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MARKOV PROPERTIES OF MEASURE–INDEXED GAUSSIAN RANDOM FIELDS

By

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Abstract

Markov Properties of Measure - indexed Gaussian Random Fields

By

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We consider the Gaussian random field $\{X_\mu, \ \mu \in M(E)\}$,where $M(E)$ is a vector space of signed Radon measures with compact support on a separable locally compact Hausdorff space E. We assume that the covariance $C(\mu, \nu) = E(X_\mu X_\nu)(\mu, \nu \in M(E))$ is bilinear. The Markov properties of $\{X_\mu, \ \mu \in M(E)\}$ are defined. The necessary and sufficient conditions for $\{X_\mu, \ \mu \in M(E)\}$ to have the Markov property in terms of the geometric and analytic structure of the reproducing kernel Hilbert space of $C(\mu, \nu)$ are given under some assumptions on the index set $M(E)$. We also define the concept of dual process and in the case that a Gaussian random field $\{X_\mu, \ \mu \in M(E)\}$ has a dual, we can simplify the necessary and sufficient conditions. Applications to generalized Gaussian random fields, to the Gaussian fields related to Dirichlet forms and to the ordinary Gaussian processes are derived.

To my parents

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Contents

Chapter 1

Introduction

The study of the Markov property for multiparameter processes was initiated by P. Lévy[15] who conjectured that Lévy Brownian motion in odd dimension has this Markov property. McKean [19] proved Lévy's conjecture and gave a precise definition of Markov property. Subsequently Pitt[24], Künsch [13], Molchan[21] and Kallianpur and Mandrekar[12] gave necessary and sufficient conditions for a Gaussian and generalized Gaussian process to have some type of Markov property. A systematic study of different Markov properties is given in [17] for Gaussian processes and in [18] for the generalized Gaussian processes. In [7], Dynkin introduced the study of the Markov property for Gaussian processes related to the Dirichlet space and studied Markov property for specific Gaussian processes indexed by measures of finite energy. This was generalized in an abstract way for Gaussian processes related to Dirichlet form of Fukushima[10] by Röchner[25].

The main techniques used by [12], [13], [21], [24] were geometric and depended on the geometric structure of reproducing kernel Hilbert spaces. In [12], [21] the conditions were simplified in the case the Gaussian process has a dual process(see also Rozanov[26]). The concept of a dual process originates in [12] and [21]. In [7] and [25], the Markov property was proved by relying heavily on probabilistic technique as the Gaussian processes considered by them are related to Green's functions of a symmetric Markov process.

Our purpose here is to establish general theorems, using pure geometric techniques, for Gaussian processes indexed by measures to have a Markov property. We begin by recalling certain facts about conditional independence and Gaussian processes from [17] and [18] in Chapter 2 and Chapter 3. As a consequence we derive some new facts in Lemmas 2.8 - 2.11. We introduce the concept of support (Lemma 2.2) for a linear functional on a vector space of measures satisfying (A.1) and (A.2). Such a concept plays a role in establishing and proving our main general theorems in Chapter 3 (Theorems 3.1, 3.2). In view of Examples 2.1 the main results of [12] and [21] follow from these general theorems.

We also need to modify the structure of the indexed sets used by Dynkin [7] and Rõchner [25]. We demonstrate that this modification does not affect the Markov property of the processes considered by them. However with this modification, we can set their problems as a spacial case of Corollary 3.3. To obtain this, we need to introduce an appropriate generalization of the concept of the dual process introduced in [12] to our setup. Finally our results give considerable strengthening of the results of [13] and [24] as well as generalizing the index set for the multiparameter processes. This is done in Chapter 5.

In Chapter 4 we recall the needed concepts from the theory of Dirichlet forms and prove a crucial analytic result (Lemm 4.7) in this setup which allows us to relate the local property of the Dirichlet space to the condition of Corollary 3.3.

We start the next chapter with preliminaries, notation and interrelations of the concepts used throughout the thesis.

Chapter 2

Notations and Preliminaries

In this chapter, we present some concepts and results needed in the rest of this work. We start first by introducing the reproducing kernel Hilbert space associated with a covariance function following Aronszajn[1].

Definition 2.1 *Let T be any set and C be a real valued function on $T \times T$. Then C is called a covariance on T if*

(a) $C(t,s) = C(s,t)$ *for all* $s,t \in T$ *and*

(b) $\sum_{t,s\in i} a_s a_t C(t,s) \geq 0$ *for all finite subsets* i *of* T *and* $\{a_s, s \in i\}$ *of* $\Re$.

Theorem 2.1 (Aronszajn [1]) *Let T be any set and C be a real valued covariance on T. Then there exists a unique Hilbert space $K(C)$ of functions on T satisfying*

$$C(\cdot,t) \in K(C) \quad \text{for each } t \in T, \tag{2.1}$$

$$(f, C(\cdot,t))_{K(C)} = f(t) \quad \text{for each } t \in T \text{ and } f \in K(C). \tag{2.2}$$

Here for each $t, C(\cdot,t)$ denotes the function of the first variable and $(\cdot,\cdot)_{K(C)}$ means the inner product in $K(C)$.

Proof. Let $\Re^T$ be the real linear space of all functions on T to $\Re$ with coordinate-wise addition and scalar multiplication and let H be the linear manifold in $\Re^T$ generated by $\{C(\cdot,t), t \in T\}$. On H define the inner product

$$(f,g) = \sum_{s\in i, s'\in i'} a_s b_{s'} C(s,s') = \sum_{s\in i} a_s g(s) = \sum_{s'\in i'} b_{s'} f(s'). \qquad (2.3)$$

Where $f = \sum_{s\in i} a_s C(\cdot,s), g = \sum_{s'\in i'} b_{s'} C(\cdot,s')$ with i, i' finite subsets of T. From the last two equalities in (2.3), we get (f,g) is independent of the representations of f and g. From properties of C we get $(f,f) \geq 0$ and (f,g) is a bilinear function on H. Also $f(t) = (f, C(\cdot,t))$ for each $t \in T$ and $f \in H$ gives $|f(t)|^2 \leq (f,f)C(t,t)$. This implies $(f,f) = 0$ iff $f(t) = 0$ for all t. Thus $(H,(\cdot,\cdot))$ is a pre-Hilbert space. Let $\overline{H}$ be the completion of H under norm $(f,f)^{\frac{1}{2}}$, define $K(C) = \{f \in \Re^T, f(t) = (C(\cdot,t), h_f) \text{ for } h_f \in \overline{H}\}$. On $K(C)$ define $(f,g) = (h_f, h_g)$. Then $K(C)$ has all the properties and is determined uniquely by C. □

Definition 2.2 *Let T be any set. A class $K(C)$ of functions on T forming a Hilbert space is called the reproducing kernel Hilbert space (for short, RKHS) of a covariance C if it satisfies (2.1) and (2.2). The above theorem gives existence and uniqueness.*

Definition 2.3 *Let T be a set and $(\Omega,\mathcal{F},P)$ be a probability space, Then a family $\{X_t : t \in T\}$ of real random variables is called a centered Gaussian* *process if every real linear combination of finite elements of $\{X_t : t \in T\}$ is a Gaussian random variable with mean zero.*

If $\{X_t : t \in T\}$ is a centered Gaussian process on a probability space $(\Omega, \mathcal{F}, P)$, then $C_X(t,t') = E_P(X_t X_{t'})$ is a covariance on T and $K(C_X) = \{f, f(t) = E_P(X_t Y_f)$ for a unique $Y_f \in H(X)\}$, where E_P is the expectation under P and $H(X)$ is the linear subspace of $L_2(\Omega, \mathcal{F}, P)$ generated by $\{X_t, t \in T\}$. Conversely we can associate a Gaussian process with a covariance.

Lemma 2.1 *Let C be a covariance on T, then there exists a Gaussian process $\{X_t, t \in T\}$ defined on a suitable probability space $(\Omega, \mathcal{F}, P)$ such that $E_P(X_t X_{t'}) = C(t,t')$.*

Proof. Let $K(C)$ be the RKHS of C and $\{e_j, j \in J\}$ be an orthonormal basis in $K(C)$. Define $\Omega = \Pi_j \Omega_j, \mathcal{F} = \otimes_j \mathcal{F}_j, P = \otimes_j P_j$ where $\Omega_j = \Re, \mathcal{F}_j = \mathcal{B}(\Re)$ and $P_j = N(0,1)$ for $j \in J$. Also let $\xi_j(\omega) = \omega_j$ with $j \in J$. For $h \in K(C), h = \sum_j (h, e_j) e_j$ define $\Pi(h) = \sum_j (h, e_j) \xi_j$. Then by Parseval's identity we get $\Pi(h)$ as a Gaussian random variable for each $h \in K(C)$ with $X_t = \Pi(C(\cdot, t))$ we get $\{X_t, t \in T\}$ as a Gaussian process with covariance C. $\square$

Remark: The map Π in the proof of Lemma 2.1 is an isometry between $K(C)$ and $H(X)$. We will be using this fact many times later on.

For a Gaussian process $\{X_t, t \in T\}$, when $T \subseteq \Re^n$, we call it a (Gaussian) random field. Let $C_0^\infty(E)$ be the space of infinitely differentiable functions with compact support in E, where E is a open sebset of $\Re^n$. When

$T = C_0^\infty(E)$, we call $\{X_t, t \in T\}$ generalized random field if the covariance function $C(\varphi, \psi) = E_p(X_\varphi X_\psi)$ is bilinear and continuous on $C_0^\infty(E)$ with Schwartz topology(see [9] or [11]). We shall also be using processes indexed by measures of bounded energy. They occurred in the works [12],[24] and [26]. For this we need some additional concepts.

Let E be a separable locally compact Hausdorff space. $M(E)$ is a set containing Radon signed measures on E with compact support. The support of a signed Radon measure μ on E is defined as the complement of the largest open set O such that $|\mu|(O) = 0$, where $|\mu|$ is the total variation measure of μ. We make the following assumptions on $M(E)$:

(A.1) $M(E)$ is a real vector space.

(A.2) $M(E)$ has the partition of unity property, namely for any $\mu \in M(E)$, if $\{O_1, O_2, .., O_n\}$ is an open covering of the support of μ (for short, suppμ) then there exist $\mu_1, \mu_2, ..., \mu_n \in M(E)$ with supp$\mu_i \subseteq O_i, i = 1, 2, .., n$ and $\mu = \mu_1 + \mu_2 + ... + \mu_n$.

(A.3) If f is a linear functional on $M(E)$, and the support of f (for short, supp$f$) is contained in $A_1 \bigcup A_2$ where A_1 and A_2 are two disjoint closed subsets of E, then $f = f_1 + f_2$, where f_1, f_2 are linear functionals on $M(E)$ with supp$f_i \subseteq A_i, i = 1, 2$. The support of a linear functional f on $M(E)$ is defined as the complement of the largest open set N of E such that $f(\mu) = 0$ for all $\mu \in M(E)$ with supp$\mu \subseteq N$.

Under the assumptions (A.1) and (A.2) the support of a linear functional on $M(E)$ is well defined, Actually we have the following:

Lemma 2.2 *Under the assumptions (A.1) and (A.2) we have*

(a) If f is a linear functional on $M(E)$, then $suppf$ = complement of $\bigcup_i O_i$, where the union is taken over all open set O_i such that $f(\mu) = 0$ for all $\mu \in M(E)$ with $supp\mu \subseteq O_i$.

(b) If f is a linear functional on $M(E)$ and $suppf$ is an empty set, then $f \equiv 0$, i.e. $f(\mu) = 0$ for all $\mu \in M(E)$.

(c) If f_1 and f_2 are two linear functionals on $M(E)$, then

$$supp(f_1 + f_2) \subseteq (suppf_1) \bigcup (suppf_2).$$

Proof. (a) We only need to show that $\bigcup_{i\in I} O_i \subseteq (\text{supp}f)^c$. Let $\mu \in M(E)$ be such that $\text{supp}\mu \subseteq \bigcup_{i\in I} O_i$. By the compactness of $\text{supp}\mu$, we may choose finite sets $O_{i1}, ..., O_{in}$ to cover $\text{supp}\mu$, using the partition of unity property (A.2) we have $\mu = \mu_1 + ... + \mu_n$ where $\mu_j \in M(E)$ and $\text{supp}\mu_j \subseteq O_{i_j}, j = 1, .., n$. Then

$$f(\mu) = f(\mu_1 + ... + \mu_n) = \sum_{i=1}^{n} f(\mu_i) = 0.$$

(b) Using the definition of support of a linear functional.

(c) Let $A_i = \text{supp}f_i, i = 1, 2$. Let $\mu \in M(E)$ with $\text{supp}\mu \subseteq (A_1 \bigcup A_2)^c = A_1^c \bigcap A_2^c$. Then $\text{supp}\mu \subset A_i^c$ for $i = 1, 2$. Hence $f_i(\mu) = 0$ for $i = 1, 2$. Then $f(\mu) = f_1(\mu) + f_2(\mu) = 0$. So $\text{supp}(f_1 + f_2) \subseteq A_1 \bigcup A_2$. □

Lemma 2.3 *Under assumptions (A.1) and (A.2), (A.3) is equivalent to the following (A.3)':*

(A.3)' *If f is a linear functional on $M(E)$ and $suppf \subseteq A_1 \bigcup A_2$ where A_1 and A_2 are two disjoint closed sets, then for any two disjoint open sets*

O_1, O_2 with $A_i \subseteq O_i, i = 1, 2$, f can be decomposed into the sum of two linear functionals f_1 and f_2 on $M(E)$ with supp$f_i \subseteq O_i, i = 1, 2$.

Proof. That (A.3) implies (A.3)' is obvious. To prove the converse, let O_1 and O_2 be two disjoint open sets of E such that $O_i \supseteq A_i, i = 1, 2$. Then $f = f_1 + f_2$ with the f_i's linear functionals on $M(E)$ and supp$f_i \subseteq O_i, i = 1, 2$. Now take another open set $O_1' \subseteq O_1$ with $O_1' \supseteq A_1$. Then $f = f_1' + f_2'$ with supp$f_1' \subseteq O_1'$ and supp$f_2' \subseteq O_2$ so $f_1 - f_1' = f_2' - f_2$. By Lemma 2.2(c) supp$(f_1 - f_1') \subseteq$(suppf_1) $\cup$ (suppf_1)$\subseteq O_1$ and supp$(f_2' - f_2) \subseteq$(suppf_2') $\cup$ (suppf_2)$\subseteq O_2$. Since $O_1 \cap O_2 = \phi$, supp$(f_1 - f_1')$ =supp$(f_2' - f_2) = \phi$. Then by Lemma 1.2(b) $f_1 - f_1' = f_2' - f_2 = 0$. So supp$f_1 \subseteq O_1'$, hence

$$\text{supp} f_1 \subseteq \bigcap_{A_1 \subseteq O \subseteq O_1} O = A_1.$$

Similarly we can show supp$f_2 \subseteq A_2$. □

We will give some examples in which our assumptions (A.1), (A.2) and (A.3) are satisfied. Before we give the examples, we need the following Lemma.

Lemma 2.4 *If E is a normal space([9],p.2) and $\{O_1, ..., O_n\}$ is an open covering of a closed set A of E then there exist open sets $U_1, U_2, ..., U_n$ such that $\overline{U}_i \subseteq O_i, i = 1, 2, ..., n$ and*

$$\bigcup_{i=1}^{n} U_i \supseteq A$$

where $\overline{U}_i$ means the closure of U_i.

We note that ([2],p.8) a separable locally compact Hausdorff space is a normal space. The proof of the lemma is based on induction on n.

Proof. If $A \subseteq O$ where A is closed and O is open, then A and O^c are two disjoint closed sets and hence by the normality of space there exist two disjoint open sets U_1, U_2 such that $A \subseteq U_1$ and $O^c \subseteq U_2$. Hence $\overline{U_1} \subseteq \overline{U_2^c} = U_2^c \subseteq O$. Then U_1 is the candidate, so the lemma is true for $n = 1$.

Assume $A \subseteq O_1 \bigcup O_2$, A is closed and O_1, O_2 are open. Then $A \bigcap O_2^c \subseteq O_1$. Since $A_1 \bigcap O_2^c$ is a closed set, there is an open set U_1 such that $\overline{U_1} \subseteq O_1$ and $A \cap O_2^c \subseteq U_1$. Then $A = (A \bigcap U_1) \bigcup (A \bigcap U_1^c)$. Since $A_1 \bigcap U_1^c$ is closed and $A_1 \bigcap U_1^c \subseteq O_2$,then there is an open set U_2 such that $\overline{U_2} \subseteq O_2$ and $A \bigcap U_1^c \subseteq U_2$. Then $A \subseteq U_1 \bigcup U_2$ so the lemma is true for $n = 2$.

Assume that the lemma is true for $n \leq k (k \geq 2)$. Then if

$$A \subseteq \bigcup_{i=1}^{k+1} O_i = (\bigcup_{i=1}^{k} O_i) \bigcup O_{k+1}$$

there exist open sets U' and U_{k+1} such that

$$\overline{U'} \subseteq \bigcup_{i=1}^{k} O_i,$$

$$\overline{U}_{k+1} \subseteq O_{k+1},$$

and

$$A \subseteq U' \bigcup U_{k+1}.$$

Now since

$$\overline{U'} \subseteq \bigcup_{i=1}^{k} O_i$$

by induction there exist open sets $U_1, ..., U_k$ such that $\overline{U_i} \subseteq O_i, i = 1, 2, ..., k$ and

$$\overline{U'} \subseteq \bigcup_{i=1}^{k} U_i.$$

Then

$$A \subseteq U' \bigcup U_{k+1} \subseteq \bigcup_{k=1}^{k+1} U_i.$$

□

Example 2.1 *(Infinitely differentiable functions)*

Let E be an open domain in $\Re^n$, $M(E) = \{\mu : d\mu = \varphi dx, \varphi \in C_0^\infty(E)\}$, where $C_0^\infty(E)$ consists of all infinitely differentiable functions with compact support on E. E is equiped with relative topology. Obviously $M(E)$ is a vector space in the sense of (A.1). If $\varphi \in C_0^\infty(E)$ and $\{O_1, O_2, ..., O_n\}$ is an open covering of suppφ, then there exist $\varphi_1, ..., \varphi_n \in C_0^\infty(E)$ with supp$\varphi_i \in O_i$ and $\varphi = \sum_{i=1}^n \varphi_i$ where for a function $\varphi \in C_0^\infty(E)$, supp$\varphi$ = closure of $\{x : x \in E, \varphi(x) \neq 0\}$. Notice that supp$\varphi$ =supp(φdx), so (A.2) is satisfied.

To verify (A.3)', let f be a linear functional on $M(E)$ and supp$f \subseteq A_1 \bigcup A_2$. Where A_1 and A_2 are two disjoint closed subsets of E, take two disjoint open sets O_1, O_2 such that $A_i \subseteq O_i, i = 1,2$. Then we can take two open sets O_1' and O_2' such that $A_i \subseteq O_i' \subseteq O_i, i = 1,2$ and $\overline{O_1'} \bigcap \overline{O_2'}$ is the empty set. Take $O_3' = (A_1 \bigcup A_2)^c$ then $\{O_1', O_2', O_3'\}$ is an open covering of $\Re^n$. Then there exist $\varphi_i \in C^\infty(E), i = 1,2,3$ such that $\varphi_i \geq 0$, supp$\varphi \subseteq O_i', i = 1,2,3$ and $\sum_{i=1}^3 \varphi_i = 1$ ([9],p.45) where $C^\infty(E)$ consists of all infinitely differentiable functions. Now for any $\varphi \in C_0^\infty(E), \varphi = \sum_{i=1}^3 \varphi\varphi_i$

$$f(\varphi) = f(\varphi\varphi_1) + f(\varphi\varphi_2) + f(\varphi\varphi_3).$$

Since supp($\varphi\varphi_3$) =supp($\varphi\varphi_3 dx$) $\subseteq O_3$ then $f(\varphi\varphi_3) = 0$. Let $f_i(\varphi) = f(\varphi\varphi_i), i = 1,2$. If $\varphi \in C_0^\infty(E)$ with supp(φdx) =supp$\varphi \subseteq$ (suppφ_i)^c

then $\varphi\varphi_i \equiv 0, i = 1,2$. So $f_i(\varphi) = 0, i = 1,2$ if supp(φdx) =supp$\varphi \subseteq$ (supp$\varphi_i)^c, i = 1,2$. Then supp$f_i \subseteq$ ((supp$\varphi_i)^c)^c$ =supp$\varphi_i \subseteq O_i, i = 1,2$. This implies (A.3)' holds.

Example 2.2 *(Measures of finite energy)*

Let $M(E)$ be a vector space having the following property. If $\mu \in M(E)$, then $1_A\mu \in M(E)$ for any $A \in \mathcal{B}(E)$ where $\mathcal{B}(E)$ is the σ-field generated by all open subsets of E and $1_A\mu$ is the measure of μ restricted to A, i.e., $1_A\mu(B) = \mu(A \bigcap B)\ \forall B \in \mathcal{B}(E)$. Then $M(E)$ satisfies (A.2) and (A.3): suppose $\mu \in M(E)$ with supp$\mu \subseteq \bigcup_{i=1}^n O_i$ where $O_i^\prime s$ are open sets.By Lemma 2.4 we can find open sets $U_i, (i = 1, ...n)$ such that $\overline{U_i} \subseteq O_i$ and supp$\mu \subseteq \bigcup_{i=1}^n U_i$. Then

$$\mu = (1_{\cup_{i=1}^n U_i})\mu = 1_{U_1}\mu + 1_{U_2 \cap U_1^c}\mu + ... + 1_{U_n \cup U_1^c \cap ... \cap U_{n-1}^c}\mu \equiv \mu_1 + \mu_2 + \ldots + \mu_n \text{ say,}$$

such an expression makes sense because μ has compact support and hence is a finite measure. Obviously supp$\mu_i \subseteq \overline{U_i} \subseteq O_i, i = 1,2,...,n$ and $\mu_i \in M(E)$ by assumption for $i = 1,2,...,n$. Hence (A.2) holds.

If f is a linear functional on $M(E)$ with supp$f \subseteq A_1 \bigcup A_2$ where A_1 and A_2 are two disjoint closed sets. Choose open sets O_1, O_2 be with $A_i \subseteq O_i, i = 1,2$ and $\overline{O_1} \bigcap \overline{O_2} = \phi$. Let $O_3 = (O_1 \bigcup O_2)^c$, then $O_1 \bigcup O_2 \bigcup O_3 = E$ and these $O_i^\prime s$ are also disjoint. Then for any $\mu \in M(E)$,

$$\mu = 1_{O_1}\mu + 1_{O_2}\mu + 1_{O_3}\mu \equiv \mu_1 + \mu_2 + \mu_3 \text{ say,}$$

then

$$f(\mu) = f(\mu_1) + f(\mu_2) + f(\mu_3) \equiv f_1(\mu) + f_2(\mu) + f_3(\mu) \text{ say,}$$

notice that f_1, f_2 and f_3 are also linear on $M(E)$. Since supp$\mu_3 \subseteq \overline{O_3} = O_3 = (O_1 \bigcup O_2)^c \subseteq (A_1 \bigcup A_2)^c$ so $f_3(\mu) = f(\mu_3) = 0$ for all $\mu \in M(E)$. Notice that for any μ with supp$\mu \subseteq \overline{O}_i^c$, $i = 1,2$, $\mu_i = 1_{O_i}\mu \equiv 0$ hence $f_i(\mu) = 0$, $i = 1,2$ for all $\mu \in M(E)$ with supp$\mu \subseteq \overline{O}_i^c$. So supp$f_i \subseteq \overline{O_i}, i = 1,2$. (A.3)' also holds.

From the above two examples we notice that it is easier to check (A.3)' instead of (A.3) when we know that (A.1) and (A.2) are already satisfied.

We are interested in the Markov Property of Gaussian processes indexed by $M(E)$, here we assume that $M(E)$ satisfies assumptions (A.1)-(A.3). For this we need some elementary concepts of conditional independence([16]) and related results from [17].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume unless stated otherwise that $\mathcal{F}$ is complete and all sub σ-fields(algebras) contain all sets of measure zero from $\mathcal{F}$.We mean by conditional expectation or conditional probability the equivalence classes of random variables. Because of the above assumptions on all sub σ - fields, the equivalence classes for different σ - fields are the statement a.e. and consider equalities in terms of equivalence classes.

Definition 2.4 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{G}$ be sub σ-fields of $\mathcal{F}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{G}$ if*

$$P(A \bigcap B|\mathcal{G}) = P(A|\mathcal{G})P(B|\mathcal{G}) \qquad A \in \mathcal{A}, B \in \mathcal{B}$$

where $P(\cdot|\mathcal{G})$ is the conditional probability given $\mathcal{G}$. We denote this by $\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}$. If $\mathcal{G} = \{\Omega, \phi\}$, then we say that $\mathcal{A}$ is independent of $\mathcal{B}$ and denote this by $\mathcal{A} \perp\!\!\!\perp \mathcal{B}$.

Lemma 2.5 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{G}$ be sub σ-fields of $\mathcal{F}$ such that $\mathcal{G} \subseteq \mathcal{B}$ then $\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}$ iff $E[f|\mathcal{G}] = E[f|\mathcal{B}]$ for all bounded $\mathcal{A}$- measurable f. In particular if $\mathcal{G} = \{\phi, \Omega\}$ then $\mathcal{A} \perp\!\!\!\perp \mathcal{B}$ iff $E[f|\mathcal{B}] = E[f]$ for all bounded $\mathcal{A}$-measurable f. Here $E[\cdot|\mathcal{G}]$ denotes the conditional expectation.*

Proof. Assume $E[f|\mathcal{G}] = E[f|\mathcal{B}]$ for all bounded $\mathcal{A}$-measurable f, by the properties of conditional expectation $E[fg|\mathcal{G}] = E[E[fg|\mathcal{B}]|\mathcal{G}]$ for g bounded $\mathcal{B}$-measurable and f bounded $\mathcal{A}$-measurable, $E[fg|\mathcal{G}] = E[gE[f|\mathcal{B}]|\mathcal{G}] = E[f|\mathcal{G}]E[g|\mathcal{G}]$, where the last equality follows since $E[f|\mathcal{G}] = E[f|\mathcal{B}]$. To prove the converse, observe for $A \in \mathcal{B}$ and f bounded $\mathcal{A}$-measurable

$$\int_A f dP = \int_A E[1_A f|\mathcal{G}]dP = \int_A E[1_A|\mathcal{G}]E[f|\mathcal{G}]dP = \int_A E[f|\mathcal{G}]dP$$

Here the second equality uses conditional independence and the last equality follows from the fact that

$$E[1_A|\mathcal{G}]E[f|\mathcal{G}] = E[1_A E[f|\mathcal{G}]|\mathcal{G}].$$

□

Using the fact that $\mathcal{B} \bigvee \mathcal{G} = \sigma\{B \bigcap A; B \in \mathcal{B}, A \in \mathcal{G}\}$ and the arguments in the proof of Lemma2.5 where $\bigvee$ denotes the σ-field generated by both $\mathcal{B}$ and $\mathcal{G}$, $\sigma\{\ldots\}$ denotes the generated σ-field, we get.

Lemma 2.6 *$\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}$ implies $E[f|\mathcal{B} \bigvee \mathcal{G}] = E[f|\mathcal{G}]$ for all f bounded $\mathcal{A}$-measurable.*

Lemma 2.7 *$\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}$ implies the following* *(a) For every $\tilde{\mathcal{G}}$ satisfying $\mathcal{G} \subseteq \tilde{\mathcal{G}} \subseteq \mathcal{G} \bigvee \mathcal{B}$, $\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\tilde{\mathcal{G}}$.*

(b) if $\tilde{\mathcal{G}} \subseteq \mathcal{G} \bigvee \mathcal{B}$, $\mathcal{A} \perp\!\!\!\perp \tilde{\mathcal{G}}|\mathcal{G}$.

Proof. Lemma 2.5 and Lemma 2.6 imply $\mathcal{A} \perp\!\!\!\perp (\mathcal{G} \bigvee \mathcal{B})|\mathcal{G}$ giving (b). To obtain (a), by Lemma 2.5 and Lemma 2.6 $E[f|\mathcal{G}] = E[f|\mathcal{B} \bigvee \mathcal{G}]$ for all f bounded $\mathcal{A}$- measurable. Hence by Lemma 2.5, $\mathcal{A}$ is conditionally independent of $\mathcal{B}$ given $\tilde{\mathcal{G}} \subseteq \mathcal{G} \bigvee \mathcal{B}$ □

Corollary 2.1 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{G}$ be sub σ-fields of $\mathcal{F}$, $\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}$. Then we have the following.*

(i) If $\mathcal{A}' \subseteq \mathcal{A} \bigvee \mathcal{G}, \mathcal{B}' \subseteq \mathcal{B} \bigvee \mathcal{G}$, then $\mathcal{A}' \perp\!\!\!\perp \mathcal{B}'|\mathcal{G}$.

(ii) If $\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{A} \bigvee \mathcal{B} \bigvee \mathcal{G}$ then $\mathcal{A} \perp\!\!\!\perp \mathcal{B}|\mathcal{G}'$.

Proof. (i) can be deduced by using Lemma2.7 (b) twice.

(ii) can be deduced by using Lemma2.7 (a) twice. □

Let $M(E)$ be a set satisfying assumptions (A.1)- (A.3) and $\{X_\mu,\ \mu \in M(E)\}$ be a Gaussian centered random field indexed by $M(E)$. We assume the covariance function of $\{X_\mu, \mu \in M(E)\}$ is bilinear, i.e., $C(\mu,\nu) = E(X_\mu X_\nu)$ as a functional on $M(E) \times M(E)$ is bilinear. An immediate consequence is that if μ_1, μ_2 and $\mu \in M(E)$ such that $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$ for $\alpha_1, \alpha_2 \in \Re^1$ then $X_\mu = \alpha_1 X_{\mu_1} + \alpha_2 X_{\mu_2}$. For a subset S of $E, \overline{S}$ means the closure of S, S^c means the complement of S, whereas ∂S means the boundary of S, that is $\partial S = \overline{S} \bigcap \overline{S^c}$. We define

$$F(S) = \sigma\{X_\mu, \mu \in M(E) \text{ and } \mathrm{supp}\mu \subseteq S\}$$

and denote

$$\Sigma(S) = \bigcap_{S \subseteq O} F(O)$$

where the intersection is taken over all open sets O.

We will define three different Markov Properties of $\{X_\mu, \mu \in M(E)\}$.

Definition 2.5 (McKean[19]) *We say that $\{X_\mu, \mu \in M(E)\}$ has Markov Property I(MPI) on an open subset S of E if for every open subset O of E with $\partial S \subseteq O$*

$$F(S) \perp\!\!\!\perp F(\overline{S}^c) | F(O).$$

Definition 2.6 (Germ Field Markov Property) *We say that $\{X_\mu, \mu \in M(E)\}$ has Markov Property II(MPII) on a subset S (not necessarily open) of E if*

$$\Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(\overline{S^c}) | \Sigma(\partial S).$$

Definition 2.7 *We say that $\{X_\mu, \mu \in M(E)\}$ has Markov Property III(MPIII) for a subset S of E if*

$$F(\overline{S}) \perp\!\!\!\perp F(\overline{S^c}) | F(\partial S).$$

Notice that MPI is only defined for open sets, whereas MPII and MPIII are defined for arbitrary sets. We will explore some relationships between these Markov properties for $\{X_\mu, \mu \in M(E)\}$.

Lemma 2.8 *For $\{X_\mu, \mu \in M(E)\}$, $F(O \bigcup O') = F(O) \bigvee F(O')$, where O, O' are open subsets of E.*

Proof. We only need to prove $F(O \bigcup O') \subseteq F(O) \bigvee F(O')$. If $\mu \in M(E)$ with support of μ contained in $O \bigcup O'$, since $M(E)$ has partition of unity property (assumption (A.2)), $\mu = \mu_1 + \mu_2$, here $\mu_1, \mu_2 \in M(E)$, supp$\mu_1 \subseteq O$ and supp$\mu_2 \subseteq O'$. Then $X_\mu = X_{\mu_1} + X_{\mu_2}$ hence X_μ is measurable with respect to $F(O) \bigvee F(O')$. □

Lemma 2.9 *For $\{X_\mu, \mu \in M(E)\}$, let S be an open subset of E and O an open set containning ∂S. Then*

$$F(S) \perp\!\!\!\perp F(\overline{S}^c) | F(O) \quad iff \quad \Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(S^c) | F(O).$$

Proof. The "if" part is easy because $F(S) \subseteq \Sigma(\overline{S})$ and $F(\overline{S}^c) \subseteq \Sigma(S^c)$. To prove the converse, notice by lemma 2.8

$$F(S) \bigvee F(O) = F(S \cup O) \supseteq \Sigma(\overline{S}) \tag{2.4}$$

and

$$F(\overline{S}^c) \bigvee F(O) = F(\overline{S}^c \cup O) \supseteq \Sigma(\overline{S}). \tag{2.5}$$

Then by Corollary 2.1 $\Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(S^c) | F(O)$ for open set $O \supseteq \partial S$. □

Lemma 2.10 *For Gaussian random field $\{X_\mu, \mu \in M(E)\}$. Markov Property I on an open set S implies Markov property II (Germ field Markov Property) on S.*

Proof. By previous lemma, we know MPI for an open set S is equivalent to

$$\Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(S^c)|F(O)$$

for all open subset $O \subseteq \partial S$.

We define a direct order on all open sets in terms of inclusion then for any set $A \in \Sigma(\overline{S})$ and $B \in \Sigma(S^c)$, we have

$$P[A \bigcap B|F(O)] = P[A|F(O)]P[B|F(O)]$$

By maringale convergence theorem with net indexed σ-fields(see [23],Chpter V)

$$\lim_O P[A|F(O)]P[B|F(O)] = P[A|\Sigma(\partial S)]P[B|\Sigma(\partial S)] \tag{2.6}$$

and

$$\lim_O P[A \bigcap B|F(O)] = P[A \bigcap B|\Sigma(\partial S)] \tag{2.7}$$

Both limits are in $L_2(\Omega, \mathcal{F}, P)$. Hence we have

$$P[A \bigcap B|\Sigma(\partial S)] = P[A|\Sigma(\partial S)]P(B|\Sigma(\partial S)]$$

for all $A \in \Sigma(\overline{S})$ and $B \in \Sigma(S^c)$, this is Markov Property II on open set S. □

Lemma 2.11 *Suppose the Gaussian random field $\{X_\mu, \mu \in M(E)\}$ has Germ Field Markov Property (MPII) on an open set S and also*

$$\Sigma(\overline{S}) \bigvee \Sigma(S^c) = F(E)$$

. Then it also has MPI on S.

Proof. MPII says $\Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(S^c)|\Sigma(\partial S)$. For any open subset O of E with $\partial S \subseteq O$

$$\Sigma(\partial S) \subseteq F(O) \subseteq F(E) = \Sigma(\overline{S}) \bigvee \Sigma S^c)$$

By (ii) of Corollary 2.1 $\Sigma(\overline{S}) \perp\!\!\!\perp \Sigma(S^c)|F(O)$ which implies

$$F(S) \perp\!\!\!\perp F(\overline{S^c})|F(O).$$

□

Relationships between MPI and MPIII will be discussed in Chapter 4 in case of Gaussian random fields related to Dirichlet Space.

Chapter 3

General Results

In the previous chapter, we already set up some basic notations and lemmas. Our goal in this chapter is to explore the relationships between the Markov property of $\{X_\mu, \mu \in M(E)\}$ and its reproducing kernel Hilbert space. We are particularly interested in the Markov property of $\{X_\mu, \mu \in M(E)\}$ for some classes of sets: the class of all open sets and the class of all pre-compact open sets (a set is called pre- compact open if its closure is compact). In the second part of this chapter, we will discuss the case in which a dual process exists. To begin with we introduce some properties of Gaussian spaces([18]Appendix).

Let $(\Omega, \mathcal{F}, P)$ be a probability space,by a (centered) Gaussian space we mean a subspace of $L_2(\Omega, \mathcal{F}, P)$ such that every finite collection of elements of this subspace is Gaussian distributed with mean zero. We assume all Gaussian spaces are closed unless otherwise mentioned.

Lemma 3.1 *Let $\{X_1, ..., X_n\}$ be a subset of a (centered) Gaussian space.*

$\{X_1, ..., X_n\}$ are independent iff $E(X_iX_j) = 0, \;\; i \neq j$.

Proof. Let $\vec{u} = (u_1, ..., u_n) \in \Re^n, \;\; \vec{X} = (X_1, ...X_n)$. Then

$$\begin{aligned}\Phi_{\vec{X}}(\vec{u}) &= E(exp(i\vec{u} \cdot \vec{X})) = exp(-\frac{1}{2}E(\sum_i u_iX_i)^2) \\ &= \prod_i \Phi_{X_i}(u_i) \quad \text{if } E(X_iX_j) = 0 \text{ for} i \neq j,\end{aligned}$$

where $\Phi_{X_i}(\cdot)$ is the characteristic function of X_i $(i = 1, 2, ..., n)$. Conversely,

$$\begin{aligned}E(X_iX_j) &= E[E[X_iX_j|\sigma(X_j)]] = E[X_jE[X_i|\sigma(X_j)]] \\ &= EX_jEX_i = 0 \quad \text{if } \; i \neq j.\end{aligned}$$

□

Lemma 3.2 *Let H_1 and H_2 be subspaces of a (centered) Gaussian space H. Then $\sigma(H_1)$ and $\sigma(H_2)$ are independent iff $H_1 \perp H_2$.*

Proof. That $\sigma(H_1)$ independent of $\sigma(H_2)$ implies $H_1 \perp H_2$ follows from Lemma 3.1. To prove the converse, let $\{\xi_{1j}, j \in J_1\}$ and $\{\xi_{2j}, j \in J_2\}$ be orthogonal bases in H_1 and H_2 respectively. Then $\{\xi_{ij}, \;\; j \in J_i \;\; i = 1, 2\}$ has every finite subfamily orthogonal and hence independent by Lemma 3.1. In particular $\sigma(H_1) = \sigma\{\xi_{1j}, j \in J_1\}$ is independent of $\sigma(H_1) = \sigma\{\xi_{2j}, j \in J_2\}$.
□

Lemma 3.3 *Let H_1 be a subspace of a Gaussian space H. Then $E[Y|\sigma(H_1)] = P_{roj\,H_1}Y$ for any $Y \in H$, where $Proj_{H_1}$ is the projection operator on H_1*

Proof. Let $Y = Y_1 + Y_2$, where $Y_1 = Proj_{H_1}Y$ and $Y_2 = Y - Y_1$. Then $E(Y_1Y_2) = 0$. Hence by Lemma 2.1, Y_1 and Y_2 are independent. Then

$$E[Y|\sigma(H_1)] = Y_1 + E[Y_2|\sigma(H_1)] = Y_1 + EY_2 = Y_1.$$

□

Lemma 3.4 *Let H_0, H_1 and H_2 be subspaces of a Gaussian space H. Then the following are equivalent:*

(a) $\sigma(H_1) \perp\!\!\!\perp \sigma(H_2)|\sigma(H_0)$.

(b) $H_1' \ominus H_0 \perp H_2' \ominus H_0$ where $H_i' = H_i \vee H_0$, $(i = 1,2)$ and $H_i' \ominus H_0$ means the subspace of $L_2(\Omega, \mathcal{F}, P)$ generated by $\{\eta - Proj_{H_0}\eta, \ \ \eta \in H_i\}$.

Proof. (a) implies $\sigma(H_1') \perp\!\!\!\perp \sigma(H_2')|\sigma(H_0)$ by Corollary 2.1. Let $X_1 \in H_1'$ and $X_2 \in H_2'$. Then

$$E(X_1X_2|\sigma(H_0)) = E(X_1|\sigma(H_0))E(X_2|\sigma(H_0)) = Proj_{H_0}X_1Proj_{H_0}X_2.$$

Hence

$$E(X_iX_j) = E(X_iProj_{H_0}X_j) = E(Proj_{H_0}X_iProj_{H_0}X_j) \quad \text{for } i \neq j$$

giving $E(X_1 - Proj_{H_0}X_1)(X_2 - Proj_{H_2}) = 0$

Conversely, from Lemma 3.2 $\sigma(H_1' \ominus H_0)$ is independent of $\sigma(H_2' \ominus H_0)$. Now $\sigma(H_i') = \sigma(\sigma(H_i' \ominus H_0) \vee \sigma(H_0))$, $(i = 1,2)$. For $A_i \in \sigma(H_i' \ominus H_0)$, $B_i \in \sigma(H_0)$, $i = 1,2$,

$$E(1_{A_1 \cap B_1}1_{A_2 \cap B_2}|\sigma(H_0)) = 1_{B_1 \cap B_2}E(1_{A_1 \cap A_2}|\sigma(H_0))$$

$$
\begin{aligned}
&= 1_{B_1 \cap B_2} P(A_1 \cap A_2) \\
&= 1_{B_1} P(A_1) 1_{B_2} P(A_2) \\
&= E(1_{A_1 \cap B_1} | \sigma(H_0)) E(1_{A_1 \cap B_1} | \sigma(H_0)).
\end{aligned}
$$

This gives $\sigma(H_1') \perp\!\!\!\perp \sigma(H_2') | \sigma(H_0)$ which gives $\sigma(H_1) \perp\!\!\!\perp \sigma(H_2) | \sigma(H_0)$. □

Definition 3.1 *Let H_0, H_1, H_2 be subspaces of a Gaussian space. We say that the space H_0 splits H_1 and H_2 if $\sigma(H_1) \perp\!\!\!\perp \sigma(H_2) | \sigma(H_0)$.*

Lemma 3.5 *Suppose H_0, H_1, H_2 are subspaces of a Gaussian space such that $H_0 \subseteq H_1 \cap H_2$. Then H_0 splits H_1 and H_2 iff $H_1 \cap H_2 = H_0$ and $H_1^\perp \perp H_2^\perp$ where $H_i^\perp$ are the orthogonal space of H_i in $H_1 \vee H_2, (i = 1, 2)$*

Proof. Under the assumption $H_0 \subseteq H_1 \cap H_2$ and from Lemma 3.4(b) H_0 splits H_1 and H_2 iff

$$H_1 \vee H_2 = (H_1 \ominus H_0) \oplus H_0 \oplus (H_2 \ominus H_0)$$

where $\oplus$ denotes the orthogonal sum of the spaces. Then obviously this is equivalent to $H_1 \cap H_2 = H_0$ and $H_1^\perp \perp H_2^\perp$. □

Corollary 3.1 *Let H_1 and H_2 be two subspaces of a Gaussian space, then $\sigma(H_1) \perp\!\!\!\perp \sigma(H_2) | \sigma(H_1 \cap H_2)$ iff $Proj_{H_1} Proj_{H_2} = P_{rojH_1 \cap H_2}$.*

Lemma 3.6 *Let H be a Gaussian subspace. Then $L_2(\Omega, \mathcal{F}_H, P)$ is generated by $M = \{expX, X \in H\}$, where $\mathcal{F}_H = \sigma\{H\}$.*

Proof. Clearly $M \subseteq L_2(\Omega, \mathcal{F}_H, P)$. Let Y be an element in $L_2(\Omega, \mathcal{F}_H, P)$ such that $EYe^X = 0$ for all $X \in H$. then $EYe^{tX} = 0$ for all $X \in H$. Since $f(t) = EYe^{tX}$ is analytic in t, if it is zero for all real t, it also vanishes for all complex t. In particular $EYe^{iX} = 0$ for all $X \in H$. Put $Z \in \mathcal{W}$ if Z is bounded and $EYZ = 0$. Then $\mathcal{W}$ contains the family $\{e^{iX},\ X \in H\}$ which is closed under multiplication. $\mathcal{W}$ is also a linear space. It contains with each function the complex conjugate of this function and with each uniformly bounded convergent sequence the limit of this sequence. This implies (see P.A. Meyer(1966),Chapter1,Theorem 2) that $\mathcal{W}$ contains all bounded function measurable with respect to the σ-algebra generated by $\{e^{iX}, X \in H\}$ which is the same as $\sigma\{H\}$. Y is orthogonal to all bounded $\mathcal{F}_H$ measurable functions, hence $Y = 0$. □

Corollary 3.2 *Let $\{X_t, t \in T\}$ be a Gaussian process. Then the algebra generated by polynomials in $\{X_t, t \in T\}$ is dense in $L_2(\Omega, \sigma(H), P)$, where H is the subspace of $L_2(\Omega, \mathcal{F}_H, P)$ generated by $\{X_t, t \in T\}$.*

Recall that we assume $M(E)$ is a vector space satisfying assumptions (A.1)-(A.3). $\{X_\mu, \mu \in M(E)\}$ is a centered Gaussian process with bilinear covariance $C(\mu, \nu) = E(X_\mu X_\nu),\ \ \mu, \nu \in M(E)$. Let $K(C)$ be the reproducing kernel Hilbert space corresponding to the covariance $C(\mu, \nu)$ and $H(X)$ be the closed linear subspace of $L_2(\Omega, \mathcal{F}, P)$ generated by $\{X_\mu, \mu \in M(E)\}$. The map Π is the isometry between $K(C)$ and $H(X)$ with $\Pi C(\cdot, \mu) = X_\mu$. For any subset S of E, define $H(S)$ the subspace of L_2 generated by all X_μ with $\text{supp}\mu \subseteq S$ and $K(S)$ is the corresponding subspace of $K(C)$ under the map

Π, namely $K(S) = \Pi^{-1}H(S)$. By Lemmas 3.1 and 3.5 we have the following lemma:

Lemma 3.7 *The Gaussian process $\{X_\mu, \mu \in M(E)\}$ has Markov Property I on an open set S of E iff one of the following holds:*

(i) $E(\eta_1 - Proj_{H(O)}\eta_1)(\eta_2 - Proj_{H(O)}\eta_2) = 0$ for any $\eta_1 \in H(S)$, $\eta_2 \in H(\overline{S}^c)$ and any open set O containing ∂S.

(ii) $H(S \cup O) \cap H(\overline{S}^c \cup O) = H(O)$ and

$$H(S \cup O)^{\perp} \perp H(\overline{S}^c \cup O)^{\perp}$$

for all open set O containg ∂S, where $H(S \cup O)^{\perp}$ and $H(S^c \cup O)^{\perp}$ are orthogonal spaces of $H(S \cup O)$ and $H(S^c \cup O)$ in $H(X)$ respectively.

Next theorem will give the relationship between the Markov Property I of $\{X_\mu, \mu \in M(E)\}$ for all open sets and the reproducing kernel Hilbert space $K(C)$ of covariance function $C(\mu, \nu)$.

Theorem 3.1 *Let $\{X_\mu, \mu \in M(E)\}$ be a Gaussian process such that $M(E)$ satisfies assumptions (A.1)-(A.3) in Chapter 2 and covariance function $C(\mu, \nu) = E(X_\mu X_\nu)$ bilinear in μ and ν. Let $K(C)$ be the RKHS of $C(\mu, \nu)$. Then $\{X_\mu, \mu \in M(E)\}$ has the Markov Property I for all open subsets of E iff the following (a) and (b) hold.*

(a) If $f_1, f_2 \in K(C)$ with $suppf_1 \cap suppf_2 = \phi$ then $(f_1, f_2)_{K(C)} = 0$, where $(f_1, f_2)_{K(C)}$ means the inner product of f_1 and f_2 in $K(C)$.

(b) If $f \in K(C)$ and $f = f_1 + f_2$ where both f_1 and f_2 are linear functionals of $M(E)$ with $(suppf_1) \cap (suppf_2) = \phi$, then $f_1, f_2 \in K(C)$.

Remark: Since each element in $K(C)$ is also a linear functional on $M(E)$, for every $f \in K(C)$, supp$f$ is well defined by (i) of Lemma 2.2.

Proof. We know that $\{X_\mu, \mu \in M(E)\}$ has Markov Property I for all open sets of E iff

$$H(S \cup O) \quad \cap \quad H(\overline{S^c} \cup O) = H(O) \tag{3.1}$$

$$(S \cup O)^\perp \quad \perp \quad H(\overline{S^c} \cup O)^\perp \tag{3.2}$$

holds for all open set S and open set O containing ∂S. We will prove that (a) and (b) of Theorem 3.1 is equivalent to (3.1) and (3.2). We separate our proof into two parts.

<u>**Sufficiency:**</u> Suppose (a) and (b) of Theorem 3.1 hold, we need to show (3.1) and (3.2). To verify (3.1) it is enough to prove $H(O) \supseteq H(S \cup O) \cap H(\overline{S^c} \cup O)$ this is equivalent to

$$H(O)^\perp \subseteq H(S \cup O)^\perp \vee H(\overline{S^c} \cup O)^\perp \tag{3.3}$$

which is the same as

$$K(O)^\perp \subseteq K(S \cup O)^\perp \vee K(\overline{S^c} \cup O)^\perp. \tag{3.4}$$

Let $f \in K(O)^\perp$, then $f = \Pi^{-1}Y$ for some $Y \in H(O)^\perp$. Then if $X_\mu \in H(O)$

$$f(\mu) = (f, C(\cdot, \mu))_{K(C)} = E(YX_\mu) = 0.$$

In particular $f(\mu) = 0$ if supp$\mu \subseteq O$. Hence supp$f \subseteq O^c$(see definiton of supp$f$ in chapter 2) observe the following facts: $S \cap O^c = \overline{S} \cap O^c$ and $\overline{S^c} \cap O^c = \overline{\overline{S^c}} \cap O^c$ because $\partial(\overline{S^c}) \subseteq \partial S = \partial \overline{S} \subseteq O$. So $S \cap O^c$ and $\overline{S^c} \cap O^c$ are two disjoint closed sets, furthermore their union is O^c. By our assumption

(A.3) $f(\mu) = f_1(\mu) + f_2(\mu)$ with f_1, f_2 being linear functionals of $M(E)$ and $\text{supp} f_1 \subseteq S \cap O^c$, $\text{supp} f_2 \subseteq \overline{S^c} \cap O^c$. By (b) of Theorem 3.1 f_1 and f_2 belong to $K(C)$. Then

$$\begin{aligned} f_1 \subseteq K((\overline{S} \cap O^c)^c)^\perp &= K(\overline{S^c} \cup O)^\perp, \\ f_2 \subseteq K((\overline{S^c} \cap O^c)^c)^\perp &= K(\overline{S} \cup O)^\perp = K(S \cup O)^\perp. \end{aligned}$$

To prove (3.2), we will show the equivalence condition as follows

$$K(S \cup O)^\perp \perp K(\overline{S^c} \cup O)^\perp. \tag{3.5}$$

This is true if we can show that

$$K(S \cup O)^\perp \subseteq \overline{\text{span}}\{f : f \in K(C), \text{supp} f \subseteq \overline{S^c}\} \tag{3.6}$$

and

$$K(\overline{S^c} \cup O)^\perp \subseteq \overline{\text{span}}\{f : f \in K(C), \text{supp} f \subseteq S\}. \tag{3.7}$$

But if $f \in K(S \cup O)^\perp$, then for every $\mu \in M(E)$ with $\text{supp}\mu \subseteq S \cup O$, $f(\mu) = (f, C(\mu, \cdot))_{K(C)} = 0$, hence $\text{supp} f \subseteq (S \cup O)^c = (\overline{S} \cup O)^c = \overline{S^c} \cap O^c \subseteq \overline{S^c}$. Similarly if $f \in K(\overline{S^c} \cup O)^\perp$ then

$$\text{supp} f \subseteq (\overline{S^c} \cup O)^c = \overline{S} \cap O^c = S \cap O^c \subseteq S.$$

<u>**Necessity:**</u> Suppose (3.1) and (3.2) hold. Let f_1 and f_2 be in $K(C)$ with disjoint support, then there exists an open set S such that $\text{supp} f_1 \subseteq S$ and $\text{supp} f_2 \subseteq \overline{S^c}$. Take $O = [(\text{supp} f_1) \cup (\text{supp} f_2)]^c$ then O is an open set containg ∂S. Since $S \cup O \subseteq (\text{supp} f_2)^c$ and $\overline{S^c} \cup O \subseteq (\text{supp} f_1)^c$, so $f_1 \in K(\overline{S^c} \cup O)^\perp$ and $f_2 \in K(S \cup O)^\perp$. Hence $(f_1, f_2)_{K(C)} = 0$ by (3.5).

Assume $f \in K(C)$ and $f = f_1 + f_2$ where f_1 and f_2 are linear functionals of $M(E)$ with suppf_1 and suppf_2 being disjoint, we choose an open set S such that supp$f_1 \subseteq S$ and supp$f_2 \subseteq \overline{S}^c$. Let $O = (\text{supp}f_1 \cup \text{supp}f_2)^c$. By(3.1) and (3.2) we have

$$H(X) = H(S \cup O)^{\perp} \oplus H(\overline{S}^c \cup O)^{\perp} \oplus H(O)$$

which is the same as

$$K(C) = K(S \cup O)^{\perp} \oplus K(\overline{S}^c \cup O)^{\perp} \oplus K(O).$$

Note that supp$f \subseteq$ supp$f_1 \cup$supp$f_2 = O^c$ so $f \in K(O)^{\perp}$, hence $f = f_1' + f_2'$ with $f_1' \in K(\overline{S}^c \cup O)^{\perp}$ and $f_2' \in K(S \cup O)^{\perp}$, then supp$f_1' \subseteq (\overline{S}^c \cup O)^c = \overline{S} \cap O^c = \overline{S} \cap (\text{supp}f_1 \cup \text{supp}f_2) = (\overline{S} \cap \text{supp}f_1) \cup (\overline{S} \cap \text{supp}f_2) = \overline{S} \cap \text{supp}f_1 \subseteq \text{supp}f_1$. Similarly supp$f_2' \subseteq$ suppf_2. Now $f = f_1 + f_2 = f_1' + f_2'$, $f_1 - f_1' = f_2' - f_2$ with supp$(f_1 - f_1') \cap$supp$(f_2' - f_2) = \phi$, so supp$(f_1 - f_1') = \phi$ which implies $f_1 = f_1'$ and $f_2 = f_2'$ by virtue of (b) of Lemma 2.2. Hence $f_1, f_2 \in K(C)$. □

We can also consider the Markov Property I of $\{X_\mu, \mu \in M(E)\}$ for all pre-compact open sets, then we have the following theorem similar to Theorem 3.1.

Theorem 3.2 *Let $\{X_\mu, \mu \in M(E)\}$ be the same as in Theorem 3.1 then it has Markov Property I for all pre-compact open sets iff the following (a) and (b) hold:*

(a) For any f_1 and $f_2 \in K(C)$ with supp$f_1 \cap$ supp$f_2 = \phi$ and at least one of the supp$f_i (i = 1, 2)$ is compact, then $(f_1, f_2)_{K(C)} = 0$

(b) If $f \in K(C)$ and $f = f_1 + f_2$ with $suppf_1 \cap suppf_2 = \phi$ and at least one of the $suppf_i (i = 1, 2)$ is compact, where f_1 and f_2 are linear functionals of $M(E)$, then f_1 and f_2 belong to $K(C)$.

Proof. All arguments in the proof of Theorem 3.1 go through, except when one of $\text{supp} f_i (i = 1, 2)$ is compact. We can choose a pre-compact open set S to cover the compact one and $\overline{S}^c$ to cover another. □

Now we extend the concept of dual process introduced in [12] for the processes $\{X_\mu, \mu \in M(E)\}$.

For the separable locally compact Hausdorff space E, we denote by $C_0(E)$ the space of all continuous functions on E with compact support. Let $G(E)$ be a subset of $C_0(E)$. ($G(E)$ need not be a subspace of $C_0(E)$). Let $\{\hat{X}_g, g \in G(E)\}$ be a Gaussian process defined on the same probability space as $\{X_\mu, \mu \in M(E)\}$. Then we have,

Definition 3.2 *The Gaussian field $\{\hat{X}_g, g \in G(E)\}$ is called a dual process of $\{X_\mu, \mu \in M(E)\}$ if*

(i) $H(X) = H(\hat{X})$.

(ii) $E(\hat{X}_g X_\mu) = \int_E g d\mu$ for all $g \in G(E)$ and $\mu \in M(E)$, where $H(X)$ and $H(\hat{X})$ are the subspaces of $L_2(\Omega, \mathcal{F}, P)$ generated by $\{X_\mu, \mu \in M(E)\}$ and $\{\hat{X}_g, g \in G(E)\}$ respectively.

Remark: (i) we denote by g with or without sub(supper)scripts as elements in $G(E)$ and f with or without sub(supper)scropts as the elements in $K(C)$.

(ii)For any $g \in G(E)$ we denote by $f_g(\cdot)$ as $f_g(\mu) = \int_E g d\mu,\ \mu \in M(E)$. Since $\hat{X}_g \in H(X), f_g(\mu) = E(\hat{X}_g X_\mu)$ and $f_g(\cdot) \in K(C)$.

For any open subset D of E, we define subspaces $M(D)$ and $\hat{M}(D)$ of $K(C)$ as following:

$$M(D) = \overline{\textbf{span}}\{f,\ f \in K(C)\,\text{supp} f \subseteq D\} \tag{3.8}$$

$$\hat{M}(D) = \overline{\textbf{span}}\{f_g(\cdot),\ g \in G(E)\,\text{supp} g \subseteq D\} \tag{3.9}$$

where for $g \in G(E)$, supp$g$ is defined in the usual sense, namely supp$g$ = closure of $\{e :\ e \in E, g(e) \neq 0\}$.

Remark: We remark that $K(D) = \Pi^{-1}(H(D))$ and is distinct from $M(D)$.

Lemma 3.8 *For any open set D, $\hat{M}(D) \subseteq M(D)$.*

Proof. Let $g \in G(E)$ such that supp$G \subseteq D$ then $g(e) = 0$ if $e \in$ (supp$g)^c$. Then $f_g(\mu) = \int g d\mu = 0$ if supp$\mu \subseteq$ (supp$g)^c$. That supp$f_g(\cdot) \subseteq$ (supp$g)^c \subseteq D$ impllies $f_g(\cdot) \in M(D)$ □

Definition 3.3 *we say that $G(E)$ has the partition of unity property if for every $g \in G(E)$, $O_1, ...O_n$ are open sets covering suppg, then $g = \sum_{i=1}^n g_i$ with $g_i \in G(E)$ and supp$g_i \subseteq O_i$, $(i = 1, ...n)$.*

Lemma 3.9 *If $G(E)$ has the partition of unity property, then*

$$\hat{M}(D_1 \cup D_2) = \hat{M}(D_1) \vee \hat{M}(D_2)$$

for every two open sets D_1 and D_2.

Proof. Let $g \in G(E)$ with supp$g \subseteq D_1 \cup D_2$, then $g = g_1 + g_2$ with $g_i \in G(E)$ and supp$g_i \subseteq D_i$ $(i = 1,2)$, hence $f_g(\mu) = f_{g_1}(\mu) + f_{g_2}(\mu)$ for $\mu \in M(E)$. But $f_{g_i}(\cdot) \in \hat{M}(D_i)$, $(i = 1,2)$, hence $f_g(\cdot) \in \hat{M}(D_1) \vee \hat{M}(D_2)$. This gives $\hat{M}(D_1 \cup D_2) = \hat{M}(D_1) \vee \hat{M}(D_2)$ □

Theorem 3.3 *Let $\{\hat{X}_g, g \in G(E)\}$ be a dual process of $\{X_\mu, \mu \in M(E)\}$. $G(E)$ has the partition of unity property and $\hat{M}(D) = M(D)$ for all open sets D. Then (a) of Theorem3.1 implies (b) of Theorem3.1.*

Proof. If (a) of Theorem3.1 holds then $M(D_1) \perp M(D_2)$ for every two disjoint open sets D_1 and D_2. By Lemma 3.9

$$\hat{M}(D_1 \cup D_2) = \hat{M}(D_1) \vee \hat{M}(D_2) = M(D_1) \oplus M(D_2)$$

together with $\hat{M}(D_1 \cup D_2) = M(D_1 \cup D_2)$ gives

$$M(D_1 \cup D_2) = M(D_1) \oplus M(D_2)$$

for every two disjoint open sets D_1 and D_2. Let $f \in K(C)$ with $f = f_1 + f_2$ where f_1 and f_2 are linear functionals of $M(E)$ with supp$f_1 \cap$supp$f_2 = \phi$, then we can choose two disjoint open sets D_1 and D_2 such that supp$f_i \subseteq D_i$ $(i = 1,2)$ and $\overline{D_1} \cap \overline{D_2} = \phi$. Let $D = D_1 \cup D_2$, then supp$f \subseteq D_1 \cup D_2$, $f \in M(D)$. We can write

$$f = Proj_{M(D)} f = Proj_{M(D_1)} f + P_{roj M(D_2)} f = f_1' + f_2' \quad \text{say,}$$

f_1' as an element in $M(D_1)$ is a limit of sequence $\tilde{f}_n$ in $K(C)$ with supp$\tilde{f}_n \subseteq D_1$. Then $f_1'(\mu) = \lim_n \tilde{f}_n(\mu) = 0$ for any $\mu \in M(E)$ with supp$\mu \subseteq \overline{D}_1^c$. So supp$f_1' \subseteq \overline{D}_1$. Similarly supp$f_2' \subseteq \overline{D}_2$. Now $f = f_1 + f_2 = f_1' + f_2'$ gives $f_1 - f_1' = f_2' - f_2$. But supp$(f_1 - f_1') \subseteq \overline{D}_1$ and supp$(f_2 - f_2') \subseteq \overline{D}_2$, so $\overline{D}_1 \cap \overline{D}_2 = \phi$ implies supp$(f_1 - f_1') = (f_2' - f_2) = \phi$. Hence $f_i = f_i' \in K(C)$ $i = 1, 2$. □

Theorem 3.4 *Let $\{\hat{X}_g, g \in G(E)\}$ be a dual process of $\{X_\mu, \mu \in M(E)\}$, $G(E)$ has the partition of unity property and $\hat{M}(D) = M(D)$ for all open sets D. Then (a) of Theorem3.2 implies (b) of Theorem3.2.*

Proof. The arguments are similar to the proof of Theorem 3.3 and hence omitted. □

The following corollary is an immediate consequence of Theorems 3.1, 3.2, 3.3 and 3.4.

Corollary 3.3 *Let $\{\hat{X}_g, g \in G(E)\}$ be a dual process of $\{X_\mu, \mu \in M(E)\}$, $G(E)$ has the partition of unity property and $\hat{M}(D) = M(D)$ for all open sets D. Then*

(a)$\{X_\mu, \mu \in M(E)\}$ has Markov Property I for all open sets iff $(f_1, f_2)_{K(C)} = 0$ for any $f_1, f_2 \in K(C)$ with supp$f_1 \cap$supp$f_2 = \phi$.

(b)$\{X_\mu, \mu \in M(E)\}$ has Markov Property I for all pre- compact open sets iff $(f_1, f_2)_{K(C)} = 0$ for any $f_1, f_2 \in K(C)$ with supp$f_1 \cap$supp$f_2 = \phi$ and at least one of supp$f_i (i = 1, 2)$ being compact.

If we impose on $G(E)$ the following assumption which is stroger than the partition of unity property, then we can explore more properties about $\{X_\mu, \mu \in M(E)\}$ and its dual process $\{\hat{X}_g, g \in G(E)\}$.

Assumption 3.1 *For any fixed $g \in G(E)$ there exists a positive number L_g such that for any open covers $O_1, ..., O_n$ of supp$g$ there exist $g_i \in G(E)$ with supp$g_i \subseteq O_i$, $i = 1, ..., n$. $g = \sum_{i=1}^{n} g_i$ and*

$$E|\hat{X}_{g_i}|^2 \leq L_g, \quad i = 1, ..., n.$$

Theorem 3.5 *Let $\{\hat{X}_g, g \in G(E)\}$ be a dual process of $\{X_\mu, \mu \in M(E)\}$ and $G(E)$ satisfy assumption3.1, then the following are equivalent,*

(a) $(f_1, f_2)_{K(C)} = 0$ for any $f_1, f_2 \in K(C)$ with supp$f_1 \cap$ supp$f_2 = \phi$.

(b) $\Pi f_g(\cdot) \in H(D)$ for every $g \in G(E)$ such that supp$g \subseteq D$, where D is open and Π is the isometry map between $K(C)$ and $H(X)$.

(c) $\hat{M}(D_1) \perp \hat{M}(D_2)$ for any disjoint open sets D_1 and D_2 and $M(D) = \hat{M}(D)$ for every open set D.

Proof. (a)$\Longrightarrow$(b). For open set D define P_{rojD} the projection of $K(C) \rightarrow K(D)$, where $K(D) = \Pi^{-1}H(D)$. Let $g \in G(E)$ with support of g contained in D, then for $\mu \in M(E)$ with supp$\mu \subseteq D$.

$$f_g(\mu) - Proj_D f_g(\mu) = (f_g(\cdot), C(\mu, \cdot))_{K(C)} - (Proj_D f_g(\cdot), C(\mu, \cdot))_{K(C)}.$$

Since $C(\mu, \cdot) = \Pi^{-1} X_\mu \in K(D)$ hence

$$(Proj_D f_g(\cdot), C(\mu, \cdot))_{K(C)} = (f_g(\cdot), C(\mu, \cdot))_{K(C)} = f_g(\mu).$$

So $f_g - Proj_D f_g(\mu) = 0$ for all $\mu \in M(E)$ with supp$\mu \subseteq D$. This implies that supp$(f_g(\cdot) - P_{rojD} f_g(\cdot)) \subseteq D^c$. We know that supp$f_g(\cdot) \subseteq$supp$g \subseteq D$. From (a) of the theorem $(f_g(\cdot) - Proj_D f_g(\cdot), f_g(\cdot))_{K(C)} = 0$ this implies $f_g(\cdot) = Proj_D f_g(\cdot) \in \Pi^{-1} H(D)$.

(b) $\Longrightarrow$(c) Let $g_1, g_2 \in G(E)$ such that $\text{supp} g_1 \cap \text{supp} g_2 = \phi$ choose open set D such that $\text{supp} g_1 \subseteq D$ and $\text{supp} g_2 \subseteq \overline{D}^c$ then $f_{g_1}(\cdot) \in \hat{M}(D) \subseteq M(D)$ and from (b) $f_{g_2}(\cdot) \in K(\overline{D}^c)$. It is easy to check that $M(D) \perp K(\overline{D}^c)$. Then

$$(f_{g_1}(\cdot), f_{g_2}(\cdot))_{K(C)} = 0, \text{ namely } E(\hat{X}_{g_1}\hat{X}_{g_2}) = 0.$$

To show $M(D) = \hat{M}(D)$ for open set D. We need to use assumption 3.1. Let $f \in M(D)$, we denote $\tilde{f} = f - Proj_{\hat{M}(D)} f \in M(D)$. Then there exists $f_n \in M(D)$ with $\text{supp} f_n \subseteq D$ such that $f_n \longrightarrow \tilde{f}$ in $K(C)$ as $n \to \infty$. Denote $D_n = (\text{supp} f_n)^c$,then for any n $\{D, D_n\}$ is an open cover of space E. So for any $g \in G(E)$ by Assumption3.1 $g = g_1^n + g_2^n$ with $g_i^n \in G(E)$ and $\text{supp} g_1^n \subseteq D$,$\text{supp} g_2^n \subseteq D_n$, furthermore $(f_{g_i^n}, f_{g_i^n})_{K(C)} \leq L_g$, where L_g is a constant only depending on g. Then

$$(\tilde{f}, f_g(\cdot))_{K(C)} = \lim_n (f_n, f_g(\cdot))_{K(C)} = \lim_n (f_n, f_{g_1^n}(\cdot))_{K(C)} + \lim_n (f_n, f_{g_2^n}(\cdot))_{K(C)}$$

But $(f_n, f_{g_2^n}(\cdot))_{K(C)} = 0$ because $f_n \perp K(D_n) = \Pi^{-1} H(D_n)$ and $f_{g_2^n}(\cdot) \in K(D_n)$ by (b). So $(\tilde{f}, f_g(\cdot))_{K(C)} = \lim_n (f_n, f_{g_1^n}(\cdot))_{K(C)} = \lim_n (f_n - \tilde{f}, f_{g_1^n}(\cdot))_{K(C)}$ The last equality holds because $\tilde{f} \perp \hat{M}(D)$ and $f_{g_1^n}(\cdot) \in \hat{M}(D)$. Then

$$|(f_n - \tilde{f}, f_{g_1^n}(\cdot))_{K(C)}| \leq \|f_n - \tilde{f}\|_{K(C)} \|f_{g_1^n}\|_{K(C)} \leq L_g \|f_n - \tilde{f}\|_{K(C)} \to 0 \text{ as } n \to \infty.$$

So $(\tilde{f}, f_g(\cdot))_{K(C)} = 0$ for all $g \in G(E)$. Since $\{f_g(\cdot), g \in G(E)\}$ is dense in $K(C) \Rightarrow \tilde{f} = 0$ then $f = Proj_{\hat{M}(D)} f \in \hat{M}(D)$.

(c)$\Longrightarrow$(a) Let $f_1, f_2 \in K(C)$ and $\text{supp} f_1 \cap \text{supp} f_2 = \phi$. Choose open set D such that $\text{supp} f_1 \subseteq D$ and $\text{supp} f_2 \subseteq \overline{D}^c$. Then $f_1 \in M(D) = \hat{M}(D), f_2 \in M(\overline{D}^c) = \hat{M}(\overline{D}^c)$. Since $\hat{M}(D) \perp \hat{M}(\overline{D}^c), (f_1, f_2)_{K(C)} = 0$. □

Chapter 4

Gaussian Processes Related to Dirichlet Forms

In this chapter, we show that the probelm considered by Röchner[25] is a special case of Corollary 3.3. In [25], Röchner considered the Gaussian random field induced from a Dirichlet form and prove that it has the Markov property III for all sets iff the underlying Direchlet form has local property. He shows that free random field studied by Nelson [22], and in some cases the "generalized random fields" studied by Kallianpur and Mandrekar[12], Molchan[21] and Rozanov[26] can be handled within his framwork. First we introduce the Dirichlet form and related potential theory. The notations and terminologies are from the basic book by Fukushima[10]. For details and further information, the reader is referred to [10].

Let E be a separable locally compact Hausdorff space and m be a positive Radon measure on E with supp$(m) = E$. According to [10](p.35), a pair $(\mathcal{F}_e, \mathcal{E})$ is called a regular extended (transient) Dirichlet space with reference measure m if the following conditions are satisfied:

$(\mathcal{F}_e.1)$ $\mathcal{F}_e$ is a real Hilbert space with inner product $\mathcal{E}$.

$(\mathcal{F}_e.2)$ There exists an m-integrable bounded function g, strictly positive $a.e.m.$ Such that $\mathcal{F}_e \subseteq L_1(E, \nu_g)$ and

$$\int |u| d\nu_g = \int |u| g dm \leq \sqrt{\mathcal{E}(u,u)} \quad \text{for every } u \in \mathcal{F}_e$$

where ν denotes the measure with density g, i.e. $d\nu_g = gdm$.

$(\mathcal{F}_e.3)$ $\mathcal{F}_e \cap C_0(E)$ is dense both in $(\mathcal{F}_e, \mathcal{E})$ and in $(C_0(E), \| \ \|_\infty)$, where $C_0(E)$ denotes the set of all real continuous functions on E with compact support and $\|f\|_\infty = \text{supp}_{x \in E}|f(x)|$ for $f \in C_0(E)$.

Let us say that funciton v is a normal contraction of a function u if $|v(x) - v(y)| \leq |u(x) - u(y)|$ and $|v(x)| \leq |u(x)|$ for all $x, y \in E$. We assume:

$(\mathcal{F}_e.4)$ Every normal contraction operates on $(\mathcal{F}_e.\mathcal{E})$. i.e. if $u \in \mathcal{F}_e$ and v is a normal contraction of u then $v \in \mathcal{F}_e$ and $\mathcal{E}(v,v) \leq \mathcal{E}(u,u)$.

The following lemma gives slight extension of ([10],p.25).

Lemma 4.1 *A regular extended (transient) Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ has the following properties:*

(i) If $u, v \in \mathcal{F}_e$, then $u \vee v, u \wedge v, u \wedge 1, u^+, u^- \in \mathcal{F}_e$. Also $u, v \in \mathcal{F}_e \cap L_\infty(E,\ m)$ implies $uv \in \mathcal{F}_e$.

(ii) Let $\{u_n, u\} \subseteq \mathcal{F}_e$ such that $u_n \to u$ in $(\mathcal{F}_e, \mathcal{E})$ as $n \to \infty$. let $\varphi(t)$ be a real function such that $\varphi(0) = 0$, $|\varphi(t) - \varphi(t')| \leq |t - t'|$ for $t, t' \in \Re$. Then $\varphi(u_n), \varphi(u) \in \mathcal{F}_e$ and $\varphi(u_n) \to \varphi(u)$ weakly with respect to $\mathcal{E}$. In addition, if $\varphi(u) = u$ then the convergence is strong with respect to the norm given by $\mathcal{E}$.

(iii) For any $u \in C_0(E)$ there exist $u_n \in \mathcal{F}_e \cap C_0(E)$, $n = 1,2,...$, such that supp$(u_n) \subseteq \{x \in E : u(x) \neq 0\}$ $n = 1,2,...$, and u_n converges to u uniformly.

Proof. The arguments are similar to the proof of Theorem 1.4.2 and Lemma 1.4.2 of [10](pp.25-26). □

Remark: For the connection between Dirichlet space on $L^2(E,m)$ and extended Dirichlet space see ([10],p.35)

Definition 4.1 *We say that $(\mathcal{F}_e, \mathcal{E})$ has local property if $\mathcal{E}(u,v) = 0$ for every $u,v \in \mathcal{F}_e \cap L_2(E,m)$ such that supp$(u \cdot dm)$ and supp$(v \cdot dm)$ are compact and disjoint.*

Definition 4.2 *A signed Radon measure μ on E is a measure of bounded energy if there exists a constant $c > 0$ such that*

$$\int |u| d|\mu| \leq c\sqrt{\mathcal{E}(u,u)} \; for \; every \; u \in \mathcal{F}_e \cap C_0(E).$$

We denote by M_e all measures of bounded energy, let

$$M(E) = \{\mu \in M_e; \text{ supp}\mu \text{ is compact}\} \tag{4.1}$$

$$M_e^+ = \{\mu; \; \mu \in M_e, \; \mu \geq 0\}$$

$$M^+(E) = M_e^+ \cap M(E)$$

Notice that $M(E)$ is a vector space and also if $\mu \in M(E)$ then $1_A\mu \in M(E)$ for any Borel set $A \subseteq E$. By Example 2.2 $M(E)$ satisfies assumptions(A.1)–(A.3).

For any $A \subseteq E$ we define

$$\mathcal{L}_A = \{u : u \in \mathcal{F}_e, u \geq 1 \ a.e.m \text{ on } A\}.$$

Definition 4.3 *We define the 0-order capacity $Cap_0(O)$ of an open set $O \subseteq E$ as*

$$Cap_0(O) = \begin{cases} \inf_{u \in \mathcal{L}_O} \mathcal{E}(u,u) & \mathcal{L}_O \neq \phi \\ \infty & \mathcal{L}_O = \phi \end{cases}$$

and for an arbitrary set $A \subseteq E$

$$Cap_0(A) = \inf_{O \supseteq A} Cap_0(O) \quad \text{for all opene set } O \tag{4.2}$$

Lemma 4.2 *The capacity defined by (4.2) is a Choquet capacity i.e.*

(i) $A \subseteq B \Rightarrow Cap_0(A) \subseteq Cap_0(B)$.

(ii) $A_n \uparrow \Rightarrow Cap_0(\bigcup_n A_n) = sup_n \, Cap_0(A_n)$.

(iii) A_n compact, $A_n \downarrow \Rightarrow Cap_0(\cap A_n) = inf_n \, Cap_0(A_n)$.

Proof. The proof is similar to that of Theorem 3.1.1 in [10]. □

Definition 4.4 *A statement depending on $x \in S \subseteq E$ is said to hold' quasi-everywhere' (for short q.e.) on S if there exists a set $N \subseteq S$ of zero capacity such that the statement is true for every $x \in S \backslash N$.*

Definition 4.5 *Let $E_\Delta = E \cup \Delta$ be the one-point compactificaiton of E. A function u defined on E is called quasi-continuous if there exists for any $\epsilon > 0$ an open set $G \subseteq E$ such that $Cap_0(G) < \epsilon$ and $u|_{E \backslash G}$ is continuous. Here $u|_{E \backslash G}$ denotes the restriction of u to $E \backslash G$. If we replace $u|_{E \backslash G}$ by $u|_{(E \cup \Delta) \backslash G}$ in the above definition then u is called quasi-continuous in the restricted sense. Here $u|_{(E \cup \Delta) \backslash G}$ denotes the restriction of u to $(E \cup \Delta) \backslash G$ with $u(\Delta) = 0$.*

Definition 4.6 *Given two functions u and v on E, v is said to be a q.e. modification of u in the restricted sense if v is quasi-continuous in the restricted sense and $v = u$ a.e m.*

Lemma 4.3 *Every element $u \in \mathcal{F}_e$ admits a q.e. modification in the restricted sense denoted by $\tilde{u}$.*

Proof. The proof is similar to that of Theorem 3.1.3 in [10]. □

Using ([10],p71) we get that for any $\mu \in M_{\mathcal{E}}^{+}$ there exists a unique element $U\mu \in \mathcal{F}_e$ such that

$$\mathcal{E}(U\mu, v) = \int \tilde{v} d\mu \quad \forall v \in \mathcal{F}_e \tag{4.3}$$

Here $\tilde{v}$ denotes any quasi-continuous modification of v in the restricted sense. Define the map

$$\begin{aligned} U: \; M_{\mathcal{E}} &\rightarrow \mathcal{F}_e \\ U\mu &= U\mu^{+} - U\mu^{-} \end{aligned}$$

where μ^{+} and μ^{-} are the positive and negative parts of μ in the Jordan decomposition. $U\mu$ is called potential of μ

Lemma 4.4 *Let $(\mathcal{F}_e, \mathcal{E})$ be a regular extended (transient) Dirichlet space, then the linear manifolds $\{U\mu - U\nu : \; \mu, \nu \in M_{\mathcal{E}}^{+}\}$ and $\{U\mu : \; \mu \in M(E)\}$ are dense in $(\mathcal{F}_e, \mathcal{E})$.*

Proof. By [10](Lemma 3.3.4 and Theorem 3.3.4) we know that

$$\mathcal{F}_e = \overline{\text{span}}^{\mathcal{E}}\{U(M_{\mathcal{E}}^{+}) - U(M_{\mathcal{E}}^{+})\}$$

The second part of the lemma is a consequence of the following general result Lemma 4.5. □

For any set $A \subseteq E$ we define

$$\begin{aligned} M_E(A) &= \{\mu;\ \mu \in M(E),\ \text{supp}\mu \subseteq A\} \\ M_{\mathcal{E}}(A) &= \{\mu:\ \mu \in M_{\mathcal{E}},\ \text{supp}\mu \subseteq A\} \end{aligned}$$

Lemma 4.5 *Let $A \subseteq E$. Then*

$$\overline{span}^{\mathcal{E}}\{U\mu, \mu \in M_{\mathcal{E}}(A)\} = \overline{span}^{\mathcal{E}}\{U\mu, \mu \in M_E(A)\} \tag{4.4}$$

Proof. We will show that if $\mu \in M_E^+(A)$ with supp$\mu \subseteq A$, we can find $\mu \in M_E^+(A)$ such that $U\mu_n \to U\mu$ in $(\mathcal{F}, \mathcal{E})$. Let K_n be a sequence of compact sets such that $K_n \uparrow E$. Let $\mu_n = 1_{K_n}\mu$ then $\mu_n \in M_E^+(A)$. We will show

$$U\mu_n \to U\mu \ in \ (\mathcal{F}, \mathcal{E}) \tag{4.5}$$

$$\|U\mu - U\mu_n\|_{\mathcal{E}}^2 = \mathcal{E}(U\mu, U\mu) - 2\mathcal{E}(U\mu, U\mu_n) + \mathcal{E}(U\mu_n, U\mu_n)$$

by(4.3)

$$\mathcal{E}(U\mu_n, U\mu_n) = \int \widetilde{U\mu_n} d\mu_n$$

Where $\widetilde{U\mu_n}$ is any q.e. modification of $U\mu_n$. From[27](p3.2), we know that $\widetilde{U\mu_n} \geq 0$ q.e. which also implies that $\widetilde{U\mu_n} \geq 0$ a.e. μ (also see[10]p.71). Hence

$$\begin{aligned} \mathcal{E}(U\mu_n, U\mu_n) &= \int_{K_n} \widetilde{U\mu_n} d\mu \leq \int \widetilde{U\mu_n} d\mu = \mathcal{E}(U\mu, U\mu_n) \\ &= \int \widetilde{U\mu} d\mu_n \leq \int \widetilde{U\mu} d\mu = \mathcal{E}(U\mu, U\mu) \end{aligned}$$

Thus

$$\|U\mu - U\mu_n\|_{\mathcal{E}}^2 \leq 2(\mathcal{E}(U\mu, U\mu) - \mathcal{E}(U\mu, U\mu_n))$$

By monotone convergence theorem

$$\mathcal{E}(U\mu, U\mu_n) = \int \widetilde{U\mu} d\mu_n = \int_{K_n} \widetilde{U\mu} d\mu \to \int \widetilde{U\mu} d\mu = \mathcal{E}(U\mu, U\mu) \text{ as } n \to \infty,$$

so (4.5) holds. This completes the proof. □

For any Borel set $A \subseteq E$, define a subspace of $(\mathcal{F}_e, \mathcal{E})$ as

$$\mathcal{F}_{E\setminus A} = \{u; \ u \in \mathcal{F}_e, \ \tilde{u} = 0 \ \ q.e. \text{ on } A\} \tag{4.6}$$

Where $q.e$ means quasi-everwhere, and $\tilde{u}$ is any $q.e$ modification of u in the restricted sense. We denote by $\mathcal{H}_0^A$ as the orthogonal complement of $\mathcal{F}_{E\setminus A}$ in $\mathcal{F}_e$, namely

$$\mathcal{H}_0^A = \mathcal{F}_{E\setminus A}^{\perp} \tag{4.7}$$

Definition 4.7 ([10],p.79) *For any $v \in \mathcal{F}_e$ we define the spectrum of v (denoted by $S(v)$) as the compelemt of the largest open set O such that $\mathcal{E}(v, u) = 0$ for any $u \in \mathcal{F}_e \cap C_0(E)$ with $suppu \subseteq O$. In particaular when $\mu \in M_\epsilon^+$, $S(U\mu) = supp\mu$.*

Lemma 4.6 ([10],p.80) *Let A be an open or closed subset of E. Then*

$$\mathcal{H}_0^A = \mathcal{W}_0^A = \overline{span}^{\mathcal{E}}\{U\mu; \mu \in M_{\mathcal{E}}^+(A)\} = \overline{span}^{\mathcal{E}}\{U\mu; \mu \in M_E^+(A)\}$$

where

$$\mathcal{W}_0^A = \overline{span}^{\mathcal{E}}\{v \in \mathcal{F}_e, \ S(v) \subseteq A\} \tag{4.8}$$

We need now the concept of ' Balayage measure '. Let P_A be the projection on $\mathcal{H}_0^A$ in $(\mathcal{F}_e, \mathcal{E})$ for any Borel set A. For $\mu \in M_E^+$, we know from[10](section3.3) that there corrosponds a potential $f = U\mu \in \mathcal{F}_e$. Let $f_A = P_A f$ then $f_A \in U(M_\mathcal{E}^+)$ and hence $f_A = U\mu^A$ with $\mu^A \in M_\mathcal{E}^+$ also supp$\mu^A \subseteq \overline{A}$. Following [10], we call μ^A the Balayage measure (or sweeping out) of μ on A.

Lemma 4.7 *Let A be a closed set and $A \subseteq D$, D is open, if $u \in \mathcal{F}_A$ then there exists a sequence $\{g_n\} \subseteq \mathcal{F}_e \cap C_0(E)$ with supp$(g_n) \subseteq D$ and*

$$g_n \to u \ \ in \ (\mathcal{F}_e, \mathcal{E}) \ \ as \ n \to \infty \tag{4.9}$$

Proof. If $u \in \mathcal{F}_A$, then by Lemma 4.1 $u^+, u^- \in \mathcal{F}_e$. But $\widetilde{u^+} \leq \widetilde{|u|} = |\tilde{u}|$ and $\widetilde{u^-} \leq \widetilde{|u|} \leq |\tilde{u}|$ so both u^+ and u^- are in $\mathcal{F}_A$. Without loss of generality, we may assume u is nonnegtive and itself quasi-continuous.
Since $\mathcal{F}_e \cap C_0(E)$ is dense in $(\mathcal{F}_e, \mathcal{E})$. There exists $\{v_n\} \in \mathcal{F}_e \cap C_0(E)$ such that $v_n \to u$ in $(\mathcal{F}_e, \mathcal{E})$. We may assume $v_n \geq 0$, because we always can replace v_n by $v_n^+ = \frac{1}{2}v_n + \frac{1}{2}|v_n|$ and $v_n^+ \to u$ in $(\mathcal{F}_e, \mathcal{E})$ by virtue of Lemma 4.1(ii). Let $h_n = v_n \wedge u = \frac{1}{2}(v_n + u) - \frac{1}{2}|v_n - u|$, using Lemma 4.1(ii) again we can show $h_n \to u$ in $(\mathcal{F}_e, \mathcal{E})$. Notice that h_n is bounded, $h_n \in \mathcal{F}_A$ and closure of $\{x;\ x \in E,\ h_n(x) \neq 0\}$ is compact, we can choose $w'_n \in C_0(E)$ and $w'_n \geq 0$ such that $w'_n \geq h_n$ q.e. and supp$w'_n \subseteq D$. Choose another $w''_n \in C_0(E)$ and $w''_n \geq 0$ such that supp$(w''_n) \subseteq D$ and $w''_n \geq w'_n + 1$ for $x \in$suppw'_n. By Lemma 4.1(iii) for each n we can find $\{w_m^n\} \subseteq \mathcal{F}_e \cap C_0(E)$ with $w_m^n \geq 0$ such that supp$(w_m^n) \subseteq \{x;\ w''_n(x) \neq 0\}$ and $\|w_m^n - w''_n\|_\infty \to 0$ as $m \to \infty$. So for each n we can find $w_n \in \mathcal{F}_e \cap C_0(E)$ such that $w_n \geq 0$ and $w_n \geq w''_n(x) - \frac{1}{2}$ for all x, then $w_n \geq h_n$ q.e.. Now for any n, select $\{u_m^n\} \subseteq \mathcal{F}_e \cap C_0(E)$ such that

$u_m^n \to v_n'$ as $m \to \infty$ in $(\mathcal{F}_e, \mathcal{E})$, let $e_m^n = w_n \wedge u_m^n = \frac{1}{2}(w_n - u_m^n) - \frac{1}{2}|w_n - u_m^n|$ and using Lemma4.1(ii) again we can show

$$e_m^n \to v_n' \text{ in } (\mathcal{F}_e, \mathcal{E}) \quad \text{as } m \to \infty$$

Notice $\text{supp}(e_m^n) \subseteq D$ and $v_n' \to u$ in $(\mathcal{F}_e, \mathcal{E})$ as $n \to \infty$. So we can find $\{g_n\} \subseteq \mathcal{F}_e \cap C_0(E)$ such that $\text{supp} g_n \subseteq D$ and

$$g_n \to u \text{ in } (\mathcal{F}_e, \mathcal{E}) \text{ as } n \to \infty$$

□

Definition 4.8 *Let $(\mathcal{F}_e, \mathcal{E})$ be a regular extended transient Dirichlet space and $M(E)$ is defined as in (4.1) . The (centered) Gaussian random field $\{X_\mu, \mu \in M(E)\}$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfying $E(X_\mu X_\nu) = \mathcal{E}(U\mu, U\nu)$ is called $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field.*

Remark: Rõchner considered the Gaussian field indexed by M_ϵ with covariance $E(X_\mu X_\nu) = \mathcal{E}(U\mu, U\nu)$.But the Markov properties of$\{X_\mu, \mu \in M(E)\}$ and $\{X_\mu, \mu \in M_\epsilon\}$ are the same by virtue of Lemma 4.5. We point out here that M_ϵ may not be a vector space because the sum of two Radon measures may not make sense.

For every $g \in \mathcal{F}_e \cap C_0(E)$ there exist $\mu_n \in M(E)$ and $U\mu_n \to g$ (by Lemma 4.4) then $\{X_{\mu_n}\}$ is Cauchy in $L_2(\Omega, \mathcal{F}, P)$. We define

$$\hat{X}_g = \lim_{n \to \infty} X_{\mu_n} \tag{4.10}$$

for every $g \in \mathcal{F}_e \cap C_0(E)$. Then $E(\hat{X}_g X_\mu) = \mathcal{E}(g, U\mu) = \int \tilde{g} d\mu = \int g d\mu$ for $\mu \in M(E)$

Let $G(E) = \mathcal{F}_e \cap C_0(E)$. Since $\mathcal{F}_e \cap C_0(E)$ is dense in $(\mathcal{F}_e, \mathcal{E})$, $\{\hat{X}_g, g \in G(E)\}$ is the dual process of $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu, \mu \in M(E)\}$ in the sense of Definition 3.2.

Lemma 4.8 $G(E) = \mathcal{F}_e \cap C_0(E)$ *has the partition of unity property.*

Proof. Suppose $v \in \mathcal{F}_e \cap C_0(E)$ with supp$v \subseteq G_1 \cup G_2$ (G_1 and G_2 are open). Then take a pre-compact open set O such that supp$v \setminus G_2 \subseteq O \subseteq \overline{O} \subseteq G_1$. By Lemma 4.1(iii) we can choose $w \in \mathcal{F}_e \cap C_0(E)$, such that $w = 1$ on O and supp$w \subseteq G_1$ then

$$v = vw + (v - vw) \equiv v_1 + v_2 \qquad \text{say,}$$

then by Lemma 4.1(i), $v_i \in \mathcal{F}_e \cap C_0(E)$ and supp$v_i \subseteq G_i$, $(i = 1, 2)$. □

Lemma 4.9 *Let* $K(C)$ *be the RKHS of* $C(\mu, \nu) = \mathcal{E}(U\mu, U\nu)$, $\mu, \nu \in M(E)$. *Let* $\{\hat{X}_g, g \in \mathcal{F}_e \cap C_0(E)\}$ *be defined as in (4.10). Then for every open set* D, $\hat{M}(D) = M(D)$.

Remark: Recall $\hat{M}(D) = \overline{\text{span}}^{K(C)}\{f_g(\cdot);\ g \in G(E) \text{ with supp} g \subseteq D\}$ and $M(D) = \overline{\text{span}}^{K(C)}\{f;\ f \in K(C) \text{ with supp} f \subseteq D\}$.(cf. (3.8) and (3.9))

Proof. We only need to prove $\hat{M}(D) \subseteq M(D)$. Let $f \in M(D)$ such that supp$f \subseteq D$, let A =supp$f$ then there exists an element $u \in \mathcal{F}_e$ such that $f(\mu) = \mathcal{E}(U\mu, u)$ then $\mathcal{E}(U\mu, u) = 0$ for any $\mu \in M(E)$ with supp$\mu \subseteq A^c$. By Lemma 4.6

$$u \in [\overline{\text{span}}^{\mathcal{E}}\{U_\mu, \mu \in M_E(A^c)\}]^\perp = (\mathcal{W}_0^{A^c})^\perp = (\mathcal{H}_0^{A^c})^\perp = \mathcal{F}_A.$$

By Lemma 4.7 there exists $\{g_n\} \subseteq \mathcal{F}_e \cap C_0(E)$ with supp$g_n \subseteq D$ such that $g_n \to u$ in $(\mathcal{F}_e, \mathcal{E})$ as $n \to \infty$. Then $\hat{X}_{g_n} \to \Pi f$ in $L_2(\Omega, \mathcal{F}, P)$, where Π is the isometry between $K(C)$ and $H(X) = \overline{\text{span}}\{X_\mu, \mu \in M(E)\}$ such that $\Pi C(\cdot, \mu) = X_\mu$. So $f_{g_n} \to f$ in $K(C)$ where $f_{g_n}(\mu) = \int g_n d\mu$. Obviously $f_{g_n} \in \hat{M}(D)$ hence $\hat{M}(D) = M(D)$. □

The next theorem characterizes the relationship of the local property of $(\mathcal{F}_e, \mathcal{E})$ (see Definition4.1) and the Markov property I of $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu, \mu \in M(E)\}$ for all open sets.

Theorem 4.1 *$(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu, \mu \in M(E)\}$ has Markov property I for all open sets iff one of the following holds:*

(a) $(\mathcal{F}_e, \mathcal{E})$ has local property, namely $\mathcal{E}(u,v) = 0$ for every $u, v \in \mathcal{F}_e \cap L_2(E,m)$ such that supp(udm) and supp(vdm) are compact and disjoint.

(b) $(f_1, f_2)_{K(C)} = 0$ for $f_1, f_2 \in K(C)$ with supp$f_1 \cap$ supp$f_2 = \phi$.

Proof. The equivalence of (b) and Markov property I of $\{X_\mu, \mu \in M(E)\}$ is the consequence of Lemma 4.8 and Lemma 4.9 and Corrollary3.3. We only need to prove (a) and(b) are equivalent.

(a) ⇒(b) Let $f_1, f_2 \in K(C)$, supp$f_1 \cap$supp$f_2 = \phi$. Since $\overline{\text{span}}^{\mathcal{E}}\{U\mu;\ \mu \in M(E)\} = \mathcal{F}_e$, there exist $u_1, u_2 \in \mathcal{F}_e$ such that $(f_1, f_2)_{K(C)} = \mathcal{E}(u_1, u_2)$ and

$$f_i(\mu) = \mathcal{E}(u_i, U\mu) = \int \tilde{u}_i d\mu$$

for all $\mu \in M(E)$, $i = 1, 2$.

Let A_i =suppf_i, $i = 1, 2$, then $\mathcal{E}(u_i, U\mu) = 0$ for any $\mu \in M(E)$ with supp$\mu \subseteq A_i^c$, $i = 1, 2$. By Lemma 4.6

$$u_i \in [\overline{\text{span}}^{\mathcal{E}}\{U\mu; \mu \in M_E(A_i^c)\}]^\perp = (\mathcal{W}_0^{A_i^c})^\perp = \mathcal{F}_{A_i}.$$

Where $\mathcal{F}_{A_i} = \{v \in \mathcal{F}_e, \tilde{v} = 0 \; q.e. \text{ on } A_i^c\}$. Let D_i be some open neighborhood of A_i such that $D_1 \cap D_2 = \phi$. By Lemma 4.7 there exist $\{g_n^i\} \subseteq \mathcal{F}_e \cap C_0(E)$ with supp$g_n^i \subseteq D_i$ and $g_n^i \to u_i$ as $n \to \infty$ in $(\mathcal{F}_e, \mathcal{E})$ for $i = 1, 2$. Then

$$(f_1, f_2)_{K(C)} = \mathcal{E}(u_1, u_2) = \lim_n \mathcal{E}(e_n^1, e_n^2) = 0$$

because $\mathcal{E}(e_n^1, e_n^2) = 0$ for each n by the local property of $(\mathcal{F}_e, \mathcal{E})$.

(b) $\Rightarrow$ (a) Let $v_1, v_2 \in \mathcal{F}_e \cap L_2(E, m)$ such that supp$(v_1 dm)$ and supp$(v_2 dm)$ are compact and disjoint. We need to show $\mathcal{E}(v_1, v_2) = 0$. But since $K(C)$ and $(\mathcal{F}_e, \mathcal{E})$ are isometry there exist $f_1, f_2 \in K(C)$ such that $\mathcal{E}(v_1, v_2) = (f_1, f_2)_{K(C)}$ and $\mathcal{E}(v_i, U\mu) = f_i(\mu)$ for $\mu \in M(E)$ and $i = 1, 2$. Let $A_i =$supp$(v_i dm)$. $v_i = 0 \; a.e, m$ on A_i^c then $v_i = 0 \; q.e$ on A_i^c, hence $v_i \in \mathcal{F}_{A_i} = (\mathcal{H}_0^{A_i^c})^\perp = (\mathcal{W}_0^{A_i^c})^\perp$ (by Lemma 4.6). So $\mathcal{E}(v_i, U\mu) = 0$ for every $\mu \in M(E)$ with supp$\mu \subseteq A_i^c$ $i = 1, 2$. Then $f_i(\mu) = 0$ for every $\mu \in M(E)$ with supp$\mu \subseteq A_i^c$, hence supp$f_i \subseteq A_i$ $i = 1, 2$. So $\mathcal{E}(v_1, v_2) = (f_1, f_2)_{K(C)} = 0$ □

From the proof we see that (a) of Theorem 4.1 is also equivalent to the following (b'):

(b') $(f_1, f_2)_{K(C)} = 0$ if $f_1, f_2 \in K(C)$, suppf_1 and suppf_1 are compact and disjoint.

Using this, Theorem 4.1 and Corollary 3.3, we get:

Theorem 4.2 *For $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu, \mu \in M(E)\}$ the following are equivalent.*

(i) $\{X_\mu \in M(E)\}$ has Markov property I for all open sets.

(ii) $\{X_\mu \in M(E)\}$ has Markov Property I for all pre-compact sets.

(iii) $(\mathcal{F}_e, \mathcal{E})$ has local property.

From Lemma 2.11 we know that for an open set D if

$$\Sigma(\overline{D}) \vee \Sigma(D^c) = F(E) \tag{4.11}$$

then MPI on D is equivalent to MPII on D. We shall prove (4.11) is the case. In fact, we have:

Lemma 4.10 *For $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu \in M(E)\}$, $F(S) \vee F(S^c) = F(E)$ holds for every open set S.*

Proof. Let $\mu \in M(E)$, then $\mu = 1_S\mu + 1_{S^c}\mu$ which gives $X_\mu = X_{1_S\mu} + X_{1_{S^c}\mu}$. Since supp$(1_{S^c}\mu) \subseteq S^c$, $X_{1_{S^c}\mu}$ is measurable $F(S^c)$. Take K_n compact and $K_n \uparrow S$ then $|\mu|(S \backslash K_n) \to 0$. We can show $\mathcal{E}(U\mu_n, U\mu_n) \to 0$, where $\mu_n = 1_{S\backslash K_n}\mu$ (see the proof of Lemma 4.5). This means

$$U(1_{K_n}\mu) \to U(1_S\mu) \quad \text{as } n \to \infty \ \text{ in } \ (\mathcal{F}_e, \mathcal{E})$$

or

$$X_{1_{K_n}\mu} \to X_{1_S\mu} \text{ in } L_2(\Omega, \mathcal{F}, P).$$

But $X_{1_{K_n}\mu} \in \mathcal{F}(S)$, so $X_{1_S\mu} \in \mathcal{F}(S)$ and hence $X_\mu \in F(S) \vee F(S^c)$. □

Corollary 4.1 *$(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu \in M(E)\}$ has MPII for all open sets iff $(\mathcal{F}_e, \mathcal{E})$ has local property.*

In the rest of this chapter, we will prove that $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu \in M(E)\}$ has MPI for all open sets is equivelent to MPIII fo all subsets of E (see definition2.7 for MPIII). First we need a few lemmas.

Lemma 4.11 *For $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu \in M(E)\}$. We have*

(i) If A, B are closed sets of E and O open sets of E then $O \cup A \supseteq B$ implies $\mathcal{F}(O) \vee \mathcal{F}(A) = \mathcal{F}(B)$.

(ii) For any closed set A, $F(A) = \sum(A) = \cap_{O \supseteq A} F(O)$, where the intersection is taken over all open sets O.

Proof. (i) is generalization of Lemma 4.10, the proof is similar to that of Lemma 4.10.

(ii) We can choose decreasing open sets U_n, $n = 1, 2, ...,$ such that $U_n \supseteq \overline{U}_{n+1}$ and $\cap_{n=1}^{\infty} U_n = A$. We will show that

$$\cap_n F(\overline{U}_n) = F(A) \tag{4.12}$$

this implies $F(A) = \sum(A)$.

Let $\mathcal{G} = \cap_n F(\overline{U}_n)$ and $\mathcal{F}_n = F(\overline{U}_n)$. Let $k \in N$, $\mu_1, \mu_2, ..., \mu_k \in M_E^+$ and $Z = \Pi_{i=1}^{k} X_{\mu_i}$. We will show that $E[Z|\mathcal{G}]$ is $\mathcal{F}(A)$ measurable. By martingale convergence theorem we know that

$$E[Z|\mathcal{G}] = \lim_n E[Z|\mathcal{F}_n].$$

For any $\mu \in M(E)$ let μ^n be the Balayaged measure of μ on $\overline{U_n}$(For Balayage measure see definition before Lemma 4.7). We denote by X_{μ^n} the projection of X_μ on $\overline{\text{span}}\{X_\nu; \nu \in M(E), \text{supp}\nu \subseteq \overline{U_n}\}$ and write $X_{\mu - \mu^n} = X_\mu - X_{\mu^n}$. Now Z can be written as a sum of terms

$$\Pi_{i=1}^{k} (X_{\mu_i^n})^{\alpha_i} \Pi_{i=1}^{k} (X_{\mu_i - \mu_i^n})^{\beta_i}, \quad \alpha_i, \beta_i \in \{0, 1\} \text{ and } \sum_i (\alpha_i + \beta_i) = k.$$

Since $U\mu_i^n \in \mathcal{H}_0^{\overline{U_n}} = \mathcal{W}_0^{\overline{U_n}}$ and $U(\mu_i - \mu_i^n) \perp \mathcal{W}_0^{\overline{U_n}}$, $\Pi_{i=1}^k (X_{\mu_i - \mu_i^n})^{\beta_i}$ is independent of $\mathcal{F}_n$ and since supp$\mu_i^n \subseteq \overline{U_n}$, so $X_{\mu_i^n} \in \mathcal{F}_n$. Then

$$E[Z|\mathcal{F}_n] = \Pi_{i=1}^k (X_{\mu_i^n})^{\alpha_i} E[\Pi_{i=1}^k (X_{\mu_i - \mu_i^n})^{\beta_i}].$$

Using (3.1.18) of [10]

$$\mathcal{W}_0^A = \bigcap_{n=1}^{\infty} \mathcal{W}_0^{U_n} = \bigcap_{n=1}^{\infty} \mathcal{W}_0^{\overline{U_n}}.$$

So $P_{\mathcal{W}_0^{\overline{U_n}}}(U\mu_i) \to P_{\mathcal{W}_0^A}(U\mu_i)$, where $P_{\mathcal{W}_0^{\overline{U_n}}}$ is the projection on $\mathcal{W}_0^{\overline{U_n}}$. This means $X_{\mu_i^n} \to X_{\mu_i^A}$ in $L_2(\Omega, \mathcal{F}, P)$, where μ_i^A is the Balayage measure of μ on A. Thus $\Pi_{i=1}^k (X_{\mu_i^n})^{\alpha_i} \to \Pi_{i=1}^k (X_{\mu_i^A})^{\alpha_i}$ in probability. Since $X_{\mu_i - \mu_i^n} \to X_{\mu_i^n - \mu_i^A}$ in $L_2(\Omega, \mathcal{F}, P)$ for $i = 1, 2, ..., k$, since $\{X_{\mu_i^n - \mu_i^n}, i = 1, 2, ..., k\}$ are joint Gaussian, by a simple argument concerning the Fourier transform of their joint distributions, we obtain $\lim_n E[\Pi_{i=1}^k (X_{\mu_i - \mu_i^n})^{\beta_i}]$ exists, thus $\lim_n E[\mathcal{Z}|\mathcal{F}_n]$ is $\mathcal{F}(A)$ measurable because $\Pi_{i=1}^k (X_{\mu_i^A})^{\alpha_i}$ is $\mathcal{F}(A)$ measurable, so $E[\mathcal{Z}|\mathcal{F}_n]$ is $\mathcal{F}(A)$ measurable. By Corrollary3.2, the polynomials in X_μ, $\mu \in M(E)$ is dense in $L_2(\Omega, \mathcal{F}(X), P)$. So $\mathcal{G} = \mathcal{F}(A)$ □

Lemma 4.12 *Let $A \subseteq E$. Then the following are equivalentf for $(\mathcal{F}_e, \mathcal{E})$ -Gaussian field $\{X_\mu,\ \mu \in M(E)\}$,*

(i) $F(\overline{A}) \perp\!\!\!\perp F(\overline{A^c}) | \mathcal{F}(\partial A)$.

(ii) $F(A) \perp\!\!\!\perp F(A^c) | F(\partial A)$.

(iii) $F(\mathring{A}) \perp\!\!\!\perp F(\mathring{A}^c) | F(\partial A)$.

Where $\mathring{A}$ and $\mathring{A}^c$ mean the interiors of A and A^c respectively. ∂A means the boundary of A.

Proof. (i)That $\Rightarrow$(iii) is trival because $F(\mathring{A}) \subseteq F(\overline{A})$ and $F(\mathring{A}^c) \subseteq F(\overline{A^c})$

(iii)$\Rightarrow$(i) Since $\mathring{A} \cup \partial A = \bar{A}$ and $\mathring{A}^c \cup \partial A = \overline{A^c}$. By Lemma 4.11(i) $F(\mathring{A}) \vee F(\partial A) = F(\bar{A})$, $F(\mathring{A}^c) \vee F(\partial A) = F(\overline{A^c})$. Then apply Corollary 2.1(i) to get (i).

Since $F(\mathring{A}) \subseteq F(A) \subseteq F(\bar{A})$ and $F(\mathring{A}^c) \subseteq F(A^c) \subseteq F(\overline{A^c})$. From (i) $\Leftrightarrow$ (iii) we get (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). □

Lemma 4.13 *For $(\mathcal{F}_e, \mathcal{E})$ - Gaussian field $\{X_\mu,\ \mu \in M(E)\}$, the following are equivalent:*

(i) $F(\overline{O}) \perp\!\!\!\perp F(\overline{O^c})|F(\partial O)$ for all open subsets O of E.

(ii) $F(\overline{A}) \perp\!\!\!\perp F(\overline{A^c})|F(\partial A)$ for all subsets A of E.

Proof. that (ii) $\Rightarrow$ (i) is trival.

(i) $\Rightarrow$ (ii): For any subset A, By previous lemma it is enough to show $F(\mathring{A}) \perp\!\!\!\perp F(\mathring{A}^c)|F(\partial A)$. But since $\mathring{A}$ is open, we have $F(\overline{\mathring{A}}) \perp\!\!\!\perp F(\overline{(\mathring{A})^c})|F(\partial \mathring{A})$. Since $\partial \mathring{A} \subseteq \partial A$ and $\overline{\mathring{A}} \cup \overline{(\mathring{A})^c} = E, F(\overline{\mathring{A}}) \vee F(\overline{(\mathring{A})^c}) = F(E) \supseteq F(\partial A)$. To apply Corollary 2.1(b), we get

$$\mathcal{F}(\overline{\mathring{A}}) \perp\!\!\!\perp F(\overline{(\mathring{A})^c})|F(\partial A).$$

This implies $F(\mathring{A}) \perp\!\!\!\perp F(\mathring{A}^c)|F(\partial A)$ and finishes the proof. □

Notice that Lemma 4.13(i) is nothing but Markov property II for all open sets (see Definition2.6 and Lemma 4.11(ii)) and that Lemma 4.13(ii) is nothing but Markov property III for all sets. Combining Theorem 4.2, Corollary 4.1 and Lemma 4.13, we have the following:

Theorem 4.3 *For $(\mathcal{F}_e, \mathcal{G})$ - Gaussian field $\{X_\mu,\ \mu \in M(E)\}$, the following are equivalent:*

(i) It has the Markov Property I for all open sets.

(ii) It has the Markov Property I for all pre-compact open sets.

(iii) It has the Markov Property II(GFMP) for all open sets.

(iv) It has the Markov Property III for all subsets.

(v) $(\mathcal{F}_e, \mathcal{E})$ has the local property.

Chapter 5

Applications to Ordinary Gaussian Processes

The Markov Property of ordinary Gaussian stochastic processes forms an important special case in our work. Let E be a separable locally compact Hausdorff space and $\{\xi_t, t \in E\}$ be a centered Gaussian random field. Then the Markov property of $\{\xi_t, t \in E\}$ can be handled within our framework and we deduce and extend the main results of Künsch[13] from our work. We consider first the case that E is an open domain of $\Re^n (n \geq 1)$ and then consider the general case.

Let T be an open subset of $\Re^n (n \geq 1)$ and $\{\xi_t, t \in E\}$ be a mean zero Gaussian process. Let A be a subset of T,$\bar{A}$ be the closure of A in T and ∂A be the boundary of A in T. Let $F(\bar{A}) = \sigma\{\xi_t, t \in \bar{A}\}$, $F(A^c) = \sigma\{\xi_t, t \in A^c\}$ and $F(\partial A) = \sigma\{\xi_t, t \in \partial A\}$. Then we say that $\{\xi_t, t \in T\}$ has the simple Markov property on A if $F(\bar{A}) \perp\!\!\!\perp F(A^c)|F(\partial A)$. It is well known that for

$n \geq 2$ such a definition is not reasonable because it turns out to be too narrow and to leave out many interesting multiparameter processes. For instant, Lévy's mutidimensional parameter Brownian motion does not have this property. Hence Lévy proposed in (1956) (see also McKean (1963)) the following definition:

Definition 5.1 *We say that $\{\xi_t, t \in E\}$ has the Markov property on a subset A of T if*

$$F(\bar{A}) \perp\!\!\!\perp F(A^c)|F(\partial A)$$

where for any set $B \subseteq T$, $F(B) = \sigma\{\xi_t, t \in B\}$ and $\sum(B) = \cap_{O \supseteq B} F(O)$. Here the intersection is taken over all open set O.

The following lemma can be found in [18].

Lemma 5.1 *The following are equivalent for a stochastic process $\{\xi_t, t \in E\}$ and a subset A of T.*

(i) $F(\bar{A}) \perp\!\!\!\perp F(\overline{A^c})|\sum(\partial A)$.

(ii) For every open set $O \supseteq \partial A$, $F(A) \perp\!\!\!\perp F(\bar{A}^c)|F(O)$.

(iii) $\sum(\bar{A}) \perp\!\!\!\perp \sum(\overline{A^c})|\sum(\partial A)$.

Notice that if A is an open set then (ii) is similar to MPI defined in chapter 1 and (iii) is MPII(GFMP).

We assume that $E(\xi_t \xi_s) = R(t,s)$ is continuous. This is equivalent to $T \to \{\xi_t, t \in T\}$ is continuous in $L_2(\Omega, P)$. Take $M(T) = \{\varphi dt,\ \varphi \in C_0^\infty(T)\}$. We know from Example that 2.1 $M(T)$ satisfies assumptions(A.1)-(A.3). We associate with it the random field

$$X_\varphi = \int_T \xi_t \varphi(t) dt \quad \varphi \in C_0^\infty(T) \tag{5.1}$$

and get a generalized Gaussian random field $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$.

Lemma 5.2 *For any open set $O \subseteq T$,*

$$H(O;X) = H(O;\xi),$$

where

$$H(O;X) = \overline{span}\{X_\varphi,\ \varphi \in C_0^\infty(T) \text{ with } supp\varphi \subseteq O\}$$

and

$$H(O;\xi) = \overline{span}\{\xi_t,\ t \in O\}.$$

Proof. If $\varphi \in C_0^\infty(T)$ with supp$\varphi \subseteq O$ then $\int_T \xi_t \varphi(t)dt$ as a limit of Riemann sums belongs to $H(O;\xi)$, so that

$$H(O,X) \subseteq H(O;\xi).$$

To prove the converse inclusion, let $t_0 \in O$ and choose N so that $\{t \in T, |t - t_0| < \frac{1}{n}\} \subseteq O$ for $n \geq N$. Let $\varepsilon > 0$ and $\varphi_{\frac{1}{\varepsilon}}$ be the function

$$\varphi_{\frac{1}{\varepsilon}} = \begin{cases} \varepsilon^{-n}\exp(\frac{\varepsilon^2}{\varepsilon^2 - |t-t_0|^2}) & |t - t_0| < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Observe $\int_T \varphi_{\frac{1}{\varepsilon}}(t)dt = 1$, $\varphi_{\frac{1}{\varepsilon}} \in C_0^\infty(T)$ and $\text{supp}(\varphi_n) \subseteq O,\ n \geq N$, Then

$$\begin{aligned} E|\int \xi_t\varphi_n(t)dt - \xi_{t_0}| &\leq E\int_T |\xi_t - \xi_{t_0}|\varphi_n(t)dt \\ &\leq \text{supp}_{|t-t_0|\leq\frac{1}{n}} E|\xi_t - \xi_{t_0}| \to 0 \ \text{ as } n \to \infty \end{aligned}$$

Hence $X_{\varphi_n} \to \xi_{t_0}$ in probability as $n \to \infty$, $\xi_{t_0} \in H(O;X)$ and $H(O;\xi) = H(O;X)$. □

From Lemma 5.1 and Lemma 5.2 we have:

Corollary 5.1 *Let $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$ be defined as in (5.1). Then for an any open set O, it has MPI on O iff it has MPII on O.*

We know that the MPI of $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$ can be characterized as some properties of the corresponding reproducing kernel Hilbert space (see Theorem 3.1 and Theorem 3.2). By lemma 5.2 we know that the Markov property of $\{\xi_t, t \in T\}$ for open sets is equivalent to MPI of $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$. So we can use Theorem 3.1 and Theorem 3.2 to get similar results for $\{\xi_t, t \in T\}$. First we shall deduce some relationships between the RKHS of $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$ and that of $\{\xi_t, t \in T\}$.

Let $K(C_X)$ and $K(C_\xi)$ be the reproducing kernel Hilbert spaces of $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$ and $\{\xi_t, t \in T\}$ respectively, and we denote by $H(X)$ and $H(\xi)$ the linear spaces in $L_2(\Omega, F, P)$ generated by $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$ and $\{\xi_t, t \in T\}$ respectively. Notice that since $R(s,t) = E(\xi_s\xi_t)$ is continuous, every element in $K(C_\xi)$ is also a continuous function on T. Define Π^{-1} and Π'^{-1} as follows

$$\begin{aligned}
\Pi^{-1} : H(X) &\rightarrow K(C_X), \\
(\Pi^{-1}Y)(\varphi) &= E(YX_\varphi)\ \ Y \in H(X). \\
\Pi'^{-1} : H(\xi) &\rightarrow K(C_\xi), \\
(\Pi'^{-1}Y)(t) &= E(Y\xi_t)\ \ Y \in H(\xi).
\end{aligned}$$

We know that both Π^{-1} and Π'^{-1} are isometric maps. Since $H(X) = H(\xi)$, hence $J = \Pi^{-1}\Pi'$ is an isometric map between $K(C_\xi)$ and $K(C_X)$. We can explicitly express the map J, if $f \in K(C_\xi)$ then

$$\begin{aligned}(Jf)(\varphi) &= (\Pi^{-1}(\Pi' f))(\varphi) = E[(\Pi' f)X_\varphi] \\ &= E[\Pi' f \int_T \xi_t \varphi(t)dt] \\ &= \int_T E(\Pi' f \xi_t)\varphi(t)dt \\ &= \int_T f(t)\varphi(t)dt.\end{aligned}$$

It can be easily checked that for f_1, f_2 and $f \in K(C_\xi)$

$$\text{supp}(Jf) = \text{supp} f \tag{5.2}$$

and

$$(Jf_1, Jf_2)_{K(C_X)} = (f_1, f_2)_{K(C_\xi)}. \tag{5.3}$$

Where supp(Jf) is defined for a linear functional Jf of $C_0^\infty(T)$(see Lemma 2.2(a)) and supp$f$ is defined as the complement in T of the largest open set O such that $f(t) = 0$ on O. Now we can state and prove the following improvement of Künsch[13] and Pitt[24].

Theorem 5.1 *Let T be an open set of $\Re^n$ and $\{\xi_t, t \in T\}$ be a Gaussian process with continuous covariance. Then it has Markov property for all open subsets of T iff the following hold.*

(a) $(f_1, f_2)_{K(C_\xi)} = 0$ if f_1 and $f_2 \in K(C_\xi)$ such that $(\text{supp} f_1) \cap (\text{supp} f_2) = \phi$.

(b) If $f \in K(C_\xi)$ and $f = f_1 + f_2$, where f_1 and f_2 are continuous and have disjoint supports. Then $f_1, f_2 \in K(C_\xi)$.

Proof. We know that $\{\xi_t, t \in T\}$ has Markov property for all open sets of T is equivalent to the MPI of $\{X_\varphi,\ \varphi \in C_0^\infty(T)\}$, where X_φ is defined in (5.1). Hence we only need to show that (a) and (b) of Theorem 3.1 are equivalent to (a) and (b) of Theorem 5.1. It is easy to see that (a) of Theorem 3.1 is equivalent to (a) of Theorem 5.1 because of (5.2) and (5.3). To show that (b) of Theorem 3.1 implies (b) of Theorem 5.1, let $f \in K(C_\xi)$ and $f = f_1 + f_2$ with f_i's being continuous and having disjoint supports. Then for any $\varphi \in C_0^\infty(T)$

$$(Jf)(\varphi) = \int_T f_1 \varphi dt + \int_T f_2 \varphi dt \equiv F_1(\varphi) + F_2(\varphi) \qquad \text{say,}$$

since F_1 and F_2 are linear functionals on $C_0^\infty(T)$ and $\text{supp}F_i = \text{supp}f_i$, hence $F_i \in K(C_X)$ by (b) of Theorem 3.1. Then $F_i(\varphi) = \int_T f_i' \varphi dt$ with $f_i' \in K(C_\xi), (i = 1,2)$. Now $\int_T f_i \varphi dt = \int_T f_i' \varphi dt$ for all $\varphi \in C_0^\infty(T)$, $(i = 1,2)$ implies $f_i = f_i'$ $(i = 1,2)$ because f_i and f_i' are continuous. So $f_i \in K(C_\xi)$. Conversely if $F \in K(C_X)$ $F = F_1 + F_2$ with F_1 and F_2 being linear on $C_0^\infty(T)$ and having disjoint supports, then $F(\varphi) = \int_T f \varphi dt$ with $f \in K(C_\xi)$. Since $\text{supp}f = \text{supp}F \subseteq (\text{supp}F_1) \cup (\text{supp}F_2)$, define for $i = 1,2$

$$f_i(t) = \begin{cases} f(t) & t \in \text{supp}F_i \\ 0 & \text{otherwise} \end{cases}$$

then we can show that f_1 and f_2 are continuous and $\text{supp}f_i \subseteq \text{supp}F_i$, $i = 1,2$, furthermore $f = f_1 + f_2$. By (b) of Theorem 5.1 $f_i \in K(C_\xi)$. Then

$$F_1(\varphi) - \int_T f_1 \varphi dt = \int_T f_2 \varphi dt - F_2(\varphi), \; \varphi \in C_0^\infty(T).$$

Both sides are linear functionals on $C_0^\infty(T)$ and have disjoint supports, so they are zero functionals by Lemma 2.2(b), hence $F_i(\varphi) = \int f_i \varphi dt$ $(i = 1,2)$ and $F_i \in K(C_X)$ $i = 1,2$. □

From the above proof, we notice that the choice of $M(T)$ is not unique. Any $M(T)$ which satisfies the following additional assumptions (5.a) and (5.b) besides (A.1) - (A.3)will do the job:

(5.a) If f is a continuous function on T and for every open set O, $\int_T f d\mu = 0$ for all $\mu \in M(T)$ with supp$\mu \subseteq O$. Then $f = 0$ on O.

(5.b) For every open subset O of $T, H(O;\xi) = H(O;X)$ where $H(O;\xi) = \overline{\text{span}}\ \{\xi_t, t \in O\}, H(O;X) = \overline{\text{span}}\ \{X_\mu, \mu \in M(T)$ with supp$\mu \subseteq O\}$ and $X_\mu = \int_T \xi_t d\mu$.

From this we can generalize T to any separable locally compact Hausdorff space.

Let E be a separable locally compact Hausdorff space and $\{\xi_t, t \in E\}$ be a Gaussian process with continuous covariance function. Let m be a positive Radon measure on E with supp(m) = E. Then we define

$$M(E) = \{\sum_{i=1}^{n} 1_{A_i}\varphi_i dm : n \geq 1, A_i \in \mathcal{B}(E), \varphi_i \in C_0(E)\}$$

where $C_0(E)$ consists of all continuous functions on E with compact support. By Example 2.2 $M(E)$ satisfies assumptions (A.1)-(A.3). Also if f is continuous and $\int_E f d\mu = 0$ for all $\mu \in M(E)$ with supp$\mu \subseteq O$ then $\int_E f\varphi dm = 0$ for all $\varphi \in C_0(E)$ with supp$\varphi \subseteq O$ this implies $f = 0$ on O because f is continuous and supp(m) = E.

Define

$$X_\mu = \int_E \xi_t d\mu \quad \mu \in M(E). \tag{5.4}$$

Then we have

Lemma 5.3 *for every open set O, $H(O;\xi) = H(O;X)$.*

Proof. It is enough to show the lemm for precompact open set O. Let $\mu \in M(E)$ with supp$\mu \subseteq O$, then since any continuous function f can be approximated by f_n on O with form $f_n(t) = \sum_{i=1}^{m_n} f(t_i)1_{A_i}(t)$ with $t_i \in O$ and $A_i \in \mathcal{B}(E)$ i.e. $f_n \to f$ pointwise on O. Let $\xi_n(t) = \sum_{i=1}^{m_n} \xi_{t_i} 1_{A_i}(t)$ with $t_i \in O$, $\xi_n(t) \to \xi_t$ on every $t \in O$ in $L_2(\Omega, F, P)$. Then $\int_E \xi_n(t)d\mu \to \int_E \xi_t d\mu = X_\mu$ in $L_2(\Omega, F, P)$ because supp$\mu \subseteq O$. But $\int_E \xi_n d\mu \in H(O;\xi)$ hence $X_\mu \in H(O;\xi)$. On the other hand let $t_0 \in O$ then we can choose precompact open set $O_n, t_0 \in O_n$ such that $\overline{O_n} \subseteq O$ and $\overline{O_n} \downarrow \{t_0\}$. Let $d\mu_n = \alpha_n 1_{O_n} dm$ with $\alpha_n = \frac{1}{m(O_n)}$ then we can see that $\mu_n \in M(E)$ with supp$\mu_n \subseteq O$ and

$$\begin{aligned}
E|\int \xi_t d\mu_n - \xi_{t_0}| &\leq E\int_{O_n} \alpha_n |\xi(t) - \xi(t_0)| dm \\
&= \alpha_n \int 1_{O_n} E|\xi(t) - \xi(t_0)| dm \\
&\leq \sup_{t \in O_n} E|\xi(t) - \xi(t_0)| \alpha_n \int_{O_n} dm \\
&= \sup_{t \in O_n} E|\xi(t) - \xi(t_0)|.
\end{aligned}$$

Since O_n are all pre-compact and contained in compact set $\overline{O_1}$, also $O_n \downarrow \{t_0\}$ by uniformly continuity of $E|\xi_t - \xi_{t_0}|$ on $\overline{O_1}$ we have

$$\text{supp}_{t \in O_n} E|\xi(t) - \xi(t_0)| \to 0 \; as \; n \to \infty.$$

Hence $\xi_{t_0} \in H(O;X)$. □

We shall have the following isometry between $K(C_\xi)$ and $K(C_X)$ by J.

$$\begin{aligned}
J: \; K(C_\xi) &\to K(C_X) \\
(Jf)(\mu) &= \int f d\mu \quad f \in K(C_\xi)
\end{aligned}$$

and similar to (5.2) and (5.3), for $f \in K(C_\xi)$, we have

$$\mathrm{supp}(Jf) = \mathrm{supp} f \tag{5.5}$$

$$\textit{and} \quad (Jf_1, Jf_2)_{K(C_X)} = (f_1, f_2)_{K(C_\xi)} \tag{5.6}$$

The following theorem is extension of Theorem 5.1 and the proof is almost the same and hence omitted.

Theorem 5.2 *Let E be a separable locally compact Hausdorff space, $\{\xi_t, t \in E\}$ be a Gaussian processes with continuous covariance and $K(C_\xi)$ be the RKHS of its covariance. Then it has the Markov property for all open sets iff*

(a) $(f_1, f_2)_K(C_\xi) = 0$ if f_1 and $f_2 \in K(C_\xi)$ with disjoint supports.

(b) If $f \in K(C_\xi)$ $f = f_1 + f_2$, where f_1 and f_2 are continuous and have disjoint supports, then $f_i \in K(C_\xi)$ $(i = 1, 2)$.

We also get the following theorem similar to Theorem 3.2.

Theorem 5.3 *Let $\{\xi_t, t \in E\}$ be the same as in Theorem 5.2, then it has the Markov property for all pre-compact open sets iff*

(a) $(f_1, f_2)_{K(C_\xi)} = 0$ if f_1 and $f_2 \in K(C_\xi)$ with disjoint supports and one of the supports is compact.

(b) If $f \in K(C_\xi)$, $f = f_1 + f_2$ with f_1 and f_2 being continuous and having disjoint supports of which one is compact. Then $f_i \in K(C_\xi)$ $(i = 1, 2)$.

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