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Mangalam Vasudaven

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WEAK CONVERGENCE OF STANDARDIZED KAPLAN–MEIER PROCESS IN L^2 SPACE

By

Mangalam Vasudaven

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ABSTRACT

WEAK CONVERGENCE OF STANDARDIZED KAPLAN–MEIER PROCESS IN L^2 SPACE

By

Mangalam Vasudaven

Let $X_1,\dots,X_n$ be i.i.d. with a distribution function F and $Y_1,\dots,Y_n$ be i.i.d. with an unknown censoring distribution function G with X_i's and Y_i's independent. In the random censorship model one observes $Z_i = X_i \wedge Y_i$ and $\delta_i = I(X_i \le Y_i)$, $1 \le i \le n$. Such models arise in clinical trials and survival analysis.

This thesis discusses the weak convergence of some standardized and rescaled versions of the Kaplan – Meier Process to a Gaussian process in $L^2(A, \mu)$ where A is the support set of F and μ is a σ–finite measure defined on A. Applications of this result to the goodness–of–fit test pertaining to F are also discussed. In addition, certain tests of H_0: $F = F_0$, F_0 a known distribution function, based on L^2– norms of a suitably scaled Kaplan – Meier Process with respect to certain random measures, are shown to be asymptotically distribution free.

To my parents and my wife Sushama

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CHAPTER 0

INTRODUCTION

A common feature of many survival studies is that the time of occurrence of the event of interest, called a death, may be prevented for an item of the sample by previous occurrence of some other event, called a loss. In other words, a censorship may be present due to which the units under study may not be completely observable. Typically this is the case in a clinical trial where patients under treatment cannot be followed up due to withdrawals from study. For example, in medical follow-up studies to determine the distribution of the survival times after an operation, contact with some of the patients may be lost before their death and others may die from causes it is desired to exclude from consideration. Similarly, observation of the life of a vacuum tube may be ended by breakage of the tube or a need to use the test facilities for some other purposes. In both examples, incomplete observation may also result from a need to get a report out within a reasonable time. The losses may be either accidental or by design, the latter resulting from a decision to terminate certain observations. There are various types of censorships. For an excellent overview of different types of censorship, see Chapter 3 of Gill (1980).

In the random right censorship model (random censorship model, for short) considered here, for each item, the only data available are the minimum of the survival time and the time of loss, and whether or not censoring is present. Kaplan and Meier (1958) suggested the Product–Limit Estimator (PLE) $\hat{F}_n$ of the true survival distribution of the lifetime when there is random censorship in the data. While the usual empirical distribution function (EDF) assigns mass $1/n$ to each of the observations, the PLE redistributes the mass of censored observations equally among the observations to the right, giving zero mass to all the censored observations. More precisely, first, one arranges the observations in the ascending order and assigns mass $1/n$ to each observation, whether or not uncensored. Next, starting from the smallest observation, one locates the first censored observation, and redistributes its mass equally among the observations to its right. Then one moves on to the next censored observation and redistributes its new mass equally among the observations to *its* right. This process is continued for all the censored observations. Note that, if the largest observation is censored, the total mass will be less than 1, thereby making the PLE a defective distribution.

This thesis studies the standardized error W_n of a version of the PLE as a process with paths in the space of square–integrable functions. It is shown that this process converges weakly to a Gaussian Process in the space mentioned above, where the integration is with respect to a measure belonging to a large class of σ–finite measures. Also proved are the weak convergence of some rescaled versions of the error process in the aforementioned space. These results are useful in developing goodness–of–fit tests pertaining to the survival distribution F and/or the censoring distribution G. It is conceivable that they may also be useful in minimum distance estimation problems.

In the uncensored case Anderson and Darling (1952) considered a Cramèr–von Mises type statistic for the goodness–of–fit problem of testing H_0: $F = F_0$ vs H_1: $F \neq F_0$, where F_0 is a completely specified continuous d.f.. The statistic was obtained by squaring the standardized E.D.F. and integrating it with respect to the σ–finite measure $dF_0/(F_0(1-F_0))$. The asymptotic distribution of this statistic was shown to be that of the integral of the square of a rescaled Brownian Bridge, where the scaling function is the standard deviation function of the Brownian Bridge so as to make the standard deviation 1 through out. Koziol and Green (1976) considered the above-mentioned testing problem for the censored data and discussed the much simpler situation when the hazard functions of the survival and the censoring distributions are proportional (herein referred to as proportional hazards model) and the integrating measure is the finite measure dF_0. Even in this case it is not clear how they prove the tightness.

In this dissertation, the result of Anderson and Darling (1952) is extended to random censorship model, where there is an additional unknown function G. Even if the integrating measure depends on F, the limiting distribution of corresponding goodness-of-fit test statistic may not be distribution-free in the case of randomly censored data. However, if one chooses the measure suitably, perhaps depending on the data and varying with n, one can get asymptotically distribution-free (A.D.F.) tests. Two Anderson-Darling type statistics are considered and it is proved that these statistics are A.D.F. and hence can be used for tests regarding F. The limiting null distribution in both cases is same as that of the Anderson-Darling statistic of the uncensored case.

Now we will state the problem more precisely. Let $X_1,\ldots,X_n$ be i.i.d. random variables (r.v.'s) with distribution function (d.f.) F on $[0, \infty)$, and $Y_1,\ldots,Y_n$ be i.i.d. r.v.'s independent of X_i's with unknown and possibly defective d.f. G (that is, G may assign positive mass to ∞) on $[0, \infty]$. We observe the pairs $\{(X_i \wedge Y_i), I(X_i \leq Y_i)\}$, $1 \leq i \leq n$, where $a \wedge b$ denotes $\min(a,b)$ and $I(A)$ denotes the indicator function of the set A. If we observe Y_i, that is, if $I(X_i \leq Y_i) = 0$, we say the observation is censored. Otherwise it is referred to as an uncensored observation. The general problem is to make inferences about F.

Let τ_H denote the supremum of the support of $X_1 \wedge Y_1$ and $\hat{W}_n(t)$ denote the scaled error $\sqrt{n}\,(\hat{F}_n(t) - F(t))$, $t \le \tau_H$. Gill (1980) proved that $\hat{W}_n \Rightarrow W$ in $D[0, \tau]$ for every $\tau < \tau_H$, where W is a Gaussian process depending on F and G, and $D[0, \tau]$ denotes the set of all bounded CADLAG functions on $[0, \tau]$. Yang (1988) extended this result to the entire set $[0, \tau_H]$.

Let μ be a σ–finite measure on $[0, \tau_H]$. It is shown in this thesis that, under suitable conditions on F, G and μ, W_n belongs to $L^2([0, \tau_H], \mu)$ and that W_n converges weakly to W in $L^2([0, \tau_H], \mu)$. For infinite μ, $\hat{W}_n$ may not belong to $L^2([0, \tau_H], \mu)$ which is the reason for considering the modified version W_n. This is discussed in more detail in Remark 1.1. For the L^2– weak convergence of W_n, the key step is to show that R_n, the mean of an estimate of the survival ratio $(1-F)/(1-G)$ is uniformly bounded on $[0, \tau_H]$.

The material is organized as follows. In Chapter 1, further notation and assumptions are stated and some preliminary results are proved. In Chapter 2 it is proved that W_n converges weakly to W on $L^2[0, \tau]$ for every $\tau < \tau_H$. Chapter 3 is devoted to proving that if the underlying model is proportional hazards model with hazard ratio less than 1, then R_n is uniformly bounded on $[0, \tau_H]$. In Chapter 4, the weak convergence of W_n to W in $L^2([0, \tau_H], \mu)$ under the proportional hazards model is proved. In Chapter 5, it is proved that the above weak convergence holds under more general assumptions on F and G. Some examples, where proportional hazards model does not hold, are given. Two A.D.F. tests are constructed in Chapter 6 and the variance of the limiting distribution is computed.

CHAPTER 1

NOTATION, ASSUMPTIONS AND PRELIMINARIES

First, we need to introduce some general notation.

(i) For any real–valued function f, $\bar{f}$ denotes $1-f$, $\|f\|_a^b$ denotes $\sup_{a \le t \le b} |f(t)|$, $f_-(t) := \lim_{s \uparrow t} f(s)$ whenever the limit exists and $f(\tau_H) := f_-(\tau_H)$. Unless otherwise mentioned, $f^{-1}(t) = 1/f(t)$ for any real–valued function f. For any d.f. K, let τ_K denote the supremum of the support of K. For any set A, $I(A)$ denotes the indicator function of A. For any interval I, $D(I)$ denotes the set of all bounded CADLAG functions on I equipped with the supremum norm.

Next, notation and assumptions for some frequently used functions and r.v.'s are introduced.

(ii) Let F be a continuous d. f. on $[0, \infty)$, G be a possibly defective censoring d.f. on $[0, \infty]$ such that $\tau_F \le \tau_G$; $\bar{H} = \bar{F}\,\bar{G}$; $X_1,\ldots,X_n$ be i.i.d. F and $Y_1,\ldots,Y_n$ be i.i.d. G such that X_i's and Y_i's are independent; $Z_i := X_i \wedge Y_i$ and $\delta_i := I(X_i \le Y_i)$. Let $Z_{(i)}$'s denote the order statistics of Z_i's, $\delta_{(i)}$'s denote the order statistics induced on δ_i's by $Z_{(i)}$'s and T_n denote $Z_{(n)}$. Let

$$C(t) := \int_0^t \frac{dF}{\bar{F}^2\,\bar{G}_-}, \quad K(t) := C(t)/(1+C(t)), \quad t \le \tau_H.$$

Next the PLE's of F and G, two versions of the standardized Kaplan–Meier Processes (KMP's) and some other related stochastic processes are defined.

(iii) Define the PLE $\hat{F}_n$ of F by

$$\bar{\hat{F}}_n(t) := \prod_{\substack{j=1 \\ Z_{(j)} \leq t}}^{n} \left(\frac{n-j}{n-j+1}\right)^{\delta_{(j)}},$$

and F_n by

$$F_n(t) := I(t < T_n)\ \bar{\hat{F}}_n(t), \quad t \leq \tau_H.$$

The PLE G_n of G is defined by

$$\bar{G}_n(t) := \prod_{\substack{j=1 \\ Z_{(j)} \leq t}}^{n} \left(\frac{n-j}{n-j+1}\right)^{1-\delta_{(j)}}, \quad t \leq \tau_H.$$

The standardized KMP's are

$$\hat{W}_n(t) := \sqrt{n}\ (\hat{F}_n(t) - F(t)), \quad W_n(t) := \sqrt{n}\ (F_n(t) - F(t)),$$

for $t \leq \tau_H$. We shall also need the following estimators of C

$$C_n(t) := \int_0^t \frac{dF}{F^2\ \bar{G}_{n-}}, \quad C_n^*(t) := \int_0^t \frac{dF_n}{F_n F_{n-} \bar{G}_{n-}},$$

and the corresponding estimators of K

$$K_n(t) ;= C_n(t)/(1+C_n(t)) \quad \text{and} \quad K_n^*(t) := C_n^*(t)/(1+C_n^*(t)) \quad \text{for} \quad t \leq \tau_H.$$

Some rescaled KMP's are defined by

$$\tilde{\xi}_n(t) := (K(t)/F(t))\cdot W_n(t), \quad \xi_n(t) := (K_n(t)/F(t))\cdot W_n(t),$$
$$\xi_n^*(t) := (K_n^*(t)/F_n(t))\cdot W_n(t), \quad t \leq \tau_H.$$

Another sequence of stochastic processes and their mean functions are defined next.

(iv) Define an estimate of the survival ratio and its mean by

$$Q_n(t) := \frac{F_{n-}(t)}{\bar{G}_{n-}(t)}\ I(t \leq T_n), \quad R_n(t) := E(Q_n(t)), \quad t \leq \tau_H.$$

Observe that $Q_n(t)$ is bounded for each n and each $t \leq \tau_H$, so the expectation is guaranteed to be finite.

Next, two key Gaussian processes are defined.

(v) Let B be a Brownian Motion on $[0, \infty)$, B_0 be a Brownian Bridge on $[0, 1]$,

$$W(t) := \bar{F}(t)\, B(C(t)), \quad t < \tau_H.$$

Finally, we state the condition on the integrating measure with respect to which the L^2 space is defined.

(vi) For any set S, any measure γ on S and any subset A of S, let $L^2(A, \gamma)$ be the set of all real-valued functions on A which are square integrable with respect to γ. Let μ be a σ-finite measure on $[0, \tau_H]$ such that

$$\int_0^{\tau_H} \bar{F}^2\, C\, d\mu < \infty.$$

__REMARK 1.1.__ Recall that, if the last observation is censored, that is, if $\delta_{(n)} = 0$, then for $t \geq T_n$, $\hat{F}_n(t) = \hat{F}_n(T_n) < 1$ so that $\hat{F}_n$ is a defective distribution. Consequently, $\hat{W}_n$ may not be in $L^2([0, \tau_H], \mu)$, for it will be a non-zero constant function of t on $[T_n, \tau_H]$ and hence non-integrable if μ is an infinite measure. To overcome this difficulty, as is customary in applications, we assign all the left-over mass to $Z_{(n)}$, the last observation, whether or not it is censored, making F_n a regular distribution. This makes $F_n \equiv 1$ on $[T_n, \tau_H]$ and enables us to prove that, under (vi), W_n belongs to $L^2([0, \tau_H], \mu)$. Note that $F_n(t)$ and $\hat{F}_n(t)$ differ only for $t \geq T_n$. As we have the condition that $\tau_F \leq \tau_G$, $F_n(\tau_H) = F(\tau_H) = 1$ and the change in the definition of PLE will not affect the uniform convergence of F_n to F on $[0, \tau_H]$. This becomes clearer after Lemma 1.1. On the other hand G_n should be defined *without* the indicator function F_n has, to assure us of the uniform convergence on $[0, \tau_H]$ when $\tau_F < \tau_G$. $G_n(\tau_H)$ cannot converge to $G(\tau_H)$ if $G_n(\tau_H)$ is always 1 and $G(\tau_H) < 1$.

We shall state and prove a few lemmas that are frequently used in this dissertation.

LEMMA 1.1.

(i) $\| F_n - F \|_0^{\tau_H} \to 0$ a.s..

(ii) $\| G_n - G \|_0^{\tau_H} \to^P 0$.

PROOF.

(i) From the Proposition of Wang (1987), since $F(\tau_H -) = 1$, we get,

(1.1) $$\sup_{t<\tau_H} |\hat{F}_n(t) - F(t)| \to 0 \text{ a.s..}$$

Since $F(T_n) \to 1$ a.s., (3) of Wang (1987) gives $\hat{F}_n(T_n) \to 1$ a.s.. So

(1.2) $$\sup_{t<\tau_H} |F_n(t) - \hat{F}_n(t)| = |1 - \hat{F}_n(T_n)| \to 0 \text{ a.s..}$$

Now, (1.1) and (1.2) together with the fact that $F_n(\tau_H) = F(\tau_H) = 1$ imply (i).

(ii) Apply (2) of Wang (1987) to $\hat{G}_n$; F and G are interchangeable for the results in Wang (1987), so

$$\sup_{t<\tau_H} | \hat{G}_n(t) - G(t) | \to^P 0.$$

By (3) of Wang (1987) applied to $\hat{G}_n$, $G_n(T_n) \to^P G(\tau_H)$ and hence

$$G_n(\tau_H) \to^P G(\tau_H). \qquad \square$$

REMARK 1.2. All the results in Wang (1987) go through even if G is defective.

LEMMA 1.2. For every $\tau < \tau_H$,

$$W_n \Rightarrow W \quad \text{on} \quad D[0,\tau].$$

PROOF. By Theorem 1.1 of Gill (1983),

$$\hat{W}_n \Rightarrow W \quad \text{on} \quad D[0,\tau].$$

$$\|W_n - \hat{W}_n\|_0^\tau = \|\sqrt{n}\,(F_n - \hat{F}_n)\|_0^\tau = |\sqrt{n}\,(1 - \hat{F}_n(T_n))|\ I(T_n \le t \le \tau)$$
$$\longrightarrow 0 \ \text{ a.s.}$$

because $T_n \longrightarrow \tau_H$ a.s. and $\tau < \tau_H$. □

REMARK 1.3. It may be interesting to know the conditions under which

$$W_n \Rightarrow W \text{ on } D[0,\tau_H]. \tag{1.3}$$

In Theorem 2.2 of Yang (1988), it is proved that $\hat{W}_n \Rightarrow W$ on $D[0,\tau_H]$ under the condition

$$\int_0^{\tau_H} \frac{dF}{\overline{G}_-} < \infty. \tag{1.4}$$

So it suffices to examine $\left\| W_n - \hat{W}_n \right\|_0^{\tau_H}$. This quantity is equal to $\sqrt{n}\ \overline{\hat{F}}_n(T_n)$. But by Gill (1983; Remark 2.2), under (1.4), $\overline{F}(t)\ B(C(t)) \longrightarrow 0$ a.s. as $t \longrightarrow \tau_H$. So the aforementioned weak convergence result of Yang also tells us that $\sqrt{n}\ [\overline{\hat{F}}_n(T_n) - \overline{F}(T_n)] \longrightarrow^P 0$, since $T_n \longrightarrow \tau_H$ a.s. as $n \longrightarrow \infty$. Therefore we need to find when $\sqrt{n}\ \overline{F}(T_n) \longrightarrow^P 0$. in Lemma 1.3, the condition under which this happens is proven to be

$$\overline{F}(t)/\overline{G}(t)) \longrightarrow 0 \quad \text{as } t \longrightarrow \tau_H. \tag{1.5}$$

So (1.4) and (1.5) together imply (1.3). Note that, though both (1.4) and (1.5) say that $F(t)$ should approach 1 as $t \longrightarrow \tau_H$ somewhat faster than $G(t)$ does, neither of these conditions implies the other.

LEMMA 1.3. A necessary and sufficient for $\sqrt{n}\ \overline{F}(T_n) \to^P 0$ is (1.5).

PROOF. $\sqrt{n}\ \overline{F}(T_n) \to^P 0 \iff P\{\sqrt{n}\ \overline{F}(T_n) > \epsilon\} \to 0 \ \forall \ \epsilon > 0$

$$\iff P\{T_n < F^{-1}(1-[\epsilon/\sqrt{n}])\} \to 0 \ \forall \ \epsilon > 0$$

$$\iff P\{T_n \le F^{-1}(1-[\epsilon/\sqrt{n}])\} \to 0 \ \forall \ \epsilon > 0$$

where $F^{-1}(t) = \inf\{x\colon F(x) \ge t\}$. Fix $\epsilon > 0$.

$$P\{T_n \le F^{-1}(1-[\epsilon/\sqrt{n}])\} = H^n[F^{-1}(1-[\epsilon/\sqrt{n}])]$$

$$= [1 - (\epsilon/\sqrt{n})\ \overline{G}(F^{-1}(1-[\epsilon/\sqrt{n}]))]^n.$$

Now recall that for a sequence $\{\theta_n\}$ such that $0 \le \theta_n \le 1$, $[1 - \theta_n]^n \to 0$ if and only if $n\ \theta_n \to \infty$. Thus $\sqrt{n}\ \overline{F}(T_n) \to^P 0$ if and only if $\sqrt{n}\ \overline{G}(F^{-1}(1-[\epsilon/\sqrt{n}])) \to \infty\ \forall\ \epsilon > 0$. If M is the smallest integer greater than or equal to $(1/\epsilon)^2$, then $\epsilon/\sqrt{n} \ge 1/\sqrt{Mn}$ and hence

$$\sqrt{n}\ \overline{G}(F^{-1}(1-[\epsilon/\sqrt{n}])) \ge \sqrt{n}\ \overline{G}(F^{-1}(1-[1/\sqrt{Mn}]))$$

$$= (1/\sqrt{M})\ \sqrt{Mn}\ \overline{G}(F^{-1}(1-[1/\sqrt{Mn}])).$$

Therefore, since $F(F^{-1}(x)) = x$ and $\tau_F = \tau_H$,

$$\sqrt{n}\ \overline{G}(F^{-1}(1-[\epsilon/\sqrt{n}])) \to \infty \ \forall \ \epsilon > 0 \iff \sqrt{n}\ \overline{G}(F^{-1}(1-[1/\sqrt{n}])) \to \infty$$

$$\iff \overline{G}(F^{-1}(x)/(1-x)) \to \infty \ \text{as}\ x \to 1$$

$$\iff \overline{G}(t)/(1-F(t)) \to \infty \ \text{as}\ t \to \tau_H.$$

This proves the desired result. □

LEMMA 1.4. Let $A \in (0, \infty]$ and let $g\colon [0, A] \longmapsto [0, \infty]$ be such that $g(t) < \infty$ for all $t < A$ and $g(t) \to \theta \in [0, \infty]$ as $t \to A$. Let μ be an infinite measure on $[0, A]$ such that $\mu[0, t] < \infty$ for all $t < A$. Then

$$(\mu[0, t])^{-1} \int_0^t g\ d\mu \to \theta \ \text{as}\ t \to A.$$

PROOF. *Case 1.* $\theta < \infty$. Given $\epsilon > 0$, $\exists\ M > 0$ such that for all

$t > M$, $|g(t) - \theta| < \epsilon$. So

$$\int_0^t g\, d\mu = \int_0^M g\, d\mu + \int_M^t g\, d\mu$$

implies

$$(\mu[0,\ t])^{-1} \int_0^M g d\mu + (\theta-\epsilon)\{\mu[M,\ t]/\mu[0,\ t]\} \le (\mu[0,\ t])^{-1} \int_0^t g\, d\mu$$
$$\le (\mu[0,\ t])^{-1} \int_0^M g\, d\mu + \theta + \epsilon$$

As $t \longrightarrow A$, $\mu[0,\ t] \longrightarrow \infty$ and $\{\mu[M,\ t]/\mu[0,\ t]\} \longrightarrow 1$ so by taking liminf in the first inequality and limsup in the second inequality we get the result.

Case 2. $\theta = \infty$. Proof is similar. □

CHAPTER 2

CONVERGENCE OF W_n IN $L^2\,[0,\,\tau]$, $\tau < \tau_H$

In this Chapter τ will denote an arbitrary but fixed number less than τ_H. All the conditions of Chapter 1 are assumed to hold, though all the functions and measures that are originally defined on $[0,\ \tau_H]$ are now restricted to $[0,\ \tau]$. Theorem 2.1 states that $W_n \Rightarrow W$ on $L^2([0,\ \tau],\ \mu)$ where μ is a σ–finite measure satisfying 1(vi). To prove this, we are going to show that a bias–adjusted version of W_n satisfies all the conditions of Lemma 9 of Koul (1984), which in turn is based on Theorem 2.2 of Parthasarathy (1967). For that, we will state and prove a few lemmas.

For $p = 1,\ 2$, $L^p(\mu)$ is short for $L^p([0,\ \tau],\ \mu)$. As $L^1(\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$, there exists a sequence $\{e_n\colon n \in \mathbb{N}\} \subseteq L^1(\mu) \cap L^2(\mu)$ such that $\{e_n\colon n \in \mathbb{N}\}$ is a complete orthonormal set for $L^2(\mu)$. Define

$$U_n(t) := I(t \geq T_n)\,\frac{\overline{\hat{F}}_n(T_n)\ \overline{F}(t)}{\overline{F}(T_n)},\quad X_n(t) := W_n(t) - \sqrt{n}\ U_n(t),\quad t \leq \tau_H.$$

<u>**LEMMA 2.1.**</u>

$$E(X_n(t)) = 0,\quad V(X_n(t)) = \overline{F}^2(t)\int_0^t \frac{R_n dF}{\overline{F}^3},\quad t \leq \tau_H$$

where R_n is as in 1(iv).

<u>**PROOF.**</u> From the identity $F_n(t) = \hat{F}_n(t) + \overline{\hat{F}}_n(T_n)\ I(t \geq T_n)$ we get

$$\begin{aligned} F_n(t) - F(t) - U_n(t) &= \hat{F}_n(t) + I(t \geq T_n)\ \overline{\hat{F}}_n(T_n) - F(t) - U_n(t) \\ &= \hat{F}_n(t) - F(t) + I(t \geq T_n)\ \overline{\hat{F}}_n(T_n)\ [1 - (\overline{F}(t)/\overline{F}(T_n))] \end{aligned}$$

$$= \overline{F}(t) \int_0^t \frac{(1-\hat{F}_n)\_J_n}{n\ \overline{F}\ \overline{H}_{n-}}\, dM_n$$

$$- I(t \geq T_n) \left[\frac{\hat{\overline{F}}_n(T_n)[F(t) - F(T_n)]}{\overline{F}(T_n)} \right]$$

$$+ I(t \geq T_n)\ \hat{\overline{F}}_n(T_n)\ [1 - (\overline{F}(t)/\overline{F}(T_n))]$$

(from 3.2.15 of Gill (1980))

(2.1) $$= \overline{F}(t) \int_0^t \frac{(1-\hat{F}_n)\_J_n}{n\ \overline{F}\ \overline{H}_{n-}}\, dM_n$$

where $\overline{H}_n = \overline{F}_n\ \overline{G}_n$, $J_n = I(\overline{H}_{n-} > 0)$, and

$$M_n(t) = \sum_{j=1}^{n} I(Z_j \leq t,\ \delta_j = 1) - \int_0^t (n\ \overline{H}_{n-}/\overline{F})\ dF.$$

(Note that in Gill (1980), $n\ \overline{H}_{n-}$ is denoted by Y_n.) By (2.1) and an argument similar to 3.2.20 of Gill (1980),

$$E(X_n(t)) = 0, \quad V(X_n(t)) = \overline{F}^2(t) \int_0^t E\left[\frac{(1-\hat{F}_n)_-^2\ J_n}{\overline{H}_{n-}}\right] \frac{dF}{\overline{F}^3},\ t \leq \tau_H.$$

Moreover,

$$\begin{aligned} (1-\hat{F}_{n-}(t))^2\ J_n(t) &= (1-\hat{F}_{n-}(t))^2\ I(t \leq T_n) \\ &= \overline{F}_{n-}^2(t)\ I(t \leq T_n) \\ &= \overline{H}_{n-}(t)\ Q_n(t). \end{aligned}$$

by 1(iv). Thus

$$V(X_n(t)) = \overline{F}^2(t) \int_0^t \frac{R_n dF}{\overline{F}^3}. \qquad \square$$

<u>**Lemma 2.2.**</u> As $n \to \infty$,

$$V(X_n(t)) \to \overline{F}^2(t)\ C(t) \quad \forall\ t \leq \tau_H$$

and

$$\int_0^\tau V(X_n(t))\, d\mu(t) \;\to\; \int_0^\tau F^2(t)\, C(t)\, d\mu(t).$$

PROOF. By Lemma 1.1, $F_n(t) \to F(t)$ a.s. and $G_n(t) \to^P G(t)$ for each $t \in [0, \tau]$. Therefore $Q_n(t) \to^P \frac{\overline{F}(t)}{\overline{G}\_(t)}$ for each fixed $t \in [0, \tau]$. We shall show that $Q_n(t)$ is uniformly integrable by showing $E(Q_n^2(t)) \le C_\tau$ for a constant C_τ depending only on τ. As $n\, \overline{H}_{n-}(t) \sim \mathrm{Bin}(n, \overline{H}\_(t))$,

$$\begin{aligned}
E(Q_n^2(t)) &\le n^2\, E\left[\frac{J_n\,(t)}{n\, \overline{H}_{n-}(t)}\right]^2 \\
&= n^2 \sum_{i=1}^{n} i^{-1}\, C_i^n\, [\overline{H}_(t)]^i\, [H_(t)]^{n-i} \\
&= n^2 \sum_{i=0}^{n-1} (i+1)^{-2}\, C_{i+1}^n\, [\overline{H}_(t)]^{i+1}\, [H_(t)]^{n-i-1} \\
&= n^3\, \overline{H}_(t)\, E[1 + \mathrm{Bin}(n-1, \overline{H}_(t))]^{-3} \\
&\le n^3\, \overline{H}_(t) \left[\overline{H}_(t)\right]^{-3} (3!)(n-1)^{-3} \\
&\le 48\, \overline{H}_-^{-2}(\tau), \quad t \in [0, \tau]
\end{aligned}$$

because $E[1 + \mathrm{Bin}(n, p)]^{-r} \le r!\, (np)^{-r}$ by Moment Lemma, Section 7 of Koul, Susarla and Van Ryzin (1981).

Consequently,

$$R_n(t) \equiv E(Q_n(t)) \;\to\; \frac{\overline{F}(t)}{\overline{G}\_(t)},$$

and

$$\frac{R_n(t)}{\overline{F}^3(t)} \;\to\; [\overline{F}^2(t) \overline{G}\_(t)]^{-1} \quad \text{for all} \quad t \in [0, \tau].$$

Now,

$$R_n(t) = E(Q_n(t)) \le [E(Q_n^2(t))]^{\frac{1}{2}} \le 7\, [\overline{H}\_(\tau)]^{-1} \quad \text{for all} \quad t \in [0, \tau].$$

Therefore by the Bounded Convergence Theorem and by 1(ii),

$$\int_0^t \frac{R_n dF}{F^3} \;\longrightarrow\; \int_0^t [F^2(s)\overline{G}\_(s)]^{-1}\, dF(s) \equiv C(t).$$

Hence by Lemma 2.1,

$$V(X_n(t)) \;\longrightarrow\; F^2(t)C(t) \quad \text{for all} \quad t \in [0, \tau].$$

Now,

$$V(X_n(t)) \;\le\; 7\,[\overline{H}\_(\tau)]^{-1}\, F^2(t) \int_0^t F^{-3}\, dF \;=\; 7\,[\overline{H}\_(\tau)]^{-1}\, F(t);$$

by 1(vi) and the fact that

$$\int_0^\tau F(t)\, d\mu(t) \le \frac{1}{F(\tau)}\int_0^\tau F(t)\, F(t)\, d\mu(t) \le \frac{1}{F(\tau)}\int_0^\tau F^2(t)\, C(t)\, d\mu(t),$$

$F(t)$ is integrable on $[0, \tau]$ with respect to μ. So by the Dominated Convergence Theorem,

$$\int_0^\tau V(X_n(t))\, d\mu(t) \;\longrightarrow\; \int_0^\tau F^2(t)\, C(t)\, d\mu(t). \qquad \square$$

__LEMMA 2.3.__ For every $m \in \mathbb{N}$,

$$\left[\int_0^\cdot X_n e_1 d\mu,\; \int_0^\cdot X_n e_2 d\mu, \ldots, \int_0^\cdot X_n e_m d\mu\right] \;\Rightarrow\; \left[\int_0^\cdot W e_1 d\mu,\; \int_0^\cdot W e_2 d\mu, \ldots, \int_0^\cdot W e_m d\mu\right].$$

__PROOF.__ By Lemma 1.2, we know that $W_n \Rightarrow W$ on $D[0, \tau]$. Next, observe that on $[T_n \le t]$, $F(t) \le F(T_n)$ and hence $U_n(t) \le I(T_n \le t)$. So,

$$\sup_{t \le \tau} |\sqrt{n}\; U_n(t)| \;\le\; \sqrt{n} \sup_{t \le \tau} |I(T_n \le t)| \;\le\; \sqrt{n}\; I(T_n \le \tau).$$

Since $T_n \longrightarrow \tau_H$ a.s. and $\tau < \tau_H$, for a.e. ω, $I(T_n(\omega) \le t) = 0$ for n sufficiently large. Therefore,

$$\sup_{t \le \tau} |\sqrt{n}\; U_n(t)| \;\longrightarrow\; 0 \quad \text{a.s.}.$$

Thus $X_n = W_n - \sqrt{n}\; U_n \Rightarrow W$ on $D[0, \tau]$.

Define $\Gamma\colon D[0, \tau] \longmapsto \mathbb{R}^m$ by

$$\Gamma(x) = \left[\int_0^\cdot x\; e_1\; d\mu\;,\; \int_0^\cdot x\; e_2\; d\mu\;, \ldots\ldots, \int_0^\cdot x\; e_m\; d\mu\right].$$

Then Γ is a continuous function since e_i's are in $L^1(\mu)$. Therefore,

$$\Gamma(X_n) \Rightarrow \Gamma(W) \quad \text{on} \quad \mathbb{R}^m. \qquad \square$$

LEMMA 2.4. For all $i \in \mathbb{N}$,

$$E\left[\int_0^\tau X_n\, e_i\, d\mu\right]^2 \to E\left[\int_0^\tau W\, e_i\, d\mu\right]^2.$$

PROOF. Fix $i \in \mathbb{N}$. Let $d\psi = e_i\, d\mu$. Then ψ is a finite signed measure and we need to show that

$$E\left[\int_0^\tau X_n\, d\psi\right]^2 \to E\left[\int_0^\tau W\, d\psi\right]^2.$$

By Theorem 3.4 of Gardiner, Susarla and Van Ryzin (1985) and Theorem 1 of Lo and Singh (1986),

$$W_n(t) = n^{-\frac{1}{2}} \sum_1^n \xi_i(t) + \sqrt{n}\, r_n(t),\ t \le \tau, \tag{2.2}$$

where ξ_i's are i.i.d. uniformly bounded r.v.'s with mean zero,

$$\sup_{t\le\tau} E|r_n(t)|^p = O(n^{-p}) \tag{2.3}$$

and

$$\sup_{t\le\tau} |r_n(t)| = O(n^{-3/4} (\log n)^{3/4}) \quad \text{a.s.}. \tag{2.4}$$

From (2.2), (2.4) and Lemma 1.2,

$$\tilde{W}_n(t) := n^{-\frac{1}{2}} \sum_1^n \xi_i(t) \Rightarrow W \quad \text{on} \quad D[0, \tau]$$

and hence

$$\int_0^\tau \tilde{W}_n\, d\psi \Rightarrow \int_0^\tau W\, d\psi. \tag{2.5}$$

Now,

$$\int_0^\tau \tilde{W}_n\, d\psi = n^{-\frac{1}{2}} \sum_1^n \int_0^\tau \xi_i(t)\, d\psi(t) = n^{-\frac{1}{2}} \sum_1^n B_i \qquad \text{(say)}$$

where B_i's are i.i.d. bounded mean zero r.v's. So by the Central Limit

Theorem,

$$(2.6) \qquad \int_0^\tau \tilde{W}_n \, d\psi \;\Rightarrow\; \text{Normal } (0,\ V(B_1))$$

where $V(X)$ indicates variance of X. By (2.5) and (2.6),

$$(2.7) \quad E\left[\int_0^\tau \tilde{W}_n d\psi\right]^2 = V\left[\int_0^\tau \tilde{W}_n d\psi\right] = V(B_1) = V\left[\int_0^\tau W d\psi\right] = E\left[\int_0^\tau W d\psi\right]^2.$$

As $W_n = \tilde{W}_n + \sqrt{n}\, r_n$, by Jensen's inequality and (2.3),

$$(2.8) E\left[\int_0^\tau W_n d\psi - \int_0^\tau \tilde{W}_n d\psi\right]^2 = n\, E\left[\int_0^\tau r_n d\psi\right]^2 \le k_1\, n \int_0^\tau E(r_n^2) d|\psi| \le k\, n^{-1}$$

for some k_1 and k. From (2.7) and (2.8) we get

$$(2.9) \qquad E\left[\int_0^\tau W_n \, d\psi\right]^2 \rightarrow E\left[\int_0^\tau W \, d\psi\right]^2.$$

Now we shall show that

$$(2.10) \qquad E\left[\int_0^\tau (W_n - X_n)\, d\psi\right]^2 \rightarrow 0$$

which will prove the lemma in view of (2.9). Note that $W_n - X_n = \sqrt{n}\, U_n$. Now,

$$E\left[\sqrt{n} \int_0^\tau U_n d\psi\right]^2 \le k\, n \int_0^\tau E(U_n^2) d|\psi| \le k\, n \int_0^\tau [\text{H\_}(\tau)]^n d|\psi|$$
$$= k\, n\, \|\psi\|\, [\text{H\_}(\tau)]^n \rightarrow 0$$

for some k by Jensen's inequality since $U_n(t) \le I(T_n \le \tau)$ for $t \le \tau$ and $P(T_n \le \tau) = [\text{H\_}(\tau)]^n$. Hence (2.10). □

THEOREM 2.1. Assume 1(i) – 1(vi) hold. Then,

$$W_n \Rightarrow W \text{ on } L^2(\mu).$$

PROOF. All the conditions of Lemma 9 of Koul (1984) are verified by Lemmas 2.1 through 2.4; so it follows that $X_n \Rightarrow W$ on $L^2(\mu)$. Now all that needs verification is that $\int_0^{\tau} n\ U_n^2\ d\mu \ \xrightarrow{P}\ 0.$

We know that $T_n \longrightarrow \tau_H$ a.s. so for almost all ω, $I(T_n < \tau)$ is equal to zero if n is sufficiently large. Therefore for almost all ω, $\exists\ N_\omega$ such that $\sqrt{n}\ U_n(\omega) = 0$ for all $n \geq N_\omega$. Therefore for almost all ω, $\exists$ N_ω such that $\int_0^{\tau} n\ U_n^2(\omega)\ d\mu = 0$ for all $n \geq N_\omega$. Thus

$$\int_0^{\tau} n\ U_n^2\ d\mu \ \longrightarrow\ 0 \ \text{a.s.}. \qquad \square$$

CHAPTER 3

UNIFORM BOUNDEDNESS OF $\frac{R_n}{F/\overline{G}}$

In view of Theorem 2.1, to show the weak convergence of W_n in $L^2[0, \tau_H]$, we need to show that W_n is 'tight at τ_H' in the sense of Theorem 4.2 of Billingsley (1968). A sufficient condition for this to hold is that the sequence $R_n \le k\ F/\overline{G}$ for some constant k. In this chapter it is proved that $\frac{R_n}{F/\overline{G}}$ is uniformly bounded on $[0, \tau_H]$ under the assumption that

(3.1) $$\overline{G} = \overline{F}^{\alpha} \quad \text{for some} \quad \alpha \in [0, 1).$$

We begin with several Lemmas.

<u>**LEMMA 3.1.**</u> Let $X \sim F$ and $Y \sim G$ be two independent r.v.'s. Let ν be $(F+G)/2$. Let $f = \frac{dF}{d\nu}$ and $g = \frac{dG}{d\nu}$. Let Z denote $X \wedge Y$ and

$$\phi(x) = \frac{f(x)\ \overline{G}_-(x)}{f(x)\overline{G}_-(x)+g(x)\overline{F}(x)}.$$

Then $\phi(Z)$ is a version of $P(X \le Y|Z)$.

<u>**PROOF.**</u> As $\phi(Z)$ is $\sigma\langle Z\rangle$ measurable, all we need to check is

$$\int_A \phi(Z)\ dP = \int_A I(X \le Y)\ dP$$

for all sets A of the form $\{Z \le z\}$, where P is the probability measure with respect to which the distributions of X and Y are F and G respectively.

$$\int_A I(X{\le}Y)dP = P(X \le Y,\ Z \le z)$$

$$= \iint_{s\le t, s\le z} dF(s)\ dG(t)$$

$$= \int_{s \le z} \overline{G}\_(s)\ dF(s).$$

$$\int_A \phi(Z)\ dP \;=\; \iint_{s \le t, s \le z} \phi(s) dF(s) dG(t) + \iint_{s > t, t \le z} \phi(t)\ dF(s)\ dG(t)$$

$$= \int_{s \le z} \phi(s)\ \overline{G}\_(s)\ dF(s) + \int_{t \le z} \phi(t)\ F(t)\ dG(t)$$

$$= \int_{s \le z} \phi(s)\ [\overline{G}\_(s) f(s) + F(s) g(s)]\ d\nu(s)$$

$$= \int_{s \le z} \overline{G}\_(s) f(s)\ d\nu(s)$$

$$= \int_{s \le z} \overline{G}\_(s)\ dF(s).$$

This proves the result. □

Lemma 3.2. Let $\{(\tilde{Z}_i, \tilde{\delta}_i), 1 \le i \le n\}$ be i.i.d. two–dimensional random vectors with the d.f. of $\tilde{Z}_1$ continuous. Let $\tilde{Z}_{(i)}$, $1 \le i \le n$ be the order statistics of $\tilde{Z}_i$'s, $\tilde{\delta}_{(i)}$ be the corresponding induced order statistics of $\tilde{\delta}_i$'s and G_z denote the conditional d.f. of $\tilde{\delta}$ given $\tilde{Z} = z$. Given $\{\tilde{Z}_i, 1 \le i \le n\}$, $\tilde{\delta}_{(i)}$, $1 \le i \le n$, are conditionally independent with d.f.'s $G_{\tilde{Z}_{(i)}}$.

Proof. See Lemma 1 of Bhattacharya (1974). □

Lemma 3.3. Let X_i, Y_i, Z_i, δ_i's and $\delta_{(i)}$'s be as in 1(ii). Assume (3.1) holds. Then $\delta_{(i)}$'s are i.i.d Bernoulli (p) and are independent of $Z_{(i)}$'s, where $p = \frac{1}{1+\alpha}$.

Proof. From Lemmas 3.1 and 3.2, all we need to show is that the function ϕ of Lemma 3.1 is identically equal to p when $\overline{G} = \overline{F}^{\alpha}$.

First note that $g := \alpha\ \overline{F}^{\alpha-1} f$ is a possible version of $\frac{dG}{d\nu}$ because

$$\int_0^t \alpha F^{\alpha-1} f\, d\nu = \int_0^t \alpha\, F^{\alpha-1}\, dF = 1 - F^{\alpha}(t) = G(t).$$

Thus

$$\phi = \frac{f\overline{G}}{f\overline{G}+g\overline{F}} = \frac{f\overline{F}^{\alpha-1}}{f\overline{F}^{\alpha}+\alpha\overline{F}^{\alpha-1}f\overline{F}} = \frac{1}{1+\alpha} = p. \qquad \square$$

In what follows, $D = \{0, 1\}$ and $D_j = \{0, 1\}^j$, $1 \le j \le n$. For $\underset{\sim}{d} \in D_n$, d_k is the k^{th} entry of $\underset{\sim}{d}$ and $\underset{\sim}{d}(0)$ and $\underset{\sim}{d}(1)$ respectively denote the number of zeros and ones in $\underset{\sim}{d}$. By $B_{ni}(q)$, we mean $P(X = i)$ when $X \sim$ Binomial (n, q) and p will always denote $\frac{1}{1+\alpha}$.

LEMMA 3.4. Let Q_n and R_n be as in 1(iv). Let $a_{n0} = 1$ and

$$a_{ni} = \prod_{j=1}^{i} \left[p\left(\frac{n-j}{n-j+1}\right) + (1-p)\left(\frac{n-j+1}{n-j}\right)\right]$$

for $1 \le i \le n-1$. Then under (3.1),

$$R_n(t) = \sum_{i=0}^{n-1} B_{ni}(H(t))a_{ni}.$$

PROOF. Note that

$$Q_n(t) = \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}$$

on $\{Z_{(i)} < t \le Z_{(i+1)},\ \delta_{(k)} = d_k,\ 1 \le k \le n-1\}$. Hence,

$$\begin{aligned} R_n(t) &= E(Q_n(t)) \\ &= \sum_{i=0}^{n-1} \sum_{\underset{\sim}{d}\in D_{n-1}} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} \\ &\qquad \cdot P\{Z_{(i)} < t \le Z_{(i+1)},\ \delta_{(j)} = d_j,\ 1 \le j \le n-1\} \end{aligned}$$

$$= \sum_{i=0}^{n-1} A_{ni} \quad \text{(say)}.$$

By Lemma 3.3 and by elementary properties of order statistics,

$$P\{Z_{(i)}<t\leq Z_{(i+1)},\ \delta_{(k)}=d_k,\ 1\leq k\leq n-1\} = P\{Z_{(i)} < t \leq Z_{(i+1)}\} \cdot P\{\delta_{(k)} = d_k,\ 1\leq k \leq n-1\}$$
$$= B_{ni}(H(t))\, p^{\underset{\sim}{d}(1)} (1-p)^{\underset{\sim}{d}(0)}.$$

Therefore,

$$A_{ni} = B_{ni}(H(t)) \sum_{\underset{\sim}{d}\in D_{n-1}} p^{\underset{\sim}{d}(1)}(1-p)^{\underset{\sim}{d}(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}.$$

Write D_{n-1} as $D_i \times D_{n-1-i}$ so that $\underset{\sim}{d}$ is written correspondingly as $(\underset{\sim}{d}_1, \underset{\sim}{d}_2)$. Then, $\underset{\sim}{d}_1(0) + \underset{\sim}{d}_2(0) = \underset{\sim}{d}(0)$ and $\underset{\sim}{d}_1(1) + \underset{\sim}{d}_2(1) = \underset{\sim}{d}(1)$. Hence

$$\alpha_{ni} := \sum_{\underset{\sim}{d}\in D_{n-1}} p^{\underset{\sim}{d}(1)}(1-p)^{\underset{\sim}{d}(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}$$
$$= \sum_{\underset{\sim}{d}_1\in D_i} p^{\underset{\sim}{d}_1(1)}(1-p)^{\underset{\sim}{d}_1(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} \sum_{\underset{\sim}{d}_2\in D_{n-1-i}} p^{\underset{\sim}{d}_2(1)}(1-p)^{\underset{\sim}{d}_2(0)}$$
$$= \sum_{\underset{\sim}{d}_1\in D_i} p^{\underset{\sim}{d}_1(1)}(1-p)^{\underset{\sim}{d}_1(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}$$

because the last sum is 1. Now, by breaking up D_i as $D_{i-1} \times D$ and proceeding as in the earlier step,

$$\alpha_{ni} = \sum_{\underset{\sim}{d}_1\in D_i} p^{\underset{\sim}{d}(1)}(1-p)^{\underset{\sim}{d}(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}$$
$$= \sum_{\underset{\sim}{d}_1\in D_{i-1}} p^{\underset{\sim}{d}_1(1)}(1-p)^{\underset{\sim}{d}_1(0)} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} \sum_{d\in D} p^{d(1)}(1-p)^{d(0)} \left(\frac{n-i}{n-i+1}\right)^{2d-1}$$
$$= \sum_{\underset{\sim}{d}_1\in D_{i-1}} p^{\underset{\sim}{d}_1(1)}(1-p)^{\underset{\sim}{d}_1(0)} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} \left[p\left(\frac{n-i}{n-i+1}\right) + (1-p)\left(\frac{n-i+1}{n-i}\right)\right]$$

$$= \alpha_{ni-1}[p(\tfrac{n-i}{n-i+1}) + (1-p)(\tfrac{n-i+1}{n-i})].$$

Since $\alpha_{n0} = 1$, iteration of the above relation gives

$$\alpha_{ni} = \prod_{j=1}^{i}[p(\tfrac{n-j}{n-j+1}) + (1-p)(\tfrac{n-j+1}{n-j})] = a_{ni}.$$

Thus $A_{ni} = B_{ni}(H(t))\ \alpha_{ni}$ and $R_n(t) = \sum_{i=0}^{n-1} B_{ni}(H(t))\ a_{ni}$. □

LEMMA 3.5. Let

$$f(x) = \log\ [p(\tfrac{n-x}{n-x+1}) + (1-p)(\tfrac{n-x+1}{n-x})], \quad x \in [0, n).$$

Then for $M \le n$,

$$\int_0^M f(x)\,dx = (n+\beta)\log[(n+\beta)^2 + \gamma] - (n-M+\beta)\log[(n-M+\beta)^2 + \gamma]$$
$$+ (n-M)\log(n-M) + (n-M+1)\log(n-M+1) - n\log n$$
$$- (n+1)\log(n+1) + 2\sqrt{\gamma}\ \{\tan^{-1}(\tfrac{n+\beta}{\sqrt{\gamma}}) - \tan^{-1}(\tfrac{n-M+\beta}{\sqrt{\gamma}})\}$$

where $\beta = 1-p$ and $\gamma = \beta(1-\beta) = p(1-p)$.

PROOF. We follow the convention that $x \log x = 0$ if $x = 0$.

$$f(x) = \log\left[\frac{p(n-x)^2 + (1-p)(n-x+1)^2}{(n-x)\ (n-x+1)}\right]$$
$$= \log\ [(n-x)^2 + 2\beta(n-x) + \beta] - \log(n-x) - \log(n-x+1).$$

So

$$\int_0^M f(x)\,dx = \int_{n-M}^{n}\log[y^2 + 2\beta y + \beta]\,dy - \int_{n-M}^{n}\log y\,dy - \int_{n-M+1}^{n}\log y\,dy$$
$$= (I) - (II) - (III) \quad \text{(say).}$$

$$(I) = \int_{n-M}^{n}\log[(y+\beta)^2 + \beta(1-\beta)]\,dy$$
$$= \int_{n-M+\beta}^{n+\beta}\log(z^2+\gamma)\,dz$$
$$= \Psi(n+\beta) - \Psi(n-M+\beta)$$

where

$$\Psi(z) = z \log (z^2+\gamma) + 2\{\sqrt{\gamma} \tan^{-1}(\frac{z}{\sqrt{\gamma}}) - z\}.$$

This will give us

$$\text{(I)} \quad = (n+\beta) \log [(n+\beta)^2 + \gamma] - (n-M+\beta) \log[(n-M+\beta)^2 + \gamma] + 2\sqrt{\gamma}\{ \tan^{-1} (\frac{n+\beta}{\sqrt{\gamma}}) - \tan^{-1}(\frac{n-M+\beta}{\sqrt{\gamma}})\} - 2M.$$

(II) + (III) is easily seen to be

$$n \log n - (n-M) \log(n-M) + (n+1) \log(n+1) - (n-M+1) \log(n-M+1) - 2M.$$

From these relations, the lemma follows. □

LEMMA 3.6. There is a constant K such that $a_{ni} \leq K\ (n-i)^{2p-1}\ n^{1-2p}$ for $0 \leq i \leq n-1$, where a_{ni}'s are as in Lemma 3.4.

PROOF. Let f be as in Lemma 3.5. One can easily see that

$$\operatorname{sgn}(f'(x)) = \operatorname{sgn}(x - n + \frac{1}{\sqrt{p/(1-p)} - 1})$$

so that $f'(x) = 0$ for at most one $x \in [0, n)$. Hence f has at most one local minimum in $[0, n)$. Thus it follows that

$$\sum_{j=1}^{i} f(j) \leq \int_0^{i+1} f(x)\, dx. \tag{3.2}$$

Note that LHS of (3.2) = $\log a_{ni}$; therefore,

$$a_{ni} \leq \exp(\textstyle\int_0^{i+1} f(x)\, dx). \tag{3.3}$$

Now, $\tan^{-1}(a) - \tan^{-1}(b) \leq \frac{\pi}{2}$ for any non-negative a and b; so

$$\exp\left[2\sqrt{p(1-p)}[\tan^{-1}\left\{\frac{n+1-p}{\sqrt{p(1-p)}}\right\} - \tan^{-1}\left\{\frac{n-i-p}{\sqrt{p(1-p)}}\right\} \right] \leq e^{\pi/2}. \tag{3.4}$$

Note that

$$\left[\frac{(n+1-p)^2 + p(1-p)}{n(n+1)} \right]^n = \left[1 + \frac{1-2p}{n} + \frac{p}{n(n+1)} \right]^n$$

$$\leq \left[1 + \frac{2-2p}{n} \right]^n$$

(3.5) $$\leq e^{2(1-p)};$$

(3.6) $$\frac{[(n+1-p)^2 + p(1-p)]^{1-p}}{n+1} \leq 4n^{1-2p};$$

and since

$$\left[\frac{M^2}{(M-p)^2+p(1-p)}\right]^{M-p} \leq \left[\frac{M^2}{(M-p)^2}\right]^{M-p}$$
$$= [1-(p/M)]^{2(p-M)},$$

we have

$$\sup_{M\geq 1} \left[\frac{M^2}{(M-p)^2+p(1-p)}\right]^{M-p} \leq (1-(p/2))^{-4} \leq 16.$$

Therefore,

$$\frac{(n-i-1)^{n-i-1}(n-i)^{n-i}}{[(n-i-p)^2+p(1-p)]^{n-i-p}} = \left[\frac{(n-i)^2}{[(n-i-p)^2+p(1-p)]}\right]^{n-i-p} \left(\frac{n-i-1}{n-i}\right)^{n-i-1}(n-i)^{2p-1}$$

(3.7) $$\leq 16\ (n-i)^{2p-1}.$$

From (3.4), (3.5), (3.6), (3.7) and Lemma 3.5 we get

$$\exp\left(\int_0^{i+1} f(x)\ dx\right) \leq 64e^{2+\pi/2}\ n^{1-2p}\ (n-i)^{2p-1}.$$

Therefore by (3.3),

$$a_{ni} \leq K\ n^{1-2p}\ (n-i)^{2p-1}. \qquad \square$$

<u>**THEOREM 3.1.**</u> Assume 1(i) – 1(vi). Then under (3.1),

(3.8) $$\frac{R_n}{\overline{F}/\overline{G}}(t) \leq K, \quad t \leq \tau_H, \quad n \in \mathbb{N}$$

where K as in Lemma 3.6.

<u>**PROOF.**</u> From the convexity of the map $x \longmapsto x^{1/r}$ for $x \in [0, \infty)$, $r \in (0, 1]$, and Jensen's inequality, it follows that for any non-negative r.v.

Y, $[E(Y)]^{1/r} \leq E(Y^{1/r})$. Applying this to $r = 2p-1$ and $Y = X^{2p-1}$ where X is Binomial $(n, \bar{H}(t))$, one gets

$$E(X^{2p-1}) \leq \left[n \bar{H}(t)\right]^{2p-1}.$$

Now, by Lemmas 3.4 and 3.6,

$$\begin{aligned}
R_n(t) &= \sum_{i=0}^{n-1} B_{ni}(H(t))\, a_{ni} \\
&\leq K \sum_{i=0}^{n-1} B_{ni}(H(t))\, (n-i)^{2p-1}\, n^{1-2p} \\
&= K \sum_{i=1}^{n} B_{ni}(\bar{H}(t))\, i^{2p-1}\, n^{1-2p} \\
&= K\, E(X^{2p-1})\, n^{1-2p} \\
&\leq K \left[n \bar{H}(t)\right]^{2p-1} n^{1-2p} \\
&= K \left[\bar{H}(t)\right]^{2p-1}.
\end{aligned}$$

Now, since $p = \frac{1}{1+\alpha}$, $\bar{H} = \bar{F}\,\bar{G}$ and (3.1) holds, $\bar{H}^{2p-1} = \bar{F}/\bar{G}$. Hence (3.8). □

CHAPTER 4

CONVERGENCE OF W_n ON $L^2[0, \tau_H]$

This chapter proves the weak convergence of W_n to W in $L^2([0, \tau_H], \mu)$ under (3.1).

LEMMA 4.1. Under the assumptions of Chapter 1,

$$W \in L^2([0, \tau_H], \mu) \quad \text{a.s..}$$

Moreover, for every n,

$$W_n \in L^2([0, \tau_H], \mu) \quad \text{a.s..}$$

PROOF. That $W \in L^2([0, \tau_H], \mu)$ a.s. is obvious since by 1(vi) its second moment is integrable with respect to μ.

Let $n_0 \in N$ be fixed. We shall show that $\int_0^{\tau_H} W_{n_0}^2(\omega) d\mu < \infty$ for almost all ω. Let $\{r_n : n \in N\}$ be an enumeration of rationals in $[0, \tau_H)$. Let

$$\Omega_{n1} = \{\omega : \int_0^{r_n} W_{n_0}^2(\omega)\, d\mu < \infty \}.$$

By Theorem 3.1, $W_n \in L^2([0, \tau], \mu)$ almost surely for all $\tau < \tau_H$, so $P(\Omega_{n1}) = 1$ for all n and hence $\Omega_{01} := \bigcap_{n=1}^{\infty} \Omega_{n1}$ has probability 1. Since F is continuous,

$$\Omega_{n2} := \{\omega : T_n(\omega) < \tau_H\}$$

has probability 1 for each n and hence so does $\Omega_{02} := \bigcap_{n=1}^{\infty} \Omega_{n2}$. Let $\Omega_0 := \Omega_{01} \cap \Omega_{02}$ so that $P(\Omega_0) = 1$. Let $\omega \in \Omega_0$. $T_{n_0}(\omega) < \tau_H$ so $\exists$ $n(\omega) \in N$ such that $r_{n(\omega)} \in (T_{n_0}(\omega), \tau_H)$ and $r_{n(\omega)} > \kappa_F$, where κ_F is the infimum of the support of F. Since $\omega \in \Omega_{n(\omega)1}$, the following hold when evaluated at ω.

First of all,

$$\int_0^{r_n} W_{n_0}^2 \, d\mu < \infty.$$

Also

$$\begin{aligned}\int_{r_n}^{\tau_H} W_{n_0}^2 \, d\mu &= n \int_{r_n}^{\tau_H} F^2 \, d\mu \\ &= n[C(r_n)]^{-1} \int_{r_n}^{\tau_H} F^2 \, d\mu \\ &< \infty.\end{aligned}$$

Thus $\int_0^{\tau_H} W_{n_0}^2(\omega) d\mu < \infty$ for all $\omega \in \Omega_0$. □

<u>**LEMMA 4.2.**</u> If ϕ, defined in Lemma 3.1, is bounded away from zero near τ_H, then

$$n \int_{T_n}^{\tau_H} F^2 d\mu \to^P 0.$$

<u>**PROOF.**</u> Let $V_n = n \int_{T_n}^{\tau_H} F^2 \, d\mu$. Since $T_n \to \tau_H$ a.s., $P(T_n < a) \to^P 0$ for all $a < \tau_H$. So it is enough to show that $\exists\, a \in (0, \tau_H)$ such that

$$V_n' := n \int_{\max(T_n,\, a)}^{\tau_H} F^2 \, d\mu \to^P 0.$$

Define

$$f(x) = \sup_{n \in \mathbb{N}} n\, x^n, \quad x \in [0, 1].$$

Simple calculus techniques show that

$$f(x) \le -(\log x)^{-1} \le (1-x)^{-1}.$$

Hence

$$n(H(t))^n \leq (\bar{H}(t))^{-1}, \quad [0, \tau_H] \tag{4.1}$$

Note that

$$C(t) = \int_0^t \phi \, dH^{-1}$$

and use Lemma 1.4 to get $\bar{H}\, C$ is bounded away from zero near τ_H. So by the continuity of F $\exists\, a \in (0, \tau_H)$ and $k \in (0, \infty)$ such that

$$\bar{H}^{-1} \leq k\, C$$

on $[a, \tau_H]$. This along with (4.1) gives us

$$nF^2(t)\,(H(t))^n \leq k\, F^2(t)\, C(t)$$

$\forall\, n$ and $\forall\, t \in [a, \tau_H]$. Since for every $t \in [a, \tau_H]$, $n\, F^2(t)\,(H(t))^n$ converges to 0 as $n \to \infty$ and is dominated by the integrable function $k\, F^2(t)\, C(t)$, Dominated Convergence Theorem gives

$$n \int_a^{\tau_H} F^2(t)(H(t))^n \, d\mu(t) \quad \to \quad 0.$$

This quantity is precisely the expectation of V_n'. □

LEMMA 4.3. If ϕ is bounded away from zero near τ_H, then

$$\int_0^{\tau_H} n\, U_n^2 \, d\mu \ \to^P \ 0.$$

PROOF.
$$\int_0^{\tau_H} n\, U_n^2\, d\mu = \int_{T_n}^{\tau_H} n\, \frac{F_n^2(T_n) F^2(t)}{F^2(T_n)} \, d\mu(t)$$

$$= \frac{F_n^2(T_n)}{F^2(T_n)} \int_{T_n}^{\tau_H} n\, F^2(t)\, d\mu(t).$$

By Theorem 3.2.1 of Gill (1980), $\dfrac{F_n^2(T_n)}{F^2(T_n)}$ is $O_p(1)$ and by Lemma 4.2,

$\int_{T_n}^{\tau_H} n\, F^2(t)\, d\mu(t) \to^p 0$. Hence the product goes to zero in probability. □

THEOREM 4.1. Assume 1(i) – 1(vi) and (3.1) hold. Then

$$W_n \Rightarrow W \text{ on } L^2([0, \tau_H], \mu).$$

PROOF. Since (3.1) holds, by Lemma 3.3, $\phi \equiv p$ so that the condition in Lemmas 4.2 and 4.3 is satisfied. Since Theorem 2.1 has already shown the weak convergence on $L^2([0, \tau], \mu)$ for every $\tau < \tau_H$, as mentioned at the beginning of Chapter 3, it is enough to show 'tightness at τ_H' in the sense of Theorem 4.2 of Billingsley (1968).

Define

$$\begin{aligned} X_{\tau n}(t) &= W_n(t) \text{ if } t \le \tau \wedge T_n \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

and

$$\begin{aligned} X_\tau(t) &= W(t) \text{ if } t \le \tau \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

$W_n \Rightarrow W$ on $L^2\big([0, \tau], \mu\big)$ by Theorem 3.1. Now if ρ_τ denotes the norm on $L^2([0, \tau], \mu)$,

$$\begin{aligned} \rho_\tau^2(W_n, X_{\tau n}) &= \int_0^\tau [W_n(t) - W_n(t \wedge T_n)]^2\, d\mu(t) \\ &= I(T_n < \tau) \int_{T_n}^\tau [W_n(t) - W_n(T_n)]^2\, d\mu(t) \\ &\to 0 \text{ a.s.} \end{aligned}$$

since $T_n \to \tau_H$ a.s. and $\tau < \tau_H$. Therefore,

$$X_{\tau n} \Rightarrow W \quad \text{on } L^2([0, \tau], \mu) \text{ as } n \to \infty.$$

This is same as saying $\forall\, \tau < \tau_H$,

$$X_{\tau n} \Rightarrow X_\tau \quad \text{on } L^2([0, \tau_H], \mu) \text{ as } n \to \infty.$$

Now we shall show that as $\tau \to \tau_H$, $X_\tau \Rightarrow W$ on $L^2([0, \tau_H], \mu)$

by showing that $\rho(X_\tau, W) \to^P 0$.

$$E[\rho^2(X_\tau,W)] = E\left\{ \int_0^{\tau_H} (X_\tau - W)^2 \, d\mu \right\}$$

$$= \int_\tau^{\tau_H} E(W^2(t)) \, d\mu(t)$$

$$= \int_\tau^{\tau_H} \bar{F}^2(t) C(t) \, d\mu(t)$$

$$\to 0 \quad \text{as} \quad \tau \to \tau_H.$$

Therefore $\rho(X_\tau,W) \to 0$ in probability.

It remains to show that

$$\lim_{\tau \to \tau_H} \limsup_{n \to \infty} P\{\rho(X_{\tau n},W_n) > \epsilon\} = 0. \tag{4.2}$$

But observe that

$$\rho^2(X_{\tau n},W_n) = \int_0^{T_n} (X_{\tau n} - W_n)^2 \, d\mu + \int_{T_n}^{\tau_H} (X_{\tau n} - W_n)^2 \, d\mu.$$

The second term $= \int_{T_n}^{\tau_H} W_n^2(t) \, d\mu(t)$

$$= n \int_{T_n}^{\tau_H} \bar{F}^2(t) \, d\mu(t)$$

$$\to^P 0 \tag{4.3}$$

by Lemma 4.2. As far as the first term is concerned,

$$\int_0^{T_n} (X_{\tau n} - W_n)^2 \, d\mu = I(T_n > \tau) \int_\tau^{T_n} W_n^2(t) \, d\mu(t)$$

$$= I(T_n > \tau) \int_\tau^{T_n} X_n^2(t) \, d\mu(t)$$

$$\leq \int_\tau^{\tau_H} X_n^2(t) \, d\mu(t) \tag{4.4}$$

because $X_n = W_n$ for $t \leq T_n$. But by Lemma 2.1,

$$E(X_n^2(t)) = V(X_n(t)) = \bar{F}^2(t) \int_0^t \frac{R_n dF}{\bar{F}^3}$$

and by Theorem 3.1, $R_n(t) \leq K \, \bar{F}(t) \, \bar{G}^{-1}(t)$. Hence

$$V(X_n(t)) \leq K\ F^2(t)\ C(t). \tag{4.5}$$

Now (4.2) follows from (4.3), (4.4), (4.5) and 1(vi). □

CHAPTER 5

MORE GENERAL CASES

In this Chapter we shall assume F and G are any two continuous d.f's on $[0,\infty)$, not necessarily satisfying (3.1). Theorem 5.1 proves the weak convergence of W_n in $L^2([0, \tau_H], \mu)$ under more general conditions. 1(i) – 1(vi) are assumed to hold.

LEMMA 5.1. Let $Z_1, Z_2, \ldots\ldots\ldots, Z_n$ be i.i.d. H and $Z_{(1)}, Z_{(2)}, \ldots\ldots\ldots, Z_{(n)}$ be the corresponding order statistics. Let ν be as in Lemma 2.1 and h be a density of H w.r.t. ν. For $0 \le i \le n-1$, define

$$A_i(t) \;:=\; \{\omega : Z_{(i)}(\omega) < t \le Z_{(i+1)}(\omega)\}.$$

Then for every i, the conditional density of $(Z_{(1)}, Z_{(2)}, \ldots\ldots\ldots, Z_{(i)})$, given $A_i(t)$, is given by

$$h_i^t(x_1,x_2,\ldots\ldots,x_i) \;=\; \frac{i!}{H^i(t)}\left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots.. x_i \le t)$$

where all the densities are w.r.t. the corresponding product measure.

PROOF. We need to show that

$$(5.1) \qquad \int_0^{z_1}\int_0^{z_2}\cdots\int_0^{z_i} h_i^t(x_1,x_2,\ldots\ldots,x_i)\; d\nu(x_1)\; d\nu(x_2)\ldots\ldots d\nu(x_i)$$
$$= \; P[Z_{(1)} \le z_1, Z_{(2)} \le z_2, \ldots\ldots\ldots Z_{(i)} \le z_i \,|\, A_i(t)].$$

Let h_i denote the joint density of $(Z_{(1)}, Z_{(2)}, \ldots\ldots\ldots, Z_{(i)})$. Using the ideas of Section 2.2 of David (1970), we get that $h_i(x_1,x_2,\ldots\ldots,x_i)$ is equal to

$$\frac{n!}{(n-i)!}\, \bar{H}^{n-i}(x_i)\left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots. \le x_i).$$

For $i = 1$ to $n-1$, define

$$h^*_{i+1}(x_1,\ldots, x_i,t) \;:=\; \int_0^t h_{i+1}(x_1,\ldots\ldots x_i,s)\; d\nu(s).$$

Note that

$$h^*_{i+1}(x_1,\ldots\ldots x_i,t) = \int_0^t \Bigg\{ \frac{n!}{(n-i-1)!} \left[\prod_{j=1}^{i} h(x_j)\right] h(s)\, \bar{H}^{n-i-1}(s) \cdot I(x_1 \le x_2 \le \ldots\ldots \le x_i \le s) \Bigg\}\, d\nu(s)$$

$$= \frac{n!}{(n-i-1)!} \left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots\ldots \le x_i) \cdot \int_{x_i}^t \bar{H}^{n-i-1}(s)\, dH(s)$$

$$= \frac{n!}{(n-i)!} \left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots\ldots \le x_i) \cdot \left\{ \bar{H}^{n-i}(x_i) - \bar{H}^{n-i}(t) \right\}$$

$$= h_i(x_1,\ldots\ldots x_i) - \frac{n!}{(n-i)!} \left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots\ldots \le x_i) \cdot \bar{H}^{n-i}(t).$$

On the other hand, because $P(A_i(t)) = \frac{n!}{i!(n-i)!} H^i(t)\, \bar{H}^{n-i}(t)$, one can rewrite $h_i^t(x_1,x_2,\ldots\ldots,x_i)$ as

$$\frac{I(x_i \le t)}{P(A_i(t))} \left\{ \frac{n!}{(n-i)!} \left[\prod_{j=1}^{i} h(x_j)\right] I(x_1 \le x_2 \le \ldots\ldots \le x_i)\; \bar{H}^{n-i}(t) \right\}.$$

Thus

$$h_i^t(x_1,x_2,\ldots\ldots,x_i) = \frac{I(x_i \le t)}{P(A_i(t))} \left[h_i(x_1,\ldots\ldots x_i) - h^*_{i+1}(x_1,\ldots\ldots x_i,t)\right].$$

$$= \frac{I(x_i \le t)}{P(A_i(t))} \left[h_i(x_1,\ldots\ldots x_i) - \int_0^t h_{i+1}(x_1,\ldots\ldots x_i,s)\; d\nu(s)\right].$$

Therefore, LHS of (5.1)

$$= [P(A_i(t)]^{-1} \left\{ \int_0^{s_1}\int_0^{s_2} \cdots \int_0^{s_i \wedge t} \left[h_i(x_1,x_2,\ldots\ldots,x_i) - \int_0^t h_{i+1}(x_1,\ldots,x_i,s) d\nu(s)\right] \prod_{j=1}^{i} d\nu(x_j) \right\}$$

$$= [P(A_i(t)]^{-1} \left\{H_i(z_1,z_2,\ldots..z_{i-1},z_i\wedge t) - H_{i+1}(z_1,z_2,\ldots..z_{i-1},z_i\wedge t,t)\right\}$$
$$= [P(A_i(t)]^{-1} P[Z_{(1)} \le z_1, Z_{(2)} \le z_2,\ldots\ldots\ldots Z_{(i)} \le z_i, Z_{(i)} \le t < Z_{(i+1)}]$$
$$= P[Z_{(1)} \le z_1, Z_{(2)} \le z_2,\ldots\ldots\ldots.Z_{(i)} \le z_i | A_i(t)]$$

where H_i is the joint d.f. corresponding to h_i. □

<u>**LEMMA 5.2.**</u> Let Q_n and R_n be as in 1(iv). Then

$$R_n(t) = \sum_{i=0}^{n-1} B_{ni}(H(t))a_{ni}(t)$$

where

$$a_{ni}(t) = \frac{i!}{H^i(t)}\int_0^t\int_0^{x_i}\int_0^{x_{i-1}}\cdots\int_0^{x_2}\left\{\prod_{j=1}^{i}[(\frac{n-j}{n-j+1})\ \phi(x_j) + (\frac{n-j+1}{n-j})\ \bar{\phi}(x_j)]\right\}\prod_{j=1}^{i}d\nu(x_j).$$

<u>**PROOF.**</u> Note that exactly as in Lemma 3.4,

$$R_n(t) = \sum_{i=0}^{n-1} \sum_{\underset{\sim}{d}\in D_{n-1}} \prod_{j=1}^{i} (\frac{n-j}{n-j+1})^{2d_j-1}$$
$$\cdot P\left\{Z_{(i)} < t \le Z_{(i+1)},\delta_{(k)} = d_k,\ 1 \le k \le n-1\right\}$$
$$= \sum_{i=0}^{n-1} \sum_{\underset{\sim}{d}\in D_{n-1}} \prod_{j=1}^{i} (\frac{n-j}{n-j+1})^{2d_j-1} P\left\{\delta_{(k)} = d_k,\ 1 \le k \le n-1 | A_i(t)\right\} P(A_i(t))$$
$$= \sum_{i=0}^{n-1} B_{ni}(H(t)) \sum_{\underset{\sim}{d}\in D_{n-1}} \prod_{j=1}^{i} (\frac{n-j}{n-j+1})^{2d_j-1} P\left\{\delta_{(k)} = d_k,\ 1 \le k \le n-1\right\}$$
$$= \sum_{i=0}^{n-1} B_{ni}(H(t))\alpha_{ni}(t) \qquad \text{(say)}.$$

We need to prove that $\alpha_{ni}(t) = a_{ni}(t)$. By writing $D_{n-1} = D_i \times D_{n-1-i}$,

$$\alpha_{ni}(t) = \sum_{\underset{\sim}{d}_1\in D_i} \sum_{\underset{\sim}{d}_2\in D_{n-1-i}} \prod_{j=1}^{i} (\frac{n-j}{n-j+1})^{2d_j-1}$$
$$\cdot P\left\{\delta_{(k)} = d_k\ 1 \le k \le n-1 | A_i(t)\right\}$$

$$= \sum_{\underset{\sim}{d}_1 \in D_i} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} \sum_{\underset{\sim}{d}_2 \in D_{n-1-i}} P\left\{\delta_{(k)} = d_k,\ 1 \le k \le n-1 \,|\, A_i(t)\right\}$$

$$= \sum_{\underset{\sim}{d}_1 \in D_i} \prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1} P\left\{\delta_{(k)} = d_k\ 1 \le k \le i \,|\, A_i(t)\right\}.$$

From Lemma 3.2, given $Z_{(j)}$'s, $\delta_{(j)}$'s have conditional distribution given by

$$P\left\{\delta_{(j)} = d_j,\ 1 \le j \le i \,|\, Z_{(1)}, Z_{(2)}, \ldots\ldots Z_{(i)}\right\} = \prod_{j=1}^{i} [\phi^{d_j}(Z_{(j)})\bar{\phi}^{1-d_j}(Z_{(j)})].$$

Thus, by Lemma 5.1,

$$P\left\{\delta_{(j)} = d_j,\ 1 \le j \le i \,|\, A_i(t)\right\}$$

$$= E\left[\prod_{j=1}^{i} [\phi^{d_j}(Z_{(j)})\bar{\phi}^{1-d_j}(Z_{(j)})] \,|\, A_i(t)\right]$$

$$= \frac{i!}{H^i(t)} \int_0^t \int_0^{x_i} \int_0^{x_{i-1}} \cdots \int_0^{x_2} \prod_{j=1}^{i} [\phi^{d_j}(x_j)\bar{\phi}^{1-d_j}(x_j)] \prod_{j=1}^{i} dH(x_j).$$

Therefore, to show $\alpha_{ni} = a_{ni}$, it is enough to show that

$$\sum_{\underset{\sim}{d}_1 \in D_i} \left[\prod_{j=1}^{i} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}\right] \left[\prod_{j=1}^{i} [\phi^{d_j}(x_j)\bar{\phi}^{1-d_j}(x_j)]\right]$$

$$= \prod_{j=1}^{i} [(\frac{n-j}{n-j+1})\ \phi(x_j) + (\frac{n-j+1}{n-j})\ \bar{\phi}(x_j).$$

We shall show this by induction on i. For $i = 1$,

$$\text{LHS} = \sum_{\underset{\sim}{d}=0,1} \left[\frac{n-1}{n}\right]^{2d-1} \phi^{d}(x_1)\ \bar{\phi}^{1-d}(x_1)$$

$$= \left[\frac{n-1}{n}\right] \phi(x_1) + \left[\frac{n}{n-1}\right] \bar{\phi}(x_1)$$

$$= \text{RHS}.$$

Assume the result for $i-1$. Write $D_i = D_{i-1} \times D$ and $\underset{\sim}{d}$ as $(\underset{\sim}{d}_1, d)$.

Then LHS is equal to

$$\sum_{\underset{\sim}{d}_1 \in D_{i-1}} \left[\prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1}\right)^{2d_j-1}\right]\left[\prod_{j=1}^{i-1}[\phi^{d_j}(x_j)\bar{\phi}^{1-d_j}(x_j)]\right] \times \sum_{d\in D}\left(\frac{n-i}{n-i+1}\right)^{2d-1}\phi^{d}(x_i)\bar{\phi}^{1-d}(x_i)$$

$$= \prod_{j=1}^{i-1} [(\tfrac{n-j}{n-j+1})\ \phi(x_j) + (\tfrac{n-j+1}{n-j})\ \bar{\phi}(x_j)] \left\{(\tfrac{n-i}{n-i+1})\ \phi(x_i) + (\tfrac{n-i+1}{n-i})\ \bar{\phi}(x_i)\right\}$$

$$= \prod_{j=1}^{i} [(\tfrac{n-j}{n-j+1})\ \phi(x_j) + (\tfrac{n-j+1}{n-j})\ \bar{\phi}(x_j)]$$

$$= \text{RHS.} \qquad \square$$

<u>**THEOREM 5.1.**</u> Assume 1(i) – 1(vi) hold. Then

$$W_n \Rightarrow W \quad \text{in} \quad L^2([0,\ \tau_H],\ \mu)$$

if any of the following conditions is satisfied.

(5.2) For some $\alpha \in [0, 1)$, $\bar{G}/\bar{F}^{\alpha}$ is non–decreasing and bounded above.

(5.3) ϕ is non–increasing and $\bar{F}^{\phi-1}\ \bar{G}^{\phi}$ is bounded above.

<u>**PROOF.**</u> First, we will prove that the condition $\bar{G}/\bar{F}^{\alpha}$ is non–decreasing is equivalent to $\phi \geq \frac{1}{1+\alpha}$ a.s. ν. Since F and G are continuous,

$$\begin{aligned}
\phi \geq \frac{1}{1+\alpha} \text{ a.s. } \nu \ &\Leftrightarrow\ [1 + (g\bar{F}/f\bar{G})]^{-1} \ \geq\ \frac{1}{1+\alpha} \text{ a.s. } \nu \\
&\Leftrightarrow\ g\bar{F}/f\bar{G} \ \leq\ \alpha \text{ a.s. } \nu \\
&\Leftrightarrow\ g/\bar{G} \ \leq\ \alpha(f/\bar{F}) \text{ a.s. } \nu \\
&\Leftrightarrow\ \int_a^b (g/\bar{G})d\nu \leq \int_a^b \alpha(f/\bar{F})d\nu \ \text{ for all } a < b \\
&\Leftrightarrow\ \log \bar{G}(a) - \log \bar{G}(b) \ \leq\ \alpha[\log \bar{F}(a) - \log \bar{F}(b)] \\
&\qquad \text{for all } a < b \\
&\Leftrightarrow\ \bar{G}(a)/\bar{G}(b) \ \leq\ [\bar{F}(a)/\bar{F}(b)]^{\alpha} \ \text{ for all } a < b \\
&\Leftrightarrow\ (\bar{G}/\bar{F}^{\alpha})(a) \ \leq\ (\bar{G}/\bar{F}^{\alpha})(b) \ \text{ for all } a < b \\
&\Leftrightarrow\ (\bar{G}/\bar{F}^{\alpha}) \text{ is non–decreasing.}
\end{aligned}$$

$\phi \geq \frac{1}{1+\alpha}$ implies

$$\left(\frac{n-j}{n-j+1}\right)\phi + \left(\frac{n-j+1}{n-j}\right)\bar{\phi} \leq \left(\frac{n-j}{n-j+1}\right)\left(\frac{1}{1+\alpha}\right) + \left(\frac{n-j+1}{n-j}\right)\left(\frac{\alpha}{1+\alpha}\right)$$

so

$\phi \geq \frac{1}{1+\alpha}$ a.s. ν implies

$$a_{ni}(t) \leq \prod_{j=1}^{i}\left\{\left(\frac{n-j}{n-j+1}\right)\left(\frac{1}{1+\alpha}\right) + \left(\frac{n-j+1}{n-j}\right)\left(\frac{\alpha}{1+\alpha}\right)\right\}$$

from Lemma 5.2 and from the fact that

$$\frac{i!}{H^{i}(t)}\int_0^t\int_0^{x_i}\int_0^{x_{i-1}}\cdots\int_0^{x_2}\prod_{j=1}^{i}dH(x_j) = 1.$$

Now from Lemma 3.6 and proof of Theorem 3.1, it follows that

$$R_n(t) \leq K\, H^{\frac{1-\alpha}{1+\alpha}}(t).$$

Therefore,

$$\frac{R_n}{\bar{F}/\bar{G}} \leq K\, F^{\frac{1-\alpha}{1+\alpha}}\, G^{\frac{1-\alpha}{1+\alpha}}\, \bar{F}^{-1}\bar{G}$$

$$= K\, F^{\frac{-2\alpha}{1+\alpha}}\, G^{\frac{2}{1+\alpha}}.$$

Now $\bar{G} \leq K_0\, \bar{F}^{\alpha}$ implies $\bar{G}^{\frac{1}{1+\alpha}} \leq K_0^{\frac{1}{1+\alpha}}\, \bar{F}^{\frac{\alpha}{1+\alpha}}$

implies $\bar{F}^{\frac{-\alpha}{1+\alpha}}\, \bar{G}^{\frac{1}{1+\alpha}} \leq K_0^{\frac{1}{1+\alpha}}$

implies $K\, \bar{F}^{\frac{-2\alpha}{1+\alpha}}\, \bar{G}^{\frac{2}{1+\alpha}} \leq K\, K_0^{\frac{2}{1+\alpha}}$

implies $\frac{R_n}{\bar{F}/\bar{G}} \leq K_1$ for some constant K_1.

From the proof of Theorem 4.1, it is clear that this is all we need to show that (5.2) is sufficient for the weak convergence of W_n to W in $L^2([0, \tau_H], \mu)$.

To show that (5.3) is sufficient for the L^2– weak convergence of W_n, as earlier, we just need to show $R_n \leq K_1\, \bar{F}/\bar{G}$ for some constant K_1.

If ϕ is non–increasing, $(\frac{n-j}{n-j+1})\,\phi + (\frac{n-j+1}{n-j})\,\bar{\phi}$ is non–decreasing; so for all j, $1 \le j \le n$,

$$(\frac{n-j}{n-j+1})\,\phi(x_j) + (\frac{n-j+1}{n-j})\,\bar{\phi}(x_j) \;\le\; (\frac{n-j}{n-j+1})\,\phi(t) + (\frac{n-j+1}{n-j})\,\bar{\phi}(t)$$

and therefore, as earlier, $a_{ni}(t) \;\le\; \prod_{j=1}^{i}\left\{(\frac{n-j}{n-j+1})\,\phi(t) + (\frac{n-j+1}{n-j})\,\bar{\phi}(t)\right\}$

Consequently $R_n(t) \;\le\; K\;\bar{H}^{2\phi(t)-1}(t)$ as in Lemma 2.5 and Theorem 2.1. Note that the bound in Lemma 2.5 is uniformly in p so the constant K is free of t.

$$\bar{H}^{2\phi(t)-1} = \bar{F}^{2\phi(t)-1}(t)\;\bar{G}^{2\phi(t)-1}(t)$$

implies
$$\begin{aligned}\frac{R_n}{F/\bar{G}}(t) &\le K\;\bar{F}^{2\phi(t)-2}(t)\;\bar{G}^{2\phi(t)}(t)\\ &= K\left\{\bar{F}^{\phi(t)-1}(t)\;\bar{G}^{2\phi(t)}\right\}^2\\ &= K\,K_0^2\\ &= K_1\end{aligned}$$

where K_0 bounds $\bar{F}^{2\phi-1}\,\bar{G}^{\phi}$. □

<u>**REMARK 5.1.**</u> Note that condition (ii) of Theorem 5.1 is the same as λ_F/λ_G being non–increasing and $\Lambda_F - \Lambda_H\cdot(\lambda_F/\lambda_G)$ being bounded above, where λ's and Λ's are hazard functions and integrated hazard functions respectively.

<u>**EXAMPLE 5.1.**</u> Let G be any distribution on $[0, \infty)$, $\delta \in (1, \infty)$ and Ψ be any bounded non–decreasing function on $[0, \infty)$ with $\Psi(0) = 1$. Define F by

$$\bar{F}(t) = [G(t)/\Psi(t)]^{\delta}.$$

It is easy to verify that these F and G do not satisfy the proportional hazards model, but do satisfy (5.2).

CHAPTER 6

ASYMPTOTICALLY DISTRIBUTION FREE TESTS

In this chapter some Anderson–Darling type A.D.F. statistics that are used to test H_0: $F = F_0$ vs. H_1: $F \neq F_0$ are discussed. It is shown that

$$\int_0^{\tau_H} [K_n/K_n] \, [W_n/F]^2 \, dK_n \Rightarrow \int_0^1 B_0^2(t)/[t(1-t)] \, dt$$

and

$$\int_0^{\tau_H} [K_n^*/K_n^*] \, [W_n/F_n]^2 \, dK_n^* \Rightarrow \int_0^1 B_0^2(t)/[t(1-t)] \, dt.$$

where K_n and K_n^* are as in 1(iii). Also discussed are the L^2 – weak convergence of $\tilde{\xi}_n$, ξ_n and ξ_n^* (as defined in 1(iii)) to W. A few preliminary results are proved first.

<u>**LEMMA 6.1 a.**</u> For each n, $0 \leq \dfrac{K_n}{F} \leq 1$ and $\dfrac{K_n}{F}$ is a monotonic decreasing function on $[0, \tau_H]$.

<u>**PROOF.**</u> From the definition both C_n and K_n are increasing and non–negative functions. Moreover, for $t \leq \tau_H$,

$$C_n(t) \geq \int_0^t \frac{dF}{F^2} = \frac{F}{F}(t); \text{ so } \frac{1}{1+C_n}(t) \leq F(t) \text{ and hence } \frac{K_n}{F}(t) \leq 1.$$

For $t \leq \tau_H$, we shall show that

$$\frac{F}{K_n}(t) = \int_0^t M_n \, dF$$

where

$$M_n = - C_n + \frac{1}{F \, G_{n-}} \geq 0,$$

which will prove that $\dfrac{F}{K_n}$ is increasing.

Using integration by parts, as F is continuous and $C_n(0) = 0$,

$$\int_0^t \frac{dF}{F\,\overline{G}_{n-}} = \int_0^t F\, dC_n = F(t)\, C_n(t) + \int_0^t C_n\, dF$$

and hence

$$\int_0^t M_n\, dF = F(t)\, C_n(t) = \frac{F}{K_n}(t).$$

But

$$C_n(t) = \int_0^t \frac{dF}{F^2\, \overline{G}_{n-}} \leq \frac{F}{F\, \overline{G}_{n-}}(t) \leq \frac{1}{F\, \overline{G}_{n-}}(t)$$

and hence $M_n(t) \geq 0$. □

<u>**Lemma 6.1 *b.***</u> For each n, $\overline{G} \leq \dfrac{K_n^*}{F_n} \leq 1$ and $\dfrac{K_n^*}{F_n}$ is a monotonic decreasing function on $[0, \tau_H]$.

<u>**Proof.**</u> Note that on $[T_n, \tau_H]$, $\dfrac{K_n^*}{F_n}$ is of the form $\dfrac{0}{0}$ so there, it is defined to be $\dfrac{K_n^*}{F_n}(T_n-)$, which is well-defined. Now, C_n^* is same as the function $\hat{C}$ defined in Gill (1983). The remarks that follow Theorem 1.2 of Gill (1983) give the result. □

<u>**Lemma 6.2 *a.***</u>

(6.1)
$$\left\| \frac{K_n}{F} - \frac{K}{F} \right\|_0^{\tau_H} \to^P 0.$$

<u>**Proof.**</u> The proof is split into two cases. First is the case when

$\overline{G}(\tau_H) > 0$. In this case, $\|\overline{G}_n^{-1} - \overline{G}^{-1}\|_0^{\tau_H} \to^P 0$. The proof below is given under the assumption that the convergence holds almost surely. If it holds only in probability, a subsequence argument will yield the result.

All the statements in this paragraph hold on a probability 1 set. Given $\epsilon > 0$, $\exists$ N such that $\forall$ $n \geq N$, $\|\overline{G}_n^{-1} - \overline{G}^{-1}\|_0^{\tau_H} \leq \epsilon$. If $n \geq N$,

$$|F(t)\ (C_n(t) - C(t))| \leq \epsilon\ F(t) \left|\int_0^t F^{-2}\ dF\right| \leq \epsilon$$

for all $t \leq \tau_H$. So it follows that $\|F(C_n - C)\|_0^{\tau_H} \to 0$. By an application of Lemma 1.4 with $d\mu = dF^{-1}$ and $g = \overline{G}_-^{-1}$, $\overline{F}(t)\ C(t)$ is seen to converge to $\overline{G}^{-1}(\tau_H)$ as $t \to \tau_H$. Since $F\ (1+C_n) \to F\ (1+C)$ uniformly on $[0,\tau_H]$ and $F\ (1+C)$ is bounded away from zero near τ_H, (6.1) follows.

Next is the case when $\overline{G}(\tau_H) = 0$. In this case $\overline{F}(t)C(t) \to \infty$ as $t \to \tau_H$ and hence $\frac{K}{F}(t) \to 0$. Now for each $\tau < \tau_H$,

$$\left\| \frac{K_n}{F} - \frac{K}{F} \right\|_0^{\tau} \to 0 \text{ a.s.}$$

because

$$\frac{1}{\overline{G}_{n-}} \to \frac{1}{\overline{G}_-} \text{ uniformly on } [0,\ \tau] \text{ a.s..}$$

(Note that $\overline{G}$ and F are bounded away from zero on $[0,\ \tau]$.) So $\exists$ Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, $\left\| \frac{K_n}{F}(\omega) - \frac{K}{F} \right\|_0^{\tau} \to 0$ for all $\tau < \tau_H$. Then the following statements hold on Ω_0.

$\frac{K_n}{F}(0) = \frac{K}{F}(0) = 1$; $\frac{K_n}{F}(\tau_H) = \frac{K}{F}(\tau_H) = 0$; $0 \leq \frac{K_n}{F} \leq 1$; $\frac{K_n}{F}$ is monotonic decreasing; and $\frac{K_n}{F}(t) \to \frac{K}{F}(t)$ pointwise. Since $\frac{K}{F}$ is continuous, by Polya's Theorem the convergence is uniform on $[0,\ \tau_H]$. $\square$

LEMMA 6.2 *b*.

(6.2) $$\left\| \frac{K_n^*}{F_n} - \frac{K}{F} \right\|_0^{\tau_H} \to^P 0.$$

PROOF. Again the proof is split into the same two cases as in Lemma 6.2 *a*. Because of Lemma 6.1 *b*, the proof in the second case is exactly the same as in the proof of Lemma 6.2 *a* so we will prove the result only in case 1. Here, the proof will go along the same lines as in Lemma 6.2 *a*, with the comment about a possible subsequence argument applicable. So all we need to prove is that $\|F_n(1+C_n^*) - F(1+C)\| \to 0$. Note that we need to define $F_n(1+C_n^*)$ to be a suitable left limit whenever it is of the form 0 times ∞.

By Lemma 6.1 *b*, $\overline{G}_n \leq K_n/F_n \leq 1$ and K_n/F_n is increasing. Hence

(6.3) $$F_n(1+C_n) \leq \overline{G}_n^{-1}$$

and for each n,

(6.4) $F_n(1+C_n)$ in a non-decreasing function.

We know that $\overline{G}_n^{-1} \to \overline{G}^{-1}$ on $[0, \tau_H]$ as $\overline{G}(\tau_H) > 0$. Also we know that $F(t)(1+C(t)) \to \overline{G}^{-1}(\tau_H)$ as $t \to \tau_H$ and that, for any $\tau < \tau_H$, $F_n(1+C_n) \to F(1+C)$ as $n \to \infty$ on $[0, \tau]$. Let $\epsilon > 0$. $\exists\ \tau < \tau_H$ such that

(6.5) $$|F(t)(1+C(t)) - \overline{G}^{-1}(\tau_H)| \leq \epsilon$$

for all $t \geq \tau$. Also, $\exists$ N such that $\forall\ n \geq N$,

(6.6) $$\|F_n(1+C_n) - F(1+C)\|_0^{\tau} \leq \epsilon$$

and

(6.7) $$\|\overline{G}_n^{-1} - \overline{G}^{-1}\| \leq \epsilon.$$

Now $\forall\ n \geq N$ and $\forall\ t \geq \tau$,

$$|\overline{G}_n^{-1}(t) - F_n(t)(1+C_n(t))| = \overline{G}_n^{-1}(t) - F_n(t)(1+C_n(t))$$

(by (6.3))

$$\begin{aligned}
&\le \overline{G}_n^{-1}(t) - \overline{F}_n(\tau)(1+C_n(\tau)) \\
&\qquad \text{(by (6.4))} \\
&\le \overline{G}^{-1}(t) - \overline{F}(\tau)(1+C(\tau)) + 2\epsilon \\
&\le 3\epsilon.
\end{aligned}$$

So by (6.5) and (6.7), $\forall n \ge N$ and $\forall\, t \ge \tau$,

$$|\overline{F}_n(t)(1+C_n(t)) - \overline{F}(t)(1+C(t))| \le 5\epsilon$$

The above, together with (6.6), gives the result. □

LEMMA 6.3 a. $\left\| \frac{K_n}{K} \right\|_0^{T_n}$ is bounded in probability. More specifically, for all $M \in (1, \infty)$,

$$P\left\{ \left\| \frac{K_n}{K} \right\|_0^{T_n} > M+1 \right\} \le \frac{1}{M}\,.$$

PROOF. Note that it is enough to show that

$$P\left\{ \left\| \frac{C}{C_n} \right\|_0^{T_n} > M \right\} \le \frac{1}{M} \quad \text{for all} \quad M \in (1,\infty)$$

because

$$\frac{K_n}{K} = \frac{1+C}{1+C_n} \le 1 + \frac{C}{C_n}\,.$$

Apply Lemma 2.6 of Gill (1983) to G_n to get

$$P\left\{ \left\| \frac{\overline{G}_{n-}}{\overline{G}_-} \right\|_0^{T_n} > M \right\} \le \frac{1}{M} \quad \text{for all} \quad M \in (1,\infty).$$

Now,

$$\begin{aligned}
C_n &\ge C_n\, I\left\{\left\| \frac{\overline{G}_{n-}}{\overline{G}_-} \right\|_0^{T_n} \le M\right\} \\
&\ge \frac{1}{M} \int_0^{\cdot} \frac{dF}{\overline{F}^2 \overline{G}_-}\; I\left\{\left\| \frac{\overline{G}_{n-}}{\overline{G}_-} \right\|_0^{T_n} \le M\right\} \\
&= \frac{C}{M}\, I\left\{\left\| \frac{\overline{G}_{n-}}{\overline{G}_-} \right\|_0^{T_n} \le M\right\}
\end{aligned}$$

on $[t \le T_n]$. Hence

$$P\left\{\left\| \frac{C}{C_n} \right\|_0^{T_n} \ge M\right\} \le P\left\{\left\| \frac{\bar{G}_{n-}}{\bar{G}_-} \right\|_0^{T_n} \ge M\right\} \le \frac{1}{M}. \quad \square$$

<u>LEMMA 6.3 *b*.</u> $\left\| \frac{K_n^*}{K} \right\|_0^{\tau_H}$ is bounded in probability.

<u>PROOF.</u> The proof will follow as in Lemma 6.3 *a* but we still need to show

$$\left\| \int_0^t \frac{dF}{F} \Big/ \int_0^t \frac{dF_n}{F_n} \right\|_0^{\tau_H} \text{ is } O_p(1). \tag{6.8}$$

By (7.7.21) of Shorack and Wellner (1986),

$$\|K_n^* - K\|_0^{\tau} \to 0 \text{ a.s. } \forall\, \tau < \tau_H$$

and hence, as $K(\tau) > 0$ for $\tau < \tau_H$,

$$\left\| \frac{K_n^*}{K} \right\|_0^{\tau} \to 1 \text{ a.s., } \tau < \tau_H.$$

Therefore, we only need to show (6.8) near τ_H. Now, if i is such that $Z_{(i)} \le t < Z_{(i+1)}$, by explicit computation we can show that

$$\int_0^t \frac{dF_n}{F_n} = \sum_{j=1}^{i} \left[\left\{\frac{n-j+1}{n-j}\right\}^{\delta_{(j)}} - 1\right] \tag{6.9}$$

Also, $\log x \le x - 1 \;\; \forall\, x \ge 1$, so

$$\sum_{j=1}^{i} \left[\left\{\frac{n-j+1}{n-j}\right\}^{\delta_{(j)}} - 1\right] \ge \sum_{j=1}^{i} \log \left\{\frac{n-j+1}{n-j}\right\}^{\delta_{(j)}} \ge -\log F_n. \tag{6.10}$$

by the definition of F_n. By Lemma 2.6 of Gill (1983) and the relation $\log (F_n/F) \,/\, (-\log F_n) \ge -1$, it follows that

$$[\log (F_n/F) \,/\, (-\log F_n)] = O_p(1) \tag{6.11}$$

near τ_H. Now (6.8) follows from (6.9), (6.10) and (6.11) $\quad \square$

LEMMA 6.4. Assume ϕ is bounded away from zero near τ_H a.s. ν. Then $\forall\ \delta > 1/2$,

$$\sqrt{n}\ K^{\delta}(T_n) \xrightarrow{P} 0. \tag{6.12}$$

PROOF. Following along the same line as in the proof of Lemma 1.3, we get that (6.12) is equivalent to

$$[1+C(t)]^{2\delta}\ \bar{H}(t) \to \infty \text{ as } t \to \tau_H.$$

So it is enough to prove that $C(t)\ \bar{H}(t)$ is bounded away from 0 near τ_H. Note that $C(t) = \int_0^t \phi\ d\bar{H}^{-1}$. So by Lemma 1.4, $C(t)\ \bar{H}(t) \to \phi(\tau_H)$ as $t \to \tau_H$. □

LEMMA 6.5 a. Assume that $\exists\ \alpha \in [0, 1)$ such that

$$\phi \geq \frac{1}{1+\alpha} \tag{6.13}$$

and

$$(\bar{G}/\bar{F}^{\alpha}) \text{ is non-decreasing.} \tag{6.14}$$

Then $\exists\ \beta \in (0, 1/2)$ such that for a special construction of X_i's, Y_i's and B,

$$\left\| \frac{K_n^{1-\beta} W_n}{\bar{F}} - \frac{K^{1-\beta} W}{\bar{F}} \right\|_0^{\tau_H} \xrightarrow{P} 0$$

where $W := \bar{F}\ B(C)$.

PROOF. Under (6.14), $\exists\ \beta \in (0,1/2)$ such that

$$\int_0^{\tau_H} \frac{dF}{\bar{F}^{2\beta}\bar{G}_-} < \infty \tag{6.15}$$

Use the construction of Theorem 7.1.1 of Shorack and Wellner (1986) and apply the theorem with $q(t) := (1-t)^{\beta}$ to get

(6.16) $$\left\| \frac{K}{F} \left[\frac{\hat{W}_n - W}{K^{\beta}} \right] \right\|_{K^{-1}(\frac{1}{2})}^{T_n} \to^P 0.$$

By (6.13), Lemma 6.4 is applicable, and hence $\hat{W}_n$ in (6.16) can be replaced by W_n. Also,

$$\left\| \frac{K}{F} \left[\frac{W_n - W}{K^{\beta}} \right] \right\|_0^{K^{-1}(\frac{1}{2})} \to^P 0$$

since F is bounded away from zero and $\left\| W_n - W \right\|_0^{K^{-1}(\frac{1}{2})} \to^P 0$. Thus we have

(6.17) $$\left\| \frac{K^{1-\beta}[W_n - W]}{F} \right\|_0^{T_n} \to^P 0.$$

From (6.15) and from Remark 2.2 of Gill (1983), it follows that $F^{1-\beta}\, B(C(t)) \to 0$ a.s. as $t \to \tau_H$. Therefore

(6.18) $$\left\| \frac{W}{F^{\beta}} \right\|_0^{\tau_H} < \infty \text{ a.s..}$$

and since $K \le F$,

(6.19) $$K^{1-\beta}\, B(C(t)) \to 0 \text{ a.s. as } t \to \tau_H.$$

From (6.12), (6.17) and (6.19) one gets

(6.20) $$\left\| \frac{K^{1-\beta}[W_n - W]}{F} \right\|_0^{\tau_H} \to^P 0.$$

Use Lemma 6.3 *a* to get

$$\left\| \frac{K_n^{1-\beta}[W_n - W]}{F} \right\|_0^{T_n} \to^P 0$$

and use the same idea as above to conclude

(6.21) $$\left\| \frac{K_n^{1-\beta}[W_n - W]}{F} \right\|_0^{\tau_H} \to^P 0.$$

So it is enough to show that

$$\left\| \frac{[K_n^{1-\beta} - K^{1-\beta}]W}{F} \right\|_0^{\tau_H} \to^P 0.$$

From Lemma 6.2 *a*, $\left\| \frac{[K_n^{1-\beta} - K^{1-\beta}]}{F^{1-\beta}} \right\|_0^{\tau_H} \to^P 0$. Now the result follows from (6.18). □

<u>**LEMMA 6.5 *b*.**</u> Assume (6.13) and (6.14) hold. Then for the β and the special construction in Lemma 6.5 *a*,

$$\left\| \frac{K_n^{*1-\beta}\, W_n}{F} - \frac{K^{1-\beta}\, W}{F} \right\|_0^{\tau_H} \to^P 0.$$

<u>**PROOF.**</u> Follows exactly as in Lemma 6.5 *a* by using *b* type lemmas instead of *a* type whenever applicable. □

<u>**REMARK 6.2.**</u> Note that if G is continuous, we need to assume only one of (6.13) and (6.14), since they are equivalent in that case by the proof of Theorem 5.1. The remark is applicable to Lemma 6.5 *a* also.

<u>**LEMMA 6.6.**</u> Fix $A > 0$ and let $\{X(t)\colon t \in [0, A]\}$ be a stochastic process such that $X(t) \to 0$ a.s. as $t \to 0$ and $\|X\|_0^A < \infty$ a.s.. Let $\{Y_n(t)\colon t \in [0, A]\}$ be a sequence of stochastic processes such that $\|Y_n\|_0^A \le D$ for a r.v. D and $\|Y_n\|_\epsilon^A \to 0$ a.s. for all $\epsilon > 0$. Then $\|X\, Y_n\|_0^A \to 0$ a.s.. If the convergence of $\|Y_n\|_\epsilon^A$ holds in probability, so does the convergence in the conclusion.

<u>**PROOF.**</u> First, note that the probability 1 set where $\|Y_n\|_\epsilon^A \to 0$ can be chosen free of ϵ by taking countable intersection of probability 1 sets. So $\exists\ \Omega_0$, $P(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$, $X(t,\omega) \to 0$ as $t \to 0$ and

$\|Y_n(\omega)\|_\epsilon^A \longrightarrow 0$ for every $\epsilon > 0$. Now the statements in the following paragraph hold on Ω_0.

Let $\delta \leq 1$. $\exists\ \epsilon > 0$ such that for all $t \leq \epsilon$, $|X(t)| \leq \delta$.

$$\begin{aligned} \|XY_n\|_0^A &\leq \|XY_n\|_0^\epsilon + \|XY_n\|_\epsilon^A \\ &\leq \delta\, D + \|X\|_0^A\ \|Y_n\|_\epsilon^A. \end{aligned}$$

Taking limsup as $n \longrightarrow \infty$, $\limsup\limits_{n \longrightarrow \infty} \|XY_n\|_0^A \leq \delta\, D$. Since δ is arbitrary, it follows that $\|XY_n\|_0^A \longrightarrow 0$.

Thus we have proved that on Ω_0, $\|XY_n\|_0^A \longrightarrow 0$. Hence the first part. For the second part, take any subsequence $\{n_k\}$ of n. By a diagonal argument, $\exists$ a further subsequence $\{n_{k_j}\}$ of $\{n_k\}$ through which $\|Y_n\|_\epsilon^A \longrightarrow 0$ a.s. simultaneously for all $\epsilon > 0$. Apply the first part of the theorem to this subsequence and this will prove the second assertion. □

Now we shall prove a variation of Helly–Bray Lemma.

<u>**Lemma 6.7.**</u> Let $a, b \in [0, \infty]$ such that $a \leq b$ and let f_n be a sequence of functions on $[a, b]$ such that $\|f_n - f\|_a^b \longrightarrow 0$ and f is continuous and bounded on $[a, b]$. Let F_n be a sequence of monotonic functions on $[a, b]$ such that $0 \leq F_n \leq 1$, F_n converges pointwise to F on $[a, b]$ and F is continuous on $[a, b]$. Then

$$\int_{(a,b]} f_n\, dF_n \longrightarrow \int_{(a,b]} f\, dF.$$

<u>**Proof.**</u> By integration by parts formula,

$$\int_{(a,b]} f\, dF_n = f(b)\, F_n(b) - f(a)\, F_n(a) - \int_{(a,b]} F_n\, df$$

because f is continuous. Also

$$\int_{(a,b]} f\, dF = f(b)\, F(b) - f(a)\, F(a) - \int_{(a,b]} F\, df.$$

Since by Polya's Theorem $\|F_n-F\|_a^b \to 0$,

$$F_n(b) \to F(b),\ F_n(a) \to F(a) \quad \text{and} \quad \int_{(a,b]} F_n\, df \to \int_{(a,b]} F\, df.$$

So

$$\int_{(a,b]} f\, dF_n \to \int_{(a,b]} f\, dF \ .$$

Let $\epsilon > 0$. $\exists\ N_1$ such that for all $n \geq N_1$, $\|f_n-f\|_a^b \leq \epsilon/2$. $\exists\ N_2$ such that for all $n \geq N_2$,

$$\int_{(a,b]} f\, dF_n \to \int_{(a,b]} f\, dF \leq \epsilon/2.$$

Let $N = N_1 \vee N_2$. Now for all $n \geq N$,

$$\begin{aligned}\left|\int_{(a,b]} f_n\, dF_n - \int_{(a,b]} f\, dF\right| &\leq \left|\int_{(a,b]} f_n\, dF_n - \int_{(a,b]} f\, dF_n\right| \\ &\quad + \left|\int_{(a,b]} f\, dF_n - \int_{(a,b]} f\, dF\right| \\ &\leq |F_n(b) - F_n(a)|\ (\epsilon/2) + \epsilon/2 \\ &\leq \epsilon. \qquad \square\end{aligned}$$

Now we shall state and prove a theorem that will provide an A.D.F. test for H_0: $F = F_0$ vs. H_1: $F \neq F_0$

<u>THEOREM 6.1 a.</u> Assume that i(i) – 1(vi), (6.11) and (6.12) hold. Then

$$\int_0^{\tau_H} [K_n/K_n]\ [W_n/F]^2\, dK_n \Rightarrow \int_0^1 B_0^2(t)/[t(1-t)]\, dt.$$

<u>PROOF.</u> It suffices to show that

(1) $$\int_0^{\tau} [K_n/K_n]\cdot[W_n/F]^2\, dK_n \Rightarrow \int_0^{\tau} B_0^2(K)/[K(1-K)]\, dK$$

and

(2) $$\int_{\tau}^{\tau_H} [K_n/K_n]\cdot[W_n/F]^2\, dK_n \Rightarrow \int_{\tau}^{\tau_H} B_0^2(K)/[K(1-K)]\, dK$$

for some $\tau \in (0, \tau_H)$ because then the result follows from (1) and (2) together with a simple change of variable.

Proof of (1): Apply (7.7.9) of Shorack and Wellner (1986) with $q(t) = (1-t)^{\beta}$ to get that for every $\beta \in [0, 1/2)$, $\exists$ a special construction of X_i's, Y_i's and W such that

$$\|[W_n^2 - W^2]/K^{2\beta}\|_0^{\tau} \rightarrow^{P} 0$$

because $0 \le \frac{K}{F}(\tau) \le \frac{K}{F}(t) \le 1$ for $t \in [0, \tau]$.

$$\begin{aligned}
\frac{K}{K_n}(t) &= \left[\frac{1+C_n}{1+C}(t)\right] \frac{C}{C_n}(t) \\
&\le [1+C_n(\tau)]\left[\int_0^t \frac{dF}{F^2\bar{G}_-}\right] \Big/ \left[\int_0^t \frac{dF}{F^2\bar{G}_{n-}}\right] \\
&\le [1+C_n(\tau)]\ [F^2(\tau)\bar{G}_-(\tau)]^{-1}
\end{aligned}$$

and $C_n(\tau) \rightarrow C(\tau)$ a.s. Therefore for all $t \in [0, \tau]$, $\frac{K}{K_n}(t) \le B_1$ for some r.v. B_1 free of n and t. Thus $\|[W_n^2 - W^2]/K_n^{2\beta}\|_0^{\tau} \rightarrow^{P} 0$. Now we shall show that

$$\|[W^2[K_n^{2\beta} - K^{2\beta}]\|_0^{\tau} \rightarrow^{P} 0. \tag{6.22}$$

As

$$\|[W^2[K_n^{2\beta} - K^{2\beta}]\|_0^{\tau} = \frac{W^2}{K^{2\beta}}\left[\frac{K^{2\beta}}{K_n^{2\beta}} - 1\right],$$

we can invoke Lemma 6.6 with $X = \frac{W^2}{K^{2\beta}}$ and $Y_n = (K^{2\beta}/K_n^{2\beta}) - 1$. We will now verify the conditions of Lemma 6.4. As

$$W = \bar{F}\ B(C) = \bar{F}\ B(K/(1-K)),$$

an application of the Law of Iterated Logarithm for Brownian motion (see Theorem 12.29 of Breiman (1968)) will show that X satisfies the conditions of Lemma 6.6. Since $K_n^{2\beta} \rightarrow K^{2\beta}$ uniformly a.s. on $[0, \tau]$ by (7.7.21) of Shorack and Wellner (1986) and $K^{2\beta}$ is bounded away from zero on $[\epsilon, \tau]$,

$$\left\| \frac{K^{2\beta}}{K_n^{2\beta}} - 1 \right\|_{\epsilon}^{\tau} \longrightarrow 0 \quad \text{a.s..}$$

As $\frac{K}{K_n}(t) \le B_1$ for all $t \in [0, \tau]$, $\left\| \frac{K^{2\beta}}{K_n^{2\beta}} - 1 \right\|_0^{\epsilon} \le B_1^{2\beta} + 1$. Hence (6.22).

Since F is bounded away from zero on $[0, \tau]$, it follows that

$$\left\| K_n \left[\frac{W_n}{F K_n^{\beta}} \right]^2 - K \left[\frac{W}{F K^{\beta}} \right]^2 \right\|_0^{\tau} \longrightarrow^{P} 0.$$

Invoke Lemma 6.7 with $f_n = \frac{K_n}{2\beta} \left[\frac{W_n}{F K_n^{\beta}} \right]^2$, $f = \frac{K}{2\beta} \left[\frac{W}{F K^{\beta}} \right]^2$, $F_n = K_n^{2\beta}$ and $F = K^{2\beta}$. This gives us

$$\int_0^{\tau} \frac{K_n}{2\beta} \left[\frac{W_n}{F K_n^{\beta}} \right]^2 dK_n^{2\beta} \longrightarrow^{P} \int_0^{\tau} \frac{K}{2\beta} \left[\frac{W}{F K^{\beta}} \right]^2 dK^{2\beta}.$$

That is,

$$\int_0^{\tau} [K_n/K_n]\, [W_n/F]^2\, dK_n \;\Rightarrow\; \int_0^{\tau} B_0^2(K)/[K(1-K)]\, dK.$$

Proof of (2): From Lemma 6.3 and the fact that K is bounded away from zero, it follows that

$$\left\| \frac{[K_n^{1-\beta}\, W_n]^2}{F^2\, K_n} - \frac{[K^{1-\beta}\, W]^2}{F^2\, K} \right\|_0^{\tau_H} \longrightarrow^{P} 0.$$

Apply Lemma 6.7. again with $f_n = \frac{[K_n^{1-\beta}\, W_n]^2}{2\beta\, F^2\, K_n}$ and $f = \frac{[K^{1-\beta}\, W]^2}{2\beta\, F^2\, K}$, $F_n = 1 - K_n^{2\beta}$, and $F = 1 - K^{2\beta}$. By (6.19), we are assured that f satisfies the required conditions. Now the result follows. □

THEOREM 6.1 *b*. Assume 1(i) – 1(vi), (6.11) and (6.12) hold. Then

$$\int_0^{\tau_H} [K_n^*/K_n^*]\,[W_n/F_n]^2\, dK_n^* \;\Rightarrow\; \int_0^1 B_0^2(t)/[t(1-t)]\, dt.$$

PROOF. Follows exactly as in Theorem 6.1 *a*. □

Now we shall prove the weak convergence of ξ_n, $\tilde{\xi}_n$ and ξ_n^* in the L^2 – space. See 1(iii) for notation.

THEOREM 6.2. Assume 1(i) – 1(vi) hold. If (5.2) or (5.3) holds, then on $L^2([0, \tau_H], \mu)$,

(i) $\xi_n \Rightarrow B_0(K)$,

(ii) $\tilde{\xi}_n \Rightarrow B_0(K)$,

(iii) $\xi_n^* \Rightarrow B_0(K)$,

where B_0 is a Brownian Bridge.

PROOF. First, note that $(\bar{K}/\bar{F})\, W$ has the same distribution as $B_0(K)$. Now, by Lemma 6.1 *a*, $\xi_n \in L^2([0, \tau_H], \mu)$. By Theorem 5.1, there is a construction of W_n and W such that $W_n - W \to 0$ a.s. in $L^2([0, \tau_H], \mu)$. So by Lemma 6.1 *a*,

$$\frac{K_n}{F}\,(W_n - W) \;\to\; 0 \quad \text{a.s.} \tag{6.23}$$

in $L^2([0, \tau_H], \mu)$. Moreover, by Lemma 6.2 *a*,

$$\left[\frac{K_n}{F} - \frac{K}{F}\right] W \;\to^{P}\; 0. \tag{6.24}$$

in $L^2([0, \tau_H], \mu)$. Now (i) follows from (6.23) and (6.24).

Proof of (ii) and (iii) are similar. □

Denote the limiting r.v. in Theorems 6.1 *a* and *b* by X. For a representation of X as a mixture of independent chi-square r.v.'s, see Anderson and Darling (1952). Also given in the paper mentioned above is the characteristic function of X. A more direct computation of the mean and the variance of X is now discussed. The mean is easily seen to be 1 by taking the expectation inside the integration. Calculation of the variance is done in the following lemma.

<u>LEMMA 6.8.</u> X has mean 1 and variance $(2\pi^2/3) - 6$.

<u>PROOF.</u> As all the quantities involved are non-negative, Fubini's Theorem justifies interchange of expectation and integration and thus the mean of X is 1. Let us denote $B_0^2(t)/t(1-t)$ by $B^*(t)$ and $E(X^2)$ by Q so that $V(X) = Q - 1$.

$$\begin{aligned} Q &= E\left[\int_0^1 B^*(s)\,ds \int_0^1 B^*(t)\,dt\right] \\ &= E\left[\int_0^1 \int_0^1 B^*(s)\,B^*(t)\,ds\,dt\right] \\ &= 2\int_0^1 \int_0^t E[B^*(s)\,B^*(t)]\,ds\,dt \end{aligned}$$

If (X,Y) has bivariate normal distribution with mean 0, variance 1 and correlation ρ, then $E(XY) = 1 + 2\rho^2$. As $(B_0(s), B_0(t))$, for $s \le t$, is bivariate normal with dispersion matrix Σ given by

$$\Sigma = \begin{bmatrix} s(1-s) & s(1-t) \\ s(1-t) & t(1-t) \end{bmatrix},$$

it follows that

$$Q = 2\int_0^1 \int_0^t \{1 + [2s(1-t)/(1-s)t]\}\,ds\,dt.$$

The above expression simplifies to $4\int_0^1 (-\log x)\,(x/(1-x))\,dx - 1$.

Integrating by parts and changing variables we get

$$\int_0^1 (-\log x)\ (x/(1-x))\ dx \ = \ 1 + \int_0^\infty y\ \exp(-y)\ \log(1-\exp(-y))\ dy$$

Using the Taylor expansion for $\log x$, integrating term by term, and making use of the fact that $\sum_1^\infty n^{-2} = \pi^2/6$, we will get that the expression above is equal to $\pi^2/6 - 1$. Thus $Q = 4(\pi^2/6 - 1) - 1 = (2\pi^2/3) - 5$ and hence $V(X) = (2\pi^2/3) - 6$. □

REFERENCES.

Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain goodness of fit criteria based on stochastic processes. *Annals of Statistics* **23** 193–212

Bhattacharya, P. K. (1974). Sums of order statistics. *Annals of Statistics* **2** 1034–1039

Billingsley, P. (1968). *Convergence of probability measures.* John Wiley and Sons, New York

Breiman, L. (1968). *Probability.* Addison Wesley.

David, H. A. (1970). *Order Statistics.* John Wiley and Sons, New York.

Gardiner, J. C., Susarla, V. and Van Ryzin, J. (1985). Estimation of the median survival time under random censorship. *Adaptive Statistical Procedures and Related Topics.* IMS Lecture Notes **8** 350–364

Gill, R. D. (1980). Censoring and stochastic integrals. *Mathematical Centre Tracts* **124**, Amsterdam.

Gill, R. D. (1983). Large sample behavior of the PLE on the whole line. *Annals of Statistics* **11** 49–58

Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete samples. *Journal of American Statistical Association* **53**, 457–481.

Koul, H. L., Susarla, V., and Van Ryzin, J. (1981). Regression analysis with randomly right–censored data. *Annals of Statistics* **9** 1276–1288.

Koul, H. L. (1984). Tests of goodness–of–fit in linear regression. *Colloquia Mathematica societatis Janos Bolyai* **45**.

Koziol, J. A. and Green, S. B. (1976). A Cramèr–von Mises statistic for randomly censored data. *Biometrika* **63** 465–474

Lo, S. – H. (1986). The Product Limit Estimator and the Bootstrap: Some Asymptotic Representations. *Probability Theory and Related Fields* 71, 455–465.

Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces*. Academic Press, New York.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*. John Wiley and Sons, New York

Wang, J. G. (1987). A note on the uniform consistency of the Kaplan – Meier estimator. *Annals of Statistics* 15 1313–1316.

Yang, S. (1988). *Minimum Hellinger distance Estimation of parameters in Random Censoring Models*. Ph. D. Thesis: Michigan State University.