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ABSTRACT

HOMOTOPY METHODS FOR SOLVING DEFICIENT POLYNOMIAL SYSTEMS

By

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By a *deficient* polynomial system of n polynomial equation in n unknowns we mean a system that has fewer solutions than that predicted by the Bézout number of the corresponding homogeneous system. Sometimes if the system is m-homogenized, the Bézout number can be considerably reduced. In this paper, we introduce homotopies for numerically determining all isolated solutions of deficient m-homogeneous systems. The homotopy $H(x,t)$ is chosen such that $H(x,0) = Q$ is a trivial system and $H(x,1) = P$ is the system to be solved and such that the *subschemes* or the *number* of solutions of $H(x,t)$ at infinity remains invariant when t varies in $[0,1)$. Thus the number of paths to be followed is reduced.

To my wife Yingni.

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Introduction

1.1 Linear Homotopies

Let $P(x) = 0$ be a system of n polynomial equations in n unknowns, where $P = (p_1, \ldots, p_n)$ and $x = (x_1, \ldots, x_n)$. We want to find all isolated solutions of

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ p_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{1.1}$$

in $\mathbf{C}^n$. The homotopy continuation method for solving this system is to find a system

$$H(x,t) = 0 \tag{1.2}$$

such that the solutions of $H(x,0) = 0$ are known, $H(x,1) = P(x)$ and then to follow the curves in the real variable t which make up the solution set of

$$H(x,t) = 0. \tag{1.3}$$

More precisely, we choose $H(x,t)$ such that the following three assumptions hold:

1. (*Triviality*) The solutions of $H(x,0) = 0$ are known.

2. (*Smoothness*) The solution set of $H(x,t) = 0$ for $0 \le t < 1$ consists of a finite number of smooth paths, each parameterized by t in $[0,1)$.

3. (*Accessibility*) Every isolated solution of $H(x,1) = P(x) = 0$ is reached by some path originating at $t = 0$. It follows that this path starts at a solution of $H(x,0) = Q(x) = 0$.

When the three assumptions hold, we try to follow the solution paths from the initial points (known because of property 1) at $t = 0$ to all solutions of the original problem $P(x) = 0$ at $t = 1$.

It is important to realize that even though Properties (1) – (3) imply that each solution of $P(x) = 0$ will lie at the end of a solution path, it is also consistent with these properties that some of the paths may diverge to infinity as the parameter t approaches 1.

This approach has the virtue of locating all isolated solutions of the system $P(x) = 0$. A typical choice of H that satisfies the three properties [9,16,21] is

$$H(x,t) = (1-t)cQ + tP, \tag{1.4}$$

where

$$q_1(x) = (x_1^{d_1} - 1)$$
$$\vdots$$
$$q_n(x) = (x_n^{d_n} - 1)$$

where $d_1, \ldots, d_n$ are the degrees of $p_1(x), \ldots, \ldots, p_n(x)$, and c is a random complex number . In this case, the number of paths which need to be followed to arrive at all solutions of $P(x) = 0$ is the product $d \equiv d_1, \cdots, d_n$. This number, often called the Bézout number of the corresponding homogeneous system, is a classical upper bound on the number of isolated solutions, counting multiplicities. However, in most practical cases that we have seen, the number of solutions of (1.1) can turn out to be smaller than d, and in some cases only a small fraction of d. Such systems are called *deficient.* When applying the homotopy continuation method to a deficient system, by sending out d paths in search of solutions, the paths which do not converge to solutions of (1.1) will go to infinity, representing wasted computation.

For deficient systems, various homotopies have been introduced ([10,11,12, 17,19]). Sometimes we can m-homogenize (1.1) , to get a smaller Bézout number ([17]) and hence if we use a homotopy with same m-homogeneous structure as P we can reduce number of paths needed to be followed. Given a polynomial p of degree d in the n variables $x_1, \dots, x_n$, we can define its homogenization

$$\tilde{p}(x_0, \dots, x_n) = (x_0)^d \, p(x_1/x_0, \dots, x_n/x_0).$$

For polynomial system $P = (p_1, \dots, p_n)$ we use $\tilde{P}$ to represent $(\tilde{p}_1, \dots, \tilde{p}_n)$. A typical suggestion in [10], [11] and [12] for deficient polynomial system is to choose $Q(x)$ in (1.4) which shares a similar type of deficiency as $P(x)$, with the basic assumption that *the zeros of* $Q(x)$ *at infinity*, i.e. zeros of $\tilde{Q}(x)$ with $x_0 = 0$, *are nonsingular.* Then, we need follow fewer solution paths to obtain all isolated solutions of P. However a flaw was discovered in one of main theorems of [19]. Basically they claimed a general result that the nonsingularity of the zeros of $Q(x)$ at infinity can be replaced by the following. Let the common zeros of P and Q in (1.4) at infinity be denoted by S. If for each $s \in S$ the multiplicity of s as a solution of $\tilde{Q}(x) = 0$ is less than or equal to that of s as a solution of $\tilde{P}(x) = 0$ and all other zeros of $\tilde{Q}(x)$ are isolated and nonsingular, then for 'almost all' $a \in \mathbf{C}$, by following the solution paths of

$$\tilde{H}(x,t) = a(1-t)\tilde{Q}(x) + t\,\tilde{P}(x) = 0 \tag{1.5}$$

starting from the isolated zeros of $\tilde{Q}(x)$ outside S, one can obtain all isolated zeros of $\tilde{P}(x) = 0$ outside S. This assertion can be shown to be in error as the following example indicates.

Example: Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be defined as

$$\begin{aligned} p_1(x_1, x_2) &= x_2^2 + x_1 \\ p_2(x_1, x_2) &= x_2^2 + x_2 \end{aligned} \tag{1.6}$$

$$\begin{aligned} q_1(x_1, x_2) &= x_2^2 - 1 \\ q_2(x_1, x_2) &= x_2^2 + x_1 x_2. \end{aligned} \tag{1.7}$$

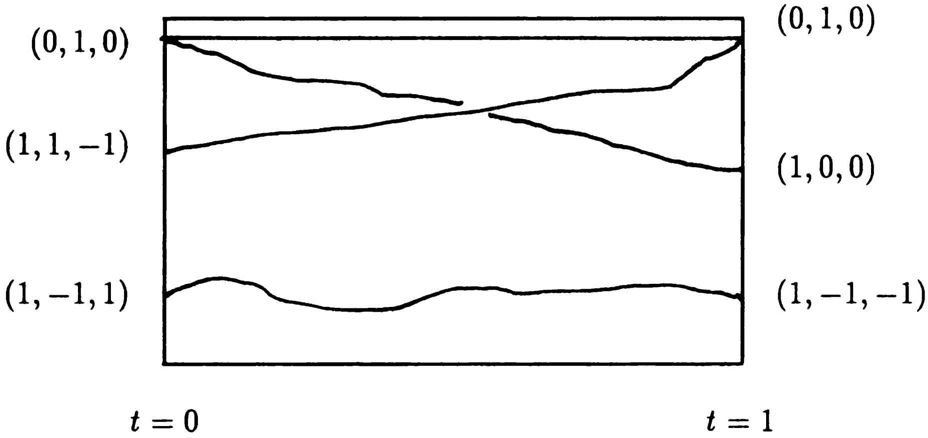

Figure 1.1: The four solution paths.

The nontrivial common solution set of $\tilde{P}(x_0, x_1, x_2) = 0$ and $\tilde{Q}(x_0, x_1, x_2) = 0$ at infinity is $(0,1,0)$ with multiplicity 2. However, for any nonzero $a \in \mathbf{C}$ which is not a negative real number, by following the solution paths of (1.4) starting from the 2 zeros of $\tilde{Q}$ in affine space $(1,-1,1)$ and $(1,1,-1)$, one can only find one of the isolated zeros of $\tilde{P}(x), (1,-1,-1)$, in affine space. For $a = .59032965 + .15799344\, i$ the computation results are shown in Figure 1.1. The solution path starting with $(1,-1,1)$ can reach $(1,-1,-1)$ and the solution path starting from $(1,1,-1)$ goes to infinity as t tends to 1. A proof of this assertion for almost all a is given in an appendix.

In view of this counterexample, we suggest an alternative which guarantees the accessibility of the homotopy for deficient polynomial systems. In our homotopy, we choose $\tilde{Q}(x)$ in such a way that its *subscheme* at infinity contains the subscheme of $\tilde{P}(x)$ at infinity. Then, for 'almost all' $a \in \mathbf{C}$, the subschemes of $\tilde{H}(x,t)$ in (1.4) at infinity remain the same for all $t \in [0,1)$. Consequently, solution paths of (1.4) originated at zeros of $\tilde{Q}(x)$ in affine space stay in affine space for all $t \in [0,1)$. As a result, the typical assumption of nonsingularity of $Q(x)$ at infinity in [10,11,12] can be dropped.

The main results are stated and proved in Chapter 2 for general m-homogeneous deficient polynomial systems . When $m = 1$ the conditions given in Theorem 2.1 and Proposition 2.1 are equivalent to the condition (2.4) of Theorem 2.3 in [12](See [14] for the proof). However, the condition here is much easier to be verified. In chapter 2, we also give several examples for which our main results apply.

1.2 Nonlinear Homotopies

Many polynomial systems in applications are a family of systems parameterized by $q \in C^s$, and for generic family members the number of solutions at infinity are the same. To deal with this kind of deficient systems, various homotopies have been introduced ([13,18]). But each of those homotopies can be used to solve only a small group of systems in that family. In Chapter 3, we suggest an alternative which can be used to solve the whole family of systems. We also show how the result can be used to solve a very important problem arising from the robot arm design.

1.3 Preliminaries

The complex n-space $\mathbf{C}^n$ can be naturally embedded in complex projective space

$$\mathbf{P}^n = \{(x_0, \dots, x_n) \in \mathbf{C}^{n+1} \backslash (0, \dots, 0)\} / \sim$$

where the equivalent relation "$\sim$" is given by $x \sim y$ if $x = cy$ for nonzero $c \in \mathbf{C}$. Similarly, the space $N = \mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_m}$ can be naturally embedded in $M = \mathbf{P}^{k_1} \times \cdots \times \mathbf{P}^{k_m}$. A point $(y_1, \dots, y_m)$ in N with $y_i = (y_1^i, \dots, y_{k_i}^i), i = 1, \dots, m$ corresponds to a point $(z_1, \dots, z_m)$ in M with $z_i = (z_0^i, \dots, z_{k_i}^i)$ and $z_0^i = 1, i = 1, \dots, m$. The set of such points in M is usually called the *affine space* in this setting. The points in M with at least one $z_0^i = 0$ are called the *points at infinity.*

Given a polynomial p in the n variables $x_1, \dots, x_n$, if we partition the variables into m groups $y_1 = (x_1^1, \dots, x_{k_1}^1), y_2 = (x_1^2, \dots, x_{k_2}^2), \dots, y_m = (x_1^m, \dots, x_{k_m}^m)$ with $k_1 + \cdots + k_m = n$ and let d_i be the degree of p with respect to y_i(more precisely, to the variables in y_i), then we can define its m-homogenization as

$$\tilde{p}(z_1, \dots, z_m) = (z_0^1)^{d_1} \times \cdots \times (z_0^m)^{d_m} p(y_1/z_0^1, \dots, y_m/z_0^m)$$

which is homogeneous with respect to each $z_i = (z_0^i, \dots, z_{k_i}^i), i = 1, \dots, m$. Here $z_j^i = x_j^i$, for $j \neq 0$. Such a polynomial is said to be *m-homogeneous.* To illustrate this definition, let us consider the polynomial

$$p(\lambda, x_1, \dots, x_n) = \lambda^2(a_1x_1 + \cdots + a_nx_n - a)$$
$$+\lambda(b_1x_1 + \cdots + b_nx_n - b) + (c_1x_1 + \cdots + c_nx_n - c).$$

We may let $y_1 = (\lambda), y_2 = (x_1, \dots, x_n)$ and $z_1 = (\lambda_0, \lambda), z_2 = (x_0, x_1, \dots, x_n)$. The degree of p is 2 with respect to y_1 and is 1 with respect to y_2. Hence its 2-homogenization is

$$\tilde{p}(\lambda_0, \lambda, x_0, x_1, \dots, x_n) = \lambda^2(a_1x_1 + \cdots + a_nx_n - ax_0)$$
$$+\lambda\lambda_0(b_1x_1 + \cdots + b_nx_n - bx_0) + \lambda_0^2(c_1x_1 + \cdots + c_nx_n - cx_0),$$

which is homogeneous with respect to (λ_0, λ) and $(x_0, x_1, \dots, x_n)$. For m-homogeneous polynomial systems $P(z) = 0$, we are interested in the solutions in $\mathbf{P}^{k_1} \times \dots \times \mathbf{P}^{k_m}$. So often, by abuse of terminology, we say $z \in \mathbf{P}^{k_1} \times \dots \times \mathbf{P}^{k_m}$. For $z_i = (z_0^i, \dots, z_{k_i}^i)$ $, i = 1, \dots, m$, let $S = \mathbf{C}[z_0^1, z_1^1, \dots, z_{k_m}^m]$ be the polynomial ring of the variables in z_i's with complex coefficients. If A is an ideal generate by m-homogeneous polynomials and T is a prime ideal of S, denote by A^T the ideal $\{f \in S \mid hf \in A \text{ for some m-homogeneous } h \notin T\}$. For a point $z = (z_1, \dots, z_m) \in \mathbf{P}^{k_1} \times \cdots \times \mathbf{P}^{k_m}$, let I_z denote the maximal ideal $\{f \in S \mid f(z) = 0\}$. If $f_1, \dots, f_n$ are m-homogeneous polynomials in the variables $(z_1, \dots, z_m)$, let $V(f_1, \dots, f_r)$ be the common zero set of $f_1, \dots, f_r$ in $\mathbf{P}^{k_1} \times \cdots \times \mathbf{P}^{k_m}$. We say a point $y \in V(f_1, \dots, f_r)$ is *nonsingular* if

$$rank\, \frac{\partial(f_1, \dots, f_r)}{\partial(z_1, \dots, z_m)} = codim_y(V(f_1, \dots, f_r), P^{k_1} \times \cdots \times P^{k_m})$$

where codim denotes complex codimension. We use $\langle f_1, \dots, f_r \rangle$ to represent the ideal generated by $f_1, \dots, f_r$. To be more precise, $\langle f_1, \dots, f_r \rangle$ is the set of all polynomials of form

$$\sum_{i=1}^{r} g_i f_i$$

where the g_i's are polynomials in S.

For a 2-homogeneous polynomial system $\tilde{P} = (\tilde{p}_1, \ldots, \tilde{p}_n)$ in the variables $z_i = (z_0^i, \ldots, z_{k_i}^i)$ $i = 1, 2$ with $k_1 = k$, $k_2 = \ell$, and $k + \ell = n$ and deg $\tilde{p}_i = (d_1^i, d_2^i)$, $i = 1, \cdots, n$ the *Bézout number* B of this system is ([6, p. 146]) the coefficient of $\alpha^k \beta^\ell$ in the product

$$\prod_{i=1}^{n} (d_1^i \alpha + d_2^i \beta). \tag{1.8}$$

That means the number of isolated solutions of the system in $\mathbf{P}^k \times \mathbf{P}^\ell$, counting multiplicities, is at most B. For $m > 2$ we have similar formula(See[17]).

Chapter 2

Linear Homotopies

2.1 Main Results

Given the system $P(x) = (p_1(x), \ldots, p_n(x))$ in (1.1) let $Q(x) = (q_1(x), \ldots, q_n(x))$ and

$$H(a, x, t) = (1-t)aQ(x) + t\,P(x), \quad a \in \mathbf{C}. \tag{2.1}$$

Here, we consider $x \in \mathbf{C}^{k_1} \times \mathbf{C}^{k_2} \times \cdots \times \mathbf{C}^{k_m}$ with $k_1 + k_2 + \cdots + k_m = n$, t real , $H : \mathbf{C}^{k_1} \times \mathbf{C}^{k_2} \times \cdots \times \mathbf{C}^{k_m} \longrightarrow \mathbf{C}^n$ and $\deg p_i = \deg q_i, i = 1, \ldots, n$, here by degree we mean the m-degree. Let

$$\tilde{H}(a, z, t) = (1-t)a\tilde{Q}(z) + t\,\tilde{P}(z) \;\; t \in [0,1], \tag{2.2}$$

which is the m-homogenization of (2.1). Let $\langle \tilde{Q} \rangle = \langle \tilde{q}_1, \ldots, \tilde{q}_n \rangle$ and $\langle \tilde{P} \rangle = \langle \tilde{p}_1, \ldots, \tilde{p}_n \rangle$ and $M = P^{k_1} \times \cdots \times P^{k_m}$. The main result is the following.

Theorem 2.1 *Suppose that the polynomial system $\tilde{Q}$ in (2.2) satisfies the following assumptions:*

(1) for every point $z \in M$ at infinity,

$$\langle \tilde{Q} \rangle^{I_z} \supseteq \langle \tilde{P} \rangle^{I_z};$$

(2) the set $T = \{$the points of $V(\tilde{q}_1, \ldots, \tilde{q}_n) \subset M$ in affine space$\}$ consists of non-singular isolated points $x^1, \ldots, x^r$.

Then, there exists an open dense subset D of $\mathbf{C}$ with full measure, such that for a^{-1} chosen from D, we have

a. *(Smoothness) For each isolated zero $x^k \in T, k = 1, \ldots, r$ there is a function $x^k(t) : [0,1] \to M$ which is analytic and contained in affine space for all t in $[0,1)$ and satisfies $\tilde{H}(a, x^k(t), t) = 0$.*

b. *(Accessibility) Each isolated solution of $P(x) = 0$ in the affine space is reached by $x^k(t)$ for some k at $t = 1$.*

Remark 2.1 *If $z \notin V(\tilde{p}_1, \ldots, \tilde{p}_n)$ then there exists $h \in \langle \tilde{p}_1, \ldots, \tilde{p}_n \rangle$ such that $h(z) \neq 0$, i.e. $h \notin I_z$. Thus, $\langle \tilde{P} \rangle^{I_z} = \{f \in S \mid fh \in \langle \tilde{P} \rangle$ for some $h \notin I_z\} = S$. Hence, condition (1) above implies that every point of $V(\tilde{q}_1, \ldots, \tilde{q}_n)$ at infinity is also a point of $V(\tilde{p}_1, \ldots, \tilde{p}_n)$. By the same argument, if $z \notin V(\tilde{q}_1, \ldots, \tilde{q}_n)$ then $\langle \tilde{Q} \rangle^{I_z} = S$ and (1) is obvious. So, in order to check the condition (1) of Theorem 2.1 one only needs to check this condition for those points at infinity which lie in $V(\tilde{q}_1, \ldots, \tilde{q}_n)$. Consequently, the condition (1) implies that the subscheme at infinity of the polynomial system $\tilde{Q}(z)$ contains the subscheme of $\tilde{P}(z)$ at infinity. (For general definitions and properties of scheme and subscheme, see [7, pp. 60-190].)*

Remark 2.2 *By a straightforward verification one can easily see that $\langle \langle \tilde{Q} \rangle^{I_z} \rangle^{I_z} = \langle \tilde{Q} \rangle^{I_z}$. Hence, if $\langle \tilde{Q} \rangle^{I_z} \supseteq \langle \tilde{P} \rangle$, then $\langle \tilde{Q} \rangle^{I_z} \supseteq \langle \tilde{P} \rangle^{I_z}$.*

Remark 2.3 *In the counterexample (1.5), (1.6) we give in Chapter 1,*

$$\begin{aligned}
\tilde{p}_1(x_0, x_1, x_2) &= x_2^2 + x_1 x_0 \\
\tilde{p}_2(x_0, x_1, x_2) &= x_2^2 + x_2 x_0 \\
\tilde{q}_1(x_0, x_1, x_2) &= x_2^2 + x_0^2 \\
\tilde{q}_2(x_0, x_1, x_2) &= x_2^2 + x_1 x_2.
\end{aligned}$$

At $z = (x_0, x_1, x_2) = (0,1,0) \in bf P^2, \langle \tilde{Q} \rangle^{I_z} \not\supseteq \langle \tilde{P} \rangle^{I_z}$. This can be shown as follows. Since $\tilde{p}_2 - \tilde{p}_1 = x_0(x_2 - x_1) \in \langle \tilde{P} \rangle^{I_z}$ and $x_2 - x_1 \neq 0$ at $z = (0,1,0), x_0 \in \langle \tilde{P} \rangle^{I_z}$. Thus $x_2^2 \in \langle \tilde{P} \rangle^{I_z}$.So $\langle \tilde{P} \rangle^{I_z} = \langle x_0, x_2^2 \rangle$. Since $(\tilde{p}_1 - \tilde{p}_2)x_2 - \tilde{p}_1 x_1 = x_0^2(x_1 + x_2) \in \langle \tilde{Q} \rangle^{I_z}$ and $x_1 + x_2 \neq 0$ at $z, x_0^2 \in \langle \tilde{Q} \rangle^{I_z}$. Therefore $\langle \tilde{Q} \rangle^{I_z} = \langle x_2, x_0^2 \rangle$. Apparently, $\langle \tilde{Q} \rangle^{I_z} \not\supseteq \langle \tilde{P} \rangle^{I_z}$.

We need several lemmas for the proof of the theorem. Let R be a ring of polynomials (perhaps a quotient of S) and q a prime ideal of R. We denote by $R_{(q)}$ the *localization* of R at q. The local ring $R_{(q)}$ is made up of "formal fractions"

$$\{\frac{f}{g} \mid f \in R,\ g \notin q,\ \deg f = \deg g \text{ with respect to each } z_i,\ i = 1, \ldots, m\}$$

such that $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ if and only if $f_1 g_2 = f_2 g_1$ in R.

Lemma 2.1 *If*

$$\langle \tilde{Q} \rangle^{I_y} \supseteq \langle \tilde{P} \rangle^{I_y} \tag{2.3}$$

for any point y at infinity, then there exists a subset D_1 of $\mathbf{C}$

$$D_1 = \{re^{i\theta} \in C \mid \theta \in [0, 2\pi) \backslash F,\ F \text{ a finite set},\ r > 0\}$$

such that for any y at infinity and $c \in D_1$

$$\langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle^{I_y} = \langle \tilde{Q} \rangle^{I_y}. \tag{2.4}$$

Proof. From (2.3), for any y at infinity we have

$$a_i^y\, \tilde{p}_i = b_{i1}^y\, \tilde{q}_1 + \ldots + b_{in}^y\, \tilde{q}_n, \qquad i = 1, \ldots, n$$

where $a_i^y,\ b_{ij}^y \in S$ and $a_i^y(y) \neq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, n$. Thus,

$$\begin{aligned} a_1^y\, (\tilde{q}_1 + c\tilde{p}_1) &= (a_1^y + cb_{11}^y)\, \tilde{q}_1 + \ldots + cb_{1n}^y\, \tilde{q}_n \\ &\vdots \\ a_n^y\, (\tilde{q}_n + c\tilde{p}_n) &= cb_{n1}^y\, \tilde{q}_1 + \ldots + (a_n^y + cb_{nn}^y)\, \tilde{q}_n \end{aligned} \tag{2.5}$$

for any $c \in \mathbf{C}$. For $f \in \langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle^{I_y}$ there exists $h \in S$, such that $h(y) \neq 0$, and

$$fh = \sum_{i=1}^{n} d_i(\tilde{q}_i + c\tilde{p}_i), \tag{2.6}$$

where $d_i \in S,\ i = 1, \ldots, n$. Multiplying both sides of (2.6) by $a_1^y \times \ldots \times a_n^y$ and using (2.5), we have $f \in \langle \tilde{Q} \rangle^{I_y}$. Hence,

$$\langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle^{I_y} \subseteq \langle \tilde{Q} \rangle^{I_y}$$

for any $c \in \mathbf{C}$. For the reverse inclusion, let

$$A_y(c,z) = \begin{bmatrix} a_1^y(z) + cb_{11}^y(z) & \cdots & cb_{1n}^y(z) \\ cb_{21}^y(z) & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ cb_{n1}^y(z) & \cdots & a_n^y(z) + cb_{nn}^y(z) \end{bmatrix} \tag{2.7}$$

then (2.5) can be written as

$$\begin{bmatrix} a_1^y(\tilde{q}_1 + c\tilde{p}_1) \\ \vdots \\ a_n^y(\tilde{q}_n + c\tilde{p}_n) \end{bmatrix} = A_y(c,z) \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_n \end{bmatrix}. \tag{2.8}$$

Let $B_y(c,z)$ be the determinant of $A_y(c,z)$ and $\overline{A}_y(c,z)$ be the adjoint matrix of $A_y(c,z)$. Multiplying both sides of (2.8) by $\overline{A}_y(c,z)$ yields,

$$\overline{A}_y(c,z) \begin{bmatrix} a_1^y(\tilde{q}_1 + c\tilde{p}_1) \\ \vdots \\ a_n^y(\tilde{q}_n + c\tilde{p}_n) \end{bmatrix} = B_y(c,z) \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_n \end{bmatrix}. \tag{2.9}$$

Consider $B_y(c,z)$ as a polynomial in $\mathbf{C} \times M$. Denote its homogenization with respect to c in $\mathbf{P}^1 \times M$ by $\tilde{B}_y(c_0, c, z)$. Let B be the ideal generated by the $\tilde{B}_y$'s. Its zero set at infinity, denoted by v, is an algebraic set. Let $\pi_1 : \mathbf{P}^1 \times M \to \mathbf{P}^1$ be the natural projection. By the proper mapping theorem ([5, p.64]) $\pi_1(v)$ is an algebraic set in $\mathbf{P}^1$. The only algebraic subsets in $\mathbf{P}^1$ are the empty set, the finite-element subsets and $\mathbf{P}^1$ itself. Since $\tilde{B}_y(1,0,y) \neq 0$ for any y at infinity, $(1,0) \notin \pi_1(v)$. So $\pi_1(v)$ is a proper algebraic set of $\mathbf{P}^1$ and hence is a finite set $\{(c_i, d_i),\ i = 1, \ldots, k\}$. Let $F_1 = \{\theta_i = \arg\,(\frac{d_i}{c_i}) \mid c_i \neq 0\}$ and $D_1 = \{re^{i\theta} \in \mathbf{C} \mid r > 0,\ \ \theta \in [0, 2\pi)\backslash F_1\}$. Then for any y at infinity and $c \in D_1$, $(1, c, y) \notin v$, that is, there exists $b \in B$ such that $b(1, c, y) \neq 0$. Since $b \in B$, $b = g_1\tilde{B}_{y_1} + \cdots + g_s\tilde{B}_{y_s}$ where $y_1, \ldots, y_s$ are some points at infinity and $g_1, \ldots, g_s$ are polynomials.

From (2.9) we see that $b\tilde{q}_i \in \langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle$, $i = 1, \ldots, n$. Hence, $\langle \tilde{Q} \rangle \subseteq \langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle^{I_y}$ and we conclude that, by Remark 2.2,

$$\langle \tilde{Q} \rangle^{I_y} \subseteq \langle \tilde{q}_1 + c\tilde{p}_1, \ldots, \tilde{q}_n + c\tilde{p}_n \rangle^{I_y}.$$

This completes the proof. □

Under the same assumption of Lemma 2.1, we have the following three corollaries:

Corollary 2.1 *For fixed nonzero a, with $a^{-1} \in D_1$ and any y at infinity*

$$\langle \tilde{H}(a,z,t) \rangle^{I_y} = \langle \tilde{Q} \rangle^{I_y} \qquad \textit{for all } t \in [0,1). \tag{2.10}$$

Proof. From (2.2),

$$\tilde{H}(a,z,t) = (1-t)a\tilde{Q}(z) + t\tilde{P}(z) = (1-t)a\,(\tilde{Q}(z) + \frac{t\tilde{P}(z)}{(1-t)a}).$$

Since $a^{-1} \in D_1$, for $t \neq 1$, $\frac{t}{(1-t)a} \in D_1$. The assertion follows. □

Corollary 2.2 *For fixed $a^{-1} \in D_1$, $t \in [0,1)$ and y at infinity, we have*

(1) the quotient rings $S/\langle \tilde{H}(a,z,t) \rangle$ and $S/\langle \tilde{Q} \rangle$ have the same localization at the maximal ideal $I_y = \{f \in S \mid f(y) = 0\}$. That is,

$$\langle S/\langle \tilde{H}(a,z,t) \rangle \rangle_{(I_y)} = \langle S/\langle \tilde{Q} \rangle \rangle_{(I_y)}.$$

Here I_y is considered as maximal ideal in $S/\langle \tilde{H}(a,z,t) \rangle$ and $S/\langle \tilde{Q} \rangle$ through canonical projections.

(2) For any prime ideal q of S, considered as prime ideal in $S/\langle \tilde{H}(a,z,t) \rangle$ and $S/\langle \tilde{Q} \rangle$, with zero set $V(q)$ lying at infinity, we have

$$\langle S/\langle \tilde{H}(a,z,t) \rangle \rangle_{(q)} = \langle S/\langle \tilde{Q} \rangle \rangle_{(q)}.$$

Proof.

(1) For any $f \in \langle S/\langle\tilde{Q}\rangle\rangle_{(I_y)}$, $f = \frac{a+q}{b}$, where $q \in \langle\tilde{Q}\rangle$ and $b(y) \neq 0$. From Corollary 2.1, $\langle\tilde{H}(a,z,t)\rangle^{I_y} = \langle\tilde{Q}\rangle^{I_y} \supset \langle\tilde{Q}\rangle$, so, there exists an m-homogeneous $r \in S$ such that $r(y) \neq 0$ and $rq \in \langle\tilde{H}(a,z,t)\rangle$. Thus,

$$f = \frac{r(a+q)}{rb} = \frac{(ra+rq)}{rb} \in \langle S/\langle\tilde{H}(a,z,t)\rangle\rangle_{(I_y)},$$

and hence, $\langle S/\langle\tilde{H}(a,z,t)\rangle\rangle_{(I_y)} \supseteq \langle S/\langle\tilde{Q}\rangle\rangle_{(I_y)}$. The reverse inclusion follows by the same reason.

(2) Let $y \in V(q)$, then $I_y \supset q$ and

$$\langle S/\langle\tilde{H}(a,z,t)\rangle\rangle_{(q)} = \langle\langle S/\langle\tilde{H}(a,z,t)\rangle_{(I_y)}\rangle_{(q)} = \langle\langle S/\langle\tilde{Q}\rangle\rangle_{(I_y)}\rangle_{(q)} = \langle S/\langle\tilde{Q}\rangle\rangle_{(q)}.$$

□

Corollary 2.3 *For $a^{-1} \in D_1$, the intersection schemes of*

$$\bigcap_{i=1}^{n} \tilde{h}_i(a,z,t)$$

at infinity are the same closed subscheme of the projective scheme

$$P^{k_1} \times \cdots \times P^{k_m}$$

for all $t \in [0,1)$.

Proof. This follows from Corollary 2.2 and the local property of a scheme. □

Lemma 2.2 *Let P and Q satisfy the conditions (1) and (2) in Theorem 2.1 and*

$$\overline{H}(\lambda_0,\lambda,z) = \lambda_0\tilde{Q} + \lambda_1\tilde{P} \tag{2.11}$$

with $(\lambda_0,\lambda_1) \in \mathbf{P}^1$. Then for each k, the irreducible component $A_k \subset M$ of $\overline{H}^{-1}(0)$ passing through x^k satisfies the following:

(1) Let N be the set of points $(\lambda_0, \lambda_1, z)$ with z at infinity, then $\pi_1(A_k \cap N) \in \mathbf{P}^1$ is a finite set, where π_1 is the natural projection;

(2) $(1,0) \notin \pi_1(A_k \cap N)$.

Proof.

(1) By exercise II. 3.12 of [7] dim $A_k = 1$, since x^k is a nonsingular point of A_k. Let B_j be any irreducible component of $A_k \cap N$. Since $B_j \neq A_k$, by Theorem 2, X.5. of [8] dim $B_j < 1$. So (1) follows.

(2) From the proof of Lemma 2.1, there exists a set $\overline{D} = \{\mathbf{C}\backslash$ a finite set$\}$ such that for $\lambda_0 = 1$ and $\lambda_1 \in \overline{D}$ the intersection schemes of $\overline{H}(1, \lambda_1, z)$ at infinity are the same. By Proposition 9.1.2 and Example 9.1.10 of [6], for $\lambda_1 \in \overline{D}$ and $\lambda_0 = 1$, the number of solutions of (2.11) in affine space are the same $(= r)$. Since there are only finitely many points not in $\overline{D}$, there exists $\varepsilon_1 > 0$ such that $0 <\mid \lambda_1 \mid< \varepsilon_1$ implies $\lambda_1 \in \overline{D}$. And since $(1, 0, x^k)$, $k = 1, \ldots, r$ are nonsingular, there exists $0 < \varepsilon < \varepsilon_1$ such that for each $0 <\mid \lambda_1 \mid< \varepsilon$, $\tilde{Q} + \lambda_1 \tilde{P} = 0$ has r isolated affine solutions $x^k(\lambda_1)$. We claim that for each k,

$$V(\tilde{Q}) \cap N \cap A_k = \phi. \tag{2.12}$$

Suppose this is not true. Then there is a point $(1, 0, z^0) \in N \cap A_k$. Since A_k is connected and dim $A_k = 1$, let $c(s)$ be a curve such that c(0)=$(1, 0, z^0)$ and c(1)= $(1, 0, x^k)$. From (1) above, there are only finitely many (λ_0, λ_1) such that (λ_0, λ_1) can be in $N \cap A_k$ for suitable z. Hence, there is $0 < s_0 < 1$ and $\varepsilon_2 > 0$ such that if $\delta \in (s_0, s_0 + \varepsilon_2)$, $c(\delta) \notin N$ and $\pi_1(c(\delta)) = (1, \lambda_1)$ with $|\lambda_1| < \varepsilon$. But then we have at least r+1 zeros of $\tilde{Q} + \lambda_1 \tilde{P}$ in affine space, which leads to a contradiction. This completes the proof. $\square$

(Proof of Theorem 2.1.)

Let $\overline{H}(\lambda, z) = \lambda_0 \tilde{Q}(z) + \lambda_1 \tilde{P}(z)$ with $\lambda = (\lambda_0, \lambda_1) \in \mathbf{P}^1$. A point (λ, z) in $\mathbf{P}^1 \times M$ is said to be *regular* if and only if *rank* $\overline{H}_z(\lambda, z) = n$. For each x^k, $k = 1, \ldots, r$ in T, let A_k be the irreducible component of $V(\overline{H})$, the zero set of $\overline{H}$ in $\mathbf{P}^1 \times M$, passing through x^k. Let B_k be the set of points in A_k which are nonregular. Nonregularity can be described in terms of vanishing subdeterminants of $\overline{H}_z(\lambda, z)$ which lead to a system of polynomial equations. Consequently, B_k is an algebraic set for each k. So $\pi_1(B_k)$ is a proper algebraic set in $\mathbf{P}^1$, because $(1,0) \notin \pi_1(B_k)$ by Lemma 2.2 and hence it is finite for each k. Let $A = \bigcup_{i=1}^{r} \pi_1(B_k) = \{(c_i', d_i') \mid i = 1, \ldots, I\}$, $F_2 = \{\theta_i = \arg\left(\frac{d_i'}{c_i'}\right), i = 1, \ldots, I \mid c_i \neq 0\}$ and $D_2 = \{re^{i\theta} \in \mathbf{C} \mid r > 0, \ \theta \in [0, 2\pi) \backslash F_2\}$. For $a \in \mathbf{C}$ with $a^{-1} \in D_2$, $\frac{t}{(1-t)a} \in D_2$ for all $t \in [0,1)$. That is, $(1, \frac{t}{(1-t)a}) \notin A$, so, $\overline{H}_z(1, \frac{t}{(1-t)a}, z)$ is of rank n for any $(1, \frac{t}{(1-t)a}, z) \notin B_k$. A repeat application of the Implicit Function Theorem on the affine representation of the homotopy

$$0 = \tilde{H}(a, z, t) = (1-t)a\tilde{Q}(z) + t\tilde{P}(z) = (1-t)a\overline{H}\left(1, \frac{t}{(1-t)a}, z\right) \qquad (2.13)$$

implies the smoothness property. For accessibility, it follows from Corollary 2.3 that for fixed $a^{-1} \in D_1$, the intersection schemes of $\tilde{H}(a, z, t)$ at infinity are the same for all $t \in [0,1)$. By Proposition 9.1.2 and Example 9.1.10 of [6], for each t in $[0,1)$ the number of solutions of (2.13) in affine space are the same $(= r)$. As a consequence, the $x^k(t)$'s are the only solutions in affine space for each $t \in [0,1)$. By a degree theory argument as in [3], or an algebraic argument as in [10], the accessibility property follows.

By choosing $D = D_1 \cap D_2$, the proof of the theorem is completed. □

The following proposition indicates that Theorem 2.1 is a generalization of the main result in [12].

Proposition 2.1 *Suppose that the polynomial systems $\tilde{Q}, \tilde{P}$ in (2.2) has the following properties:*

(1) every point of $V(\tilde{Q})$ at infinity is also a point of $V(\tilde{P})$;

(2) the set $T = \{$the points of $V(\tilde{Q})$ in affine space$\}$ consists of nonsingular isolated points.

Then for a nonsingular point z of $V(\tilde{Q})$ at infinity, $\langle\tilde{Q}\rangle^{I_z} \supseteq \langle\tilde{P}\rangle^{I_z}$.

Proof For $f \in \langle\tilde{P}\rangle^{I_z}$ there exists $h \in S$ such that $h(z) \neq 0$, and $fh \in \langle\tilde{P}\rangle$. From condition (b), fh vanishes on the set of zeros of $\langle\tilde{Q}\rangle$ at infinity. Let $x^i = (x_1^i, \ldots, x_n^i)$ $i = 1, \ldots, r$, be the isolated zeros of Q in $\mathbf{C}^n$, and

$$F(x) = \prod_{j=1}^{r} \sum_{i=1}^{n} e_i(x_i - x_i^j)$$

where $e_i \in \mathbf{C}$, $i = 1, \ldots, n$ are chosen such that $\tilde{F}(z) \neq 0$. It is easy to see that $\tilde{F}(z^i) = 0$ for each $i = 1, \ldots, r$, where z^i is the corresponding point of x^i in M, and $\tilde{F}hf$ vanishes on $V(\tilde{Q})$. By the Nullstellensatz, $(\tilde{F}hf)^k \in \langle\tilde{Q}\rangle$ for some positive integer k. Since $(\tilde{F}h)^k(z) \neq 0$, $f^k \in \langle\tilde{Q}\rangle^{I_z}$. By Theorem 48 of [15], $\langle\tilde{Q}\rangle^{I_z}$ is a prime ideal. Hence, $f \in \langle\tilde{Q}\rangle^{I_z}$ which completes the proof. □

2.2 Applications

Example 2.1. Suppose we want to solve the system

$$\begin{aligned} p_1(x,y) &= xy + y + 1 = 0 \\ p_2(x,y) &= x^3y^2 - xy + 1 = 0. \end{aligned} \tag{2.14}$$

By considering $(x, y) \in C^1 \times C^1$, we may 2-homogenize (2.14) as

$$\begin{aligned} \tilde{p}_1(x_0, x, y_0, y) &= xy + x_0y + x_0y_0 = 0 \\ \tilde{p}_2(x_0, x, y_0, y) &= x^3y^2 - xx_0^2yy_0 + x_0^3y_0^2 = 0 \end{aligned} \tag{2.15}$$

where $(x_0, x, y_0, y) \in \mathbf{P}^1 \times \mathbf{P}^1$. Then this system $\tilde{P} = (\tilde{p}_1, \tilde{p}_2)$ has 1 solution $(0, 1, 1, 0)$ at infinity with multiplicity 2 and 3 affine solutions. Our starting system $Q = (q_1, q_2)$ can be chosen as

$$\begin{aligned} q_1(x,y) &= xy + y + 1 = 0 \\ q_2(x,y) &= x^3y^2 - xy = 0. \end{aligned} \tag{2.16}$$

Parameter $a = -.13960695 - .6281187i$				
Starting Point			**Solution Reached**	
	x	y	x	y
1.	0	-1	$-.4301591$	-1.7648765
2.	$-1/2 + \sqrt{3}\,i/2$	$-1/2 + \sqrt{3}\,i/2$	$-.78492 + 1.307138i$	$-.1225614 + 7448609i$
3.	$-1/2 - \sqrt{3}\,i/2$	$-1/2 - \sqrt{3}\,i/2$	$-.78492 - 1.3071413i$	$-.1225611 - .7448618i$

Table 2.1: Solutions to (2.14)

Its 2-homogenization is

$$\begin{aligned} \tilde{q}_1(x_0, x, y_0, y) &= xy + x_0 y + x_0 y_0 = 0 \\ \tilde{q}_2(x_0, x, y_0, y) &= x^3 y^2 - x x_0^2 y y_0 = 0. \end{aligned}$$

The system Q has 3 nonsingular solutions $(x, y) = (0, -1), (-1/2 + \sqrt{3}\,i/2, -1/2 + \sqrt{3}\,i/2)$ and $(-1/2 - \sqrt{3}\,i/2, -1/2 - \sqrt{3}\,i/2)$ and its solution at infinity is the same as that of P. Write $\tilde{q}_2 = xg$ with $g = x^2y^2 - x_0^2 y_0 y$. Since $x \neq 0$ at $z = (0, 1, 1, 0), x \notin I_z$, so, $g \in \langle \tilde{Q} \rangle^{I_z}$. Further, $\tilde{q}_1 xy - g = x_0 y(x_0 y_0 + xy + x y_0) \in \langle \tilde{Q} \rangle^{I_z}$ and $x_0 y_0 + xy + x y_0 \neq 0$ at z imply $x_0 y \in \langle \tilde{Q} \rangle^{I_z}$. Since $x_0 y_0 \tilde{q}_1 \in \langle \tilde{Q} \rangle^{I_z}$ we have $x_0^2 y_0^2 = x_0 y_0 \tilde{q}_1 - x_0 y(x y_0 + x_0 y_0) \in \langle \tilde{Q} \rangle^{I_z}$ and thus $\tilde{p}_2 = \tilde{q}_2 + x_0(x_0^2 y_0^2) \in \langle \tilde{Q} \rangle^{I_z}$. Along with $\tilde{p}_1 = \tilde{q}_1 \in \langle \tilde{Q} \rangle^{I_z}$, we have $\langle \tilde{Q} \rangle^{I_z} \supseteq \langle \tilde{P} \rangle^{I_z}$. So Theorem 2.1 applies. It provides a homotopy and 3 paths, beginning from the roots of (2.16), which lead to all roots of (2.14).

The table 2.1 shows the computing result.

The notion of m-homogeneous when $m = 1$ is the same as homogeneous. For homogeneous polynomials $f_1, \ldots, f_r$ we use $\langle f_1, \ldots, f_r \rangle_e$ to denote the subset of $\langle f_1, \ldots, f_r \rangle$ consisting of homogeneous polynomials of degree e. In [12], the following condition of $\tilde{P}, \tilde{Q}$ in (2.2) is used to guarantee the accessibility of the "random product homotopy" paths:

For each positive integer k,

$$\langle \tilde{q}_1, \ldots, \tilde{q}_n, x_0^k \rangle_e \supseteq \langle \tilde{p}_1, \ldots, \tilde{p}_n, x_0^k \rangle_e \tag{2.17}$$

for all sufficiently large e.

The condition (2.17) is equivalent to condition (1) in Theorem 2.1 when $m = 1$(See [14]). However, we shall illustrate in Example 2.2 that condition (1) in Theorem 2.1 can be much easier to verify.

Example 2.2. The following system is the mathematical model of a lumped-parameter chemically reacting system [2].

$$\begin{aligned} p_1(x_1,x_2,x_3,x_4) &= -a_1x_1(1-x_3-x_4)+a_2x_3-(x_1-b_1) \\ p_2(x_1,x_2,x_3,x_4) &= -a_3x_2(1-x_3-x_4)+a_4x_4-(x_2-b_2) \\ p_3(x_1,x_2,x_3,x_4) &= a_1x_1(1-x_3-x_4)-a_2x_3-a_5x_3x_4 \\ p_4(x_1,x_2,x_3,x_4) &= a_3x_2(1-x_3-x_4)-a_4x_4-a_5x_3x_4. \end{aligned} \tag{2.18}$$

While the Bézout number of the corresponding homogeneous system of $P = (p_1, p_2, p_3, p_4)$ is 16, for generic a_i's and b_i's there are only 4 zeros of (2.18) in $\mathbf{C}^4$([2]). Define $Q = (q_1, q_2, q_3, q_4)$ by

$$\begin{aligned} q_1(x_1,x_2,x_3,x_4) &= (x_1-1)(1-x_3-x_4) \\ q_2(x_1,x_2,x_3,x_4) &= (x_2-i)(2-x_3-x_4) \\ q_3(x_1,x_2,x_3,x_4) &= x_1(2-x_3-x_4)+x_3x_4 \\ q_4(x_1,x_2,x_3,x_4) &= (x_2+1)(1-x_3-x_4)+x_3x_4. \end{aligned} \tag{2.19}$$

The homogenization $\tilde{Q}$ of Q is

$$\begin{aligned} \tilde{q}_1(x_0,x_1,x_2,x_3,x_4) &= (x_1-x_0)(x_0-x_3-x_4) \\ \tilde{q}_2(x_0,x_1,x_2,x_3,x_4) &= (x_2-ix_0)(2x_0-x_3-x_4) \\ \tilde{q}_3(x_0,x_1,x_2,x_3,x_4) &= x_1(2x_0-x_3-x_4)+x_3x_4 \\ \tilde{q}_4(x_0,x_1,x_2,x_3,x_4) &= (x_2+x_0)(x_0-x_3-x_4)+x_3x_4. \end{aligned} \tag{2.20}$$

The points of $V(\tilde{Q})$ at infinity are $(x_0, x_1, x_2, x_3, x_4) = (0,0,0,0,1), (0,0,0,1,0)$ and a line $I = (0, x_1, x_2, 0, 0)$. The rank of

$$\left.\frac{\partial(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4)}{\partial(x_0,x_1,x_2,x_3,x_4)}\right|_{(0,0,0,0,1)} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ i & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

is 4 and hence, $(0,0,0,0,1)$ is nonsingular, similarly for $(0,0,0,1,0)$. These two points also belong to $V(\tilde{P})$. Further, the system (2.19) has 4 nonsingular isolated zeros: $(x_1,x_2,x_3,x_4) = (1,-1,0,2),(1,-1,2,0),(0,i,0,1)$ and $(0,i,1,0)$. So, Proposition 2.1 applies. That is,

$$\langle \tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4\rangle^{I_z} \supseteq \langle \tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4\rangle^{I_z}$$

for $z = (0,0,0,0,1),(0,0,0,1,0)$. For $z \in I = (0,x_1,x_2,0,0)$, either $x_1 \neq 0$ or $x_2 \neq 0$, say $x_1 \neq 0$. Then $x_1 - x_0 \neq 0$ at z. So, from $\tilde{q}_1$,

$$x_0 - x_3 - x_4 \in \langle \tilde{Q}\rangle^{I_z}. \tag{2.21}$$

It follows, from $\tilde{q}_4, x_3x_4 \in \langle \tilde{Q}\rangle^{I_z}$ and hence $x_1(2x_0 - x_3 - x_4) \in \langle \tilde{Q}\rangle^{I_z}$ from $\tilde{q}_3$. Since $x_1 \neq 0$ at $z, 2x_0 - x_3 - x_4 \in \langle \tilde{Q}\rangle^{I_z}$. Comparing with (2.21), yields $x_0 \in \langle \tilde{Q}\rangle^{I_z}$. Accordingly, it is easy to see that $\langle \tilde{Q}\rangle^{I_z} \supseteq \langle \tilde{P}\rangle$. Thus by Remark 2.2 we have $\langle \tilde{Q}\rangle^{I_z} \supseteq \langle \tilde{P}\rangle^{I_z}$. So Theorem 2.1 provides a homotopy and 4 paths which lead to all roots of (2.18). The table 2.2 shows a computing result with given parameters.

Parameters		
$a_1 = .76771879 + .32820278i$	$a_2 = .54890949 + .1093949i$	$a_3 = .33010021 + .89058417i$
$a_4 = .11129092 + .67177492i$	$a_5 = .89248163 + .45296562i$	$b_1 = .04796080 + .88868678i$
$b_2 = .82915151 + .66987747i$	$a = .59527814 + .71154547i$	

Example 2.3. For generalized eigenvalue problems (or λ-matrix problem), the system P has the following form:

$$\begin{aligned} &\lambda^k B_0 x + \lambda^{k-1} B_1 x + \cdots + B_k x = 0 \\ &1 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \end{aligned} \tag{2.22}$$

Starting Points			Solution Reached	
1.	$x_1 = 1$	$x_2 = -1$	$x_1 = -.731938 + .3453089i$	$x_2 = .049258 + .1264814i$
	$x_3 = 0$	$x_4 = 2$	$x_3 = 2.9605355 + 7.79467i$	$x_4 = .0547949 - .0998576i$
2.	$x_1 = 1$	$x_2 = -1$	$x_1 = -2.214090 + 1.074372i$	$x_2 = -1.43290 + .8555617i$
	$x_3 = 2$	$x_4 = 0$	$x_3 = .6873539 - 1.0286067i$	$x_4 = 1.6660806 + .7643948i$
3.	$x_1 = 0$	$x_2 = i$	$x_1 = .0433732 + .750366i$	$x_2 = .8245659 + .5315566i$
	$x_3 = 0$	$x_4 = 1$	$x_3 = .1027011 + .2787743i$	$x_4 = .4602318 - .0694782i$
4.	$x_1 = 0$	$x_2 = i$	$x_1 = .8908184 - .8041871i$	$x_2 = 1.6720086 - 1.0229946i$
	$x_3 = 1$	$x_2 = 0$	$x_3 = 1.6573049 - 1.607046i$	$x_4 = -.5652349 + .591969i$

Table 2.2: Solutions to (2.18) with given parameters

where $x = (x_1, \ldots, x_n), k > 1$ and $B_0, \ldots, B_k$ are $n \times n$ matrices. Consider $(\lambda, x_1, \ldots, x_n) \in C^1 \times C^n$. With 2-homogenization, (2.22) becomes

$$\lambda^k B_0 x + \lambda^{k-1}\lambda_0 B_1 x + \cdots + (\lambda_0)^k B_k x = 0 \tag{2.23}$$
$$x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$$

with $(\lambda_0, \lambda, x_0, \ldots, x_n) \in \mathbf{P}^1 \times \mathbf{P}^n$. If B_0 is a nonsingular matrix, it is quite obvious that (2.23) has kn solutions for generic α_i's. In [4], a homotopy is given for nonsingular B_0, which provides kn paths leading to all roots of (2.22).

Here, we give an example on which Theorem 2.1 can be applied when B_0 is singular. For $n = 3$ and $k = 2$, let

$$B_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\alpha_1 = \cdots = \alpha_n = 1$. Then (2.23) becomes,

$$\begin{aligned} \tilde{p}_1 &= \lambda^2 x_2 + \lambda\lambda_0 x_2 + \lambda_0^2 x_1 = 0 \\ \tilde{p}_2 &= \lambda^2 x_2 + \lambda\lambda_0 x_3 + \lambda_0^2 x_3 = 0 \\ \tilde{p}_3 &= \lambda^2 x_3 + \lambda\lambda_0 x_1 + \lambda_0^2 x_2 = 0 \\ \tilde{p}_4 &= x_0 + x_1 + x_2 + x_3 = 0 \end{aligned} \tag{2.24}$$

Parameter $a = -.74127114 + .70628309i$				
	x_1	x_2	x_3	λ
1.	-4.0795970	3.075972	0	$.7548779$
2.	$-.4602 - .1825814i$	$-.5397982 + .1825814i$	0	$-.8774412 + .7448597i$
3.	$-.4602 + .1825814i$	$-.5397982 - .1825814i$	0	$-.8774414 - .7448597i$
4.	$-.33333333$	$-.333333333$	$-.3333333$	$-.5 - .866025i$
5.	$-.33333333$	$-.333333333$	$-.3333333$	$-.5 + .866025i$

Table 2.3: Solutions to (2.24)

and the solution set at infinity is $v = \{(\lambda_0, \lambda, x_0, x_1, x_2, x_3) \mid \lambda_0 = 0,\ \lambda = 1,\ x_2 = 0,\ x_0 + x_1 = 0,\ x_3 = 0\}$. Define $Q = (q_1, q_2, q_3, q_4)$ by

$$\begin{aligned} q_1 &= (\lambda - 1)(\lambda - 2)x_2 = 0 \\ q_2 &= (\lambda - 3)(\lambda - 4)x_3 = 0 \\ q_3 &= (\lambda - 3)(\lambda - 4)x_3 + \lambda x_1 = 0 \\ q_4 &= 1 + x_1 + x_2 + x_3 = 0. \end{aligned} \tag{2.25}$$

It is easy to check that the zero set at infinity of

$$\begin{aligned} \tilde{q}_1 &= (\lambda - \lambda_0)(\lambda - 2\lambda_0)x_2 = 0 \\ \tilde{q}_2 &= (\lambda - 3\lambda_0)(\lambda - 4\lambda_0)x_3 = 0 \\ \tilde{q}_3 &= (\lambda - 3\lambda_0)(\lambda - 4\lambda_0)x_3 + \lambda\lambda_0 x_1 = 0 \\ \tilde{q}_4 &= x_0 + x_1 + x_2 + x_3 = 0 \end{aligned} \tag{2.26}$$

is the same as that of (2.24). The system (2.25) has 5 nonsingular solutions $(\lambda, x_1, x_2, x_3) = (0, -1, 0, 0)$, $(1, 0, -1, 0)$, $(2, 0, -1, 0)$, $(3, 0, 0, -1)$ and $(4, 0, 0, -1)$. For any $z \in v$, $\lambda_0 = 0$ and $\lambda = 1$ hence $(\lambda - 3\lambda_0)(\lambda - 4\lambda_0) \neq 0$ and $(\lambda - \lambda_0)(\lambda - 2\lambda_0) \neq 0$. From $\tilde{q}_1$ and $\tilde{q}_2$ both x_2 and x_3 are in $\langle\tilde{Q}\rangle^{I_z}$ and from $\tilde{q}_3$, $\lambda_0 x_1 \in \langle\tilde{Q}\rangle^{I_z}$. In summary, $\langle\tilde{Q}\rangle^{I_z} \supseteq \langle\tilde{P}\rangle^{I_z}$ and Theorem 2.1 applies. The table 2.3 shows our computing result.

Chapter 3

Nonlinear Homotopies

It occurs sometimes in practice that the polynomial system P(x) is associated with a set of parameters $q = (q_1, ..., q_j) \in \mathbf{C}^j$. In this situation we write $P(q, x)$. It is often the case that the system needs to be solved repeatedly with varying parameters of q's. In this chapter we give a procedure that begins with solving $P(q, x)$ for a particular parameter q^0. Then for each subsequent choice of the parameters q of the system, a nonlinear homotopy is used to find all isolated solutions with amount of computation approximately proportional to the actual number of solutions. The theorem on which the method is based is described in section 3.1. As an application, we show, in section 3.2, how to compute the 32 solutions of the Tsai-Morgan manipulator problem in [20] by following only 32 solution paths.

3.1 Main Results

Definition 3.1 *We say that a property* K *holds for generic* $q \in C^s$ (P^s), *if there exists an algebraic subset* $E \subset C^s(P^s)$ *of dimension* $< s$ *such that* $q \notin E$ *implies* K *hold.*

Theorem 3.1 *Suppose that the polynomial system* $P(q, z)$ *of* n *equations with* $z \in \mathbf{M} = \mathbf{P}^{k_1} \times ... \times \mathbf{P}^{k_m}$ *and* $q \in \mathbf{C}^j$, *which is m-homogeneous with respect to z, satisfies the following conditions:*

1. *For generic* $q \in \mathbf{C}^j$, *the solutions of* $P(q,z) = 0$ *at infinity are isolated and its number, counting multiplicities, equals* B - r, *where* B *is the Bézout number of the system* $P(q,z)$;

2. There is a fixed generic point q^0 such that the solution set of $P(q^0, z) = 0$ in the affine space consists of nonsingular isolated points $x^1, ..., x^r$.

For any point q^1 in $\mathbf{C}^j$, let

$$H(a, z, t) = P((1 - t + t(1 - t)a)q^0 + (t(1 - a(1 - t))q^1, z). \tag{3.1}$$

Then there exists an open dense full measure subset D of $\mathbf{C}$, such that for $a \in D$, we have

a. (Smoothness) For each isolated zero $x^k \in T, k = 1, \ldots, r$ there is a function $x^k(t)$: $[0, 1] \to \mathbf{M}$ which is analytic and contained in affine space for all t in $[0, 1)$ and satisfies $H(a, x^k(t), t) = 0$.

b. (Accessibility) Each isolated affine solution of $P(q^1, x) = 0$ is reached by $x^k(t)$ for some k at $t = 1$.

Before proving Theorem 3.1, we need the following lemma. Let E be the algebraic set associated with the genericity of the parameter q in the condition (1) above. That is, if $q \notin E$, then the system $P(q, z)$ satisfies condition (1).

Lemma 3.1 *Suppose that q^0 and q^1 are as in Theorem 3.1 and consider the homotopy*

$$\bar{H}(\lambda, z) = P((1 - \lambda)q^0 + \lambda q^1, z) = 0, \tag{3.2}$$

where $\lambda \in \mathbf{C}$. Let $\tilde{H}(\lambda_0, \lambda, z)$, $(\lambda_0, \lambda, z) \in \mathbf{P}^1 \times \mathbf{M}$, be the homogenization of $\bar{H}$ with respect to λ. Then for each k=1,...,r, the irreducible component of $\tilde{H}^{-1}(0)$ passing through x^k satisfies the following:

(1) Let N be the set of points $(\lambda_0, \lambda_1, z)$ with z at infinity, then $\pi_1(A_k \cap N) \in \mathbf{P}^1$ is a finite set, where π_1 is the natural projection;

(2) Let

$$J = \{\lambda \in \mathbf{C} | (1 - \lambda)q^0 + \lambda q^1 \in E\}. \tag{3.3}$$

Then J is a finite set.

(3) (1,0) $\notin \pi_1(A_k \cap N)$.

Proof. The proof of (1) follows the same line of arguments in Lemma 2.2 , so we omit it.

(2) Without loss of generality we may assume $q^0 = (0, ..., 0)$ and $q^1 = (1, 0, ..., 0)$. Because E is an algebraic set , it is the set of common zeros of polynomials $f_1, ..., f_t$. Since $q^0 \notin E$, $f_i(0, ..., 0) \neq 0$ for some i. Thus $f_i(\lambda, 0, ..., 0) \not\equiv 0$. Hence there are only finite many solutions to $g(\lambda) \equiv f_i(\lambda, 0, ..., 0) = 0$. Therefore the line $(1-\lambda)q^0 + \lambda q^1 = (\lambda, 0, ..., 0)$ intersects with E at finitely many points. This completes the proof of (2).

(3) From the proof of (2), there exists a set $\overline{D} = \{\mathbf{C}\backslash$ a finite set$\}$ such that for $\lambda_0 = 1$ and $\lambda_1 \in \overline{D}$ the number of solutions of

$$\tilde{H}(1, \lambda_1, z) = 0 \tag{3.4}$$

at infinity are the same. The rest of the proof is the same as the proof of Lemma 2.2, so we omit it.

For each λ in J, the set $I_\lambda = \{(1, \lambda, z) \in \mathbf{P}^1 \times \mathbf{M} | z \in \mathbf{M}\}$ is an algebraic set, so is its union $\bigcup_{\lambda \in J} J_\lambda$ since J is finite.

Proof of Theorem 3.1

For each x^k, $k = 1, \ldots, r$ in T, let A_k be the irreducible component of $V(\tilde{H})$, the zero set of $\tilde{H}$ in $\mathbf{P}^1 \times \mathbf{M}$, passing through x^k. Let B_k be the set of points (λ_0, λ, z) in A_k which are nonregular or at infinity or $\pi_1(\lambda_0, \lambda, z) = (1, \bar{\lambda})$ with $\bar{\lambda}$ in J. B_k is an algebraic set for each k. So $\pi_1(B_k)$ is a proper algebraic set in $\mathbf{P}^1$, because $(1, 0) \notin \pi_1(B_k)$ by Lemma 3.1 and hence it is finite for each k. Let $A = \bigcup_{k=1}^{r} \pi_1(B_k) = \{(c_i, d_i) \mid i = 1, \ldots, I\}$, $F = \{a_i | a_i = \frac{d_i'}{c_i'}, (c_i', d_i') \in A, c_i' \neq 0\}$. As long as $a \notin D_1 = \{(t - a_i)/t(1-t) | a_i \in F, t \in [0, 1)\}$, $(1, t(1 - a(1-t))) \neq (1, a_i)$ for any i and t$\in (0, 1)$. Thus $(1, t(1 - a(1-t)), z) \notin B_k$ for any k, $t \in (0, 1)$ and $z \in \mathbf{M}$. Let D $= C\backslash D_1$. By a similar argument as in the proof of theorem 2.1, the smoothness property and the accessibility property follow.

When the polynomial system $P(q, z)$ satisfies the property that

$$P(q, z) = 0 \text{ implies } P(aq, z) = 0 \quad \text{for any } a \in \mathbf{C}$$

then, in Lemma 3.1 and Theorem 3.1, instead of considering the line $(1-\lambda)q^0 + \lambda q^1$ and the quadratic curves $[(1 - t + t(1-t)a]q^0 + [t - t(1-t)a]q^1$ through q^0 and q^1 we may consider $\lambda_0 q^0 + \lambda q^1$ and $(1-t)\lambda_0 q^0 + t\lambda q^1$ and we can take the homotopy

$$H(z, t) \quad = \quad P((1-t)aq^0 + tq^1, z)$$

to solve $P(q^1, z) = 0$. In this situation we have a linear homotopy. In particular, we have

Theorem 3.2 *Suppose that for generic (a,b)$\in \mathbf{P}^1$ the solutions to*

$$\tilde{H}(a,b,z) = a\tilde{Q}(z) + b\tilde{P}(z) \tag{3.5}$$

at infinity are isolated and the number of them , counting the multiplicities, equals B-r, where B is the Bézout number of the system . And the set $T = \{$ the points of $V(\tilde{Q})$ in affine space $\}$ consists of nonsingular isolated points $x^1, \ldots, x^r$.Then, there exists an open dense subset D of $\mathbf{C}$ *with full measure, such that for a^{-1} chosen from D, we have*

a. *(Smoothness) For each isolated zero $x^k \in T, k = 1, \ldots, r$ there is a function $x^k(t)$: $[0,1] \to \mathbf{M}$ which is*

   analytic and contained in affine space for all t in $[0,1)$ and satisfies

$$\tilde{H}(a, x^k(t), t) = (1-t)a\tilde{Q}(x^k(t)) + t\tilde{P}(x^k(t)) = 0. \tag{3.6}$$

b. *(Accessibility) Each isolated solution of $\tilde{P}(z) = 0$ in affine space is reached by $x^k(t)$ for some k at $t = 1$.*

3.2 Applications

The very important inverse kinematics problem for the 6R manipulator of general geometry (robot arm design) was reduced by Tsai and Morgan ([20]) to the solution of a polynomial system of 8 equations in 8 unknowns.The system is as follows.

$$\begin{aligned} p_1 \;=\;& -x_1x_3\lambda_1\mu_2 q + x_1x_4\mu_2 p + x_2x_3\lambda_1\mu_2 p + x_2x_4\mu_2 q - \\ & x_5x_8\mu_3\lambda_4 a_5 - x_6x_7\mu_3 a_5 - x_1\lambda_2\mu_1 q + x_2\lambda_2\mu_1 p - \\ & x_3\mu_1\mu_2(r-d_1) - x_4\mu_2 a_1 + x_5\mu_3\mu_4 d_5 - x_6\mu_3 a_4 - \\ & x_8\mu_4\lambda_3 a_5 + \lambda_1\lambda_2 r - \lambda_1\lambda_2 d_1 - \lambda_2 d_2 - d_3 - \lambda_3 d_4 - \\ & \lambda_3\lambda_4 d_5 = 0 \\ p_2 \;=\;& -x_1x_3\lambda_1\mu_2 v + x_1x_4\mu_2 u + x_2x_4\mu_2 v + x_2x_3\lambda_1\mu_2 u + \\ & x_5x_7\mu_3\lambda_4\mu_5 - x_6x_8\mu_3\mu_5 - x_1\lambda_2\mu_1 v + x_2\lambda_2\mu_1 u - \\ & x_3\mu_1\mu_2 w + x_5\mu_3\mu_4\lambda_5 + x_7\lambda_3\mu_4\mu_5 + \lambda_1\lambda_2 w \\ & -\lambda_3\lambda_4\lambda_5 = 0 \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
p_3 &= x_1x_3a_2u + x_1x_4\lambda_1a_2v + x_2x_3a_2v - x_2x_4\lambda_1a_2u - \\
&\quad x_5x_7\mu_3\lambda_4\mu_5d_3 + x_5x_8\mu_5a_3 + x_6x_7\mu_5\lambda_4a_3 + \\
&\quad x_6x_8\mu_3\mu_5d_3 - x_1(-a_1u + \mu_1d_2v) + x_2(a_1v + \mu_1d_2u) + \\
&\quad x_4\mu_1a_2w - x_5\mu_3\mu_4\lambda_5d_3 + x_6\mu_4\lambda_5a_3 - x_7(\mu_4\mu_5d_4 + \\
&\quad \lambda_3\mu_4\mu_5d_3) + x_8\mu_5a_4 + d_1w + \lambda_1d_2w + \lambda_3\lambda_4\lambda_5d_3 + \\
&\quad \lambda_4\lambda_5d_4 + \lambda_5d_5 - pu - qv - rw = 0 \\
p_4 &= x_1x_3a_2p + x_1x_4\lambda_1a_2q + x_2x_3a_2q - x_2x_4\lambda_1a_2p + \\
&\quad x_5x_7a_3a_5 + x_5x_8\mu_3\lambda_4a_5d_3 + x_6x_7\mu_3a_5d_3 - \\
&\quad x_6x_8\lambda_4a_3a_5 + x_1(a_1p - \mu_1d_2q) + x_2(a_1q + \mu_1d_2p) - \\
&\quad x_3a_1a_2 + x_4(-\mu_1a_2d_1 + \mu_1a_2r) + x_5(a_3a_4 - \mu_3\mu_4d_3d_5) \\
&\quad +x_6(\mu_4d_5a_3 + \mu_3a_4d_3) + x_7a_4a_5 + x_8(\mu_4a_5d_4 + \lambda_3\mu_4a_5d_3) \\
&\quad +d_1r + \lambda_1d_2r - \lambda_1d_1d_2 + \lambda_3d_3d_4 + \lambda_3\lambda_4d_3d_5 + \lambda_4d_4d_5 \\
&\quad +0.5(-a_1^2 - d_1^2 - a_2^2 - d_2^2 + a_3^2 + d_3^2 + a_4^2 + d_4^2 + a_5^2 \\
&\quad +d_5^2 - p^2 - q^2 - r^2) = 0 \\
p_5 &= x_1^2 + x_2^2 - 1 = 0 \\
p_6 &= x_3^2 + x_4^2 - 1 = 0 \\
p_7 &= x_5^2 + x_6^2 - 1 = 0 \\
p_8 &= x_7^2 + x_8^2 - 1 = 0
\end{aligned}
$$

where x_i, $i = 1, 2, \cdots, 8$ are the variables and the others are parameters. From various computing experiences, it has been predicted ([13, 19]) that this system has 32 isolated solutions for generic parameters. In this section, we shall prove this assertion and give an algorithm, via homotopy continuation method, for finding all 32 isolated solutions with minimal computation efforts. By letting $(x_0, x_1, x_2, x_5, x_6) \in P^4$ and $(y_0, x_3, x_4, x_7, x_8) \in P^4$, we may 2-homogenize (3.7) and obtain (introduced in [19]),

$$
\begin{aligned}
\tilde{p}_1 &= -x_1x_3\lambda_1\mu_2q + x_1x_4\mu_2p + x_2x_3\lambda_1\mu_2p + x_2x_4\mu_2q - \\
&\quad x_5x_8\mu_3\lambda_4a_5 - x_6x_7\mu_3a_5 - x_1\lambda_2\mu_1qy_0 + x_2\lambda_2\mu_1py_0 - \\
&\quad x_3\mu_1\mu_2(r - d_1)x_0 - x_4\mu_2a_1x_0 + x_5\mu_3\mu_4d_5y_0 - x_6\mu_3a_4y_0 -
\end{aligned}
$$

$$
\begin{aligned}
& x_8\mu_4\lambda_3 a_5 x_0 + (\lambda_1\lambda_2 r - \lambda_1\lambda_2 d_1 - \lambda_2 d_2 - d_3 - \lambda_3 d_4 - \\
& \lambda_3\lambda_4 d_5)x_0 y_0 = 0 \\
\tilde{p}_2 \;=\; & -x_1x_3\lambda_1\mu_2 v + x_1x_4\mu_2 u + x_2x_4\mu_2 v + x_2x_3\lambda_1\mu_2 u + \\
& x_5x_7\mu_3\lambda_4\mu_5 - x_6x_8\mu_3\mu_5 - x_1\lambda_2\mu_1 v y_0 + x_2\lambda_2\mu_1 u y_0 - \\
& x_3\mu_1\mu_2 w x_0 + x_5\mu_3\mu_4\lambda_5 y_0 + x_7\lambda_3\mu_4\mu_5 x_0 + (\lambda_1\lambda_2 w \\
& -\lambda_3\lambda_4\lambda_5)x_0 y_0 = 0 \\
\tilde{p}_3 \;=\; & x_1x_3a_2u + x_1x_4\lambda_1a_2v + x_2x_3a_2v - x_2x_4\lambda_1a_2u - \\
& x_5x_7\mu_3\lambda_4\mu_5 d_3 + x_5x_8\mu_5a_3 + x_6x_7\mu_5\lambda_4a_3 + \\
& x_6x_8\mu_3\mu_5d_3 - x_1(-a_1u + \mu_1d_2v)y_0 + x_2(a_1v + \mu_1d_2u)y_0 + \\
& x_4\mu_1a_2wx_0 - x_5\mu_3\mu_4\lambda_5d_3y_0 + x_6\mu_4\lambda_5a_3y_0 - x_7(\mu_4\mu_5d_4 + \\
& \lambda_3\mu_4\mu_5d_3)x_0 + x_8\mu_5a_4x_0 + (d_1w + \lambda_1d_2w + \lambda_3\lambda_4\lambda_5d_3 + \\
& \lambda_4\lambda_5d_4 + \lambda_5d_5 - pu - qv - rw)x_0y_0 = 0 \\
\tilde{p}_4 \;=\; & x_1x_3a_2p + x_1x_4\lambda_1a_2q + x_2x_3a_2q - x_2x_4\lambda_1a_2p + \\
& x_5x_7a_3a_5 + x_5x_8\mu_3\lambda_4a_5d_3 + x_6x_7\mu_3a_5d_3 - \\
& x_6x_8\lambda_4a_3a_5 + x_1(a_1p - \mu_1d_2q)y_0 + x_2(a_1q + \mu_1d_2p)y_0 - \\
& x_3a_1a_2x_0 + x_4(-\mu_1a_2d_1 + \mu_1a_2r)x_0 + x_5(a_3a_4 - \mu_3\mu_4d_3d_5)y_0 \\
& +x_6(\mu_4d_5a_3 + \mu_3a_4d_3) + x_7a_4a_5 + x_8(\mu_4a_5d_4 + \lambda_3\mu_4a_5d_3) \\
& +(d_1r + \lambda_1d_2r - \lambda_1d_1d_2 + \lambda_3d_3d_4 + \lambda_3\lambda_4d_3d_5 + \lambda_4d_5)x_0y_0 \\
& +0.5(-a_1^2 - d_1^2 - a_2^2 - d_2^2 + a_3^2 + d_3^2 + a_4^2 + d_4^2 + a_5^2 \\
& +d_5^2 - p^2 - q^2 - r^2)x_0y_0 = 0 \\
\tilde{p}_5 \;=\; & x_1^2 + x_2^2 - x_0^2 = 0 \\
\tilde{p}_6 \;=\; & x_3^2 + x_4^2 - y_0^2 = 0 \\
\tilde{p}_7 \;=\; & x_5^2 + x_6^2 - x_0^2 = 0 \\
\tilde{p}_8 \;=\; & x_7^2 + x_8^2 - y_0^2 = 0
\end{aligned}
\tag{3.8}
$$

It is easily seen that for the system (3.8), $d_1^i = d_2^i = 1$, $i = 1, \cdots, 4$, $d_1^5 = d_1^7 = d_2^6 = d_2^8 = 2$ and $d_2^5 = d_2^7 = d_1^6 = d_1^8 = 0$. Accordingly, from (1.7), the Bézout number of the system is the coefficient of $\alpha^4\beta^4$ in the product

$$(\alpha + \beta)^4(2\alpha)^2(2\beta)^2,$$

which equals 96. It was proved in [19] that this system has at most 64 isolated solutions in $\mathbf{P}^4 \times P^4$. We shall prove that the number of zeros at infinity, counting multiplicities, is 64, and consequently, the number of isolated zeros in affine space is at most 32. The points at infinity consists of 3 categories, that is, (i) $x_0 = y_0 = 0$, (ii) $x_0 = 0,\ y_0 = 1$, (iii) $x_0 = 1,\ y_0 = 0$. We shall discuss each case separately.

(i) $x_0 = y_0 = 0$.

The last 4 equations in (3.8) gives,

$$\begin{aligned}
\tilde{p}_5 &= x_1^2 + x_2^2 = 0 \\
\tilde{p}_6 &= x_3^2 + x_4^2 = 0 \\
\tilde{p}_7 &= x_5^2 + x_6^2 = 0 \\
\tilde{p}_8 &= x_7^2 + x_8^2 = 0,
\end{aligned}$$

and hence $x_2 = \pm i x_1$, $x_4 = \pm i x_3$, $x_6 = \pm i x_5$ and $x_8 = \pm i x_7$. There are 16 combinations. For a typical combination, say, $x_2 = ix_1$, $x_4 = ix_3$, $x_6 = ix_5$, $x_8 = ix_7$, the first 4 equations of (3.8) gives

$$\begin{aligned}
\tilde{p}_1 &= \mu_2(1+\lambda_1)(-q+ip)x_1x_3 - ia_5\mu_3x_5x_7(1+\lambda_4) \\
\tilde{p}_2 &= \mu_2(1+\lambda_1)(-v+iu)x_1x_3 + \mu_3\mu_5x_5x_7(1+\lambda_4) \\
\tilde{p}_3 &= a_2(-i)(1+\lambda_1)(-v+iu)x_1x_3 + \mu_5x_5x_7(1+\lambda_4)(ia_3 - \mu_3d_3) \\
\tilde{p}_4 &= a_2i(1+\lambda_1)(q-ip)x_1x_3 + a_5x_5x_7(1+\lambda_4)(a_3+i\mu_3)
\end{aligned} \tag{3.9}$$

If we regard (3.9) as a linear system with unknowns x_1x_3 and x_5x_7, then it is easy to see that the only solutions are $x_1x_3 = 0$ and $x_5x_7 = 0$ for generic parameters. However, x_1 and x_5 cannot be zero simultaneously, for otherwise $(x_0, x_1, x_2, x_5, x_6) = (0,0,0,0,0)$ which is not in P^4. Similarly, x_3 and x_7 cannot be zero simultaneously. Thus, only 2 solutions left, that is, $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (1, i, 0, 0, 0, 0, 1, i)$ and $(0, 0, 1, i, 1, i, 0, 0)$.

Counting multiplicities, there are 32 isolated solutions in total in this case.

(ii) $x_0 = 0,\ y_0 = 1$

From (3.8), we have

$$\begin{aligned}
\tilde{p}_5 &= x_1^2 + x_2^2 = 0 \\
\tilde{p}_7 &= x_5^2 + x_6^2 = 0.
\end{aligned}$$

Hence, $x_2 = \pm i x_1$, $x_6 = \pm i x_5$. There are 4 combinations. Let us consider a typical situation: $x_2 = -ix_1$, $x_6 = ix_5$. It follows from (3.8),

$$\begin{aligned}
\tilde{p}_1 &= -x_1(q+ip)f_1(x_3,x_4) - \mu_3 x_5 g_1(x_7,x_8) = 0 \\
\tilde{p}_2 &= -x_1(v+iu)f_1(x_3,x_4) + \mu_3 x_5 g_2(x_7,x_8) = 0 \\
\tilde{p}_3 &= x_1(v+iu)f_2(x_3,x_4) + (ia_3 - \mu_3 d_3)x_5 g_2(x_7,x_8) = 0 \\
\tilde{p}_4 &= x_1(q+ip)f_2(x_3,x_4) - (ia_3 - \mu_3 d_3)x_5 g_1(x_7,x_8) = 0
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
f_1(x_3,x_4) &= \mu_2(\lambda_1 x_3 + ix_4) + \lambda_2\mu_1, \\
f_2(x_3,x_4) &= a_2(-ix_3 + \lambda_1 x_4) - (ia_1 + d_2\mu_1)
\end{aligned}$$

and

$$\begin{aligned}
g_1(x_7,x_8) &= a_5(ix_7 + \lambda_4 x_8) - (\mu_4 d_5 - ia_4), \\
g_2(x_7,x_8) &= \mu_5(\lambda_4 x_7 - ix_8) + \mu_4\lambda_5.
\end{aligned}$$

Now,

$$\begin{aligned}
&(v+iu)\times\tilde{p}_1 - (q+ip)\times\tilde{p}_2 \\
&\quad = -\mu_3 x_5[(v+iu)g_1(x_7,x_8) + (q+ip)g_2(x_7,x_8)] = 0.
\end{aligned}$$

If $x_5 = 0$, then $x_6 = 0$ and $x_1 \neq 0$. We then have an overdetermined system of x_3 and x_4 for generic parameters. That is,

$$\begin{aligned}
f_1(x_3,x_4) &= \mu_2(\lambda_1 x_3 + ix_4) + \lambda_2\mu_1 = 0 \\
f_2(x_3,x_4) &= a_2(-ix_3 + \lambda_1 x_4) - (ia_1 + d_2\mu_1) = 0 \\
\tilde{p}_6 &= x_3^2 + x_4^2 - 1 = 0,
\end{aligned}$$

which has no solution in general. Hence, $x_5 \neq 0$, and

$$(v+iu)g_1(x_7,x_8) + (q+ip)g_2(x_7,x_8) = 0.$$

Combining this linear equation with $\tilde{p}_8 = x_7^2 + x_8^2 - 1 = 0$, we arrive at 2 solutions for x_7 and x_8. On the other hand,

$$\begin{aligned}
&(d_3\mu_3 - ia_3)\times\tilde{p}_1 + \mu_3\tilde{p}_4 \\
&\quad = x_1(q+ip)[(ia_3 - d_3\mu_3)f_1(x_3,x_4) + \mu_3 f_2(x_3,x_4)] = 0.
\end{aligned}$$

By a similar argument, $x_1 \neq 0$ for generic parameters, and combining the linear equation

$$(ia_3 - d_3\mu_3)f_1(x_3, x_4) + \mu_3 f_2(x_3, x_4) = 0$$

with $\tilde{p}_6 = x_3^2 + x_4^2 - 1 = 0$, we arrive at 2 solutions for x_3 and x_4. Substituting any combination of x_3, x_4, x_7, x_8 we have obtained back to $\tilde{p}_1 = 0, \tilde{p}_2 = 0$ in (3.10), unique solution of x_1, x_5 can be found. As a consequence, there are 4 solutions in this case. And there are 16 solutions in total.

(iii) $x_0 = 1, \quad y_0 = 0$

From (3.8), we have

$$\begin{aligned} \tilde{p}_6 &= x_3^2 + x_4^2 = 0 \\ \tilde{p}_8 &= x_7^2 + x_8^2 = 0 \end{aligned}$$

Along the same line of argument as in (ii), we consider a typical combination: $x_4 = -ix_3$ and $x_8 = ix_7$. It follows that

$$\begin{aligned} \tilde{p}_1 &= \mu_2 x_3 f_3(x_1, x_2) - ia_5 x_7 g_3(x_5, x_6) = 0 \\ \tilde{p}_2 &= \mu_2 x_3 f_4(x_1, x_2) + \mu_5 x_7 g_3(x_5, x_6) = 0 \\ \tilde{p}_3 &= ia_2 x_3 f_4(x_1, x_2) - \mu_5 x_7 g_4(x_5, x_6) = 0 \\ \tilde{p}_4 &= ia_2 x_3 f_3(x_1, x_2) + ia_5 x_7 g_4(x_5, x_6) = 0 \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} f_3(x_1, x_2) &= -(ip + \lambda_1 q)x_1 + (\lambda_1 p - iq)x_2 + (ia_1 - \mu_1(r - d_1)) \\ f_4(x_1, x_2) &= -(iu + \lambda_1 v)x_1 + (\lambda_1 u - iv)x_2 - \mu_1 w \\ g_3(x_5, x_6) &= \lambda_4 \mu_3 x_5 - i\mu_3 x_6 + \lambda_3 \mu_4 \\ g_4(x_5, x_6) &= (d_3 \lambda_4 \mu_5 - ia_3)x_5 - (id_3 \mu_3 + \lambda_4 a_3)x_6 \\ &\quad + d_3 \lambda_3 \mu_4 + \mu_4 d_4 - ia_4. \end{aligned}$$

And,

$$(-u_5 i) \times \tilde{p}_1 + a_5 \times \tilde{p}_2 = \mu_2 x_3 (a_5 f_4 - \mu_5 i f_3) = 0.$$

It can be shown that $x_3 \neq 0$ for generic parameters. And combining the linear equation

$$a_5 f_4(x_1, x_2) - \mu_5 i f_3(x_1, x_2) = 0$$

with $\tilde{p}_5 = x_1^2 + x_2^2 - 1 = 0$, we have two solutions for x_1 and x_2. On the other hand,

$$a_2 \times \tilde{p}_2 + (\mu_2 i) \times \tilde{p}_3 = \mu_5 x_7 (a_2 g_3 - \mu_2 i g_4) = 0.$$

Again, for generic parameters, $x_7 \neq 0$. The linear equation

$$a_2 g_3(x_5, x_6) - \mu_2 i g_4(x_5, x_6) = 0$$

and

$$\tilde{p}_7 = x_5^2 + x_6^2 - 1 = 0$$

yields 2 solutions for x_5 and x_6. Substituting any combination of x_1, x_2, x_5, x_6 back to (3.11), we have a unique solution of x_3 and x_7. Hence, we have 4 solutions in this case and 16 solutions in total.

There are 26 parameters in the system (3.7). Let $E \in C^{26}$ be the vector representing all these parameters and write $P(x) = P(E, x)$. To find a particular system with easily calculated isolated zeros, we proceed as follows. First, we add more parameters $B = (b_1, b_2, b_3, b_4)$ into (3.7) by defining $\bar{P}(B, E, x) = (\bar{p}_1, \cdots, \bar{p}_8)$ as:

$$\begin{aligned}
\bar{p}_1 &= p_1 + b_1 \\
\bar{p}_2 &= p_2 \\
\bar{p}_3 &= p_3 + b_2 \\
\bar{p}_4 &= p_4 + b_3 \\
\bar{p}_5 &= p_5 \\
\bar{p}_6 &= p_6 \\
\bar{p}_7 &= p_7 \\
\bar{p}_8 &= p_8 + b_4
\end{aligned} \tag{3.12}$$

When we 2-homogenize the system (3.12) and look at the zeros at infinity of (3.12) as we did on the system (3.7), all the parameters in B will disappear. Thus, for generic parameters B and E, $\bar{P}(B, E, x) = 0$ has the same number of solutions as $\bar{P}(0, E, x) = P(E, x)$. Namely, for generic parameters B and E, $\bar{P}(B, E, x) = 0$ has at most 32 isolated solutions. For a particular choice of B and E, denoted by (B_0, E_0), with $r = w = v = q = d_1 = d_3 = d_5 = a_4 = \lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = \mu_2 = 0$, $u = p = a_1 = a_2 = a_3 = a_5 = d_2 = d_4 = \mu_1 = \mu_3 =$

$\mu_4 = \mu_5 = \lambda_2 = 1$ and $b_1 = 1, b_2 = -4, b_3 = -4.5, b_4 = -1$, the system (3.12) becomes,

$$
\begin{aligned}
\bar{p}_1 &= -x_6x_7 + x_2 \\
\bar{p}_2 &= -x_6x_8 + x_2 \\
\bar{p}_3 &= x_1x_3 + x_5x_8 + x_1 + x_2 - x_7 - 5 \\
\bar{p}_4 &= x_1x_3 + x_5x_7 + x_1 + x_2 - x_3 + x_8 - 5 \\
\bar{p}_5 &= x_1^2 + x_2^2 - 1 \\
\bar{p}_6 &= x_3^2 + x_4^2 - 1 \\
\bar{p}_7 &= x_5^2 + x_6^2 - 1 \\
\bar{p}_8 &= x_7^2 + x_8^2 - 2
\end{aligned} \tag{3.13}
$$

There are exactly 32 isolated zeros of the system above, and each one of them can be easily obtained by straightforward eliminations. We shall briefly describe the calculations here. From $\bar{p}_1 - \bar{p}_2 = 0$, we have

$$x_6(x_7 - x_8) = 0$$

which implies (i) $x_6 = 0$, or (ii) $x_7 = x_8$.

(i) $x_6 = 0$.

In this case, $x_2 = 0$. Consequently, $x_5 = \pm 1$ and $x_1 = \pm 1$. A typical combination, say, $x_1 = 1$ and $x_5 = 1$ yields, from $\bar{p}_4 = 0$,

$$x_7 + x_8 - 4 = 0.$$

Together with $\bar{p}_8 = x_7^2 + x_8^2 - 2 = 0$, two solutions of x_7 and x_8 are obtained. That is, $x_7 = 2 \pm \sqrt{3}i$ and $x_8 = 2 \mp \sqrt{3}i$. Substituting back to $\bar{p}_3 = 0$, we have $x_3 = 4 \pm 2\sqrt{3}i$. And, from $\bar{p}_5 = x_3^2 + x_4^2 - 1 = 0$, two solutions of x_4 can be calculated for each value of x_3.

In total, we have 16 solutions in this category.

(ii) $x_7 = x_8$.

From $\bar{p}_8 = x_7^2 + x_8^2 - 2 = 0$, we have $x_7 = x_8 = \pm 1$, and, from $\bar{p}_1 = 0$, $x_2 = \pm x_6$. Consequently, $x_1 = \pm x_5$. Also, $\bar{P}_4 - \bar{P}_3 = 0$ yields, $x_3 = 2$, and hence, $x_4 = \pm\sqrt{3}i$. A typical combination, say, $x_7 = x_8 = 1$, $x_2 = x_6$, $x_1 = x_5$ and $x_3 = 2$, gives, from $\bar{p}_3 = 0$,

$$4x_1 + x_2 = 6.$$

Together with $\bar{p}_5 = x_1^2 + x_2^2 - 1 = 0$, two solutions of x_1 and x_2 become available. That is, $x_1 = \frac{24 \pm \sqrt{19}i}{17}$ and $x_2 = \frac{24 \mp \sqrt{19}i}{17}$.

In total, we also have 16 solutions in this case.

Now, consider the homotopy

$$\begin{aligned} H(a,x,t) = \bar{P}([(1-t) + t(1-t)a](B_0, E_0) \\ +[t(1-a(1-t))](0,E), x) = 0, \end{aligned} \tag{3.14}$$

where $a \in C$. When $t = 0$, we have $H(a,x,0) = \bar{P}(B_0, E_0, x) = 0$, which is the system (3.13) and the solutions are known. When $t = 1$, $H(a,x,1) = \bar{P}(0,E,x) = P(E,x) = 0$ which is the system (3.7). By the Theorem 3.1, we have the following,

Proposition 3.1 *For any given parameter set $E \in C^{26}$ and a randomly chosen $a \in C$, the homotopy (3.14) satisfies the smoothness and accessibility properties.*

From this proposition, every isolated solution of $P(E,x) = 0$ in (3.7) can be reached by some solution path of $H(a,x,t) = 0$ in (3.14), originating at $t = 0$. The path can be parameterized by t in $[0,1)$ and starts at a solution of $H(a,x,0) = \bar{P}(B_0, E_0, x) = 0$ in (3.13). Notice that there are exactly 32 paths, which is less than all the existing homotopies for this problem by at least a half. The algorithm has been implemented and executed without any failure for different sets of parameters we tried. The table 3.1 shows a typical computing result.

An application of theorem 3.2 can be found in following example.

Suppose we want to solve the system

$$\begin{aligned} p_1 &= x^3y^3 - 1 \\ p_2 &= xy + x + y + 1 \end{aligned}$$

Let

$$\begin{aligned} q_1 &= (x-1)(x-2)(x-3)(y-1)(y-2)(y-3) \\ q_1 &= xy + 1 \end{aligned}$$

and $G(a,b,x,y) = aQ + bP$. If $\tilde{G}$ is the 2-homogenization of G, then the Bézout number of the system $\tilde{G}$ is 12. For generic $(a,b) \in P^1((a,b) \neq (1,\text{-}1))$, the system has two solutions

Parameters						
a		u	v	w	μ_4	μ_2
.336675 -.28947i		.2727448	.57121	.0000	.3663635	-.5088737
a_1	d_1	μ_1	λ_1	a_2	d_2	q
.982733	.4798967	-.9464922	.3227267	.7639239	.25918	.7449497
a_3	d_3	μ_3	λ_3	a_4	d_4	p
.5451146	.04037	-.071255	.997458	.3263	.8215618	.963759
a_5	d_5	μ_5	λ_5	λ_2	λ_4	r
.107496	.6027525	.803982	.594653	.8608412	.9304718	.52614

Solutions Found							
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
.4479964	-.894035	.6380733	.7699757	-.938979	.3439749	-.159925	-.9871292
.2218439	-.975082	.4720904	.8815501	.0203477	.999793	-.260256	.9655395
-.135018	-.990843	.0696905	.9975684	-.022661	.9997432	.0982504	.9951615
.4540939	-.890954	.5255288	.8507758	-.733651	-.6795284	-.34666	-.937991
$-.540165\mp$ .0356i	$.842621\mp$ .022824i	$-2.32259\mp$ 5.6207i	$5.69644\mp$ 2.291703i	$10.09924\mp$ 15.663703i	$-15.6863\mp$ 10.08472i	$5.74658\mp$ 2.22103i	$2.251062\pm$ 5.66893i
$1.661705\pm$ .471729i	$-.573575\pm$ 1.3666474i	$-1.15989\mp$ .359879i	$-.56853\pm$ .734212i	$-8.53055\mp$ 7.486027i	$-7.515156\pm$ 8.497486i	$-1.669081\mp$ .843605i	$.98484\mp$ 1.429713i
$1.004864\pm$ .024479i	$-.14298\pm$ .172036i	$3.19571\pm$ 8.393040i	$8.44499\mp$ 3.176051i	$-2.379306\pm$ 7.513149i	$-7.573479\mp$ 2.360352i	$2.41357\mp$ 2.215251i	$-2.324815\mp$ 2.297895i
$1.155795\pm$ .0001i	$.0002\mp$ .579536i	$-6.719234\mp$ .223521i	$.226035\mp$ 6.644488i	$-9.35885\pm$ 1.87464i	$-1.88501\mp$ 9.307368i	$-1.964046\mp$ 4.038446i	$-4.13826\pm$ 1.916653i
$1.423549\mp$ .3663253i	$-0.490041\mp$ 1.0641598i	$1.082638\mp$ .457099i	$-0.716682\mp$ .6905057i	$1.00763\pm$ 12.771532i	$-12.81038\pm$ 1.004575i	$-1.6951\pm$ 1.166219i	$1.31788\pm$ 1.50003i
$2.955148\mp$ .728427i	$.770925\pm$ 2.792242i	$-1.185142\pm$ .7125328i	$-.947416\mp$ .8913217i	$1.687669\mp$ 4.544349i	$-4.64051\mp$ 1.6526986i	$-1.166774\mp$ 2.494902i	$2.65762\mp$ 1.095336
$.896764\mp$ .054138i	$-.458226\mp$ .10595i	$1.946745\pm$ .802069i	$.906342\mp$ 1.7227758i	$-6.9181\pm$ 1.891155i	$1.909785\pm$ 6.850612i	$-1.035343\mp$ .7526668i	$1.0319154\mp$ .7551665i
$4.368425\mp$ .22472i	$.230832\pm$ 4.252755i	$-1.292526\pm$ .3702095i	$-.530158\mp$ .902571i	$-.947854\pm$ .952823i	$1.240695\pm$ .7279286i	$-1.076679\mp$ 1.971413i	$2.16523\mp$ .9803015i
$-.278632\mp$ .1481645i	$-.972686\pm$ .0424426i	$-.28221\pm$ .1118686i	$.966405\pm$ .032668i	$-1.6677\pm$ .5240848i	$.632894\pm$ 1.380983i	$.447261\pm$ .3815813i	$-.9876348\pm$ .172803i
$-.992321\pm$ .000862i	$.123889\pm$ .006907i	$.508913\mp$ 1.7531098i	$2.003189\mp$ .44538i	$-4.387059\mp$ 9.8143i	$9.856731\mp$ 4.3681736i	$2.722861\mp$ .976045i	$-1.03913\mp$ 2.557557i
$3.728055\mp$ .059146i	$.061396\pm$ 3.59147i	$-1.270256\pm$ .4070247i	$-.582666\mp$ .887345i	$-.979966\pm$ .009874i	$.204924\pm$ .0472206i	$-.51712\mp$ -1.749672i	$-1.999667\pm$ .4524707i
$.038208\pm$ .1618099i	$-1.012304\pm$ .006107i	$.276872\pm$ .131319i	$.970562\mp$ .037461i	$.832789\mp$ .209601i	$-.650007\mp$ .2685406i	$.260632\mp$ .1590995i	$.9793755\pm$ .0423396i
$1.183174\pm$ .288567i	$-.466935\pm$ .7312i	$-3.58459\mp$ .801433i	$-.832769\pm$ 3.44971i	$-10.39366\pm$ 9.467553i	$-9.491538\mp$ 10.36739i	$-1.77677\pm$ .4623247i	$.548376\pm$ 1.4979576i
$2.863816\mp$ .291212i	$-1.36066\mp$ 2.717648i	$-1.0717\mp$ .272529i	$-.507232\pm$ .5758127i	$-3.735566\mp$ 2.622366i	$-2.686579\pm$ 3.64628i	$-.96021\pm$ 2.042529i	$-2.23979\mp$ .8756418i

Table 3.1: Solutions to (3.7) with given parameters

(0,1,0) and (0,0,1) at infinity, each one with multiplicity 3. Thus by theorem 3.1 we can use the homotopy $H(x,t) = (1-t)aQ + tP$ to find all isolated solutions of P by following 6 solution paths.

Appendix

For $P = (p_1, p_2)$ in (1.5) and $Q = (q_1, q_2)$ in (1.6), let $H = (h_1, h_2) = (1-t)aQ + tP$, where a is any nonzero number in $\mathbf{C}$, which is not a negative real number. To be precise,

$$(1) \qquad h_1(a, x_1, x_2, t) = (1-t)a(x_2^2 - 1) + t(x_2^2 + x_1) = 0$$

$$(2) \qquad h_2(a, x_1, x_2, t) = (1-t)a(x_2^2 + x_1x_2) + t(x_2^2 + x_2) = 0.$$

Multiplying (1) by $x_2(1-t)a$ and subtracting $t \times (2)$, yields,

$$(3) \qquad x_2^3\,[(1-t)at + (1-t)^2a^2] + x_2^2\,[-(1-t)at - t^2] + x_2[-(1-t)^2a^2 - t^2] = 0.$$

From (3), we can see that for each fixed $a \in \mathbf{C}$ and $t \in (0,1)$, the zero set of $H(a, x_1, x_2, t)$ are $(x_1, x_2) = (\frac{(1-t)a}{t}, 0)$, (d_1, e_1) and (d_2, e_2), where

$$(4) \qquad e_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \qquad e_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

$$(5) \qquad d_i = -\frac{t(e_i + 1)}{(1-t)a} - e_i \qquad i = 1, 2,$$

or

$$(6) \qquad d_i = -\frac{[((e_i)^2 - 1)(1-t)a - (e_i)^2 t]}{t} \qquad i = 1, 2,$$

with

$$b = \frac{-t}{(1-t)a}, \qquad c = \frac{-(1-t)^2a^2 - t^2}{(1-t)a[t + (1-t)a]}.$$

It is easy to see that as $t \to 0$, $b \to 0$, $c \to -1$. Hence, from (4) and (5), $(d_1, e_1) \to (-1, 1)$ and $(d_2, e_2) \to (1, -1)$. When $t \to 1$, $\frac{4c}{b^2} \to 0$, $\frac{c}{b}$ is bounded and

$$f \equiv -b + \sqrt{b^2 - 4c} = b(-1 + \sqrt{1 - \frac{4c}{b^2}}) = b(-1 + 1 - \frac{2c}{b^2} + o(\frac{4c}{b^2})) = \frac{-2c}{b} + o(\frac{4c}{b}).$$

However,

$$\frac{c}{b} = \frac{[t^2 + (1-t)^2a^2]}{t[t + (1-t)a]} \to 1 \text{ as } t \to 1.$$

Hence $f \to -2$, and $e_1 \to -1$, as $t \to 1$. From (6), $(d_1, e_1) \to (-1, -1)$. Similarly,

$$\begin{aligned} g &\equiv -b - \sqrt{b^2 - 4c} = b(-1 - \sqrt{1 - \frac{4c}{b^2}}) \\ &= b(-1 - [1 - \frac{2c}{b^2} + o(\frac{4c}{b^2})]) \\ &= -2b + \frac{2c}{b} + o(\frac{4c}{b^2}). \end{aligned}$$

When $t \to 1$, $b \to +\infty$, hence, $g \to +\infty$ and $e_2 \to +\infty$. Therefore, $(d_2, e_2) \to (+\infty, +\infty)$ from (6).

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