argued that research on teaching mathematics should incorporate an analysis of the mathematics content into their studies of teaching. Recently research has emerged that studies how mathematics teachers' views about subject matter, teaching, and learning influence their classroom behavior (Madison-Nason & Lanier, 1986; Carpenter et al.,1986). Teachers' thoughts, beliefs, and cognition, as well as students' thought and actions have been emerging as important areas of inquiry in the recent research on teaching and learning (Clark & Peterson, 1986; Wittock, 1986). But most of this research emphasizes primarily content-free generic cognitive processes and instruction. Brophy (1986) stated that research is needed on teacher effectiveness within specific subject matter areas. Shulman (1986) labeled the absence of the focus on the subject matter to be taught in this new cognitive research on teaching as a "missing paradigm."

Research on learning and learners, and research on teaching and teachers, have been conducted separately from each other for a long time. There is not much research in mathematics education that integrates knowledge from both bodies of work. But teaching cannot be successful unless learning takes place and the process of learning is influenced directly by the different approaches to teaching and the teacher who is doing the teaching.

Learning and teaching, learners and teachers are two sides of a coin. The desirability of research that integrates research on teaching and teachers and research on learning and learners has been realized.

Teachers' Subject Matter Knowledge'

Recently, Shulman (1986) has proposed more qualitative analyses of teachers' knowledge. He suggested that a teacher should have three categories of knowledge: subject matter knowledge, curricular knowledge, and pedagogical content knowledge. The study of teachers' subject matter knowledge has thus come to represent a new focus in research on teaching (Shulman, 1986) and teacher education (National Center for Research on Teacher Education, 1988). From a variety of perspectives and with a variety of approaches researchers increasingly focus on the subject matter knowledge of teachers and its role in teaching. In research on the teaching of mathematics, some researchers investigate teachers' and prospective teachers' beliefs about the subject or their notions about teaching it. Other researchers focus on teachers' and prospective teachers' understandings of specific topics (Ball, 1988, Ball & McDiarmid, 1988; Davis, 1986; Dreyfus & Vinner, 1982; Even, 1989; Linhardt & Smith, 1985; Hershkowitz & Vinner, 1984). They explore how teachers think about their mathematical knowledge and how they understand (or misunderstand) specific ideas.

Due to advanced technology, we live in an information-driven society. In order to provide a workforce to meet the needs for this society, mathematics teaching can no longer be confined as the providing of skill acquisition, but rather should be perceived in terms of thinking processes (Balacheff, 1990). Teachers need a knowledge that, as stated by Carpenter (1989) "May provide teachers with a basis to more effectively assess their students' knowledge and make decisions about appropriate instruction." Teachers' beliefs and knowledge about a specific topic in mathematics may have a profound effect on how they teach and as a consequence on the learning of students in their classrooms. *The National Council of Teachers of Mathematics' Professional Standards for Teaching Mathematics* recommended that the image of mathematics teaching should include the following:

1. Selecting mathematical tasks to engage students' interests and intellect;
2. Providing opportunities to deepen their understanding of the mathematics being studied and its' applications;
3. Orchestrating classroom discourse in ways that promote the investigation and growth of mathematical ideas;
4. Using, and helping students use, technology and other tools to pursue mathematical investigations;
5. Seeking, and helping students seek, connections to previous and developing knowledge;
6. Guiding individual, small-group, and whole-class work (p.1).

The concept of limit has had profound influence on the development of mathematics in general and on the rigorous foundation of calculus in particular. It is a concept that provides connections to previous and further developing mathematical knowledge.

Purpose of this Study

The purpose of this study, therefore, is to investigate prospective secondary mathematics teachers' understanding about the mathematical limit in terms of subject matter knowledge, curriculum knowledge and pedagogical content knowledge (Shulman, 1986). In the understanding of subject matter knowledge, this researcher intends to find out how well prospective teachers understand the limit concept and what their misconceptions, difficulties, and errors about the limit concept are. In curriculum knowledge, this researcher intends to find out what prospective teachers thought the role of limit concept in the K-12 mathematics curriculum should be and what activities for helping younger students to learn this limit concept could be developed. In pedagogical content knowledge, this researcher intends to find out how they think about teaching this limit concept when provided a teaching situation, and how much they know about students' misconceptions, difficulties, and errors; and what are their strategies for helping their students to overcome these misconceptions, difficulties, and errors. The purposes of the present study are:

1. To examine prospective secondary mathematics teachers' understanding about the limit concept.
2. To investigate prospective secondary mathematics teachers' misconceptions, difficulties, and errors and their model of the limit concept.
3. To describe prospective secondary mathematics teachers' opinions about the connection of the limit concept to K-12 mathematics curriculum.
4. To explore prospective secondary mathematics teachers' understanding about students' misconceptions, difficulties, and errors concerning the limit concept.

Based on the above purposes this researcher intends to investigate the following research questions:

1. How well do prospective teachers understand the concept of limits?
2. What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the concept of limits?
3. What are prospective teachers' opinions about the involvement of the concept of limits in k-12 mathematics curriculum?
4. What are the possible misconceptions, difficulties, and errors the prospective teachers anticipate in teaching the concept of limits?

The Structure of this Dissertation

The second chapter focuses on the limit concept and the development of the limit from a historical point of view based on books and articles written by Boyer (1949), Cajori (1915, 1923), Confrey (1980), Edwards (1979), and Kline (1970, 1972, 1980). The discussion focuses on how the limit concept has evolved over time, and on how the concept has been perceived by mathematicians in different times. What turned the ancient mathematicians' minds towards the origins of the limit concept? What phenomena in the real world motivated the mathematicians, philosophers, and physicists to develop an intuitive understanding of this limit notion? How did this intuitive understanding create difficulties in the development of the limit concept? What prevented the development of a rigorous formulation of the limit concept for such a long period of time? This chapter, on the development of limit concept, shows that many ancient mathematicians who lacked a rigorous understanding could often solve the computational work, which is similar to many students in present-day classrooms. In this chapter we also see that the difficulties encountered by many students today confused many ancient mathematicians and that the misconceptions prevalent among today's students were common to some of the ancient mathematicians. Errors made by students were made by some of the ancient mathematicians too. Perhaps, as an educator, one should start to think about mathematics history and how history of mathematics can help in teaching mathematics.

The third chapter is the literature review, which consists of four different categories of studies. The first category of literature review will focus on the studies related to students' notion of limit in the following areas: 1) studies pertaining to the psychological question: can students at the high school level learn the limit concept, 2) studies pertaining to the philosophical question: assuming that the high school students can learn the limit concept, should it be taught, 3) studies pertaining to the pragmatic question: what is the best method for teaching the limit concept, and 4) studies pertaining to the students' misconceptions: what are the students' misconceptions, what causes these misconceptions, could these misconceptions be changed by providing an adequate teaching situation (Confrey, 1980; Davis, 1984, 1985; Davis & Vinner, 1986; Dreyfus, 1990; Fischbein et al., 1979; Fless, 1988; Orton, 1983a, 1983b, 1986,1987; Orton & Reynold, 1986; Sierpinska, 1987; Tall, 1981,1985; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1989, 1991). The second category of the literature review will focus on studies pertaining to the teachers' misconceptions: What are the teachers' misconceptions, what causes these misconceptions, and how might these misconceptions affect their teaching? The third category of literature review will focus on studies pertaining to levels of understanding of specific mathematics topics. The last category of literature review focuses on the teachers' knowledge: How knowledge is categorized? What are different representations of knowledge? How is teacher's knowledge stored ?

Chapter Four includes the design of this study, the theoretical model of five-category of understanding for analyzing the data, the construction of the questionnaire test items, the design of the interview questions, the background of the subjects, the models of limit held by the subjects, the procedures of data collection, and data analysis.

The fifth chapter consists a report the analyses of the data. The analysis will focus on discussing the results of these four research questions: 1) How well do prospective teachers understand the concept of limits? 2) What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the concept of limits? 3) What are prospective teachers' opinions about the role of the concept of limits in k-12 mathematics curriculum? and 4) What are the possible misconceptions, difficulties, and errors prospective teachers anticipate in teaching the concept of limits? These four research questions are intended to find out about prospective teachers' subject matter knowledge, curriculum knowledge, and pedagogical content knowledge. The discussion of the first two research questions will describe what prospective teachers' subject matter knowledge about limit concept is. Results related to the third research question will describe what prospective teachers' curriculum knowledge looks like. And the discussion of the fourth research question will describe what prospective teachers' pedagogical content knowledge is. Because of the small number of responses on the survey questions and the fact that only four prospective secondary teachers were interviewed, there was inadequate data to come to any significant conclusions related to research questions three and four concerning the curriculum knowledge and pedagogical content knowledge; however, the data collected from the survey and excerpts from transcripts of the interviews are presented in order to perhaps motivate or provide baseline information for future studies that may examine these types of teachers' knowledge. The discussion to the first two research questions will be based on the responses of the participants provided on the questionnaire question items Part II number 1 through number 10. A scale like the Guttman scalogram scale was used to scale prospective teachers' performances at 70%, 80%, 90% criterion, and a constructed theoretical model of categories of understanding has been developed, and will be used as the frame to discuss the understanding about the limit concept. The responses of the participants will be gathered, grouped and analyzed based on the constructed theoretical model of five categories of understanding.

The third research question: What are prospective teachers' opinions about the role of the concept of limits in K-12 mathematics curriculum? was embedded in the first part of the questionnaire. In there, the subjects were asked to respond to the same open-ended question. The responses showing whether or not the prospective secondary mathematics teachers are aware of the role of the limit concept in K-12 mathematics curriculum were gathered. First this researcher will list the activities that the participants provided in the survey as examples of what they thought would be good activities for K-2 and 4-5 grade ranges of children. Next, the transcripts of four subjects' interview data were presented and followed by a short summary.

The last research question: How much do the prospective secondary mathematics teachers anticipate the students' misconceptions, errors, and difficulties? was embedded in the first part of the questionnaire. The responses will be listed and grouped. First, the researcher will write down the participants' statistical results on the survey, and then qualitatively explain what misconceptions, difficulties, and errors the participants thought most often occur. Next, the researcher will describe what the participants thought would be a good method to alleviate these difficulties, misconceptions, and errors. Finally there follows an in-depth discussion based on the transcripts from the four interviewees.

The final chapter contains a short summary, defines the limitations of this study, gives some recommendations for teaching the notion of limit, and mentions some possible further research directions.

CHAPTER TWO

THE CONCEPT OF THE LIMIT: AN HISTORICAL PERSPECTIVE'

The notion of limit is a central idea in many different branches in mathematics. Basically, some version of the limit concept must appear whenever some mathematical object (e.g. a geometric figure or a number) is represented as somehow being the end result of some infinite process. Limits are necessary in mathematics because finite processes are not sufficient to enable one to advance very far in certain vital mathematical analyses. The limit concept is the logical cornerstone of the calculus, which means that students who do not understand this concept never advance beyond a superficial understanding of calculus. This is why researchers have been interested in trying to investigate what it is about the limit concept that makes it so particularly difficult to grasp.

Researchers on students' understanding of limits have found that errors and mistakes seem to broadly follow certain patterns; there seem to be certain misconceptions and concept image difficulties that are common to many students (Confrey, 1980; Davis, 1984, 1985; Davis & Vinner, 1986; Dreyfus, 1990; Fischbein et al., 1979; Fless, 1988; Orton, 1983a, 1983b, 1986, 1987; Orton & Reynold, 1986; Sierpinska, 1987; Tall, 1981, 1985; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1989, 1991). A study of the historical development of the limit can afford valuable insight into why people think the way they do. As Confrey (1980) remarked, "One way to discover and investigate a conceptual change problem is through examining the historical development of a concept or system of concepts." This researcher believes that to a large extent each student must create mathematical knowledge from the very beginning for him/herself, experiencing mankind's mathematical development in microcosm. Morris Kline (1970) argued the following:

> There is not much doubt that the difficulties the great mathematicians encountered are precisely the stumbling blocks that students experience and that no attempt to smother these difficulties with logical verbiage will succeed. ... Moreover, students will have to master these difficulties in about the same way that the mathematicians did, by gradually accustoming themselves to the new concepts, by working with them and by taking advantage of all the intuitive support that the teacher can muster (p.270).

This study of the history behind the limit concept has, in fact, been fruitful. We find many of the concept-image problems of present-day students troubling mathematicians of the past, many of them among the greatest thinkers of their time. This is suggestive, at least, that there are certain conceptual difficulties and hindrances to absorbing the limit concept that are somewhat "natural" and "inevitable" features of the human mind. There may be certain successive stages of understanding that almost everyone must go through. If this is indeed the case, it would certainly be desirable if teachers were sensitive to these stages and could devise strategies for helping students get through them.

The history of the continuum concept parallels the history of the limit concept. The mathematical concept of the continuum, though difficult in some ways to understand, and impossible to understand without talking about limits, actually makes many problems easier. The mechanics of solid bodies (a branch of physics), would be made much more difficult by thinking of each body as made up as a very large but finite number of atoms, each of which had to be mathematically described separately, rather than as a continuous spread of mass. In studying the history of the development of the notion of limits, we will be able to see how mathematicians of the past were unable to answer satisfactorily questions of great importance in the development of science because the concept of the limit and techniques in infinite processes were missing. We can see that on many occasions mathematicians came close to understanding limits but were unable to formulate their concepts with the necessary precision. We will have a better understanding of why the final formulation of the limit concept, as given by Weierstrass in the 1870's, needed to take the form it did. It seems that these insights concerning the limit concept: its necessity in mathematics and science; why it proved so elusive; why it took the final form it did, are actually insights about why the calculus is so difficult for students to grasp.

In this historical survey, we shall find that many of the questions that puzzle present-day students concerning the nature of infinite processes, limits, and the number system (Confrey, 1980; Taback, 1975; Williams, 1989) also bothered many of the greatest thinkers since antiquity. In fact, many of the concept images of the participants in this study, as evidenced by questionnaire responses and interviews, were shared by a large number of mathematicians in the past. We have used the term "concept image" above, and it is wise at this point to clarify what we mean by this term. Tall and Vinner (1981) gave the following definitions for concept image and concept definition:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experience of all kinds, changing as the individual meets new stimuli and matures.
>
> We shall regard the concept definition to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him or constructed by himself, he may vary it from time to time. In this way a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large (p.152).

It seems as though mankind as a whole, in coming to understand the foundations of calculus, has had to go through essentially the same sequence of understanding as every present day student must go through if he/she is to truly master calculus. The root cause of all the difficulties with the limit concept is that the naive human mind cannot conceive of infinity; everything one experiences is finite. Yet in mathematics, as one shall see, the infinite intrudes at a very early stage of development. What was needed was a way to bridge the gap between the inability to visualize the whole of an infinite process and the need to be able to work with entities which could only be realized as the end results of some infinite process. This needed bridge is the concept of the limit.

Pre-Hellenic Mathematics: Greek Notion of Mathematics as a Deductive System

The pre-Hellenic Egyptians produced a large body of numerical and spatial relations as a result of empirical investigations, and even found the formula for volume of square pyramids, probably as an extrapolation from empirical work (Boyer, 1949). The Babylonian astronomers studied problems involving continuous variation, by tabulating values of certain functions (such as brightness of the moon) and then inferring the (approximate) maximum of the function (Boyer, 1949). These accomplishments seem primitive to anyone with some knowledge of modern mathematics, but one has to remember that these people were organizing knowledge out of the raw material presented by sensory perception. Boyer (1949) remarked,

> More fundamental than this lack of deductive proofs of inferred results is the fact that in all this Egyptian work the rules were applied to concrete cases with definite numbers only. There was no conception in their geometry of a triangle as representative of all triangles, an abstract generalization necessary for the elaboration of a deductive system (p.15).

Thus the pre-Hellenic Egyptians had not reached a stage of sophistication where it would occur to them to make a general statement about, for example, <u>all</u> triangles, and then try to prove the statement for all triangles at once.

The ancient Greek mathematicians adopted the attitude that mathematics was to be a closed deductive system, and this largely determined for all later generations the principle characteristic of the area of activity known as mathematics. This seems to have happened very early in Greek mathematics with Thales (Boyer, 1949). It is possible that Thales was influenced by Egyptian and Babylonian thinkers, but information about this period is fragmentary. A further reason for continuing to maintain mathematics as a closed deductive system was the Greek belief in the unity and reasonableness of nature, a belief that led them to think that the laws of nature could be deduced as part of mathematics (Boyer, 1949). Going right along with this, in fact probably inseparable from it, is the philosophical attitude that the geometric figures to be considered must be abstractions rather than actual material objects. As Morris Kline (1972) remarked:

> One of the great Greek contributions to the very concept of mathematics was the conscious recognition and emphasis of the fact that mathematical entities, numbers, and geometrical figures are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures.... Moreover, geometrical thinking in all pre-Greek civilizations was definitely tied to matter. To the Egyptians, for example, a line was no more than either a stretched rope or the edge of a field and the rectangle was the boundary of a field (p.29).

Greek Concepts

One of the necessities for the development of modern mathematical analysis (by which one means calculus and its various extensions and applications) is an adequate concept of the number system. The real number system one uses today is not directly observable. What is more or less observable is the counting numbers; that is, the set of positive integers. Any more elaborate number system must be in large part an invention of the human mind. "The integers come from God, all else is the work of man" as Leopold Kronecker once remarked (Kline, 1972, p.979). The ancient Greek's did not regard the fraction 2/3, say, as a single number, but thought of it as a proportion 2:3 (Boyer, 1949; Kline, 1972). They developed a theory of proportions, a proportion being really an ordered pair of integers. Thus the Greek's number system was restricted to what we would call the rationals, although they did not think of them in the same way as we do. At the same time, the Greek mathematicians were working on the problems of area and length. Since area and length in general could not be defined non-tautologically, (that is, defined in terms of other notions which are already understood.) they based their theory on the notion of *application*, by which they basically meant placing one figure on the top of another and seeing if one fit inside or coincided with the other. The Greek geometers did not think of a single figure, such as rectangle or a circle, as having an area, say, or a length, but rather thought in terms of ratios of two figures. Two figures A and A' (e.g. segments or rectangles) were called *commensurable* if a third figure B could be found such that A and A' could each be chopped up into an integral number of copies of B. Any pair of line segments having rational lengths (or any pair of rectangles having rational sides) would be commensurable. If all line segments were of rational length then any pair of line segments would be commensurable. It was discovered, to the considerable dismay of Greek geometers, that certain pairs of line segments were incommensurable (Boyer, 1949; Browne, 1934; Cajori, 1915, 1923; Edwards, 1979; Kline, 1972). Indeed, it was found that the diagonal of a square is not commensurable with its sides. Thus, as a corollary, not all lengths were rational multiples of each other.

This incommensureability phenomenon meant that for the Greek mathematicians there was no exact correspondence between numbers and geometric quantities like length and area, and this hampered the development of Greek mathematics. They talked of geometric magnitudes, but could not regard magnitudes and numbers as the same thing, because they only recognized what now would be called the rational numbers and therefore any pair of numbers would have to be commensurable.

Some attempts were made to remedy the incommensureablility dilemma by introducing a kind of infinitely small segment which could be used as a common unit of measure for the side and diagonal of a square, but Greek mathematicians generally did not take kindly to infinitesimals. Indeed, admitting infinitesimals was correctly seen to be equivalent to admitting infinity, and Greek thinkers had, as Boyer and Edwards remarked several times, a "horror of infinity." Kline (1972) informed us that good and evil were associated with limited and unlimited respectively (p.175). Since the Greek thinkers had a great deal of faith in the reasonableness of nature and believed that mathematics mirrors nature closely, they rejected such things as infinite sets and infinite processes. Aristotle distinguished between a "potential infinity" and an "actual infinity." (Boyer, 1949; Fischbein et al. 1979; Kline, 1972; Tall, 1981) "Potential infinity" occurs in a situation in which no matter where one is one can go another step; for example, given any positive integer one can always think of a larger one. Thus the set of whole numbers can be regarded as potential infinity in that, if one stops at one million, he/she can always consider one more, two more, and so forth. However, the set of whole numbers viewed as an existing totality is actual infinity (Kline, 1980). Aristotle denied the existence of the actual infinity. Aristotle said, "In point of fact they (mathematicians) do not need the infinite and do not use it (Boyer, p.41)."

Greek mathematicians also grappled with trying to understand the nature of time and space. Many ascribed to an atomistic view, maintaining that time and/or space is made up of individual units, something like "atoms" of time or space. Others saw time and space as connected, and Plato seems to have tried to fuse the two viewpoints by thinking of them as "generated by the flowing of the apeiron" (Boyer, p.28). The relation of the discrete to the continuous was in all a puzzle to the Greek, as noted by Kline (1972), who goes on to paraphrase Aristotle on the relation between points and lines, "Though he admits points are on lines, he says that a line is not made up of points and that the continuous cannot be made up of the discrete ... (p.175-176)."

The weakness in Greek concepts of number, time and space were pointed up by Zeno, who propounded his famous four paradoxes. The first two paradoxes, known as the Dichotomy and the Achilles, seem to show it is logically inconsistent to hold that time and space are infinitely divisible, and the third and fourth paradoxes, known as the Arrow and the Stade, go in the other direction by seeming to show that the atomistic view of time and space is self-contradictory. (For a brief description and discussion of the paradoxes, see Boyer, 1949, p.24 and Kline, 1972, p.35-37. As an example, let us briefly sketch one of these paradoxes, the paradox commonly known as the Dichotomy. The Dichotomy goes as follows: in order to traverse a line segment AB, one must first arrive at the midpoint C of AB; to arrive at C one must first arrive at the midpoint D of AC; and so forth. "In other words, on the assumption that space is infinitely divisible and therefore that a finite length contains an infinitely number of points, it is impossible to cover even a finite length in a finite time (Kline, 1972, p.35)."). The Greek philosophers were not able to really resolve these paradoxes, and in fact they puzzled such people as Galileo many centuries later (Boyer, 1949). As Boyer (1949) pointed out the uneasiness caused by the paradox known as the Dichotomy is caused by the process by which an infinite series converges to a finite sum and he continued,

> It is clear that the answers to Zeno's paradoxes involve the notions of continuity, limits, and infinite aggregates-abstractions (all related to that of number) to which the Greeks had not risen ... (p.25).

Perhaps the root of the problem was the difficulty understanding how a finite interval of numbers could be divided into infinitely many subsets. The paradoxes of Zeno (Boyer, 1949; Cajori, 1915; Kilmister, 1980; Kline, 1972) seemed to have discouraged Greek mathematicians from trying to quantitatively describe and analyze variable phenomena, in particular motion. In fact, throughout the history of mathematics the understanding of motion seems to have remained harder to achieve than the understanding of shape and of number. In this regard, Boyer (1949) remarked that "as long as Aristotle and the Greeks considered motion continuous and number discontinuous, a rigorous mathematical analysis and a satisfactory science of dynamics were difficult of achievement (p.43)."

The possibility of finding inscribed and circumscribed regular polygons which fit the circle closely was the basis for estimating its area and proving statements about its area. It seems to have even been hoped by early geometers (Boyer, 1949, p.32) that by successively doubling the number of sides of the inscribed regular polygon one could eventually reach a polygon which would coincide with the circle. Although later geometers know that there is no way to reach a "last polygon," the sequence of inscribed and circumscribed regular polygons one gets by successively doubling the number of sides is a very useful sequence, for it is possible to compute the bases and altitudes of the constituent triangles, and so compute their areas. One has then a sequence $\{P_n\}$, where P_n is a regular polygon having 2^n sides, and an analogous sequence $\{Q_n\}$ of circumscribed regular polygons, and one has

$$\text{Area of } P_n < \text{Area of } C < \text{Area of } Q_n.$$

If one calculates the first and the third quantities for a large value of n, one has a good approximation to the area of the circle. The reader who wishes to draw a circle with inscribed and circumscribed regular polygons of 4 sides (n=2), then 8 sides (n=3), then 16 sides (n=4), will see why it seems reasonable that these regular polygons will fit the circle more and more closely and will in fact fit the circle very closely for very moderate values of n.

There is of course no value of n so large that P_n and Q_n actually coincide with the circle, but one can understand how this possibility would occur to a geometer who thought of the points on the circumference as having a certain size, like an "atom of arc length."

This sequence of regular polygons may well be the first example of a really natural and useful sequence. A modern mathematician would think of the area of C as being the limit, as n tends to infinity, of the numbers {Area of P_n}, perhaps taking this to define what is meant by the term "area of the circle." One has to remember that the Greek mathematicians would not have regarded an infinite process as being completed, but one is, one way or another, thinking of the circle as being the end result of some infinite process when one considers computing its length or area. The mathematicians and philosophers of Greek antiquity for the most part did not believe in the infinitely large or infinitely small, and did not depart far from what could be perceived by the senses. For example, Boyer (1949) stated that "... Aristotle ... did not go beyond what is clearly representable in the mind. In consequence he denied altogether the existence of the actual infinite and restricted the use of the term to indicate a potentiality only (p.40)." To a mathematician of that era, it would have been invalid to think of a circle as being a limit of polygons unless one could visualize the whole process by which the polygons "eventually become" the circle, and such a visualization is impossible. Explaining how a sequence of polygons can eventually become a circle is very much like answering the Zeno paradox called the Achilles.

Although the geometry problems dealt with by the ancients frequently involved infinite processes, implicitly or otherwise, they were frequently able to sidestep a direct confrontation with the infinite (and thus with limits) and still make progress by using a technique known as the "Method of Exhaustion." This method was used also by later geometers. Basically the method can be described as applying (often in ingenious ways), the Axiom of Eudoxis, which can be stated as follows:

> Axiom of Eudoxis: Let M_0 and ε be given magnitudes and suppose that M_1 is formed by throwing away at least half of M_0 (so that $M_1 \leq \frac{M_0}{2}$) and similarly, M_2 is formed by throwing away at least half of M_1 ($M_2 \leq \frac{M_1}{2}$), and that this process is continued , in such a way that we always have $M_j \leq \frac{M_{j-1}}{2}$. Then there is a natural number n for which $M_n < \varepsilon$.

In more colloquial language, if we successively bisect some magnitude we eventually reach a stage where what is left is less than any pre-assigned magnitude. By using this method, the Greek mathematicians were able to avoid having to regard infinite processes as being completed, but there was a rather heavy price; it seems the logic of the situation is such that this method ends up involving a double reductio ad absurdum. This meant that their proofs could often be quite cumbersome and difficult. To appreciate this point one can read (Edwards, 1979, p.16-27), in which several examples of the method of exhaustion are given. As Edwards (1979) remarked:

> A logically complete indirect proof is thereby obtained without explicit reference to limits. The mystery which the Greeks attached to the infinite

and, in particular, to what we call the limit concept is absorbed (if not obviated) in Eudoxus' principle (p.27).

Although the purpose of Eudoxis' Axiom was to obviate the difficulties of infinite processes, one can see the seeds of the modern limit concept in it. First, there is clearly an intuitive recognition of the fact that the sequence $(1/2)^n$ tends to zero. More striking is the conception that the magnitudes M_n eventually become less than any pre-assigned fixed magnitude.

As stated earlier, the Greek mathematicians had a "horror of infinity." However, Achimedes used infinitely small quantities (infinitesimals), but only as an intuitive working tool. Boyer (1949) explained that "using this heuristic method, Archimedes was able to anticipate the integral calculus in achieving a number of remarkable results" (p.50). However, Boyer thought that "Archimedes employed his heuristic method... as an investigation preliminary to the rigorous demonstration by the method of exhaustion (p.51)." A particularly remarkable achievement of Archimedes was his discovery of the formula for area of a parabolic segment (Boyer, 1949). Achimedes was led to consider the sequence of sums

$$A,\ A(1+1/4),\ A(1+1/4+1/4^2),\ \ldots,\ A(1+1/4+\ldots+(1/4)^n),\ \ldots$$

and by using the double reductio ad absurdum of the method of exhaustion inferred that the area could be neither greater than nor less than (3/4) A. (A being the area of a certain triangle inscribed in the parabolic segment.)

Medieval Times

With the collapse of the Roman Empire, about 500 AD, began a period of several hundred years when scholarly activity and the development of science in Western Europe basically stopped. The Greek heritage was not entirely lost because of the Byzantine and Muslim civilizations which preserved it (Edwards, 1972). As Edwards (1972) remarked,

> For approximately four centuries the Muslim world preserved the Greek mathematical tradition and enriched it with the addition of Eastern elements of arithmetic and algebra. By the end of the twelfth century Arabic science had begun to decline but fortunately, Western Europe had emerged from its dark ages with an appetite for new knowledge (p.85).

Thus there is a long gap in time when not much happened that is directly relevant to our history, but it is fortunate that the ancients' contributions were not lost.

During the thirteenth and fourteenth centuries the scientific works of Aristotle were freely circulated, and stimulated much discussion on the nature of time and space, and the notion of indivisible (Boyer, 1949). Some scholars claimed that time was made up of indivisible, an hour containing 22,560 such instants, or "atoms of time." Others believed time was composed of infinitely many continua.

During the second quarter of the fourteenth century the problem of quantifying change was attacked by scholars at Merton College in Oxford (Edwards, 1979), including Thomas Brandywine and Richard Swineshead (the latter of whom was also known as Richard Suiseth, and as the Calculator). In particular they addressed themselves to trying to quantify motion. This was an important new direction in mathematics. We recall that the Greek mathematicians, perhaps dismayed by Zeno's paradoxes, were unable to apply any kind of quantitative reasoning to motion, and regarded it as a quality rather than a quantity (Boyer, 1949). Thus the Merton College scholars were probably the first people to attempt a mathematical analysis of changing quantities. These scholars came up with suitable definitions of uniform speed and uniform acceleration, and formulated the "mean speed theorem," which says that under uniform acceleration the average speed during any time interval is the arithmetic average of the initial and terminal speeds. Thus it was not until the fourteenth century that mathematicians began to rise toward a descriptive deductive understanding of time and motion, so necessary to the development of any science of dynamics. Other thinkers during medieval times also concerned themselves with understanding of the nature of change and motion, of whom perhaps the most notable was

Nicole Oresme, whose dates are approximately 1320-1380. Oresme was perhaps the first to make use of graphical representations (Boyer, 1949; Edwards, 1979), very much as a modern mathematician would. Oresme made "...most effective use of geometrical diagrams and intuition, and of a coordinate system, to give his demonstrations a convincing simplicity (Boyer, p.80)," in contrast to the "tediously verbose" (Boyer, p.78) proofs given by Calculator. Work on geometry continued also during the medieval period. This work apparently did not go significantly beyond what was known to the Greeks, but at least all this activity kept the ancient knowledge alive.

Boyer (1949) essentially summed up the medieval period as follows:

> It has been remarked that the medieval period added little to the classical Greek works in geometry or to the theory of algebra. Its contributions were chiefly in the form of speculations, largely from the philosophical point of view, on the infinite, the infinitesimal, and continuity, as well as new points of view with reference to the study of motion and variability. Such disquisitions were to play a not insignificant part in the development of the methods and concepts of the calculus, for they were to lead the early founders of the subject to associate with the static geometry of the Greeks the graphical representation of variables and the idea of functionality (p.94).

The medieval mathematicians encountered and sometimes solved problems involving infinite summations (usually called infinite series in present day mathematics). Edwards (1979) informed us that "the subject of infinite series fascinated medieval philosophers and mathematicians, appealing to both their interest in the infinite and their disputatious delight with apparent paradoxes (p.91)."

Calculator considered, and intuitively solved by means of a "long and tedious verbal proof," the question of average velocity for a point which moves throughout the first half of a certain time interval with a certain initial velocity, throughout the following eighth at triple the initial velocity, and so on ad infinitum; in other words, he summed the series

$$1/2 + 2/4 + 3/8 + \ldots + n/2^n + \ldots .$$

Oresme also solved some problems involving infinite summations. Edwards (1979) said,

> The study of infinite series continued during the fifteenth and sixteenth centuries in the mode of Swineshead and Oresme, without significant advance over their exclusively verbal and geometric techniques. The principle contribution of these early infinite series investigations lay not in the particular results obtained, but in the encouragement of a new point of view--the free acceptance of infinite processes in mathematics (p.93).

From our point of view, then, there were two extremely important things that happened during the thirteenth and fourteenth centuries. One was the free acceptance of infinite processes in mathematics. Mathematicians had begun to overcome the Greek "horror of the infinity," a necessary step toward the eventual development of the limit concept. The other was the initiation of a quantitative analysis of variable quantities. Thus mathematicians no longer needed to be confined to a static unchanging world as in the time of the ancient Greeks.

The Sixteenth and Seventeenth Centuries

Some of the events of the sixteenth and seventeenth centuries contributed indirectly to the development of the limit concept and to calculus, by simplifying and at the same time enriching the mathematical environment in which the necessary discoveries were later to be made. Mathematicians began to accept irrational and negative quantities, though grudgingly, as bonafide numbers (Boyer, 1949). Francois Viete (1540-1603) invented an efficient system of notation, including a literal designation that clearly distinguished between knowns and unknowns in an equation (Edwards, 1979). This evolved in such a way that Rene Descartes (1596-1650) used essentially today's standard algebraic notation (Edwards, 1979). This made it possible for mathematicians to communicate both the statements and the solutions of problems with a brevity and clarity not possible for earlier mathematicians, who necessarily had to use cumbersome verbal statements and expositions, and doubtless made it easier to perceive common elements between problems. Descartes and Pierre de Fermat (1601-1665) originated analytic geometry, and they also emphasized the notion of a variable, which was "indispensable to the development of the calculus (Edwards, p.97)." These improvements created an environment in which it was possible to do all mathematics, including the mathematics relevant to this history, more efficiently.

The works of Archimedes became widely available and popular during the latter sixteenth century (Boyer, 1949) and soon mathematicians had reached the point where they could make original extensions of the Greek work, solving new problems involving areas, volumes, centroids and their geometric properties. The later sixteenth and earlier seventeenth centuries saw a great deal of activity of this type. During this period dissatisfaction with the ancient method of exhaustion, with its ubiquitous and often cumbersome double reductio ad absurdum began to manifest itself. Many mathematicians, notably Simon Stevin about 1585 and Luca Valerio about 1600, attempted to find modifications of Archimedes' procedure which would obviate the need for always going through the complete reductio ad absurdum. Another school of investigators, notably Kepler, Galileo, and Galileo's student Cavalieri, essentially ignored the method of exhaustion and achieved heuristic non-rigorous solutions of many problems by developing a "method of indivisible" which is somewhat reminiscent of the intuitive explanation of various formulas for area and volume offered to students of integral calculus today, and also is similar to the infinitesimals method used as a preliminary exploratory tool by Achimedes. Although Cavalieri was able to find the correct answers to many geometric problems by his methods while avoiding the method of exhaustion, he was unable to answer such questions as how a solid could be built up from entities that had no thickness and how one could compare the indivisible of different solids (Boyer, 1949). Cavalieri was interested in results, not rigor (Boyer, 1949). Torricelli (1608-1647), a disciple of Galileo and Cavalieri (Edwards, 1979) also made adept and creative use of indivisible, but was very much aware of the logical difficulties involved in the method (Boyer, 1949). As Boyer (1949) pointed out there was no "safe guide" in the use of indivisible.

The necessary "safe guide" in using any infinite process in mathematics, such as building up a solid from indivisible as done by Cavalieri and others, is of course a properly evolved theory of limits. Lacking such concept and theory, the geometers of these times could choose one of two unsatisfactory alternatives. They could go back to the method of exhaustion, which was logically unassailable but so cumbersome to apply that it seemed to actually slow down the process of discovery; or they could use the method of indivisible which was built on shaky logical premises and thus not trusted by everyone.

Besides problems concerning area and volumes and centroids, progress was being made on another important geometric problem, the problem of finding the line tangent to a curve at a given point. The problem of finding tangent lines to curves would later become of fundamental importance in calculus. Workers on this problem include Torricelli , Fermat, Descartes, and Isaac Barrow, who was one of Newton's teachers, and others (Boyer, 1949; Edwards, 1979). The methods used were intuitive, the logic behind them unclear and open to objections. Nonetheless heuristic solutions, later found to be correct, were found to a variety of problems. As Gauss put it. "I have got my result but I do not know yet how to get it (Kline, 1970, p.271)." The art had advanced by the 1650's to the point where Hudde and Sluse and Huygens (independently) (Boyer, 1949; Edwards, 1979) were able to write down algorithms for finding tangent lines to any rational curve.

In addition to the above accomplishments of the sixteenth and seventeenth centuries, Galileo and Torricelli (and no doubt others) had made some progress on understanding the nature of velocity and motion (Boyer, 1949). Galileo found a rule for the distance traveled by a falling body. Galileo also had interesting thoughts on the nature of infinity (Boyer, 1949). Galileo observed that infinite sets could be put into one-to-one correspondence with proper subsets of themselves; for instance the natural numbers can be put into one-to-one correspondence with the perfect squares.

1	2	3	4	5	6	7	8	...
1	4	9	16	25	36	49	64	...

Galileo made an uncritical use of indivisible in his work. Galileo felt that "infinity and indivisibility are in their very nature incomprehensible to us (Boyer, 1949, p.115)." In trying to understand the continuum he suggested that there might be a third possible kind of set between the finite and infinite. This notion of a set "between" finite and infinite seems naive today, because by definition infinite means "not finite," but may be a reflection of the ancients' notion of "potential" and/or "actual" infinity, and perhaps some such notion is what lies behind the idea of many present-day students that there is a kind of last term in a sequence just before it reaches its limit (Davis & Vinner, 1986; Tall & Vinner, 1981). Galileo pictured the continuum as made up of indivisible parts, which yet merged together as a kind of fluid, an analogy which Boyer (1949) called "a beautiful illustration of the effort which men made to picture in some way the transition from the finite to the infinite (p.116)."

This period which has been under discussion, the century or so preceding Newton and Leibniz, must then have been an exciting period of discovery. In terms of the limit concept, however, mathematicians had not advanced to a clear understanding, though they often exhibited amazing ingenuity in calculating quantities which can only be properly characterized as the end results of some limiting process. Many of them were aware of the possibility of paradoxes in the methods they used, and understood the need for a firmer logical foundation. A few were reaching very close to the limit concept. Thus Roberval in reconciling demonstrations by indivisible with those of the ancient geometers, "first... showed that the unknown quantity lies between inscribed and circumscribed figures which differ 'by less than every known quantity proposed' (Boyer, 1949, p.146)," the phrase quoted being reminiscent of the language used today to define the limit concept. One should also mention Gregory of St. Vincent, who "gave perhaps the first explicit statement that an infinite series *defines in itself* a magnitude which may be called the limit of the series" (Boyer, p.137). (Our italics: in other words, a series might be used as the definition of a quantity not a priori known.)

Newton and Leibniz

Newton and Leibniz synthesized much of the work of their predecessors into what become in effect a new field of mathematics; the calculus, and discovered many of the theorems and applications of the calculus. However, in spite of their great achievements, Newton and Leibniz do not seem to have advanced past their predecessors in grasping the limit concept. Newton worked with the difference quotients which are used to (in modern terminology) define the derivative (which gives a precise meaning to the notion of instantaneous rate of change). To define the derivative of a function f, one forms an expression $\frac{f(x+h)-f(x)}{h}$ (which of course only makes sense if h is non-zero) then, usually, simplifies algebraically as much as possible (still under the assumption that h is non-zero) and then finally takes the limit as h tends to zero. Thus one has to deal with quotients of variable quantities both of which are tending to zero. This was extremely puzzling to Newton's contemporaries. If we let h actually become zero, our quotient becomes $\frac{0}{0}$, an algebraically meaningless quotient. (To see that it is meaningless, observe that equation $\frac{0}{0} = x$ is equivalent to $0 = x \times 0$; but this latter is true for any number x, and so there is no possibility of assigning a clear-cut meaning to the symbol $\frac{0}{0}$). Newton himself, though no doubt having a powerful intuitive understanding of his methods, was unable to explain them clearly because a clear formulation of what it means to "take a limit" had not yet come into mathematics. As Boyer remarked, we see that Newton first had in mind infinitely small quantities which are not finite nor yet precisely zero. "'Ghosts of departed quantities' (The phrase "ghosts of departed quantities" seems to be originally due to Berkeley (Edwards, 1979).) they were fittingly called by the critics of the method in the following century (Boyer, 1949)." Again, Boyer (1949) remarked,

> The meaning of the terms "evanescent quantities" and "prime and ultimate ratio" had not been clearly explained by Newton, his answers being equivalent to tautologies: "But the answer is easy, for by the ultimate velocity is meant that, with which the body is moved, neither before it

> arrives at its last place, when the motion ceases, nor after; but at the very instant when it arrives.... And, in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor after, but that with which they vanish" (p.216).

A similar vagueness occurs in the attempted explanations of Leibniz; for instance, Boyer (1949) quoted the following line from one of Leibniz's letters, "we conceive the infinitely small not as a simple and absolute zero, but as a relative zero.... that is as an evanescent quantity which yet retains the character of that which is disappearing (p.218-219)."

Newton and Leibniz derived the basic algorithmic rules of differential calculus, many of the techniques used in integration, derived some important series expansions, and carried out many ingenious calculations concerning rather special curves and functions (Edwards, 1979). This tradition of computational mastery was carried on later, most notably by Leonhard Euler (1707-1783) (Edwards, 1979). However, the mainstream of the development of calculus took a different, less computationally oriented, direction after the time of Newton and Leibniz. Boyer (1949) stated that "throughout the whole of the eighteenth century there was general doubt as to the nature of the foundations of the methods of fluxions and the differential calculus (p.224)." The new differential calculus clearly had much that was right about it, since it had successfully answered many scientific and mathematical questions, but the logical foundations were not clear. If the new mathematics could not be placed on a firm logical foundation, it was possible that internal inconsistencies could arise, as well as erroneous results when the new mathematics was applied to the sciences. A number of mathematical thinkers began to voice skepticism. The most effective and influential critic was probably George Berkeley. Berkeley wrote a book criticizing Newton's work in terms that both Boyer (Boyer, 1949) and Edwards (Edwards, 1979) seemed to consider as perfectly fair and reasonable. Berkeley criticized Newton on grounds both of lack of clarity and of occasional inconsistencies. In particular, Berkeley pointed out that when Newton wishes to calculate a derivative he begins by writing down an expression involving the two quantities x and h in which h is specifically required to be different from zero, simplifying appropriately then dividing by h (which is only permissible if h is non-zero), then finally setting h=0. Berkeley objects, quite correctly, that if one puts h equal to zero at the end of the derivation, then one is contradicting oneself in that one specifically required h to be non-zero at the beginning of the derivation.

With the publication of Berkeley's book it became clear that the logic behind Newton's work needed to be clarified, and mathematicians became interested in remedying the situation. There were at first a number of "spirited rejoinders" to Berkeley, "most of which proved only that their authors hardly understood Berkeley, much less the calculus... (Edwards, p.295)." Benjamin Robins, responding to a pamphlet by James Jurin, made a respectable but ultimately unsatisfactory attempt to clarify Newton's thought (Boyer, 1949). Robins recognized that the term "ultimate ratio" was a figurative expression, and referred to some

> Fixed quantity which some varying quantity, by a continual argumentation or diminution, shall perpetually approach, ..., provided the varying quantity can be made in its approach to the other to differ from it by less than any quantity how minute soever, that can be assigned, ... though it can never be made absolutely equal to it (Boyer, 1949, p.230).

The problem with this, besides its wordiness, is the insistence that the limit can never be reached, which for instance would prevent constant sequences from having a limit. Boyer's account of the controversy between Jurin and Robins (both attempting to defend Newton but disagreeing with each other on how to do it) sheds interesting light on the conceptual difficulties at that time. Boyer (1949) said that in this controversy

> The question as to whether a variable was to be considered as necessarily reaching its limit played a large part. Robins upheld the negative side; Jurin insisted that there are variables which reach their limits and vigorously accused his opponent of misinterpreting Newton's true meaning. It is difficult to judge what he meant. The phrase 'ultimate ratio' certainly favors Jurin's interpretation, but to avoid the logical difficulties inherent in questions of infinitesimals and the meaning of 0/0, it was necessary at the time to accept the more logical view of Robins that the variable need not attain its limit (p.231).

Elsewhere Boyer seemed to ascribe to Robins the view that the variable not only need not, but actually must not, reach its limit. As Boyer (1949) pointed out near the end of this discussion, "the question as to whether a variable reaches its limit or not has no significance, in the light of modern definitions (Boyer, p.232)."

Other scholars developed their own approaches to differential calculus, trying to deduce the laws of differential calculus from other beginning premises than those of Newton or of Leibniz, but it seemed that all such attempts suffered their own unclarities and logical difficulties. Lagrange attempted to found differential calculus on the use of infinite series expansions (Boyer, 1949), using the coefficients of these series to define the fluxions and differentials of Newton and Leibniz; however, assuming a priori the existence of such series expansions is at least as large a leap of faith as anything in the work of Newton or Leibniz (besides which, the usual manipulations valid for finite sums can not be applied uncritically to infinite series). According to Edwards (1979), Lagrange wished to "expunge from the calculus all trace of infinitesimals and limit concept..." John Landen published a book in 1758 called the *Residual Analysis* which seemed to have been based on applying algebraic rules uncritically to indeterminate quotients in which both numerator and denominator vanish (Boyer, 1949). Landen wished to found the calculus only on known principles of algebra and geometry, without introducing "foreign ones (principles) relating to an imaginary motion or incomprehensible infinitesimals." (The term "imaginary motion" is probably a reference to Newton's "fluxions.")

All this is not to say that the work done during the eighteenth century was without value. For instance, d'Alembert essentially gave the correct definition of the derivative as the limit of a ratio of increments (Boyer, 1949; Edwards, 1979) and made an effort to "present a satisfactory idea of the notion of limit" (Boyer, 1949, p.253), although d'Alembert's definition lacked the clarity necessary to make it a substitute for the somewhat mystical infinitesimal interpretation. d'Alembert stated that one calls a magnitude a limit of another magnitude:

> When the second may approximate to the first closer than by a given magnitude, however small one may suppose it, without however that the approaching magnitude may surpass the magnitude that it approaches, in such a manner that the difference of such a quantity and its limit is absolutely unassignable (Cajori, 1923, p.224).

Boyer (1949) commented,

> Because of his geometric ideology, d'Alembert's elaboration of the limit concept lacked the clear-cut phraseology necessary to make it acceptable as a substitute fro the infinitesimal interpretation. Thus to say with d'Alembert that the secant becomes the tangent when the two points are one and that it is therefore the limit of the secant imposes the necessity of visualizing the process by which two points become one, thus leaving the interpretation open to Zeno's criticisms (p.249).

Lagrange, although evidently somewhat dubious about limits, attempted a definition as follows "veritable limits are quantities which one can not pass, although one can approach them as near as one wishes (Cajori, 1923, p.223)." The language suggests that the approach to limits must be one-sided.

Simon L'Huilier, in 1787, published an exposition of calculus in which the limit concept was basic (Boyer, p.255). His definition of a limit made, "The variables always greater or always less than the limit; the variable could not oscillate to values above and below the limit (Cajori, 1923, p.228)." Very interestingly, he warned against trying to interpret the numerator and denominator separately in the symbol $\frac{d^2y}{dx^2}$. He considered $\lim\frac{\Delta y}{\Delta x}$ as the limit of one variable, not the ratio of the zero-limits of two variables (Cajori, 1923).

The way was in part prepared for these developments earlier in the eighteenth century by Euler. Euler published his volume *Introductio in Analysis Infinitorum* in 1748 and in this work, although he occasionally is overly creative in his use of infinitely small numbers, nonetheless makes many extremely valuable contributions. The most valuable of Euler's contributions in the *Introduction* is probably that he uses explicitly the concept of a function (that is, a rule of correspondence between two variables), making functions rather than geometric figures the primary object of study (Boyer, 1949; Edwards, 1979). This was eventually to be important in arriving at a precise concept of limit because, as Edwards said, "It was the identification of functions, rather than curves, as the principal objects of study, that permitted the arithmetization of geometry,... (p.270)."

Why is this "arithmetization" important? One shall see that when, toward the end of the century following the one now under review, mathematicians were finally able to achieve a rigorous formulation of the limit concept, the formal definition was given entirely in terms of numbers. In the formal logic of the limit definition, geometric considerations play no role whatsoever. Thus it is possible to explain precisely what is meant by saying "this sequence of numbers has that number as its limit," but not possible to explain directly what precisely is meant by saying "this sequence of curves approaches that curve" or "this region is the limit of that sequence of regions." What one really means in the back of one's mind by such statements, if anything, is that there is some numerical characteristic of the geometric figures in the sequence (area, say, or arc-length) which as a sequence of numbers has the corresponding numerical characteristic of the "limiting figure" as its limit. (Indeed, there is even a danger of tautological thinking in such statements; in many cases the numerical characteristic (area, say) of the so-called "limiting figure" may not be a priori a well-defined quantity, and one might actually need to *define* the numerical characteristic of the limit figure as the limit of the corresponding numerical characteristic of "approaching" figures.) Boyer (1949) commented several times on the need for arithmetization (p.104, 124, 153, 235, 271). At least until modern times, mathematicians tended to think geometrically rather than arithmetically. This of course provided them with interesting problems to work on, problems which would eventually lead to calculus because calculus was necessary for their solution, but at the same time this tendency to think geometrically often obscured the limit concept from them at times when it was almost in reach.

Cauchy and Weierstrass

The nineteenth century was to see, finally, the rigorous formulation of the limit concept, which had as an immediate consequence the placing of differential calculus on a solid logical foundation, as well as the resolution of paradoxes which had puzzled mathematicians since the time of the ancient Greeks. With calculus on a firm foundation, one can also consider the problem of quantitatively explaining motion as solved; the derivative gives a clear, unambiguous definition of velocity which agrees with all intuitions of what velocity should mean. The two most important figures in this final step to provide a rigorous understanding of the limit concept were Augustine-Louis Cauchy (1789-1857) and Karl Weierstrass (1815-1897).

Cauchy published three treatises on the calculus in the years 1821 to 1829, in which he gave a definition of the limit that is close to the modern one. As Boyer pointed out, previous writers had prepared the ground for Cauchy by popularizing the limit idea and demonstrating its potential usefulness, but their conception remained largely geometrical, hence intuitive rather than precise. Cauchy's definition of a limit was arithmetic rather than geometrical and stated as the following in which no reference is made to geometric figures, "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others (Boyer, 1949, p.272)."

Using this definition of the limit, Cauchy was able to say precisely what is meant by "summing" an infinite series, which gives a satisfactory resolution to the troublesome Zeno paradox known as "the Achilles."

Boyer (1949) remarked that "in spite of the care with which Cauchy worked, there were a number of phrases in his exposition which required further explanation (p.284)." Examples are the phrases "approach indefinitely" and "as little as one wishes which" suggested difficulties which had been raised in the preceding century. The very idea of a variable approaching a limit called forth vague intuitions of motion and the generation of quantities (Boyer, p.284). There is still an element of dynamism in Cauchy's definition, a hint of the ancient difficulty of feeling we are forced to try to visualize the "final stages" of an infinite process. Cauchy's definitions of function, limit, continuity, and derivative were essentially correct but the language he used was by modern standards imprecise.

It was Weierstrass, in the 1870's who finally gave the modern, completely arithmetical, definition of limit (Boyer, 1949). Weierstrass had proved in 1872 a result that ran completely counter to most mathematicians' geometric intuition (i.e., the existence of continuous curves which failed at every point to have a well-defined tangent line) and therefore decided that geometric intuition could not be relied upon, and hence that the bases of calculus had to be made as rigorous and purely formal as possible. Weierstrass' definition of limit for a sequence, if adapted from the definition in the context of functions as given by Boyer (1949), would read as follows:

> L is the limit of the sequence S_n if and only if it holds that for every positive number ε there is a positive integer n_0 such that if $n>n_0$ then the difference $L-S_n$ is less in absolute value than ε (p.287).

Save perhaps for trivial changes in wording this is the definition that would appear in a present-day calculus textbook. The definition of limit for a function of a continuous variable is analogous.

The Weierstrass definition is the long needed "safe guide" for working with limits, and is the definition still used in mathematics today. It obviates completely the need to try to visualize how an infinite process "approaches" its "final state." It is purely static, free from all intuition of motion. In deciding whether a given sequence has a given number L as its limit, one need only consider the algebraic problem of whether certain inequalities have solutions, and is not required to try to visualize anything at all. The definition also has a kind of built-in "versatility" in that it lends itself easily to generalizations; thus with trivial and obvious modifications in the definition one can say precisely what is meant by a limit for functions of several variables, and can even quite easily extend the concept to mappings between abstract metric spaces.

A feature of Weierstrass' definition that must immediately strike the reader is that it is hard to understand. That single jaw-breaking sentence is hard to absorb. This is of course the price that has to be paid for making the formal definition independent of intuition. A mathematical robot could use the definition in a mechanical way, without feeling any need to have any visual representation or see any "meaning" to what it is doing, but human mathematicians, especially students, feel a need to "understand" such mathematical activity in the sense of being able to connect it with experience or sensory impressions. In order to be able to "make sense" out of the definition, the student would probably have to have a strong intuition based on having seen examples and discussed the limit concept in informal exploratory ways. The Weierstrass definition can be made a little less forbidding by introducing some logical symbols. Thus the necessary and sufficient condition for L to be the limit of the sequence S_n is often rendered as follows:

Given $\varepsilon>0$, there exists n_o
such that $|S_n - L| < \varepsilon$, for all $n>n_o$.

Another occasionally helpful phrasing is:

For every $\varepsilon>0$, we have

$|S_n-L| < \varepsilon$ for all but finitely many natural numbers n.

At the same time that he gave the rigorous definition of the limit concept, Weierstrass recognized the need to obviate another difficulty; the concept of number had to be clarified. In attempting to bridge the gap between the rational numbers, which are easily understood, to the more difficult-to-visualize irrational numbers, Cauchy had stated that irrational numbers could be defined as the limits of sequences of rational numbers. However there is a circularity in this kind of definition that Cauchy seems to have overlooked. As Boyer (1949) explained the difficulty very clearly:

> Since a limit is defined as a number to which the terms of the sequence approach in such a way that ultimately the difference between this number and the terms of the sequence can be made less than any given number, the existence of the irrational number depends, in the definition of limit, upon the known existence, and hence the prior definition, of the very quantity whose definition is being attempted (p.281).

Thus to put calculus on a solid footing, it was necessary to find a way to explain, without recourse to the limit concept, what is meant by the notion of real number. Weierstrass did this, basing his development on the idea of thinking of a real number as a collection of decimal places. Another axiomatic development of the real number system was given by Dedekind (Boyer, 1949), based on his "Cut Axiom," which states that if the reals are divided into two sets L and R such that every member of L is less than every member of R then either L has a last member or R has a first member. Dedekind's approach perhaps establishes most firmly that one can think of the real numbers as corresponding in a natural way to points on a line. The other mathematician who should be mentioned here is G. Cantor, whose theory of infinite sets, based on the apparently simple but highly original idea of comparing the "size" of infinite sets according to whether or not they could be put into one-to-one correspondence with each other, has given us a better understanding of the nature of the real line.

The "method of indivisible," which in various forms was used by Cavalieri and goes back to Archimedes, was the forerunner of integral calculus. The principle conceptual difficulty of the method of indivisible; that is questions as to how one could find the area of a two-dimensional region by considering it as a union of pieces which are one-dimensional and therefore have zero area, is resolved in the foundations of integral calculus. In 1854, G. B. F. Riemann, improving on work of Cauchy, gave a definition of the definite integral that is still used today. To define $\int_a^b f(x)\,dx$, we mark off partition points

$$a < x_1 < x_2 < \dots < x_{n-1} < b,$$

choose arbitrarily a point x_i^* in the interval (x_{i-1}, x_i), then form the sum

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

The definite integral is then to be understood as the limiting value of such sums as we ran through a sequence of progressively finer partitions of the interval [a, b].

Summary

The pre-Hellenic development of mathematics was due mostly to the Egyptians and Babylonians. Their mathematics consisted of arithmetical operations applied to whole numbers and ratios of whole numbers, and some empirical geometrical observations.

The ancient Greeks established the tradition that mathematics was to be a deductive system; that is, all mathematical propositions were to be deduced logically from certain small set of axioms. The numbers as known to the Greeks consisted of whole numbers, ratios of whole numbers, and the operations with them; irrational numbers did not exist for them. Thus they did not consider the length of the diagonal of a square of side one as being a number, but a "magnitude." The inability to invent a number system that could adequately represent geometrical magnitudes was a weakness of Greeks mathematics.

The ancient Greek mathematicians did not stray far from what can be clearly visualized, and thus could never regard an infinite process as completed. This would have restricted Greek geometers to the study of straight-line figures, had they not discovered the "method of exhaustion." The method of exhaustion, based on the Axiom of Eudoxis, obviated any necessity of considering an infinite process as being completed, but at a price; it led to cumbersome proofs. The relation between the discrete and the continuous was a perplexing puzzle to the ancients. The paradoxes of Zeno on the nature of motion could not be answered in terms of the concepts available to the ancient Greeks. The paradoxes of Zeno seem to have discouraged Greek mathematicians from trying to quantitatively study motion and in general variable phenomena. The mathematician Archimedes anticipated integral calculus by using the notion of infinitely small piece of regions (infinitesimals), but purely as a working tool.

During medieval times, the problem of quantifying motion was first attacked, with some success. The sixteenth and seventeenth centuries saw more progress toward understanding the nature of motion, and also more work in geometry. Many geometers tried to find substitute for cumbersome "method of exhaustion." Answers were found to many geometrical questions, but the methods were intuitive, the logical foundations of the methods shaky and thus not entirely trusted by many scholars. There was a great deal of speculation on the nature of the infinite, and on the nature of the continuum. Mathematicians lost some of their reluctance to work with infinite processes.

Newton and Leibniz synthesized much of the work of their predecessors into what became in affect a new field of mathematics; the calculus, and demonstrated its problem solving effectiveness. They were unable however to clearly explain the logical bases of the calculus (for instance, what does it means to take the ratio of two quantities which are becoming arbitrarily close to zero?). Mathematicians became interested in strengthening the logical foundations of the calculus. During the eighteenth century much effort was expended in trying to understand and explain Newton's work, or in trying to find alternate approaches to the calculus. In time, a number of mathematicians came to realize that calculus had to be founded on the notion of limits. Various attempts were made to formulate exactly what is meant by a limit.

The satisfactory definition of limit, the one that is still used, was finally given by Weierstrass in the 1870's. A concomitant of Weierstrass' definition is a need to clarify the concept of number, in such a way that limit are not used to define irrational numbers (so as to avoid logical circularity). Weierstrass accomplished this. Later, a different axiomatic development of the real number system was given by Dedekind.

CHAPTER THREE

REVIEW OF LITERATURE

After the launch of the "sputnik", the United States started to evaluate its educational objectives and curriculum. In the 60s high school mathematics curricula underwent significant revision. Providing solid and rigorous mathematics and science seemed like a good way to prepare students. "New math" and science projects were introduced by many universities and research institutes. The so-called modern approach to mathematics had been well established. But exactly which courses should be offered to the advanced mathematics students had not been firmly established. There were many studies focused on what should be taught in high school (Buchanan, 1964, 1965; Cambridge Conference on School mathematics, 1963; Shelton, 1965). The controversy was basically a debate on whether or not calculus should be incorporated into the high school curriculum. Among those who opposed the teaching of calculus in the high schools, some recommended that concentration should be given to subject matter which will prepare the students for a rigorous college calculus course (Taylor, 1969, Commissions Report, 1959, Buchanan, 1965). Among those who advocated the study of calculus in high school, the controversy was basically a debate on whether or not the students will be mature enough to learn the calculus; and if they are able to learn the limit concept, then what are the best instruction methods. No matter which side of the coin, the center focus is the study of the limit concept. Thus the studies about the limit concept started from this period of time.

Many studies have been undertaken on students' understanding of the limit concept. Some researchers were interested in whether or not students are mature enough or capable enough to learn the limit concept (Smith, 1959, 1961; Taback, 1975). Some of the researchers examined whether or not the schools should teach the limit concept (Buchanan, 1964; Chaney, 1967). Due to the results of the above research, some researchers investigated alternate ways to introduce the limit concept in either high schools or colleges (Isaac, 1967; Lackner, 1968; Lytle, 1073; Macey, 1970; Pavlick, 1968; Shelton, 1965;). The research on limit concept tends to shift from the "New Math" movement to "Back to Basic" (Fey, 1978) to the more recent "concept definition and concept image" studies which focus on students' misconceptions (Confrey, 1980; Davis, 1984, 1985; Davis & Vinner, 1986; Dreyfus, 1990; Fischbein et al., 1979; Fless, 1988; Orton, 1983a, 1983b, 1986, 1987; Orton & Reynold, 1986; Sierpinska, 1987; Tall, 1981, 1985; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1989, 1991).

In this literature review the studies are divided into four categories. The first category will focus on the studies related to students' notion of limit as follows: 1) studies pertinent to the psychological question: can students at the high school level learn the limit concept, 2) studies pertinent to the philosophical question: assuming that the high school students can learn the limit concept, should it be taught, 3) studies pertinent to the pragmatic question: what is the best method for teaching the limit concept, and 4) studies pertinent to the students' misconceptions: what are the students' misconceptions, what causes these misconceptions, could these misconceptions be changed by providing an adequate teaching situation. The second category will focus on studies pertinent to the teachers' misconceptions: What are the teachers' misconceptions, what causes these misconceptions, and how might these misconceptions affect their teaching? The third category will focus on studies pertinent to the level of understanding of a specific mathematics topic. The last category focuses on the teachers' knowledge and how this knowledge is represented.

Studies Pertinent to the Psychological Question: Can Students at High School Level Learn the Limit Concept?

Smith (1959) studied 578 pupils from grades seven through twelve in an effort to determine whether or not secondary school students can profit from a study of the limit concept. He conducted an experiment that provided students with three hours of special experience, consisting of discussion of the solutions to a variety of problems involving a limiting process. In order to teach students the limit concept, many questions dealing with limiting situations were presented, such as logical physical situation, geometrical situation, and numerical situation:

1. Logical physical situations: For example, a frog in a well starts jumping up to the wall of the well to get out. On the first jump he gets half way out. On the second he gets half of the rest of the way out. Each time he jumps, he lands half between where he is and the top. If he keeps on jumping, what will happen?
2. Geometrical situations: For example, examine the figures (see Fig.3.1), and ask if we were to increase the number of sides, what would happen to the size of the shaded area? If we continue increasing the number of sides, the shaded area will get nearer and nearer to what size?
3. Numerical situations: For example, 1/1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, what is the next fraction? If we were to continue writing new fractions in this fashion indefinitely, the value of the new fractions would get nearer and nearer to what value?

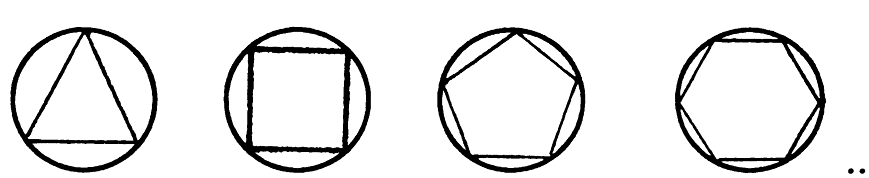


Figure 3.1 -- Geometrical Situation

One of his assumptions was that intuitive understandings of limit concepts must precede and are fundamental to understanding the more abstract symbolic notions of mathematical limit such as delta-epsilon relations. Strong evidence was found that first, students at secondary school level can conceptualize the limit concept if the students are provided with the appropriate instruction, and secondly that students' background and experience were important for predicting success in conceptualizing the limit concept. In particular, he concluded, "Three hours of experience did produce a difference whereas a year of added maturity did not (p.55)."

Taback (1975) investigated the child's intuitive understanding of limiting processes. Eight tasks embodying a limiting process were developed for presentation across three age levels, namely eight, ten and twelve years. One task, Halfway Rabbit Task, is similar to the grasshopper problem in the questionnaire. In it, the subjects were asked what will happen if a segment was infinitely often divided into halves? The concepts under investigation were: 1) rule of a function, 2) neighborhood of a point, 3) convergence and divergence, and 4) limit point. The students' understanding of the nature of the convergence or divergence involved in a particular task was questioned on both the concrete and abstract levels. On the concrete level, the student was presented with some concrete objects such as a circle, a loop of string, or a rectangular strip of paper around which he could organize his thoughts. On the abstract level the student was to organize his thoughts without the help of concrete materials. These tasks were administered individually to each subject through a tape-recorded interview. Clear understanding, uncertain understanding, and no understanding were the categories used to rate the responses.

In terms of Halfway Rabbit Task, clear understanding response would look like:

> The rabbit won't reach B, because the hops get shorter and shorter but he'll still be able to make it (The "it" here means another jump rather than reach point B).

The uncertain understanding response would look like:

He won't reach B, because the hops are getting too small.

The no understanding response would look like:

Yes, he'll reach B, because no matter how small, he's moving over and he's still going forward.

He found significant differences in performance between 8 and 10 year-olds, and comparatively small differences between 10 and 12 year-olds on most of the tasks. In summary, Taback writes:

> In general, the eight-year-olds could do little more than follow a simple rule of correspondence. The ten- and twelve- year-olds were much more successful on all four concepts examined. However, with few exceptions, only twelve-year-old subjects could conceptualize an infinite process. Thus, judging from the intuitive understanding of the sample, instruction relating to the mathematical concept of limit seems feasible only for the twelve-year-old age group (p.143).

Summary and Discussion

In Smith's study, secondary school students were asked to deal with limiting situations involving the following types: logical physical situations, geometrical situations, and numerical situations. It seemed that students could conceptualize the limit concept if the students were provided with the appropriate instruction and experience. Taback tried to identify young children's intuitive understanding of the limiting processes. Although Smith and Taback appear to come to different conclusions as to the relative importance of maturity versus special instructional experience in being able to understand the notion of limits, it is however interesting to notice that both investigators seem to find that children younger than about 12 are generally not yet ready to conceptualize infinite processes. Studies concerning the developmental readiness of children to understand concepts associated with limits would be of marginal interest if there were no reasons to believe they actually should learn these ideas. Thus many mathematics educators have considered the question: Should the limit concept be taught in high school (or perhaps even earlier)?

Studies Pertinent to the Philosophical Question: Should the limit concept be taught in high school?

Buchanan (1964) sent a questionnaire to the chairman of each departments of mathematics, applied mathematics, and statistics of 233 colleges and universities to survey the following question: Should a unit on limits be taught in the twelfth grade? A unit on limits for the twelfth-year course in mathematics was favored by a majority of the college teachers of mathematics, whereas a course on calculus was not favored. Thus Buchanan (1964) conducted an experimental study on a twelfth-grade mathematics class and used a four week unit on limits of sequences approach as the introduction.

Buchanan concluded that the students were favorably disposed toward the inclusion of such a unit in the twelfth-year, being cognizant of the variety of benefits to be derived from an understanding of the limit concept. However, somewhat longer than four weeks was required to adequately present the material of the limit concept. It was found that the greatest understanding occurred during the fourth and fifth weeks (pp.223-225).

Chaney (1967) designed a study in which he tested for the effects of a formal experience with the limit in high school on grades of first year calculus students at the University of Kansas. Three groups of students were chosen from among those who were seniors in high school during the spring of 1965 and were enrolled in Calculus and Analytic Geometry I at University of Kansas during the fall of 1965. The first group had studied the limit concept for sequences and functions, continuity, and also had studied conic sections. The second group had the same analytic geometry, but nothing on limits. The third group had none of the above experiences. Chaney concluded:

> When the final grade in calculus was employed as criterion variable, a significant F-value was obtained at the .05 level. Appropriate t-tests were computed for groups by pairs and it was found that group A, The Limit and Analytic Group, performed significantly better than Group B, the Non-Limits and Analytic Group, Group B, however, did not perform significantly better than Group C. These results imply that of those students who have had the basic analytical geometry described in this study, the previous study of the limit concept in high school is of significant advantage in college calculus. Those students who had studied analytical geometry in high school did not appear to have a significant advantage in college calculus over those students who had not studied analytical geometry (p.56-67).

Summary and discussion

Buchanan's conclusion indicated that students were able to conceptualize the limit concept if provided enough instruction time. Especially he mentioned that the greatest understanding occurred during the fourth and fifth weeks of exposure to the limit concept. Chaney's study showed that students who had experience with limits in high school did better in college than those who were not exposed to the limit concept. Apparently students benefited from the learning of the limit concept whether in high school or further in college. The lack of enough learning time ascribed by Buchanan seemed to affect the understanding of limit concept. Now that we know the answer to the philosophical question is affirmative, it seems that the limit concept ought to be taught in high school (or maybe even introduced earlier than that). The researchers agreed that limits should be taught in high school, but what is confronted next is the question of how mathematical limits ought to be taught. This question, referred to as the "pragmatic question," is not easy because the limit concept is evidently difficult for students to grasp.

Studies Pertinent to the Pragmatic Question: What is the best method for teaching the limit concept?

Isaac (1967) investigated the following questions: 1) What treatment should be given to limits? 2) Are epsilon-delta absolute value methods appropriate? and 3) What degree of rigor should be used? He wrote a unit on limits for an experimental group of 51 twelfth-grade students, in which he employed an open interval approach, basing the concept of limit of a function upon the idea of limit of a sequence. The control group of 91 twelfth-grade students studied a delta-epsilon unit on limits from their textbook. Seventeen school days were used for the experiment including the administration of the examination. The examination constructed by the investigator consisted of 18 multiple-choice items. This instrument was divided into two parts. The first part consisted of items not generally emphasized in a calculus preparatory presentation. The second part consisted of more traditional types of limit questions generally found in calculus textbooks. An item analysis was performed; the following are his conclusions:

1. The classroom experience and results of the examination indicated that the open interval approach to the limit concept is practicable for a twelfth year mathematics course.

2. Even in an informal discussion of the meaning of the limit of a sequence, care must be taken not to use the concept that a sequence approaches its limit. Rather than the sequence approaching its limit, the concept should be that the sequence has the property of being contained by any ε open interval about its limit.... The students had difficulty in accepting the concept that 1 +.9 +.09 +.009 +.0009 +... is another name for 2. The students' reaction was that the series approached 2 and never equaled 2. The approaching concept of a limit hinders their understanding of the meaning of the sum of an infinite series. Furthermore, the approach concept is confusing if the sequence contains its limit or is a constant sequence.

3. When discussing the infinite series, it should be emphasized that it can be used as a numeral. The number 1/3 may be named by 2/6, 1/3, .333..., or .3 +.03 +.003 +.0003 +... . As a consequence, the number 2, which may be used to measure the length of a line segment, can be named by many numerals among them being 1+ 1/2 +1/4 +1/8 +... . The fact that a number such as 2, measuring the length of a line segment, can be named by the infinite series 1 +1/2 +1/4 +1/8 +... is a consequence of the assumption of

the infinite divisibility of a line segment. It should not cause any more wonder than the assumption that a line segment is composed of an infinite number of smaller and smaller segments.

4. Time should be made available while discussing the limit of sequences to emphasize that the limit of a sequence often measures something different from that which the members of the sequence measure.

    a. The measure of the area of a circle is the limit of a sequence of measures of the areas of the inscribed and circumscribed polygons.

    b. The slope of the tangent line to a curve at a point as the limit of a sequence of slopes of secant lines passing through the point.

    c. The instantaneous velocity at a point as the limit of a sequence of average velocities between that point and other points.

5. The epsilon-delta and the open interval definitions of limit f(x) as x approaches a are in static, motionless terms. For this reason the language used by the instructor and class should be changed to coincide with the definition. The limit of f(x) as x approaches a should be the limit of f(x) for every sequence whose limit is a.

6. The time devoted to teaching this unit should be increased by three to four days to allow more time for developing the relationships involved in multiplying and dividing sequences and developing the concept of the limit of f(x) for every sequence x whose limit is a (p.92-95).

Lackner (1968) duplicated Shelton's experiment and instruments to conduct a research which consisted of three studies. One was a study comparing inductive (discovery) and deductive (expository) approaches in teaching the limit concept. A second study made the same comparisons in teaching the derivative. The third study was a total treatment comparing the four paired, ordered combinations of the two concepts together: inductive limit-inductive derivative, inductive limit- deductive derivative, deductive limit-inductive derivative, and deductive limit-deductive derivative. Lackner states that:

> For the limit study no difference in teaching methods was found. For the derivative and total treatment studies a difference in teaching method was found, favoring the deductive approach for the derivative concepts and the deductive limit-deductive derivative combination in the total treatment study.
>
> Further correlation and multiple regression analysis showed that a student's prior mathematical knowledge was a determining factor in learning. For the limit concept the pre-test score was the important determinant in learning.

> The pre-test and limit test scores were the primary predictors in learning the derivative concept, even though the deductive derivative teaching method was found superior to the inductive.
>
> The results of this study are somewhat startling. From pilot and related research studies, if a difference in inductive and deductive teaching methods existed, the inductive (discovery) method was favored. However, in teaching the derivative concept and the limit and derivative concepts together to high school students by the units used in this study, the deductive (expository) approach was superior (p.83-90).

Lytle (1973) was trying to find the effectiveness of two learning sequences of the limit concept. Lytle investigated whether a student can profit later by studies of limits in high school. Two learning models consisting of hierarchies of subprinciples and concepts within the limit principle were defined. This was done by means of a logical analysis of mathematical dependencies between topics in presentations of real-value limits. One hierarchy was defined for an approach based upon the limit of a sequence, and another for an approach based upon the limit of a function with at least semi-continuous domain. Clearly the students in the algebra groups attributed much less importance to the study of limits than did those in the analysis groups. In over sixty percent of the cases where the analysis student felt the units were important he was planning on going into a field of study requiring more mathematics (p.153).

Macey (1970) investigated whether instructing students in selected topics of logic would better prepare them to understand the definition of the limit of a sequence and its application in proving statements regarding limits of sequences. Macey stated that studies are needed which determine the prerequisites for the learning of certain ideas in the calculus, particularly those ideas relating to limits. Macey conducted an experimental study designed to investigate the above question and expressed hope it could make a significant contribution to mathematics education.

Macey pointed out that in order to understand the statement of definition of a limit of a sequence, first a student must understand what is meant by each of the four component phrases: 1) for each $\varepsilon > 0$, 2) there exists a positive integer N, 3) for each integer $n \geq N$, and 4) we have $|a_n - L| < \varepsilon$. Observe that these four phrases involve quantifiers and open sentences. The next step is that of assimilating the four component phrases into the compound statement "for each $\varepsilon > 0$ there exists a positive integer N such that for each integer $n \geq N$, it follows that $|a_n - L| < \varepsilon$.

Thus Macey developed the criterion test and tried to measure the student's understanding of the definition of the limit of a sequence and its application in proving convergence of sequences. Macey found that the beginning calculus student generally does not understand how to go about choosing the positive integer N.

Pavlick (1968) attempted to measure, on the basis of an achievement test, the difference in learning between students presented limit theory through the δ-ε approach (Treatment T) and those presented an advanced set approach (Treatment A). Also, he investigated the variation between a whole-part versus part-whole learning, with the advanced set approach serving as the whole-part method, and the traditional approach representing the "by parts" method of teaching limits. The traditional instruction on the limit concept treats the definition of limit of sequences, limit of functions and limit of functions with deleted neighborhood domain as separate entities. Students have to learn basically the same definition three different times. The advanced set approach seems to put these three in one definition and expands to different domains of functions. Pavlick indicated that no matter how complicatedly one concept disguises itself in different problem situations, if one has really mastered the integrated concept, one can always find a way to solve the problems. Thus he stated that his study leads him to believe that:

> The efficiency of the advanced set approach to limits depends on the ability and achievement level of the calculus student. Students at a high level of achievement and ability when taught by the advanced set (whole-part) approach should do at least as well as when taught by the traditional approach to limits. However, students of average or below average achievement and ability would profit more from the traditional (by-parts) method of instruction (p.54).

Shelton (1965) investigated the difference between a concrete inductive approach (defined as a sequence of items leading from specific numerical examples to the general case) and an abstract deductive approach (defined as a sequence of items leading from the abstract to the particular). The content of each approach consisted of the definition of limit of a sequence and limit of a function, the development of the usual theorems, and applications such as the sequences formed by the inscribed and circumscribed polygons of a unit circle, the geometric sequences and the sequences formed by the partial sums of a given sequence and others. The two groups were presented with six hours of programmed materials. The subjects were divided into high and low level groups based on a pre-test over items judged to be helpful in learning the limit concept. The results showed as follows:

> There were no significant differences in achievement between the two treatment groups.
>
> There were no significant differences in achievement between the two levels.
>
> There was no statistically significant interaction between treatments and levels (p.58-59).

Summary and Conclusion

In summary, the above studies have one common thread; that is they are all trying to find out a best way to introduce the limit concept. They all show that the formal epsilon-delta definition for limit is too abstract for students to understand, and to accept. The alternative instruction methods mentioned for teaching limits were: inductive (discovery) and deductive (expository); the open interval approach; the preparation for prerequisites to the study of limits; and the advanced set approach (the whole-part method). In all these different approaches whether from inductive approach or deductive approach, the open interval approach or the neighborhood approach, deriving limit of a function from limit of a sequence or vice versa, from the "whole-to-part", the advanced set approach, or the "part-to-whole" approach, the same result was produced; that is, in terms of the effectiveness of learning the limit concept, there is not much difference. These studies showed that there were many alternative approaches to introducing the limit concept, but did not explain why our students do not have a real understanding of the limit concept. In the following section, I will present some of the studies of why students have difficulty understanding the limit concept.

Studies Pertinent to the Students' Misconceptions

In teaching any subject matter, an expert instructor is sensitive to the errors and conceptual difficulties commonly experienced by students, and is able to come up with cures for these difficulties in advance. Conceptual difficulties are particularly likely to occur among students studying limits because no one has ever directly experienced an infinite process and our intuition is likely to be faulty. Indeed the difficulty of conceptualizing infinite processes is what lies behind the confusing paradoxes of Zeno. Sometimes the misconceptions are very hard to change. Therefore, many educators interested in the teaching of limits have studied the concept images and misconceptions common to students studying the subject of limits.

Confrey's (1980) empirical study was intended to be primarily descriptive and exploratory. In order to illustrate eleven college students' conceptual change related to the concept areas of number concept and calculus, she conducted a clinical interview to explore student's number concepts and to see how these changed over a period of three weeks when the students were required to solve certain problems, such as "Harey rabbit" (the infinite divisibility of a segment into halves) problem and the 0.999...=1 problem. By using the number concepts and calculus as an example of conceptual change she revealed that the view of conceptual change in mathematics and mathematics education is not only plausible but exceptionally fruitful in providing new perspectives. In her study, Confrey suspected,

> Students are unable to grasp calculus because they hold a discrete concept of number; the analogy used was an infinite deck of cards. When in calculus, they are introduced to limits, even though they had not yet perceived a need for the concept. The difficulties inherent in attaching numbers to continuous quantities such as time, motion and area had not been adequately considered. The switch to a continuous concept of number in which infinite sequences are mentally completed and points are not always precisely determined had not been achieved (p.228).

In conclusion, she stated:

> There were, however, pieces of various concepts revealed, such as attaching discrete objects more readily to numbers, feeling dissatisfied equating finite and infinite number symbols, wanting to locate numbers as specific points on a number line and considering how numbers behave in equations (p.232).

Davis and Vinner (1986) conducted a study on college students' notion of limits. They argued that traditional skill acquisition was taught and learned by rote as a ritual to be performed in a certain way, entirely divorced from meaning. Hence students who have studied calculus are often unable to define limit correctly, or to explain why the concept of limit is fundamental to calculus. They also argued that if "understanding" can be taught earlier, then it will make a difference to how and what the students learn. In order to test their hypothesis, they implemented a special 2-year calculus course in a university high school. At the beginning of the second year, an unannounced written test was given to the students. The students responses were analyzed, looking for correct and incorrect ideas. They summarize the most prevailing misconceptions as follows:

1. A sequence "must not reach its limit."
2. Implicit monotonicity for a_n--regarding the phrase "going toward a limit" as having its everyday literal meaning.
3. Confusing limit with bound, requiring that a limit be an upper or lower bound for all a_n in the sequence.

4. Assuming that the sequence has a "last" term, a sort of a_{∞}.

5. Assuming that you somehow can "go through infinitely many terms" of the sequence.

6. Confusing $f(x_0)$ with $\lim_{x \to x0} f(x)$.

7. Assuming that sequences must have some obvious, consistent pattern (or even a simple algebraic formula for a_n), so that sequences such as:

   $$1,0,1,1,0,0,1,1,1,0,0,0,...$$

   and

   $$1,2,3,1,1/2,1/3,1/4,1/5,...$$

   are immediately excluded.

8. Neglect of the important role of temporal order. For the definitions to work, one must first be given an $\varepsilon > 0$, after which one determines an appropriate cut-point N. One cannot (in general) select an N and promise that, for n>N,

   $L - \varepsilon \leq a_n \leq L + \varepsilon$ will be true for any positive ε.

9. Confusing the fact that n does not reach infinity and the question of whether a_n may possibly "reach" the number L (p.294-296).

Sierpinska, A. (1987) worked four 45 minutes sessions with a group of 17 year old humanities students. The aim was to explore the possibilities of elaborating didactical situations that would help the students overcome epistemological obstacles related to limits. These obstacles were enumerated as follows:

1. The Eudoxis obstacle: moving to the limit is not a mathematical operation but a rigorous method of proving certain relations between quantities.

2. The Fermat obstacle: moving to the limit is a mathematical operation which consists in affecting numbers to variables and omitting values negligible with respect to others.

3. The heuristic obstacle: moving to the limit is not a mathematical operation but a heuristic method that leads to discoveries thanks to a reasoning based upon incomplete induction.

4. The heuristic static obstacle: intuition free from the idea of movement: finding the limit is finding something of which only approximations are known.

5. The heuristic kinetic obstacle: intuition linked with the idea of movement: finding the limit is finding something as we are approaching infinity (p.372).

She claimed that obstacles related to four notions seem to be the main sources of epistemological obstacles concerning limits: scientific knowledge, infinity, function, and real number.

She continued specifically that when students are confronted with 0.999..., they may feel uneasy about the result 0.999...=1 and/or the proof. The student's concept of infinity seems to be that of "potential infinity", something that can never be completed. Therefore, 0.999... is not really a number, and thus cannot be said to equal 1. In conclusion, Sierpinska presented eight models of students' conceptions of limits:

1. The "intuitive definitist" model of a limit: all sequences are finite and the number of their terms are well determined; "0.999..." denotes the number 0.999...9 which is an approximation of the number 1; 0.999...=1- ε, where $\varepsilon > 0$.

2. The "discursive definitist" model of limit: all bounded sequences are finite and the numbers of their terms are well determined; "0.999..." denotes the number 0.999...9 which is an approximation of the number 1; 0.999...=1- ε, where $\varepsilon > 0$.

3. The "intuitive indefinitist" model of limit: all sequences are finite but sometimes it is possible to determine the number of the terms; the true limit of a sequence is its last term; if it is impossible to determine the last term, one agrees on an approximation of the true limit.

4. The "discursive indefinitist" model of limit: all bounded sequences are finite but sometimes it is possible to determine the number of the terms; the true limit of a sequence is its last term; if it is impossible to determine the last term, one agrees on an approximation of the true limit.

5. The "potentialist" model of limit: the limit of a sequence is what that sequence is infinitely approaching without ever reaching it; the impossibility of reaching the limit is implied by the impossibility of running through infinity in a finite time; in particular, the number 0.999... is an infinite sequence which is being constructed in time; it is a number than tends to 1 without ever reaching it.

6. The "potential actualist" model of limit: one can admit that after an infinite time the infinite sequence will be fulfilled, and all its terms will be available; the limit of a sequence is its ultimate term; the number 0.999... is considered as arising in time; when all its terms are there then it is 1 (or the last number before one).

7. The "boundist" model of limit: a sequence is a set which may be bounded or boundless; one can speak of the bounds of a sequence; sometimes one of

the bounds can be distinguished, e.g. there are two bounds to the sequence 1,0,1,0,1,0,... but 2 can be a distinguished bound to the sequence 1, 1.9, 1.99, ...

8. The "infinitesimalist" model of limit: g is the limit of a sequence A if the difference between A and g is infinitely small (or if one can get from any term of A by adding an infinite number of infinitely small quantities); the equality 0.999... = 1 is a convention which results by way of a mathematical proof from conventions established before (p.384-389).

Tall and Schwarzenberger (1978) asked first year university students in mathematics the following question: "Is .999... equal to 1, or just less than 1?" The majority of students thought that 0.999... is less than 1, because the process of getting closer to one *goes on forever* without one ever being completed. Schwarzenberger and Tall suspected that the majority of teachers would think so too, but they did not conduct a study to find that answer. What the authors did was to find out what causes students' confusion. First they presented four methods to show that 0.999... =1.

1. Since $\frac{1}{3} = 0.333\ldots$, then $3 \times (\frac{1}{3}) = 0.999\ldots = 1$

2. By long division, $\frac{1}{9} = 0.111\ldots$, $\frac{2}{9} = 0.222\ldots$, ..., $\frac{8}{9} = 0.888\ldots$, so $\frac{9}{9} = 0.999\ldots$. The latter is sometimes proved by a slightly different argument represented by "90 divided by 9 is 9 with remainder 9," so that the long division quotient is

```
      0.999...
9    |9.0
      81
      90
      81
       9...
          etc.
```

3. The product $10 \times 0.999\ldots = 9.999\ldots$. Show that the difference $10 \times 0.999\ldots - 0.999\ldots$ is equal to 9, and so $9 \times 0.999\ldots = 9$. Dividing by 9 we obtain $0.999\ldots = 1$.

4. An alternative "legal" way of seeing that the statement 0.999... = 1 is consistent with the processes of arithmetic, is to say: if $\frac{(1+a)}{2} = a$, then on simplification, a = 1. Now take a + 0.999... and divide 1.999... by 2 using the usual process of long division:

```
     0.999...
  2 |1.999...
     1.8
      19
      18
        1 ...
              etc.
```

Hence a = 0.999... = 1 (p.46-47).

Then the authors conclude the following reasons for students' confusion of thinking that 0.999... is not equal to 1:

1. The lack of understanding of the limit concept.
2. The misinterpretation of the symbol 0.999... as a large but finite number of 9s.
3. The intrusion of infinitesmals (infinitely close but not equal).
4. The belief that there should be a one-to-one correspondence between infinite decimals and real numbers (p.44).

Finally, the authors concluded that there are three types of conflicts existing in the students' thinking about the limit concept and suggested the cures for these conflicts.

> In some cases, the cause of the conflict can be seen to arise from a purely linguistic infelicity and the conflict can be cured by a more careful choice of motivation or definition. In other cases, the conflict arises from a genuine mathematical distinction, for example, between sequences and series, where we advocate removing the initial conflict by concentrating on sequences first, introducing the term series later. In other cases again, the conflict arises from particular events in the past experience of an individual pupil, and can be cured only by a sensitive teacher aware of the total situation (p.49).

Tall and Vinner (1981) used a questionnaire to survey university students' mathematical knowledge of the notion of limit. Students were asked to write down the definition of the limit of a sequence; either with a formal definition or an informal definition. They claim that students often form a concept image of "a_n -->L" to imply a_n approaches L, but never actually reaches there. The verbal definition of a limit "a_n--> L" which says "we can make a_n as close to L as we please, provided that we take n sufficiently large" induces in many individual, the notion that a_n cannot be equal to L.

Thus the students believed that 0.999...<1 and can never be equal to 1. However, the students accepted that

$$\lim_{n \to \infty} \left(1+\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\ldots+\frac{9}{10^n}\right) = 2.$$

Thus if a student has a concept image that does not allow a_n to equal L (because it gets "closer" to L as n increases) then he or she may not absorb an example such as the one below;

$$a_n = \begin{cases} n / n+1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases}.$$

The students insisted that the above sequence was not one sequence, but two. The odd terms tended to one and the even terms were equal to one. The authors also claimed that students have great initial difficulties with the use of quantifiers "all" and "some" and the standard definitions of limits and continuity all can present problems to the students (p.160).

Williams (1989) conducted a combination of a survey and a clinical interview study to investigate the understanding of the limit concept in college calculus students. Ten themes for investigating students' model of thinking about the limit concepts were used:

1 Estimate: A limit is a sort of estimate of a given value the function attains within a given amount of tolerance. You can get better and better estimates by restricting x, so the tolerance gets smaller and smaller, but a function never reaches its limit. A limit is really an approximation, not an exact number. If you plug in numbers close to s, you can get close to the limit, but not beyond it.

2. Unreachable: A limit is a number or point that the function has values close to but never exactly equal to. If you take the limit of f(x) as x->s, you can make the function as close as you want to the limit, but it will never actually equal the limit, just like x never actually equals s. When you take a limit, you don't care if x is ever really equal to s, just that it's close. Same with f(x). It doesn't really matter if f(x) is bigger or smaller than the limit, but just that it's close. If f(x) ever equals the limit, you don't really have a limit.
3. Bounded: The limit is the maximum (or minimum) of a function as x approaches some number. As you get close to that number, the values of the function are trapped by the limit number. For example, when a function grows really fast but then levels off to an asymptote, the limit is the value of

the line. So a limit is a point or a number past which values of the function will not go; in fact, the values never even reach the limit, but they do get close.

4. Delta-epsilon: A function f has a limit L as x->s if the values of numbers near s are near L. Specifically, for any tiny interval you draw around L, you can find an interval around s so that all x values in the interval around s have function values somewhere in the interval around L.

5. Asymptote: A limit means that when x moves closer to some s, f(x) is moving closer to the limit. The function gets infinitely close to the limit but never touches it. It's like an asymptote that the function might cross over a few times (or even infinitely many times) but will never get closer and closer to.

6. Monotonicity: When a function moves toward a certain number and gets closer and closer to it, that number is the limit. So a limit is a number or point that a function grows toward, but doesn't go past. It is like walking halfway to a wall, then halfway again, and so forth. You keep moving closer, and the wall is like the limit. Eventually, you reach the limit, just like you reach the wall.

7. Closeness: What's important about limits is the idea of "closeness." When you say limit as x approaches s, it means that if x is "close to" s, then f(x) is 'close to' the limit. That is what the definition is trying to say. The idea of the definition is proving you can get "as close as you want". I say I can make f(x) as close as you want to the limit by making x close enough to s, and I prove it by telling you how close x has to be to s whenever you tell me how close you want f(x) to be to the limit. That is what all the delta-epsilon stuff is about.

8. Close enough: You can't evaluate a limit by just plugging in points close to the number, because you can only plug in a finite number of points, and that isn't enough to tell you what the function is really doing. It might be different when you get closer to the point. You really have to prove that you can get as close as you want to the number and the function is still close to the limit. That's why you need the limit theorems.

9. Plugging in: Finding a limit is a lot easier than understanding the definition. When you need to find a limit, you just plug the number in. Like, to find the limit of f(x) as x approaches 0, you plug 0 into f(x). If it doesn't work, you do some algebra, try to cancel some stuff out, and try again. The definition talks about "getting close" and all that, but when you work the problems, the limit turns out to be what you get when you plug the value in.

10. Movement: If you want to picture a limit, picture x moving closer and closer to some number, and the point on the graph above it moving along the graph, getting closer and closer to the limit. You're just approaching a point on the graph that the function goes through. The function goes through the limit point, and the points are just moving along the graph toward the point. There's no restriction on how close they can get and eventually, when x reaches the number, the function will reach the limit.(p.108-110).

A questionnaire based on these themes was administered to students in 18 discussion sections of second semester calculus classes and 341 questionnaires were collected. Answers from this questionnaire were examined. Twelve students were chosen to participate in the clinical interview and ten students completed all sessions. There were five sessions in total. During the first interview session, baseline data was gathered in the form of a repertory grid. During the middle three sessions, an opportunity to work anomalous problems, read alternative viewpoints of limit, and discuss how the differing viewpoints accounted for the anomalous problems were discussed. During the last session, repertory grid data was again gathered. As results of the summary, he concluded:

> Students viewed mathematical knowledge as consisting of separate components. Procedural knowledge was applied in a situationally dependent way. In large part, students' conceptual knowledge was applied in much the same way. Statements about limits were described as being true or false for specific functions but not for others. Little sense of generality, of mathematical truth as meaning truth in every instance, emerged from the data (p.243).
>
> Both students and teachers must come to be concerned with conceptual as well as procedural knowledge if such knowledge is to be valued and communicated. Students should be introduced to an analogy for limit, compatible with the formal definition, which has explanatory power for the tasks which they must perform (p.250).

Williams (1991) published an article on identifying models of limit held by college calculus students as a result of his dissertation. Six statements were listed on the initial questionnaire, #3 was picked by a majority of the 341 students as true. The three most popular views of limit seem to be a view that is basically dynamic, a view that sees limit as unreachable, and a view that echoes the formal definition. These three (#1, #4, and #3) choices also were selected most frequently as the best description of a limit. The following Table 3.1. consists of the questionnaire items and the percentage responses.

Table 3.1. Questionnaire Questions and Percentage of Subject Indicating each Statement as True, False, or Best on the Initial Questionnaire

A. Please mark the following six statements about limits as being true or false:

1 T F A limit describes how a function moves as x moves toward a certain point.

2 T F A limit is a number or point past which a function cannot go.

3 T F A limit is a number that y-values of a function can be made arbitrarily close to by restricting x-values.

4 T F A limit is a number or point the function gets close to but never reaches.

5 T F A limit is an approximation that can be made as accurate as you wish.

6 T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B Which of the above statements best describes a limit as you understand it? (Circle one)

1 2 3 4 5 6 None

C Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as x->s is some number L.

Question number	Statement Type	Original Sample (N=341)			Students indicating Statement 3 as true (n=226)		
		True	False	Best	True	false	Best
1	Dynamic-Theoretical	80	19	30	82	18	29
2	Boundary	33	67	3	28	72	2
3	Formal	66	31	19	100	0	29
4	Unreachable	70	30	36	65	35	31
5	Approximation	49	50	4	53	46	3
6	Dynamic-Practical	43	57	5	45	55	5

Note. In some rows, responses for true and false do not sum to 100% because of nonresponses. Responses for best statement do not sum to 100% because of nonresponses.

In summary, the studies above show that students possess many misconceptions and difficulties about the notion of limit. There are also many concept images held by many students which seem very hard to change and which in turn hinders the learning of mathematics in general and the notion of limit in particular. One of the most popular questions asked by these researchers is " Is 0.999...equal to one or just less than one?" Most of the students answer "less than one". With this common denominator among students, we would like to know where does this common denominator come from? Since the students' subject matter knowledge come from their teachers, some of the researchers are interested in teachers' misconceptions and difficulties. The following section will review these types of studies.

Literature Review on Teachers' Misconceptions

Arcavi et al. (1987) designed a course to help prospective teachers to learn about the irrational numbers through the study of history of mathematics.. They tried to assess teachers' previous knowledge, conceptions and/or misconceptions about irrationals. They claimed that students in colleges and universities are often presented with mathematical definitions divorced from the context which gave rise to them, and that this context can contribute to the feeling of logical necessity for such definitions. They suggested that, for this population, one of the sources of confusion between rational and irrational numbers is the common use of a rational approximation to an irrational as their irrational itself. Although this is the way many practical problems are solved the distinction should be very clear-certainly to the teacher (p.19).

Civil, M. (1990) investigated four prospective teachers' views about mathematics and about "do math" in mathematics. The investigator was interested in finding out the answers to the following questions: What do these subjects view as doing mathematics? Applying procedures? Getting the answer? Thinking the problem through? The investigator claimed that,

> The subjects did not believe in their own ways of doing mathematics. They felt more confident when they could solve the problem using some pre-established mathematical procedures (even if they might not quite understand it) (p.8).

This statement is parallel to Davis' (1986) "a wrong view of mathematics," which is the view that doing mathematics would only refer to the actual writing down of equations, and other symbolic operations, rather than to the thinking process involved.

Galbraith (1982) conducted a comparison study on the mathematical vitality of secondary mathematics graduates and prospective teachers. In this article, the author links two areas of contemporary interest in mathematics education. These are, respectively, mathematical characteristics of prospective teachers and the notion of levels of understanding. The author claimed that the type of understanding sought was not enhanced merely by taking more mathematics courses; it was the problem of recycling of attitudes and mathematical misconceptions within the secondary teaching structure, and the question of the approach to the study of mathematics at both secondary and tertiary level. The problem was that teachers had in many cases never learned to learn; their university and college preparation had turned them into absorbers of pre-digested information. They had not been encouraged or trained to learn and create mathematics by themselves. They had been trained to accept what was offered to them. They had not been encouraged or trained to question the criteria underlying the selection and methods of presentation of the material, and they had not learned to view mathematics as an on-going activity.

Although these students had been "exposed to" the calculus and "passed" limits in various guises, and most of them were able to formulate a simple argument in analysis, they seriously possessed almost no intuition on the subject at all. Early unfortunate experiences in mathematics learning have a permanent disabling effect. Many ideas, fundamental to the whole of mathematics, are difficult to teach explicitly and are rarely examined. Misconceptions, misguided and underdeveloped methods, undefined intuition tend to remain. Misconceptions and habits of working seem to be very resistant to change. An understanding of fundamental notions of analysis such as the distinction between the limit and value of a function, and between continuous and differentiable, seem to be missing. The results will be discussed in terms of the implications for mathematics teaching as follows:

1. That misconceptions and misunderstandings received at secondary level tend to remain and are resistant to correction through the agency of advanced level courses.
2. That the level of mathematical vitality achieved by secondary graduates is independent of local contexts with regard to precise syllabus content and curricular emphases.
3. That the level of mathematical vitality achieved by graduates is independent of particular institutions or mathematics units studied. (p.107)

Graeber et al. (1989) were interested in finding out what are preservice teachers' misconceptions in solving verbal problems in multiplication and division. They believed that in recent years much has been written about children's and adolescents' misconceptions concerning the operations needed to solve multiplication and division word problems. The purpose of their study was to explore whether prospective elementary teachers have the same misconceptions. Graeber et al. claimed that if the preservice teachers hold these misconceptions, they are not likely to recognize the related errors students make. And their instruction might inadvertently contribute to perpetuating the misconceptions. Thus their study was concerned with noting whether the prospective teachers would exhibit other misconceptions and the extent to which such misconceptions were similar to those previously noted among children.

A test which was used by Fischbein et al. (1985), modified slightly, was administrated. An interview was scheduled to obtain more information about the conceptions the preservice teachers held and the reasoning they used. Thirty-three preservice teachers were selected for interviews. All these prospective teachers had given incorrect answers to one or more of the eight most commonly missed problems (p.96). They concluded,

> The results clearly suggest that preservice teachers are influenced by the same primitive behavioral models for multiplication and division that influenced the 10- to 15-year-old students in Fischbein et al. (1985) study. Further, the most common errors made by both groups are quite similar. Because today's preservice teachers are tomorrow's teachers, the learning/teaching cycle may perpetuate misconceptions and misunderstandings about multiplication and division. Efficient strategies are needed for training teachers to monitor and control the impact that misconceptions and primitive models have on their thinking and their students' thinking (p.100).

Loef and Lehrer (1990) conducted a study to understand teachers' knowledge of fractions. In their study teachers were presented three fraction problems, two of which were more similar to each other and different from the third in terms of the content, to find, to investigate how students think about the problems, and the pedagogical actions that they associate with the particular problems. They claimed,

> Many teachers have little knowledge of fractions and those who do often possess misunderstandings about fraction concepts and procedures. Teachers could not solve many of the fraction problems given and of those who could solve them, only a minority could adequately explain their solutions (p.6-7).

Steinbrenner (1955) investigated the concept of continuity in teachers of secondary school mathematics. The purpose of the study was to bring to light knowledge of teachers and trainers of teachers regarding the place of continuity in mathematics. The method used is an analysis of the historical development, and of mathematical definitions of continuity, and of the material in elementary textbooks pertaining to continuity. Steinbrenner claimed,

> The concept of continuity and related concepts of irrational number and limits constitute valuable material for the teachers of secondary mathematics teachers to know. The teacher should understand these concepts in their rigorous form and be able to give a correct informal discussion consistent with the formal approach. ... A disadvantage of the informal approach is that inaccuracies in statements and definitions may pass unnoticed more readily than in a formal approach. Such inaccuracies may give rise to misconceptions in the mind of a student. To guard against this difficulty, a

> teacher must understand correct, rigorous definition and know what is lacking in the simplified versions (p.152-153).

Thipkong (1988) investigated preservice elementary teachers' misconceptions in interpreting units and solving multiplication and division decimal word problems. The author indicated that it is important to know preservice teachers' weakness in order to help them become better in their subject matter in preparation for teaching students since today's preservice teachers are tomorrow's teachers. The purposes of this study were to:

1. investigate preservice elementary teachers' interpretations and misconceptions of decimal notation involving subunits based on ten and not based on ten, and with familiar and unfamiliar decimals;
2. describe the processes preservice elementary teachers use in solving decimal word problems involving multiplication and division with familiar and unfamiliar decimals on subunits based on ten and not based on ten; and
3. analyze how preservice elementary teachers' interpretations and performances were affected by misconceptions of decimals numbers.

The instrument was a 45-item written test. Nineteen preservice teachers were interviewed based on their written test scores. The results from this study showed that

> preservice teachers who had more experience in solving problems by taking more mathematics courses in high schools and colleges tended to get high scores on the test and preservice teachers who had good attitudes towards mathematics also had high scores. In terms of misconceptions in concepts, the results showed that some preservice teachers were not able to interpret decimals as points on number lines. In terms of the misconceptions in word problems, the results showed that some preservice teachers could not give correct interpretations for problems involving concepts of decimals and the units of conversion (pp. 112-114).

From the above studies of teachers' misconceptions, we see that there exists a common thread; that is, teachers do possess misconceptions in different topics. In Arcavi et al.'s study, some of prospective teachers were unable to distinguish between rational numbers and irrational numbers. Civil claimed that prospective teachers' view of doing mathematics is doing some pre-established mathematical procedures. In other words doing mathematics would only refer to the actual writing down of the equations and symbolic operations rather than the thinking process involved. Galbraith concluded that prospective teachers never "learned to learn" and become a vehicle for recycling their attitudes and mathematical misconceptions within the secondary teaching structure. Gralber et al. found in their study that prospective teachers have the same primitive behavioral models for multiplication and division as 10- to 15-year-old students. Hershkowitz and Vinner's study examined elementary teachers understanding of geometry and concluded that teachers have similar geometrical concept image structures to those of children. Loef and Lehrer concluded that teachers know how to do fraction problems but could not explain their solutions. Thipkong investigated preservice teachers' notion of decimals and found out that preservice teachers would not be able to interpret decimals as points on number lines. These researchers all mentioned that teachers have knowledge about how to do problems but are unable to explain why and how they get the answers. This learning and teaching cycle if unavoided may perpetuate misconceptions and misunderstandings about different topics in mathematics. In the next section, this researcher reviews the literature on the levels of understanding in different topics, hoping to come up with patterns that will help to examine knowledge of different topics in mathematics.

Literature Review on the Levels of Understanding

The Thomas Stages of Attainment in Function Concept

Thomas (1975) proposed a five stages model of the attainment of a concept of function. The description of the stages and of behaviors relative to each stage formed a hierarchy in understanding the notion of functions. The operational definition of this model is given below:

Stage I: Finding images in a mapping of the whole numbers to the whole numbers, using simple arithmetic and linear algebraic forms of a rule for the mapping.

+9
7------>? n------>3n+5

Identification of the object assigned to an element by a mapping as the *Image* of that element or by some other appropriate terminology. Simple interpretations of arrow notations.

Stage II: Identification of instances of mappings with finite domains, involving:

1. Objects familiar to the student's experience, with the assignments given by a description of a physical or arithmetic process.

2. Assignments given by an explicit display of the ordered pairs.

3. Assignments to pairs of whole numbers by the usual operations of arithmetic or processes involving them (infinite domains may be considered here).

Stage III: Operational ability in finding images, pre-images, range (or set of images), and domain (or the set of elements assigned images), where the mappings are given by some display of the set of ordered pairs. Finite domains only.

Stage IV: Identification of non-instances of mappings with finite domains. Assignments as in Stage II.

Stage V: Composition of mappings and the translation from one representation of a mapping to another.

1. Assignment of images under composition where individual assignments are given and where an algebraic (linear) rule is given for each mapping to be composed.

2. Determining the algebraic rule for the composition where algebraic rules(linear) are given for the mappings to be composed.

3. Translation from a rule for a mapping to a line-to-line graph or to a Cartesian graph of a mapping.

4. Translation from a Cartesian graph to a line-to-line graph of a mapping, or vice versa (p.155).

The Guttman scalogram scale method (Dunn-Rankin, 1983) was used to analyze the data. Each stage-subtest was treated as an item in a scale, with dichotomous categories of response. Guttman's coefficient of reproducibility was obtained as a measure of the adequacy of this set of items as a scale. Although a value of 0.921 of the coefficient of reproducibility was obtained, this analysis did not resolve the difficulties with the concept identification Stages II and IV and with Stage III. The non-perfect response patterns associated with subtests 3 and 4 were those which make the distribution of response pattern appear non-random.

The Fless Levels of Understanding in Calculus

Fless (1988) conducted a survey to investigate introductory calculus students' understanding of limits and derivatives. Two questionnaires consisting of four levels of question items on each concept area of limits and derivatives were developed. The theoretical framework of the five-level model with the operational definitions similar to the van Hiele levels in geometry was designed. The key words for these five levels in the model are: computational, intuitive, transitional, rigorous, and abstract. The behavioral definition of the five levels comprising the model are given below:

Level I

> Students are able to perform the basic operations of calculus such as finding limits, derivatives, and integrals of functions. This level is essentially algorithmic, rule-oriented, or computational in nature. Students functioning only at this level, however, do not understand the concepts which underlie these operations and often cannot recognize which concepts are needed to solve applied problems. In short, students lack what might be described as an intuitive understanding of calculus concepts.

Level II

> At this level, students possess an intuitive understanding of calculus concepts which enables them to explain their meaning and use them to solve applied problems. For example, students understand that a derivative can be interpreted as the slope of a tangent line to a curve or that an integral can be interpreted as the area under a curve. They can also solve maximum-minimum problems, sketch curves, find equations of tangent lines, and calculate arc lengths, area, and volumes.
>
> Performance at this level is also characterized by the ability to do calculus-related tasks which do not require an understanding, or even a knowledge, of formal definitions. For instance, a student at this level would be able to determine the slope of a tangent line to a curve, at least to any specified degree of accuracy, by calculating the slopes of approaching secant lines. Likewise, a student could find the area under a curve by summing the areas

of approximating rectangles. Students who have reached only level II in their development, however, fail to understand formal definitions of calculus concepts, and notation and terminology, such as epsilons and deltas.

Level III

This is a transitional level linking levels II and IV. At this stage of development, students understand formal definitions and can use precise notation and terminology in a meaningful way. Thus, students can not only state the definitions of limit, derivative, and integral, as well as their negations, but also explain their meanings in terms of a graph. Students now see how these definitions capture or formalize the corresponding intuitive notions of limit, derivative, and integral.

Given a stated condition involving the terminology and notation associated with a formal definition, a student can also determine, for example, whether that condition is stronger or weaker than the actual definition. Although students can state what is required by definition to prove a certain proposition, and perhaps even suggest a strategy for doing so, they do not generally understand formal proofs in calculus and cannot construct them.

Level IV

Students at this level of development understand and can construct formal proofs which involve the various concepts of calculus. Results of calculus which were before understood only intuitively, can now be proven rigorously. Students are now ready to begin studying the extensions of many of these results to more abstract settings, such as a metric topological space.

Level V

This level can be described as the ability to do calculus in an abstract environment, such as a generalized metric or topological space. For instance, the facts that "real-valued continuous functions on a compact metric space achieve both a maximum and minimum value" and "limits in a Hausdorff topological space are unique" are extensions of similar results in first-year calculus. Understanding at this level would be desirable for students of intermediate or advanced calculus, but content characteristic of this level is seldom encountered in introductory calculus.

Two research questions were under investigation: 1) how well do students understand the concept of calculus? and 2) what kinds of misconceptions, difficulties, and errors do students have concerning the content of calculus? The Guttman scalogram scale method was used to analyze the data. Each level-subtest was treated as an item in a scale, with dichotomous categories of response. Guttman's coefficient of reproducibility was obtained as a measure of the adequacy of this set of items as a scale. Although all values at least of 0.961 of the coefficient of reproducibility were obtained, Fless concludes:

> As with the van Hiele Levels in geometry, however, few students performed to even low criterion at Level III and IV of the model in either concept area. Performance at level II in both concept areas, although better, also indicated substantial room for improvement. With relatively minor exceptions, performance at Level I for each concept was satisfactory, at least in comparison to performance at the other levels (p.186).

The van Hiele Levels of Development in Geometry

In 1976, Wirszup introduced the van Hiele levels of development in geometry to the United States (Wirszup, 1976). This was work of Pierre M. van Hiele, who introduced a "levels-of-understanding" model for analyzing school children's knowledge of basic geometry. Van Hiele became aware of levels of thinking because his students' learning processes got stuck at the same places every year. Apparently the levels of thinking correspond with plateaus of a very special character in the learning curve. The levels have the properties that there is a certain discontinuity between one level and the next, and that they have a hierarchic nature in that one cannot possess understanding at a particular level without having achieved the preceding levels of understanding. In all accounts, the first level is described as a level of visualization or of getting familiar with the domain of study; thus a child studying geometry first learns to recognize squares and parallelograms at sight without being able to explain how he/she knows what they are. At a second, descriptive level, the pupil is able to clearly describe the properties of the figures. At a third level the pupil is able to appreciate the role of definitions and use them to, for instance, distinguish a square from a general rhombus. Deduction takes place at higher level. Clearly it would not be possible to function at this third level (using formal definitions to distinguish between figures) without having achieved the second level (being able to recognize and formulate properties of geometric figures). In the same way, one clearly could not function at a deductive level (level four or higher) without having achieved the third level in which the role of definitions is understood. Van Hiele makes the point that passing from one of his levels to the next is not the result of a biological maturation process, like Piaget's stages of development, but the result of a learning process. The levels of understanding introduced by van Hiele have a characteristic of inevitability; it seems that every pupil must pass through all of them.

As stated earlier, van Hiele concluded from teaching the geometry classes that children reached different well-defined levels of understanding in geometry and that there are ways to ascend from one level to the next and the teacher can help the pupil to find these ways. An operational definition of these five levels is given below.

Level I

> The initial level is characterized by the perception of geometric figures in their totality as entities. Figures are judged according to their appearance. The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves. They cannot even compare figures with common properties with one another. The children who reason at this level distinguish figures by their shape as a whole. They recognize, for example, a rectangle, a square, and other figures. They conceive of the rectangle, however, as completely different from the square. When a six-year old is shown what a rhombus, a rectangle, a square, and a parallelogram are, he is capable of reproducing these figures without error on a "geoboard of Gattegno," even in difficult arrangements. The child can memorize the names of these figures relatively quickly, recognizing the figures by their shapes alone, but he does not recognize the square as a rhombus, or the rhombus as a parallelogram. To him, these figures are still completely distinct.

Level II

> The pupil who has reached the second level begins to discern the components of the figures; he also establishes relationships among these components and relationships between individual figures. At this level, he is therefore able to make an analysis of the figures perceived. This takes place in the process (and with the help) of observations, measurements, drawings, and model-making. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. At this stage, the figures act as the bearers of their properties, and the student recognizes them by their properties. That a figure is a rectangle means that it has four right angles, that the diagonals are equal, and that the opposite sides are equal. However, these properties are still not connected with one another. For example, the pupil notices that in both the rectangle and the parallelogram of general type the opposite sides are equal to one another, but he does not yet conclude that a rectangle is a parallelogram.

Level III

Students who have reached this level of geometric development establish relations among the properties of a figure and among figures themselves. At this level there occurs a logical ordering of the properties of a figure and of classes of figures. The pupil is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level the student still does not grasp the meaning of deduction as a whole. The order of logical conclusion is established with the help of the textbook or the teacher. The child himself does not yet understand how it could be possible to modify this order, nor does he see the possibility of constructing the theory proceeding from different premises. He does not yet understand the role of axioms, and cannot yet see the methods appear in conjunction with experimentation, thus permitting other properties to be obtained by reasoning from some experimentally obtained properties. At the third level a square is already viewed as a rectangle and as a parallelogram.

Level IV

At the fourth level, the students grasp the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the pupils' understanding of the role and the essence of axioms, definitions, and theorems; of the logical structure of a proof; and of the analysis of the logical relationships between concepts and statements. The students can now see the various possibilities for developing a theory proceeding from various premises. For example, the pupil can now examine the whole system of properties and features of the parallelogram by using the textbook definition of a parallelogram: A parallelogram is a quadrilateral in which the opposite sides are parallel. But he can also construct another system based, say, on the following definition: A parallelogram is a quadrilateral, two opposite sides of which are equal and parallel.

Level V

This level of intellectual development in geometry corresponds to the modern (Hilbertian) standard of rigor. At this level, one attains an abstraction from the concrete nature of objects and from the concrete meaning of the relations connecting these objects. A person at this level develops a theory without making any concrete interpretation. Here geometry acquires a general character and broader applications. For example, several objects, phenomena or conditions serve as "points," and any set of "points" serve as a "figure," and so on.

The three bodies of literature just reviewed show that some topics in mathematics are characterized by levels of understanding. Thomas proposed a five stages model for the attainment of the concept of function. Fless hypothesized five levels of understanding in calculus. Van Hiele constructed a five levels model of development in geometry. Maybe a better understanding of the steps one must go through to acquire mastery of some important topics (or ideas) in mathematics will provide insight into how to go about teaching and learning those specific topics. How do we understand teachers' knowledge about any specific important topic in mathematics? In the next section, the literature review will focus on teachers' knowledge.

Literature Review On Teachers' Knowledge

Making teachers' knowledge a focus of study has been a recent trend in the research on the teaching and learning community. Researchers in the area of teacher thinking are interested in teachers' subject matter knowledge and the role that it plays in teaching (Wilson & Shulman, 1987). They are interested in how the teachers' knowledge is organized, justified, validated in their own heads, and how to represent, transform, and foster it in their students' heads. In the next sections, this researcher will review the literature about how the researchers define knowledge, how knowledge is organized, and how could knowledge be represented. The first part concerns the studies of procedural and conceptual knowledge. The second part concerns the study of subject matter knowledge and pedagogical content knowledge. The third part concerns the studies of concept definition and concept image.

Conceptual and Procedural Knowledge

Teachers' thoughts are based on their knowledge and reflected by their actions. How is teachers' knowledge represented? Gagné (1985) described two types of knowledge representation: one is declarative knowledge-- knowing that something is the case, and the other is procedural knowledge-- knowing how to do something. A common method for representing declarative knowledge is in the form of propositional networks in which propositions (ideas) are nodes that are linked closely together according to some relationships between them. A common method for representing procedural knowledge is through the use of production rules that specify the conditions under which some actions can take place.

Hiebert and Lefevre (1986) claimed that mathematics knowledge is represented by conceptual knowledge (similar to Gagné's declarative knowledge) and procedural knowledge. Both representations play an important role in knowledge acquisition. In mathematics, for instance, procedural knowledge consists of knowledge of formal mathematical symbols (e.g., numerals and signs for operators) and the procedures (e.g., algorithms) that operate on these symbols to complete mathematical tasks, while conceptual knowledge in mathematics (as in all fields) is rich in relationships, a "connected web of knowledge" achieved by constructing relationships between pieces of information. In terms of the limit concept, the procedural knowledge consists of symbols such as, lim, n-->∞, lim a_n, a_n, a_n-L and $\varepsilon > 0$, n>N, $| a_n - L | < \varepsilon$, etc. The conceptual knowledge of limit concept would be making connections among these symbols. What is the meaning of each symbol? How are these symbols related or interrelated? How could one interpret these symbols to someone who has no ideas about these symbols? How these two knowledge representations, conceptual and procedural, are interrelated with each other is an important debate in cognitive psychological research. In recent research on mathematics education, the specific debate topics are how the two are interrelated; how to achieve an appropriate balance between the two; or whether one has to come before the other (Nesher,1986; Romberg & Carpenter, 1986; Putnam, 1987). In most mathematics instruction, the main focus has been on computational skills, which often did not produce understanding. Researchers on mathematics teaching and learning have started to investigate whether teaching understanding will enhance procedural competence (Leinhardt and Smith, 1985; Lampert, 1986; Nesher, 1986; Putnam, 1987).

Nesher (1986) described two studies that focus on whether the conceptual and procedural knowledge enhance each other. One was a study done by her student which involved dealing with the question of the relationship between algorithmic performance and understanding in decimals. The other, which was carried out by Resnick, Omanson and Peled, dealt with understanding place value concepts and performance on the subtraction algorithm. But neither of these two studies supports the idea that in learning a certain algorithm one needs a prior understanding. Nesher argued:

> Understanding is never finite and complete, rather it is open ended. We do not know precisely enough what the states that lead to understanding are, and we cannot acquire understanding in a mechanistic manner that will assure the pre-determined outcomes (p.3).

Putnam (1987) pointed out that conceptual knowledge does not automatically produce procedural competence. But this rich conceptual knowledge which constitutes mathematical understanding should link with the procedural knowledge, which has meaning and is understood based on this rich conceptual base. It is these links that allow procedures to be applied appropriately to problem solving and the acquisition of other mathematical concepts.

Subject Matter Knowledge and Pedagogical Content Knowledge

What the teacher knows about the subject matter to be taught, the ways to communicate knowledge to students, and knowledge of how to help his or her students come to understanding the subject matter influence one another. Shulman (1986) addressed the fact that investigators ignored one central aspect of classroom life: the subject matter, and that

> No one asked how subject matter was transformed from the knowledge of the teacher into the content of instruction. Nor did they ask how particular formulations of that content related to what students came to know or misconstrue (p.6).

How do prospective secondary mathematics teachers represent their knowledge and what do they need in teaching? Shulman (1986) claimed that teaching should emphasize comprehension and reasoning, transformation and reflection. In answering what are the sources of the knowledge for teaching, Wilson and Shulman (1987) proposed a knowledge base for teaching, Even (1989) restated as follows:

1. Subject matter content knowledge, which is the understanding of the subject matter structure. It consists of the knowledge, understanding, skill, and disposition which are taught and to be learned by school children. The knowledge about why, how and what should be taught, how those topics are related to each other, not only knowing how something is the case, but also knowing how to do something.
2. Pedagogical content knowledge represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of the learner, and presented for instruction. This knowledge includes an understanding of what it means to teach a particular topic and knowledge of the principles and techniques for doing so.
3. Curriculum knowledge represents an understanding of the curricular alternatives available for instruction; familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school (p.49-50).

Even (1989, 1990) tried to identify important aspects of subject matter knowledge for teaching the concept of function and to describe kinds of knowledge prospective teachers have with respect to these aspects. Her data was gathered in two phases: an open-ended questionnaire followed by an interview. There were 152 prospective secondary mathematics teachers participating in her survey and 10 among them been interviewed. The results showed that there were discrepancies between the participants' concept image and concept definition of function, difficulties with translations between different representations of function, lack of rich relationships between the informal meaning of inverse as "undoing" and the formal definition of function, and incomplete understanding of the role of the notion of function in the curriculum.

Even proposed six aspects of teachers' subject matter knowledge about functions, they are:

1. What is a function? (includes image and definition of the concept of function, univalent property of functions, and arbitrariness of function).
2. Different representations of function.
3. inverse function and composition of functions.
4. Functions of the high school curriculum.
5. Different ways of approaching functions: Point-wise, interval-wise, global and as entities.
6. Different kinds of knowledge and understanding of function and mathematics (p.5-6).

In addition she added two aspects of pedagogical content knowledge:

1. Teaching toward different kinds of knowledge and understanding of functions and mathematics. This aspect includes teaching with emphasis on conceptual or procedural knowledge, meaning or rote learning, teaching for understanding or emphasis on following rules.
2. Students' mistakes--what they do and why? This includes knowledge about students' common mistakes about functions and their sources.(p.5-6)

Leinhardt and Smith (1985) argued that we can come close to a definition of mathematical understanding if we think of it as a collection of different ways of knowing mathematics and an appreciation of the connections among them. In order to test their theory, they have contrasted expert and novice teachers to examine the knowledge that teaching requires. They argue that teaching is based on two bodies of knowledge: knowledge of lesson structure and knowledge of subject matter. One consists of general teaching skills and strategies, the other consists of subject matter information necessary for the content presentation. They described teachers' knowledge of arithmetic as follows:

> Subject matter knowledge includes conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of student errors, and curriculum presentation (p.247).

In order to help students come to understand mathematical notions, the prospective secondary mathematics teachers need not only to master the subject matter, but to know what prior existing misconceptions their students have. Most important of all, they themselves must not have these misconceptions. In the next section, the literature review will concern the discussion of discrepancies between teachers' concept definition and concept image.

Concept Image and Concept Definition

The other type of description of the knowledge that students have in mathematics and science involves the terminologies of "concept image" and "concept definition." Concept images and concept definitions were discussed in several papers recently (Davis & Vinner, 1986; Dreyfus & Vinner,1982; Even, 1989; Hershkowitz and Vinner,1984; Tall, 1989; Tall and Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989). Almost all except the most primitive mathematical concepts have formal definitions. Many of these definitions are taught at one time or other to high school or college students. On the other hand, all concepts, even nonmathematical ones (like concepts in science), have accompanying concept images. Each individual's concept image is his or her own mental picture, representations, and understanding of related properties of that concept. Revealing the concept images of teachers as well as of students becomes very important in teaching and learning; not only might it give us a better understanding of teachers' knowledge and how well students learn, but also it might suggest some improvements in the teaching and learning to prevent recycle the wrong concept images. In the previous literature review, this researcher already mentioned studies on students' concept images. In the present section, we consider only studies concerned with teachers' concept images.

Dreyfus and Vinner (1982, 1989) examined the notion of function in 271 college students and 36 junior high school teachers. They found that concept images played a crucial role in learning and teaching. Even though a student may have a correct concept definition his or her concept images (sometimes wrong) may interfere with correct performance. Sometimes, the concept image produces an answer contradictory to the concept definition. This causes difficulty in learning mathematics. For example, one of questionnaire questions (see Figure 3.2) in this study asked students whether there exists a function the graph of which is:

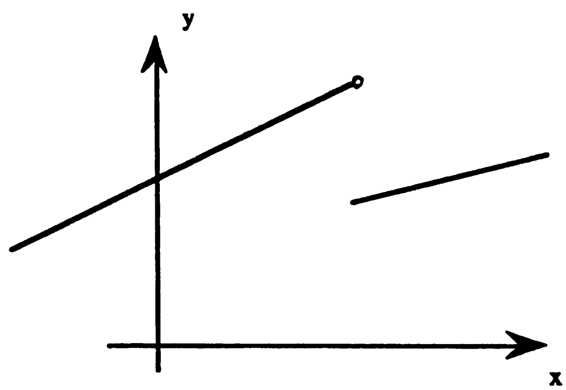


Figure 3.2 -- Discontinuous Function

Some negative answers were explained by saying that the graph is discontinuous and therefore it cannot be the graph of a function. On the other hand, the explanations for the positive answers stated that discontinuous functions are legitimate members of the function family. The definition of function does not preclude a function having a graph which has jumps or breaks, and in fact the answer to the above question is affirmative. Most people have the idea that functions should behave smoothly.

Hershkowitz and Vinner (1984) investigated basic geometrical concepts in school children in a series of studies. The purpose of their studies was to obtain a better view of the processes of concept formation in children and of the factors affecting their knowledge acquisition by studying these same concepts in elementary school teachers. The subjects of these studies were asked in one of the questionnaire question to identify the interior points in an angle (see Fig.3.3.a). Both the students and the teachers had the misconception that the sides of the angle are segments instead of rays. This concept image comes from the usual drawing of an angle. Thus, they would say that P is an interior point, but Q is not. Actually, if the sides of the angle extended longer (see Fig. 3.3.b), we could see that Q is an interior point.

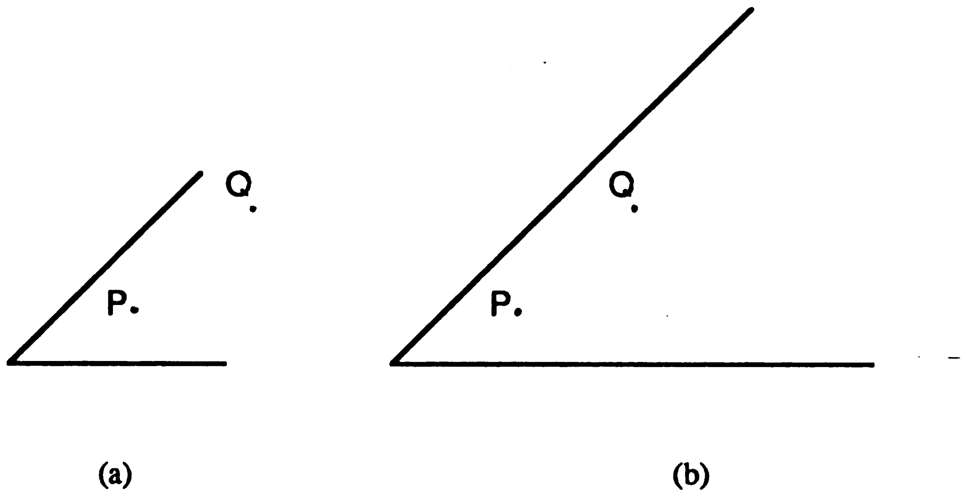


Figure 3.3 -- Interior Point of an Angle: Two Drawings

Due to the similarity of teachers' and students' concept images and performance, this study suggests that a teacher's incomplete or incorrect concept image will probably be repeated in the students' geometrical thinking.

In summary, knowledge can be represented as procedural and conceptual knowledge. Procedural knowledge means knowing how to do something, while conceptual knowledge means knowing that something is the case. Teachers' knowledge can be represented as subject matter knowledge which includes both conceptual and procedural knowledge, in addition to which teachers' knowledge also includes pedagogical knowledge and curriculum knowledge. Mostly teachers' knowledge is seen as a collection of procedures divorced from the underlying conceptual understanding. Besides the separation of the conceptual knowledge and the procedural knowledge, the literature shows that there exists a discrepancy between teachers' concept definition and concept image. How to avoid the teaching and learning recycling of misconceptions is something the teacher training institutions should think about.

Conclusion

In this chapter, the literature review reveals that researchers are in general agreed that the limit concept should be taught at least in high school. Although there were many studies which investigated which is the best method to teach the limit concept, there seemed no difference in students' performances. No matter what instruction methods were used: whether the limit concept was taught by inductive or deductive reasoning, whether the limit concept was taught by limits of functions followed by limits of sequences or vice versa, whether the limit concept was taught by advanced set method or by logical preparation method, the difficulties, misconceptions, and errors made by students are consistently identified by many studies. On the other hand, the research on teachers' knowledge of specific topics also exhibits that there exist pervasive misconceptions, difficulties and errors common to teachers The discrepancy between the mathematical concept definition and concept image interferes with knowledge acquisition in the teaching and learning of mathematics. What needs to be done, first, is to identify more teachers' misconceptions, difficulties, and errors in different important mathematical topics, and then the teacher training institutions can help prospective teachers undo the damage caused by the wrong concept image.

From the above literature review, one can see that there seem to be no studies investigating the inservice teachers who teach the limit concept or research specific to preservice teachers who are going to teach the limit concept. Prospective secondary mathematics teachers are a very interesting group to study, because they are still student-teachers, but will shortly become the new professional teachers themselves. What they know and understand now will probably reflect how they are going to teach the limit concept. By studying their understanding about the limit concept and how they teach the limit concept when provided a teaching situation, we will certainly learn not only how well prospective teachers learn their knowledge from the teacher training institutions, but also how are they going to teach in mathematics classrooms.

In the next chapter, this researcher will describe in detail the procedure, the methodology, and the data collection of the present study.

CHAPTER FOUR

THE STUDY: PURPOSE AND DESIGN

Purpose

Limit is a central concept in analysis. Most students are formally introduced to this concept in a calculus or precalculus class. Studies of students' understanding of limits indicate there are not only difficulties in the learning this concept but also report many students lack understanding and hold a variety of misconceptions. One of the variables relating to student outcomes in understanding the limit concept is teacher knowledge. Shulman (1986) proposed a knowledge base for teachers. He stated the knowledge base a teacher needs to perform the teaching profession includes: subject matter knowledge, pedagogical content knowledge, and curriculum knowledge. The subject matter knowledge is a common personal knowledge of the subject matter (for instance, in the present study it is the mathematical notion of limit), pedagogical content knowledge is knowledge of how to help someone else develop an understanding of the subject matter (in this study it is how to teach the notion of limit when provided with a teaching situation; knowing what are students misconceptions, difficulties and errors; and what are the methods for overcoming these misconceptions), and curriculum knowledge is knowledge about the range of instructional materials available to teach a particular subject (in this study knowing where the limit concept comes from and where it leads to; knowing what is the role of the limit concept in K-2 mathematics curriculum; knowing what kinds of activities will contribute to

K-12 ?

helping students learn the limit concept at each different grade level). In order to investigate the prospective teachers' knowledge about the notion of limit, the following research questions were formulated:

1. How well do prospective teachers understand the concept of limits?
2. What kinds of misconceptions, difficulties, and errors do prospective teachers have with regard to the concept of limits?
3. What are prospective teachers' opinions about the role of the concept of limits in K-12 mathematics curriculum?
4. What are the possible misconceptions, difficulties, and errors prospective teachers anticipate in teaching the concept of limits?

The first two questions are related to prospective teachers' subject matter knowledge. The third research question is designed to examine prospective teachers' curriculum knowledge about the limit concept. The last research question leads to exploring prospective teachers' pedagogical content knowledge. In order to be able to address these research questions, this researcher constructed a five-category theoretical model of understanding as a data analysis framework.

Design

The Theoretical Model for this Study

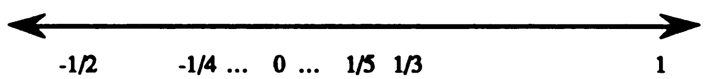
In order to investigate the research questions, this researcher constructed a theoretical model for describing prospective secondary mathematics teachers' understanding about the limit concept. The form of this model was mainly suggested from the following: 1) readings on the history of the limit concept and the literature, 2) the van Hiele levels of development in geometry, 3) Fless's levels of understanding in calculus, and 4) Thomas' stages of attainment regarding functions. In addition, the results of a pilot study conducted by this researcher was used. The operational definitions of the constructed five categories comprising the model of this study are given below.

Category I: Basic Understanding

Category I, Basic Understanding, includes the ability to conjecture correctly at the existence or non-existence of the limit for an infinite sequence evolving according to a sufficiently simple numerical, graphical, or geometrical pattern without necessarily understanding what a limit is in a formal sense. A person functioning in this category, for example, would look at the pattern

$$1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots$$

and infer that the rule is n --> 1/n, would observe that the values of the terms get smaller and smaller, and would thus recognize that the limit is 0. When presented with the following diagram



a person at Category I would observe that the terms are clustering toward the number 0, and would thus recognize that what is being presented is a sequence converging to 0. In Category I, one would know that sequences such as

$$1, 2, 3, 4, 5, \ldots, n, \ldots$$

or

$$1, 4, 9, 16, 25, \ldots, n^2, \ldots$$

do not have limits because the values of the successive terms get larger and larger without bound. A person in this category would recognize that the sequence

$$1, 1/2, 1/2^2, 1/2^3, 1/2^4, 1/2^5, \ldots, 1/2^n, \ldots$$

has limit zero and, perhaps, that the sequence n/(n + 1) has limit 1.

Category II: Computational Understanding

Category II, Computational Understanding, includes the ability to use algebraic operations and basic theorems on how limits and algebraic operations interact (the theorems for limits of sums, differences, products, and quotients) to find limits of many sequences through computation. This category is essentially algorithmic, rule-oriented, or computational in nature. A person with what we call computational understanding could certainly find, for example,

$$\lim_{n\to\infty} \frac{n^2 + 1}{(2n + 1)^2},$$

and even perhaps compute limits for much more complicated sequences. However, it is possible to apply the rules of the algebra of limits in a very mechanical way and still be quite successful in computing limits without having a clear understanding of why the computed result is actually the limit of the given sequence or even having a clear conception of what the term "limit" means. Thus computational understanding is still a fairly primitive category of understanding. In this category what one needs to know includes also the basic understanding, however, for the following reasons: 1) a person functioning at this category would have to know or believe that $\lim_{n\to\infty} \frac{1}{n}$ equals zero and $\lim_{n\to\infty} \frac{1}{2^n}$ equals zero, 2) the theorems on algebra of limits do not enable us to compute limits for all sequences, but instead are used to reduce a given limit problem to finding limits of simpler sequences, such as mentioned in 1), thus to do any problems a person would need to have some basic skills to complete the problem after the algebraic reduction has been made, and 3) a person would hardly be said to have computational understanding if one writes statements such as

$$\lim_{n\to\infty} \frac{n}{n + 1} = \frac{\infty}{\infty} = 1$$

or

$$\lim_{n\to\infty} \frac{1/n^2}{1/n} = \frac{0}{0} = 0$$

as answers to a problem, although both answers happen to be correct.

Category III: Transitional Understanding

Category III, Transitional Understanding, is when one has achieved the skills of the proceeding categories and is beginning to have some theoretical grasp. In this category one can state the formal definition of limit, and explain the notations and terminologies verbally and graphically. One is aware of the important role of ε and N and the necessity of taking them in the proper order and can do problems involving explicitly finding N in terms of ε for given sequences. One would know some theorems, beyond the simple algebra of limits, concerning when limits exist; for example, one would recognize when to use the Squeeze Theorem to find the limit of a given sequence. One would be able to identify under what situation "the sum of the limits is the limit of sums," rather to follow the routine computational manipulations. One would know that the limit (if it exists) is a real number, so one can divide the limit by a non-zero real number. However, a person functioning in this category might not be able to provide rigorous proofs of statements about limits; and might not be able to state and to prove the negation of the definition of the limit.

Category IV: Rigorous Understanding

Category IV, Rigorous Understanding, comprises not only basic and computational understanding of limits, but also the ability to rigorously prove statements about limits. In this category one can: 1) apply the skills of categories I and II to find limits, 2) state and explain the formal and informal definition of the term "limit", 3) provide proofs that the initial conjecture at the limit of a sequence, derived basically or computationally, actually is that sequence's limit according to the formal definition, 4) formulate and explain the negation of the definition of limit, and use it in proofs, and 5) prove and use the standard theorems about limits.

Category V: Abstract Understanding

Category V, Abstract Understanding, is a developmental stage of "feeling comfortable with" the limit concept. A person functioning at this category of development would have all the skills associated with the proceeding categories. In addition, in this category one is aware of the important role of the limit concept in mathematics; of where it shows up in the standard K-12 mathematics curriculum; and what mathematical knowledge can be generalized from here. As a teacher in this category, one is able to tie in the limit concept by using different examples or non-examples to different age students, either intuitively or rigorously. In this category one is comfortable enough with the limit concept to be able to handle its generalizations to more abstract settings, such as encountered in topology or the study of function spaces. Finally, in this category one is free of the usual misconcepts about limits, which often cause people to perceive the subject of limits as being beset by "Zeno-like" paradoxes. Ideally, prospective secondary mathematics teachers should be aware of these misconceptions (which seem to have a somewhat inevitable character) and thus be able to help their students work through these misconceptions.

This researcher believes it is reasonable, and illuminating, to draw a rough parallel between the history of the limit concept and hierarchical stages of understanding models such as used in this study. The knowledge possessed by the ancient Greek geometers, an intuitive recognition that successive bisections produce a sequence tending to zero and that the circle was in some sense the limit of a sequence of inscribed polygons, for instance, might be thought of as corresponding to Category I. Of course the ancients could not have stated that the circle was the limit of a sequence of inscribed polygons because the concept of the limit had not been invented. An interesting period exhibiting highly ingenious computational skills extends through the time of Cavalieri and even Newton and Leibniz. During this time many problems were solved, correctly, but the solvers were not able to explain their reasonings clearly or to answer perfectly reasonable objections to their logic.

When it became clear that the calculus was important, that there were certain logical problems with its foundations, and that limits were somehow the key idea, there was a kind of transitional era during which mathematicians worked on coming up with a satisfactory logically sound explanation of limits. The full rigorous understanding came with Weierstrass' definition; a definition which is still found to be satisfactory and which is clear and flexible enough to allow of easy generalization to contexts other than the real number system.

These five categories are used as the framework of the theoretical model for data analysis and the design of the questionnaire test items.

Category I and II of understanding would correspond pretty exactly to what is called procedural knowledge; the remaining categories certainly involve conceptual knowledge (Hiebert & Lefevre, 1986).

The Pilot Study

As the literature review in Chapter Three revealed, very little research has focused on the prospective teachers' understanding of the limit concept. There were studies on other mathematical topics which tend to show that prospective teachers do possess the same kind of misconceptions as their students. But what about the prospective teachers' notion of limits? Are prospective teachers' conceptions of the limit concept similar to the students'? In an attempt to determine how information about prospective teachers' understanding of the limit concept could be obtained, a pilot study was conducted. While teaching in Taiwan during fall of 1990, this researcher devised an instrument, in the form of a questionnaire based on the theoretical five categories model of understanding. Copies of the questionnaire and the description of the theoretical model of understanding were sent to several mathematics professors at the National Normal University in Taiwan, and they agreed generally on the description of the model and the instrument. The instrument was then given to the faculty at Wu Feng Junior College of Technology to check for length and difficulty. After a minor revision, the revised questionnaire was then administered to 25 prospective secondary mathematics teachers, in the National Normal University. The instruments from the pilot study were scored, and examined to see if the questions reflected the desired understanding as they were intended to.

In winter 1991, the researcher presented a copy of the instrument, together with the theoretical five categories model of understanding of the limit, a statement of purpose, the answer sheet of the test items, and the grading policy of the test items to a panel of 11 Michigan State University mathematics and mathematics education professors, from the mathematics department and the teacher education department. Comments were sought and received from this panel. There was wide agreement that the categories of understanding were appropriately chosen and adequately described, and the test items also indicated the categories of understanding. There were several detailed comments on the questionnaire indicating that some test items needed rewording.

Several changes were made in the questionnaire as a result of the panel's comments. A few questions were re-phrased for greater clarity, some have been deleted, and new questions were included. And the questionnaire was shortened so as not to appear intimidating or overly time-consuming to the subjects, and so that it could be completed in an hour.

The new questionnaire was then given to two prospective secondary mathematics teacher volunteers to check that it could reasonably be completed in an hour. The results seemed to satisfy the researcher's intention.

This Study

Instrumentation

Questionnaire

As the goals of this study stated earlier, this study was intended to reveal the kind of subject matter knowledge of limits prospective teachers have as well as to point out misconceptions, difficulties, and errors in their subject matter knowledge, to explore their curriculum knowledge, and to describe their pedagogical content knowledge. The questionnaire was then designed to examine how well prospective teachers understand the topic of limits and how to teach it. Therefore, an instrument was developed that would measure their understanding based on the constructed five-category model. In accordance with this model, the subject matter problems presented to the participants were chosen to be both standard and non-standard problems.

This questionnaire (appearing in Appendix A), which included 24 test items and was expected to be finished within 45-60 minutes, was administrated to the participants in their class. They were asked to sign a consent form, but were asked not to sign their names on the questionnaire. The questionnaire consisted of two parts. Part I included the demographic background data of the subjects obtained from item #1 to item #4. Based on William's (1989) questionnaire items on function limits, the researcher constructed a parallel True-False question on limits of sequences (Part I #5). This question was formed by eight statements of which only one among them is the mathematically correct answer for the definition of limit of a sequence. The responses to this multiple-choice question will first provide answers for what the subjects think about limit. Then, Part I, item #6 was designed for the subjects to identify what statement in test item #5 matches their best description of limit. Items #7 and #8 were intended to find out about prospective teachers' (formal or informal, correct or incorrect) way of defining what they think is the meaning of the limit concept. Item number 9 asked the prospective teachers to provide activities which implicitly or explicitly involve the limit concept in K-2 and 4-5 grade ranges. Finally, item number 10 was intended to explore their ability to recognize the difficulties, misconceptions, and errors they encountered and how to teach their students to overcome these problems.

Part II includes 14 test items which divide into 5 categories based on the framework of the theoretical model. The classification of test items by category is given in Table 4.1:

Table 4.1. Classification of Test Items by Category

Category	Description	Test Item Number
I	Basic	1(a), 1(b), 1(c), 1(d), 1(e), 1(f), 2(a), 2(b), and 3(a)
II	Computational	4(a), 4(b), 4(c), and 4(d)
III	Transitional	3(b), 5(a), 5(b), 6, and 7
IV	Rigorous	5(c), 8, 9(a), 9(b), and 10
V	Abstract	11, 12, 13, and 14

Only the first 10 test items were graded and scored based on the scoring system. There were very few responses to the fifth category test items, so this researcher decided there was no reason to discuss them.

Interview

Information gathered from a written question is sufficient for a general description of some facets of the prospective teachers' knowledge about the notion of limit and teaching, but is limited and sometimes hard to interpret. In order to overcome these difficulties, an in-depth interview was included in this study. The interview was conducted by providing questions and asking the subjects to explain what they think and why; asking their reactions as teachers as to how to either identify students' misconceptions or try to help students to overcome these misconceptions; asking what they think should be the role of limit concept in K-12 mathematics curriculum and how the limit concept is related to different topics in mathematics; asking them to provide an activity that could explain the limit concept to a very young child and then why they thought that activity could help the young child learn the limit concept; and asking them whether the mathematics curriculum should include intuitive presentations of the limit concept in early grades. All these types of questions will provide further information about prospective teachers' knowledge about the limit and teaching about limits, which could not be provided by the written questionnaire. Thus, the individual interviews were conducted in order to provide the following data about the subjects:

1. Information which could not be supplied by the written questionnaire.
2. Subjects' conceptual understanding and transfer of that understanding into verbal communication.
3. Subjects' ideas on how to teach the concept of limit when provided with a teaching situation.

Therefore, the interview was used to get information that the questionnaire could not possibly give. The interview questions are presented in Appendix B.

Population and Sample

General Background

The participant subjects in this study were 38 prospective secondary mathematics teachers in the last stage of their professional education. They were finishing or had already finished their mathematics methods class. This group was selected so that the description of their knowledge would reflect the knowledge teachers have gained during their college education, but before they start teaching. The subjects came from six universities: Western Michigan University, University of Iowa, University of Wisconsin-- Madison, Michigan State University, University of Texas-- Austin, and Oral Roberts University. The subjects were prospective teachers enrolled in a mathematics methods class. A mathematics educator and professor at Michigan State University contacted some of his colleagues in some Mid-western universities. First he sent them a letter to ask whether they were willing to let prospective secondary mathematics teachers in their methods course participate in this study. If they agreed to participate in this study, then the required number of questionnaires were sent to them. The selection of the university was made according to the mathematics method instructors' cooperation and willingness to devote one hour of their class time to administering the questionnaire to their students. The distribution of subjects by university, by sex, and by age is given by Table 4.2.

Academic Background

Over 80% of the subjects (who provided the information) in the first phase of the study had an over all college grade point average above 3.0 (on a scale of 0 to 4). The mathematics grade average point was 5% lower -- about 75% of the subjects had a mathematics grade point average between 3.0 and 4.0. Table 4.3 shows the distribution of subjects by universities, by grade point average in general and grade point average in mathematics.

Table 4.2 -- Distribution of Subjects by Universities, by Sex and by Age

	Sex				Age					
Univ.	Male	Female	N/R	Total	19-23	24-29	30-35	over 35	N/R	Total
1	2	6	--	8	5	2	--	--	1	8
2	3	9	--	12	8	2	1	1	--	12
3	3	--	--	3	1	1	--	1	--	3
4	1	2	--	3	2	--	1	--	--	3
5	4	3	--	7	3	2	--	2	--	7
6	3	2	--	5	4	1	--	--	--	5
Total	16	22	0	38	23	8	2	4	1	38

Note: N/R indicates no response.

Table 4.3 -- Distribution of Subjects by Universities, by GPA and by MGPA

	GPA							MGPA						
Univ.	2.0	2.5	3.0	3.5	4.0	N/R	Total	2.0	2.5	3.0	3.5	4.0	N/R	Total
1	--	1	2	4	--	1	8	1	--	2	3	--	2	8
2	--	3	3	5	--	1	12	1	3	5	1	--	2	12
3	--	1	1	1	--	--	3	--	1	--	1	1	--	3
4	--	--	1	2	--	--	3	--	--	2	1	--	--	3
5	--	1	1	3	--	2	7	--	1	1	3	--	2	7
6	--	--	2	2	--	1	5	--	1	1	1	1	1	5
Total	--	6--	10	17	--	5	38	2	5	1	10	2	7	38

Note. N/R indicates no response.

Model of Limit Held by Subjects

In Questionnaire Part I question number 5, the subjects were asked to identify eight statements as true or false. Among the eight statements only #5-c parallels the formal definition of a limit of a sequence. Then in question #6, the subjects were asked to identify which statement is the best description of a limit of a sequence. Eighty-two percent of the subjects chose #5-c as a true statement. The four most popular views of limit seem to be a view that echoes the formal definition (82%), a view that is dynamic-theoretical (63%), a view that sees limit as boundary (58%), and a view that sees limits as unreachable (55%). However, this unreachability statement (#5-d) was selected most frequently as the best description of a limit. Table 4.4 shows the percentages of subjects indicating each statement as true, false, or best on the questionnaire.

Table 4.4--Model of Limit Held by Subjects

Question number	Model of limit	Original Sample (N=38)			Subjects Indicating Statement 5-c as true (n=31)		
		True	False	Best	True	false	Best
5-a	Dynamic-Theoretical	63	37	13	65	35	16
5-b	Boundary	58	42	3	58	42	3
5-c	Formal	82	18	18	100	0	23
5-d	Unreachable	55	45	26	58	42	26
5-e	Approximation	50	50	5	52	48	3
5-f	Dynamic-Practical	32	68	0	39	61	0
5-g	The Last Term	39	61	11	48	52	13
5-h	Limit as a Variable	47	50	11	52	45	6
Total				87			90

Note: Responses for best statement do not sum to 100% because of nonresponses.

Procedure and Data Collection

Data collection for this study was conducted from April 1991 to February 1992. The administration of the questionnaire to the 38 subjects took place between April 1991 to August 1991 in a regular mathematics method course by regular instructors of that class. Data collection for the interviews was conducted from November 1991 to February 1992 by this researcher herself.

The following data were collected from the whole sample of 42 (38 subjects took the questionnaire and four were interviewed) in six universities.

General Information

This information was collected in order to describe some demographical characteristics and academic background information about participants. It included gender, age, university grade point average in general and in mathematics, and mathematics courses taken at university level, and models of the limit concept held by this group of prospective teachers.

Questionnaire

This contained standard mathematics problems from the ordinary textbooks as well as non-standard mathematics problems gleaned from the history of mathematics and problems designed based on students' misconceptions found in other studies. The participants were not allowed to use any outside resources and had 45-60 minutes to complete answering. This information could provide insight into the general knowledge and understanding of the notion of limit and teaching the limit concept that prospective teachers have.

In addition, due to the fact that there were fewer participants than this researcher expected, an interview session was added to collect more information. This included four subjects from one university.

Interview

The interview questions (appearing in Appendix B) resulted from the analysis of questionnaires of these thirty eight subjects. Since this researcher could not find subjects willing to devote three hours to participate in this study, the interviewees only answered the interview questions rather than answering both the questionnaire and being interviewed. The interview lasted about one hour on the average. In the processes of interviewing, the thinking aloud technique was conducted in order to find out why people said or did what they said or did. The interview questions can probe the depth of prospective teachers' conceptual understanding, and explore how they externalize their understanding through the thinking aloud method. Their interpretations about the notion of limits were collected as information to indicate how they teach the mathematical limit concept and what were their misconceptions and their concept images. Probing was an important component of the interview processes. But, since this researcher has been a classroom teacher for so long, it had become an habit to provide hints for the students, thus the interview questions were designed as a structured interview. That means, every interviewee answers the same sets of questions. Unless this researcher was uncertain about the responses of the subjects, she was quiet most of the time. The interviewee read the interview questions and provided the answers they thought best explained the questions that been asked. The interview session was audiotaped to assure an accurate record of what was said.

Data Analysis: Scoring

For the first research question: How well do prospective teachers understand the concept of limits?

An answer sheet for question items number 1 to number 10 on Part II of the questionnaire was reviewed by the panel of eleven mathematicians and mathematics educators and a sample is provided in an appendix C. Based on this answer sheet the researcher designed a scoring system which was also reviewed and approved by the panel.

Raw scores of distribution in each category were then obtained by three graders first adding up the scores of each item in each category and then dividing the total scores of that particular category's question items, giving a percentage score (appears in Appendix D) at each level of understanding. Since the researcher provided an answer sheet, the results of the raw scores the graders provided were within 95% of the agreement. Based on this percentage score a scale like the Guttman Scale was formed. This scale will be used as an indicator of how well prospective teachers did at 90%, 80%, and 70% performance criteria.

For the second research question: What kinds of misconceptions, difficulties, and errors do prospective teachers have with regard to the concept of limits?

The responses of each test item in questionnaire Part II number 1 to number 10, were first collected separately. Then the responses were gathered into groups, each group representing the same core response. Next all the subgroups of each group were gathered together to form the main responses for each category test item. For each category, a chart is provided which shows the number of subjects who gave a response from each of a particular set of common responses for that category. This chart could be used as an indication of the subject's difficulties, misconceptions, and errors made for the particular categories. From the above information, the second research question was addressed.

For the third research question: What are prospective teachers' opinions about the role of the concept of limits in K-12 mathematics curriculum?

In order to answer the third research question: what are prospective teachers' opinions regarding the role of the limit concept in K-12 mathematics curriculum, the researcher designed an open-ended question (Part I, #9). In this question, the researcher asked the prospective teachers to describe an activity that would introduce the notion of limit to children in (a) K-2 grade range and (b) 4-5 grade range. How many activities could they mention? What were the connections they made between the notion of limit and the activity they presented? Did they realize when the activity they presented was too easy or too hard for that group of children? Furthermore, there are the in-depth interview transcripts which will be discussed in more detail.

For the fourth research question: What are the possible misconceptions, difficulties, and errors prospective teachers anticipate in teaching the concept of limits?

Here, this researcher intended to investigate what difficulties prospective teachers encountered while learning about limits and how would they as teachers help their own students to overcome these difficulties. Based on the responses collected from this question, the information could be analyzed quantitatively and qualitatively in terms of particular types of misconceptions, difficulties and errors identified by this particular group of subjects. An in-depth interview transcript will add to the qualitative analysis in order to answer this last research question. The discussion to these four research questions will be addressed in next chapter.

CHAPTER FIVE

ANALYSIS OF DATA

The responses to the questionnaire test items were analyzed in order to provide a framework for discussion of the four research questions. The first research question related to prospective teachers' subject matter knowledge--the concept of limit and the second research question was designed to identify their misconceptions, difficulties, and errors concerning their subject matter knowledge. These two research questions will be addressed based on the framework of the theoretical model of five categories of understanding discussed in the previous chapter. The key words for describing the understanding of limit concept in this model are: basic, computational, transitional, rigorous, and abstract. The third research question was intended to investigate prospective teachers' curriculum knowledge regarding the role of the limit concept in K-12 mathematics curriculum. In order to address this research question, an open-ended item was embedded in questionnaire Part I, and the similar responses were collected and grouped and discussed. Similarly, the last research question was also embedded also in the questionnaire Part I. The subjects' responses were also collected and grouped. In addition to the two embedded questions in the questionnaire Part I, four interviewees' transcripts were added hoping to provide a better picture of prospective secondary teachers' curriculum knowledge and pedagogical content knowledge.

Teachers' subject matter knowledge about the limit concept is intended to be addressed by the responses to the first two research questions: How well do prospective teachers understand the concept of limits? and What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the concept of limits? The two research questions are addressed within the theoretical framework of the five-category model of understanding discussed in the previous chapter.

Question 1: How well do prospective teachers understand the concept of limits?

Category I: Basic Understanding

Basic Understanding includes the ability to conjecture correctly at the limit or non-existence of the limit for an infinite sequence evolving according to a sufficiently simple numerical, graphical, rule-oriented, or geometrical pattern without necessarily understanding the underlying concepts of what a limit is. Thus, basic understanding of the limit concept includes the ability to find the limit of different representations of infinite sequences intuitively. The test items were divided into four different representations of sequences. First of all, these included a numerical representation of a sequence which just simply lists the first few terms of the sequence successively according to a specific rule which is not provided, such as in test item numbers #1-a and #1-b. The second representation of a sequence is generated by a formula, such as in test item numbers #1-c and #1-d. The third representation of a sequence is through a graph either one dimensional, such as in the test item #1-e, or two dimensional, such as in test item number #1-f. In problem number one, the subjects had four choices: (A) The indicated limit is 0, (B) the indicated limit is 1, (C) the indicated limit is -1, and (D) the sequence does not have a limit (which includes ∞ and $-\infty$); of which only one among them was the correct answer. The first question in the Questionnaire Part II is provided in Table 5.1 and is followed by a scoring system with some examples. The results of these test items in question #1 based on this scoring system are given by frequency, relative frequency, and mean score in Table 5.2. A short discussion will summarize these results.

Table 5.1 -- Question #1 Test Items and Scoring System.

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1. In the following infinite sequences (a) - (f), select exactly one of the following answers:

   (A) The indicated limit is 0.

   (B) The indicated limit is 1.

   (C) The indicated limit is -1.

   (D) The sequence does not have a limit (which includes ∞ and -∞).

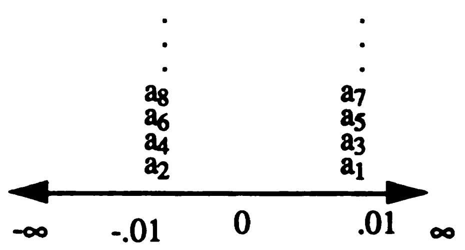
   a) 1, -1, 1, -1, 1, -1,...

   b) 3/4, 9/16, 27/64, 81/256, 243/1024, ...

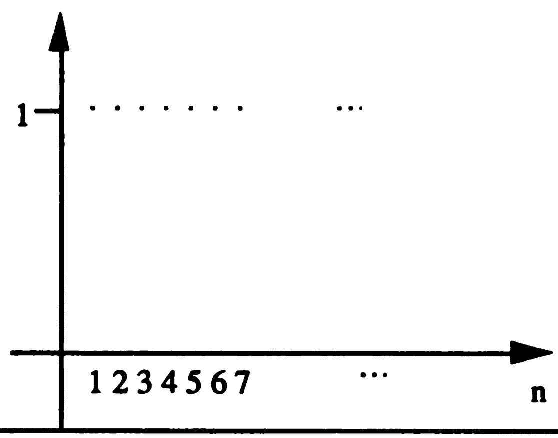
   c) $a_n = 1 + \frac{(-1)^n}{n}$

   d) $a_n = \begin{cases} n/n+1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases}$

   e)



   f)



Scoring System:

0 pt.-- Incorrect choice of A, B, C, and D.
1 pt.-- Correct choice of A, B, C, and D.
Examples:
If on #1-a a subject chooses D, or
If on #1-b a subject chooses A, or
If on #1-c a subject chooses B, or
If on #1-d a subject chooses B, or
If on #1-e a subject chooses D, or
If on #1-f a subject chooses B.

Table 5.2.-- Distribution of Raw Scores on Question #1

Item	0 points		1 point		Mean Score
	f.	r.f.	f.	r.f.	
#1-a	6	0.16	32	0.84	0.84
#1-b	7	0.18	31	0.82	0.82
#1-c	13	0.34	25	0.66	0.66
#1-d	19	0.50	19	0.50	0.50
#1-e	6	0.16	32	0.84	0.84
#1-f	8	0.21	30	0.79	0.79

From Table 5.2 we can see that prospective teachers did well on #1-a, #1-b, #1-e, and #1-f; the relative frequencies of these four items are 84%, 82%, 84% and 79% respectively. This is an indication that prospective teachers are familiar with numerical and graphical representations of sequences, and thus it is easy for them to identify the limits intuitively. However, only 66% of the subjects could identify the limit for #1-c and 50% could identify the limit for #1-d, indicating that the subjects had not done well on the rule-oriented sequences. This indicates that prospective teachers are troubled by the rule-oriented representation of sequences, although this formula type of representation is the most common kind of textbook exercise for limit problems. Examination of responses to the sequence #1-c: $\{a_n=1+\frac{(-1)^n}{n}\}$ revealed that 34% of the subjects did not recognize that the limit exists. One of the possible reasons could be that subjects confused this sequence with the divergent sequence #4-b: $\{a_n= (-1)^n + \frac{1}{n}\}$. In both cases the components are the combination of 1, $(-1)^n$, $\frac{(-1)^n}{n}$, and $\frac{1}{n}$; but the results are different. The terms of the first sequence clustered inward towards 1 from above and from below and two subsequences formed by the even term and odd terms both had the same limit 1. The terms of the second sequence clustered towards 1 and -1 and the two subsequences formed by the odd terms and even terms had different limits, namely 1 and -1. When examining #1-d, in which the sequence is given by the rule

$$a_n = \begin{cases} n/n+1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases},$$

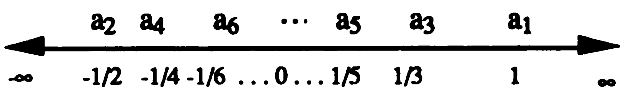
the other popular choice given by 42% of the subjects was also (D) which says the limit does not exist. Although in test #1 there is no indication of the underlying reason for this choice, this researcher suspects that it might be that rule-oriented sequences are not intuitively understood easily, unless the subjects were willing to list the numerical terms as in #1-a and #1-b or to draw the graphs as in #1-e and #1-f which makes it easy to reach the right conclusion; otherwise the subjects need to have a very good comprehension in order to be able to identify the required limits.

Question number two has two test items. Both of them are graphically represented; #2-a is one dimensional and #2-b is a two dimensional graph. In these items, not only did the subjects have to choose between having a limit or not, they also needed to provide a reasoning for that choice. The responses to the request for supporting reasons will provide evidence on whether the answer was pure guessing, which was all that was necessary to answer the previous #1 items, or there was some in-depth understanding there. Table 5.3 provides the test items of question #2 and the scoring system for this question. The raw scores on how well did the subjects perform on this question, based on the scoring system explained above, are given by the frequency, relative frequency, and means score in Table 5.4.

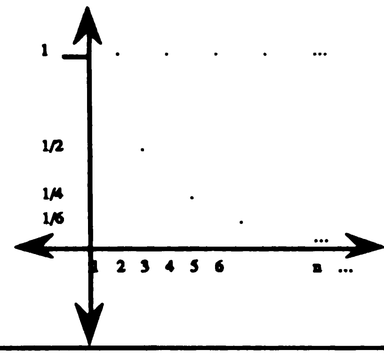
Table 5.3 -- Question #2 Test Items and Scoring System.

2. The following infinite sequences (a) - (b) are described by giving their graphs. Find what the limit is (if there is one) or indicate there is no limit. In both cases, please explain why.

a)



b)



Scoring System:

0 pt.-- No response or incorrect choice of limit exists or does not exist.
 Examples:
 If on #2-a a subject chooses a limit does not exist, or
 If on #2-b a subject chooses a limit does exist.

1 pt.-- Correct choice, but providing no explanation why L is the limit or why the given sequence does not have a limit.
 Examples:
 If on #2-a a subject chooses the answer that the limit exists and gives no response what the limit is, or

 If on #2-b a subject chooses the answer that the limit does not exist and provides the incorrect explanation that "not defined on all points"

2 pt.-- Correct choice with reasonable explanation for that choice.
 Examples:
 If on #2-a a subject chooses the answer that the sequence has limit 0, and provides the reason that because "We can see that we can get a_n as close to 0 as we want by taking n large enough", or

 If on #2-a a subject gives the general rule $a_n = (-1)^n \frac{1}{n}$, and provides the reason that as n-->∞, 1/n->0, or

 If on #2-b a subject says the limit does not exist because the given sequence does not converges to any one number, or

 If on #2-b a subject says the limit does not exist because when n is even, a_n decreases and when n is odd, a_n =1

Table 5.4--The distribution of raw score on question #2

Item	0 points		1 point		2 points		Mean Score
	f.	r.f.	f.	r.f.	f.	r.f.	
#2-a	6	0.16	3	0.08	29	0.76	1.61
#2-b	24	0.63	1	0.03	13	0.34	0.71

The relative frequency of the correct response being 76% in #2-a indicates that prospective teachers did well on finding the limit from the given one dimensional graph. However, 63% of the subjects could not recognize that the two dimensional graph given in #2-b did not have a limit. This shows that recognizing when a sequence does not have a limit is a relatively harder problem than guessing correctly at the limit of a convergent sequence, because a higher understanding of the underlying concepts is involved. In the present case in #2-b, the graph consists of two parts, one part formed by the even terms and the other part formed by the odd terms. One of the basic theorems about limits states that if a limit exists it has to be unique. Thus in order for this given sequence to have a limit, the two parts of this graph need to converge to the same value. But from the graph we can see that when n is even the dots decrease to zero in height and when n is odd all the dots are the same positive height. Only 37% of the subjects could provide the correct response for #2-b indicating that prospective teachers were not familiar with multiple descriptions of sequences. This finding matches Davis & Vinner's (1986) and Tall & Schwarzenberger's (1978) conclusions about students finding it difficult to deal with split domain sequences.

Test item #3 was the last kind of representation of sequences in this category, namely, geometrical representation. The subjects were asked to write down a sequence based on the fraction bars given in #3-a and were asked to find its limit. Two subjects provided a harmonic series instead of a sequence, and since it also fit the description of the question asked, both of the sequence and series were considered correct responses. Geometrical representations should be one of the components of mathematical knowledge that prospective teachers are most familiar with because finding the area of an irregular shape is one major task for the definite integral in calculus. Since #3-b asked the subjects to find the sum of an infinite series, which is something that can not be intuitively understood, test item #3-b was excluded from basic understanding and was included in the transitional understanding category. Table 5.5 provides the question of test #3-b with the scoring system. The raw scores on how well the subjects performed on this question based on the scoring system are given by the frequency, relative frequency, and means score in Table 5.6.

Table 5.5 -- Question #3-a Test Item and Scoring System.

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3. Figure (A) below illustrates a fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:
   a) Write down the infinite sequence formed by the individual shaded fraction bars in figure (B), and what is its limit?

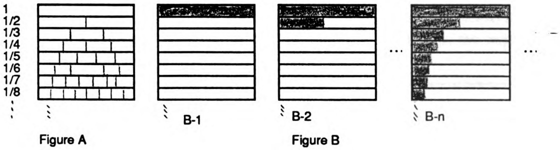

Table 5.5 -- Continued.

Scoring System:

0 pt.-- No response or incorrect response.
Examples:

If a subject responds that the sequence is {B-1/n} and the limit is B, or

If a subject responds that the sequence is {1/2, 2/3, 3/4, 4/5, 5/6, ...} and the limit is 1.

1 pt.-- Providing the correct sequence with incorrect limit or with no limit number given.
Examples:

If a subject states that the sequence is $\{a_n=1/n\}$, but the limit is 2, or some other finite number rather than the true limit which is 0; or

If a subject states that the sequence is $\{a_n=\sum_{k=1}^{k=n} 1/k\}$, but the limit is 2, or some other finite number rather than that this sequence is divergent; or

If the subject states that the sequence is $\{a_n=1/n\}$ with no limit value given, or

If a subject states that the sequence is $\{a_n=\sum_{k=1}^{k=n} 1/k\}$ with no limit value given.

1 pt.-- Providing the correct limit with incorrect sequence.
Example:
If a subject states that the sequence is $\{a_n=1/2^n\}$, but the limit is 0 which is true for both sequences. (Although the given sequence was not the one asked still the subject shows the ability to find limit, thus we score one point.)

2 pt.-- Providing the correct sequence with correct matching limit.
Examples:
If a subject states the harmonic sequence $\{a_n=1/n\}$ and says its limit is 0, or
If a subject states the harmonic series $\{a_n=\sum_{k=1}^{k=n} 1/k\}$ and says this sequence diverges or its limit is infinity.

Table 5.6 -- Distribution of Raw Scores on question #3-a in

Item	0 points		1 point		2 points		Mean Score
	f.	r.f.	f.	r.f.	f.	r.f.	
3-a	10	0.26	11	0.29	17	0.45	1.18

Less than half of the responses on #3-a were correct, indicating that subjects lack knowledge about geometrical representations. One reason could be the unfamiliarity of the type of representation, and the other could be the subjects are not familiar with the activity of fraction bars. There were 26% of the subjects who got the right sequence but provided the wrong limit and one subject got the right limit with the wrong sequence and thus scored one point. In the second research question, this researcher will discuss what might be the reasons that the subjects did not do well on some of the test items.

Category II: Computational Understanding

Computational Understanding includes the ability to use algebraic operations and basic theorems on how limits and algebraic operations interact (the theorems for limits of sums, differences, products, and quotients) to find limits of many sequences through computation. This ability is essentially algorithmic, rule-oriented, and computational in nature. The computational understanding of the limit concept should include being able to find limits of different types of sequences generated by a specific formula. The formulas which defined sequences included in present study can be categorized as: simple formula, rational formula, exponential formula, and radical formula. Of course, these do not exhaust all types of formula, but certainly include the formulas most commonly encountered in high schools. Table 5.7 provides the question of test number four with the scoring system. The raw scores on how well the subjects performed on this question based on the scoring system are given by the frequency, relative frequency, and means score in Table 5.8.

Table 5.7 -- Question #4 Test Items and Scoring System.

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4. In (a)- (e), select exactly one of the following answers: (Show your work or give explanation!)

   (A) The indicated limit is a finite number L. In this case, state specifically what the number is.
   (B) The indicated limit is ∞.
   (C) The indicated limit is - ∞.
   (D) The sequence does not have a limit (Which excludes ∞ and $-\infty$).

   a) $\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1}$

   b) $\lim_{n \to \infty} \{ (-1)^n + \frac{1}{n} \}$

   c) $\lim_{n \to \infty} \frac{3^{1-n}}{4^{1-n}}$

   d) $\lim_{n \to \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$

<u>Scoring System:</u>

0 pt.-- Incorrect choice with either wrong computation (#4-a, and #4-d) or with incorrect explanation (for #4-b and 4#-c), or no response.

Examples:

If on #4-a a subject chooses B and gives the incomplete computation

$\frac{3+5}{6+1} = \frac{8}{7}, \frac{12+10}{24+1} = \frac{22}{25}$, or

If on #4-b a subject chooses C and gives the following explanation

"As you plug ∞ in you get $-\infty + \frac{1}{\infty} = -\infty + 0 = -\infty$.", or

If on #4-c a subject chooses A and gives 1 for L with the computation:

$\frac{3^0}{4^0} = 1$, or

If on #4-d a subject chooses C and gives the following computation:

$$\lim_{n \to \infty} [(n^2+n)^{1/2} - (n^2-10n)^{1/2}]$$
$$= \lim_{n \to \infty} (n^2+n)^{1/2} - \lim_{n \to \infty} (n^2-10n)^{1/2}$$
$$= \lim_{n \to \infty} (1+\frac{1}{n})^{1/2} - \lim_{n \to \infty} (1-\frac{10}{n})^{1/2}$$
$$= 1-1 = 0$$

Table 5.7--Continued.

Scoring System:

1 pt.-- Correct choice (choose "A" for #4-a, "D" for #4-b, "B" for #4-c, and "A" for #4-d) with either wrong computation or incorrect explanation or no computation and no explanation.
Examples:

If on #4-a a subject chooses A and says $\frac{3+5}{6} = \frac{8}{6} = \frac{4}{3}$ for L, or

If on #4-b a subject chooses D and says the sequence does not converge as an explanation which did not provide any explanation at all, or
If on #4-c a subject chooses B and gives no explanation, or
If on #4-d a subject chooses A and gives 0 for L with no computation.

1 pt.-- Incorrect choice with correct computation or correct explanation.
Examples:
If on #4-b a subject chooses C, but provides the right sequence: {0, 3/2, -2/3, 5/4, -4/5,...}

If on #4-c a subject chooses D and provides the following computation:

$$\left(\frac{3}{4}\right)^{1-n} = \left(\frac{4}{3}\right)^{n-1}$$

$$r = \frac{4}{3} \quad |r| > 0$$

so the geometric sequence diverges.

2 pt.-- Correct choice, and correct number L (for #4-a) or correct explanation (for #4-b and #4-c) or for #4-d one last crucial computational error.
Examples:

If on #4-a a subject chooses A and gives $\frac{1}{2}$ for L, or

If on #4-b a subject chooses D, since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, or
If on #4-c a subject chooses B and gives the explanation that "the geometric sequence with ratio bigger than 1 is divergent", or

If on #4-d a subject chooses C and gives the following computation:

$$\frac{n^2+n-n^2-10n}{\sqrt{n^2+n} + \sqrt{1+10n}}$$

$$= \frac{-9n}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n}} + \sqrt{1+\frac{10}{n}}}$$

$$= \frac{-9}{2}n$$

Table 5.7 -- Continued.

Scoring System:

3 pt.-- Correct choice, correct number L, and correct computation (for #4-a and #4-d).
Examples:

If on #4-a a subject chooses A and gives $\frac{1}{2}$ for L, and the following computation:

$$\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1} = \lim_{n \to \infty} \frac{3 + 5/n}{6 + 1/n^2} = \frac{3 + \lim_{n \to \infty} 5/n}{6 + \lim_{n \to \infty} 1/n^2} = \frac{3}{6} = \frac{1}{2}$$

Since both $\lim_{n \to \infty} 5/n = 0$ and $\lim_{n \to \infty} 1/n^2 = 0$, or

If on #4-d a subject chooses A, and gives -9/2 for L, and the following computation:

$$\lim_{n \to \infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 + 10n} \right)$$

$$= \lim_{n \to \infty} \frac{\left(\sqrt{n^2 + n} - \sqrt{n^2 + 10n} \right)\left(\sqrt{n^2 + n} + \sqrt{n^2 + 10n} \right)}{\left(\sqrt{n^2 + n} + \sqrt{n^2 + 10n} \right)}$$

$$= \lim_{n \to \infty} \frac{(n^2 + n) - (n^2 + 10n)}{\left(\sqrt{n^2 + n} + \sqrt{n^2 + 10n} \right)}$$

$$= \lim_{n \to \infty} \frac{-9n}{\left(\sqrt{n^2 + n} + \sqrt{n^2 + 10n} \right)}$$

$$= \lim_{n \to \infty} \frac{-9}{\left(\sqrt{1 + 1/n} + \sqrt{1 + 10/n} \right)}$$

$$= \frac{-9}{\left(\sqrt{1 + \lim_{n \to \infty} 1/n} + \sqrt{1 + \lim_{n \to \infty} 10/n} \right)}$$

$$= -\frac{9}{2}$$

Since both $\lim_{n \to \infty} 1/n = 0$ and $\lim_{n \to \infty} 10/n = 0$

Table 5.8 -- Distribution of Raw Scores on #4 Test Items

Item	0 points		1 point		2 points		3 points		Mean Score
	f.	r.f.	f.	r.f.	f.	r.f.	f.	r.f.	
#4-a	14	.37	2	.05	3	.08	19	.50	1.71
#4-b	11	.29	4	.011	23	.61			1.32
#4-c	19	.50	4	.11	15	.39			0.89
#4-d	23	.60	12	.32	1	.03	2	.05	0.47

The decreasing of the mean scores of responses on items of test number four shows the relative difficulties of these four items. The relative frequency of prospective teachers who were able to solve #4-a: $\{\lim_{n\to\infty} \frac{3n^2 + 5n}{6n^2 + 1}\}$ is 50% and 61% were able to solve #4-b: $\lim_{n\to\infty} \{1 + \frac{(-1)^n}{n}\}$. As mentioned earlier #4-b is similar to #1-c: $\lim_{n\to\infty} \{1 + \frac{(-1)^n}{n}\}$ and the relative frequencies for both were quite close, namely, 61% and 66%, respectively. This researcher suspects that the 5% difference is due to the familiarity of the computational work. Only 39% of the prospective teachers could solve item #4-c: $\{\lim_{n\to\infty} \frac{3^{1-n}}{4^{1-n}}\}$. The difficulty of this item was to recognize the negative exponent. Lack of transfer knowledge (Putnam, 1987) probably was the main reason for the lower relative frequency. That only 5% of the prospective teachers solved #4-d: $\{\lim_{n\to\infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})\}$ indicated that most subjects were unfamiliar with radical forms of representation and/or did not know how to rationalize a radical form (Davis, 1982).

Category III: Transitional Understanding

Being able to compute and find the limit of a given function does not guarantee one will be able to understand the underlying concept of limit. Nowadays, calculators and computers can compute and find some limits faster than human beings. The basic understanding is sometimes not reliable because it is really only a conjecture based on looking at the first few terms of a given sequence. The computational understanding (or procedural knowledge) only enables one to compute the results mechanically. Transitional understanding enables one to provide conceptual knowledge for certain methods of finding the limit, for instance, the usual way for computing the limit of a rational formula by dividing the numerator and the denominator of the formula by the highest exponent terms. The transitional understanding should provide an adequate knowledge for explaining why they do problems the way they do. Thus in this category, the knowledge will make one familiar with some underlying concepts and prepare one to achieve more rigorous understanding. In this category, the subjects do not need to formally state what the definition of a limit is or to prove the theorems that have been used. But definitely, an informal definition should be possible for them to give. The subjects should be able to identify the symbolic meaning of the underlying subconcepts of each representation in the formal definition of limit. For example, given the $\varepsilon>0$, the subjects should be able to find the natural number N and be able to recognize that N is a function of ε. That is, they should understand that one needs to know ε in order to be able to find N. The subject should know when to use the Squeeze Theorem. The questionnaire test items for this category and the scoring system are given in Table 5.9. and the distribution of raw scores is given in Table 5.10.

Table 5.9 -- Test Items in Category III and Scoring System.

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3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:
   b) Write down the infinite sequence formed by the partial sums of the sequence in (a), and what is its limit?

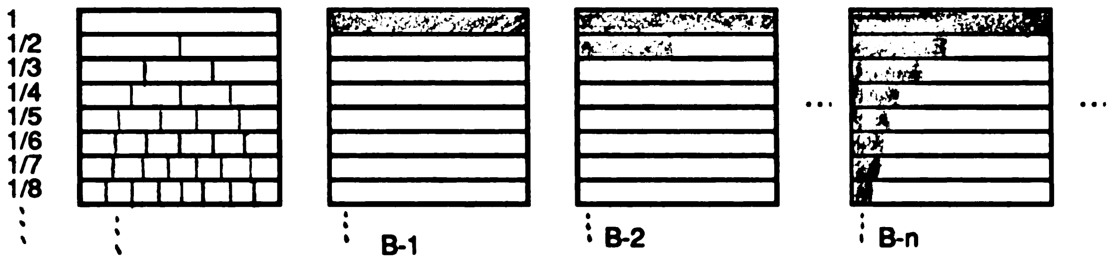


Figure A   Figure B

<u>Scoring System:</u>

0 pt.-- Incorrect sequence and incorrect limit.
Examples:

If a subject gives the general term of the sequence as $a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ and gives 0 for L, or

If a subject gives $a_n = \Sigma_{k=1}^{k=n} \frac{1}{2k}$ and gives 2 for the limit, or

If a subject gives $a_n = \frac{1}{n-1}$ and gives the expression $\lim_{n \to \infty} \frac{1}{n-1}$ for the limit, or

If a subject gives the sequence as: $\frac{1}{2} + \frac{2}{3}, \frac{3}{4} + \frac{4}{5}, \frac{5}{6} + \frac{6}{7}, \ldots$ and gives 2 for the limit.

1 pt.-- Correct sequence and incorrect limit.
Examples:

If a subject gives the sequence $\{a_n = \Sigma_{k=1}^{k=n} \frac{1}{k}\}$ and gives 2 for the limit or other finite numbers, or

If a subject gives the sequence $\{a_n = \Sigma_{k=1}^{k=n} \frac{1}{k}\}$ and gives no limit.

2 pt.-- Correct sequence and correct limit.
Examples:

If a subject gives the sequence is $\{a_n = \Sigma_{k=1}^{k=n} \frac{1}{k}\}$ and the sequence is divergent and has no limit, or

If a subject gives the sequence in the numerical representation as
1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4,..., 1+1/2+1/3+...+1/n,...
and gives the limit is positive infinity.

Table 5.10 -- Question #5 Test Items and Scoring System

5. The formal definition of the phrase " $\lim_{n \to \infty} a_n = L$, L is a finite real number" is as follows:

"For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L| < \varepsilon$ whenever $n > N$".

a) Illustrate the meaning of this definition, by using the sequence $\{a_n = \frac{2n}{n+1}\}$ with $\lim_{n \to \infty} a_n = 2$ on a graph.

b) According to the <u>formal</u> definition of limit, what would one have to show in order to prove $\lim_{n \to \infty} \frac{2n}{n+1} = 2$?

<u>Scoring System:</u>

0 pt.-- Incorrect graph (for #5-a) and incorrect response (for #5-b) or no response

Examples:

If on #5-a a subject gives an incorrect graph of the sequence and incorrect labeling of the limit on that graph, e.g. graph (a) in Fig 5.1, or

If on #5-b a subject gives an incorrect statement like "the limit of n approaches ∞"

1 pt.-- Continuous graph (for #5-a) and explanation with some ideas in it.

Examples:

If on #5-a a subject draws a continuos graph, e.g. graph (b) in Fig 5.1, rather a discrete graph, or

If on #5-b a subject gives the following formal definition explanation:

"For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L | < \varepsilon$ whenever $n > N$".

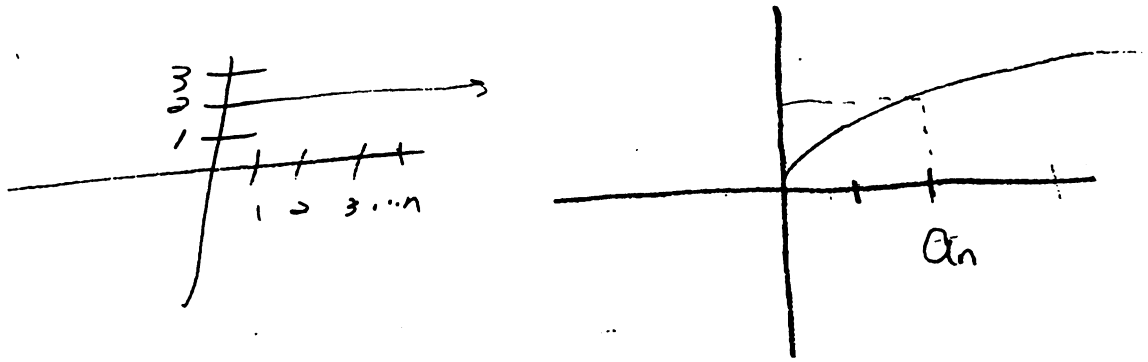
2 pt.-- Correct graph (for #5-a) and correct explanation.

Examples:

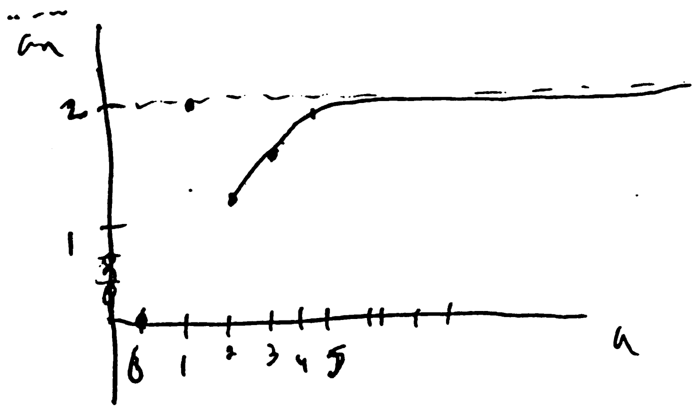
If on #5-a a subject draws the correct graph, or

If on #5-b a subject responds that " for any positive real number ε there is a natural number N such that $| \frac{2n}{n+1} - 2 | < \varepsilon$ if $n>N$", or

If a subject gives the following explanation "for $n>N$, the terms of the sequence all lie within ε units of 2".



(a)



(b)

Figure 5.1 -- Subjects' Graphs for Test Item #5-b

Table 5.11-- Question #6 Test Item and Scoring System

---

6. The infinite sequence a_n is defined by $a_n = \frac{6n-(-1)^n}{2n}$. Which of the following is the smallest N such that for n>N, a_n will be contained in an open interval of radius 1/500 about 3. (Show your work!)

____a)N=1000 ____b) N=500 ____c) N=250 ____d) N=125 ____e) N=100

Scoring System:

0 pt.-- No response or incorrect response.
Examples:

If a subject chooses N (=500), or

If a subject chooses N (=125), or

If a subject chooses N (=100).

1 pt.-- Correct choice for N (=250) with no work shown.

2 pt.-- Correct choice for N (=250) and correct work.
Examples:

If a subject chooses N (=250), and shows the following correct computation,

$|a_n - 3| = \frac{1}{2n}$

So $|a_n - 3| < \frac{1}{500}$ iff $\frac{1}{2n} < \frac{1}{500}$

Iff $2n > 500$

Iff $n > 250$

So N = 250, or

If a subject chooses N (=250) by plugging all the possible N's and concludes by looking at the patterns.

Table 5.12 -- Question #7 Test Item and Scoring System

---

7. Find $\lim_{n \to \infty} a_n$, given the information that the sequence $\{a_n\}$ satisfies

$$3n - 1 < n\,a_n < 3n + 2.$$

Scoring System:

0 pt.-- No response or incorrect response.
Examples:

If a subject says the limit is infinity, or

If a subject gives the following answer:

$$3n - 1 < n\,a_n < 3n + 2 \Longleftrightarrow 3\,(\infty) - 1 < (\infty)\, a_{(\infty)} < 3\,(\infty) + 2$$

$$3 < \infty\, a_\infty < 3 \Longleftrightarrow \text{This is a contradiction.}$$

1 pt.-- Gives answer $\lim_{n \to \infty} a_n = 3$ with no work shown.

2 pt.-- Correct limit found by using half the inequality.
Example:

If a subject gives the following computation:

$$a_n < \frac{3n+2}{n}\ ;\ \lim_{n \to \infty} \frac{3n+2}{n} = \lim_{n \to \infty} 3 + \frac{2}{n} = 3 + \lim_{n \to \infty} \frac{2}{n} = 3 + 0 = 3.$$

3 pt.-- Correct limit with correct computation.
Examples:

If a subject gives the following expression:

$\frac{3n-1}{n} < a_n < \frac{3n+2}{n}$ and looks for patterns by plugging in different values for n,
or

If a subject finds the limit and gives work by using the Squeeze Theorem.

Table 5.13 Distribution of Raw Score of Test Items on Category III

Item	0 points		1 point		2 points		3 points		Mean
	f.	r.f.	f.	r.f.	f.	r.f.	f.	r.f.	Score
#3-b	22	.58	10	.26	6	.16			0.58
#5-a	23	.61	10	.26	5	.13			0.53
#5-b	22	.58	9	.24	7	.18			0.61
#6	29	.76	1	.03	8	.21			0.45
#7	25	.66	2	.05	2	.05	9	.24	0.87

The test items asked in this category were preparatory knowledge for rigorous understanding of limit concept. The relative frequency on item #3-b, the fraction wall problem, indicated that only 16% of the prospective teachers could identify the harmonic series from the geometrical representation of a sequence. Actually 42% of the subjects did come up with the harmonic series, but they hold a finite view about the sum of harmonic series. That is, 26% of the subjects thought the sum of the harmonic series ($\sum_{k=1}^{k=n} \frac{1}{k}$ =1+1/2+1/3+1/4+...) is finite. The most agreed-on finite sum is 2. Thirteen percent of the prospective teachers were able to draw a correct graph to illustrate the meaning of existence of the limit, as in test item number #5-a. As a matter of fact, 39% of the subjects did draw a graph to illustrate the meaning of the sequence, but 26% of the subjects provided a continuous graph and thus scored only one point. That is, over one quarter of the subjects provided the continuous graph of a function rather than the discrete graph of a sequence. Eighteen percent of the subjects were able to state informally what one needs to know in order to prove a limit exists, as in test item number #5-b. And 24% of the subjects did provide an informal statement of what one needs to know in order to prove the statement of #5-b, but they were unable to distinguish between a general case versus a specific case.

Thus, what they did was to write down the formal definition stated in the question asked and they were scored one point for that. Twenty one percent of the subjects were aware of the importance of the choice of temporal order, and were able to produce an N when provided an ε, such as in test item #6. Twenty four percent of prospective teachers were able to apply the Squeeze Theorem and knew when and where to use it, as in test item #7. It seems that prospective teachers could do better dealing with the Squeeze Theorem than dealing with the temporal order.

Category IV: Rigorous Understanding

The rigorous understanding of the limit concept enables one to state the formal definition of a limit and the negation of the definition of limit as well as to use the definitions to prove certain sequences have or do not have limits. It also enables one to actually believe that the intuitively conjectured and computationally found limit of an infinite sequence can indeed be proved to be the limit. Sometimes students can provide a memorized proof of a statement, but do not really "buy it" in the sense of having been convinced of the result. Thus people who have rigorous understanding of the limit concept possess knowledge that not only enables them to understand the artificially designed problems provided by the mathematicians or the curriculum development, but also understand how these problems could be mathematically proved. Table 5.14, Table 5.15, Table 5.16, and Table 5.17 shows questionnaire test items in Category IV with the scoring system. The distribution of raw scores of the rigorous understanding is given in Table 5.18.

Table 5.14 -- Question #5 Test Item (c) and Scoring System.

---

5. The formal definition of the phrase " $\lim_{n\to\infty} a_n = L$, L is a finite real number" is as follows:

   "For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L | < \varepsilon$ whenever $n > N$".

   c) Using the <u>formal</u> definition of limit, prove that $\lim_{n\to\infty} \frac{2n}{n+1} = 2$.

<u>Scoring System:</u>

0 pt.-- No proof or incorrect proof or merely finding the limit of a given sequence rather than a proof.

Examples:

If a subject gives the following proof

$$\lim_{n\to\infty} \frac{2n}{n+1} = \frac{\lim_{n\to\infty} 2n}{\lim_{n\to\infty} n+1} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} 1+1/n} = \frac{2}{1} = 2, \text{ or}$$

If a subject gives $\int_{-\infty}^{\infty} 2n(\frac{1}{n+1}) = 2\int_{-\infty}^{\infty} n(\frac{1}{n+1}) =$, or

If a subject gives $\frac{2\infty}{\infty+1} = \frac{2\infty}{\infty} = 2$

1 pt.-- Incomplete proof

Examples:

If a subject shows part of the proof as

$$\lim_{n\to\infty} \frac{2n}{n+1} - \frac{2\,(n+1)}{n+1} = 0;\ \lim_{n\to\infty} \frac{-1}{n+1} = 0, \text{ or}$$

If a subject shows that $\left| \frac{2n}{n+1} - 2 \right| = \left| \frac{2n-2n-2}{n+1} \right| = \left| \frac{-2}{n+1} \right|$

2 pt.-- Slightly incorrect proof.

Examples:

If a subject shows how to find N for a specific choice of ε (ε=0.01) but not how to find N for general ε

3 pt.-- Correct proof.

Examples:

If a subject proves that $\left| \frac{2n}{n+1} - 2 \right| = \frac{2n+2-2n}{n+1} = \frac{2}{n+1} < \varepsilon$

$n+1<2/\varepsilon \qquad n<2/\varepsilon - 1$

$N=[2/\varepsilon-1]$

Table 5.15--Question #8 Test Item and Scoring System

---

8. Write down the formal definition of the negation of the limit of a sequence, that is, " $\lim_{n \to \infty} a_n \neq L$, where L is a finite real number".

<u>Scoring System:</u>

0 pt.-- Incorrect statement of definition.
Examples:

If a subject states that the definition is "The negation of a limit exists when a sequence approaches one value from below but a different one from above", or

If a subject states that "If the limit goes to L then the limit a_n is not equal to L".

1 pt.-- Statement with two quantifiers wrong.
Example:

If a subject states that "if there exists an ε such that there exists a natural number N such that $| a_n - L | \leq \varepsilon$ for each $n > N$"

2 pt.-- Statement with one quantifier wrong.
Example:

If a subject states that "there is an $\varepsilon > 0$ s.t. for all N, $| a_n - L | \geq \varepsilon$ when $n > N$."

3 pt.-- Correct statement.

Table 5.16 -- Question #9 Test Items and Scoring System

---

9. Suppose $a_n = \begin{cases} 1 \text{ for n odd} \\ -1 \text{ for n even} \end{cases}$

a) According to the formal definition of limit, what would one have to show in order to prove that $\lim_{n\to\infty} a_n$ does not exist?

b) Using the formal definition of limit, prove that $\lim_{n\to\infty} a_n$ does not exist.

Scoring System:

0 pt.-- Incorrect statement (for #9-a) and incorrect proof (for #9-b).
Examples:

If on #9-a a subject states that "showing that there is no one value for a_n for $n\to\infty$", or

If on #9-b a subject proves that:

$$\left|\frac{a_{n+1}}{a_n}\right| \neq \frac{a_n+1}{a_n} \quad \text{as } n\text{--}>\infty$$

1 pt.-- Statement and proof not by definition
Examples:

If on #9-a a subject states that "The limit of a_n would have to be $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} a_n = -1$. Since $1 \neq -1$, the limit would not be a unique one as it must be", or

If on #9-b a subject states that

$$\left| L - \begin{matrix} 1 \text{ for n odd} \\ -1 \text{ for n even} \end{matrix} \right| < \varepsilon$$

$L-1 < \varepsilon$ $\quad$ $L < \varepsilon + 1$
$L+1 < \varepsilon$ $\quad$ $L < \varepsilon - 1$
There is no N for which this will work.

2 pt.-- Statement and proof with one quantifier missing.
Example:

If on #9-a a subject states that "there exists $\varepsilon > 0$ s.t. no natural number N exists that satisfies $|a_n - L| < \varepsilon$."

3 pt.--Correct statement and correct proof.

Table 5.17--Question #10 Test Item and Scoring System

---

10. Using the formal definition of limit prove the following statement:

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} (a_n + b_n)$ exists and

$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Scoring System:

0 pt.-- Incorrect proof.

Examples:

If a subject argues as follows:

Assume $\lim a_n=0$ and $\lim b_n=0$

If we add $\lim a_n + \lim b_n = 0$ because $0 + 0 = 0$

Therefore $\lim (a_n+b_n) = 0$

So $\lim (a_n+b_n) = \lim a_n + \lim b_n$

If a subject argues as follows:

$\lim (a_n+b_n)$ exists: Since both limits exist you can combine them to make a true statement. However, if one were false, you could not do this.

$\text{Lim } (a_n+b_n) = \lim a_n + \lim b_n$: This is just using the distributive property. It is like saying: $A(x+y) = Ax + Ay$.

1 pt.-- Incomplete proof.

Example:

If a subject argues as follows:

$| a_n-L | < \varepsilon \; | b_n-L | < \varepsilon$

$| a_n+b_n | = | a_n-L+L-b_n | \leq | a_n-L | + | L-b_n | < \varepsilon/2 + \varepsilon/2 < \varepsilon$

2 pt.--Proof with one quantifier wrong.

Example:

If a subject argues as follows:

Let $\lim a_n = A$ then $| a_n-A | < \varepsilon/2$ when $n > N_1$

Let $\lim b_n = B$ and $| b_n-B | < \varepsilon/2$ when $n > N_2$

Let $N = \max(N_1, N_2)$

Let $\varepsilon > 0$ need to show $| (a_n+b_n)-(A+B) | < \varepsilon$

$| (a_n+b_n)-(A+B) | = | a_n-A + b_n-B | \leq | a_n-A | + | b_n+B | < \varepsilon/2+\varepsilon/2 = \varepsilon$ whenever $n>N$

3 pt.--Correct proof.

Table 5.18 -- Distribution of Raw Score of Test Items in Category IV

Item	0 points		1 point		2 points		3 points		Mean
	f.	r.f.	f.	r.f.	f.	r.f.	f.	r.f.	Score
#5-c	30	0.79	2	0.05	4	0.11	2	0.05	0.42
#8	34	0.89	2	0.05	2	0.05	0	0.00	0.16
#9-a	27	0.71	10	0.26	1	0.03	0	0.00	0.32
#9-b	36	0.95	2	0.05	0	0.00	0	0.00	0.05
#10	31	0.82	3	0.08	1	0.03	0	0.00	0.13

The relative frequencies on test items in category IV indicated a very low success. This result seems to match the conclusion from Fless and van Hiele that the fourth level is rarely reached by the majority of subjects. None of the subjects in this study were able to produce a correct response for items #8, #9, and #10. Five percent of the subjects did provide a perfect proof for item #5-c. There existed two common wrong methods for solving problems in the subjects' responses: the first one is that subjects used the algorithmic method for finding limit as a tool to prove a value is the required limit and the second common mistake is that most of the subjects were confused about the temporal order. That is, they did not really understand the relationship between ε and N. Particularly, they were confused by the quantifiers such as "all" and "some". It seems to them that in order to prove the theorem all one has to do is perform certain procedures which did not require any understanding of the underlying concept. This indicated that most subjects need to reinforce their knowledge in terms of understanding the rigorous definition of the limit concept.

The raw scores were then used to calculate percentage scores for each subject in each category. For example, subject number 1 scored 1 point on item #1-a, 1 on #1-b, 1 on #1-c, 0 on #1-d, 1 on #1-e, 1 on #1-f, 2 on #2-a, 0 on #2-b, and 2 on #3-a. Thus, his/her percentage score on category I of the test was

$$(1+1+1+0+1+1+2+0+2)/(1+1+1+1+1+1+2+2+2) \times 100$$

$$=(9/12) \times 100 = 75\%.$$

The percentage raw scores are given in Appendix D The frequencies, relative frequencies, and averages of the percentage scores for each category are given in Table 5.19. The Guttman scalogram scale method was used to form a scale of the data. Each category-subtest was treated as an item in a scale, with dichotomous categories of response. Based on these percentage raw scores, a scale like the Guttman scales with dichotomous categories of response was generated and is given in Table 5.20.

Table 5.19 -- Distribution of Percentage Scores By Categories

Percentage Score	Category I	Category II	Category III	Category IV
90-100	9	2	2	0
80-89	7	5	2	0
70-79	5	7	2	0
60-69	2	1	0	0
50-59	7	5	2	1
40-49	2	2	1	1
30-39	1	6	5	0
20-29	3	0	4	5
10-19	0	2	3	4
0- 9	2	8	17	28
Total	38	38	38	38
Mean(%)	66	44.5	27.5	7.72

Table 5.20 -- A Scale Like Guttman Scale Based on 90%, 80%, and 70% Performance Criterion

Response Pattern	Performance Criteria					
	90%		80%		70%	
	f.	r.f.	f.	r.f.	f.	r.f.
1 1 1 1	0	0.00	0	0.00	0	0.00
1 1 1 0	0	0.00	1	0.03	3	0.08
1 1 0 0	1	0.03	2	0.05	8	0.21
1 0 0 0	7	0.18	11	0.29	7	0.18
0 0 0 0	27	0.71	18	0.47	14	0.37
1 0 1 0	1	0.03	1	0.03	3	0.08
0 1 1 0	0	0.00	1	0.03	0	0.00
0 1 0 0	1	0.03	3	0.08	3	0.08
0 0 1 0	1	0.03	1	0.03	0	0.00
Total	38	1.01	38	1.01	38	1.00

In the present study the subjects at 70%, 80% and 90% performance criteria showed that:

1. None of the subjects' performances reached 70% criterion for all categories.
2. Only three subjects could reach 70% criteria for the first three categories test items.
3. Eight subjects could reach above 70% criterion for the first two categories.
4. Seven subjects could reach above 70% criterion for the first category test items.

Question 2: What kind of misconceptions, difficulties and errors do prospective secondary mathematics teachers have?

Errors do not occur randomly, but originate in a consistent conceptual framework based on earlier acquired knowledge (Nesher, 1987; Schwarzenberger, 1984). From the historical development of the concept of the limit and literature reviews, we know some misconceptions, difficulties, and errors were widespread among ancient mathematicians and present day students. The surprising discovery about misconceptions, difficulties, and errors is that they do not occur at random but in general occur for good reasons. Thus, this researcher was interested in finding out whether one of the reasons is due to the fact that prospective teachers possess the same kind of misconceptions, difficulties, and errors as students which were reported in several research studies. Hence the second research question is intended to investigate prospective teachers' misconceptions, difficulties, and errors on the subject matter knowledge of limit concept and is also addressed within the framework of the five-category theoretical model of understanding.

Category I: Basic Understanding

The nine basic test items on the questionnaire Part II from test number #1-a to #3-a were intended to investigate prospective teachers misconceptions, difficulties, and errors regarding the limit concept. The nine test items were designed to find how well prospective teachers understand the four different types of presentation of sequences, namely, 1) numerical representation by listing the first few terms of the sequence, 2) rule-oriented representation by giving a formula of the sequence, 3) graphical representation by giving the graph of a sequence either in one dimensional format or two dimensional format, and 4) geometrical representation by giving geometrical shapes of a sequence. Next the subjects representation, whether the sequences had limits or not and why. In calculus textbooks, the types encountered by students the most are the first two. Among these nine items there were some convergent sequences and there were some divergent sequences. For those sequences that were divergent, there were two types of sequences. One type was where the limit of a sequence is infinite and the other type was where the limit does not exist in the extended real number system. The items in the test could provide information about whether the subjects were able to distinguish between the limit not existing and the limit being infinity or negative infinity. There were items with a split domain, which usually caused trouble for recognizing the limit of the given sequence for many students (Davis, 1986; Tall, 1980). ("Split domain" means that the domain of a sequence was split into two sub-domains; for example, when the sequence is { $(-1)^n$ } then the terms of this sequence are divided into two groups, namely the terms having value positive number one and the terms having value negative number one. When using the representation of listing the terms, then the sequence is displayed as follow:

-1, 1, -1, 1, -1, 1, ...

On the other hand, the sequence could also be defined by a multiple description (or split domain) formula instead:

$$a_n = \begin{cases} -1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases}.)$$

From the analysis of the first research question, we know that 80% of the responses for question #1 test items were correct except for #1-c, and #1-d; the relative frequencies of correct answers for these two items were 66%, and 50%, respectively. Test items #1-c and #1-d were two sequences generated by formulas, which is the most common type of exercise in the usual chapter on infinite sequences. In Table 5.21 the responses to question #1 items were grouped and the correct responses were marked with asterisks *.

Table 5.21--The Distribution of Responses to Question #1

1. In the following infinite sequences (a) - (f), select exactly one of the following answers:
   (A) The indicated limit is 0.
   (B) The indicated limit is 1.
   (C) The indicated limit is -1.
   (D) The sequence does not have a limit (which includes ∞ and -∞).

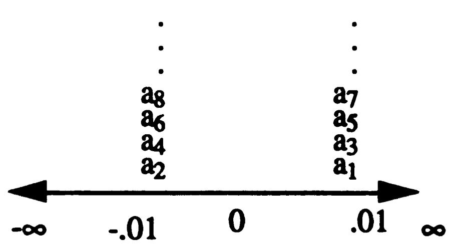
   a) 1, -1, 1, -1, 1, -1,...

   b) 3/4, 9/16, 27/64, 81/256, 243/1024, ...

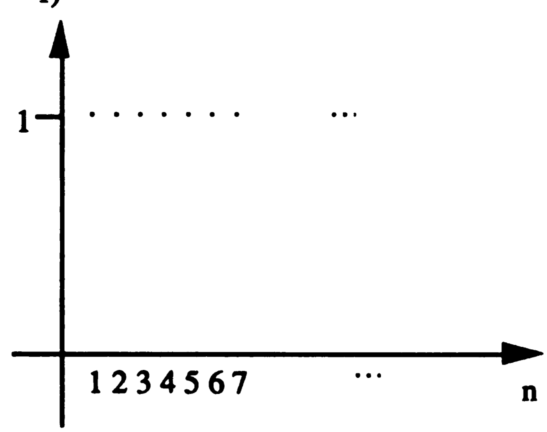
   c) 1 + 1/2, 1 - 1/3, 1 + 1/4, 1 - 1/5, 1 + 1/6, 1 - 1/7, ...

   d) $a_n = \begin{cases} n/n+1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases}$

   e)



   f)



Response	#1-a f.	r.f.	#1-b f.	r.f.	#1-c f.	r.f.	#1-d f.	r.f.	#1-e f.	r.f.	#1-f f.	r.f.
A	5	.13	31*	.82	2	.05	1	.03	2	.05	2	.05
B	1	.03	3	.08	25*	.66	19*	.50	0	.00	30*	.79
C	0	.00	0	.00	1	.03	1	.03	1	.03	0	.00
D	32*	.84	3	.08	9	.24	16	.42	32*	.84	5	.13
No response	0	.00	1	.03	1	.03	1	.03	3	.08	1	.0
Total	38	1	38	1	38	1	38	1	38	1	38	1

Note: * indicates the correct response.

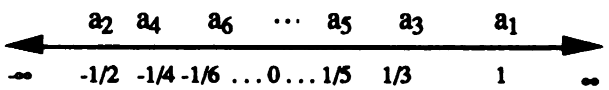
The relatively fewer correct responses for #1-c and #1-d were probably due to the fact that formulas were not intuitively understood easily. The other reason for #1-d is probably that the sequence was defined with split domain (in other word with multiple descriptions by more than one formula) which caused trouble in recognizing the limits. But since in question #1 the subjects were not asked to give reasons for their choices, this researcher could only conjecture based on the responses.

In order to overcome the above uncertainty, this researcher designed question #2, in which the subjects not only needed to find out whether the limits of the given sequences exist or not, but also needed to provide explanations. Table 5.22 presents the question #2 and the distribution of the responses.

Table 5.22 -- The Distribution of Responses for Question #2

2. The following infinite sequences (a) - (b) are described by giving their graphs. Find what the limit is (if there is one) or indicate there is no limit. In both cases, please explain why.

a)



b)

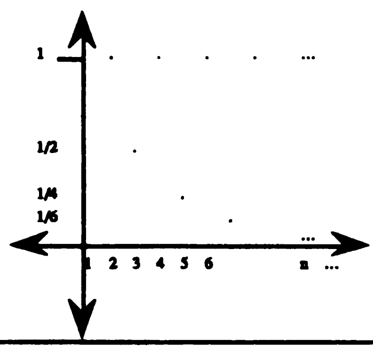


Table 5.22 -- Continued

Response	Item #2-a		Item #2-b	
	f.	r.f.	f.	r.f.
Limit Exists				
L=0 with computation	28*	0.74	13	0.34
L=0 without computation	4	0.11	2	0.05
L=1			1	0.03
Limit is infinity	2	0.05	2	0.05
Limit does not exist				
With explanation			13*	0.34
Without explanation			1	0.03
No response	4	0.11	6	0.16
Total	38	1.01	38	1

Note: * indicates the correct response.

Four subjects did not respond for question #2-a. Two subjects stated that the limit for the given sequence in #2-a is infinity. Among these two, one did not provide any reason, and the other stated that the limit was infinity because "they go on forever." The error made here perhaps was the subject might be thinking in the reverse direction, thinking of the terms as clustering inwards to zero instead of outwards to infinity.

In #2-b, 47% of the subjects provided an incorrect response. Among them 39% of the subjects responded that the given sequence has a limit 0 which is the limit of the sub-sequence formed by the even terms, and some of their reasonings were as follows:

1. The limit is 0, because "the dots get closer and closer to the x-axis (x=0)." (Here, the x-axis is represented by the function y=0, but instead the subject wrote x=0, which is an error. The other error is neglecting the other parts of the graph in which every term is 1.)

2. The limit is 0, because "as n goes to infinity, a_n goes to 0. Although will never be zero. Will get arbitrarily close to it, $\frac{1}{1000000}$. (Again the subject was neglecting the other part of the graph, same as the previous one.)

3. The limit is 0, because "the sequence is derived from 1/n. As n-->∞, 1/n gets smaller and smaller and eventually goes to zero." (Again the same error as above.)

For this group of subjects, the neglecting of part of the graph was the major reason for having the wrong response. This matches the research findings about the split domain functions causing learning troubles for students.

Two subjects thought the limit of the given sequence in #2-b is infinity which is incorrect, and they provided the following reasons:

1. The limit does not exist, because "you get closer and closer but never reach the x axis."

2. The limit does not exist, because "gets closer to n but doesn't ever touch."

For these two subjects, they had difficulties to distinguish between the limit is infinity and the limit does not exist in the extended real number system.

The above reasons definitely exhibited the dynamic viewpoint of limit concept which prevents prospective teachers from approaching the right understanding. This result is similarly to Tall & Schwarzenberger (1978) and Davis & Vinner's (1986) research findings on students.

From the analysis of the first research question, the researcher concludes that the geometrical representation of a sequence is unfamiliar to the students. Probably one of the reasons for #3-a having fewer correct responses is that the subjects had not seen this type of representation of sequences in textbooks. Table 5.23 presents the distribution of responses of test items #3-a in category I.

Table 5.23 --The Distribution of Response for Question #3-a

3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:
   a) Write down the infinite sequence formed by the individual shaded fraction bars in figure (B), and what is its limit?

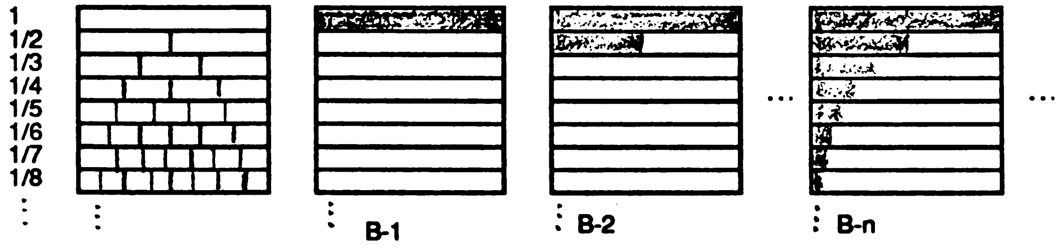


Figure A　　　　Figure B

Responses	Item #3-a	
	f	r.f.
The given sequence is $\{a_n=1/n\}$ (value of L)		
L=0	16*	0.42
L=2	1	0.03
The given sequence is $\{a_n = \Sigma_{k=1}^{k=n} \frac{1}{k}\}$ (value of L)		
L=∞	2*	0.05
L=2	4	0.11
No limit is provided	3	0.08
L=3	1	0.03
Either incorrect sequence and/or incorrect limit	5	0.13
No response	6	0.16
Total	38	1.01

Note: * indicates the numbers of the correct response.

The above table shows that 16% of the subjects provided no response. There were only two who thought the given sequence was a harmonic series and provided the answer that the limit is either infinity or does not exist in the extended real number system. Thirty five percent of the subjects thought that the harmonic series has a finite limit, which is incorrect. Thirteen percent of the subjects came up with an incorrect sequence by examining the geometrical representation. These incorrect responses were as follows:

1. The given sequence is {1/2, 2/3, 3/4, 4/5, 5/6, 6/7, ...} and its limit is 1.
2. The given sequence is B-(1/n) and its limit is B.
3. The given sequence is {1, 1/2, 1/4, 1/8, ..., $1/2^n$} and its limit is 0.
4. The given sequence is $\{\frac{1}{1+n}\}$ (and did not provide a limit).
5. Providing no sequence but giving the answer that the limit is 1.

One of the subjects knew the given sequence is {1, 1/2, 1/3, 1/4, ...}, but provided an answer that the limit is 2 based on the following figure:

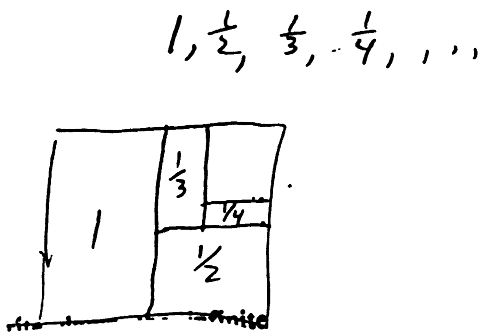


Figure 5.2 -- Subject's Drawing to Illustrate $\sum_{k=1}^{k=n}\frac{1}{k}=2$

What could be concluded from this figure? The researcher suspects that the reason was the subject confused the sequence $\{a_n = 1/2^n\}$ with the harmonic sequence like one other subject did. This confusion between two given sequences $\{a_n = 1/n\}$ and $\{a_n = 1/2^n\}$ made 11% of the subjects come to the incorrect conclusion. Probably the following usual diagram for illustrating of this geometrical series contribute the central confusion like one of subject did when providing the above figure.

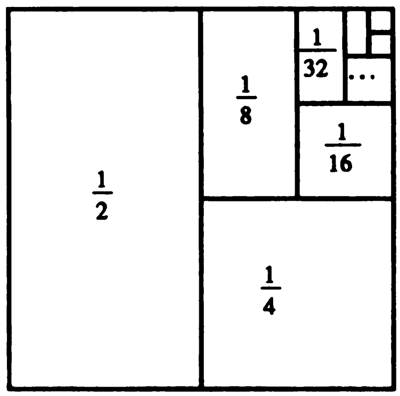


Figure 5.3 -- Geometrical Expression of $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1$

Category II: Computational Understanding

Items in this category were intended to measure the computational understanding of the limit concept. That is, ability to find the limit based on the theorems concerning the algebraic operations on limits. Question number 4 consisted of four sub-items, designed according to different laws for sequences. Usually, in mathematics a situation can be interpreted by a function or a sequence and we can represent this function (or sequence) by its general rule. The general rule of functions can be written as combinations of different algebraic formulas. Based on the different combinations, we name them as constant function, linear function, rational function, exponential function, radical function, quadratic function and etc. In this computational category, the test items only included four types. Item 4-a was a rational form, and was the kind most frequently encountered in textbook exercises. This is a typical limit problem. In order to find the limit, students usually have to divide the numerator and the denominator by the highest exponent term and then set 1/n and/or $1/n^2$ or the like equal to zero. This is one of the techniques for dealing with limit problems that students learn in a standard calculus course. The relative frequency of correct responses for this item was 53%, 10% of the subjects provided partial answers, 21% of the subjects made errors, and 16% of them were unable to make any response. The errors they made are listed as follows:

1. Plugging in a specific value for n:

   a) $\frac{3+5}{6} = \frac{8}{6} = \frac{4}{3}$, or

   b) Plugging ∞ into the equation and getting $\frac{\infty}{\infty} = \infty$.

2. Factoring n out: as $n\left(\frac{3n+5}{6n^2+1}\right)$ which gives the answer that sequence diverges.

3. Basing answer on a wrong conclusion to the pattern recognition step:
   One subject based his answer on these two terms, $\frac{3+5}{6+1} = \frac{8}{7}$
   $\frac{12+10}{24-1} = \frac{22}{25}$ and concluded that the sequence diverged.

Item #4-b was a sequence formed by the sum of two simple sequences, namely $a_n=(-1)^n$ and $b_n=1/n$. There were several ways for finding the limit of this sequence. One was listing the first few individual terms of this sequence and then intuitively deciding from there; another was using the theorem: the limit of the sum of the sequences is the sum of the limits of the sequences, provided the limits of the component sequences exist. Of course, there exist other methods such as using L'Hopital's rule. The relative frequency of the correct responses was 61%, 5% provided partial answers, 16% made errors and 18% were unable to provide responses. Eight percent of the subjects chose the answer that the limit of $\{ (-1)^n + \frac{1}{n} \}$ exists and one provided the answer that the limit is one. The reason given was that $\frac{1}{n}$ ->0. The other 8% chose the answer that the limit is -∞. The following were their reasons:

1 As you plug ∞ in you get -∞ +1/∞ =-∞ + 0 =-∞.

2 (-1) + 1/1=0, 1+1/2=3/2, -1+1/3=-2/3, +1+1/4=5/4, -1+1/5= -4/5. (Based on this pattern, this subject concluded that the limit is -∞.)

Item #4-c was a geometric sequence. Students also learned the formula for finding the limit by investigating the ratio. If the absolute value of the ratio is less than one, then this geometric sequence is convergent and the limit is zero, otherwise this geometric sequence is divergent and the limit is infinity. However, this test item was designed differently from the usual format; because the exponent was a negative number, the subjects had to convert the ratio into one with a positive exponent in order to find the result. Thus the relative frequency of correct responses compared with #4-a and #4-b was sharply decreased. It was 39% for the correct responses, 11% for the partially correct answers, 29% made errors, and 21% were unable to respond. In this item, over one quarter of subjects chose the answer that the limit of this sequence { $\frac{3^{1-n}}{4^{1-n}}$ } is finite. The errors made here are stated below:

1. $\frac{3^{1-n}}{4^{1-n}} = \frac{3^0}{4^0} = 1$

2. $\frac{3^{1-n}}{4^{1-n}} = (\frac{3}{4})^{1-n} = o \quad |\frac{3}{4}| < 1$

3. $\frac{3^{1-n}}{4^{1-n}} = \frac{3^\infty}{4^\infty} = 1$

Item #4-d was one of the common textbook exercises, but since the problem was in a radical form, it was necessary to find the conjugate first, and then to rationalize the expression next. Most students in algebra classes have difficulty simplifying this type of radical form. This was indeed the case with our subjects too; the relative frequency of the correct responses was only 5%. Thirty-two percent of the subjects were able to identify the limit as a finite number, but were unable to provide an adequate computation or explanation. Thirty-seven percent of the subjects tried to solve with no success, and 26% gave up trying. The common error made here is most of the subjects did not realize one needs to rationalize the radical form before finding the limit (Davis, 1982). Table 5.24 provides the distribution of the responses to test items in category II.

Table 5.24. Distribution of Responses to Question #4

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4. In (a)- (e), select exactly one of the following answers: (Show your work or give explanation!)
   (A) The indicated limit is a finite number L. In this case, state specifically what the number is.

   (B) The indicated limit is ∞.

   (C) The indicated limit is $-\infty$.

   (D) The sequence does not have a limit (which excludes ∞ and $-\infty$).

   a) $\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1}$

   b) $\lim_{n \to \infty} \{ (-1)^n + \frac{1}{n} \}$

   c) $\lim_{n \to \infty} \frac{3^{1-n}}{4^{1-n}}$

   d) $\lim_{n \to \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$

Table 5.24 -- Continued

Responses	#4-a		#4-b		#4-c		#4-d	
	f.	r.f.	f.	r.f.	f.	r.f.	f.	r.f.
A, 1/2, R	20*	.53						
A, 1/2, b	2	.05						
A, b, b	2	.05					12	.32
D, WE			23*	.62				
D, b			2	.05				
B, WE					15*	.39		
B, b					4	.11		
A, -4.5, R							2*	.05
None of the above	8	.21	6	.16	11	.29	14	.37
No response	6	.16	7	.18	8	.21	10	.26
Total	38	1	38	1	38	1	38	1

Note: 1 WE means with explanation; b means either wrong explanation or blank; R means right computation.
2 * indicates the correct response.

Besides the errors mentioned above, some errors produced by the subjects will be analyzed in detail. One of the most common errors was that subjects would ignore the indeterminate forms of some of the given sequences. "Indeterminate forms" are expressions involving the symbol ∞ in which the result can not be immediately determined by applying ordinary algebraic rules to the extended real number system; for example, one can not immediately decide what is the result of $\infty+\infty$, or $\infty-\infty$. The results of these operations could be any extended real numbers, sometimes a positive finite number, sometimes a negative finite number, sometimes positive infinity, sometimes zero, and sometimes negative infinity. That is why they are called indeterminate. Not all expressions involving the symbol ∞ are truly indeterminate; for instance it is perfectly legitimate to say $1/\infty = 0$ or $1/0 = \infty$. But in the present study, for example, in #4-a: $\frac{3n^2 + 5n}{6n^2 + 1}$, a subject got the answer ∞ by plugging ∞ into the equation to get $\infty + \infty = \infty$ which was wrong.

The following are more examples from some responses in item #4-d: ($\sqrt{n^2 + n}$ - $\sqrt{n^2 + 10n}$):

1 Subject simply substituted the infinity notation into the formula to get the answer 0 by $\infty - \infty = 0$, or

2 Subject wrote that $\sqrt{\infty} - \sqrt{\infty} = 0$, or

3 Subject first factored out the $\sqrt{n}$, then the equation was written as:

$$\sqrt{n}\,[\,\sqrt{n+1} - \sqrt{n+10}\,] = \infty \times (-\infty) = -\infty, \text{ or}$$

4 Subject wrote $\infty - 10\infty = -\infty$.

All the above examples exemplify some of the prospective teachers' notion of infinity as being such that the finite number operations of finite number algebra can be used on it.

The other error was the misuse of the technique for finding the limit. One subject used the technique mentioned earlier to divide both the numerator and the denominator by the highest exponent term, forgetting that this method only works for rational forms like #4-a: $\frac{3n^2 + 5n}{6n^2 + 1}$. This method does not work for #4-d, as in what one of the subjects wrote:

$$\lim\,[(n^2+n)^{1/2} - (n^2+10n)^{1/2}] = \lim\,(n^2+n)^{1/2} - \lim\,(n^2+10n)^{1/2}$$
$$= \lim\,\{(1+1/n)^{1/2} - \lim\,(1+10/n)^{1/2} = 1 - 1 = 0.$$

The third type error found in this study is carelessness of the subjects. For example, in doing #4-c, a subject wrote:

$$3^{1-n}/4^{1-n} = 3/4^{(1-n-1+n)} = (3/4)^0 = 1.$$

The researcher was sure that the subject was familiar with the exponential law; the error here was purely careless to think that the numerator and the denominator have the same base. In a second example, in working with #4-c the subject stated that "any fraction less than one to the power infinity will approach zero as n goes to infinity," but instead the subject wrote:

$$3^{1-n}/4^{1-n} = (3/4)^{(1-n)} = 0.$$

The exponent in this problem was negative rather than positive, so the negative sign in front of n means the exponent goes to negative infinity instead. This (-n) going to negative infinity causes the fraction to go to infinity rather than zero.

Category III: Transitional Understanding

Transitional Understanding, we recall, is a bridge or link lying between basic understanding and computational understanding on one hand and rigorous understanding on the other. A person possessing transitional understanding would know some of the basic theorems and would be able to apply them to specific examples. Thus these test items were designed to measure the prospective teachers' understanding of the component concepts of the ε-N definition for the limit of a sequence and the underlying concepts related to the limit operations. These are the concepts that formalize the basic notion of limit and serve as the basis for rigorous proofs involving limits.

Consider, for instance, the possible responses to test item #4-b where, as mentioned earlier, the sequence is generated as the sum of two sequences. There were many methods to study the convergence behavior of this sequence. Some of them might depend on the basic understanding; that is, the subject could list the first few terms and then intuitively examine the behavior of the sequence. What is lacking here was that the subject had no way to be sure that his/her answer was right, it was pure conjecture based on examining the first few terms. If the subject possesses a rigorous understanding, then he/she has the ability to conclude that the limit does not exist because one of the component sequences does not converge, i.e., limit of $(-1)^n$ does not exist. The second response involves making use of some theory, and reaching a conclusion with certainty, thus would be indicative of at least transitional understanding.

Students in calculus class often confuse infinite sequences with infinite series (Davis, 1982). Some of the subjects in this study found it difficult to differentiate between these two. Two of them claimed that they did not know what the term "partial sums" means. Thus #3-b, although it was given in a geometrical representation form, was considered as a transitional understanding problem, because it neither can be intuitively understood, nor can be computationally calculated. The relative frequency of the correct response indicated that only 16% of the subjects could find the limit when given a geometrical representation. Twenty-six percent of the subjects recognized that the sequence formed by the partial sums is the partial sum sequence for the harmonic series, but were unable to find the limit for this sequence or stated that the limit is a finite number. Eleven percent of the subjects were unable to identify the sequence and 47% gave up on providing information. The most common error in this item was for the subjects to hold a view that the harmonic series has a finite sum (See figure 5.1). One of the reasons for this view is probably because the subjects confused the harmonic series with the geometric series $\Sigma(1/2)^n$. Thus they provided the geometric series answer, which is 2, for the harmonic series. Actually when one compares the terms of these following sequences:

1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, ...

and

1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, ... ,

it is obvious that the values of the terms of the geometric sequence are getting smaller faster than those of the harmonic sequence. This suggests that the other probable reason for answering this item wrongly was that because the rate of increase of partial sums in the harmonic series is so slow, the subject did not think that in the long run the sum will diverge. Based on the comparison of these two series, some of the subjects concluded that

the sum of the harmonic series is finite. Table 5.25 presents the responses on test #3-b in Category III.

Table 5.25-- Distribution of Responses to Items #3-b

3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:
   b) Write down the infinite sequence formed by the partial sums of the sequence in (a), and what is its limit?

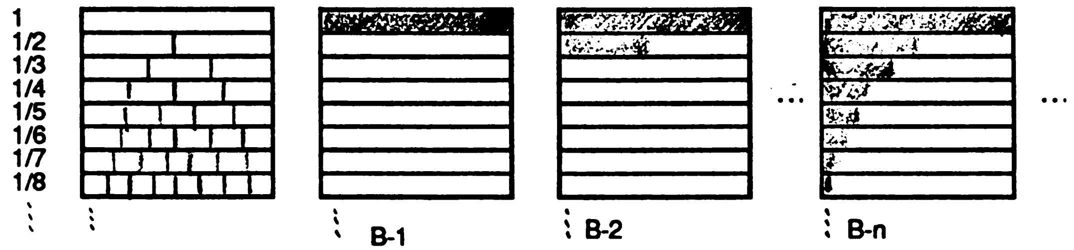


Figure A Figure B

Responses	#3-b f.	r.f.
The required sequence is $a_n = \sum_{k=1}^{k=n} \frac{1}{k}$		
L = infinity	6*	.16
L = 2	3	.08
L = 4	1	.03
No value for L	6	.16
Incorrect sequence		
$A_n = \sum_{k=1}^{k=n} 1/2^k$ with L= 2	1	.03
The sequence is 1/2+2/3, 3/4+4/5, 5/6+6/7,... with L=2	1	.03
$A_n = \lim_{n\text{->}20} \frac{1}{1-n}$	1	.03
No response	19	.50
Total	38	1.02

Note: * indicates the number of the correct response.

The ε -N definition of limit of a sequence involves many notations (ε, N, lim, n--> ∞, a_n, lim a_n, $\lim_{n->\infty} a_n$, >, <, $|a_n - L| < \varepsilon$), logical interpretations (for every ε, there exists, the relation between ε and N, the relation of the difference between a_n and L and ε), the set of positive numbers (n-->∞, what does n-->∞ means, the relation between ε and the positive numbers, the relation of the positive number N and ε , the relation between n and N), the quantifiers (when to choose "all" and "some"), and infinity (the relation between n and ∞, does n ever reach ∞, does whether n reaches infinity affect whether the limit exists or not?). In a typical calculus course usually the definition is given, the examples are written on the blackboard, the operation theorems are presented, the homework is assigned, and the learning of the limit concept is assumed to be completed (Carpenter & Romberg; 1986). In order to learn the definition of the limit of sequences, one needs prerequisites which include some of the sub-concepts mentioned above. Test items #5-a, #5-b, #6, and #7 were designed to measure this transitional understanding which provides the links between these prerequisites and the rigorous understanding.

The definition of limit of sequences was stated in item #5, because this definition is not easy to understand and/or to remember, as I stated in the previous paragraph. Besides it is not my intention to measure whether the subjects could state the formal definition, the intention was to find whether the subjects could provide another representation to explain a specific problem by the formal definition. Thus this item #5-a was intended to investigate whether the prospective teachers could graphically explain the limit concept. That is, based on the given formal definition will the prospective teacher be able to illustrate the limit concept for a specific sequence graphically. The relative frequency of the correct response was 21%. Twenty-four percent of the subjects were able to draw a graph to illustrate the meaning of the limit concept, but their graphs were continuous. Some of the examples are presented in Figure 5.4. That means one-third of the subjects were unable to differentiate between the domain of the positive integers and the domain of the real numbers. As Confrey (1980) states in her study, one of the earlier ancient mathematicians' deficiencies was inability to distinguish between the discrete and the continuous. She concludes in her study that, similarly, the students were unable to make this distinction. That so many prospective teachers drew continuous graphs is probably due to the practice of drawing functions in algebra and/or calculus courses. Five percent of the subjects drew wrong graphs and 50% provided no graphs. Table 5.26 presents the distribution of responses to test items #5-a and #5-b in category III.

Table 5.26 -- Distribution of Responses to Items #5-a and #5-b

5. The formal definition of the phrase "$\lim_{n \to \infty} a_n = L$, L is a finite real number" is as follows:

"For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L | < \varepsilon$ whenever $n > N$".

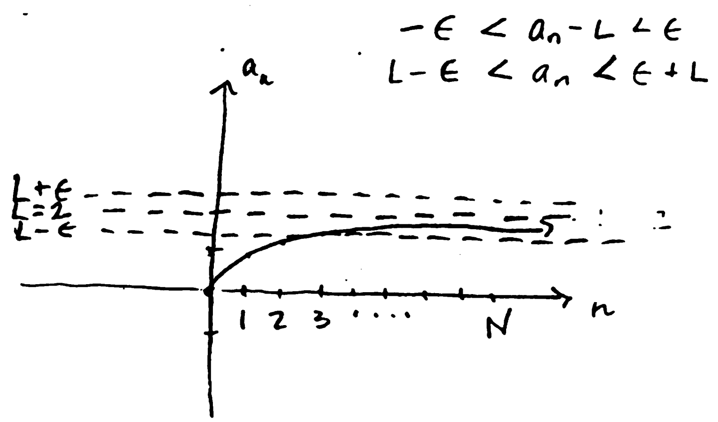
a) Illustrate the meaning of this definition, by using the sequence $\{a_n = \frac{2n}{n+1}\}$ with $\lim_{n \to \infty} a_n = 2$ on a graph.

b) According to the formal definition of limit, what would one have to show in order to prove $\lim_{n \to \infty} \frac{2n}{n+1} = 2$?

Responses	#5-a		#5-b	
	f.	r.f.	f.	r.f.
The correct graph	8*	.21		
Continuous but otherwise correct graph	9	.24		
Incorrect graph	2	.05		
Correct statement			7*	.18
Incomplete statement			9	.24
Incorrect statement			3	.08
No response	19	.50	19	.50
Total	38	1.00	38	1.00

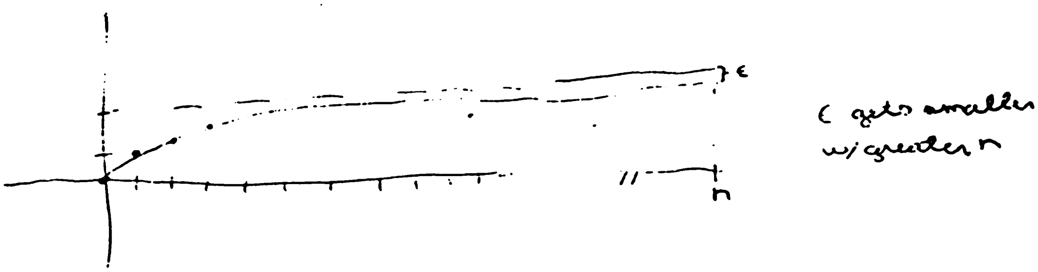
Note: * indicates the number of the correct response.

(a)

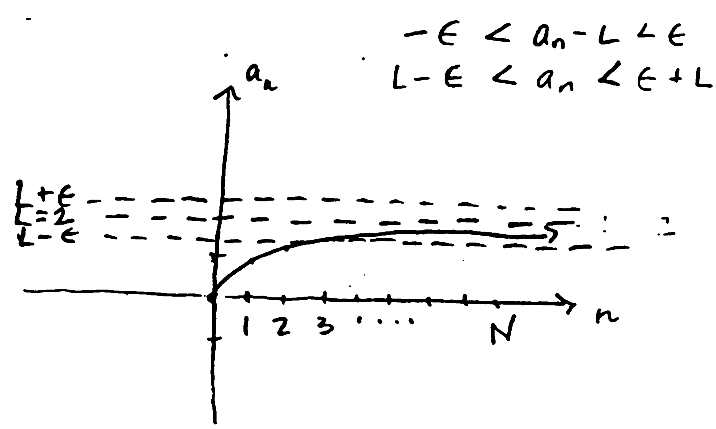


(b)

Figure 5.4 -- Continuous Graphs of a Discrete Function



(a)



(b)

Figure 5.4 -- Continuous Graphs of a Discrete Function

Item #5-b was intended to examine whether when provided a formal definition prospective teachers would be able to state it informally. It turned out that only 18% of the subjects could explain the given problem informally based on the formal definition. Twenty four percent repeated the formal definition given without being able to transfer the general case to a specific sequence, 8% presented wrong explanations and half of the subjects provided no responses. Being unable to explain the limit concept informally in a specific sequence indicated lack of understanding of the formal definition of the limit. Probably the knowledge the prospective teachers possessed was more of the kind for which Hilbert and Lefevre (1986) use the term "procedural". Thus when the subjects encountered the limit problem in a different guise, they were not able to use the usual procedures well.

Test #6 was intended to explore whether the prospective teachers understand the importance of the temporal order in the formal definition. One of the important components in the formal definition of limit is the choice of ε and N. Although most calculus students know the definition that "for every e greater than zero, there exists a natural number N... ", few really understood what that statement means. That is, students do not understand the relationship between ε and N. Especially when dealing with the proofs of propositions about limits, students are usually confused over which one should be specified first. From the research findings this was a critical deficiency for understanding the limit concept among students. The question arises; is there a discrepancy between these two teaching and learning groups? Thus this researcher designed test #6; in it, the formula of the sequence was given, the limit was given and the e was given, the subject was asked to find the natural number N. The responses of the subjects should provide information about how well the relationship of ε and N is understood. The relative frequency of the correct responses was 21%, while 3% were able to provide a partial answer. Nineteen percent of the subjects failed to provide the right relationships and 59% of the subjects did not respond. The present study's findings on deficiency in realizing the importance of the temporal order were similar to the research findings on students (Davis & Vinner, 1986). Table 5.27 shows the distribution of responses to test #6.

Table 5.27 -- Distribution of Responses to Test Item #6

6. The infinite sequence a_n is defined by $a_n = \frac{6n - (-1)^n}{2n}$. Which of the following is the smallest N such that for n>N, a_n will be contained in an open interval of radius 1/500 about 3. (Show your work!)

_____a)N=1000 ____b) N=500 ____c) N=250 ____d) N=125 ____e) N=100

Responses	#6 f.	#6 r.f.
N=250		
With the correct computation	8*	.21
With no computation or explanation	1	.03
N=500	4	.11
N=125	1	.03
N=100	2	.05
No response	22	.59
Total	38	1.02

Note: * indicates the frequency to the correct response.

Test #7 was designed to examine one of the basic theorems -- the Squeeze Theorem (also called Sandwich Theorem) which is a simple but useful idea. What this theorem says is that if sequences $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ satisfy $A_n < B_n < C_n$ (which means the value of A_n is less than B_n which in turn is less than C_n), and if the limits of sequences A_n and C_n exist and both equal L, then the limit of the middle sequence B_n is also L. The ancient Greeks used this idea in their inscribed and circumscribed polygons approach to the problem of approximating the area of a unit circle. The definite integral used in calculus was evolved from this idea along with the notion of the limit concept. Davis (1985) conducted a study to investigate students' notion of integration and found that one subject, named Lucy, could find the upper sum but was unable to find the lower sum. This indicated a deficiency in understanding the content of the Squeeze Theorem. In the same way, in the present study one of the subjects could find the limit in the sequence given in item #7, but from one side only rather than from both sides by using the Squeeze Theorem. The relative frequency of the correct responses was 26%, 3% using the one-side method to find the required limit is 3, and 5% only provided answer 3 without explaining how the answer was derived. Fourteen percent of the subjects could not perform to get the right response and 55% left the item blank. Table 5.28 shows the frequency of the responses to item #7.

Table 5.28 -- Distribution of Responses to Test Item #7.

7. Find $\lim_{n \to \infty} a_n$, given the information that the sequence $\{a_n\}$ satisfies $3n - 1 < n\, a_n < 3n + 2$.

Responses	#7	
	f.	r.f.
The required limit is 3		
With correct computation	10*	.26
With half the inequality	1	.03
With no computation	2	.05
The limit is infinity	3	.08
Other incorrect response	2	.05
No response	20	.53
Total	38	1.00

Note: * indicates the percentage of the correct response.

Category IV: Rigorous Understanding

Rigorous understanding enables one to use the precise definitions with an emphasis on exact deductive mathematical proofs. Test Items in this category required the subjects to prove a specific sequence is convergent (item #5-c), to give the formal definition of the negation of the limit concept (item #8), to state what one would have to say informally in order to prove a specific sequence is divergent (item #9-a) and formally prove a given sequence is divergent (item #9-b), and finally, to prove the limit of a sum of sequences was the sum of the limits (item #10). Since the definition of limit is so complicated and involves so many prior concepts, the prospective teachers as well as students in other studies (Davis & Vinner, 1986; Fless, 1988; Orton, 1983a, 1983b; Williams, 1989), demonstrated little understanding of the formal definition of limit. The relative frequency of the correct responses on these five items #5-c, #8, #9-a, #9-b and #10 were 5%, 0%, 0%, 0% and 0%, respectively. No completely valid proofs were produced by the subjects on item #8, item #9, and item #10. Eleven percent of the subjects produced proofs on item #5-c with one wrong quantifier, and 5% demonstrated proofs with two wrong quantifiers. Twenty-one percent produced completely incorrect proofs, and 58% produced no response. Table 5.29 presents the distribution of responses to test #5-c in Category IV.

Table 5.29 -- Distribution of Responses to Items #5-c

5. The formal definition of the phrase "$\lim_{n \to \infty} a_n = L$, L is a finite real number" is as follows:
"For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L | < \varepsilon$ whenever $n > N$".

c) Using the formal definition of limit, prove that $\lim_{n \to \infty} \frac{2n}{n + 1} = 2$.

Table 5.29 -- Continued

Responses	#5-c f.	r.f.
Correct proof	2*	.05
Incomplete proof with one wrong quantifier	4	.11
Incomplete proof with two wrong quantifiers	2	.05
Completely incorrect proof	8	.21
No response	22	.58
Total	38	1.00

Note: * indicates the numbers of the correct response.

As for question #8, 5% of the subjects stated the negation of the limit definition with one wrong quantifier, 5% stated the negation of the limit definition with two wrong quantifiers, 13% tried to state the negation of limit definition with no success and 76% provided no response. The statement of the negation of the limit definition was assumed to be easy when provided with the formal definition of limit, but this turned out not to be the case. Probably due to the lack of conceptual understanding (Hilbert & Lefevre, 1986), none of the subjects could transfer the knowledge (Putnam, 1987) on the limit definition to the negation of limit definition. Table 5.30. shows the distribution of responses to test item #8.

Table 5.30. Distribution of Responses to Test Item #8

8. Write down the formal definition of the negation of the limit of a sequence, that is, "$\lim_{n \to \infty} a_n \neq L$, where L is a finite real number".

Table 5.30 -- Continued

Responses	#8	r.f.
	f.	.00
Correct definition	0*	.05
Incomplete definition with one wrong quantifier	2	.05
Incomplete definition with two wrong quantifiers	2	.13
Completely incorrect definition	5	.76
No response	29	1.00
Total	38	

Note: * indicates the numbers of the correct response.

Test item #9-a, was intended to investigate prospective teachers' understanding of the informal description of the negation of the limit definition as applied to a specific sequence. The ability to explain informally, based on the formal negation of the definition of the limit, why a specific sequence does not have a limit is crucial for the prospective teachers. This could link the intuitive notion of divergence with the rigorous definition of the negation. None of the subjects could state in an informal way why this specific sequence did not have a limit. Eight percent of the subjects provided statement of what should one have to show in order to prove the given sequence does no have a limit with one wrong quantifier, and 21% of the subjects provided response with two wrong quantifiers. One subject's response was like this, "there exists $\varepsilon > 0$ such that no natural number N exists that satisfies $|a_n - L| < \varepsilon$", which we scored one point because this subject did not mention any relationship between n and N which this researcher considered as two wrong quantifiers. Eleven percent of the subjects provided an incorrect statement and/or a statement with at least three wrong quantifiers, and 61% gave no response. Table 5.31 presents the distribution of responses for test item #9.

Table 5.31 -- Distribution of Responses to Test Item #9

9. Suppose $a_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ -1 & \text{for } n \text{ even} \end{cases}$

a) According to the formal definition of limit, what would one have to show in order to prove that $\lim_{n \to \infty} a_n$ does not exist?

b) Using the formal definition of limit, prove that $\lim_{n \to \infty} a_n$ does not exist.

Response	#9-a f.	#9-a r.f.	#9-b f.	#9-b r.f.
Correct statement	0*	0.00		
Incomplete statement with one wrong quantifier	3	0.08		
Incomplete statement with two wrong quantifiers	8	0.21		
Completely incorrect statement	4	0.11		
Correct proof			0*	0.00
Incomplete proof with one wrong quantifier			0	0.00
Incomplete proof with two wrong quantifiers			2	0.05
Completely incorrect proof			5	0.13
No response	23	0.61	31	0.82
Total	38	1.01	38	1.00

Note: * indicated the correct frequency of the responses.

None of the subjects could produce a correct proof for #9-b, and none of them even come close to the right proof. Five percent of the subjects constructed proofs with two wrong quantifiers, 13% of the subjects failed to produce the correct proofs and 82% of the subjects provided no response.

The last test item #10 asked the subjects to prove the very basic theorem that the limit of a sum is the sum of the limits. One subject constructed a proof with one wrong quantifier, 8% of the subjects produced a proof with two wrong quantifiers, 18% produced completely wrong proof, and 71% of the subjects provided no response. Table 5.32 presents the distribution of responses to test items #10 in category IV.

Table 5.32. Distribution of Responses to Test Item #10

10. Using the formal definition of limit prove the following statement:

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty}(a_n + b_n)$ exists and

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Responses	#10 f.	r.f.
Correct proof	0*	.00
Incomplete proof with two wrong quantifiers	1	.03
Incomplete proof with three wrong quantifiers	3	.08
Completely incorrect proof	7	.18
No response	27	.71
Total	38	1.00

Note: * indicates the frequency of the correct response.

The relative frequencies of no response in this category of #5-c, #8, #9-a, #9-b and #10 were 58%, 76%, 61%, 82% and 71%, respectively. The very high frequencies of no response in category IV indicates probably that in mathematics courses students were more familiar with verifying the limits than proving the theorems. The other possibility is that prospective teachers, similarly to the students, find rigorous understanding of the limit concept difficulty to attain (Fless, 1988; Williams, 1989).

Summary

In this section, the results of the responses to the questionnaire have been analyzed. The following research question was discussed within the framework of the theoretical model constructed in Chapter Four: What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the limit concept?

In regard to this question, numerous misconceptions, difficulties, and errors were observed in all categories. In category I: Basic Understanding, some misconceptions, difficulties, and errors involved in the notion of limit were as follows:

1. The difficulty in realizing the rule of a given sequence can have a multiple description. Thus when provided with a split domain problem, the subjects tended to pick out half of the problem and overlook the other half of the problem.
2. The difficulty in interpreting information from a graphical representation. This was probably due to the intrusion of the split domain difficulty, but certainly the lack of knowledge to interpret the graphical representation was demonstrated in the subjects' responses.
3. The difficulty in interpreting information from a geometrical representation. For example, in #3-a, subjects could not interpret the geometrical representation, either in terms of the individual fraction bars, or in terms of the shaded areas.
4. The difficulty in differentiating between when the limit is infinite and when the limit does not exist in the extended real number system. Although in the textbooks, the definition of divergent is "not convergent," which means the limit is not a finite number, in later chapters in the calculus textbooks the textbook authors do introduce the concept of a sequence having limit infinity, which means the terms of the sequence eventually exceed any pre-assigned number. Based on the definition of convergence, it means the terms of a

sequence converge to a fixed number. When the limit of a sequence is infinity, the terms of the sequence do converge to infinity. But, for example, the limit of sequence $(-1)^n$ does not exist because it does not converge to any fixed number or to infinity. Thus "the limit is infinity" is a different statement from "the limit does not exist."

In Category II: Computational Understanding, the nature of the misconceptions, difficulties, and errors were analyzed based on the common types of errors they produced.

1. A major difficulty involved in finding the limit for all items in this category was lack of the ability to solve the indeterminate forms appearing in the problems. It seems that some subjects were not bothered at all by the indeterminate forms. Infinity, to them, is a fixed number that can be added, subtracted, multiplied and divided by. These four operations on infinity seemed so natural they became part of the daily algebraic operations.
2. Another difficulty seems to be misuse of the technique of dividing both numerator and denominator by the highest exponent term.
3. The third difficulty seems to be not realizing the need for rationalizing the radical form.
4. Some intrusion of the dynamic expression (as $n \to \infty$, a_n gets closer and closer to L but never reaches it) of the limit concept can be seen in some subjects' reasoning. This intrusion prevents some of the subjects from grasping the static form of the formal definition. Thus, they only see part of the graph instead of a global view.

In general, given a stated condition involving the terminology and notation associated with a formal definition, prospective teachers should be able to use the precise notation and terminology in a meaningful way. But in Category III: Transitional Understanding, it turned out that is not the case. The errors made by prospective teachers are described as follows:

1. Inability to distinguish between infinite sequences and infinite series.
2. The continuous but otherwise correct graph illustrated the error they made by misinterpreting the graph of the function whose domain is the set of positive numbers.
3. Not being sure when the Squeeze Theorem could be used.
4. Inability to grasp how formal definitions capture the intuitive notion of the limit concept. Therefore their informal notion of limit concept demonstrated misconceptions about the temporal order, which means they tended to

choose N first instead of ε. Their responses showed it was difficult for them to produce N when provided with ε.

Similarly, the inability to grasp the meaning of the formal definition of limit produced a relatively low frequency of correct responses in category IV: Rigorous Understanding. As mentioned earlier, and as this researcher intends to say again, the important kind of understanding of a concept is <u>not</u> learning by memory the formal definition of that concept, but understanding the underlying components of that concept. What one needs is not just to be able to state the formal definition of the limit, but rather to be able to explain the meaning of the definition. So, in this category, the major error was being unable to distinguish the important role of the temporal order, thus most of the subjects could not provide exact correct quantifiers which constitutes the definition of the limit. Thus how to choose ε, N the quantifiers "all" and "some", and solving the inequality become obstacles for constructing the correct proofs.

Question 3: What are prospective teachers' opinions about the role of the limit concept in K-12 mathematics curriculum

In this section, the focuses are: what is the role of limit concept in K-12 mathematics curriculum, where and how the limit concept is, implicitly and explicitly, revealing itself in K-12 mathematics curriculum, and what is prospective teachers' understanding about the curriculum knowledge regarding the limit concept. In order to address the third research question, an open-ended question was embedded in the survey. Subjects were asked to provide an activity that could be used to introduce the intuitive notion of limit to 1) K-2 grade range and 4-5 grade range children. Among the thirty-eight prospective secondary teachers who took the questionnaire, there were only thirteen subjects who responded to the K-2 grade range question and fourteen of them who responded to the 4-5 grade range question. The data received will be presented but there were not enough responses to make accurate generalizations related to this research question. Prospective teachers' responses after grouping is given in Table 5.33.

Table 5.33 The Distribution of Responses of Test Item #9, Part I

Description of Activity	Grade K-2		Grade 4-5	
	f.	r.f.	f.	r.f.
Half Division	4	0.11	3	0.08
Binary Tree	2	0.05	0	0.00
Science Activity	1	0.03	3	0.08
Counting to Infinity	3	0.08	1	0.03
Game of Hopscotch	1	0.03	0	0.00
Graph	1	0.03	2	0.05
Series	0	0.00	1	0.03
Fraction	0	0.00	2	0.05
Irrational number π	0	0.00	1	0.03
I don't know	1	0.03	1	0.03
No response	25	0.66	24	0.63
Total	38	1.02	38	1.01

Next, I will describe the categories in Table 5.33. The "Half Division" activity means that the subjects introduced an activity involving something like the grasshopper activity in questionnaire Part II item number 13. This half division activity could involve cutting a piece of paper in half repeatedly, or cutting a piece of pie diagram in half, one fourth and one eighth and etc., or taking away half of a fixed amount over and over, or walking half the distance to a wall, then half again and so on. Actually this is a very rich activity. It not only provides an opportunity for the young children to visualize the individual pieces getting smaller and smaller, but also at the same time introduces the notion of infinity because the numbers of the pieces are getting more and more. The main thing missing in the responses for this category is that none of the subjects mentioned anything about follow-up questions regarding this activity. For instance, can this process go on forever? What will happen if this process goes on and on? If this process goes on and on, what the piece will look like? How many of these individual pieces exist?

The "Binary Tree" activity mentioned by two subjects, both of them using the family tree to illustrate the activity to K-2 grade range, involved the binary growth of the family which in turn produces a divergent sequence, because if parents have children, who have children, who have children, etc., as this process continues it produces an exponential growth, which is a growth without bound. Both of the subjects mentioned using this family tree activity to introduce the notion of infinity to K-2 grade range children. This binary tree activity is indeed an excellent activity for young children, first because the exponential growth is very fast, so that the child can see the numbers getting big quickly, and, secondly, children are familiar with the family tree which directly relates to their daily experiences.

The "Science Activities" mentioned here were activities related to science experiments. For instance, blowing up a balloon until it bursts, adding a drop of water at a time to an empty swimming pool, determining how much water can be put into an 8 ounce paper cup, and putting two mirrors facing each other and letting the children observe the continuous reflections, were four responses. These four activities provide different time schedules in turn providing different intuitive understandings of the notion of limit. The blowing balloon and filling cup activities can be immediately accomplished. This will introduce the idea that in a way the limit is a bound; when that boundary is exceeded either the balloon will burst or the water will overflow. The adding a drop of water at a time to an empty swimming pool activity could be a "thought experiment." Most of the children know that is possible because they can perform this activity at home by observing that the sink or bath tub were filled by drops of water, it is only a matter of time. The last activity, the reflecting mirrors, really captures the essence of the notion of limit, introducing that there is an infinite process going on and on, and the children can experience it. It will even better exhibit the limit concept, if one object is placed in front of the mirrors. This will provide the opportunity to observe the object getting smaller and smaller and/or getting further and further away in the reflections.

The "Counting to Infinity Activities" were activities of counting numbers that never stop. For example, counting the natural numbers, or adding the counting numbers, or showing a sequence on the number line and showing where one would end up on the number line if one keeps going and going. These activities definitely are good examples to introduce the notion of infinity.

Of those thirteen one subject mentioned using the "Game Activity" which says "compare the limit to a game of hopscotch where the final square is very far away and they keep getting closer and closer, but it is always just beyond their reach."

One subject mentioned graphing as an activity in both K-2 grade ranges and 4-5 grade range and presented the graph activities by using statistics data. For example, in responding to the question about the K-2 grade range the statistical data mentioned were:

1. graph the number of hot lunches in their class or the number of absences,
2. discuss what is the most one could have,
3. discuss what is the least, and
4. discuss the range of students in class.

In regard to 4-5 grade range the statistics data mentioned were:

1. graph the daily temperature,
2. discuss the hottest it ever gets in this geographical area, and

3. graph record highs and lows.

The statistical data about maximum, minimum, range, and graphs can help children understand about functions with finite domain, but it cannot help them to understand the notion of limit because there is no continuous process going on. The other subject mentioned the graphing exercise by using Logo computer, but how to use the Logo computer and what activity can help children learn the notion of limit were not mentioned.

Two subjects mentioned the "Fraction Activity." One of them suggested "when studying fractions present the following:

$$1+1/2+1/4+1/8+1/16+1/32.$$

Does this sequence ever reach 2? Explore this using manipulatives." This subject used the sum of finite number of terms to introduce the notion of limit, and forgot to state "and so on" after the term 1/32. The other suggested "asking them (4-5 grade range children) to name numbers between 0 and 1 and responding to theirs with one closer to one every time." This indeed was an excellent activity, because the children can really capture the feeling of "closeness."

Among the responses, only one subject made the connection between the notion of limit and the irrational number π. This subject suggested to show "the sequence π, 3.14..." to 4-5 grade range children.

One subject mentioned that series can be used as an activity but did not provide any detailed description of how the series can be used to introduce the notion of limit to 4-5 grade range. The other mentioned introducing infinity to 4-5 grade range students as an activity, but similarly did not provide any description of such activity.

In summary, only when there is an infinite process can the limit concept be relevant. When there are finite numbers, finite graphs, finite tables, and finite number experiments, there is no limit concept involved. Most of the subjects who provided activities, did not make connection with how the activities they produced could be used to introduce the notion of limit. The missing tie to the notion of limit is the infinite process, probably because the participants were trying to come up with an activity and forgot to mention the importance of infinite processes involving the limit concept.

In order to get more information than was provided by one or two sentences in the survey of thirteen participants, I conducted an interview based on a structured list of subquestions related to the third research question. The interviews were audio-taped and then transcribed. The four participants in this study are pre-service secondary mathematics teachers who were not among the 13 who responded. Curriculum knowledge involves an understanding of the curricular alternatives available for instruction and familiarity with the topics and issues that have been and will be taught in the same subject during all the preceding and later years in school, so I wished to find out at how young an age this group of preservice teachers think the children will be knowledgeable enough to informally learn the notion of limit. Thus the two lowest grade ranges were the focus. In order to make the discussion easy to follow, the four subjects were assigned the names Anna, Bill, Mary, and John. The first subquestion the subjects were asked to respond to was: Do they think K-2 grade children will be knowledgeable enough to informally learn the notion of limit? The four responses, summarized from the transcript, were as follows:

Anna responded, "I think so. You could probably do things dealing with numbers. Numbers are pretty easy, if you say count one higher, count one higher and they might be able to kinda grasp what infinity might mean."

Bill said, "K-2 range, I've never thought about math much below the junior high level."

Mary said, "Well, they could not have the idea of definition until they get in college or anything like that, but they could get, I think they would be knowledgeable enough to learn like that things don't always end, there's the infinity, I think they can do some basic things, look at patterns, numbers, try to think where the pattern ends or what it would get nearer to, I think they could do that."

John responded, "I would say yeah, in fact I think that as far as I know at this point in time, there isn't a lot being done with getting, helping the younger grades get a handle on this stuff and I can't see why it would be a bad idea to put this stuff in their hands."

Except for Bill who has never thought much about mathematics below the junior high school level, all agreed that the K-2 grade children are knowledgeable enough to learn the informal notion of limit. Anna and Mary suggested some activities that could be used to introduce the informal notion of limit, such as; observing or discovering that the number can get larger and larger as well as that things don't always end (leading to the notion of infinity), and looking for patterns and where the pattern ends or what the pattern would get nearer to.

The second question originally was designed for those who thought that K-2 grade children were not knowledgeable enough to learn the notion of limit, so the question asked the participants to respond on whether they think that more mature students such as 4-5 graders might be able to learn the informal notion of limit. However, all of the four interviewees responded to this question as well.

Anna said, " ... so I do think definitely the 4th and 5th grade."

Bill hesitated at first, and said, " No I don't ... " and then continued and said, "they would be knowledgeable."

Mary said, "They could too, they always could."

John said, "No doubt."

Since this was a structured interview, I did not probe each individual much about their short responses. The next question was; assuming they think K-2 and 4-5 grade children are knowledgeable enough to learn the notion of limit, then what kind of activities they could produce that might be appropriate to introduce the notion of limit. Their responses were as follows:

Anna said, "We were just talking about the one there with numbers if you just always add one, it is always going to be one bigger. Maybe if you did something with grains of sand, peas or something or M&M's, infinitely many M&M's and you always add one more that way they can see it more."

Bill said, "I'm always straining to think of that activities to introduce students to any kind of subject in math." Then he continued to provide activity anyway, "You could ask like this is graphed, or how this equation, well, how will this tend to act over time, ..., like the temperature, you take a steamy hot cup of coffee out of oven, ... you put it down, you wonder what is the temperature over time. ... And over time, what is its temperature and will it ever actually approach room temperature, will it ever actually be at room temperature and theoretically, it won't be. And so you can say it's limit then."

Mary said, "I think have them do exploratory work, maybe have them make up their own patterns and try to see what they are, what they are going to get near."

John said, "I would go once again to geometric, or the idea of just taking halves of a pie. ... The idea that this is a pie and gets smaller and smaller as we halve all the pieces"

All subjects except Bill repeated the old activities. Although Bill did not provide much information before, here he provided a science activity which he had seen done in a science class. The activity he mentioned is examining how the temperature of a steamy cup of coffee reacted over time and he claimed that the room temperature is its limit. Up to now every participant had provided activities. I was interested in finding out whether they considered how the activities they presented could be tied to the specific school grade, thus the question asked next was: What grade range children will be able to accept your activity? The responses were as follows:

Anna said, "Probably any, if you are just talking about that activity with adding one each time. Really any grade level because you start to count in kindergarten...."

Bill said, "And the grade range,... I don't know,... I'm not sure how advanced, it's been so long since I have seen or worked with any, you know, any elementary school kids. I don't if they would be. It would sound like a good science project."

Mary said, "I'm not sure with elementary school children, how would they act. The middle school students that I have, they could do something like these. I'm pretty sure they could do it, if you make them do it." And she continued, "...I don't see why younger students with the right instruction and if they just you know understand it, simple thing, I think they can do it too. So I think pretty much even kindergartners they could do basic."

John said, "And I think for that particular activity, perhaps K-2 may be too early, but I really don't believe that it's not impossible. I wouldn't say that it's impossible to get that understanding across. They can see the pie slices getting smaller and smaller."

Based on the activities they provided, except Bill, they all agreed that their activities could be accepted by even the kindergartners. Since it was designed to be a structured interview, the questions of how the activities they presented could be used to teach kindergartners as well as fifth graders, or what topics in the fifth mathematics curriculum could be related to the activities they presented were not investigated. For example, I could have asked Anna, "How do you think your 'adding one activity' could be used to teach the kindergartners as well as the fifth graders? Is the teaching strategy the same with these two groups of children? What is the notion of limit you expected these two group children learned from this activity?" Unfortunately, I did not probe. What I did ask next was: Do they think we should informally introduce the notion of limit as early as possible? Their responses were as follows:

Anna said, "I don't see any problem in it. To me it might make it more acceptable once they do, students do get into calculus to see something before, you know, that you can even start talking about easy sequences, just things about easy sequences or things about infinity."

Bill said, "I think you could bring that up in a science class. 5th or 6th grade and take temperature of things or even younger than that. They could work on that, but the mathematical set for, it might get a little confusing maybe for less than an advanced group at that age. And it would be hard to represent that even a sequence or an equation or anything like that at such a young age before high school."

Mary said, "I think it's good for people to see, you know patterns and how patterns have something they get nearer, if they don't, they might go near to infinity. It's important to understand what infinity is. It might be hard for them to think there is no biggest number, but I think it's good place to try it. I mean its' not going to hurt them."

John said, "Yes. I don't see any reason whatsoever with hiding the bigger concepts of mathematics from children. Let them have them. Get their hands on them."

With regard to the above question, the participants all agreed that the notion of limit should be introduced early. Based on her own learning experience, Anna claimed that "it took me the longest time to figure out what that stuff (the limit) was." Although she knew how to find the limits and the derivatives, and she took higher level mathematics, nonetheless she claimed, "Even now, obviously, I still don't have a great understanding of the limit." Although John did not directly refer to the learning of the limit concept, he pointed out how the fact that his teacher prevented him from learning about logarithms in high school hindered his learning of logarithms in college. Mary said, "It is not going to hurt the children." "Don't hide the bigger concepts of mathematics from children," was said by John, "let them have them." The participants talked about whether there exist some activities that could be used to introduce the notion of limit; they all agreed that K-2 and 4-5 graders are knowledgeable enough to learn the limit concepts, and that the notion of limit should be taught early; now the last question asked was: When do they think is the best time to introduce the notion of limit? They responded as follows:

> Anna said, "You mean this, just intuitively? Ahm, probably as soon as they as soon as the students are able to I don't know, as soon as they can understand, how can you say, oh as soon as I can understand it, how do you know if they understand it. I don't know. Probably when they start doing more things with math, I mean, do they, they've got math in elementary, I mean in second and first grade, right. Maybe second or third grade when they start doing things with numbers more than just counting. Second or third, mostly in kindergarten you just count I think and maybe you start to add a little bit but then..."

> Bill said, "Yeah, when they are working on graphing, maybe when you first introduce the idea of asymptote maybe then you could bring up the idea of limit. It's the first time it seems logically to flow from what you are doing in class already. "

> Mary said, "...but I think it is important that they start to get an idea about different things early on not wait until they get to college and not even if you don't go to college."

> John said, "...I really don't know when the best time to introduce the notion of limit is."

Except Bill, the other interviewees seemed stumbling on this question. Although Anna said second or third grade, but she was not sure. The words she used were "probably" or "maybe" which indicated uncertainty. Mary said "not wait until they go to college" which indicated 12 years' difference. The reason she provided was "I'd never seen much of it (the notion of limit) before, and that's probably the hardest thing I've seen first in calculus, I did really bad on that, because I'd never seen it before in high school." John said that he really does not know, because he is not a curriculum master. Bill is the only one who pointed out that when students are working on graphing or when they were introduced to the notion of asymptote, maybe then the notion of limit could be brought up.

Summary

Four prospective teachers who were interviewed all agreed that the limit concept should be exposed to children as early as possible. They could provide activities, but could not provide much information on when would be the right time, what would be the right topics in which to introduce the limit concept, and where the limit concept is revealed in the K-12 mathematics curriculum. These four prospective teachers had their first exposure to the limit concept when taking their first calculus course, in which they were taught the usual formal ε–δ definition. It seems they all found their experience in learning the limit concept painful and frustrating, which is probably why they favor introducing the limit concept on an intuitive level much earlier in the curriculum. They did not make much connection of the limit concept with other branches of mathematics, and as a result the activities they introduced exhibited little variety or creativity.

Question 4: What are the possible misconceptions, difficulties, and errors the prospective teachers anticipate in teaching the concept of limits?

In order to address this research question, an open-ended question was embedded in the questionnaire. In this question the subjects were asked to respond: What are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how would you help your own students to overcome them? Fourteen subjects responded to this question. Although there were not enough subjects that responded to make significant conclusions related to this research question, I believe that the information

Table 5.34. Results of Item #10 in Questionnaire Part I

Description of Difficulty	Teaching Strategy
To understand the concept	Provide concrete examples
To understand the formal definition	Using examples instead of ε, n, N etc.
Very abstract idea	Come up with some more concrete examples
Definition was too intellectual	Graphs seem to help
That sometimes when the denominator is 0, the limit is zero, when to use what rule to prove the limit	N/R
Vocabulary: the meaning of "formal" vocabulary	Teach for concept
Finding limits seemed inconsistent Involved infinity Too abstract	N/R
To understand the importance and reason behind limits	N/R
Description of Misconception	**Teaching Strategy**
Sequence, such as, n+1, or $n^2/(n^2+1)$, as an interval	Showing sequence may jump around
Evaluating limits by just plugging in the value x tends to	Evaluating one-sided limits first or showing counter examples
In all cases a limit cannot be reached	Give students many examples
In $\lim_{n \to 2} n^2+5=?$ the students will tend to look to this problem as just substituting 2 for n and the result is 9	Pick numbers closer to 2 on either side and graph the results
Can't recall much about learning about limits	N/R
N/R	Try to use everyday events and maybe things that the kids do often to try and help them

Note: N/R means no response.

has the potential of providing some baseline data for further investigation. With this in mind the responses were then analyzed and grouped according to the description of difficulty and misconception as well as the teaching strategy to overcome these difficulties and misconceptions. The results are given in above Table 5.34.

Table 5.34 consists of three parts. In the first part, the responses which describe difficulties are grouped together. In the second part, I group together the responses which describe misconceptions. The third part is responses not belonging to the first two types. Fourteen of the 38 subjects responded to the survey question: What are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how would you help your own students to overcome them? Eight provided descriptions of the difficulty in the limit concept which mostly focused on the idea that the limit concept itself is too abstract to understand. Four subjects provided descriptions of misconceptions of which two were actually related to difficulties in computational techniques rather than misconceptions. The first one was that in all cases the limit cannot be reached. This indeed is a prevailing misconception not only among students as stated in the research findings (Davis & Vinner, 1986; Schwarzenberger & Tall, 1978); but also among ancient mathematicians as we discovered in studying the historical development of the limit concept. Students who possess this unreachability model of limit will reject the true statement that a constant function has a limit. This unreachability model of limit is related to one of the debates on the attainment of limits by earlier mathematicians as reported in Chapter Two. The reason for this misconception being so prevalent is probably because most of the examples of sequences in the real-world situation seem to exemplify the unreachability model of limits. Another possible explanation is that this misconception is due to the "intrusion of potential infinity", a notion which entails the impossibility of an infinite process reaching its limit. Another misconception mentioned by one subject was that the limit of a function is found simply by plugging the given n value into the given equation. When asked how to help students to overcome these difficulties, six of them thought that providing more concrete examples would help to overcome the abstractness of the limit concept. Two subjects thought that graphs might help. Two subjects, talking about the limit of functions rather than the limit of sequences, stated that showing students the one-sided limits first then showing counter examples might help students to understand the limit concept. Four of them provided no response for a teaching strategy to overcome the difficulty. One responded that ,"can't recall much about learning about limits." The other subject provided teaching strategy for learning but provided no misconception or difficulty.

In order to learn more about prospective mathematics teachers' understanding of the limit concept in terms of students' learning difficulties, I conducted an interview, the interview has given to four prospective secondary mathematics teachers and these results follow. One of the interview questions was designed to extend the survey question: What are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how did you overcome them? along with nine subquestions related to the main question which was under consideration. The following were the four interviewees' responses in terms of this research question.

When asked to respond: what do you think are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how did you overcome them? Their responses were as follows:

> Anna said, "I have a lot, like I said, I don't really feel like I have a very good understanding of what limits are so to me as a, I'm going to be a teacher, I am going to teach this maybe someday, that's pretty I don't know, scary, or whatever, but ahm, let me think, I don't really know if I know enough about limits to know what my misconceptions were. I don't know if I had a certain misconception or just didn't understand the whole concept.

> Bill said, "... when they, I think it was my advanced math, pre-calc, whatever was trig in high school that they brought up limits. I just remember the epsilons and the deltas, or whatever they used and just those symbols right there threw me off. It was it scared me because it was really confusing and a little intimidating I guess because I'd never seen anything like that before. And I couldn't understand it right away. It didn't seem to follow from what we had been doing earlier. So

that's that was one of the difficulties for me and still the difficulty if you are trying to realize what depended on what in your statement, what was it we were trying to find out within this neighborhood and what limit shows and what we chose depended on that. "

Mary said, "Ahm, I guess when I think about limits when I first found them I'm talking about the limit of functions and things like that, and you know, we were told all we got to learn to simplify in certain ways and maybe I don't understand, you know, why you could do that, or sometimes you can take out a factor of x, but you couldn't always and just to me it seemed like a bunch of rules that I didn't understand. I was told the rules and I forget them and I wouldn't know when you could do it and there were certain things and all of a sudden at one point you could just, ahm, you were supposed to just be able to see it, you know, or if it was as n approaches some number, you could just stick the number in it at a certain point and at other points, you couldn't do that because you would get division by zero, and things like that and it kinda bothers me you know, I didn't know. And I think the first thing they did was try to explain it to me with the epsilon, delta definition and the graph and that totally threw me. I understood it as they went through it, when I saw it, and I went home and I didn't know what it was. So maybe, maybe if they started you off slower and showed the patterns things like that instead of just saying, you know, the epsilon delta, draw the boxes from certain point, its going to get so close maybe a little bit slower or just because I'd never had it before, it would have been more helpful if I had had it.

John said, "... I don't remember that I encountered it but that what we were discussing earlier infinity plus a that I may struggle with that until I go to the grave, unless I run into somebody who can really explain that, well, because the idea of infinity being out there further and further and then adding to it, you know, that's like telling me that, no infinity isn't infinite, it's finite. It's like wait a minute. But I don't really recall any frustrations."

Anna confessed either there were misconceptions that she did not know or she did not fully understand the notion of limit, as well as it being a scary thing to think about how someday she is going to teach the limit concept. Bill and Mary mentioned the symbolic notation of delta and epsilon in the formal definition of limit scaring them off. In the formal definition of limit we have to have ε first then find N, and this temporal order is difficult to grasp. Bill thought the ε and N are very confusing because he could not remember which one depends on the other. Besides the notations in the formal definition of a limit, Mary was more concerned about not remembering the computational techniques for finding the limit. She mentioned that the techniques taught for finding the limits sometimes were applicable and sometimes were not, which indicated understanding of how to do the limit problem but not why the methods worked. She was the only one who mentioned how the limit concept should be taught. She said "maybe start off slower and show the patterns rather than the epsilon and delta definition." John is quite confident about his understanding of the limit concept and replied that he had not encountered any difficulty, but he stated that the notion of infinity is contradictory in nature. This contradiction, then, he said, might follow him to his grave. He also said that his calculus teacher taught him well and made sure that he understood that "the limit can never be reached." He said, "It gets closer and closer and may never ever get there." Apparently John himself did not realize that he possesses the most prevailing misconception; this unreachability model of limit (Actually, because of the ambiguity of the use of the word "may" in this sort of sentence, John's second statement given one possible interpretation would be correct, but we know from other evidences that John actually holds the misconception that it is impossible for a sequence to ever reach its limit. For instance, he also says later that "it ... never ever touches it (the limit).") After the subjects thought what probably were their own misconceptions, difficulties, and errors, they were asked to respond as to what could be students' misconceptions, difficulties, and errors by the following question: What do you anticipate are the misconceptions, difficulties, and errors that students will encounter most while learning about limits? Their responses were as follows:

> Anna said, "I think a lot of it has to do with notation. ... But I have a hard time getting general form of it. That really, that took me a long time and I still have a hard time, you know, playing with the sequence and trying to find the general form. Ahm, I think that was tough. Ahm, then when we talked about ... whether the sequence is diverged or converged and we had to look at the sequences and ... remembering anything. ... Like, trying to think, if the limit came out to be infinity over infinity, it wouldn't be one even though you know if you had n over n it would be one. If you limit with infinity, it would diverge because you could never figure which, how big those infinities were you know. Ahm, so I think, I don't know, I think the whole of that whole kinds of converging and diverging that just blew right by me and I knew you know I could figure out some of the problems because I could work like you know they give you the method to find it out or you know, there's ahm yeah, right, you know the rules to do by never I never looked to understand what the rules meant. I always looked to just to figure out what the rules were you know and apply the rules and I never had any conceptual understanding of what you know what the meant so that was difficult."

> Bill said, "Yeah, that was one problem for me. And ah."

Mary said, "Well, I guess I don't know, I don't know if I still have misconceptions, I'm sure I have misconceptions on it. ... I guess maybe they might think the number can never get bigger than the limit. I don't know, I think it can, I think it can be on either side of limit, but sometimes I'm thinking I have infinity too. ... So, ahm, things like this, sequences can bounce back and forth, but they are always going to approach something but it doesn't mean it can't be bigger than that. That might be hard. They might think the limit, you know speed limit. They can go over 55, or something and they can't be over that, that's the limit. They might think that over. That the graph never got over that point so I'm going to call that the limit."

John said, " I think that one thing they really need to understand and perhaps now that I think about it the ideas that I had most difficulty accepting was that it may never reach the limit. You know, but that's why I made the point that things get as infinitely small as well as infinitely large and it is interesting because I didn't understand this idea until I was in real analysis, you know until I started thinking about the sequence 1 over N, I never thought about the idea that there was an infinitely small as well as infinitely large."

Anna who thought the notation was not easy to understand for her, mentioned that 1) the notation made the limit concept difficult for students to learn; 2) finding the general rule of a given sequence is also very difficult; 3) it is hard to distinguish whether a given sequence is divergent or convergent; 5) the inconsistency between n divided by n giving you the answer one but when infinity is divided by infinity you do not get one, as being something else that confused the students; 6) one could not figure out what infinity is, because of not being able to decide "how big those infinities were."; and 7) the problem of knowing when and how the rules for finding limits work. Bill did not respond much to this subquestion, he said "that was one problem for me." Mary thought that thinking a limit can not be passed was probably one of the difficulties, a misconception caused in part by the fact that the ordinary usage of the word "limit" is different from the mathematical meaning of limit. She also mentioned that 1) bouncing sequences or functions could be used as an example to show that sequences can bounce back and forth and 2) the graph of a function can also used to show that "it approaches that line (limit) it can go on either sides." John thought that "a limit can never be reached" is one difficulty for him, although previously he claimed that he had never encountered any difficulty. John continued that the mathematical meaning of infinitely large and infinitely small were ideas that he did not understand until he was taking Real Analysis and he learned about the sequence $\{\frac{1}{n}\}$ where no terms of the sequence are equal to the limit 0. And he used this example to convince himself that a limit can not be reached. John said he had a hard time accepting the statement that "a limit can not be reached", but he finally accepted it because of the one sequence $\{\frac{1}{n}\}$. It seems this unreachability model of limit was really imprinted in his mind and this unreachability theme is going to be passed on to his students. When I probed with the following question: Are there more difficulties, misconceptions, and errors? their responses were as follows:

> Anna said, "Well, obviously the ones I have ... would be hard for students. ... It took me a long time to see when you have like the limit ... is going to be L, ... but this epsilon can be as big or small as you want it to be. It took me a long time to figure that out, ... and so I don't even know, students are just taking calculus in high school aren't even really going to see this unless the teacher explains it more. So, I don't know, that took me a long time to understand too."

> Bill skipped this question, because he responded earlier that presenting difficulties was one problem for him.

> Mary said, "Sometimes they just define, like , ... this one, sin x over x, they define it to be one, their definition like that, but , ... they don't explain it why it is like that. You just memorizing them and they can go into more why is like that or just mathematicians come up that's what we are going to define it to be to make it easier or you know. They can go and explain it more, some one of them like that, you know I saw one."

> John said, "That would have to be the idea that students have a hard time understanding that you can get infinitely small as well as infinitely large. Now when I'm explaining certain concepts to students, I always make sure that they understand that it (the limit) can get closer and closer but it never touches it (the limit). And I make sure that they understand the idea that you can get infinitely small as well as infinitely large and never make it. You may never equal that."

In order to respond to this probe question, Anna provided another difficulty of her own which was the difficulty to determine "the size of epsilon." In the informal definition of limit, the statement usually is like "you can pick a positive number as small as you wish, then ... ". To students, "how small is small enough" is one difficulty to figure out. Bill did not respond to this follow up question, because in the previous response he already mentioned that pointing out students' difficulties is one difficulty for him. Mary did not provide another difficulty, but she raised an important question about teaching. She said that when learning about the function $\frac{\sin x}{x}$, "they (the teachers) don't explain why it is like that, students have to memorize it." She suggested that the teachers can go and explain it more, like one teacher she saw. John suggested that "infinitely large" as well as "infinitely small' would have to be an idea that students have a hard time understanding. Again, John mentioned that he wanted to make sure that his students understand that "the limit can get closer and closer but can never be reached" As I mentioned before, since this was designed to be a structured interview, I neither probe more why "infinitely large and infinitely small" were ideas hard for students to understand nor presented an example of sequence which does reach its limit to ask him to explain. However, I did ask the general follow up question: Why do you think these (misconceptions, difficulties, and errors) cause trouble? Their responses were as follows:

Anna said, "I think because they are abstract, ... when I was in high school and my first couple of years of college, I just wanted ... to know how to do the problems. I didn't care what they meant. ... I just wanted to know how to do them so that I got my answers right and I know that students in high school think that too because, ... right now I'm student teaching and I've got these kids who come up and say, I don't care about ... what it means, I want to know the formula and I want to know how to pop those numbers into that formula so I can get correct answers. And so I don't know I think the more abstract you have to think more about you know their meanings, their meanings aren't so easily, it's not like you can say, Oh this is, put it into this formula and that is what you will get. "

Bill said, "Because trouble for me just because that statement (the formal definition of a limit), it is all a bunch of symbols and that can look like a foreign language to some students completely and ah, I mean I didn't have a good enough representation in my mind to think what followed from what or what it was depending on."

Mary said, "Ahm, like the one thing I said was the word limit causes trouble, because people used to think limit, you can't exceed, like don't exceed your limit. Don't go past that, that is what limit means, like the city limit. ... the way I've learned it would be something we can't go past it, the

> bound. Ahm. for speed you could have a 40 mile limit, speed limit, is there any limit on the size which may pass my limit for sure you know too far. You know, like in "don't push me past my limit." You don't, you know push too far. There's always something above, it's not something below, or even equal to, its always something unreachable."

John said, "... That's the one that I really worry about that I consider that students will have the most difficulty understanding is that it may not actually ever equal it.

When these four interviewees were asked why they think the difficulties they mentioned earlier caused trouble for learning the limit concept, the major response was generally pointed to the abstractness of the limit concept. The symbolic notation was seemingly attached to no meaning and was a foreign language to the students. The presentation of the limit concept was based on the abstract appearing epsilon-delta definition which had no connection with the prior mathematics.

The word limit has twofold meanings to the students; one is daily usage and the other is the mathematical meaning. It seems that ordinary language strongly dominates the thinking of mathematical language. Especially the mathematical limit was considered a "bound", which is one of the misconceptions mentioned by Davis & Vinner (1986).

The intrusion of infinity is another factor causing learning problems. As mentioned in the historical literature review, there are two kinds of infinity, namely, potential infinity and actual infinity. Most people possess the potential viewpoint of infinity (Fischbein et al., 1979). That is, potential infinity occurs in a situation in which no matter where one is, one can go another step; for example, given any positive integer one can always think of a larger one.

The last factor causing learning difficulties is the attitude of students who want the problems done quickly and the answers to check correctly, and who do not care much about the underlying concepts that they were learning. Again, I did not probe Anna, I should perhaps have asked her how she is going to deal with students who have this attitude about learning; or can she come up a better strategy for teaching to change students' attitude; or how to teach the limit concept in such a way that plugging into the formula to get the right answers was not the focus. The designed follow up question instead was: Is there a way to eliminate these misconceptions, difficulties, and errors? Their responses were as follows:

Anna said, "And I don't know is there a way to eliminate these? Maybe ... if you taught the calculus so the limit or whatever that was not just to get an answer, it was to know what limit meant, actually, you know, that the concept of limit, not so much the answers you get if you try to find out the limit or try to find the limit of a number."

Bill said, "... I'm not sure of a new way. That would be a way to start. Pictures always helped me I think. Like a lot of visuals."

Mary said, "O.K. the word limit, I guess. ... I don't know, try to explain this is not the same limit that we are used to, it can be bigger."

John said, "I think that using the fractions example, 1 over n is really a nice sequence. You know, it's the best explanation that I've seen yet for getting infinitely small. Because when you label, it's more or less the definition of a limit. You get a student to label okay, this, let them say, this is as small as it can get and say that .00000...1 with a hundred zeros preceding the one is as small as it can get. Then you introduce the number with 101 zeros preceding the one and they realize, I can still get smaller, so that's the best the 1 over n is a beauty for explaining that. "

The subjects could easily point out their own weaknesses as an indication of students' weaknesses, but it seemed difficult for them to come up with a cure for these weaknesses. Anna thought that changing students' attitudes towards understanding the underlying concept rather than the correct answers might be helpful. Bill suggested that graphical representation or visualization might be helpful, at least they are helpful for him. Mary suggested explaining that the ordinary usage of limit is different from the mathematical limit. John mentioned the sequence $\{\frac{1}{n}\}$ might help and he thought that the sequence $\{\frac{1}{n}\}$ is a beauty for explaining how something can get smaller and smaller. When the interviewees were pushed by asked by the following question: Are there other methods (to eliminate the cause of troubles)? their responses were as follows:

Anna said, "Yeah. I don't know, I'm sure there are. Because people can probably think you know different ways to teach different things you know different examples to use might make things easier."

Bill said, "When we taught it, I think it was thrown out the formulas there were given, they would give us the formula. I mean we never did any work or anything like that before hand. They just gave us the formulas to work from. If we worked on something like that, we were given the question and how this sequence, or whatever would act over time and then on our kinda of if we might ask why, well why does this happen or kinda of on our own come up with some reasonings ourselves. Yeah, that formal stuff should come later on. So, but just the ideas of things that we are approaching over time."

Mary said, "Maybe just show them many examples, like to show them the sin x over x, or to show them one over x, it always above zero, one over x, you know, they can draw the sequence 1, 1/2, 1/3, 1/4, one over one million, whatever, its not going to be zero, it always above the limit, it doesn't mean you know just keep showing them different examples, have a lot of different ways to show them, so after a while, they'll understand this limit as a mathematical limit not the speed limit or the limitations put on something you're used to."

John skipped this question and went directly to the response to the next question.

Only three subject responded to this subquestion. Anna was quite sure there were other methods that could be used to help students but she could not think of any. Bill suggested that maybe we should let the students 1) become familiar with the notion of "change over time", 2) come up with reasoning for why, and 3) explore what will happen over time rather than giving them the formula to work with. Mary used the concrete examples $\frac{1}{x}$ to show students the difference between the daily language of limit and the mathematical limit. Mary suggested that we can let x get large in $\frac{1}{x}$, but $\frac{1}{x}$ will never be zero. However, this actually is not a good example to explain her notion of limit not serving as a "bound", because the limit of this sequence is 0 while the terms of $\{\frac{1}{n}\}$ are all bigger than 0. What she thought was that no term of the sequence $\{\frac{1}{x}\}$ will be equal to zero, but that does not imply that the limit does not exist. The main issue here, as mentioned in Chapter Two, is not whether the terms of the sequence are equal to the limit, it should be whether the given sequence has a limit or not. The issue here should be the realization that if a given sequence converges, then the limit of this sequence might or might not be one of the terms. If the limit is one of the terms (perhaps even appearing more than once), then the sequence reaches its limit. If the limit is not one of the terms, then the given sequence does not attain its limit. In finding the limit, the main concern is whether the given sequence is convergent, not whether the terms of the given sequence are equal to the limit or not. The focus on the process rather than the end product of an infinite process sometimes caused difficulties for students. The next question was focused on the participants' opinions about the nature of the limit concept and/or the teaching of this limit concept: Do you think these misconceptions, difficulties, and errors are caused by the abstractness of the limit concept, or due to the teaching? Explain.

> Anna said, "I think it is probably both. Because it's hard when you are first getting to enter things that are abstract. To even train your mind to think that way so I think that's what makes the limit pretty difficult is that when you get into more I guess abstract and the teaching, if you teach it so that ... you can understand what is going on, then I think you are going to understand the whole idea better obviously and you are going to be able to do better. But if you just teach it so that its this is how you crank out the answers then I mean, especially over the limit, you can't really do that."

> Bill said, "Well, the abstractness was part of it for me. The teacher ... went over it so many times, wanted to make sure we knew it. But, it could have been a little different representation or something. Might have worked, and the abstractness I think hurt me."

> Mary said, "Probably both, ahm, the limit, ... I'm totally clear on it, you know, so maybe one thing must apply to a lot of the teachers that are doing it, I didn't, I shouldn't say it, it applies to some of them, who might not know, might not be that clear in their own heads, so they just kinda of ahm, this is what it is, they might do that too, it maybe a little bit of threat to the teachers so, you know. I think it is both the limit is abstract, they are some teachers they are really understand it and can teach it well, but others they might have troubles explaining it. I'll say it's both. It is an abstract concept at a time, but it's not something that, I think if a teacher had good grasp of it, I think they can make it clear to their students."

> John said, "Ahm, I would say more the teaching than the abstractness of the limit concept. Because if it's taught properly, the abstractness becomes less important. It's not the, it's not as overwhelming. I think that as I was saying before, we need to get students to understand that you can get infinitely close to a particular place without reaching it. "

The responses on this subquestion were divided into three categories. Anna and Mary claimed the difficulty in learning the notion of limit is both due to the abstractness of the limit concept and the way it is usually taught. John thought it was the teaching because "if the limit concept was taught properly, the abstractness becomes less important." Bill thought his teacher did a very good job trying to explain the mathematical content, but "the abstractness" he thought "hurt". Mary stated that there are some teachers out there, "who might not know, might not be that clear in their own heads, so they just kinda of ahm, this is what it is, they might do that too, it maybe a little bit of threat to the teachers so, you know." John responded again that "you can get infinitely close to a particular place without reaching it (the limit)." Next question is pointed toward finding out what participants think about the limit concept by asking them to respond: Is the limit concept easy to learn? Explain. Their responses were as follows:

Anna said, "I don't think it is. It might be if it were taught very well, you know, if it were taught very well it might be easy to learn."

Bill said, "Concept could be, I think, the formal law or rule, the formal definition is a little bit harder."

Brenda asked, "You said the notation is abstract and it is hard, but the concept is easy. What do you mean, why you think it is an easy concept?"

Bill continued, "You could hum, just with drawings, it makes sense how things can get closer over time. Just something as simple as 1 over x. Something like that, it makes sense that smaller and smaller numbers get bigger and bigger and bigger. As it gets closer to zero, on the positive side, it, it'll just go on forever from infinity, while this x approaches infinity, this thing is going to get closer and closer to zero, but it will never be zero, you could have 1 over, you know, as big a number as you could possibly think, and that makes sense, you can kinda picture that, or understand that, but the formal definition for that, it would scare off some kids. You can teach some easy algebra concepts in class, you can the teach but maybe give them a formal why afterwards. That's what scares them afterwards."

Mary said, "I think the concept ... is what it approaches as it gets really large, what the sequence gets near. I think that concept is really easy, but I think when they start putting, ... when epsilon greater than zero, there exists this..., when you start doing that, not to me, but to students, it kinda seem like, so that part doesn't seem easy."

John said, "I don't know. I could say yes it is easy to learn and there's going to be a student I deal with, ... where I was sure that they had a concept and two days later they come back you know, I felt that they left with an understanding of it and two days later they come back and they don't understand again. So I would say, no, it's not an easy concept to learn because you are going to find someone that has difficulty with it."

All subjects generally agreed that the notion of limit is easy to learn, but the formal definition and the notation involved might not be easy to learn. Especially when the formal definition was introduced and the epsilons and deltas scared them off. When Bill mentioned the function $f(x)=\frac{1}{x}$, he said, "As it gets closer to zero, on the positive side, it'll just go on forever from infinity, while this x approaches infinity, this thing is going to get closer and closer to zero, but it will never be zero " So, the focus on the infinite process, and the intrusion of the potential infinity were exhibited in Bill's statement. John talked about his teaching belief, that, "If we are to teach to each student, then, we need to be aware that it's not going to be an easy concept and we should never sell ... the idea that it's easy." He continued, "we should never sell the idea that it's easy because there's going to be that one person and that's going to frustrate them that much more if we tell them, well this is easy, why don't you get it. That's an approach to teaching that I consider a real sin." After discussion from the learning point of view about the limit concept, the question was asked: Is the limit concept easy to teach? Explain.

Anna said, "... I don't think it is the same thing, I mean, it's hard for me even to really understand, like I can't even imagine trying to get up and teach it."

Bill said, "Not if you don't have any good representations or you can't think of a better representation for it. I don't feel it would be easy to teach."

Mary said, "Ahm, no, I wouldn't think it would be easy to teach either, unless, if you had a good understand of it...."

John said, "I would say that it could, it has the possibilities of being easy to teach, but then ah the teaching is only as easy as the learner, you know, if we run into that learner who's got some kind of wall between them and the concept, then it's not going to be easy to teach, but you will have to continue to come up with new ideas, something fresh, you know, I mean, just like today as we've talked, as we've discussed this, I almost exclusively used 1 over n. Then what happens when I run into a student who doesn't understand the concept of 1 over n. Getting smaller and smaller. Then I have to come up with some way to teach that student other than that. You know, I use the geometric figures which is really, you know, hinging on the 1 over n, but you know, you always have to have some idea laying back in your mind or need to find, need to be able to resource so I would say, no that its not easy to teach either."

Three subjects thought that this is not an easy concept to teach, although they all agreed that this is an easy concept to learn. Anna claimed that the limit is hard for her to learn and she can't image trying to get up and teach it. Bill thought that teachers need good representations for teaching the limit concept. Mary thought the limit concept is not easy to teach because students might get it or might not. However, she said, "if the children started at young age, and hold better understanding of it, when they get into high level classes it would be easier to teach." John also responded that students could be a factor in ease of teaching, thus he concluded that it is hard to tell whether the concept itself is easy to learn or to teach. However, I did not probe what he meant by that statement. The last question: As a teacher, how are you going to teach the limit concept? was intended to find out if they were a calculus teacher how were they going to teach the limit concept. Their responses were as follows:

> Anna said, "Oh boy. I'd really like to teach it, teach the concept to be understood so it so you understand the idea and not teach the concept so that you can get the right answers. I haven't really thought about how I'm going to teach it."

> Bill said, "I haven't thought about that. That's not something I have really thought about."

> Mary said, "Ahm, right now I'm just doing junior high but I like to do a lot of different exploratory work and have them like have them see patterns and different things and have them make conjectures things like that don't just start off the first day and say this is the definition of limit because that will you know, scare some off and they won't pay attention because they'll think they are lost. You got to start them off real basic and then maybe maybe even have them kinda come up with idea of their own draw out of them, so it's kinda of part of them, things like that, I guess."

> John said, "Very carefully. ...that won't approaches it (the limit) and I will make that same effort (to make sure students understand you can never reach the limit)."

All the participants underwent the student teaching, one taught algebra, one geometry, and the other taught junior high school mathematics. The last subject is teaching college level preparatory algebra courses. Since all of them have teaching experience, this researcher expected to have rich responses on this subquestion, but it turned out differently. All they were concerned with was the present mathematics courses they are teaching now. They could provide general pedagogical knowledge, that is, they could provide a list of things to do as a teacher, for example, letting the students find patterns, conjectures, giving reasoning, teaching for understanding, and teaching "very carefully." Since I did not probe I was unable to find out what "very carefully" meant, nor could I find out how their generic teaching methods work for the specific situation of teaching the notion of limit

Summary

In summary, as a group, 42 subjects in this study could mention, based on their own learning experiences, many misconceptions, difficulties, and errors which are similar to those in the research findings reported in several studies (Davis &Vinner, 1986; Fless, 1988; Williams, 1989). The following were misconceptions, difficulties, and errors mentioned by the participants in this study:

1. The mathematical limit, as in daily usage of the word, serves as an upper bound for a given sequence.
2. The notations of the definition of limit are highly abstract and look like a foreign language.
3. The limit is something that can never be reached and the limit can only get closer and closer.
4. The intrusion of the notion of infinity. These can be categorized as the "actual" infinity and "potential" infinity as well as the infinitely small and infinitely large.
5. The inconsistency of $n/n=1$ on one hand, $\infty/\infty \neq 1$ on the other hand could be confusing to most students.
6. Sometimes you could plug the number in a rule, but at other times you could not.

CHAPTER SIX

SUMMARY AND CONCLUSION

In Chapter Five, data on prospective teachers' subject matter knowledge were analyzed within the framework of a five-category theoretical model. Data on prospective teachers' curriculum and pedagogical content knowledge were analyzed quantitatively and qualitatively based on the collection of the responses to the embedded test items and the excerpts from the transcripts of the four interviews. In this chapter the research questions will be discussed in the following way: first, reasonable expectations of the teachers' subject matter knowledge of the limit concept are presented; then prospective teachers in this study; their subject matter knowledge is summarized and conclusions reached based on these four categories of understanding: basic, computational, transitional and rigorous. Second, reasonable expectations of teachers' curriculum knowledge and pedagogical content knowledge are discussed. Due to the sparseness of questionnaire responses relevant to these considerations and the small number of interviewees I was able to work with, I do not feel firm conclusions are possible regarding prospective teachers' curriculum knowledge and pedagogical content knowledge in general. However, the tentative results reported here suggested some questions and provided baseline for future research. In addition, some suggestions for getting a "headstart" on teaching the limit concept in the K-12 mathematics curriculum are discussed. Finally, limitations of this study and some recommendations for further research are outlined.

Reasonable Expectation of Prospective Teachers' Subject Matter Knowledge, Curriculum Knowledge, and Pedagogical Content Knowledge

Shulman (1986) proposed that teachers' knowledge in general should at least include subject matter knowledge, curriculum knowledge, and pedagogical content knowledge. There is other knowledge teachers need to have, for example, knowledge about the learners, knowledge about psychology of teaching and learning, etc., but those are beyond the scope of the present study. Prospective teachers' subject matter knowledge of the limit concept is addressed in the first two research questions: 1) How well do prospective teachers understand the concept of limits? and 2) What kinds of misconceptions, difficulties, and errors do prospective teachers have concerning the concept of limits? The intent of the first research question was to investigate prospective teachers' subject matter knowledge; in other words, to investigate how well they understand the limit concept. The discussion to the second research question will provide a more detailed picture of what prospective teachers do not understand about the limit concept. In the following section, prospective teachers' knowledge about the limit concept will be addressed based on this researcher's theoretical model of understanding.

Subject Matter Knowledge

This researcher would expect prospective teachers' subject matter knowledge of the limit concept to at least include the first four categories of understanding: basic understanding, computational understanding, transitional understanding, and rigorous understanding. The discussion will be based on this ordering of the four categories.

<u>Category I: Basic Understanding</u>

In the category of basic understanding, teachers should be familiar with different representations of sequences in order to be able to find the limits of sequences in different modes of representations. That is, secondary mathematics teachers are expected to be knowledgeable enough to recognize different representations of sequences. For example, the first kind of representation of sequences is numerical representation, that is, a given sequence can be represented as an infinite list of numbers. In this case, teachers should be critical about whether the general term of the sequence is presented or not. If the general term of a sequence is not presented, then nobody could be able to find the limit by looking at the first few terms or from the first million terms. The reason is that there exist many sequences whose first few terms are exactly the same, and the limits of the given two sequences need not be the same. For example, when provided the first few terms as, $\frac{1}{1}$, $\frac{4}{3}$, $\frac{9}{5}$, $\frac{16}{7}$, and $\frac{25}{9}$, ... , the most likely sixth term is $\frac{36}{11}$ with the general rule being $\frac{n^2}{2n-1}$; however, there is a general rule which yields the same five given terms and a sixth term which is different from 36/11: $\{a_n\} = \{\frac{n^2+(n-1)(n-2)(n-3)(n-4)(n-5)}{2n-1}\}$, using this general rule, $a_6 = \frac{156}{11}$. The formal way of saying this is that by definition there could be finitely many terms outside any epsilon neighborhood (or interval). Basic understanding of the limit concept should be correctly matched with the formal notion of limit.

The second kind of representation of sequences is through a general rule (or formula), and this is the most common form of representation. Because of the existence of the general rule one will be able to distinguish between two sequences with first few terms having the same values. This rule is usually called the general term of the given sequence. Teachers should be expected to be able to generate the most likely rule by looking at the list of numbers given, if the sequence is a reasonably simple one given by first kind of representation.

The third kind of representation of sequences is the graphical representation. Usually students are more familiar with the reverse order; that is, they are asked to draw the graph of a sequence (or function) rather than to look at the graph to determine the sequence, and then consider the limit of this sequence. Teachers should expect to have more than one way to draw graphs. That is, they should be able to exhibit one dimensional and two dimensional drawings, be able to read information from the graphs, be able to come up with the general rules, and be able to identify the limits in given graphs.

The fourth kind of representation is geometrical representation. Teachers should be expected to know the analytical way of representing numbers. We know there is a one-to-one correspondence between numbers and points on the number line. Prospective secondary teachers should expect to be able to invert these two settings. For example, we all know square numbers, but the square numbers also can be represented by square figures (which is why these numbers are called square numbers). Thus geometrical representations of sequences could provide a different perspective on the limit concept, and making the connections between the numerical and geometrical representations of sequences could enhance the basic understanding of the limit concept.

The participants in this study did well on this basic understanding in general. Their overall average percentage score is 66.4. The rule-oriented and geometrical representations cause problems for recognizing the existence or non-existence of a limit. One of the difficulties most participants shared is having trouble grasping the multiple domain test items. Over-simplifying the split domain problems caused the low percentage scores.

<u>Category II: Computational Understanding</u>

The second category is computational understanding. Usually when one is asked to find limits of sequences, the sequence is represented by a rule (or general term). The first common kind of general term of sequences could be presented as a simple rule, such as 3n-6, $\frac{1}{n-1}$, or $(-1)^n + \frac{1}{n}$. The second frequently appearing kind of general term of a sequence could be presented as a rational expression, such as

$\frac{n-2}{3n+5}$, $\frac{3n^2 + 5n}{6n^2 + 1}$, or $\frac{2n^3+5n^2-6}{n^3-4n-2}$.

The third common kind of sequences usually are called geometric sequences and are expressed as, e.g., $(\frac{1}{2})^n$, $\frac{3^n}{4^n}$, or $\frac{3^{1-n}}{4^{1-n}}$. A fourth frequent kind of rule representing sequences is radical expressions, such as, $\frac{\sqrt{n-1}}{\sqrt{n+2}}$, $\sqrt{n-1} + \sqrt{n+2}$, or $\sqrt{n^2+n} - \sqrt{n^2+10n}$. Of course there are other types of rules for sequences. These are the four types shown in the questionnaire of this study. The order of appearance in the questionnaire was rational, combination of two simple rules, geometric rule and radical rule. The reason for this order was because 1) the rational rule shows up most in textbooks and students have been taught techniques for dealing with it, so it appeared first; 2) although the second rule appearing in the questionnaire is the combination of two simple rules, it is a divergent sequence which is relatively harder compared with simple convergent sequences; 3) in order to find the limits of geometrical sequences, subjects have to realize that the ratio determines the existence or nonexistence of the limit; when the ratio is less than 1 the given sequence has a limit, when the ratio is bigger than 1 the given sequence goes to infinity, when the ratio is equal to 1 the limit is one; or when the ratio is -1 the limit does not exists, so different ratios will produce different results, and so no unique technique works for all; and 4) radical rule is comparatively difficult because subjects usually need to find its conjugate first, then rationalize it next, and perform some computations afterward. Teachers are expected not only to understand what techniques work for finding the limits when presented with these different expressions for sequences, but also why those techniques work. Teachers should be expected to understand why the indeterminate forms are indeterminate. They should not only expect to know that L'Hopital's Rule is a method for evaluating the indeterminate forms, but also to understand the underlying reasons.

In the computational category, the participants' mean percentage score is 45. This group of prospective teachers who worked on the rational function and simple functions problems did reasonably well, but not well on the exponential and radical functions problems. It seems only 5% of the participants could solve the radical function problems.

Not realizing that ∞ (the symbol for "infinity") cannot be operated on algebraically is common to many prospective teachers.

Category III: Transitional Understanding

The third category is the Transitional Understanding. Being able to compute and find the limit of a given function does not guarantee one understands the underlying concept of limit. Thus, prospective teachers should be expected to be aware that basic understanding is sometimes not reliable because it is really only a conjecture based on finitely many terms of a given sequence, and that the computational understanding only enables one to compute the results mechanically. Transitional understanding enables teachers to provide reasons for why certain methods of finding the limit work; for instance, the usual way for computing the limit of a rational formula by dividing the numerator and the denominator of the formula by the highest power is valid because we know that the limit of 1/n equals zero. The transitional understanding should provide teachers an adequate knowledge for explaining why they do problems the way they do. Teachers with transitional understanding will make their students familiar with some underlying concepts and prepare their students to achieve more rigorous understanding, such as knowing, for example, why the Squeeze Theorem is important in finding limits and why the temporal order is important in the definition of the limit.

The subjects did not perform well on this category. The percentage mean score is 27.5. The most frequent error in this category is inability to distinguish the temporal order; they do not know how to find N when ε is given. Half of the subjects could not draw the graph of a given sequence at all, those who did answer provided a continuous graph and treated the graph of sequence as same as the function. Only 18% of the participants could informally explain what it meant to say a given sequence has a limit.

Category IV: Rigorous Understanding

The fourth category is Rigorous Understanding. Teachers with rigorous understanding of the limit concept should be expected to be able to not only understand the underlying meaning of the formal definition of the limit of a sequence as well as the negation of the definition of the limit, but also to be able to use the definition to verify whether a conjectured limit is indeed a limit or not by providing rigorous proof. They should be able to prove certain assertions not by merely memorizing the technique format, but to really understand why this is the way it is.

Teachers with rigorous understanding can use their mastery of the formal definition to prove to their students that certain sequences have or do not have limits, or to convince them that the intuitively conjectured or computationally found limit of a sequence indeed is the limit. Teachers with rigorous understanding can help students who fall into the common position of being able to provide a memorized proof of a statement but not being able to really "buy it" in the sense of being truly convinced of the result.

Apparently fewer subjects possess this rigorous understanding. The mean percentage score is 7.72. The overall low mean percentage score was due to the inability to state the negation of the definition of a limit and to do proofs. The difficulty is being unable to recognize the importance of the temporal order, the choice of quantifiers, and the relation between ε and N.

Category V: Abstract Understanding

The last category is the Abstract Understanding. In this category of understanding, mastery of a global view of the limit concept is necessary. Being able to apply the limit concept in real world situations, and understanding the application of limits in various branches of mathematics in general and calculus in particular are required.

The results of the previous chapter, in particular the percentage scores of prospective teachers' performance on these four categories test items are 66, 44.5, 27.5, and 7.7 respectively, shows that it apparently is really difficult for the subjects to reach the third and fourth categories, similarly to results reported by Fless (1988) and results of studies based on the van Hiele's model (Wirszup, 1976).

Curriculum Knowledge

Curriculum knowledge is investigated under the question: What are prospective teachers' opinions about the involvement of the concept of limits in K-12 mathematics curriculum? The curriculum knowledge involves an understanding of the curricular alternatives available for instruction and familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school (Shulman, 1986; Wilson el al; 1987; Even 1989). In terms of the limit concept, curriculum knowledge possessed by a teacher ought to include an appreciation of the fact that the limit is a central notion in almost all branches in mathematics. Thus, the open-ended question embedded in the questionnaire and the interview questions were focused on the K-2 and 4-5 grade children and activities that might be introduced to teach these two groups of children. The question might seem unfair to participants in this study, because their professional training was not for teaching these groups of children. However, it is desirable that teachers have a sense of the important ideas in mathematics, and with understanding comes creativity in presenting complicated ideas on a simple level. So, what are the important concepts that are central in K-12 mathematics curriculum? Where and how do these central ideas reveal themselves? How are we to connect the central concepts in mathematics with the rest of the concepts? Where do these important concepts come from and where do they lead? In considering the curriculum knowledge about the concept of limits, teachers at least should know what lower grade range students can do to start to learn the basic understanding of the limit concept, what kind of activities could be introduced to young students, and what could be expected from them. Then when the students get into higher grades, what are the next topics that are related to limit concept or can be taught to introduce the limit concept? What are the activities in, say, the 4-5 grade range, that could be used not only to enhance the learning of mathematics, but also used to introduce the notion of limit? Should we teach the limit concept from time to time, or all as one shot in calculus? Last, but not least, why should the limit concept be taught early? A thoughtful answer to all these issues should be part of the teachers' repertoire in relation to curriculum knowledge. Thus the curriculum knowledge teachers should have ought to contain an appreciation of the interrelation and connection between the limit concept with the other branches of mathematics, such as real numbers and limits, decimals and limits, areas and limits, geometry and limits, sequences and limits, series and limits, graphs and limits, statistics and probability as well as limits, rates of change and limits, and interest rates and limits. They should know how these topics in mathematics intertwine and enhance the teaching about limits and vice versa. How the limit concept is interrelated and connected with other branches in mathematics will be discussed in the section on implications for teaching.

Thirteen and fourteen prospective teachers did suggest some activities that might connect with the K-2 and 4-5 grade ranges mathematics curriculum, respectively in the survey. For example, they mentioned the "Half Division" activity, the counting to infinity, the "Binary Tree" activity, the fractions, the graphs, which all involve the notion of limit. However, these activities are isolated "bits and pieces" which have no relation or make no connection with any specific topic taught in K-5 mathematics. For example, the "Half Division" activity, how does it differ when presented to K-2 or 4-5 group children? What unit or topic could be tied in with this activity? What kinds of representations are suitable for these two groups of children? All these kinds of connections are not shown in this group of prospective teachers' responses to the survey or to the structured interview questions.

Pedagogical Content Knowledge

Pedagogical content knowledge is investigated with the question: What are the possible misconceptions, difficulties, and errors the prospective teachers anticipate in teaching the concept of limits? Pedagogical content knowledge represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of the learner and presented for instruction (Shulman, 1986; Wilson et al.,1987; Even, 1989). Teachers' pedagogical content knowledge in terms of limit should at least include the following: knowledge about ways to explain why limit is a central idea in mathematics in general and calculus in particular to students, knowledge of what a limit really is rather than just being able to state the formal definition to the students, knowledge of how to explain the distinction between an unending process and the product of an unending process to students, and knowledge about what are students' misconceptions, difficulties, and errors as well as why students make these kind of mistakes and how to help students to overcome these mistakes. Mistakes in mathematics are as important and as significant as correct answers and in some cases they are more significant. Mistakes can aid the process of mathematical discovery and assist our mathematical understanding. The mistakes students make can tell teachers more about what might be happening in students' minds than any number of correct answers. As a matter of fact, the correct answers may happen for many reasons, such as coincidence, blind mimicry, memorization, or blatant cheating which provide no insight for teaching and learning (Schwarzenberger, 1984).

What is a limit?

Pedagogical content knowledge of limits should enable teachers to present in some way to their students both formally and informally what a limit is. When asked what is a limit, mathematics teachers and students often tend to give the formal definition of limit as an answer. But the pedagogical content knowledge a teacher ought to have not only includes the formal definition but also different examples, counterexamples, explanations, and representations, which provide a feeling for what is the nature of limits. As Tall & Schwarzenberger (1978) stated in their article, the limit of a sequence is a number (in a numerical setting) or a point (in an analytical setting). It is a very simple object. Since in the numerical setting it is a number, we can add two limits, subtract one limit from another limit, multiply one limit by another limit, and of course, we can divide one limit by another limit provided the limit we divide by is not zero. A limit is just a number, but it is often confused with the process leading to it.

Process or Product

One thing that seems to confuse people about the notion of limit is the emphasis on the unending process rather than the product of that unending process. The notion of limit is exactly the method to get across to the result of the unending process. But ordinarily, people are more interested in the unending process itself, focusing on the unending process, trying to follow this unending process step by step, or trying to visualize each term of the unending process consecutively, and totally forgetting that what we are after is the "what will happen" rather than if the process "goes on and on". This dynamic view of limit was inherent from Greeks' time until Weierstrass. The static point of view given by Weierstrass established the sound, rigorous foundation of the limit concept. This static viewpoint of limit should be included in teachers' pedagogical content knowledge of limit concept.

Students' mistakes--what and why

The other pedagogical content knowledge a teacher should have in terms of the limit concept includes being able to anticipate the students' misconceptions, difficulties, and errors. In order to identify what are the misconceptions, teachers should have thorough understanding about the notion of limit. That is, not only understand the limit concept from an informal point of view as the end product of an unending process, but also be able to understand the formal definition of limit. Thus in the formal definition of limit, what are the subconcepts that constitute the notion of limit which cause difficulties for students? Do the meanings of those symbols representing the subconcepts go beyond human comprehension? What are the logical relationships between those symbols? What are examples, counter-examples, and other representations that will help to explain the formal definition to a student? For example, if a student believes the statement in Questionnaire Part I, test item #5-(a): "a limit describes how a sequence moves as n moves toward infinity," he/she might be presented with the following examples,

$$a_n = 1+\frac{1}{n} \quad \text{and} \quad a_n = 1+\frac{(-1)^n}{n}.$$

These both converge to 1, but the given sequences move toward their limits in different ways.

If a student believes "a number L is the limit of a sequence if the terms of the sequence are always getting closer to L", the following example might help him/her to think through his/her error.

$$2,\ 1\frac{1}{2},\ 1\frac{1}{4},\ 1\frac{1}{8},\ 1\frac{1}{16},\ \ldots;$$

Thus students can consider this statement: "A number L=0 is the limit of the above sequence if the terms of the sequence are always getting closer to that number L=0." Maybe students could find out that "the terms get closer to 0, but 0 is not the limit." This observation will help them to conclude that the above statement is false. Similarly, each of the following sequences serves as counter-examples for the statements in Questionnaire, Part I, test item #5.

5-(b)--"A limit is a number or point past which a sequence cannot go."

$$1+1,\ 1-\frac{1}{2},\ 1+\frac{1}{3},\ 1-\frac{1}{4},\ 1+\frac{1}{5},\ \ldots;\ \text{with } L=1.$$

This sequence fluctuates between numbers bigger and smaller than one.

5-(d)--"A limit is a number or point the sequence gets close to but never reaches."

$1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{1}{16}, \ldots$; with L=0.

The limit of this sequence is zero and half of the terms reach the limit zero.

5-(e)--"A limit is an approximation that can be made as accurate as you wish."

3, 3.1, 3.14, 3.141, 3.1416, 3.14159, ...; with L=π.

π is a fixed real number, not an approximation.

5-(f)--"A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached."

$1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots 1, 1/n, \ldots$; with L=0.

So when we plug in bigger and bigger numbers in the denominator, what we get is "$1/\infty = 0$" and zero is not the limit of this given sequence.

5-(g)--"A limit is the value of the nth term of a sequence, when n equals to infinity."

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$; with L=0.

How can it be said that $\frac{1}{2n}$ (or $\frac{1}{2\infty}$) actually "goes to" 0, if n never "equals to infinity"?

5-(h)--"The limit approaches to a fixed number, when n tends to infinity."

Limit is a fixed number or point itself, which therefore can not vary and <u>approach</u> any number. It is the <u>terms</u> of the given sequence which vary, that approach a fixed number provided the given sequence converges.

There are more examples that might help to clear up some other misconceptions which were not stated as questionnaire test items:

1. A number L is the limit of a sequence if the sequence gets infinitely close to the number and as it gets closer it never gets farther away again:

   $1, 5, \frac{1}{2}, \frac{5}{2}, \frac{1}{4}, \frac{5}{4}, \frac{1}{8}, \frac{5}{8}, \frac{1}{16}, \frac{5}{16}, \ldots$; with L=0.

2. A number L is the limit of a sequence if as you go through the sequence you find numbers closer and closer to that number:

   $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, 1, \frac{1}{16}, \ldots$; with L=0

   Throughout the list there exist numbers that get closer and closer to 0, but 0 is *not* the limit because there is no limit for this sequence.

3. As n approaches infinity, the terms of a sequence are moving closer and closer to the limit L
   (or as $n \to \infty$, $a_n \to L$):
   1, 1, 1, 1, 1, 1, ... ; with L=1
   As n is approaching to the infinity, none of the terms of the above sequence moves toward one, actually every one of the terms is equal to one.

In order to teach about limits and be creative inventors of teaching activities the teachers need to understand exactly what a limit means. In the formal definition of the limit of a sequence, we usually state that: "For every epsilon greater than zero, there exists a natural number N such that whenever n>N the absolute value of the difference between a_n and L is less than epsilon." What does this statement mean? Does the word "every" mean we have to try all the numbers? What is this "epsilon"? Is it a real number? Is it a positive number? or what? a Greek letter symbol for what? Why do we need epsilon to be greater than zero, can it be a negative number? Then follows the next phrase: there exists a natural number N; in this phrase what does the word "exist" mean? Does it mean one N will suffice? or must it be the smallest N? What is the relationship between epsilon and capital N? Next comes the phrase: such that whenever n>N: what does it mean? Are the words "such that" conditional words? What is the logical order in "such that"? What is the lower case n and what is its relation with the capital N? Why do we need lower case n to be bigger than the capital N? The last phrase in this definition is: "the absolute value of the difference between a_n and L is less than epsilon." What is absolute value? What is the difference? Why do we want to find out the difference between a_n and L? What is a_n? and what is L? What does $|a_n-L|$ less than epsilon mean? What does an inequality stand for? What is the relationship between the difference of a_n and L as well as epsilon. What does it mean to be "less than", what will happen if it is greater than? Just think; in this definition so many questions come out. So many mathematical concepts are involved. We have different notations to understand; what meaning does each symbol represent? We need to understand the basic topological properties of the real number system and the set of positive numbers, we need to understand the choice of temporal orders, we need to understand inequalities and be able to solve inequalities, we need to understand the choice between epsilon and N as well as that N is a function of epsilon, we need to find n, we need to understand the notions of intervals and the neighborhoods. All of these are included in the formal definition of the limit of a sequence. Teachers' pedagogical content knowledge regarding limit should include clear answers for these questions and ability to pass that knowledge to their students. A thorough understanding of the formal definition of limit of a sequence, as we stated above, is not just being able to state the definition, but is also pedagogical clarity. Instead of concentrating on "n very large," we should concentrate on "a_n and L are practically indistinguishable." There are three conditions which are involved in the formal definition of a limit:

1. a_n must vary according to some law.
2. The difference a_n-L must become numerically less than any pre-assigned number.
3. As a_n continues to vary the difference a_n-L must remain less in absolute value than this pre-assigned number.

As mentioned above we know that the definition of limit is intellectually difficult (Emch, 1902; Huntington, 1916; Roe, 1910) to comprehend due to the fact that so many subconcepts are involved. So one can ask what are the hardest parts that might be the parts causing problems for students. This type of analysis enhances the power of pedagogical content knowledge of teachers. One must first identify students' weaknesses and then one will be able to come up with solutions. The ability to recognize students' misconceptions, difficulties, and errors as well as their weaknesses should be strongly recommended to be included in teachers' pedagogical content knowledge.

This group of prospective teachers' conceptual knowledge about the notion of limits is inferior compared with their procedural knowledge. For example, they could find the limit for a specific sequence, but could not prove that a given number is indeed the limit. They know the formal definition, but could not understand the underlying concept of the formal definition of a limit. Thus, when provided a formal definition, they could not transfer the formal definition to a specific case. They know the definition of limit by words not by underlying meaning. Thus, they could not use their knowledge of definition in proving the existence of a limit or proving theorems. Most of them thought limit can never be attained. In their minds, there always exists some very small number which provides a gap to reaching the limit. Due to the intrusion of potential infinity, the infinite process can go on and on. Apparently, they confuse the product with the process. They provided some examples of misconceptions, difficulties, and errors students might have based on their own experiences, but they could not provide teaching strategies for overcoming these, neither could they provide what are the reasons behind these misconceptions, difficulties, and errors. While either solving limit problems, or explaining the limit situations, often they exhibited misconceptions. For example, one subject said that he wants to make sure that his students know that one can never ever reach the limit, only get closer and closer. This exhibited the teaching and learning recycling of misconceptions.

From the description of teachers' knowledge regarding the notion of limit, apparently the formal definition is a difficult one for students to learn, especially difficult for those students who have never been exposed to this concept before. But from our considerations of the curriculum knowledge of limit concept, we believe that teachers could provide an early headstart for learning the limit concept. In the following section, the focus will be on suggestions for teaching the notion of limit in K-12 mathematics curriculum.

Implications For Teaching

The limit concept is a fundamental concept in mathematics in general and calculus in particular. I will demonstrate the important role of the limit concept in different branches of mathematics. In the following sections, first, I will exhibit how the limit concept reveals itself implicitly or explicitly in different mathematics topics: number theory, fractions, decimals, areas, algebra, statistics and probability, graphs, geometry, conic sections, etc. These topics and activities are gathered from the writings of the following authors: Buchanan (1966), Fletcher (1980), Gardiner (1980, 1985), Hall (1971), Jochusch & McLoughlin (1990), Orton (1984, 1985, 1987), Orton & Reynold (1986). Next, I will discuss the implications for calculus teachers at university or college level.

Numbers

When students, in K-2 grade range, first start to learn the counting numbers, what will happen? Let the students find out that this counting process seems to never end. Playing a game with the students, asking one student to count to a certain number, we can always add 1 to that number and continue to count. This is the first chance to give the students a feeling about what will happen, if this counting process goes on and on? We do not expect them to come up with the notion of infinity, but at least we might expect them to start to think about whether there is a biggest number if this counting process goes on and on. Or we could provide experiences for the pupil themselves to explore.

Fractions

When the students start to learn about the fractions introduce them to the fraction bars. Maybe ask students to make their own fraction bars. Ask them to color different portions of their fraction bars. One activity we could ask students to demonstrate to themselves, is to observe the decreasing of the fraction 1/n. They could notice that the portions from each fraction bar are getting smaller. Probably this activity could help the students to eliminate one of the common mistakes of thinking "1/2 is smaller than 1/3" based on the false analogy that 2 is smaller than 3. After the students are familiar with this activity, we could ask them to think what will happen if n gets bigger? Will the portions get smaller and smaller? Will the pieces get more and more in number? This will get them to think of the infinitely many, infinitely small, and infinitely large. Not only does this fraction bar activity provide some good mathematical practice, as we said before, but it also has real world aesthetic value. The students could be asked to design a brick wall for their fences or their houses based on different size of fraction bars with different colors and different units.

Another concept which could be taught with the use of fraction bars activity is equivalent fractions. The students can observe that

1/2=2/4=3/6=4/8=5/10=6/12=7/14=8/16=9/18=10/20=...

When they list the denominators and numerators of these equivalent fractions, they have

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,...

2, 4, 6, 8,10,12,14,16,18, 20,...

and these two lists of numbers form two sequences. The teacher could start to ask the students the same old question again, what will happen if these lists of numbers go on and on? What is the relation between the first sequence (if the teacher does not wish to introduce the terminology "sequence", he/she can always use the word "list") and the second sequence? Can we assign a rule for them to show their relation? Maybe then students can have an early start on finding the general rule for sequences.

Folded paper strips can also lead to discussion of sums such as

1/2+1/4+1/8+1/16+... and

1/3+1/9+1/27+...

Consideration of sequences of partial sums of fractions such as

3/10, 3/10+3/100, 3/10+3/100+3/1000, ...

is appropriate in discussion of fractions and decimals in relation to the adding up an endless terms.

We can also ask questions about $\frac{a}{b}$, such as, what happens as a gets big, b gets big, or both a and b get big. Encourage the students to explore and experience the changes. That might give them a feeling for closeness and for infinite processes.

Decimals

After students learn about fractions, the teacher usually tries to make a connection between fractions and decimals. The most common connection is converting a decimal into a fraction. The usual method was the following:

Let $x = 0.3333\ldots$ (1)

Then $10x = 3.3333\ldots$ (2)

Subtracting (1) from (2), we get

$9x = 3$

Thus $x = 1/3$

That is $1/3 = 0.333\ldots$

Usually, students will accept this proof and believe that 1/3=0.333... But when they are asked to transfer that technique in converting 0.999..., a cognitive conflict shows up. That is, they do not believe in the validity of this technique. Although the result shows that 0.999... =1, there remains a doubt or distrust. They could visualize the result of 1+ 3 = 0.333... by the following long division:

```
   0.333...
 3 |1 0
    9
    10
     9
     10
      9
      1 ...
```

etc.

But how could there exist a number which when divided by 1 will produce 0.999... . It is a mystery to students, and they believe that mathematicians made that up. Probably from this point some students start to think mathematics is not made to be believed, mathematics is not practical, mathematics is something to trouble their mind, mathematics is a subject to let them down, so why bother. Probably this is a time to let them think, what does that three dots, i.e.,"..." mean? also what does 0.999... mean? Is 0.999... a fixed number? Is 0.999... a symbol? What is the relationship between 1 and 0.999...? Are they the same or why are they different? Can one draw 0.999... graphically on the number line? Maybe we do not want to confuse students with the formal definition of limit concept, but we certainly can let them think about the above questions. Maybe we could start them with different representations of 0.999...:

1. $0.999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$

2. $0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

3. $0.999\ldots = \lim_{n\to\infty} a_n$, where a_1=0.9, a_2=0.99, a_3=0.999, ..., a_n=n 9's after the decimal point

4. $0.999\ldots = \lim_{n\to\infty} (a_1+a_2+a_3+\ldots+a_n)$, where a_1=0.9, a_2=0.99, a_3=0.999, ..., a_n=n 9's after the decimal point

5. $0.999\ldots = \lim_{n\to\infty} (\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n})$

6. $0.999\ldots = \lim_{k\to\infty} \sum_{n=1}^{n=k} (\frac{9}{10^n})$

7. $0.999\ldots = 1$

I do not suggest to provide all the representations at once, what is exhibited here is different ways of representing 0.999.... Probably along the line the students will be able to pick the underlying idea of 0.999.... Maybe spiral ways of gradually showing these different representations at appropriate times might help students to realize that mathematics is not an isolated, compartmentalized, disjointed, uninterrelated, unconnected man-made subject, but rich, interrelated, connected, and intertwined.

The other instructional activity here is using the decimals to introduce the notion of sequence. Use the above 0.333... example, which most students accept as being equal to 1/3. We could let the students explore ways of representing this 0.333.... One representation could make the connection between the significant decimal digits with the notion of sequence, for example, if we want one digit significant, we have 0.3, if we want two digits significant, we have 0.33, if we want three digits significant, we have 0.333. Then the question could be asked if we want n digits significant, how many 3's do we get? If the number of significant digits is getting bigger and bigger, what will happen? Write down the list of significant digits values, and we get,

0.3, 0.33, 0.333, ..., 0.333...333, ...

n of them

This activity not only gets the students familiar with the notion of sequences, either finite or infinite, but also makes them to start to think about this unending process of adding new 3's. Then the same old question again, what will happen if this process goes on and on? One important idea for relating repeating decimals and fractions is to use the fact that one has to convert repeating decimals into fractions in order to do multiplication and division, sometimes this is true with addition and subtraction also. For example, we can add 0.2 to 0.3 and 0.22 +0.33, 0.222+0.333, how about adding 0.222... and .0333... , we all know that sum is possible to get by looking the pattern of the sequence we get. But when we are confronted with the following addition:

$$\begin{array}{r} 0.555555\ldots \\ +\ \underline{0.888888\ldots} \end{array}$$

what can we conclude about the sum of these two repeating decimals? Students sometimes are confused and wonder which of these is the so-called right answer; 0.121212121212... , 1.444443..., or 1.444444...3. Then, when they can not find the results of division and multiplication of repeating decimals, probably it is time for students to explore and to find out why this conversion of decimals into fractions is necessary. In addition this is a really good headstart opportunity for students to start thinking about the intuitive notion of the limit.

Areas

Either in elementary arithmetic, or in geometry, finding area of a circle, and the circumference of a circle, both involve the notion of π. Thus students could be introduced to the following grid activity as an exploratory work for π. At the same time, the notion of limit, the Squeeze Theorem, as well as the sequences formed by the upper sums and lower sums of the approximations of the area of a circle (see Fig.6.1) or an irregular shape (see Fig.6.2), and the idea of definite integral could also be shown in this activity.

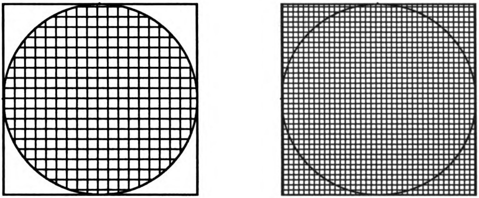


Figure 6.1 -- Grid Activity For π

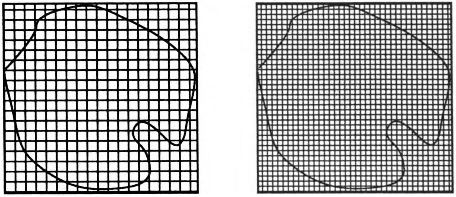
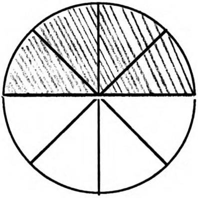


Figure 6.2 -- Grid Activity For Irregular Shape

The other activity also involves the introduction of the area of a unit circle and the circumference of a unit circle. How should we help the child to learn that the area of a circle is πr^2, where r is the radius? The following elegant method which is frequently suggested, even sometimes in primary school books, involves the idea of limit. For example, if a circle is divided into sectors and the sectors are then rearranged in a line, and the process is taken to a limit so that we have more and more sectors with smaller and smaller angles at the center (see figure 6.3), we may eventually appreciate that the required area is the same as for a rectangle of length πr and breadth r.



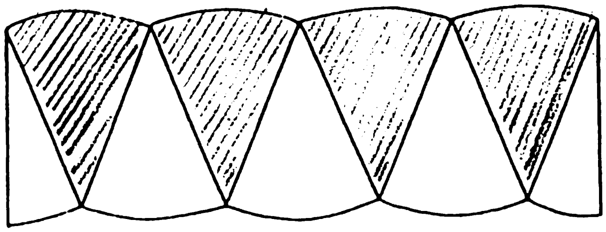


Figure 6.3 -- Slicing and Rearranging a Circle

<u>Sequences and series</u>

In fact, there are many ways in which children might be asked to think about limiting processes in connection with more elementary work. For example, what happens if we have a sequence of regular polygons, starting with a triangle, so that each new polygon has one more side than the previous polygon in the sequence? What happens if we have a sequence of prisms so that each new prism has a cross-section with one more edge than the previous one? What happens with the equivalent sequence of pyramids? In each case the idea of a never-ending process and of limit may be discussed. Supposing we are using a spiral curriculum, then by the time we start to introduce sequences and series, students already have many examples in hand. All the topics mentioned above could provide a handful of examples related to sequences. When we add the terms of the sequence successively, we create a series. Again, we could talk about the old same question: What will happen if we add two terms, three terms, four terms, and add forever and ever? Let the students use calculators or even write their own computer programs to add the terms of a given sequence and explore what will happen, conjecture the results. By drawing the following figures (See Fig.6.4), looking for patterns, maybe they will get a real feeling about the limit concept by now.

The proof of the convergence of a geometric series is as follows:

Let $S=1+r+r^2+r^3+\dots$.

Then multiplying this series by r and subtracting afterwards,

we get,

$$S-rS=1 \qquad \text{or } S = \frac{1}{1-r}.$$

The multiplication of $(1+r+r^2+r^3+\dots)$ by r together with the subtraction of one series from the other, gives the results; but it does not give understanding of how the continuing series approaches this value in its growth. Real understanding proceeds by considering what happens in the growth of the series and derives the law of this growth which lead to the limit. We can discuss why it is necessary to have $-1 < r < 1$ for the above conclusion to be valid.

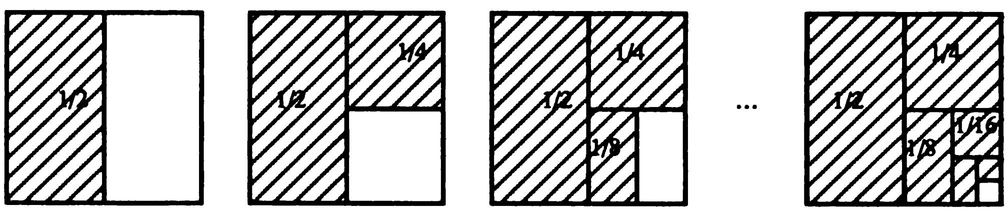


Figure 6.4 -- An Informal Approach to Infinite Series

Statistics and Probability

In NCTM's Standard for Curriculum and Evaluation, statistics and probability is recommended to be taught early in the mathematics curriculum. No matter when teaching of this subject is started, students should have the chance to explore and make connections to the notion of limit. Carrying out probability experiments and collecting the results from around the class may be used to introduce the idea of limit. The following is an activity that could connect the notion of limit in finding the probability of a given outcome when tossing a coin. What is the probability the head shows if we toss the coin once, twice, n times, and finally discuss what will happen if it is infinitely many times? This is then used to prompt discussion about the theoretical probability. The coin tossing results below (See Table 6.1) could be collected by going round in a class and adding on the new individual results each time. Discussion of this results leads quite naturally to the limit concept.

Table 6.1 -- Sets of Results of Tossing a Coin

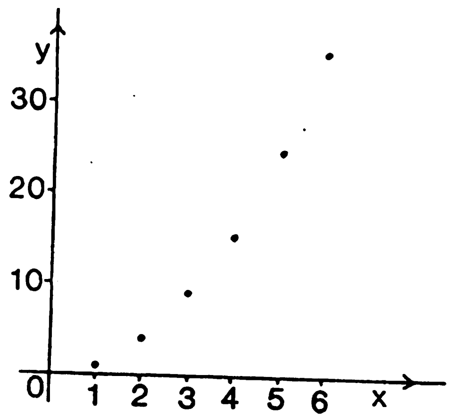
Number of sets of results	1	2	3 ...
Proportion of Heads	0.47	0.46	0.49 ...

Plotting points in graphs

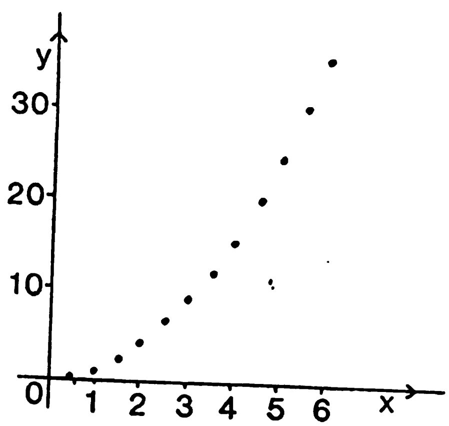
The notion of function is recommended in NCTM's Standard for Curriculum and Evaluation as an important concept which should be taught early in elementary schools. One of the exercises on functions students deal with is to draw the graph of a given function. Plotting points for drawing curves and curves sketching are part of the mathematics curriculum which ought to be taught so as to be related to the limit concept.

Plotting graphs of lines and curves involves a limiting process, for the more points which are plotted the closer we are to the best representation of the complete relationship. This plotting points activity could provide students with a feeling for the limit concept. For example, when looking at the graphs of function: $f(x)=x^2$, x in [0,6]; what will happen, and what the final graph will look like? The more points plotted in, the more accuracy of the graph (see figure 6.5).

x	0	1	2	3	4	5	6	...
y	0	1	4	9	16	25	36	...



x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0	0.25	1	2.25	4	6.25	9	12.25	16	20.25	25



x	3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4
y	9	9.61	10.24	10.89	11.56	12.25	12.96	13.69	14.44	15.21	16

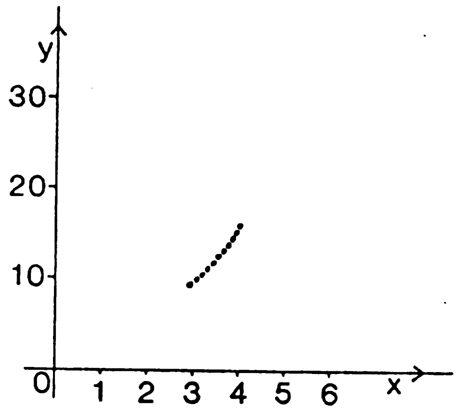


Figure 6.5 -- Plotting Points in Graph

Geometry

In geometry the relationship of the circle to polygons, of the cylinder to prisms, of the cone to pyramids and of the sphere to polyhedrons in general ought to be taught in an manner related to the idea of a limit. Some definitions, such as the definition of an asymptote, can be fully understood only when it is presented as a limit situation, and related to other limit situations to show that the definition really makes sense.

Compound Interest

People are concerned about how their money earns interest for them. Students usually are given the formula for calculating the interest, but they do not know why that formula works. We could introduce the following activity by asking what will happen if our interest is calculated yearly, semi-yearly, quarterly, monthly, weekly, daily, hourly, minutely, and continuously? Let the students explore the interests when calculated differently; is there a pattern? will the interest increase when calculated more frequently? what will happen if it is calculated continuously? This would involve a discussion of the notion of limit, and if the students are old enough, they could be explicitly introduced to the limit concept and the irrational number e as a limit of the sequence of rational approximations of e. Maybe now, the students could realize the real world application of the irrational numbers.

If we put \$100,000 dollars at the beginning in the bank with 8% interest rate annually, then the following Table 6.2 shows what exactly happens as we shorten the time interval when the interest is compounded.

Table 6.2. Amount Of Capital Gain With Compounded Interest

Compounded	Amount After One Year
Annually	$\$100,000(1+0.08)=108,000.00$
Semi-annually	$\$100,000(1+\frac{0.08}{2})^2=\$108,160.00$
Quarterly	$\$100,000(1+\frac{0.08}{4})^4=\$108,243.22$
Monthly	$\$100,000(1+\frac{0.08}{12})^{12}=\$108,299.95$
Daily	$\$100,000(1+\frac{0.08}{365})^{365}=\$108,327.75$
Hourly	$\$100,000(1+\frac{0.08}{8760})^{8760}=\$108,328.69$
Each minute	$\$100,000(1+\frac{0.08}{525600})^{525600}=\$108,328.7068$
Continuously	$\$100,000(1+\frac{0.08}{k})^k=\$100,000\ e^{0.08} \approx \$108,328.7069$ where $\lim_{k\to\infty}(1+\frac{r}{k})^k = e^r$

Irrational numbers

Students learn to add, subtract, divide and multiply irrational numbers, but they have no idea what these numbers are related to in the real world. They know they could find the point corresponding to $\sqrt{2}$ on the number line, but they could hardly distinguish between $\sqrt{2}$ with 1.414 on the number line. Maybe the student can construct irrational numbers by exploring the following figure (see Fig.6.6) based on the Pythagorean Theorem.

What they do is this; first, using the length one as the equal sides of a right isosceles triangle construct the hypotenuse of length $\sqrt{2}$; then, using the hypotenuse and side of length one to form a new right triangle construct the second hypotenuse, and the process goes on. Students then have created a finite list of irrational numbers. As long as students believe the truth of the Pythagorean Theorem they have no doubt about the construction of irrational numbers. But because they are lacking real world applications, they believe that irrational numbers are only useful in number manipulation in mathematics. What we should teach by letting students explore is that $\sqrt{2}$ is the limit of a sequence of rational approximations of $\sqrt{2}$. Either at the time when decimals are introduced or when the notion of sequences is introduced, we should let the students become familiar with the ideas that the irrational numbers are hardly distinguishable from rational numbers on the number line, and they could not be definitely written in either a repeating decimal form or decimal forms of finitely many digits. One representation of irrational numbers is that irrational numbers can be treated as the limits of some sequences of rational approximations.

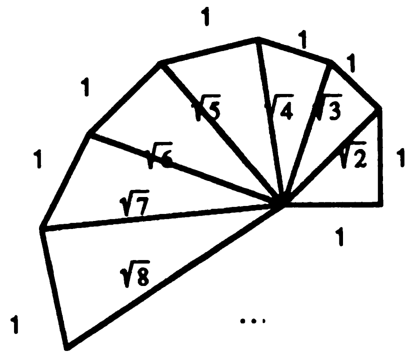


Figure 6.6 -- Construction of Irrational Numbers

Teaching Calculus

From the above discussion we know that the notion of limit reveals itself implicitly or explicitly in many topics throughout the k-12 mathematics curriculum. Typically, however, the student is first exposed to the notion of limits not through any of these topics or activities related to them, but in calculus with the $\varepsilon-\delta$ definition quickly introduced. As one of the prospective secondary teachers testified it not only frustrates them, but "scares them off". Students can easily fall into the error of thinking the limit concept is an isolated concept appearing only in calculus. This researcher believes the usually one or two days "quickie course" in limits presented in a calculus course should not be the first exposure to the notion of limits, but that students ought to be exposed to the intuitive notion of limits earlier in their K-12 through appropriate activities and topics.

This leads to one of my suggestions for mathematics methods courses. It would be desirable for prospective secondary teachers to be more familiar with the connections between important mathematical ideas in different branches of mathematics. Some material presented from a historical perspective might be helpful in developing pedagogical insight. Finally, it is possible that college and university mathematics departments could find ways to design courses or instructional packages that would be more responsive to the special needs of prospective secondary teachers. Blending of subject matter knowledge, curriculum knowledge, and pedagogical content knowledge I believe is a desirable goal in preparing prospective secondary teachers.

Limitations

This researcher remarks on two types of limitation of this study that will be discussed in the following section. The first limitation is the design of the instruments. The second limitation is that with fewer than 40 participants and lots of missing data, this study can only provide a small piece of what teachers' knowledge about limit is.

It seems that there is no other study done on teachers' knowledge on limit concept. This researcher has designed a questionnaire to investigate teachers' knowledge about the mathematical notion of limit. Although there were several pilots conducted separately, still some very important issues were neglected. For example, the questionnaire maybe was too lengthy, and the test items maybe too much resembled school examinations. One reason for this is that the pilots were conducted in Taiwan. Most of the pilot subjects either were college teachers or they were prospective secondary teachers who were accustomed to all kinds of tests because they had to undergo several entrance examinations at various stages of their schooling. Cultural difference might thus affect the content of the design and the timing of the written instrument.

The second part of the instrument is the interview. Because of the structured nature of the interview questions and subquestions, the probing technique was not used during the interview, which might affect the collection of data.

The other limitation is the sample size. Only 42 subjects participated in this study with very high percentage of missing data. Thus, I could not make a generalization about teachers' understanding about the limit concept, especially with regard to the curriculum knowledge and pedagogical content knowledge.

Implications for further research

As stated in Chapter One, the limit concept serves an important and unique role in understanding different branches of mathematics in general and calculus in particular. Improving the teaching and thus enhancing the learning of the limit concept is of considerable importance. Based on what I learned from this study and readings from the literature, some suggestions were made for teaching the limit concept by various related topics in mathematics as reported in the previous section. Certainly other related issues need to be investigated as well. Some of these are discussed below:

1. A study similar to this study needs to be made, probably with longer test times provided. A research question: Does the five categories model of understanding proposed by this researcher could provide an appropriate instrument for investigating prospective teachers' knowledge of the notion of limit? With further thought and experience it is probable that improved questionnaires could be designed, so there is no reason why this study should be the last of its kind.

2. A similar study might be conducted based on each category separately, and their correlation among separate categories could be examined. That might provide a closer understanding of knowledge of the limit concept described in each different category. In particular, does the present model provide an appropriate hierarchical level like that described by Fless (1988)?

3. A repeat similar study based on the theoretical model of this study could be tried on a larger group of prospective teachers using the clinical interview method rather than this structured interview. The probing techniques might provide a better picture about what are prospective teachers' strengths and weaknesses in terms of subject matter knowledge, curriculum knowledge and pedagogical content knowledge.

4. A study of teacher knowledge of other types of limits , based on this five category model of understanding could be made. In particular, the

knowledge of the concept of limit for functions and the related concept of continuity could be investigated.

5. A study could be made for investigating how the misconceptions possessed by both the students and prospective teachers concerning the limit concept could be changed over time as reported in Confrey's (1980) and Williams' (1989) conceptual change studies.

6. Classroom researchers might study whether teaching activities about the limit concept can be tried out for classroom teachers to help them find ways to teach the limit concept to younger students.

7. For a longitudinal research study, the researcher can design the curriculum, follow the students through the years of learning, and observe how this way of teaching influences the calculus-learning performance either in high schools or after entering colleges.

8. Those who are interested in curriculum development could research ways to implement making connections of the limit concept with other mathematics topics.

9. Ways to include basic limit ideas in elementary school mathematics could be investigated.

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APPENDICES

APPENDIX A

Questionnaire

Please try to answer ALL questions and show your work. Please do not erase any of your work, but instead simply draw one line through anything that you later decide not to use. Do all work in the spaces provided and feel free to use the back sides of sheets, if necessary. Thank you very much.

Part I

1. Sex: Male:______ Female:______

2. Age: (19-23):___ (24-29):____ (30-35):____(over 35):______

3. College GPA:_____________ College GPA in mathematics:________________

4. Mathematics courses which have been taken at college or university level:

   Differential calculus:________Integral calculus:_____Advanced calculus:________

   Differential equations:_______Euclidean geometry:_____ Projective geometry:_______

   Theory of numbers:________Theory of matrices:_____ Concepts of algebra:________

   Numerical analysis:_________Statistics & probability:_______ Topology:__________

   History of mathematics:______Vector & tensor analysis:_______ Logic:___________

   Applied discrete math:_______Others:___________________________________

5. Please mark the following eight statements about limits as being true or false:

   T___ F___ a) A limit describes how a sequence moves as n moves toward infinity.

   T___ F ___b) A limit is a number or point past which a sequence cannot go.

   T ___F ___c) A limit is a number that the value of n^{th} term of a sequence can be made arbitrarily close to by letting n go to infinity.

   T___ F ___d) A limit is a number or point the sequence gets close to but never reaches.

T ___F ___e) A limit is an approximation that can be made as accurate as you wish.

T ___F ___f) A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

T ___ F___g) A limit is the value of the n^{th} term of a sequence, when n equals to infinity.

T___ F___h) The limit approaches to a fixed number, when n tends to infinity.

6. Which of the above statements best describes a limit as you understand it (circle one).
   a b c d e f g h none

7. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that $\lim_{n \to \infty} a_n = L$.

8. Please write down a rigorous definition of limit for sequences.

9. Describe an activity that would introduce the idea of limit (of an infinite sequence or series) to a group of children in:
   (a) K-2 grade range:

   (b) 4-5 grade range:

10. What are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how would you help your own students to overcome them?

Part II

1. Find the limit of the following infinite sequences (a) - (f), select exactly one of the following answers:
   (A) The indicated limit is 0.
   (B) The indicated limit is 1.
   (C) The indicated limit is -1.
   (D) The sequence does not have a limit (which includes ∞ and -∞).
   a) 1, -1, 1, -1, 1, -1,...

   Choice (A, B, C, & D):________.

   b) 3/4, 9/16, 27/64, 81/256, 243/1024, ...

   Choice (A, B, C, & D):________.

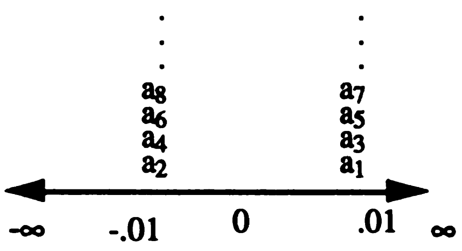
   c) $a_n = 1 + \frac{(-1)^n}{n}$

   Choice (A, B, C, & D):________.

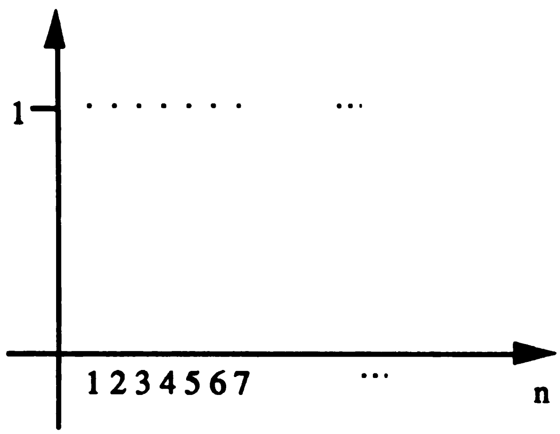
   d) $a_n = \begin{cases} n/n+1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$

   Choice (A, B, C, & D):________.

   e)

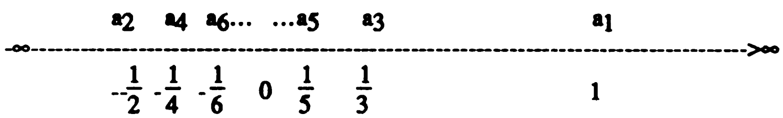


   f)



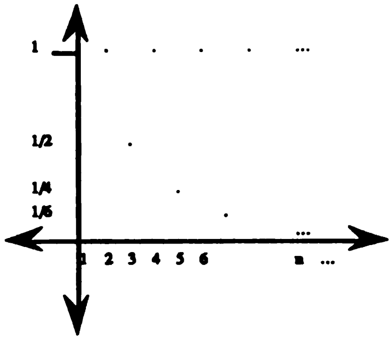
Choice (A, B, C, & D):________.   Choice (A, B, C, & D):________.

2. The following infinite sequences (a) - (b) are described by giving their graphs, find what the limit is (if there is one) or indicate there is no limit. <u>In both cases, please explain why.</u>
   a)   The limit is ______, because:



The limit does not exist, because:

b)



The limit is _______, because:

The limit does not exist, because:

3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:

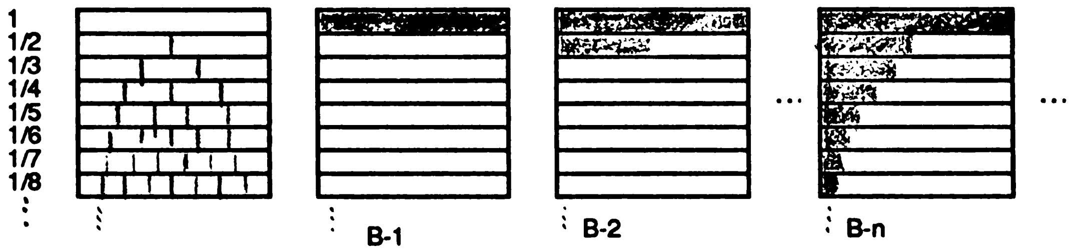


Figure A Figure B

a) Write down the infinite sequence formed by the individual shaded fraction bars in figure (B), and what is its limit?

b) Write down the infinite sequence formed by the partial sums of the sequence in (a), and what is its limit?

4. In (a)- (e), select exactly one of the following answers: **(Show your work or give explanation!)**

(A) The indicated limit is a finite number L. In this case, state specifically what the number is.
(B) The indicated limit is ∞.
(C) The indicated limit is $-\infty$.
(D) The sequence does not have a limit (Which excludes ∞ and $-\infty$).

a) $\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1}$

Choice (A, B, C, & D):________. For choice of A, L= _______
For choice of D, because

b) $\lim_{n \to \infty} \{ (-1)^n + \frac{1}{n} \}$

Choice (A, B, C, & D):________. For choice of A, L= _______
For choice of D, because

c) $\lim_{n \to \infty} \frac{3^{1-n}}{4^{1-n}}$

Choice (A, B, C, & D):________. For choice of A, L= _______
For choice of D, because

d) $\lim_{n \to \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$

Choice (A, B, C, & D):________. For choice of A, L= _______
For choice of D, because

5. The formal definition of the phrase "$\lim_{n\to\infty} a_n = L$, L is a finite real number" is as follows:

"For each $\varepsilon > 0$, there is a natural number N such that $|a_n - L| < \varepsilon$ whenever $n > N$".

a) Illustrate the meaning of this definition, by using the sequence $\{a_n = \frac{2n}{n+1}\}$ with $\lim_{n\to\infty} a_n = 2$ on a graph.

b) According to the <u>formal</u> definition of limit, what would one have to show in order to prove $\lim_{n\to\infty} \frac{2n}{n+1} = 2$?

c) Using the <u>formal</u> definition of limit, prove that $\lim_{n\to\infty} \frac{2n}{n+1} = 2$.

6. The infinite sequence a_n is defined by $a_n = \frac{6n - (-1)^n}{2n}$. Which of the following is the smallest N such that for $n > N$, a_n will be contained in an open interval of radius 1/500 about 3. **<u>(Show your work!)</u>**

_____a) N=1000   _____b) N=500   _____c) N=250   _____d) N=125   _____e) N=100

7. Find $\lim_{n\to\infty} a_n$, given the information that the sequence $\{a_n\}$ satisfies

$$3n - 1 < n\,a_n < 3n + 2.$$

8. Write down the formal definition of the <u>negation</u> of the limit of a sequence, that is, "$\lim_{n\to\infty} a_n \neq L$, where L is a finite real number".

9. Suppose $a_n = \begin{cases} 1 & \text{for n odd} \\ -1 & \text{for n even} \end{cases}$

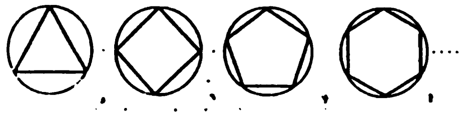
a) According to the <u>formal</u> definition of limit, what would one have to show in order to prove that $\lim_{n\to\infty} a_n$ does not exist?

b) Using the <u>formal</u> definition of limit, prove that $\lim_{n\to\infty} a_n$ does not exist.

10. Using the <u>formal</u> definition of limit prove the following statement:

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} (a_n + b_n)$ exists and $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

11. Consider the following sequence formed by geometric figures below:



a) As the number of sides tends to infinity what does the regular polygon look more like and how many polygons are formed during this process? Why?

b) Describe in words what is the sequence formed by the geometric figures above and what is its limit?

c) Does the limit of this sequence possess the same properties as the terms of the sequence? Why?

12. Given the decimal expansion 0.9999...,

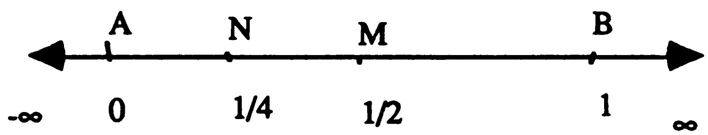
a) Please explain what real number is represented by 0.999... and why?

b) Please explain the meaning of 0.999... in terms of an infinite sequence and what is its limit?

c) Please explain the meaning of 0.999... in terms of an infinite series and what is its limit?

d) If one of your students said that 0.999... is less than 1, would you support him in his conclusion if so, why so; if not, why not?

13. A grasshopper (think of the grasshopper as a point having no length), starting at point A, jumps toward point B. On his first hop he lands at M, the midpoint of the segment AB. On his second hop he lands at N, the midpoint of the segment MB. He keeps hopping, each time landing at the midpoint of the remaining segment.



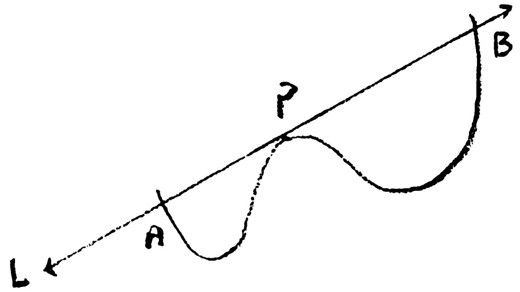
a) Write down a sequence describing the length of the hops and what is its limit?

b) Write down a sequence to describe the total distance travelled at the nth stage of this process and what is its limit?

c) Draw a geometric figure to describe both the length of the separate hops and the total distance travelled by the grasshopper.

d) Does the grasshopper ever reach the point B? Why?

14. For a given point P on the circle, the tangent to the circle at P was probably defined to be the line which passes through point P and only point P on the circle. However, for curves which are not circles, this definition would not suffice, as the curve in figure below illustrates.



In this figure, line L, which is tangent to the curve at point P, passes through points of the curve other than P, namely A and B. Moreover, there are infinitely many lines which are not tangent to the curve at P and yet which pass through P and only point P of the curve. Now, try to define "the tangent to a curve at a given point P" on the curve to your algebra II students as an example to introduce the idea of the limit of an infinite sequence.

APPENDIX B

Interview Questions

1. Please explain why the following eight statements about limits are true or false:

   a) A limit describes how a sequence moves as n moves toward infinity.

   b) A limit is a number or point past which a sequence cannot go.

   c) A limit is a number that the value of the n^{th} term of a sequence can be made arbitrarily close to by letting n go to infinity.

   d) A limit is a number or point the sequence gets close to but never reaches.

   e) A limit is an approximation that can be made as accurate as you wish.

   f) A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

   g) A limit is the value of the n^{th} term of a sequence, when n equals to infinity.

   h) The limit approaches to a fixed number, when n tends to infinity.

2. Which of the above statements best describes a limit as you understand it. Explain.

3. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that $\lim_{n \to \infty} a_n = L$.

   a. What does "lim" stand for?

   b. What does " n--> ∞" stand for? Does n ever equal infinity?

      If not, how can we write "∞+m"?

      "∞ +m = ∞ -m" is this true or not? Explain.

   c. What does " $\lim_{n \to \infty} a_n$ " stand for?

      What does "L" stand for? Is L a fixed number or a symbol?

What is the relationship between "L" and " $\lim_{n\to\infty} a_n$"?

d. What is the relationship between "a_n" and $\lim_{n\to\infty} a_n$"?

e. What is the difference between "a_n=L" and " $\lim_{n\to\infty} a_n = L$"?
Do the "equals signs" have the same meaning in both statements?

f. When we make the statement " $\lim_{n\to\infty} a_n = L$", does n equal infinity?

g. In the statement "as n-->∞, a_n-->L", what does the symbol "-->" mean? Does "a_n --> L" mean a_n will never equal L as long as n only approaches infinity?

4. We usually write the definition of limit for sequences as follows:

" $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \ni |a_n - L| < \varepsilon$, for all $n > N$ "

a. Please explain what each of the following logical symbols stands for?

b. What is the relationship between ε and N?

c. Does the order of ε and N matter in the process of testing for them?

d. Could you provide another way to define the notion of limit for sequences?

5. a. Do you think the notion of limit is an important concept in mathematics? Explain.
b. Why do we need the notion of limit in mathematics?
c. We need the notion of limit in order to solve what kind of problems?

6. What prior knowledge (or mathematical concepts) are needed for studying the notion of limit? Explain.

7. Describe an activity that would introduce the idea of limit (of an infinite sequence or series) to a group of children in:
k-2 grade range:
4-5 grade range:

a. Do you think the k-2 grade range children will be knowledgeable enough to informally learn the notion of limit?

b. How about 4-5 grade range?

c. What kind of activity do you think is appropriate to use to introduce the notion of limit?

d. What grade range will be able to accept this activity? Explain.

e. Do you think we should informally introduce the notion of limit as early as possible? Explain.

f. When do you think is the best time to introduce the notion of limit?

8. What are the possible misconceptions, difficulties, and errors you encountered while learning about limits, and how did you overcome them?

   a. What do you anticipate are the misconceptions, difficulties, and errors that students will encounter most while learning about limits? Explain.

   b. Are there more?

   c. Why do you think these cause trouble?

   d. Is there a way to eliminate these ?

   e. Are there other methods?

   f. Do you think these are caused by the abstractness of the limit concept, or due to the teaching? Explain.

   g. Is the limit concept easy to learn? Explain.

   h. Is the limit concept easy to teach? Explain.

   i. As a teacher, how are you going to teach the limit concept?

9. a. Given only the first few terms (as in the examples below), will the sequence be uniquely determined?

   i) 1, 1/2, 1/3, 1/4, 1/5, ...

   ii) 1/1, 4/3, 9/5, 16/7, 25/9,...

   b. Will we be able to find the limit of a sequence with only the first few terms (finitely many terms) known?

10. a. One student found the limit of the sequence, $\{a_n=(3/4)^n\}$, by writing down the following statements:

    $(3/4)^n = 3^n/4^n \dashrightarrow \infty/\infty=1.$

    What do you think?

    b. The other student stated that $a_n=(3/4)^n=3^n/4^n$ and as n goes to infinity, the infinity of the denominator is bigger than the infinity of the numerator. Thus the limit is zero. What is your comment?

APPENDIX C

ANSWER SHEETS AND SCORING SYSTEM

Basic Understanding

****

1. In the following infinite sequences (a) - (f), select exactly one of the following answers:
   (A) The indicated limit is 0.
   (B) The indicated limit is 1.
   (C) The indicated limit is -1.
   (D) The sequence does not have a limit (which includes ∞ and -∞).

****

Correst Answer:

a) 1, -1, 1, -1, 1, -1,...

   Choice (A, B, C, & D):_____D___.

b) 3/4, 9/16, 27/64, 81/256, 243/1024, ...

   Choice (A, B, C, & D):_____A___.

c) $a_n = 1 + \frac{(-1)^n}{n}$

   Choice (A, B, C, & D):___B_____.

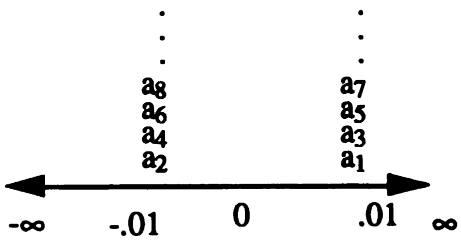
d) $a_n = \begin{cases} n/n+1 & \text{for n odd} \\ 1 & \text{for n even} \end{cases}$

   Choice (A, B, C, & D):____B____.

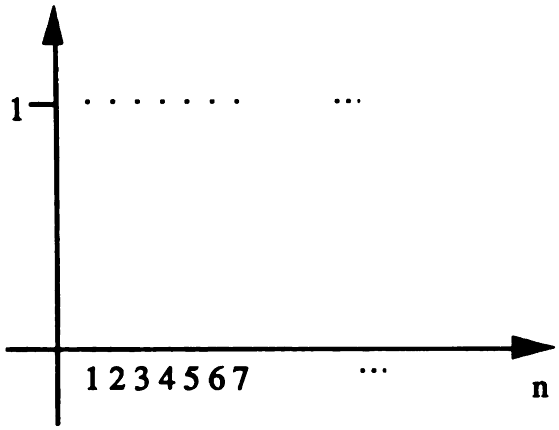
****

***

e)



f)



Choice (A, B, C, & D):____D____.

Choice (A, B, C, & D):____B____.

***

***

Scoring System For Question #1:

0 pt.-- Incorrect choice of A, B, C, and D.
1 pt.-- Correct choice of A, B, C, and D.

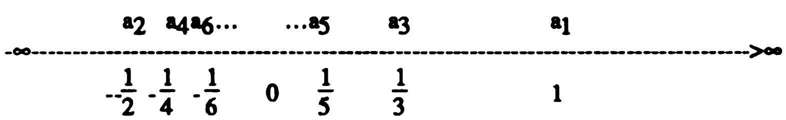
***

2. The following infinite sequences (a) - (b) is described by giving its graph, find what the limit is (if there is one) or indicate there is no limit. In both cases, please explain why.

***

Correct answer:

a)

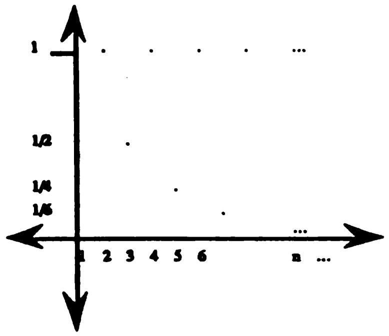


The limit is 0, because $| a_n - 0 | = \frac{1}{n}$, and $\frac{1}{n}$ gets arbitrarily small as n goes arbitrarily large or

We can see that we can get a_n as close to 0 as we want by taking n large enough.

***

b)



The limit does not exist, because for odd n a_n is equal to 1, for even n a_n is getting close to 0, but there isn't any single number the a_n are all getting close to.

Scoring System For Question #2:

0 pt.-- No response.

0 pt.-- Incorrect choice of limit exists or does not exist. For example, choose that 2-a does not have a limit, or choose that 2-b has a limit. In both cases the choice was wrong.

1 pt.-- Incorrect choice , but provide adequate explanation why the given sequence is convergent or does convergent.

1 pt.-- Correct choice, but providing no explanation why L is the limit or why the given sequence does not have a limit.

2 pt.-- Correct choice with reasonable explanation for that choice. For example, choose 2-a has limit 0, because We can see that we can get a_n as close to 0 as we want by taking n large enough.

3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:

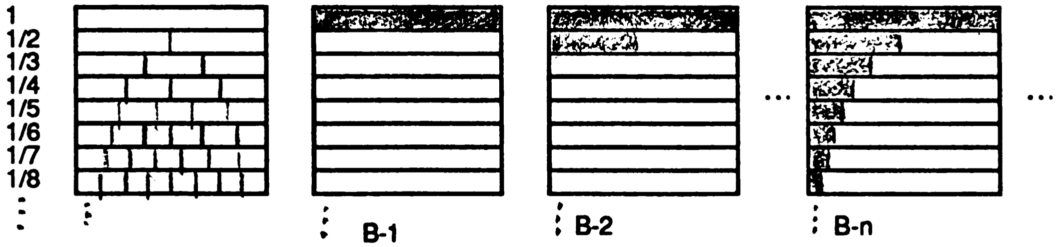


Figure A

Figure B

***

a)Write down the infinite sequence formed by the individual shaded fraction bars in figure (B), and what is its limit?

***

Correct Answer:

The sequence is {1/n}: 1, 1/2, 1/3, 1/4, ..., 1/n, ... or

$A_n=1/n$. The limit of this sequence is 0, or

$A_n = \Sigma_{k=1}^{k=n} 1/k$, the limit is infinity.

***

Scoring System For Test Item #3-a:

0 pt.-- No response.
0 pt.-- Incorrect response. For example,the sequence is {B-1/n} and the limit is B.

1 pt.-- Providing the correct sequence with incorrect limit or with no limit number given. For example, stated that $\{a_n=1/n\}$ but the limit is 2,or other finite number rather than the true limit which is 0; or stated that $\{a_n=\Sigma_{k=1}^{k=n} 1/k\}$,but the limit is 2, or other finite number rather than this sequence is divergent; or the sequence is 1/n with no limit value given.

1 pt.-- Providing the correct limit with incorrect sequence. For example, stated that $a_n=1/2^n$, but the limit is 0 which is true for both sequences.

2 pt.-- Providing the correct sequence with correct matching limit. For example, both the harmonic sequence $\{a_n=1/n\}$ and harmonic series $\{a_n=\Sigma_{k=1}^{k=n} \frac{1}{k}\}$ is considered as correct responses, and their limits are 0 and ∞, repectively.

***

Category II: Computational Understanding

4. In (a)- (e), select exactly one of the following answers: <u>(Show your work or give explanation!)</u>

(A) The indicated limit is a finite number L. In this case, state specifically what the number is.
(B) The indicated limit is ∞.
(C) The indicated limit is - ∞.
(D) The sequence does not have a limit (Which excludes ∞ and $-\infty$).

***

****

Correct Answer:

a) $\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1}$

Choice (A, B, C, & D):__A___. For choice of A, L= 1/2.

$$\lim_{n \to \infty} \frac{3n^2 + 5n}{6n^2 + 1} = \lim_{n \to \infty} \frac{3 + 5/n}{6 + 1/n^2} = \frac{3 + \lim_{n \to \infty} 5/n}{6 + \lim_{n \to \infty} 1/n^2} = \frac{3}{6} = \frac{1}{2}$$

Since both $\lim_{n \to \infty} 5/n = 0$ and $\lim_{n \to \infty} 1/n^2 = 0$

b) $\lim_{n \to \infty} \{ (-1)^n + \frac{1}{n} \}$

Choice (A, B, C, & D):__D___. For choice of A, L= _______

(1) For odd n, a_n tends to -1, and for even n, a_n tends to 1
(2) But there is no <u>single</u> number the a_n are getting close to.

c) $\lim_{n \to \infty} \frac{3^{1-n}}{4^{1-n}}$

Choice (A, B, C, & D):__B___. For choice of A, L= _______

Either $\frac{3^{1-n}}{4^{1-n}} = (\frac{4}{3})^{n-1}$, or since $\frac{4}{3}$ is larger than 1, the successive power of $\frac{4}{3}$ go to positive infinity.

d) $\lim_{n \to \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$

Choice (A, B, C, & D): A. For choice of A, L= $-\frac{9}{2}$

$$\lim_{n \to \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$$

$$= \lim_{n \to \infty} \frac{(\sqrt{n^2 + n} - \sqrt{n^2 + 10n}) (\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n \to \infty} \frac{(n^2 + n) - (n^2 + 10n)}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n \to \infty} \frac{-9n}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n \to \infty} \frac{-9}{(\sqrt{1 + 1/n} + \sqrt{1 + 10/n})}$$

$$= \frac{-9}{(\sqrt{1 + \lim_{n \to \infty} 1/n} + \sqrt{1 + \lim_{n \to \infty} 10/n})}$$

$$= -\frac{9}{2}$$

Since both $\lim_{n \to \infty} 1/n = 0$ and $\lim_{n \to \infty} 10/n = 0$

****

Scoring System For Question #4:

0 pt.-- Incorrect choice with either wrong computation (#4-a, and #4-d) or with incorrect explanation (for #4-b and 4#-c), or no response.

Examples:

If on #4-a a subject chooses B and gives the incomplete computation

$\frac{3+5}{6+1}=\frac{8}{7}, \frac{12+10}{24+1}=\frac{22}{25}$, or

If on #4-b a subject chooses C and gives the following explanation

"As you plug ∞ in you get $-\infty+\frac{1}{\infty} = -\infty + 0 = -\infty$.", or

If on #4-c a subject chooses A and gives 1 for L with the computation:

$\frac{3^0}{4^0} = 1$, or

If on #4-d a subject chooses C and gives the following computation:

$$\lim_{n\to\infty} [(n^2+n)^{1/2} - (n^2-10n)^{1/2}]$$
$$= \lim_{n\to\infty} (n^2+n)^{1/2} - \lim_{n\to\infty} (n^2-10n)^{1/2}$$
$$= \lim_{n\to\infty} (1+\frac{1}{n})^{1/2} - \lim_{n\to\infty} (1-\frac{10}{n})^{1/2}$$
$$= 1-1 = 0$$

2 pt.-- Correct choice, and correct number L (for #4-a) or correct explanation (for #4-b and #4-c) or for #4-d one last crutial computational error.

Examples:

If on #4-a a subject chooses A and gives $\frac{1}{2}$ for L, or

If on #4-b a subject chooses D, since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, or

If on #4-c a subject chooses B and gives the explanation that "the geometric sequence with ratio bigger than 1 is divergent", or

If on #4-d a subject chooses C and gives the following computation:

$$\frac{n^2+n-n^2-10n}{\sqrt{n^2+n} + \sqrt{1+10n}}$$

$$= \frac{-9n}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n}} + \sqrt{1+\frac{10}{n}}}$$

$$= \frac{-9}{2} n$$

****

Continued

****

3 pt.-- Correct choice, correct number L, and correct computation (for #4-a and #4-d).
Examples:

If on #4-a a subject chooses A and gives $\frac{1}{2}$ for L, and the following computation:

$$\lim_{n\to\infty} \frac{3n^2 + 5n}{6n^2 + 1} = \lim_{n\to\infty} \frac{3 + 5/n}{6 + 1/n^2} = \frac{3 + \lim_{n\to\infty} 5/n}{6 + \lim_{n\to\infty} 1/n^2} = \frac{3}{6} = \frac{1}{2}$$

Since both $\lim_{n\to\infty} 5/n = 0$ and $\lim_{n\to\infty} 1/n^2 = 0$, or

If on #4-d a subject chooses A, and gives -9/2 for L, and the following computation:

$$\lim_{n\to\infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10n})$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n^2 + n} - \sqrt{n^2 + 10n})(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n\to\infty} \frac{(n^2 + n) - (n^2 + 10n)}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n\to\infty} \frac{-9n}{(\sqrt{n^2 + n} + \sqrt{n^2 + 10n})}$$

$$= \lim_{n\to\infty} \frac{-9}{(\sqrt{1 + 1/n} + \sqrt{1 + 10/n})}$$

$$= \frac{-9}{(\sqrt{1 + \lim_{n\to\infty} 1/n} + \sqrt{1 + \lim_{n\to\infty} 10/n})}$$

$$= -\frac{9}{2}$$

Since both $\lim_{n\to\infty} 1/n = 0$ and $\lim_{n\to\infty} 10/n = 0$

****

Category III: Transitional Understanding

3. Figure (A) below illustrates the fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:

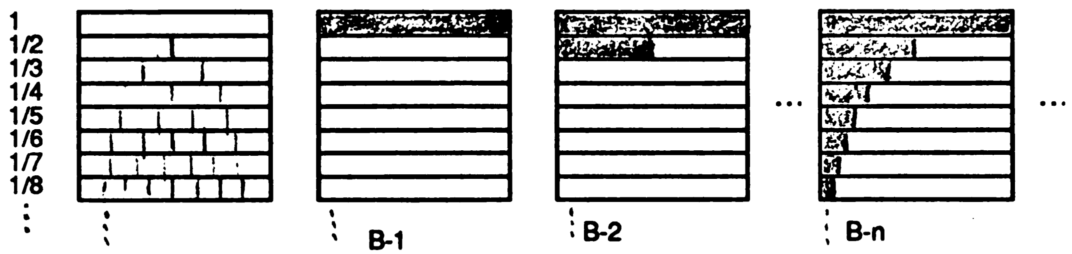


Figure A

Figure B

b) Write down the infinite sequence formed by the partial sums of the sequence in (a) and what is its limit?

****

Correct Answer:

This sequence is $\{a_n = \sum_{k=1}^{k=n} \frac{1}{k}\}$: 1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4,..., 1+1/2+1/3+...+1/n,...

The sequence is divergent and has no limit. Or the alternative answer is the limit is positive infinity.

****

****

Scoring System For Test Item #3-b:

0 pt.-- Incorrect sequence and incorrect limit.

Examples:

If a subject gives the general term of the sequence as $a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ and gives 0 for L, or

If a subject gives $a_n = \sum_{k=1}^{k=n} \frac{1}{2k}$ and gives 2 for the limit,or

If a subject gives $a_n = \frac{1}{n-1}$ and gives the expression $\lim_{n->\infty} \frac{1}{n-1}$ for the limit, or

If a subject gives the sequence as: $\frac{1}{2} + \frac{2}{3}$, $\frac{3}{4} + \frac{4}{5}$, $\frac{5}{6} + \frac{6}{7}$, ... and gives 2 for the limit.

****

Continued

****

1 pt.-- Correct sequence and incorrect limit.

Examples:

If a subject gives the sequence $\{a_n=\Sigma_{k=1}^{k=n}\frac{1}{k}\}$ and gives 2 for the limit or other finite numbers, or

If a subject gives the sequence $\{a_n=\Sigma_{k=1}^{k=n}\frac{1}{k}\}$ and gives no limit.

2 pt.-- Correct sequence and correct limit.

Examples:

If a subject gives the sequence is $\{a_n = \Sigma_{k=1}^{k=n}\frac{1}{k}\}$ and the sequence is divergent and has no limit, or

If a subject gives the sequence in the numerical representation as
1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4,..., 1+1/2+1/3+...+1/n,...
and gives the limit is positive infinity.

****

5. The formal definition of the phrase " $\lim_{n\to\infty} a_n = L$, L is a finite real number" is as follows:

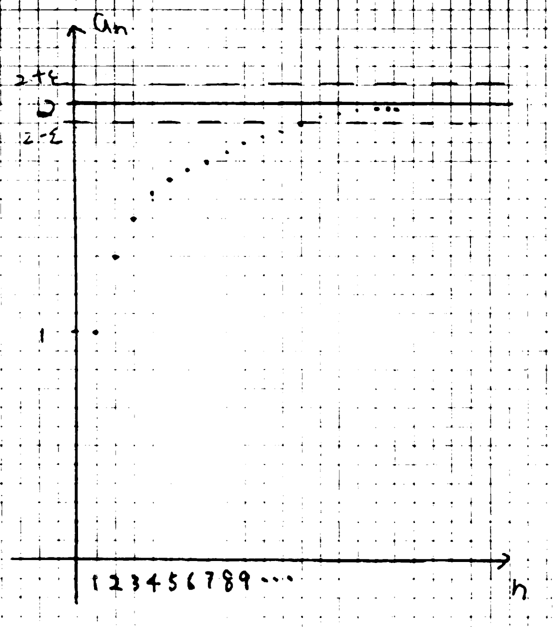
"For each $\varepsilon > 0$, there is a natural number N such that $|a_n - L| < \varepsilon$ whenever $n > N$".

a) Illustrate the meaning of this definition, by using the sequence $\{a_n = \frac{2n}{n+1}\}$ with $\lim_{n\to\infty} a_n = 2$ on a graph.

****

Correct Answer For #5-a:

n	a_n
1	1
2	$\frac{4}{3}$
3	$\frac{6}{4}$
4	$\frac{8}{5}$
5	$\frac{10}{6}$
6	$\frac{12}{7}$
7	$\frac{14}{8}$
⋮	⋮



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b) According to the formal definition of limit, what would one have to show in order to prove $\lim_{n \to \infty} \frac{2n}{n+1} = 2$?

****

Correct Answer For #5-b:

One would have to prove that for any positive real number ε there is a natural number N such that $\left| \frac{2n}{n+1} - 2 \right| < \varepsilon$ if $n > N$.

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Scoring System For Question #5:

0 pt.-- Incorrect graph (for #5-a) and incorrect response (for #5-b) or no response

Examples:

If on #5-a a subject gives an incorrect graph of the sequence and incorrect labelling of the limit on that graph, e.g. graph (a) in Fig 5.1, or

If on #5-b a subject gives an incorrect statement like "the limit of n approaches ∞"

1 pt.-- Continuous graph (for #5-a) and explanation with some ideas in it.

Examples:

If on #5-a a subject draws a continous graph, e.g. graph (b) in Fig 5.1, rather a discrete graph, or

If on #5-b a subject gives the following formal definition explanation:

"For each $\varepsilon > 0$, there is a natural number N such that $| a_n - L | < \varepsilon$ whenever $n > N$".

2 pt.-- Correct graph (for #5-a) and correct explanation.

Examples:

If on #5-a a subject draws the correct graph, or

If on #5-b a subject responds that " for any positive real number ε there is a natural number N such that $\left| \frac{2n}{n+1} - 2 \right| < \varepsilon$ if $n>N$", or

If a subject gives the following explanation "for n>N, the terms of the sequence all lie within e units of 2".

****

6. The infinite sequence a_n is defined by $a_n = \frac{6n - (-1)^n}{2n}$. Which of the following is the smallest N such that for n>N, a_n will be contained in an open interval of radius 1/500 about 3. (Show your work!)
____a)N=1000 ____b) N=500 __*__c) N=250 ____d) N=125 ____e) N=100

****

Correct Answer:

$|a_n - 3| = \frac{1}{2n}$

So $|a_n - 3| < \frac{1}{500}$ iff $\frac{1}{2n} < \frac{1}{500}$
iff $2n > 500$
Iff $n > 250$.

****

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Scoring System For Question #6:

0 pt.-- No response or incorrect response.
Examples:
If a subject chooses N (=500), or

If a subject chooses N (=125), or

If a subject chooses N (=100).

1 pt.-- Correct choice for N (=250) with no work shown.

2 pt.-- Correct choice for N (=250) and correct work.
Examples:
If a subject chooses N (=250), and shows the following correct computation,

$| a_n - 3 | = \frac{1}{2n}$

So $| a_n - 3 | < \frac{1}{500}$ iff $\frac{1}{2n} < \frac{1}{500}$
Iff $2n > 500$
Iff $n > 250$
So N = 250, or

If a subject chooses N (=250) by plugging all the possible N's and concludes by looking at the patterns.

****

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7. Find $\lim_{n\to\infty} a_n$, given the information that the sequence $\{a_n\}$ satisfies

$$3n - 1 < n\, a_n < 3n + 2.$$

****

Correct Answer:

$$\frac{3n-1}{n} < a_n < \frac{3n+2}{n}$$

$$3 - \frac{1}{n} < a_n < 3 + \frac{2}{n}$$

Both $3 - \frac{1}{n}$ and $3 + \frac{2}{n}$ tend to 3, so by the "squeeze Law" $\lim_{n\to\infty} a_n = 3$.

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<u>Scoring System For Question #7:</u>

0 pt.-- No response or incorrect response.

Examples:

If a subject says the limit is infinity, or

If a subject gives the following answer:

$3n - 1 < n\, a_n < 3n + 2 \Longleftrightarrow 3(\infty) - 1 < (\infty)\, a_{(\infty)} < 3(\infty) + 2$

$3 < \infty\, a_\infty < 3 \Longleftrightarrow$ This is a contradiction.

1 pt.-- Gives answer $\lim_{n\to\infty} a_n = 3$ with no work shown.

2 pt.-- Correct limit found by using half the inequality.

Example:

If a subject gives the following computation:

$$a_n < \frac{3n+2}{n}\ ;\ \lim_{n\to\infty} \frac{3n+2}{n} = \lim_{n\to\infty} 3 + \frac{2}{n} = 3 + \lim_{n\to\infty} \frac{2}{n} = 3 + 0 = 3.$$

3 pt.-- Correct limit with correct computation.

Examples:

If a subject gives the following expression: $\frac{3n-1}{n} < a_n < \frac{3n+2}{n}$ and looks for patterns by plugging in different values for n, or

If a subject finds the limit and gives work by using the Squeeze Theorem.

****

Category IV: Rigorous Understanding

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5. The formal definition of the phrase " $\lim_{n\to\infty} a_n = L$, L is a finite real number" is as follows:

"For each e > 0, there is a natural number N such that $|a_n - L| < e$ whenever n > N".

c) Using the <u>formal</u> definition of limit, prove that $\lim_{n\to\infty} \frac{2n}{n+1} = 2$.

****

Correct Answer:

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n-2n-2}{n+1}\right| = \frac{2}{n+1}$$

Let e >0 be given. Then $\left|\frac{2n}{n+1} - 2\right| < \varepsilon$ if $\frac{2}{n+1} < \varepsilon$

which is true if $\frac{n+1}{2} > \frac{1}{\varepsilon}$

which is true if $n > \frac{2}{\varepsilon} - 1$

So the first integer past $\frac{2}{\varepsilon} - 1$ would do for $N(\varepsilon)$.

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<u>Scoring System For Test Item #5-c:</u>

0 pt.-- No proof or incorrect proof or merely finding the limit of a given sequence rather than a proof.

Examples:

If a subject gives the following proof

$$\lim_{n\to\infty} \frac{2n}{n+1} = \frac{\lim_{n\to\infty} 2n}{\lim_{n\to\infty} n+1} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} 1+1/n} = \frac{2}{1} = 2, \text{ or}$$

If a subject gives $\int_{-\infty}^{\infty} 2n(\frac{1}{n+1}) = 2\int_{-\infty}^{\infty} n(\frac{1}{n+1}) =$, or

If a subject gives $\frac{2\infty}{\infty+1} = \frac{2\infty}{\infty} = 2$

1 pt.-- Incomplete proof

Examples:

If a subject shows part of the proof as

$$\lim_{n\to\infty} \frac{2n}{n+1} - \frac{2(n+1)}{n+1} = 0; \quad \lim_{n\to\infty} \frac{-1}{n+1} = 0 \text{ ,or}$$

If a subject shows that $\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n-2n-2}{n+1}\right| = \left|\frac{-2}{n+1}\right|$

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Continued

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2 pt.-- Slightly incorrect proof.

Examples:

If a subject shows how to find N for a specific choice of ε (ε=0.01) but not how to find N for general ε, or

3 pt.-- Correct proof.

Examples:

If a subject proves that $\left| \frac{2n}{n+1} - 2 \right| = \frac{2n+2-2n}{n+1} = \frac{2}{n+1} < \varepsilon$

$n+1<2/\varepsilon \implies n<2/\varepsilon - 1$

$N=[2/\varepsilon-1]$

****

8. Write down the formal definition of the negation of the limit of a sequence, that is, " $\lim_{n\to\infty} a_n \neq L$, where L is a finite real number".

****

Correct Answer:

There exists a positive number ε such that given a natural number N there exists n>N such that $|a_n-L| > \varepsilon$

Alternative: There exists a positive number ε such that $|a_n-L| > \varepsilon$ for infinitely many n.

****

****

Scoring System For Question #8:

0 pt.-- Incorrect statement of definition.

Examples:

If a subject states that the definition is "The negation of a limit exists when a sequence approaches one value from below but a different one from above", or

If a subject states that "If the limit goes to L then the limit a_n is not equal to L".

1 pt.-- Statement with two quantifiers wrong.

Example:

If a subject states that "if there exists an ε such that there exists a natural number N such that $| a_n-L | \leq \varepsilon$ for each n > N"

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Continued

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2 pt.-- Statement with one quantifier wrong.

Example:

If a subject states that "there is an $\varepsilon > 0$ s.t. for all N $|a_n - L| \geq \varepsilon$ when n >N."

3 pt.-- Correct statement.

****

9. Suppose $a_n = \begin{cases} 1 & \text{for n odd} \\ -1 & \text{for n even} \end{cases}$

   a) According to the formal definition of limit, what would one have to show in order to prove that $\lim_{n \to \infty} a_n$ does not exist?

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Correct Answer For #9-a:

One would have to prove that for any number L there is a positive number ε such that $|a_n - L| < \varepsilon$ fails to be true for infinitely many n.

****

   b) Using the formal definition of limit, prove that $\lim_{n \to \infty} a_n$ does not exist.

****

Correct answer For #9-b:

Let L be the proposed limit, and take e =1/2,
Then there is an N such that $|a_n - L| < 1/2$ if n>N.
That is, $L - 1/2 < a_n < L + 1/2$ if n>N.
Taking n odd we have $L - 1/2 < -1 < L + 1/2$ implies that $-3/2 < L < -1/2$
Taking n even we have $L - 1/2 < 1 < L + 1/2$ implies that $1/2 < L < 3/2$
Obviously L can't not lie in both these intervals.

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Scoring System For Question #9:

0 pt.-- Incorrect statement (for #9-a) and incorrect proof (for #9-b).

Examples:

If on #9-a a subject states that "showing that there is no one value for a_n for n->∞", or

If on #9-b a subject proves that:

$$\left| \frac{a_{n+1}}{a_n} \right| \neq \frac{a_{n+1}}{a_n} \quad \text{as } n \to \infty$$

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Continued

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1 pt.-- Statement and proof not by definition

Examples:

If on #9-a a subject states that "The limit of a_n would have to be $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} a_n = -1$. Since $1 \neq -1$, the limit would not be a unique one as it must be", or

If on #9-b a subject states that

$$\left| L - \begin{matrix} 1 \text{ for n odd} \\ -1 \text{ for n even} \end{matrix} \right| < \varepsilon$$

$L-1 < \varepsilon$ $\quad$ $L < \varepsilon + 1$

$L+1 < \varepsilon$ $\quad$ $L < \varepsilon - 1$

There is no N for which this will work.

2 pt.-- Statement and proof with one quantifier missing.

Example:

If on #9-a a subject states that "there exists $\varepsilon > 0$ s.t. no natural number N exists that satisfies $|a_n - L| < \varepsilon$."

3 pt.--Correct statement and correct proof

****

10. Using the formal definition of limit prove the following statement:

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist,

then $\lim_{n\to\infty}(a_n + b_n)$ exists and $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

****

Correct Answer:

Let $\varepsilon > 0$ be given. Since both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, for convience, put $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$

We want to prove there is an N such that $|(a_n - b_n) - (A-B)| < \varepsilon$ if $n > N$.

Since n--> A, there is an N_1 such that $|a_n - A| < \varepsilon/2$ if $n > N_1$.

Since n--> B, there is an N_2 such that $|b_n - B| < \varepsilon/2$ if $n > N_2$

Let $N = \text{Max}\{N_1, N_2\}$. Then for n>N we have

$$|(a_n - b_n) - (A-B)| = |(a_n - A) + (b_n - B)| < |a_n - A| + |b_n - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

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Scoring System For Question #10:

0 pt.-- Incorrect proof.

Examples:

If a subject argues as follows:

Assume $\lim a_n=0$ and $\lim b_n=0$

If we add $\lim a_n + \lim b_n = 0$ because $0 + 0 = 0$

Therefore $\lim (a_n+b_n) = 0$

So $\lim (a_n+b_n) = \lim a_n + \lim b_n$

If a subject argues as follows:

$\lim (a_n+b_n)$ exists: Since both limits exist you can combine them to make a true statement. However, if one were false, you could not do this.

$\text{Lim } (a_n+b_n) = \lim a_n + \lim b_n$: This is just using the distributive property. It is like saying: $A (x+y) = Ax + Ay$.

1 pt.-- Incomplete proof.

Example:

If a subject argues as follows:

$| a_n\text{-}L | < \varepsilon \quad | b_n\text{-}L | < \varepsilon$

$| a_n+b_n | = | a_n\text{-}L+L\text{-}b_n | \leq | a_n\text{-}L | + | L\text{-}b_n | < \varepsilon /2 + \varepsilon /2 < \varepsilon$

2 pt.--Proof with one quantifier wrong.

Example:

If a subject argues as follows:

Let $\lim a_n = A$ then $| a_n\text{-}A | < \varepsilon /2$ when $n > N1$

Let $\lim b_n = B$ and $| b_n\text{-}B | < \varepsilon /2$ when $n > N_2$

Let $N= \max (N_1,N_2)$

Let $\varepsilon > 0$ need to show $| (a_n+b_n)\text{-}(A+B) | < \varepsilon$

$| (a_n+b_n)\text{-}(A+B) | = | a_n\text{-}A + b_n\text{-}B | \leq | a_n\text{-}A | + | b_n+B |$

$< \varepsilon /2+\varepsilon /2 = \varepsilon$ whenever $n>N$

3 pt.--Correct proof.

****

APPENDIX D

Subjects Raw Scores, Percentage Scores and Mean Scores

Category I: Basic Understanding

Subj.	1-a 1	1-b 1	1-c 1	1-d 1	1-e 1	1-f 1	2-a 2	2-b 2	3-a 2	Total 12	Percent
1	1	1	1	0	1	1	2	0	2	9	75
2	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	1	1	2	0	0	5	42
4	1	1	0	0	1	0	2	0	2	7	58
5	1	0	1	0	0	1	0	0	0	3	25
6	0	1	0	0	0	1	1	0	0	3	25
7	1	1	0	0	1	0	0	0	1	4	33
8	0	1	0	0	0	1	2	2	1	7	58
9	1	1	1	1	1	1	2	2	1	11	92
10	1	1	1	0	1	1	2	2	2	11	92
11	1	1	0	0	1	1	2	0	1	7	58
12	1	1	1	1	1	1	2	0	2	10	83
13	1	0	0	0	0	0	0	0	0	1	8
14	1	1	1	1	1	1	2	2	2	12	100
15	1	1	1	1	1	1	2	2	0	10	83
16	1	1	1	0	1	1	2	0	2	9	75
17	1	0	0	0	1	1	0	0	0	3	25
18	0	1	1	0	1	1	2	0	0	6	50
19	1	1	0	1	1	1	2	2	2	11	92
20	1	1	1	1	1	1	2	0	2	10	83
21	1	1	1	1	1	1	2	0	2	10	83
22	1	1	1	1	1	1	2	0	2	10	83
23	1	1	1	1	1	0	2	0	2	9	75
24	1	1	1	1	1	1	2	2	0	10	83
25	1	1	1	0	1	1	2	2	2	11	92
26	1	1	1	1	1	1	2	2	2	12	100
27	1	1	1	1	1	1	2	2	1	11	92
28	1	1	1	0	1	0	2	1	2	9	75
29	1	1	1	1	1	1	2	2	1	11	92
30	1	1	0	1	1	1	2	0	2	9	75
31	0	0	1	1	1	1	2	0	1	7	58
32	1	1	1	0	1	1	1	0	0	6	50
33	1	1	0	1	1	1	2	0	1	8	67
34	1	1	0	0	0	1	1	0	2	6	50
35	1	0	1	1	1	0	0	0	1	5	42
36	1	1	1	0	1	1	2	0	1	8	67
37	1	1	1	1	1	1	2	2	2	12	100
38	1	1	1	0	1	1	2	2	1	10	83
Mean	0.8	0.8	0.7	0.5	0.8	0.8	1.6	0.7	1.2	7.97	66.5

Category II: Computational Understanding

Subjects	4-a 3	4-b 2	4-c 2	4-d 3	Total 10	Percent
1	3	0	2	1	6	60
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	1	1	10
6	0	0	0	0	0	0
7	0	2	0	1	3	30
8	0	1	0	0	1	10
9	3	2	2	1	8	80
10	3	2	2	0	7	70
11	1	2	0	1	4	40
12	3	2	2	0	7	70
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	3	2	2	1	8	80
16	2	2	0	0	4	40
17	3	0	0	0	3	30
18	0	2	1	0	3	30
19	3	1	0	1	5	50
20	3	2	0	1	6	60
21	0	2	1	0	3	30
22	3	2	2	1	8	80
23	3	2	2	1	8	80
24	3	2	0	0	5	50
25	2	1	2	2	7	70
26	3	2	2	3	10	100
27	0	0	0	0	0	0
28	1	2	0	0	3	30
29	3	2	2	0	7	70
30	3	2	2	1	8	80
31	2	2	2	1	7	70
32	3	2	2	0	7	70
33	0	2	1	0	3	30
34	3	2	0	0	5	50
35	0	0	0	0	0	0
36	3	2	2	3	10	100
37	3	1	1	0	5	50
38	3	2	2	0	7	70
Mean	1.71	1.32	0.89	0.53	4.45	44.48

Category III: Transitional Understanding

Subjects	3-b 2	5-a 2	5-b 2	6 2	7 3	Total 11	Percent
1	2	0	0	2	0	4	36
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	1	0	0	0	0	1	9
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	1	1	9
8	0	1	0	0	0	1	9
9	0	0	0	0	2	2	18
10	2	1	0	0	0	3	27
11	0	2	0	0	0	2	18
12	0	0	1	0	3	4	36
13	0	0	0	0	3	3	27
14	1	2	2	2	3	10	91
15	1	1	2	2	0	6	55
16	2	1	2	2	2	9	82
17	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0
19	1	1	0	0	3	5	45
20	1	0	0	0	0	1	9
21	0	0	0	0	0	0	0
22	1	1	0	1	1	4	36
23	2	2	1	2	3	10	91
24	0	0	1	2	3	6	55
25	1	0	2	2	3	8	73
26	0	2	2	2	3	9	82
27	0	0	1	0	0	1	9
28	2	1	1	0	0	4	36
29	0	0	0	0	0	0	0
30	1	1	1	0	0	3	27
31	0	0	1	0	0	1	9
32	1	1	2	0	0	4	36
33	0	0	0	0	0	0	0
34	1	0	1	0	0	2	18
35	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0
37	2	1	2	0	3	8	73
38	0	2	1	0	0	3	27
Mean	0.58	0.53	0.61	0.45	0.87	3.03	27.51

Category IV: Rigorous Understanding

Subjects	5-c 3	8 3	9-a 3	9-b 3	10 3	Total 15	Percent
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	1	1	7
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0
10	2	0	0	0	0	2	13
11	0	0	0	0	0	0	0
12	0	0	1	0	0	1	7
13	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0
15	0	0	1	0	0	1	7
16	2	0	1	0	0	3	20
17	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0
22	0	0	1	0	0	1	7
23	0	0	2	0	0	2	13
24	2	2	1	0	1	6	40
25	3	2	2	1	0	8	53
26	2	0	0	0	2	4	27
27	3	0	0	0	0	3	20
28	0	0	2	0	0	2	13
29	0	0	0	0	0	0	0
30	0	1	0	0	0	1	7
31	1	0	0	0	0	1	7
32	0	0	1	0	0	1	7
33	0	0	0	0	0	0	0
34	0	0	1	0	0	1	7
35	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0
37	1	0	0	1	1	3	20
38	0	1	1	0	1	3	20
Mean	0.42	0.16	0.37	0.05	0.16	1.16	7.72