ESTIMATION AND CONTROL OF NONLINEAR SYSTEMS USING EXTENDED HIGH-GAIN OBSERVERS

By

Almuatazbellah Muftah Boker

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Electrical Engineering - Doctor of Philosophy

2013

ABSTRACT

ESTIMATION AND CONTROL OF NONLINEAR SYSTEMS USING EXTENDED HIGH-GAIN OBSERVERS

By

Almuatazbellah Muftah Boker

Providing accurate state estimation is important for many systems. This is especially the case for systems whose states may be very difficult or even impossible to measure or require expensive or unreliable sensors. From another prospective, making all the states available simplifies considerably the control design and helps in providing practical and economic solutions to many control problems. The first part of our work involves the design of an observer for a class of nonlinear systems that can potentially admit unstable zero dynamics. The structure of the observer is composed of an Extended High-Gain Observer (EHGO), for the estimation of the derivatives of the output, augmented with an Extended Kalman Filter for the estimation of the states of the internal dynamics. The EHGO is also utilized to estimate a signal that is used as a virtual output to an auxiliary system comprised of the internal dynamics. We demonstrate the efficacy of the observer in two examples; namely, a synchronous generator connected to an infinite bus and a Translating Oscillator with a Rotating Actuator (TORA) system.

In the special case of the system being linear in the states of the internal dynamics, we achieve semi-global asymptotic convergence of the estimation error. We also solve for this class of systems, which may have unstable zero dynamics, the problem of output feedback stabilization. We allow the use of any globally stabilizing full state feedback control scheme. We then recover its performance using an observer-based output feedback control. As a demonstration of the efficacy of this control scheme, we design an output feedback stabilizing control for the DC-to-DC boost converter system.

We also consider the problem of output feedback tracking of possibly non-minimum phase nonlinear systems where the internal dynamics have a full relative degree with respect to a virtual output. In this case, the internal dynamics can be represented in the chain-of-integrators form. This allows the use of a high-gain observer to estimate the states of the internal dynamics, and hence, making all the system states available for the controller. We allow the use of any globally stabilizing full state feedback control and we show that it is possible to recover its stability properties and trajectory performance. Finally, as a demonstration of the design procedure, we solve the problem of output feedback tracking control of flexible joint manipulators where the link angle is the output to be controlled and the motor angle is the measured output. We demonstrate the effectiveness of the proposed scheme in the single link case, and where the zero dynamics are not asymptotically stable.

To my mother Naiema, father Muftah, wife Mona and my kids Jood and Muhammad.

ACKNOWLEDGMENTS

First and foremost, thanks to God for his endless bounties.

This work would not have been possible without the help and guidance of Professor Hassan Khalil. I am extremely fortunate to have him as my mentor and advisor. His deep understanding of the subject, patience and hard work have been, and will always be, sources of inspiration for me. I will always be grateful for his unwavering support and encouragement.

I would also like to thank my other committee members Prof. Tan, Prof. Zhu and Prof. Mukherjee for their insightful feedback.

I owe my parents everything. Their unconditional love and support is limitless. My mother's love, care and patience is like a guardian accompanying me everywhere. My father's virtues, leading by example and love to serve others have always led me to be the best i can be.

Last but not least, this journey would have been really difficult without the support and love of my friend, partner and wife Mona. I also would like to thank her for being a fabulous mom for our two beloved children Jood and Muhammad.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Nonlinear Observers

While the literature on linear observer theory may have reached a saturation point, research on observers for nonlinear systems is far from complete. In fact, a unified approach to observer design for nonlinear systems still seems to be hard to formulate. In addition, one of the difficult properties to achieve in this context is the arbitrary enlargement of the region of attraction of the observer stability. In general, there have been a number of different approaches to this problem. The first approach is based on the extension of the Leunberger observers [1] and Kalman filters [2], [3] to nonlinear systems. This approach is based on linearization and is appealing due to the simplicity of the observer design regardless of the complexity of the system. However, the drawback of this approach is that it guarantees only local stability. Section 2.1 of this dissertation will have more discussion on the use of Kalman filters in the estimation of nonlinear systems. The second approach utilizes the idea of linearizing the error dynamics, by the use of state transformation, so that the nonlinearities may only depend on the inputs and outputs, see e.g. [4], [5], [6]. The third approach is based on the use of Linear Matrix Inequalities (LMI) techniques, see e.g. [7], [8], [9]. The systems considered by this approach are composed of a linear part and a nonlinear part that satisfies some monotonic properties. The feasibility of this approach is closely linked to the feasibility of the solution of the LMI. The fourth main approach is based on the use of high-gain observers [10] and sliding mode observers [11]. These observers deal with systems that are in the uniformly observable normal form, where the nonlinearities appear in a lower triangular structure. This approach has gained popularity due to its robustness properties. The above approaches are by no means exclusive and, in fact, there have been results that make use of techniques from different approaches. Just to give an example, the paper by [12] merges the techniques of the extended Kalman filter and high-gain observer to solve the problem of state estimation for systems in the normal form with stable linear internal dynamics. It should also be mentioned that there have been other results that may not fall into the above approaches such as [13], [14], [15].

1.1.1 High-Gain and Extended High-Gain Observers: Brief Background

High-gain observers constitute a key part in the work presented in this dissertation. Therefore, we will start by giving a brief background about their design. Consider the following third order system

$$\dot{x}_1 = x_2 \tag{1.1}$$

$$\dot{x}_2 = f_1(x_1, x_2, x_3) + u \tag{1.2}$$

$$\dot{x}_3 = x_1 + x_2 - x_3 \tag{1.3}$$

$$y = x_1 \tag{1.4}$$

where $x \in R^3$ is the state vector, u is the control input and y is the output. We assume that the function f_1 satisfies $f_1(0,0,x_3) = 0$ and can be unknown. This system is minimum phase with zero dynamics $\dot{x}_3 = -x_3$. If we are interested in estimating the state x_2 and assuming the presence of a stabilizing control u, we can design the following high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{x}_1) \tag{1.5}$$

$$\dot{\hat{x}}_2 = \bar{f}_1(\hat{x}_1, \hat{x}_2) + u + \frac{\alpha_2}{\varepsilon^2}(y - \hat{x}_1) \tag{1.6}$$

In this case $\bar{f}_1(.,.)$ could take any nominal value and can even be set to zero. The observer constants α_1 and α_2 are chosen such that the polynomial $s^2 + \alpha_1 s + \alpha_2$ is Hurwitz, and $\varepsilon > 0$ is a small parameter. This observer can be augmented with an open loop observer to estimate the state x_3. This allows the design of a full information output feedback control.

In general, it can be shown that high-gain observers can provide estimates of the output and its derivatives, see e.g. [16]. Therefore, it can estimate the right hand side of (1.2), and thus, can estimate the unknown function f_1. In this case, the observer is called an Extended High Gain Observer (EHGO). To illustrate this idea, let $\sigma = f_1(x_1, x_2, x_3)$, and extend the dimension of (1.1)-(1.2) by adding σ as an extra state variable as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sigma + u \\
\dot{\sigma} &= f_2(x_1, x_2, x_3, u)
\end{aligned} \tag{1.7}$$

where

$$f_2 = \frac{df_1}{dt} = \frac{\partial f_1}{\partial x_1} x_2 + \frac{\partial f_1}{\partial x_2}[f_1(x_1, x_2, x_3) + u] + \frac{\partial f_1}{\partial x_3}[x_1 + x_2 - x_3].$$

Consequently, a high-gain observer for (1.7) can be designed as

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{\sigma} + u + \frac{\alpha_2}{\varepsilon^2}(y - \hat{x}_1) \\ \dot{\hat{\sigma}} &= f_2(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) + \frac{\alpha_3}{\varepsilon^3}(y - \hat{x}_1)\end{aligned} \tag{1.8}$$

and $\hat{f}_1 = \hat{\sigma}$. It is also worth mentioning that the effect of f_2 is attenuated as $\varepsilon \to 0$, thus the observer can tolerate f_2 being unknown and achieve an estimation error of the order $O(\varepsilon)$.

In general, high-gain observers have proved useful in nonlinear feedback control. The reference [17] provides a survey of the development of high-gain observers over the past two decades. Basically, there have been two independent schools of research on this subject. The first school, lead mostly by French researchers (Gauthier, Hammouri, and others) focused on deriving global results by considering global Lipschitz conditions; see e.g. [18], [19], [20], [21]. In the context of output feedback and in the absence of global Lipschitz condition, the second school, lead by Khalil, realized the destabilizing effect of the *peaking phenomenon* and proposed a solution for it [22]. The solution was simply to ensure global boundedness of the control over a region of interest, which can be done by saturating the state estimates.

High-gain observers have been successfully employed in partial state output feedback stabilization schemes [22], [23], and output tracking using sliding mode control [24]. Previous work on full order high-gain observers is limited to minimum phase systems. Reference [25], for instance, proposed a full order observer that employs an open loop observer for the internal dynamics, which limits the validity of the technique to minimum phase systems. Another paper [12] proposed the use of an EKF-based high-gain observer for the estimation of the full state vector of minimum phase systems with linear internal dynamics driven only by the output.

Extended high-gain observers have been used in the literature to serve different objectives. They have been used to provide estimates of unknown signals that represent model uncertainties or external inputs so that they could be canceled by the controller [26]. Within the framework of designing a full information state feedback control, similar approach was also used in [27] where a first order high-gain observer was used to estimate matched uncertainties so are then canceled by the control. Around the same time, [28] introduced an output feedback control strategy that utilizes an inner loop control based on the use of high-gain observer to estimate the inverse of a nominal model of the closed loop behavior of the plant, and an outer loop controller to shape the transient response. A very important feature that was achieved by [26], [27] and [28] is the ability to recover desired transient performance by the use of the nonlinear control, despite the presence of matched model uncertainties and disturbances. Indeed the high-gain observer was instrumental in achieving this feature. EHGO is also used to develop a Lyapunov-based switching control strategy [29], [30]. More recently, the work in [31] utilizes the EHGO to estimate a signal that is observable to the zero dynamics of a non-minimum phase system. This allowed the design of a stabilizing controller.

The wide use of high-gain observers in control applications is mainly due to a number of properties that may not be provided by other observers. The first property is the simplicity of design, where there is no need for complex gain formulas and solving LMI or partial differential equations. Secondly, the observer provides the ability to recover the state trajectories when used in output feedback control. This property is rarely considered in the literature and is very useful to the designer as it helps in shaping the transient response. Another useful property is the ability of the high-gain observer to robustly estimate the states in the presence of disturbances or model uncertainties.

1.2 Non-minimum Phase Nonlinear Systems

In recent years, more attention was directed towards the study of non-minimum phase nonlinear systems. This has been motivated by many reasons, one of them is the fact that unstable zero dynamics is an intrinsic feature of a wide variety of systems. Examples of these systems include flexible-joint robotic manipulators, electromechanical systems, under-actuated systems and chemical reactors. The focus on non-minimum phase systems also comes after many advances in the control of minimum phase systems, such as the success that has been enjoyed by the use of feedback linearization techniques.

1.2.1 Observers for Non-minimum Phase Nonlinear System

There have been a few techniques that dealt with observer design for non-minimum phase nonlinear systems and achieved non-local convergence results. For example, a method for designing observers for systems affine in the unmeasured states was introduced in [32] and for a more general class of nonlinear systems in [14]. The key idea in these results is the construction of an invariant and attractive manifold. This has to be achieved by solving a set of partial differential equations. Reference [33] proposed a Higher Order Sliding Mode Observer to estimate the full state vector with a vector of unknown inputs for non-minimum phase nonlinear systems. It considered the case when the internal dynamics are quasilinear and the forcing term can be piece-wise modeled as the output of a dynamical process given by an unknown linear system with a known order.

1.2.2 Output Feedback Control of Non-minimum Phase Nonlinear Systems

In the linear case the separation principle guarantees the global stabilization by any linear state feedback control can be achieved by output feedback when the states are replaced by their estimates. However, this principle is not valid for nonlinear observer-based output feedback systems. Furthermore, the results that has been reported in this regard, such as the ones reported in [34], [35] [36], are not valid for any observer design. One of the observers that satisfies this principle, nevertheless, is the high-gain observer.

One of the first results on the control of non-minimum phase nonlinear systems is by A. Isidori. In his paper [37], he proved semi-global stabilization for a general class of non-minimum phase nonlinear systems assuming the existence of a dynamic stabilizing controller for an auxiliary system. The same problem was considered in [31], where robust semi-global stabilization was achieved under similar assumptions but with an extended high-gain observer based output feedback controller. This paper showed the potential of using the extended high-gain observer as an alternative for the high-gain feedback scheme of [37].

Papers [38] and [39] consider a special case of the normal form, called the output feedback form, where the internal dynamics depend only on the output. Paper [38] allows the presence of disturbances and paper [39] allows model uncertainty. They both require various stabilizability conditions on the internal dynamics. Paper [40] also considers systems in the output feedback form and assumes the system will be minimum phase with respect to a new output, defined as a linear combination of the state variables. Paper [6] solves the stabilization problem for the output feedback form with linear zero dynamics. It uses backstepping technique and an observer with linear error dynamics to achieve semi-global stabilizability result. Another result reported in [41] deals with a special case of the normal form where the internal dynamics are modeled as a chain of integrators. This work uses adaptive output feedback controller based on Neural Networks and a linear state observer to achieve ultimate boundedness of the states in the presence of model uncertainties. Basically, observer and controller designs are performed for a linearized model of the system, then, Neural Networks are used to represent modeling uncertainties. Moreover, adaptive laws for the NN and adaptive gains are obtained from using Lyapunov's direct method.

1.3 Importance and Motivations

In this dissertation, we solve the problem of estimating all the states of nonlinear systems represented in the normal form. This allows us to solve the problem of output feedback control for the same class of systems. A key tool that helps in solving this problem is the high-gain observer. As eluded to in Section 1.1.1, high-gain observers have been mostly used in partial state feedback and have been limited to minimum phase systems. We, on the other hand, show that by using this tool we can solve problems where estimates of all the states are needed and the zero dynamics may not be stable. As a result, the full order observer proposed in this dissertation is characterized by the simplicity of design relative to what have been offered in the literature. Furthermore, the proposed observer-based output feedback schemes have the capability to partially recover the trajectories under state feedback.[1] This sets the proposed output feedback scheme apart from others in the literature.

The motivations of this work can be summarized by the following points

1. Solve the problem of estimation and control of a general class of nonlinear systems. This class of systems can include non-minimum phase systems. This problem is not widely studied in the literature and is related to many interesting applications. The objective is to allow flexibility in control design.

2. Introduce a simple and constructive observer design without the need for complex formulas, solving partial differential equations or the need for Lyapunov functions.

[1]In Chapter 4, we show that, for systems with internal dynamics modeled as a chain-of-integrators, the proposed output feedback control fully recovers the performance of the state feedback.

3. The possibility to recover the state trajectories in output feedback control.

1.4 An Overview of the Dissertation

This dissertation is divided into two parts. The first part deals with the design of full order observers for a class of nonlinear systems represented in the normal form. This part is mostly presented in Chapter 2 with an overlap with Chapter 4. Chapter 2 provides the main principle behind the observer design, which allows flexibility in choosing the observer for the internal dynamics. In this chapter, we used the extended Kalman filter to estimate the internal states. This chapter also solves the observer design problem for a special class of nonlinear systems that are linear in the internal (unmeasured) state. Two Examples of observer design for nonlinear systems are included in this chapter; namely, a synchronous generator connected to an infinite bus and the translating oscillator with a rotating actuator (TORA) system.

The second part, presented in Chapters 3 and 4, deals with output feedback control of special classes of the normal form. Chapter 3 makes use of the observer presented in Chapter 2 and deals with the stabilization of the special class of nonlinear systems when the system is linear in the internal dynamics. This chapter includes the design of an output feedback control to stabilize a DC-to-DC converter system. Chapter 4 tackles the problem of output feedback tracking of a class of nonlinear systems, when the internal dynamics can be represented by a chain-of-integrators model. For this purpose, this chapter includes a design of a new observer that shares the same principle presented in Chapter 2. This observer depends on the use of the high-gain observer to estimate the internal dynamics. Included also in this chapter an example of designing an output feedback tracking controller for flexible-joint robotic manipulators. Finally, Chapter 5 includes conclusions and some directions for future work.

Chapter 2

Nonlinear Observers Comprising High-Gain Observers and Extended Kalman Filters

2.1 Introduction

This chapter is concerned with the design of nonlinear observers for systems represented in the normal form. We use extended high-gain observers to provide estimates of the derivatives of the output in addition to a signal that is used as a virtual output to an auxiliary system based on the internal dynamics. This is indeed possible because of the relative high speed of the EHGO. We choose to use the extended Kalman filter as an observer for the internal dynamics due to its simplicity and applicability to a wide range of nonlinear systems. In fact, any other suitable observer can be used to estimate the state of the internal dynamics, providing flexibility for the overall observer design.

The Extended Kalman Filter (EKF) is one of the popular approaches for the design of observers for a general class of nonlinear systems. Since the 1970s, EKF have been successfully used for state estimation of nonlinear stochastic systems [42], [43]. The early result [3] showed that deterministic observers can be constructed as asymptotic limits of filters. Subsequent results on the convergence properties of the EKF for deterministic systems appeared in [44],[45], [46], [47], [19], [48]. The popularity of such an approach comes from the simplicity of the design and implementation of the observer regardless of the system complexity. It is also probably due to the time varying feature of the observer gain, which could give the observer some robustness properties. The main drawback of the EKF, however, is the need for linearization. This in turn could limit the region of attraction of the observer stability. Some ideas have been proposed to expand the region of attraction relying mostly on high-gain techniques [19], [44].

Observers, similar to Kalman filters for linear time-varying systems, have been successfully used for systems affine in the unmeasured states (see for example [49], [50] and [51]). These results do not require linearization. However, like the EKF, they require the solution of a Riccati equation to exist and be bounded for all time. Essentially, there are three main methods to prove this condition. The first method relies on either assuming [48] or proving [50] that the system is uniformly observable along the estimated trajectories and then using the classical result of Bucy [52]. The second method is based on the assumption that the system has a particular input excitation properties in the form of a bounded integral [49],[53],[51]. Finally, there have been another line of research that proves this property for certain classes of systems that have lower triangular structure [20],[54], [55]. The problem with the former two methods is that it makes it difficult to verify the boundedness of the Riccati equation priori.

For systems represented in the normal form, we achieve a convergence result that is local in the estimation error of the internal dynamics but non-local in estimating the chain-of-integrators variables. In the special case when the system is affine in the internal state, we achieve semi-global convergence. This was also a motivation for using the EKF, as it allows us to exploit the linearity in the internal dynamics to achieve a non-local result.

The remainder of the chapter is organized as follows. Section 2.2 states the problem formulation and a description of the considered class of systems. Section 2.3 discusses the problem of designing a full order EHGO observer for linear systems. This serves as a motivation for the main result presented in Sections 2.4 and 2.5 that deal with observer design for nonlinear systems. Section 2.4 tackles the observer design problem for a general class of nonlinear systems, and includes an examples of designing a full order observer for a synchronous generator connected to an infinite bus system. Section 2.5 deals with systems linear in the internal state and includes a design example of designing an observer for the TORA system. Section 2.6 includes some conclusions.

2.2 Problem formulation

We consider a single-input, single-output nonlinear system with a well defined relative degree ρ represented in the form [56]:

$$\dot{\eta} = \phi(\eta, \xi) \tag{2.1}$$

$$\dot{\xi}_i = \xi_{i+1}, \quad 1 \leq i \leq \rho - 1 \tag{2.2}$$

$$\dot{\xi}_\rho = b(\eta, \xi) + a(\xi, u) \tag{2.3}$$

$$y = \xi_1 \tag{2.4}$$

where $\eta \in R^{n-\rho}, \xi \in R^\rho$, y is the measured output and u is the control input. Equations (2.1)-(2.4) can be written in a compact form

$$\dot{\eta} = \phi(\eta, \xi) \tag{2.5}$$

$$\dot{\xi} = A\xi + B[b(\eta, \xi) + a(\xi, u)] \tag{2.6}$$

$$y = C\xi \tag{2.7}$$

where the $\rho \times \rho$ matrix A, the $\rho \times 1$ matrix B and the $1 \times \rho$ matrix C represent a chain of ρ integrators.

Assumption 2.1 *The functions $\phi(\eta, \xi)$, $a(\xi, u)$ and $b(\eta, \xi)$ are known and continuously differentiable with local Lipschitz derivatives.*

Assumption 2.2 *The system trajectories $\eta(t), \xi(t), u(t)$ belong to known compact sets for all $t \geq 0$.*

Assumption 2.2 is needed because, in this chapter, we only study observer design rather than feedback control.

The objective is to design an observer for the system (2.1)-(2.4) to provide the estimates $(\hat{\eta}, \hat{\xi})$ such that the origin of the estimation error dynamics is asymptotically stable.

2.3 Linear Systems

We briefly visit the problem of designing a full-order extended high-gain observer for linear systems as a motivation for the main results. We consider a single-input single-output linear system

$$\begin{aligned} \dot{x} &= A_l x + B_l u \\ y &= C_l x \end{aligned} \tag{2.8}$$

where the dimension of the system is n and its relative degree is ρ. It is always possible to represent this system in the normal form [57]

$$
\begin{aligned}
\dot{\eta} &= A_{00}\eta + A_{01}\xi_1 \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= A_{\rho 0}\eta + A_{\rho 1}\xi + bu \\
y &= \xi_1
\end{aligned}
\tag{2.9}
$$

We assume that the pair (A_l, C_l) is observable. This is equivalent to the observability of the pair $(A_{00}, A_{\rho 0})$. We further assume that all the system trajectories are bounded. We divide the problem into two parts. The first part concerns the design of an observer for the auxiliary system

$$
\begin{aligned}
\dot{\eta} &= A_{00}\eta + A_{01}\xi_1 \\
\sigma &= A_{\rho 0}\eta
\end{aligned}
\tag{2.10}
$$

We assume that ξ and σ are available as we anticipate that we shall utilize an EHGO to estimate them. Consequently, an observer for (2.10) is given by

$$
\dot{\hat{\eta}} = A_{00}\hat{\eta} + A_{01}\xi_1 + L_1(\sigma - A_{\rho 0}\hat{\eta}) \tag{2.11}
$$

The error dynamics are, therefore, given by

$$
\dot{\tilde{\eta}} = (A_{00} - L_1 A_{\rho 0})\tilde{\eta} \tag{2.12}
$$

where $\tilde{\eta} = \eta - \hat{\eta}$. The observer gain L_1 can be designed so that the matrix $(A_{00} - L_1 A_{\rho 0})$ has desired eigenvalues in the open left half plane. We turn now to the second part of the problem, where we need to design an observer to provide the estimates of ξ and σ. We use the EHGO

$$\begin{aligned}
\dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + (\alpha_i/\varepsilon^i)(y - \hat{\xi}_1), \quad 1 \le i \le \rho - 1 \\
\dot{\hat{\xi}}_\rho &= \hat{\sigma} + A_{\rho 1}\hat{\xi} + bu + (\alpha_\rho/\varepsilon^\rho)(y - \hat{\xi}_1) \\
\dot{\hat{\sigma}} &= (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - \hat{\xi}_1)
\end{aligned} \tag{2.13}$$

where $\alpha_1, ..., \alpha_\rho, \alpha_{\rho+1}$ are chosen such that the polynomial $s^{\rho+1} + \alpha_1 s^\rho + ... + \alpha_{\rho+1}$ is Hurwitz, and $\varepsilon > 0$ is a small parameter. Augmenting (2.11) with (2.13) yields the full order observer

$$\begin{aligned}
\dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + (\alpha_i/\varepsilon^i)(y - \hat{\xi}_1), \quad 1 \le i \le \rho - 1 \\
\dot{\hat{\xi}}_\rho &= \hat{\sigma} + A_{\rho 1}\hat{\xi} + bu + (\alpha_\rho/\varepsilon^\rho)(y - \hat{\xi}_1) \\
\dot{\hat{\sigma}} &= (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - \hat{\xi}_1) \\
\dot{\hat{\eta}} &= A_{00}\hat{\eta} + A_{01}\hat{\xi}_1 + L_1(\hat{\sigma} - A_{\rho 0}\hat{\eta})
\end{aligned} \tag{2.14}$$

It is worth noting that the eigenvalues of this observer are clustered into a group of $\rho + 1$ fast eigenvalues assigned by the EHGO (2.13), and a group of $n - \rho$ slow eigenvalues assigned by the observer (2.11).

2.4 General case

We consider the system (2.1)-(2.4). We begin by first considering an observer, we call it the internal observer, for the auxiliary system

$$\dot{\eta} = \phi(\eta, \xi), \quad \sigma = b(\eta, \xi) \tag{2.15}$$

in which ξ is considered as a known input. We shall utilize an EHGO to estimate the state vector ξ and the signal σ. Since we anticipate that the EHGO will provide these signals in a relatively fast time, we can assume that they are available for the internal observer. We choose the EKF as an observer for this system. Thus, the internal observer is given by

$$\dot{\hat{\eta}} = \phi(\hat{\eta}, \xi) + L(t)(\sigma - b(\hat{\eta}, \xi)) \tag{2.16}$$

where

$$L(t) = P(t)C_1(t)^T R^{-1}(t) \tag{2.17}$$

and $P(t)$ is the solution of the Riccati equation

$$\dot{P} = A_1 P + P A_1^T + Q - P C_1^T R^{-1} C_1 P, \tag{2.18}$$

and $P(t_0) = P_0 > 0$. The time varying matrices $A_1(t)$ and $C_1(t)$ are given by

$$A_1(t) = \frac{\partial \phi}{\partial \eta}(\hat{\eta}(t), \xi(t))$$

and

$$C_1(t) = \frac{\partial b}{\partial \eta}(\hat{\eta}(t), \xi(t)),$$

and $R(t)$ and $Q(t)$ are symmetric positive definite matrices that satisfy

$$0 < r_1 \leq R(t) \leq r_2 \tag{2.19}$$

$$0 < q_1 I_{n-\rho} \leq Q(t) \leq q_2 I_{n-\rho} \tag{2.20}$$

The EHGO is given by

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} + a(\hat{\xi}, u)] + H(\varepsilon)(y - C\hat{\xi}) \tag{2.21}$$

$$\dot{\hat{\sigma}} = \dot{b}(\hat{\eta}, \hat{\xi}, u) + (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - C\hat{\xi}) \tag{2.22}$$

where

$$\dot{b}(\hat{\eta}, \hat{\xi}, u) = \frac{d}{dt}(b(\eta, \xi))|_{(\hat{\eta}, \hat{\xi})}, \text{ and}$$
$$\frac{d}{dt}(b(\eta, \xi)) = \frac{\partial b}{\partial \eta}\phi(\xi, \eta) + \frac{\partial b}{\partial \xi}(A\xi + B[b(\eta, \xi) + a(\xi, u)]).$$

The observer gain $H(\varepsilon) = [\alpha_1/\varepsilon, ..., \alpha_\rho/\varepsilon^\rho]^T$ and $\alpha_1, ..., \alpha_\rho, \alpha_{\rho+1}$ are chosen such that the polynomial $s^{\rho+1} + \alpha_1 s^\rho + ... + \alpha_{\rho+1}$ is Hurwitz, furthermore, $\varepsilon > 0$ is a small parameter.

Combining the internal observer (2.16) with the EHGO (2.21)-(2.22) yields the full order observer

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} + a(\hat{\xi}, u)] + H(\varepsilon)(y - C\hat{\xi}) \tag{2.23}$$

$$\dot{\hat{\sigma}} = \dot{b}(\hat{\eta}, \hat{\xi}, u) + (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - C\hat{\xi}) \tag{2.24}$$

$$\dot{\hat{\eta}} = \phi(\hat{\eta}, \hat{\xi}) + L(t)(\hat{\sigma} - b(\hat{\eta}, \hat{\xi})) \tag{2.25}$$

The time Varying matrices A_1 and C_1 are now given by

$$A_1(t) = \frac{\partial \phi}{\partial \eta}(\hat{\eta}(t), \hat{\xi}(t)) \tag{2.26}$$

$$C_1(t) = \frac{\partial b}{\partial \eta}(\hat{\eta}(t), \hat{\xi}(t)) \tag{2.27}$$

The states ξ and σ are replaced by saturated versions of $\hat{\xi}$ and $\hat{\sigma}$ outside the compact sets that they belong to, according to Assumption 2.2, when used in $a(.,.), \dot{b}(.,.,.)$ and (2.25). This ensures that the observer is protected from peaking [22].

Assumption 2.3 *There is $c_0 > 0$ such that*

$$||C_1(t)|| \leq c_0, \ \forall\, t \geq 0. \tag{2.28}$$

It is worth noting that Assumption 2.3 is automatically satisfied if $b(.,.)$ is globally Lipschitz. Alternatively, it can be satisfied if, in addition to saturating $\hat{\xi}$, we saturate $\hat{\eta}$ before it is used in $b(.,.)$. The saturation should occur outside the compact set that η belongs to.

Assumption 2.4 *The Riccati equation* (2.18) *has a positive definite solution that satisfies the inequality*

$$0 < p_1 I_{n-\rho} \leq P^{-1}(t) \leq p_2 I_{n-\rho} \tag{2.29}$$

It was shown in [58] that this assumption is satisfied if the auxiliary system (2.15) has a uniform detectability property. Paper [55] also showed that Assumption 2.4 is satisfied if the auxiliary system (2.15) is uniformly observable for any input. A definition of uniform observability for any input is given in [20]. There have been a number of results in the literature that identify conditions for this property. For example, in [21] a necessary and sufficient condition was established for the special case when the system is affine in the input.

Assumption 2.4 is essential for the stability of the observer, as will be shown in the proof of Theorem 2.1. As shown in [58] and [55], the satisfaction of this assumption depends on observability properties of the auxiliary system, which comprises the zero dynamics, not on its stability properties. This indicates the capability of the proposed observer to handle non-minimum phase systems.

Remark 2.1 *The observer equations* (2.23)-(2.25) *and the Riccati equation* (2.18) *have to be solved simultaneously in real time because* $A_1(t)$ *and* $C_1(t)$ *depend on both* $\hat{\eta}(t)$ *and* $\hat{\xi}(t)$*.*

Remark 2.2 *The effect of* $\dot{b}(\hat{\eta}, \hat{\xi}, u)$ *in* (2.24) *is asymptotically attenuated as* $\varepsilon \to 0$*. However, including it is needed to prove convergence of the estimation error to zero. Without it we could only show that the error would eventually be of the order* $O(\varepsilon)$*.*

Consider the scaled estimation error

$$\tilde{\eta} = \eta - \hat{\eta} \tag{2.30}$$

$$\chi_i = (\xi_i - \hat{\xi}_i)/\varepsilon^{\rho+1-i}, \qquad 1 \leq i \leq \rho \tag{2.31}$$

$$\chi_{\rho+1} = b(\eta, \xi) - \hat{\sigma}. \tag{2.32}$$

Let $\varphi = [\chi_1, \chi_2, ..., \chi_\rho]^T$ and $D(\varepsilon) = \text{diag}[\varepsilon^\rho, \ldots, \varepsilon]$, thus we have $D(\varepsilon)\varphi = \xi - \hat{\xi}$. Consequently, equations (2.31) and (2.32) can be written compactly as $D_1(\varepsilon)\chi = [(\xi - \hat{\xi})^T \, (b(\eta, \xi) - \hat{\sigma})]^T$, where $\chi = [\varphi^T \ \chi_{\rho+1}]^T$ and $D_1(\varepsilon) = \text{diag}[D, 1]$. Using (2.1) – (2.3), (2.23) – (2.25) and (2.30) – (2.32), the estimation error dynamics can be written as

$$\dot{\tilde{\eta}} = \phi(\tilde{\eta} + \hat{\eta}, \hat{\xi} + D\varphi) - \phi(\hat{\eta}, \hat{\xi}) - L(t)[(b(\tilde{\eta} + \hat{\eta}, \hat{\xi} + D\varphi) - \chi_{\rho+1}) - b(\hat{\eta}, \hat{\xi})] \tag{2.33}$$

$$\triangleq f_r(\hat{\eta}, \hat{\xi}, \tilde{\eta}, D_1\chi, t) \tag{2.34}$$

$$\varepsilon\dot{\chi} = \Lambda\chi + \varepsilon[\bar{B}_1\Delta\dot{b} + \bar{B}_2\bar{\Delta}a] \tag{2.35}$$

where $\Delta\dot{b} = \dot{b}(\tilde{\eta}+\hat{\eta},\hat{\xi}+D\varphi,u) - \dot{b}(\hat{\eta},\hat{\xi},u)$, $\bar{\Delta}a = \Delta a/\varepsilon$, $\Delta a = a(\hat{\xi}+D\varphi,u) - a(\hat{\xi},u)$, and

$$\Lambda = \begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ -\alpha_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ -\alpha_\rho & 0 & 0 & \ddots & 1 \\ -\alpha_{\rho+1} & 0 & 0 & \dots & 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 \\ B \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

It can be seen from (2.34) and (2.35) that the observer takes a singular perturbation structure, where the estimation error due to the EKF is the slow variable and estimation error due to the EHGO is the fast variable. The design variable ε determines the speed of the EHGO, thus can allow the EHGO to provide the state σ to be used as an output for the EKF in a relatively fast time.

Let us define the initial states as $\tilde{\eta}(0) \in \mathscr{M}$ and $(\hat{\xi}(0), \hat{\sigma}(0)) \in \mathscr{N}$, where $\mathscr{M}$ is a compact set, containing the origin, defined by the region of attraction of the system (2.34); hence, it can not be arbitrarily chosen, and $\mathscr{N}$ is an arbitrarily chosen compact subset of $R^{\rho+1}$. Thus, we have $\hat{\eta}(0) = \eta(0) - \tilde{\eta}(0), \varphi(0) = D^{-1}(\varepsilon)[\xi(0) - \hat{\xi}(0)]$ and $\chi_{\rho+1}(0) = b(\eta(0), \xi(0)) - \hat{\sigma}(0)$. We now present the first theorem of our work that describes the stability of the proposed observer.

Theorem 2.1 *Consider the full order observer* (2.23)-(2.25) *for the system* (2.1)-(2.4). *Let Assumptions 2.1-2.4 hold. Then given a sufficiently small compact set $\mathscr{M} \subset R^{n-\rho}$ containing the origin and any compact set $\mathscr{N} \subset R^{\rho+1}$, for all $\tilde{\eta} \in \mathscr{M}$ and $(\hat{\xi}, \hat{\sigma}) \in \mathscr{N}$, there exists ε^* such that, for all $0 < \varepsilon < \varepsilon^*$, the estimation error converges exponentially to zero as $t \to \infty$.*

Proof:

To make the proof easy to follow, we will use the same way as in [35]. We start by showing boundedness of trajectories before we show local exponential stability.

First, because the nonlinear function $a(\xi,u)$ is continuously differentiable, with locally Lipschitz derivatives, and globally bounded, we can deduce that $\bar{\Delta}a(\xi,\hat{\xi},u)$ in (2.35) is locally Lipschitz in its arguments, uniformly in ε and bounded from above by a linear in $||\chi||$ function. This can be seen from

$$\begin{aligned}\frac{1}{\varepsilon}[a(\xi,u)-a(\hat{\xi},u)] &= \frac{1}{\varepsilon}[\int_0^1 \frac{\partial a}{\partial \xi}(\lambda(\xi-\hat{\xi})+\hat{\xi},u).d\lambda\ (\xi-\hat{\xi})] \\ &= \int_0^1 \frac{\partial a}{\partial \xi}(\lambda(\xi-\hat{\xi})+\hat{\xi},u).d\lambda\ [\varepsilon^{\rho-1}\chi_1,\ldots,\varepsilon\chi_{\rho-1},\chi_\rho]^T \\ &\leq \int_0^1 \left|\left|\frac{\partial a}{\partial \xi}(\lambda(\xi-\hat{\xi})+\hat{\xi},u)\right|\right|.d\lambda\ ||\chi||\end{aligned}$$

The matrix Λ in (2.35) is a $\rho+1\times\rho+1$ Hurwitz matrix by design. System (2.34)-(2.35) is in the standard singularly perturbed form and has an equilibrium point at the origin. Setting $\varepsilon=0$ yields $\chi=0$, and hence, the slow system is given by

$$\dot{\tilde{\eta}} = \phi(\tilde{\eta}+\hat{\eta},\hat{\xi})-\phi(\hat{\eta},\hat{\xi})-L(t)[b(\tilde{\eta}+\hat{\eta},\hat{\xi})-b(\hat{\eta},\hat{\xi})] \tag{2.36}$$

It should be noted that it is typical in singular perturbation analysis to use ξ instead of $\hat{\xi}$ in the reduced system equation, however, we chose to use $\hat{\xi}$ because the matrices A_1 and C_1, which will be used henceforth, are defined for $\hat{\xi}$. We now write (2.36) as

$$\dot{\tilde{\eta}} = [A_1(t)-L(t)C_1(t)]\tilde{\eta}+\psi(\tilde{\eta},t) \tag{2.37}$$

where $L(t)$, $A_1(t)$ and $C_1(t)$ are given by (2.17), (2.26) and (2.27), respectively, and $\psi(\tilde{\eta},t) =$

$\psi_1(\tilde{\eta},t) - L(t)\psi_2(\tilde{\eta},t)$, in which

$$\psi_1(\tilde{\eta},t) = \phi(\eta,\hat{\xi}) - \phi(\hat{\eta},\hat{\xi}) - \frac{\partial \phi}{\partial \eta}(\hat{\eta},\hat{\xi})\tilde{\eta},$$

$$\text{and} \qquad \psi_2(\tilde{\eta},t) = b(\eta,\hat{\xi}) - b(\hat{\eta},\hat{\xi}) - \frac{\partial b}{\partial \eta}(\hat{\eta},\hat{\xi})\tilde{\eta}$$

Notice that $\psi(0,t) = 0$. From Assumption 2.1 and using the bound (2.28), it can be shown that

$$||\psi(\tilde{\eta},t)|| \leq k_1 ||\tilde{\eta}||^2 \tag{2.38}$$

where k_1 is a positive constant proportional to the Lipschitz constants of $\phi(\cdot)$ and $b(\cdot)$.

The boundary layer system is obtained by applying the change of variables $\tau = t/\varepsilon$ to (2.34) and (2.35), and setting $\varepsilon = 0$ to get

$$\frac{d\chi}{d\tau} = \Lambda\chi \tag{2.39}$$

To show boundedness, we show that any trajectory starting in the compact set $\mathscr{M} \times \mathscr{N}$ enters an appropriately defined positively invariant set $\mathscr{S}$ in finite time. To this end, let $V_1(t,\tilde{\eta}) = \tilde{\eta}^T P^{-1} \tilde{\eta}$ be a Lyapunov function candidate for (2.34). It can be seen using (2.29) that V_1 satisfies

$$p_1 ||\tilde{\eta}||^2 \leq V_1(t,\tilde{\eta}) \leq p_2 ||\tilde{\eta}||^2 \tag{2.40}$$

Using the bounds (2.19), (2.20), (2.29) and (2.38), it can be shown that

$$\frac{\partial V_1}{\partial \tilde{\eta}} f_r(\hat{\eta},\hat{\xi},\tilde{\eta},0,t) \leq -a_1 ||\tilde{\eta}||^2, \quad \forall ||\tilde{\eta}|| \leq c_1, \forall t \geq t_0 \tag{2.41}$$

where a_1 and c_1 are positive constants independent of ε and c_1 is inversely proportional to k_1. We now define the set $\mathscr{M}$ so that the inequalities (2.40) and (2.41) are valid. Define the sets B_{c_1} and $\Omega_{t,c}$ by $B_{c_1} = \{||\tilde{\eta}|| \leq c_1\}$ and $\Omega_{t,c} = \{V_1(t,\tilde{\eta}) \leq c\}$ with $c = p_1 c_1^2$. Take $\mathscr{M} = \{||\tilde{\eta}|| \leq c_2\}$, where $c_2 \leq c_1\sqrt{p_1/p_2}$. This ensures that $\mathscr{M} \subset \Omega_{t,c} \subset B_{c_1} \subset R^{n-\rho}$.

For the boundary-layer system (2.39), the Lyapunov function $W(\chi) = \chi^T P_0 \chi$, where P_0 is the positive definite solution of $P_0\Lambda + \Lambda^T P_0 = -I$, satisfies

$$\lambda_{\min}(P_0)\,||\chi||^2 \leq W(\chi) \leq \lambda_{\max}(P_0)\,||\chi||^2 \tag{2.42}$$

$$\frac{\partial W}{\partial \chi}\Lambda\chi \leq -\,||\chi||^2 \tag{2.43}$$

Let $\Sigma = \{W(\chi) \leq \rho\varepsilon^2\}$ and $\mathscr{S} = \Omega_{t,c} \times \Sigma$, where ρ is a positive constant independent of ε. Since we are saturating $\hat{\xi}$ outside the compact set given in Assumption 2.2, this implies global boundedness of $C_1(\hat{\eta},\hat{\xi})$ and $\Delta\dot{b}(\eta,\xi,\hat{\eta},\hat{\xi},u)$ in $\hat{\xi}$. Using this fact, (2.41), Assumptions 2.1 and 2.2 and the fact that continuous functions are bounded on compact sets, for all $(\tilde{\eta},\chi) \in \mathscr{S}$ it can be shown that

$$\dot{V}_1 \leq -a_1\,||\tilde{\eta}||^2 + k_2\,||\chi|| \tag{2.44}$$

$$\dot{W} \leq -(\frac{1}{\varepsilon} - k_4)\,||\chi||^2 + k_3\,||\chi||\,||P_0|| \tag{2.45}$$

where k_2, k_3 and k_4 are positive constants independent of ε. Using analysis similar to [35], it can be shown that for an appropriate choice of ρ, there exists $\varepsilon_a > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_a$, the set $\mathscr{S}$ is positively invariant. Consider now the initial state $(\tilde{\eta}(0), \hat{\xi}(0), \hat{\sigma}(0)) \in \mathscr{M} \times \mathscr{N}$. Due to the fact that the right hand side of (2.34) is continuous and $\tilde{\eta}$ is in the interior of $\Omega_{t,c}$, we can show that there exists a finite time T_0, independent of ε, such that $\tilde{\eta}(t,\varepsilon) \in \Omega_{t,c}$, $\forall\, t \in [0, T_0]$. During this time period, χ will be bounded by an $O(1/\varepsilon^\rho)$ value. Furthermore, it can be verified that there exists ε_b, such that for all $0 < \varepsilon \leq \varepsilon_b$, $W(\chi(T(\varepsilon)),\varepsilon) \leq \rho\varepsilon^2$, where $T(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $T(\varepsilon) < T_0$. Choose $\varepsilon_1 = \min\{\varepsilon_a, \varepsilon_b\}$, then, for all $0 < \varepsilon \leq \varepsilon_1$, the trajectories enter the set $\mathscr{S}$ in a finite time period $[0, T(\varepsilon)]$ and remain thereafter. As a result, all trajectories are bounded.

From Assumption 2.1, $\dot{b}(\eta, \xi, u)$ is locally Lipschitz in its arguments. Moreover, it can be verified that for any $0 < \varepsilon_2 \leq 1$, there are positive constants L_1, L_2 and L_3, independent of ε, such that for all $(\tilde{\eta}, \chi) \in \mathscr{S}$ and every $0 < \varepsilon \leq \varepsilon_2$, we have

$$\left|\left|\Delta\dot{b}\right|\right| \leq L_1 \left|\left|\tilde{\eta}\right|\right| + L_2 \left|\left|\varphi\right|\right| \leq L_1 \left|\left|\tilde{\eta}\right|\right| + L_2 \left|\left|\chi\right|\right| \tag{2.46}$$

$$\left|\left| f_r(\hat{\eta}, \hat{\xi}, \tilde{\eta}, D_1\chi, t) - f_r(\hat{\eta}, \hat{\xi}, \tilde{\eta}, 0, t) \right|\right| \leq L_3 \left|\left|\chi\right|\right| \tag{2.47}$$

Consequently, by the use of (2.41), (2.46) and (2.47) and the fact that $\bar{\Delta}a$ is bounded from above by a linear function in $||\chi||$, it can be shown that

$$\dot{W} \leq -\frac{1}{\varepsilon} \left|\left|\chi\right|\right|^2 + b_1 \left|\left|\chi\right|\right|^2 + b_2 \left|\left|\chi\right|\right| \left|\left|\tilde{\eta}\right|\right| \tag{2.48}$$

$$\dot{V}_1 \leq -a_1 \left|\left|\tilde{\eta}\right|\right|^2 + a_2 \left|\left|\tilde{\eta}\right|\right| \left|\left|\chi\right|\right| \tag{2.49}$$

where b_1, b_2, a_1 and a_2 are positive constants. Consider the composite Lyapunov function $V_2(t, \tilde{\eta}, \chi) = V_1(t, \tilde{\eta}) + W(\chi)$. Using (2.48) and (2.49) we get $\dot{V}_2 \leq -\mathscr{Y}^T \Gamma \mathscr{Y}$, where

$$\Gamma = \begin{bmatrix} a_1 & -(b_2 + a_2)/2 \\ -(b_2 + a_2)/2 & 1/\varepsilon - b_1 \end{bmatrix} \quad \text{and} \quad \mathscr{Y} = \begin{bmatrix} ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}.$$

The matrix Γ will be positive definite for small ε_3, such that $0 < \varepsilon \leq \varepsilon_3$. Take $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Hence, for all $0 < \varepsilon \leq \varepsilon^*$, $t \geq t_0$ and $(\tilde{\eta}(0), \hat{\xi}(0), \hat{\sigma}(0))$ starting in $\mathscr{M} \times \mathscr{N}$, the estimation error $(\tilde{\eta}, \chi)$ converges exponentially to the origin. $\square$

2.4.1 Observer Design For a Synchronous Generator Connected to an Infinite Bus System

Consider the problem of estimating the angle and the speed of a synchronous generator connected to an infinite bus using only measurement of the field voltage. The model of the system is represented by [59]

$$\dot{\eta}_1 = \eta_2 \tag{2.50}$$

$$\dot{\eta}_2 = \frac{P}{M} - \frac{D}{M}\eta_2 - \frac{h_1}{M}\xi_1 \sin\eta_1 \tag{2.51}$$

$$\dot{\xi}_1 = -\frac{h_2}{\tau}\xi_1 + \frac{h_3}{\tau}\cos\eta_1 + \frac{u}{\tau} \tag{2.52}$$

where η_1 is the angle in radians, η_2 is the angular velocity, ξ_1 is the voltage (output), P is the mechanical input power, u is the field voltage (input), D is the damping coefficient, M is inertia, τ is time constant, and h_1, h_2, and h_3 are constant parameters. The relative degree of the system is one and it is not minimum phase. The auxiliary system is given by (2.50)-(2.51) with the output $\sigma = \frac{h_3}{\tau}\cos\eta_1$.

Following the procedure described in this section, a full order observer for the system (2.50)-(2.52) is designed as follows

$$\begin{aligned}
\dot{\hat{\eta}}_1 &= \hat{\eta}_2 + k_1(\hat{\sigma} - \frac{h_3}{\tau}\cos\hat{\eta}_1) \\
\dot{\hat{\eta}}_2 &= \frac{P}{M} - \frac{D}{M}\hat{\eta}_2 - \frac{h_1}{M}y\sin\hat{\eta}_1 + k_2(\hat{\sigma} - \frac{h_3}{\tau}\cos\hat{\eta}_1) \\
\dot{\hat{\xi}}_1 &= -\frac{h_2}{\tau}\hat{\xi}_1 + \hat{\sigma} + \frac{u}{\tau} + \frac{\alpha_1}{\varepsilon}(y - \hat{\xi}_1) \\
\dot{\hat{\sigma}} &= \phi_1(\hat{\eta}, \hat{\xi}) + \frac{\alpha_2}{\varepsilon^2}(y - \hat{\xi}_1)
\end{aligned}$$

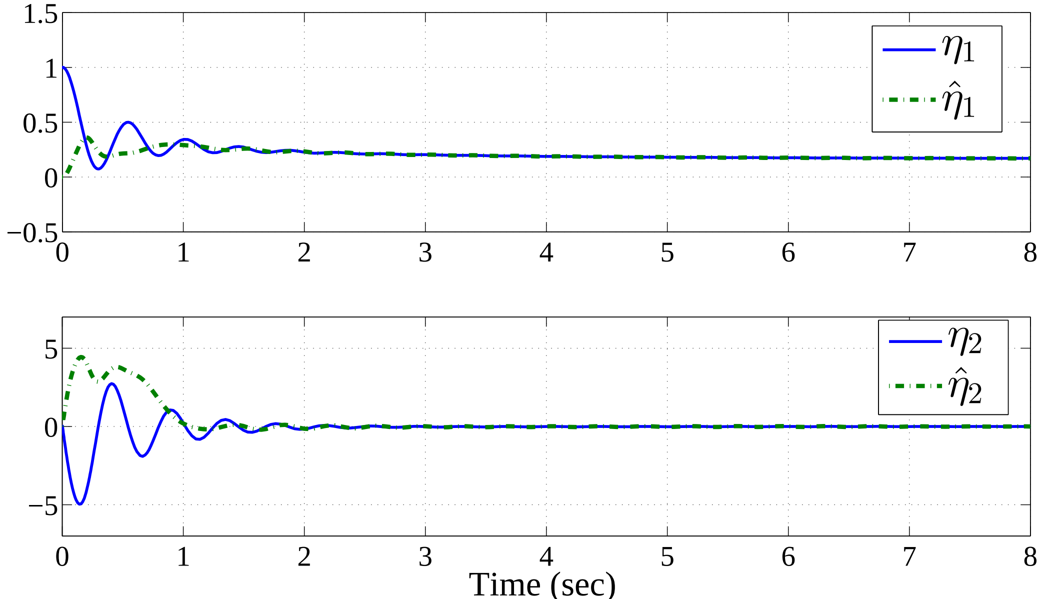

Figure 2.1: Estimation of the states η_1 and η_2. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

where the observer gain $L = [k_1 k_2]^T$ is given by (2.17) and (2.18),

$$\phi_1 = \frac{d\sigma}{dt}|_{(\hat{\eta},\hat{\xi})} = -\frac{h_3}{\tau}\hat{\eta}_2 \sin\hat{\eta}_1,$$

and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -(h_1/M)\hat{\xi}_1 \cos\hat{\eta}_1 & -D/M \end{bmatrix}, C_1 = \begin{bmatrix} -(h_3/\tau)\sin\hat{\eta}_1 & 0 \end{bmatrix}.$$

The system parameters are: $P = 0.815, h_1 = 2.0, h_2 = 2.7, h_3 = 1.7, \tau = 6.6, M = 0.0147$, and $D/M = 4$. The system was simulated for a constant input voltage $u = 5$. The initial states are $\eta_1(0) = 1, \eta_2(0) = 0.1, y(0) = 1, \hat{\xi}_1(0) = 0.5, \hat{\eta}_1(0) = 0$ and $\hat{\eta}_2(0) = 0$. The observer parameters were chosen to be $R = 1$ and $Q = \begin{bmatrix} 500 & 0 \\ 0 & 500 \end{bmatrix}, \alpha_1 = 5, \alpha_2 = 1$, and $\varepsilon = 0.05$.

It was observed that for $y(0) = [-35, 35], \eta_1(0) = [-4\pi, 4\pi]$ and $\eta_2(0) = [-1, 1]$, the system has bounded trajectories. For these ranges, it was found that ξ_1 satisfies $\xi_1 < |40|$ and σ satisfies

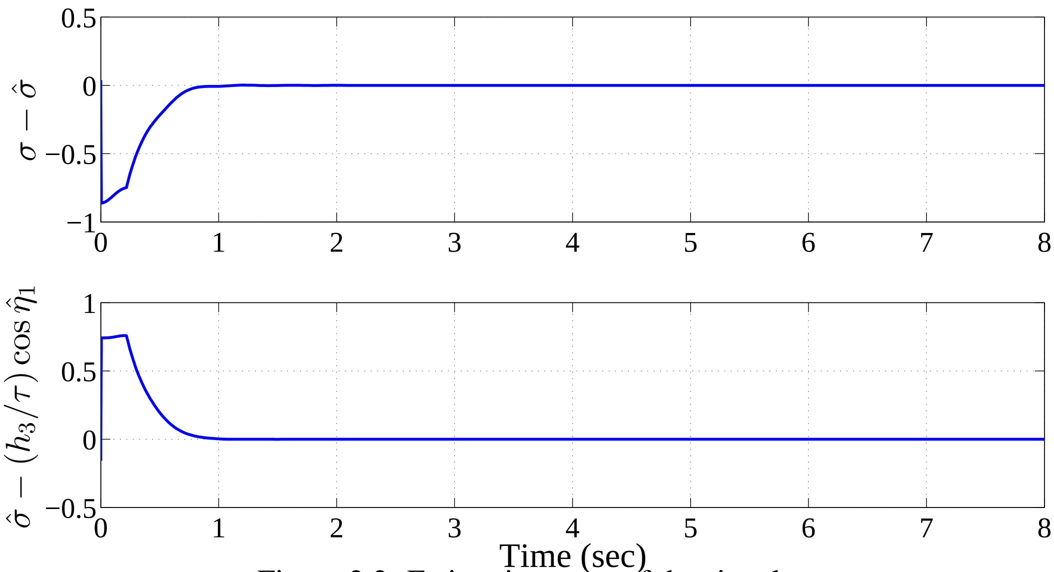

Figure 2.2: Estimation error of the signal σ.

$\sigma < |0.5|$. Therefore, in order to protect the observer from peaking, we saturate $\hat{\xi}_1$ and $\hat{\sigma}$ at ± 50 and ± 1, respectively. Fig. 2.1 shows the states η_1 and η_2 versus their estimates. It is clear that the estimated states converged to their actual values in a fairly fast time. Fig. 2.2 shows the estimation error $(\sigma - \hat{\sigma})$ and the estimation error $(\hat{\sigma} - (h_3/\tau)\cos\hat{\eta}_1)$, which is the innovation term of the internal observer.

2.5 Special case: System linear in the internal state η

We consider a special case of the normal form $(2.5)-(2.7)$, where the system is linear in η as follows

$$\dot{\eta} = A_1(\xi)\eta + \phi_0(\xi) \tag{2.53}$$

$$\dot{\xi} = A\xi + B[C_1(\xi)\eta + a(\xi, u)] \tag{2.54}$$

$$y = C\xi \tag{2.55}$$

Assumption 2.5 *The functions $A_1(\xi)$ and $\phi_0(\xi)$ are known and locally Lipschitz. Furthermore, the functions $C_1(\xi)$ and $a(\xi,u)$ are known and continuously differentiable with local Lipschitz derivatives.*

In this case, the auxiliary system is of the form

$$\dot{\eta} = A_1(\xi)\eta + \phi_0(\xi), \quad \sigma = C_1(\xi)\eta \tag{2.56}$$

In a similar way to the general case, we utilize the EHGO to provide estimates of ξ and σ. This allows us to assume that these states are available for the internal observer. Since the functions A_1 and C_1 are continuously dependent on the state ξ, and hence time varying, we again choose to use a Kalman-like observer for the auxiliary problem. The full-order observer takes the form

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} + a(\hat{\xi},u)] + H(\varepsilon)(y - C\hat{\xi}) \tag{2.57}$$

$$\dot{\hat{\sigma}} = \phi_1(\hat{\eta},\hat{\xi},u) + (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - C\hat{\xi}) \tag{2.58}$$

$$\dot{\hat{\eta}} = A_1(\hat{\xi})\hat{\eta} + \phi_0(\hat{\xi}) + L(\hat{\sigma} - C_1(\hat{\xi})\hat{\eta}) \tag{2.59}$$

where

$$\phi_2(\hat{\eta},\hat{\xi},u) = \frac{d}{dt}[C_2(\xi)\eta]|_{(\hat{\eta},\hat{\xi})}.$$

It should be mentioned that Remark 2.2 is applicable to the function ϕ_2. The observer gains $L(t)$ and $H(\varepsilon)$ with $\varepsilon_{\rho+1}$ and $\alpha_{\rho+1}$ are designed in the same way as in the previous section. In this case we also need Assumption 2.4.

In a similar discussion to the one that followed Assumption 2.4, there have been several results in the literature that dealt with verifying that assumption for systems similar to (2.56). For instance, in [50] Assumption 2.4 was shown to be satisfied for systems in which A_1 is dependent only on the output in a lower triangular way, C_1 is constant and the function $\phi_0(.)$ depends on the output and affine in the input. Another example is given in [54], where it was shown, in the context of designing an adaptive high gain EKF, that Assumption 2.4 is satisfied for a special class of systems where the pair (A_1, C_1) represents a chain of integrators that are dependent, on bounded away from zero, nonlinear functions of the input, and the function $\phi_0(.)$ is a lower triangular nonlinear function dependent on both the state and the input.

Notice that in this case we do not need to linearize with respect to η around the point $(\hat{\eta}, \hat{\xi})$. Consequently, we are able to achieve a semi-global stabilization result for the full order observer. The following theorem summarizes this finding.

Theorem 2.2 *Consider the full order observer* (2.57)-(2.59) *for the system* (2.53)-(2.55). *Let Assumptions 2.2, 2.4 and 2.5 hold. Then given any compact sets $\mathscr{M} \subset R^{n-\rho}$ containing the origin and $\mathscr{N} \subset R^{\rho+1}$, for all $\tilde{\eta} \in \mathscr{M}$, $(\hat{\xi}, \hat{\sigma}) \in \mathscr{N}$, there exists ε^* such that, for all $0 < \varepsilon < \varepsilon^*$, the estimation error converges exponentially to zero as $t \to \infty$.*

Proof:

The proof of this theorem follows closely the proof of Theorem 2.1. Consequently, we will mostly emphasize the differences. The estimation error equations are given by

$$\begin{aligned}\dot{\tilde{\eta}} =& A_1(\hat{\xi} + D\varphi)[\tilde{\eta} + \hat{\eta}] - A_1(\hat{\xi})\hat{\eta} + \phi_0(\hat{\xi} + D\varphi) - \phi_0(\hat{\xi}) - L(t)[C_1(\hat{\xi} + D\varphi)[\tilde{\eta} + \hat{\eta}] \\ & - \chi_{\rho+1} - C_1(\hat{\xi})\hat{\eta})]\end{aligned} \tag{2.60}$$

$$\triangleq f_r(\hat{\eta}, \hat{\xi}, \tilde{\eta}, D_1\chi, t) \tag{2.61}$$

$$\varepsilon\dot{\chi} = \Lambda\chi + \varepsilon[\bar{B}_1\Delta\phi_2 + \bar{B}_2\bar{\Delta}a] \tag{2.62}$$

where $\tilde{\eta}, \chi, \varphi, D, D_1, \bar{B}_1, \bar{B}_2$, $\bar{\Delta a}$ and Λ are defined in Section 2.4, $\chi_{\rho+1} = C_1(\xi)\eta - \hat{\sigma}$ and $\Delta\phi_2 = \phi_2(\tilde{\eta} + \hat{\eta}, \hat{\xi} + D\varphi, u) - \phi_2(\hat{\eta}, \hat{\xi}, u)$. Setting $\varepsilon = 0$, we get $\chi = 0$, and the reduced system $\dot{\tilde{\eta}} = [A_1(\hat{\xi}) - LC_1(\hat{\xi})]\tilde{\eta}$. Using the bounds (2.19), (2.20) and (2.29), it can be shown that $V_1(t, \tilde{\eta}) = \tilde{\eta}^T P^{-1} \tilde{\eta}$ satisfies

$$\frac{\partial V_1}{\partial \tilde{\eta}} f_r(\tilde{\eta}, 0, t) \le -c_1 \|\tilde{\eta}\|^2, \forall \tilde{\eta} \in R^{n-\rho}, \forall t \ge t_0 \tag{2.63}$$

where c_1 is a positive constant independent of ε. Choose a positive constant c such that $c > \max_{\tilde{\eta} \in \mathscr{M}} V_1(t, \tilde{\eta})$. This yields $\mathscr{M} \subset \Omega_{t,c} = \{V_1(t, \tilde{\eta}) \le c\} \subset R^{n-\rho}$. The rest of the proof follows identical arguments to the proof of Theorem 2.1. □

2.5.1 Observer Design For The Translating Oscillator With a Rotating Actuator System

Consider the problem of designing a full order observer for a Translating Oscillator with a Rotating Actuator (TORA) system. The model equations in the normal form are given by [60], [61]

$$\dot{\eta}_1 = \eta_2 \tag{2.64}$$

$$\dot{\eta}_2 = \frac{k}{m_r + m_c}\left(\frac{m_r l_r \sin \xi_1}{m_r + m_c} - \eta_1\right) \tag{2.65}$$

$$\dot{\xi}_1 = \xi_2 \tag{2.66}$$

$$\dot{\xi}_2 = \frac{1}{\delta(\xi_1)}\left\{(m_r + m_c)u - m_r l_r \cos \xi_1 \left[m_r l_r \xi_2^2 \sin \xi_1 - k\left(\eta_1 - \frac{m_r l_r \sin \xi_1}{m_r + m_c}\right)\right]\right\} \tag{2.67}$$

$$y = \xi_1 \tag{2.68}$$

where $\delta(\xi_1) = (I + m_r l_r^2)(m_r + m_c) - m_r^2 l_r^2 \cos^2 \xi_1 > 0$. m_r and I are the mass and moment of inertia of the rotating proof mass, respectively, l_r is the distance from its rotational axis, m_c is the mass of the platform, and k is the spring constant. The normal form (2.64)-(2.68) was obtained using the change of variables

$$\xi_1 = \theta, \quad \xi_2 = \dot{\theta}, \quad \eta_1 = x_c + \frac{m_r l_r \sin\theta}{m_r + m_c} \quad \text{and} \quad \eta_2 = \dot{x}_c + \frac{m_r l_r \dot{\theta}\cos\theta}{m_r + m_c},$$

where θ is the angular position of the proof mass (measured counter clock wise) and x_c is the translational position of the platform. We assume that y is the only measured variable. This system has the special form (2.53)-(2.55) and is not minimum phase. The auxiliary system is given by (2.64)-(2.65) with the output

$$\sigma = \frac{k m_r l_r \cos y}{\delta(y)}\eta_1.$$

Following the technique presented in this section, a full order observer for the system (2.64)-(2.68) can be taken as

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + (\alpha_1/\varepsilon)(y - \hat{\xi}_1) \tag{2.69}$$

$$\dot{\hat{\xi}}_2 = \hat{\sigma} + a(\hat{\xi}, u) + (\alpha_2/\varepsilon^2)(y - \hat{\xi}_1) \tag{2.70}$$

$$\dot{\hat{\sigma}} = \phi_2(\hat{\eta}, \hat{\xi}) + (\alpha_3/\varepsilon^3)(y - \hat{\xi}_1) \tag{2.71}$$

$$\dot{\hat{\eta}}_1 = \hat{\eta}_2 + L_1(\hat{\sigma} - \frac{k m_r l_r \cos y}{\delta(y)}\hat{\eta}_1) \tag{2.72}$$

$$\dot{\hat{\eta}}_2 = \frac{k}{m_r + m_c}(\frac{m_r l_r \sin y}{m_r + m_c} - \hat{\eta}_1) + L_2(\hat{\sigma} - \frac{k m_r l_r \cos y}{\delta(y)}\hat{\eta}_1), \tag{2.73}$$

where

$$a(\hat{\xi}, u) = \frac{1}{\delta(y)}[(m_r + m_c)u - \frac{m_r^2 l_r^2 k \sin y \cos y}{m_r + m_c} - m_r^2 l_r^2 \hat{\xi}_2^2 \sin y \cos y],$$

and

$$\phi_2(\hat{\eta}, \hat{\xi}) = \frac{k m_r l_r \hat{\eta}_2 \cos y}{\delta(y)} - \frac{k m_r l_r \hat{\eta}_1 \hat{\xi}_2}{\delta(y)^2}[\delta(y)\sin y + 2m_r^2 l_r^2 \cos^2 y \sin y].$$

The observer gain $L = [L_1\, L_2]^T$, is given by (2.17), and P is the solution of (2.18). The matrices A_1 and C_1 are

$$A_1 = \begin{bmatrix} 0 & 1 \\ -k/(m_r+m_c) & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} (km_r l_r \cos y)/\delta(y) & 0 \end{bmatrix}.$$

We will examine the performance of the observer when the system is stabilized by a control u that is given by [61]

$$u = -\beta\, \mathrm{sat}\left(\frac{\hat{\xi}_2 + k_2 y - k_1(\hat{\eta}_1 - (m_r l_r \sin y)/(m_r+m_c))\cos y}{\mu} \right) \tag{2.74}$$

where the positive constants k_1, k_2, β and μ are design parameters and $\mathrm{sat}(\cdot)$ is the saturation function. After an exhaustive search it was found that the system (2.64)-(2.68) with (2.74), under state feedback, has a reasonable performance for the following ranges of the initial states: $\eta_1(0) = [-0.1, 0.1], \eta_2(0) = [-0.1, 0.1], \xi_1(0) = [-2\pi, 2\pi]$ and $\xi_2(0) = [-10, 10]$. We also found that, for these ranges, $\xi_1 \in [-10, 10]$, $\xi_2 \in [-55, 55]$ and $\sigma \in [-140, 140]$. Consequently, we chose to saturate $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{\sigma}$ at the values $\pm 15, \pm 60$ and ± 145, respectively, to guard against peaking. The system was simulated using the following system parameters [61]: $m_c = 1.3608\,kg$, $m_r = 0.096\,kg$, $l_r = 0.0592\,m$, $I = 0.0002175\,kg\,m^2$, $k = 186.3\,N/m$. The controller parameters are chosen as: $k_1 = 1000$, $k_2 = 5$, $\beta = 0.2$ and $\mu = 0.1$, and the observer parameters are chosen to be: $\varepsilon = 0.003$, $\alpha_1 = 3$, $\alpha_2 = 3$, $\alpha_3 = 1$, $R = 1$ and $Q = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$. The initial values are chosen as: $\xi_1(0) = 0.5$, $\xi_2(0) = 1$, $\eta_1(0) = 0.05$, $\eta_2(0) = 0.05$, $\hat{\xi}_1(0) = 0$, $\hat{\xi}_2(0) = 0$, $\hat{\eta}_1(0) = 0$, $\hat{\eta}_2(0) = 0$ and $P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Figs. 2.3 and 2.4 show the original system states ($\dot{\theta}$, x_c, $\dot{x}_c$) and their estimates. It is clear that the observer reconstructed the system states fairly quickly.

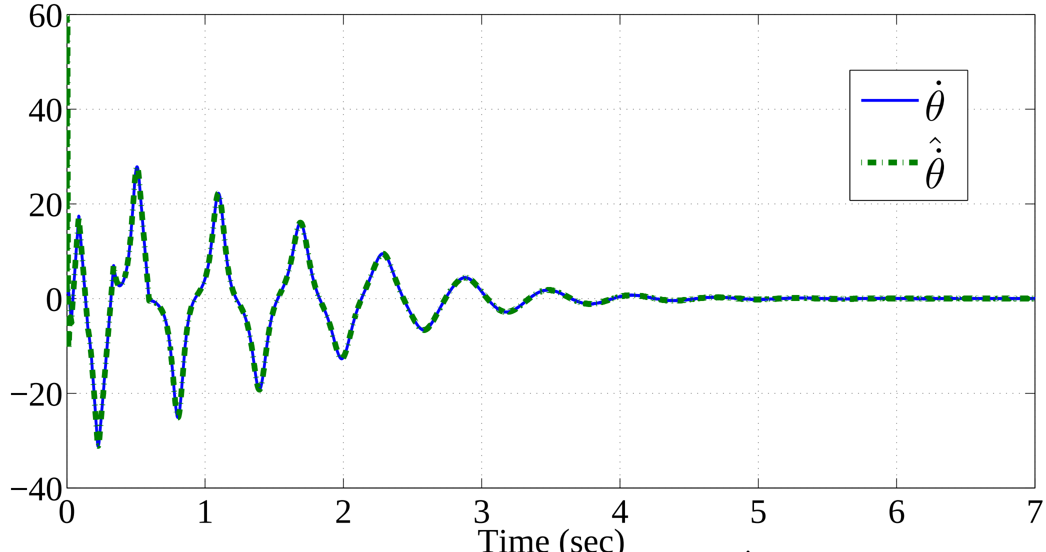

Figure 2.3: Estimation of the state $\dot{\theta}$.

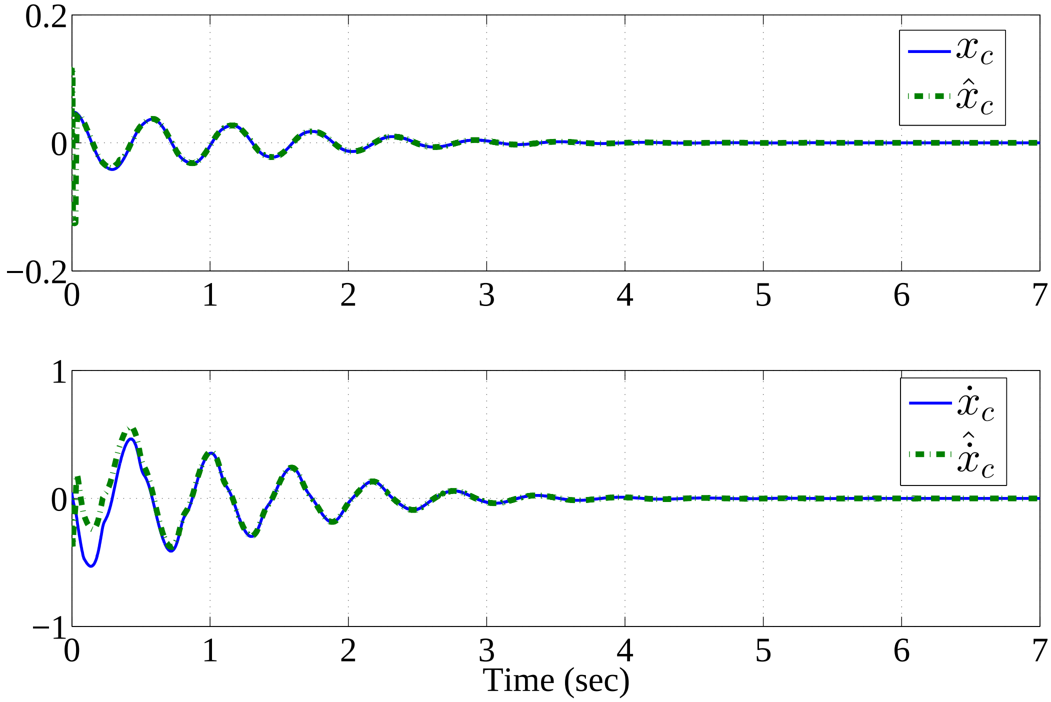

Figure 2.4: Estimation of the internal states x_c and $\dot{x}_c$.

2.6 Conclusions

We have proposed a full order observer for a class of nonlinear systems that could be non-minimum phase. The observer is based on the use of the EHGO to estimate, in a relatively fast time, the derivatives of the output and a signal that is used as a virtual output to an auxiliary system comprised of the internal dynamics. Accordingly, we chose to design an EKF observer for this system. The reason for this choice is primarily because of the EKF's simplicity and wide use in practice. If the system is linear in the internal state we achieve non-local convergence results. In the general case, the convergence result is local in the estimation error of the internal states. Finally, we demonstrated the effectiveness of the proposed observer by using it to estimate the full state vector for two systems, namely, a synchronous generator connected to an infinite bus and the TORA system.

Chapter 3

Output Feedback Stabilization of Possibly Non-Minimum Phase Nonlinear Systems

3.1 Introduction

There have been many results in the literature regarding the problem of output feedback stabilization of nonlinear systems. In recent years, more attention was directed towards the study of non-minimum phase nonlinear systems. One of the early papers to study stabilization of non-minimum phase nonlinear systems is [37]. This paper proves semi-global stabilization for a general class of non-minimum phase nonlinear systems assuming the existence of a dynamic stabilizing controller for an auxiliary system. The same problem was considered in [31], where robust semi-global stabilization was achieved under similar assumptions but with an extended high-gain observer-based output feedback controller.

Papers [39], [40], [38] and [6] solve the problem of output feedback stabilization for nonlinear systems that can have unstable zero dynamics. Paper [39] allows model uncertainty and is based on the observer introduced in [32]. Paper [40] assumes the system will be minimum phase with respect to a new output, defined as a linear combination of the state variables. Paper [38] assumes the knowledge of an observer and provides different approaches to design the control law. Papers [39], [40] and [38] achieve global stabilization results under various stabilizability conditions on the internal dynamics. Paper [6] solves the stabilization problem for systems with linear zero dynamics. It uses the backstepping technique and an observer with linear error dynamics to achieve semi-global stabilization. Another result reported in [41] deals with a special case of the normal form where the internal dynamics are modelled as a chain of integrators. It uses an adaptive output feedback controller, based on Neural Networks and a linear state observer, to achieve ultimate boundedness of the states in the presence of model uncertainties.

We make use of the Extended High-Gain Observer-Extended Kalman Filter (EHGO-EKF) approach, proposed in Chapter 2, to solve the problem of output feedback stabilization. In Chapter 2, we showed that we could use the extended high-gain observer to provide the derivatives of the measured output plus an extra derivative. This extra state is then used to provide a virtual output that can be used to make the internal dynamics observable. An extended Kalman filter can then use the virtual output to provide estimates of the remaining states; this is indeed possible thanks to the difference in time scale provided by the EHGO. The advantage of this observer is that it allows us to design the feedback control as if all the state variables were available. This is in contrast to previous high-gain observer results, which were mostly limited to partial state feedback and hence only to minimum phase systems.

In Chapter 2, we considered systems in the general normal form and proved local convergence for the EHGO-EKF observer. In the special case when the system is linear in the internal states, we proved a semi-global convergence result. The work presented in this chapter is related to this special case.[1] We show that when the observer is used in feedback control we can achieve semi-global stabilization. We allow the use of any globally stabilizing state feedback control. Furthermore, the class of systems considered includes non-minimum-phase systems. The proposed

[1]In this chapter, however, we allow the system to depend on the control in a more general manner.

controller has the ability to recover the state feedback response of an auxiliary system that is based on the original system. This is beneficial as we can shape the response of the auxiliary system using the state feedback control as desired, with the knowledge that we will be able to recover it using the proposed output feedback.

The rest of the chapter is organized as follows. Section 3.2 describes the class of systems and formulates the problem. In Section 3.3, we describe the output feedback control and state two theorems that describe the stability and trajectory recovery properties of the closed loop system. Section 3.4 presents an example of stabilizing a DC-to-DC boost converter system to illustrate the design procedure with simulation. Finally, Section 3.5 includes some concluding remarks.

3.2 Problem Formulation

We consider a single-input, single-output nonlinear system in the form

$$\dot{\eta} = A_1(\xi,u)\eta + \phi_0(\xi,u) \tag{3.1}$$

$$\dot{\xi} = A\xi + B[C_1(\xi,u)\eta + a(\xi,u)] \tag{3.2}$$

$$y = C\xi \tag{3.3}$$

where $\eta \in R^{n-\rho}, \xi \in R^{\rho}$, y is the measured output and u is the control input. The $\rho \times \rho$ matrix A, the $\rho \times 1$ matrix B and the $1 \times \rho$ matrix C represent a chain of ρ integrators and the functions $A_1(.,.), \phi_0(.,.), C_1(.,.)$ and $a(.,.)$ are known.

In the case that the nonlinear functions A_1, ϕ_0 and C_1 are independent of the control u, (3.1)-(3.3) is a special case of the normal form [56], where the system dynamics are linear in the internal state η. An example of systems of this type is the Translating Oscillator with a Rotating Actuator

(TORA) system [60].

Assumption 3.1 *The functions $A_1(\xi,u)$, $\phi_0(\xi,u)$, $a(\xi,u)$ and $C_1(\xi,u)$ are sufficiently smooth. Furthermore, $\phi_0(0,0) = 0$ and $a(0,0) = 0$.*

The goal is to stabilize the origin of the system (3.1)-(3.3) using only the measured output y.

3.3 State Feedback

We consider the system (3.1)-(3.3), and assume the existence of globally stabilizing state feedback control of the form

$$u = \gamma(\eta,\xi) \tag{3.4}$$

Rewrite the full state vector as $\vartheta = [\eta^T \xi^T]^T$, so the closed loop system will be

$$\dot{\vartheta} = f(\vartheta, \gamma(\eta,\xi)) \tag{3.5}$$

The control should have the following properties.

Assumption 3.2 1. *γ is continuously differentiable with locally Lipschitz derivatives and $\gamma(0,0) = 0$.*

2. *The origin of the closed-loop system* (3.5) *is globally asymptotically stable.*

3.4 Output Feedback

In this section, we will design an output feedback controller that is based on the observer

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} + a(\hat{\xi}, u)] + H(\varepsilon)(y - C\hat{\xi}) \tag{3.6}$$

$$\dot{\hat{\sigma}} = \phi_1(\hat{\eta}, \hat{\xi}) + (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - C\hat{\xi}) \tag{3.7}$$

$$\dot{\hat{\eta}} = A_1(\hat{\xi}, u)\hat{\eta} + \phi_0(\hat{\xi}, u) + L(t)(M_1 \mathrm{sat}(\frac{\hat{\sigma}}{M_1}) - C_1(\hat{\xi}, u)\hat{\eta}) \tag{3.8}$$

where $\phi_1(\eta, \xi)$ is the derivative of $C_1(\xi, u)\eta$ under the state feedback control $u = \gamma(\eta, \xi)$, i.e.,

$$\begin{aligned}\phi_1(\eta, \xi) = & \frac{\partial[C_1(\xi, \gamma(\eta, \xi))\eta]}{\partial \eta}[A_1(\xi, \gamma(\eta, \xi))\eta + \phi_0(\xi, \gamma(\eta, \xi))] \\ & + \frac{\partial[C_1(\xi, \gamma(\eta, \xi))\eta]}{\partial \xi}[A\xi + B[C_1(\xi, \gamma(\eta, \xi))\eta + a(\xi, \gamma(\eta, \xi))]]\end{aligned}$$

The observer gain $H(\varepsilon)$ is given by

$$H(\varepsilon) = [\alpha_1/\varepsilon, ..., \alpha_\rho/\varepsilon^\rho]^T.$$

The constants $\alpha_1, ..., \alpha_{\rho+1}$ are chosen such that the polynomial $s^{\rho+1} + \alpha_1 s^\rho + ... + \alpha_{\rho+1}$ is Hurwitz and $\varepsilon > 0$ is a small parameter. The observer gain $L(t, \varepsilon)$ is given by

$$L = PC_1^T R^{-1} \tag{3.9}$$

and $P(t, \varepsilon)$ is the solution of the Riccati equation

$$\dot{P} = A_1 P + PA_1^T + Q - PC_1^T R^{-1} C_1 P, \quad P(t_0) = P_0 > 0. \tag{3.10}$$

where $A_1(.,.)$ and $C_1(.,.)$ are functions of $\hat{\xi}$ and $\gamma(\hat{\eta},\hat{\xi})$, $R(t)$ satisfies

$$0 < r_1 \leq R(t) \leq r_2 \tag{3.11}$$

and $Q(t)$ is a symmetric positive definite matrices that satisfies

$$0 < q_1 I_{n-\rho} \leq Q(t) \leq q_2 I_{n-\rho} \tag{3.12}$$

The observer (3.6)-(3.8) is to be used with the control

$$u = \gamma(\hat{\eta},\hat{\xi}). \tag{3.13}$$

Assumption 3.3 *The nonlinear functions* $A_1(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))$, $\phi_0(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))$, $a(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))$, $C_1(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))$, $\gamma(\hat{\eta},\hat{\xi})$ *and* $\phi_1(\hat{\eta},\hat{\xi})$ *are globally bounded in* $\hat{\xi}$.

The assumption of global boundedness in $\hat{\xi}$, which comes from the extended high-gain observer, is essential to protect against peaking. It is not restrictive, as it can be done by using saturation outside compact sets to which the trajectories belong to under state feedback [22]. The saturation functions need to be also continuously differentiable.[2] The variable $\hat{\sigma}$ is saturated in (2.59) using the standard sat(.) function. This is to prevent peaking in $\hat{\eta}$. The saturation level M_1 is determined such that $M_1 \geq \max|C_1(\xi,u)\eta|$ under state feedback. This procedure is illustrated in the example presented in Section 3.5.

[2]Reference [26] shows an example of a continuously differentiable saturating function and also an approach on how to analyze the system in the case of the standard piecewise-continuous saturation function.

Assumption 3.4 *The Riccati equation* (3.10) *has a positive definite solution that satisfies*

$$0 < p_1 I_{n-\rho} \leq P^{-1}(t) \leq p_2 I_{n-\rho}, \quad \forall t \geq 0 \tag{3.14}$$

This assumption is essential for the stability of the EHGO-EKF observer as will be shown in the proof of Theorem 3.1.[3]

We will study the properties of the closed loop system with the estimation error dynamics. To that end, let us use the following rescaling of the estimation error:

$$\tilde{\eta} = \eta - \hat{\eta} \tag{3.15}$$

$$\chi_i = (\xi_i - \hat{\xi}_i)/\varepsilon^{\rho+1-i}, \qquad 1 \leq i \leq \rho \tag{3.16}$$

$$\chi_{\rho+1} = C_1(\xi, \gamma(\hat{\eta}, \xi))\eta - \hat{\sigma} \tag{3.17}$$

Let $\varphi = [\chi_1, \chi_2, ..., \chi_\rho]^T$ and $D(\varepsilon) = \text{diag}[\varepsilon^\rho, \varepsilon^{\rho-1}, ..., \varepsilon]$ so that (3.16) becomes

$$D(\varepsilon)\varphi = \xi - \hat{\xi}. \tag{3.18}$$

Moreover, let $\chi = [\varphi^T \ \chi_{\rho+1}]^T$ and $D_1(\varepsilon) = \text{diag}[D, 1]$, thus we have,

$$D_1(\varepsilon)\chi = \begin{bmatrix} \xi - \hat{\xi} \\ C_1(\xi, \gamma(\hat{\eta}, \xi))\eta - \hat{\sigma} \end{bmatrix}.$$

[3]Examples of classes of systems where this assumption is satisfied can be found in [55], [50] and in the context of designing an adaptive high gain observer in [54].

Therefore, the closed loop system under the output feedback control (3.13) takes the form

$$\dot{\eta} = A_1(\xi, \gamma(\hat{\eta}, \hat{\xi}))\eta + \phi_0(\xi, \gamma(\hat{\eta}, \hat{\xi})) \tag{3.19}$$

$$\dot{\xi} = A\xi + B[C_1(\xi, \gamma(\hat{\eta}, \hat{\xi}))\eta + a(\xi, \gamma(\hat{\eta}, \hat{\xi}))] \tag{3.20}$$

$$\begin{aligned}\dot{\hat{\eta}} =& A_1(\xi, \gamma(\hat{\eta}, \hat{\xi}))\eta - A_1(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi}))\hat{\eta} + \phi_0(\xi, \gamma(\hat{\eta}, \hat{\xi})) - \phi_0(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi})) \\ & - P(t, \varepsilon)C_1(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi}))^T R^{-1}[M_1 \mathrm{sat}\left(\frac{C_1(\xi, \gamma(\hat{\eta}, \xi))\eta - \chi_{\rho+1}}{M_1}\right) - C_1(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi}))\hat{\eta}]\end{aligned} \tag{3.21}$$

$$\varepsilon\dot{\chi} = \Lambda\chi + \varepsilon[\bar{B}_1\Delta\phi_1 + \bar{B}_2\Delta\bar{\phi}_2] \tag{3.22}$$

where

$$\Lambda = \begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ -\alpha_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ -\alpha_\rho & 0 & 0 & \ddots & 1 \\ -\alpha_{\rho+1} & 0 & 0 & \dots & 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 \\ B \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$
\begin{aligned}
\Delta\phi_1 =& \bar{\phi}_1(\eta,\xi,\hat{\eta},\hat{\xi},\varepsilon) - \phi_1(\hat{\eta},\hat{\xi}),\\
\bar{\phi}_1(\eta,\xi,\hat{\eta},\hat{\xi},\varepsilon) =& \frac{[\partial C_1(\xi,\gamma(\hat{\eta},\xi))\eta]}{\partial \xi}[A\xi + B[C_1(\xi,\gamma(\hat{\eta},\hat{\xi}))\eta + a(\xi,\gamma(\hat{\eta},\hat{\xi}))]]\\
&+ C_1(\xi,\gamma(\hat{\eta},\xi))[A_1(\xi,\gamma(\hat{\eta},\hat{\xi}))\eta + \phi_0(\xi,\gamma(\hat{\eta},\hat{\xi}))]\\
&+ \frac{[\partial C_1(\xi,\gamma(\hat{\eta},\xi))\eta]}{\partial \hat{\eta}}[A_1(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))\hat{\eta} + \phi_0(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))\\
&\quad + P(t)C_1(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))^T R^{-1}\times\\
&\qquad [M_1 \mathrm{sat}\left(\frac{C_1(\xi,\gamma(\hat{\eta},\xi))\eta - \chi_{\rho+1}}{M_1}\right) - C_1(\hat{\xi},\gamma(\hat{\eta},\hat{\xi}))\hat{\eta}]],\\
\Delta\phi_2 =& \Delta\phi_3/\varepsilon,\\
\Delta\phi_3(\eta,\xi,\hat{\eta},\hat{\xi}) =& a(\xi,\gamma(\hat{\eta},\hat{\xi})) - a(\hat{\xi},\gamma(\hat{\eta},\hat{\xi})) + C_1(\xi,\gamma(\hat{\eta},\hat{\xi}))\eta - C_1(\xi,\gamma(\hat{\eta},\xi))\eta.
\end{aligned}
$$

It can be shown that, for all $(\eta,\xi,\tilde{\eta},D(\varepsilon)\varphi) \in Q \subset R^{2n}$ where Q is a compact set, $\Delta\phi_2$ is locally Lipschitz and bounded from above by linear in $||\chi||$ function, uniformly in ε. The matrix Λ is $\rho+1\times\rho+1$ Hurwitz by design. Using (3.15) and (3.18), it can be seen that the closed-loop system has an equilibrium point at the origin $\eta = \hat{\eta} = 0$ and $\xi = \hat{\xi} = 0$. We also notice that the sat(.) function has no effect at the origin. Therefore, it is not difficult to show that $\Delta\phi_1 = 0$ and $\Delta\phi_2 = 0$ at the origin. Based on the above observations, we see that the system (3.19)-(3.22) is in the standard singularly perturbed form and has an equilibrium point at the origin. It should be noted that (3.10) is part of the estimation error dynamics and has the property mentioned in Assumption 3.4. However, in what follows, we will only show stability of the origin of the system (3.19)-(3.22).

3.4.1 Stability Recovery

Let us define the initial states as $(\eta(0), \xi(0), \hat{\eta}(0)) \in \mathscr{M}$ and $(\hat{\xi}(0), \hat{\sigma}(0)) \in \mathscr{N}$, where $\mathscr{M}$, containing the origin, and $\mathscr{N}$ are any known compact subsets of $R^{2n-\rho}$ and $R^{\rho+1}$, respectively. Thus, we have

$$\tilde{\eta}(0) = \eta(0) - \hat{\eta}(0), \quad \varphi(0) = D^{-1}(\varepsilon)[\xi(0) - \hat{\xi}(0)]$$

and

$$\chi_{\rho+1}(0) = C_1(\xi(0), \gamma(\hat{\eta}(0), \xi(0)))\eta(0) - \hat{\sigma}(0).$$

We are now ready to present the following theorem, which describes the stability properties of the closed-loop system under output feedback control.

Theorem 3.1 *Consider the closed loop system* (3.19)-(3.22)*. Let Assumptions 3.1-3.4 hold. Then for trajectories* $(\eta, \xi, \tilde{\eta}) \times (\hat{\xi}, \hat{\sigma})$*, starting in* $\mathscr{M} \times \mathscr{N}$*, we have the following*

- *There exists* ε_1^**, such that for all* $0 < \varepsilon < \varepsilon_1^*$*, the origin of the closed loop system is asymptotically stable and* $\mathscr{M} \times \mathscr{N}$ *is a subset of its region of attraction.*

- *If the origin of the system* (3.5) *is exponentially stable then, there exists* ε_2^**, such that for all* $0 < \varepsilon < \varepsilon_2^*$*, the origin of the closed loop system is exponentially stable and* $\mathscr{M} \times \mathscr{N}$ *is a subset of its region of attraction.*

Proof:

For convenience, rewrite (3.19)-(3.20) as $\dot{\vartheta} \triangleq f_a(\vartheta, \tilde{\eta}, D\varphi)$ and (3.21) as $\dot{\tilde{\eta}} = f_b(\vartheta, \tilde{\eta}, t, D_1\chi)$. Assumption 3.2 and the converse Lyapunov theorem, see [[61], Th. 4.17], imply the existence of a smooth positive definite radially unbounded function $V_1(\vartheta)$, and a continuous positive definite function $\alpha(\vartheta)$, such that

$$\frac{\partial V_1}{\partial \vartheta} f_a(\vartheta, 0, 0) \leq -\alpha(\vartheta), \quad \forall \vartheta \in R^n \tag{3.23}$$

Recall that $\varsigma = [\eta^T \xi^T \tilde{\eta}^T]^T$ and consider the composite Lyapunov function candidate $V_3(t,\varsigma) = qV_1(\vartheta) + (\tilde{\eta}^T P^{-1} \tilde{\eta})^{1/2}$ for the system (3.19)-(3.21), where q is a positive constant to be chosen. It can also be argued from the positive definiteness of V_1, see [[61], Lemma 4.3], and (3.14) that there exist class $\mathscr{K}_\infty$ functions β_1 and β_2 such that V_3 satisfies

$$\beta_1(||\varsigma||) \leq V_3(t,\varsigma) \leq \beta_2(||\varsigma||) \tag{3.24}$$

Define $\Omega \triangleq \{V_3(t,\tilde{\eta}) \leq c\}$. For any $c > 0$, $\{\beta_2(||\varsigma||) \leq c\}$ is a compact subset of $R^{2n-\rho}$. Now choose $c > \max_{(\vartheta,\tilde{\eta}) \in \mathscr{M}} \beta_2(||\varsigma||)$ so that $\mathscr{M} \in \beta_2(||\varsigma||) \in \Omega \in R^{2n-\rho}$.

Consider, for the system (2.35), the Lyapunov function candidate

$$W(\chi) = \chi^T P_0 \chi, \tag{3.25}$$

where P_0 is the positive definite solution of $P_0\Lambda + \Lambda^T P_0 = -I$. This Lyapunov function satisfies

$$\lambda_{\min}(P_0)\,||\chi||^2 \leq W(\chi) \leq \lambda_{\max}(P_0)\,||\chi||^2,$$

where $\lambda_{min}(P_0)$ and $\lambda_{max}(P_0)$ are the minimum and maximum eigenvalues of P_0, respectively.

Let $\Sigma = \{W(\chi) \leq \beta\varepsilon^2\}$ and $\mathscr{S} = \Omega \times \Sigma$, where β is a positive constant independent of ε to be defined later. In what follows, we prove that the set $\mathscr{S}$ is positively invariant and then we will prove that all the trajectories enter this set in finite time. Furthermore, we will prove the stability of the origin over this set. For this purpose, we notice that for any $0 < \varepsilon_1 \leq 1$, there are L_1 and L_2, independent of ε, such that for all $(\varsigma,\chi)\in\mathscr{S}$ and every $0<\varepsilon\leq\varepsilon_1$ and $t\geq 0$, we have

$$||f_a(\vartheta,\tilde{\eta},D\varphi)-f_a(\vartheta,\tilde{\eta},0)||\leq L_1\,||\chi|| \tag{3.26}$$

$$||f_b(\vartheta,\tilde{\eta},t,D_1\chi)-f_b(\vartheta,\tilde{\eta},t,0)||\leq L_2\,||\chi|| \tag{3.27}$$

$$||f_a(\vartheta,\tilde{\eta},D\varphi)-f_a(\vartheta,0,D\varphi)||\leq L_3\,||\tilde{\eta}|| \tag{3.28}$$

where we used the facts $||D_1||\leq 1,||\varphi||\leq||\chi||$ and sat(.) is globally Lipschitz. From Assumption 3.1 and for all $\varsigma\in\Omega$ (knowing the fact that continuous functions are bounded over compact sets) and from Assumption 3.3 and for all $\chi\in R^{\rho+1}$ (noting that we are saturating $\hat{\sigma}$ outside the set Ω), it can be verified that

$$||\Delta\phi_1(\varsigma,D_1\chi)||\leq k_5 \tag{3.29}$$

where k_5 is positive constant independent of ε. It is worth emphasizing that the function $\Delta\phi_1$ is bounded uniformly in ε because the saturation of $\hat{\sigma}$ occurs outside a compact set that is independent of ε. Using (3.11), (3.12), (3.14), (3.26), (3.28) (3.27), (3.29), continuous differentiability of V_1, Lipschitz properties of f_a and f_b and noting that $\frac{P^{-1}}{dt}=-P^{-1}\dot{P}P^{-1}$, it can be shown that

$$\dot{V}_3\leq -q\alpha_3(||\vartheta||)+(qL_3d-\frac{k_1}{2\sqrt{p_1}})\,||\tilde{\eta}||+k_2\varepsilon \tag{3.30}$$

$$\dot{W}\leq -(\frac{h_1}{\varepsilon}-h_3)W+h_2\sqrt{W} \tag{3.31}$$

where k_1 is positive constant dependent on the bounds (3.11), (3.12) and (3.14), $k_2=(qdL_1+(p_2L_2)/(2\sqrt{p_1}))\sqrt{\beta/\lambda_{\min}(P_0)}$ and d is positive constant such that $d>||\partial V_1/\partial\vartheta||$ over Ω, and h_1, h_2 and h_3 are positive constants independent of ε. It can be shown that there exists $\varepsilon_a>0$ and β large enough such that, for all $0<\varepsilon\leq\varepsilon_a$ and $(\varsigma,\chi)\in\Omega\times\{W=\beta\varepsilon^2\}$, (3.31) is negative definite.

For this value of β, it can be shown that, for $q < k_1/(2L_3 d\sqrt{p_1})$ there exists $\varepsilon_b > 0$ such that, for every $0 < \varepsilon < \varepsilon_b$, and $(\varsigma, \chi) \in \{V_3 = c\} \times \Sigma$, we have $\dot{V}_3 \leq 0$. Choose $\varepsilon_2 = \min\{\varepsilon_a, \varepsilon_b\}$, then the set $\mathscr{S}$ is positively invariant for all $0 < \varepsilon \leq \varepsilon_2$.

Consider the trajectories $(\varsigma(t), \hat{\bar{\xi}}(t), \hat{\sigma}(t))$ starting in the set $\mathscr{M} \times \mathscr{N}$, it can be verified that there exists a finite time T_0 independent of ε such that $\varsigma(t) \in \Omega$ for all $t \in [0, T_0]$. During this time, χ will be bounded by an $O(1/\varepsilon^\rho)$ value. Moreover, there exists $\varepsilon_3 > 0$ and $T(\varepsilon) > 0$ with $T(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $W(\chi(T(\varepsilon))) \leq \beta\varepsilon^2$ for every $0 < \varepsilon < \varepsilon_3$. Therefore, taking $\varepsilon_0^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ ensures that the trajectory (ς, χ) enters the set $\mathscr{S}$ during the time interval $[0, T(\varepsilon)]$, for every $0 < \varepsilon < \varepsilon_0^*$, and does not leave thereafter.

We can now work locally inside the set $\mathscr{S}$ to prove asymptotic stability of the origin. To this end, we first consider the Lyapunov function candidate $V_4(t, \tilde{\eta}, \chi) = V_1(\vartheta) + \theta_1\sqrt{\tilde{\eta}^T P^{-1} \tilde{\eta}} + \sqrt{W(\chi)}$, where $\theta_1 > 0$ to be determined. Using the continuous differentiability of V_1, the Lipschitz properties of f_a and f_b, $\Delta\phi_1$ and $\Delta\phi_2$, it can be shown that, for all $(\vartheta, \tilde{\eta}, \chi) \in \mathscr{S}$, we have

$$\dot{V}_4 \leq -\alpha_3(||\vartheta||) - [a_1\theta_1 - a_2]\,||\tilde{\eta}|| - [a_3(\frac{1}{\varepsilon} - a_4) - \theta_1 a_5]\,||\chi|| \tag{3.32}$$

where a_1 to a_5 are positive constants. Choosing $\theta_1 > a_2/a_1$ and $0 < \varepsilon_1^* \leq \varepsilon_0^*$ such that $\varepsilon_1^* < 1/((\theta_1 a_5)/a_3 + a_4)$, then, for all $0 < \varepsilon \leq \varepsilon_1^*$, $\dot{V}_4$ is negative definite. This proves the first bullet.

To prove the second bullet, we define a ball $B(0, r_1)$, for some radius $r_1 > 0$ inside the set $\mathscr{S}$ and around the origin $(\varsigma, \chi) = (0, 0)$. Since the origin of the closed loop system (3.5) is exponentially stable, there exists a Lyapunov function V_5 that satisfies the following inequalities for all $\vartheta \in$

$B(0, r_1)$ [[61], Th. 4.14]

$$b_1 ||\vartheta||^2 \le V_5(\vartheta) \le b_2 ||\vartheta||^2 \tag{3.33}$$

$$\frac{\partial V_7}{\partial \vartheta} f(\vartheta, \gamma(\eta, \xi)) \le -b_3 ||\vartheta||^2 \tag{3.34}$$

$$\left|\left|\frac{\partial V_5}{\partial \vartheta}\right|\right| \le b_4 ||\vartheta|| \tag{3.35}$$

for some positive constants b_1, b_2, b_3 and b_4. Consider now the composite Lyapunov function $V_6(t, \vartheta, \tilde{\eta}) = V_5(\vartheta) + \theta_1 \tilde{\eta}^T P^{-1} \tilde{\eta} + W(\chi)$ with $\theta_1 > 0$. Choose $r_2 < r_1$, then it can be shown, using (3.45), (3.34), (3.35), Lipschitz properties of f_a and f_b that for all $(\vartheta, \tilde{\eta}, \chi) \in B(0, r_2) \times \{||\tilde{\eta}|| \le r_2\} \times \{||\chi|| \le r_2\}$, we have

$$\begin{aligned}
\dot{V}_6 \le & -b_3 ||\vartheta||^2 + c_1 ||\vartheta|| \, ||\tilde{\eta}|| + c_2 ||\vartheta|| \, ||\chi|| - \theta_1 c_3 ||\tilde{\eta}||^2 + \theta_1 c_4 ||\tilde{\eta}|| \, ||\chi|| \\
& - [\frac{1}{\varepsilon} - c_5] ||\chi||^2 + c_6 ||\tilde{\eta}|| \, ||\chi|| \\
\le & - \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}^T \begin{bmatrix} b_3 & -c_1/2 & -c_2/2 \\ -c_1/2 & \theta_1 c_3 & -(\theta_1 c_4 + c_6)/2 \\ -c_2/2 & -(\theta_1 c_4 + c_6)/2 & [1/\varepsilon - c_5] \end{bmatrix} \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix} \\
\triangleq & - \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}^T \Gamma \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix},
\end{aligned}$$

where c_1 to c_6 are positive constants. Choose θ_1 to make $z \triangleq \theta_1 b_3 c_3 - c_1^2/4$ positive and $\varepsilon_2^* > 0$ small enough such that, for all $0 < \varepsilon \le \varepsilon_2^*$, the determinant of Γ is positive. This makes Γ a positive definite constant matrix, which makes $\dot{V}_6$ negative definite. This completes the proof. □

3.4.2 Trajectory Recovery

In this section, we will show that the output feedback system proposed in this chapter has the ability to recover the performance of an auxiliary system working under state feedback. To that end, let $\varsigma(t,\varepsilon)$ be the solution of the system (3.19)-(3.21), where $\varsigma = [\eta^T \xi^T \tilde{\eta}^T]^T = [\vartheta^T \tilde{\eta}^T]^T$, and $\varsigma_r(t)$ be the solution of

$$\dot{\vartheta} = f(\vartheta, \gamma(\eta - \tilde{\eta}, \xi)) \tag{3.36}$$

$$\dot{\tilde{\eta}} = [A_1(\xi, \gamma(\eta - \tilde{\eta}, \xi)) - \bar{L}(t)C_1(\xi, \gamma(\eta - \tilde{\eta}, \xi))]\tilde{\eta} \tag{3.37}$$

starting from $\varsigma(0)$, where $\bar{L}(t) = \bar{P}(t)C_1(\xi, \gamma(\eta - \tilde{\eta}, \xi))R^{-1}$ and $\bar{P}(t)$ is the solution of the Riccati equation

$$\dot{\bar{P}} = A_1\bar{P} + \bar{P}A_1^T + Q - \bar{P}C_1^T R^{-1} C_1 \bar{P}, \quad \bar{P}(t_0) = P_0 \tag{3.38}$$

where $A_1(.,.)$ and $C_1(.,.)$ are functions of ξ and $\gamma(\hat{\eta}, \xi)$, P_0 is the initial condition of (3.10) and $R(t)$ and $Q(t)$ satisfy (3.11) and (3.12), respectively. Notice that this Riccati equation is different than (3.10), which makes use of both $\hat{\eta}$ and $\hat{\xi}$.

Assumption 3.5 *The Riccati equation* (3.38) *has a positive definite solution that satisfies*

$$0 < \bar{p}_1 I_{n-\rho} \le \bar{P}^{-1}(t) \le \bar{p}_2 I_{n-\rho}, \quad \forall t \ge 0 \tag{3.39}$$

and a steady state solution $\bar{P}_{ss}$.

Define

$$P_{error}(t) = \bar{P}(t) - \bar{P}_{ss}(t) = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1(n-\rho)} \\ \vdots & \vdots & \vdots & \vdots \\ P_{(n-\rho)1} & P_{(n-\rho)2} & \dots & P_{(n-\rho)(n-\rho)} \end{bmatrix}_{(n-\rho)\times(n-\rho)},$$

and stack the elements of P_{error} to form the following vector

$$\tilde{\bar{P}} = [P_{11} \dots P_{1(n-\rho)} \dots P_{i1} \dots P_{i(n-\rho)} \dots P_{(n-\rho)1} \dots P_{(n-\rho)(n-\rho)}]^T,$$

where $1 < i < n - \rho$.

Assumption 3.6 *The origin of the system*

$$\dot{\tilde{\bar{P}}} = f_c(t, \tilde{\bar{P}}, \hat{\eta}, \xi) \tag{3.40}$$

is exponentially stable.

We are now ready to have the following result.

Theorem 3.2 *Let Assumptions 3.1-3.6 hold. If the origin of the system* (3.5) *is exponentially stable, then, given any* $\mu > 0$*, there exists* $\varepsilon_3^* > 0$ *such that, for sufficiently small* $\left|\left|\tilde{\bar{P}}(0)\right|\right|$ *and for trajectories starting in* $\mathscr{M} \times \mathscr{N}$*, where* $\mathscr{M}$ *is sufficiently small, and for every* $0 < \varepsilon \leq \varepsilon_3^*$*, we have*

$$||\varsigma(t,\varepsilon) - \varsigma_r(t)|| \leq \mu, \quad \forall t \geq 0. \tag{3.41}$$

Remark 3.1 *If all the conditions hold globally, then Theorem 3.2 is valid for trajectories starting in any* $\mathscr{M} \times \mathscr{N}$.

Proof:[4]

Consider the change of variables $P_*(t) = P(t) - \bar{P}_{ss}(t)$. Then we can observe that the closed loop system (3.19)-(3.22) with

$$\dot{\tilde{P}} = f_g(t, \tilde{P}, \vartheta, \tilde{\eta}, D\varphi), \tag{3.42}$$

where $\tilde{P}$ is a vector that contains the elements of $P_*(t)$, is in the singularly perturbed form, where (3.19)-(3.21) and (3.42) constitute the slow dynamics and (3.22) constitutes the fast dynamics. We also observe that the system (3.36), (3.37) and (3.40) is nothing but the reduced system of (3.19)-(3.22) and (3.42) when $\varepsilon = 0$; this is because $\varepsilon = 0$ yields $\chi = 0$, which gives $\varphi = 0$. We know from the proof of Theorem 3.1 that, there exists $\varepsilon_0^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0^*$, the trajectories are inside the set $\mathscr{S}$ for all $t \geq T(\varepsilon)$, where $\mathscr{S}$ is $O(\varepsilon)$ in the direction of the variable χ. We will start by proving the stability of the reduced system (3.36), (3.37) and (3.40), and then to accomplish the proof we will use the same approach used to prove Theorem 3 in [35].

For convenience, rewrite (3.36) as $\dot{\vartheta} \triangleq f_d(\vartheta, \tilde{\eta}) = f_a(\vartheta, \tilde{\eta}, 0)$, (3.37) as $\dot{\tilde{\eta}} \triangleq f_e(\vartheta, \tilde{\eta}, t) = f_b(\vartheta, \tilde{\eta}, t, 0)$, $\bar{\varsigma}(t, \varepsilon) \triangleq [\varsigma^T(t, \varepsilon)\ \dot{\tilde{P}}^T(t)]^T$, $\bar{\varsigma}_r(t) \triangleq [\varsigma_r^T(t)\ \tilde{\bar{P}}^T(t)]^T$, $F(t, \bar{\varsigma}(t, \varepsilon), D_1(\varepsilon)\chi) \triangleq \begin{bmatrix} f_a \\ f_b \\ f_g \end{bmatrix}$ and $\bar{\varsigma}(0) = [\varsigma(0)^T\ \tilde{P}^T(0)]$.

For (3.37), we define the Lyapunov function candidate

$$V_2(t, \tilde{\eta}) = \tilde{\eta}^T \bar{P}^{-1} \tilde{\eta}. \tag{3.43}$$

[4]we use in this proof the same definitions used in the proof of Theorem 3.1, unless otherwise stated.

It can be seen using (3.39) that V_2 satisfies

$$\bar{p}_1 \|\tilde{\eta}\|^2 \leq V_2(t,\tilde{\eta}) \leq \bar{p}_2 \|\tilde{\eta}\|^2 \tag{3.44}$$

Using the bounds (3.11), (3.12), (3.39), and $\frac{d\bar{P}^{-1}}{dt} = -\bar{P}^{-1}\dot{\bar{P}}\bar{P}^{-1}$, we obtain

$$\begin{aligned} \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial \tilde{\eta}} f_e(\vartheta,\tilde{\eta},t) &= \dot{\tilde{\eta}}^T \bar{P}^{-1}\tilde{\eta} + \tilde{\eta}^T \frac{d\bar{P}^{-1}}{dt}\tilde{\eta} + \tilde{\eta}^T \bar{P}^{-1}\dot{\tilde{\eta}} \\ &\leq -k_1 \|\tilde{\eta}\|^2, \forall \tilde{\eta} \in R^{n-\rho}, \forall t \geq 0, \end{aligned} \tag{3.45}$$

where k_1 is a positive constant.

We define a ball $B(0,r_3)$, for some radius $r_3 > 0$ around the origin $(\vartheta,\tilde{\eta},\tilde{\bar{P}}) = (0,0,0)$. Using Assumption 3.6 and the converse Lyapunov theorem [[61], Th. 4.14], we know that there exists a Lyapunov function V_7 that satisfies the following inequalities for all $\tilde{\bar{P}} \in B(0,r_3)$

$$e_1 \left\|\tilde{\bar{P}}\right\|^2 \leq V_7(t,\tilde{\bar{P}}) \leq e_2 \left\|\tilde{P}\right\|^2 \tag{3.46}$$

$$\frac{\partial V_7}{\partial t} + \frac{\partial V_7}{\partial \tilde{\bar{P}}} f_c(t,\tilde{\bar{P}},\hat{\eta},\xi) \leq -e_3 \left\|\tilde{\bar{P}}\right\|^2 \tag{3.47}$$

$$\left\|\frac{\partial V_7}{\partial \tilde{\bar{P}}}\right\| \leq e_4 \left\|\tilde{\bar{P}}\right\| \tag{3.48}$$

for some positive constants e_1, e_2, e_3 and e_4. Consider now the composite Lyapunov function $V_8(t,\vartheta,\tilde{\eta},\tilde{\bar{P}}) = \theta V_2(t,\tilde{\eta}) + V_5(\vartheta) + V_7(t,\tilde{\bar{P}})$, where V_2 is given by (3.43), θ is a positive constant to be chosen and V_5 satisfies inequalities (3.33)-(3.35) over the ball $B(0,r_4)$ for some radius $r_4 > 0$ around the origin $(\vartheta,\tilde{\eta},\tilde{\bar{P}}) = (0,0,0)$. Notice that $f_d(\vartheta,0) = f(\vartheta,\gamma(\eta,\xi))$. Choose $r_5 < \min\{r_3,r_4\}$, then it can be shown, using (3.45), (4.64) and (3.47) that, for all $(\vartheta,\tilde{\eta},\tilde{\bar{P}})$

$\in B(0,r_5) \times \{||\tilde{\eta}|| \leq r_5\} \times B(0,r_5)$,

$$\begin{aligned}
\dot{V}_8 =& \frac{\partial V_5}{\partial \vartheta} f_d(\vartheta,0) + \frac{\partial V_5}{\partial \vartheta}[f_d(\vartheta,\tilde{\eta}) - f_d(\vartheta,0)] + \theta \dot{V}_2 + \dot{V}_7 \\
\leq& - b_3 ||\vartheta||^2 + b_4 \tilde{L} ||\vartheta|| \, ||\tilde{\eta}|| - \theta k_1 ||\tilde{\eta}||^2 - e_3 \left|\left|\tilde{\bar{P}}\right|\right|^2 \\
\triangleq& - \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \end{bmatrix}^T \begin{bmatrix} b_3 & -(b_4\tilde{L})/2 \\ -(b_4\tilde{L})/2 & \theta k_1 \end{bmatrix} \begin{bmatrix} ||\vartheta|| \\ ||\tilde{\eta}|| \end{bmatrix} - e_3 \left|\left|\tilde{\bar{P}}\right|\right|^2
\end{aligned}$$

where $\tilde{L}$ is the Lipschitz constant of f_d over $\{||\tilde{\eta}|| \leq r_5\}$. Choosing $\theta > (b_4^2 \tilde{L}^2)/(4k_1 b_3)$ makes $\dot{V}_8$ negative definite. This implies that the origin of the system (3.36),(3.37) and (3.40) is exponentially stable.

We now divide the interval $[0,\infty]$ into three intervals $[0,T(\varepsilon)]$, $[T(\varepsilon),T_2]$ and $[T_2,\infty]$, where T_2 is to be determined, and show (3.41) for each interval.

1- The interval $[0,T(\varepsilon)]$.

From Assumptions 3.4 and 3.5, we know that $P(t)$ and $\bar{P}(t)$ are bounded for all $t \geq 0$, therefore, we can deduce that there exists compact sets Z_1 and Z_2 such that $\tilde{P} \in Z_1$ and $\tilde{\bar{P}} \in Z_2$ for all $t \geq 0$. . Since the vector field F is continuous, we can write

$$\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}(0) = \int_0^t F(t, \bar{\varsigma}(\tau,\varepsilon), D_1(\varepsilon)\chi).d\tau$$

Using similar arguments to the ones leading to the inequality (3.29), we can argue that $||F(.,.,.)|| \leq k_9$, where k_9 is some positive constant independent of ε. Consequently, since $\bar{\varsigma}(0)$ is in the interior of a compact set, we have

$$||\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}(0)|| \leq k_9 t, \quad \forall \bar{\varsigma}(t,\varepsilon) \in \Omega \times Z_1$$

In the same way, we can show, during the same interval, that

$$||\bar{\varsigma}_r(t) - \bar{\varsigma}(0)|| \leq k_9 t.$$

Hence,

$$||\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}_r(t)|| \leq 2k_9 T(\varepsilon), \quad \forall t \in [0, T(\varepsilon)]. \tag{3.49}$$

Since $T(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists $0 < \varepsilon_4 \leq \varepsilon_1^*$ such that, for every $0 < \varepsilon \leq \varepsilon_4$, we have

$$||\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}_r(t)|| \leq \mu, \quad \forall t \in [0, T(\varepsilon)]. \tag{3.50}$$

2- The interval $[T(\varepsilon), T_2]$.

During this interval $\bar{\varsigma}(t,\varepsilon)$ satisfies $\dot{\bar{\varsigma}} = F(t, \bar{\varsigma}, D_1\chi)$, with initial condition $\bar{\varsigma}(T(\varepsilon),\varepsilon)$ and $D_1\chi$ is $O(\varepsilon)$, and $\bar{\varsigma}_r(t)$ satisfies $\dot{\bar{\varsigma}} = F(t, \bar{\varsigma}, 0)$, with initial condition $\bar{\varsigma}_r(T(\varepsilon))$. From (3.49), we know that $||\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}_r(t)|| \leq 2k_9 T(\varepsilon) \triangleq \delta(\varepsilon)$, where $\delta \to 0$ as $\varepsilon \to 0$. Therefore, by [Theorem 3.5,[61]], we conclude that there exists $0 < \varepsilon_5 \leq \varepsilon_1^*$ such that for every $0 < \varepsilon \leq \varepsilon_5$, we have

$$||\bar{\varsigma}(t,\varepsilon) - \bar{\varsigma}_r(t)|| \leq \mu, \quad \forall t \in [T(\varepsilon), T_2]. \tag{3.51}$$

3- The interval $[T_2, \infty)$.

We can show that for any $\mu > 0$, there exists $\varepsilon_4^* > 0$ and $\tilde{T}_2 \geq T(\varepsilon) > 0$, both dependent on μ, such that, for every $0 < \varepsilon \leq \varepsilon_4^*$, we have

$$||\varsigma(t,\varepsilon)|| \leq \mu/2, \quad \forall t \geq \tilde{T}_2. \tag{3.52}$$

From the exponential stability of the reduced system, we know that there exits a finite time $\bar{T}_2$, independent of ε, such that

$$||\varsigma_r(t)|| \leq \mu/2, \quad \forall t \geq \bar{T}_2. \tag{3.53}$$

Take $T_2 = \max\{\tilde{T}_2, \bar{T}_2\}$. Then, using the triangular inequality, and from (3.52) and (3.53), we conclude that for every $0 < \varepsilon \leq \varepsilon_4^*$, we have

$$||\varsigma(t,\varepsilon) - \varsigma_r(t)|| \leq \mu, \quad \forall t \geq T_2. \tag{3.54}$$

Take $\varepsilon_3^* = \min\{\varepsilon_4, \varepsilon_5, \varepsilon_4^*\}$, then using (3.50),(3.51) and (3.54) we conclude (3.41). □

Theorem 3.2 shows that the output feedback control, due to the use of the extended high-gain observer (3.6)-(3.7), can asymptotically recover the trajectories of the system (3.36)-(3.37). This is beneficial as, first, it allows us to assume that the states ξ, η and $\tilde{\eta}$ are available in the state feedback design stage, and thus, simplifies the design procedure. In addition, this result can allow for tuning the control u to satisfy certain performance requirements in the state feedback design stage. This can be done by simulating equations (3.36)-(3.37) and checking if the trajectories satisfy the design specifications. Second, it shows that the control design has to take into consideration the estimation error $\tilde{\eta}$, which is considered as an input to the closed-loop state feedback system. Therefore, by using equation (3.37), one can deduce the properties of this input under output feedback, and hence, can design a state feedback control accordingly.

3.5 Output Feedback Control of a DC-to-DC Boost Converter System

Consider the problem of output feedback stabilization of a DC-to-DC boost converter. The averaged model of the system is described by [62], [63]

$$\begin{aligned}
\dot{\bar{\eta}} &= -\frac{1}{L}\bar{u}\bar{\xi} + \frac{E}{L} \\
\dot{\bar{\xi}} &= -\frac{G}{C}\bar{\xi} + \frac{1}{C}\bar{u}\bar{\eta} \\
\bar{y} &= \bar{\xi}
\end{aligned} \tag{3.55}$$

where $\bar{\eta}$, $\bar{\xi}$ represent the inductor current and the capacitor voltage, respectively, and $\bar{u}$ is a continuous signal representing the slew rate of a PWM circuit controlling the switch position in the converter. The inductor, capacitor, load conductance and voltage source are represented by the positive constants L, C, G and E, respectively.

The objective is to regulate the output voltage $\bar{\xi}$ to a desired value V_*. This problem can also be cast as stabilizing the equilibrium point $(\bar{\eta}_*, \bar{\xi}_*) = (GV_*^2/E, V_*)$ and the corresponding control $\bar{u}_* = E/V_*$.

We first use the change of variables $\eta = \bar{\eta} - \bar{\eta}_*$, $\xi = \bar{\xi} - \bar{\xi}_*$ and $u = \bar{u} - \bar{u}_*$ to transfer the equilibrium point to the origin. Accordingly, the system takes the form

$$\begin{aligned}
\dot{\eta} &= -\frac{\xi}{L}[u + \frac{E}{V_*}] - \frac{uV_*}{L} \\
\dot{\xi} &= -\frac{G\xi}{C} + \frac{\eta}{C}[u + \frac{E}{V_*}] + \frac{GV_*^2 u}{EC} \\
y &= \xi
\end{aligned} \tag{3.56}$$

The system (3.56) is non-minimum phase, because the zero dynamics of the linearized system around the origin are $\dot{\eta} = (E^2/(LGV_*^2))\eta$, which are unstable. We will solve the problem of output feedback control of the boost converter using the methodology presented in Section 3.4. We first start by considering the state feedback control problem. We use the control [63]

$$\bar{u} = u_* + \lambda \frac{E\eta - GV_*\xi}{1 + (E\eta - GV_*\xi)^2} \tag{3.57}$$

where $\lambda = \lambda(\eta, \xi)$ is any nonnegative function such that $0 \leq \lambda < 2\min(u_*, 1 - u_*)$. It is shown in [63] that, for any $V_* > E$ and using the Lyapunov function $V = (1/2)L\eta^2 + (1/2)C\xi^2$, the control (3.57) renders the equilibrium point $(\bar{\eta}_*, \bar{\xi}_*)$ globally asymptotically stable. Moreover, the control satisfies $\bar{u}(t) \in (0, 1)$ for all $t > 0$. It is worth noting that the open loop control $\bar{u} = u_*$ solves the stabilization problem at hand. However, the absence of feedback makes the system highly sensitive to unavoidable parasitic effects and noise [64]. Furthermore, the control (3.57) adds damping to the system, and hence, can improve the transient response. Linearizing the closed loop system (3.56) with (3.57) around the origin, yields the characteristic equation

$$s^2 + \left[\frac{GE + G^2V_*^3\lambda}{EC} + \frac{\lambda V_* E}{L}\right] s + \frac{E^2 - G^2V_*^6\lambda^2}{CLV_*^2} + \frac{\lambda V_* EG + G^2V_*^3\lambda^2}{LC} = 0.$$

If $E > \lambda GV_*^3$, the characteristic equation has negative roots, which means that the origin of the closed-loop system is exponentially stable.

We turn now to the design of the observer. Following the procedure described in Section 3.4,

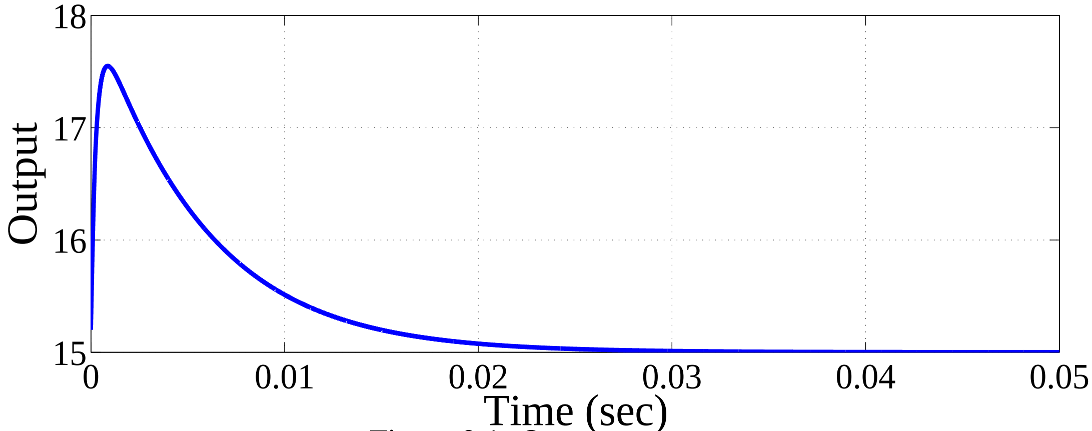

Figure 3.1: Output response.

an observer for the system (3.56) is given by

$$\dot{\hat{\xi}} = -\frac{G\hat{\xi}}{C} + \hat{\sigma} + \frac{GV_*^2 u}{EC} + \frac{\alpha_1}{\varepsilon}(y - \hat{\xi}) \tag{3.58}$$

$$\dot{\hat{\sigma}} = \phi_1(\hat{\eta}, \hat{\xi}) + \frac{\alpha_2}{\varepsilon^2}(y - \hat{\xi}) \tag{3.59}$$

$$\dot{\hat{\eta}} = -\frac{\hat{\xi}}{L}(u + \frac{E}{V_*}) - \frac{uV_*}{L} + \frac{1}{R}PC_1(\hat{\sigma} - C_1\hat{\eta}) \tag{3.60}$$

where, assuming, for convenience, that λ is constant,

$$\begin{aligned}\phi_1(\hat{\eta}, \hat{\xi}) =& (\frac{\lambda E\hat{\eta}}{C(1 + (E\hat{\eta} - GV_*\hat{\xi})^2)} + \frac{uV_* + E}{CV_*} - \frac{2u^2\hat{\eta}E}{\lambda C})(\frac{-\hat{\xi}}{L}(u + \frac{E}{V_*}) - \frac{uV_*}{L}) \\ &+ (\frac{-\lambda GV_*\hat{\eta}}{C(1 + (E\hat{\eta} - GV_*\hat{\xi})^2)} + \frac{2u^2GV_*\hat{\eta}}{C\lambda})(-\frac{G\xi}{C} + \frac{\eta}{C}(u + \frac{E}{V_*}) + \frac{GV_*^2u}{EC}),\end{aligned}$$

P is the solution of (3.10), $A_1 = 0$ and $C_1 = \frac{1}{C}(u + \frac{E}{V_*})$.

It should be mentioned that in this example, where η is scalar, it would be sufficient to use the extended high-gain observer (3.58)-(3.59) to provide the signal $\hat{\sigma}$, which in turn provides $\hat{\eta} = \frac{\hat{\sigma}}{C_1}$. However, we designed the full observer as in Section 3.4 for the purpose of demonstrating the results of this chapter.

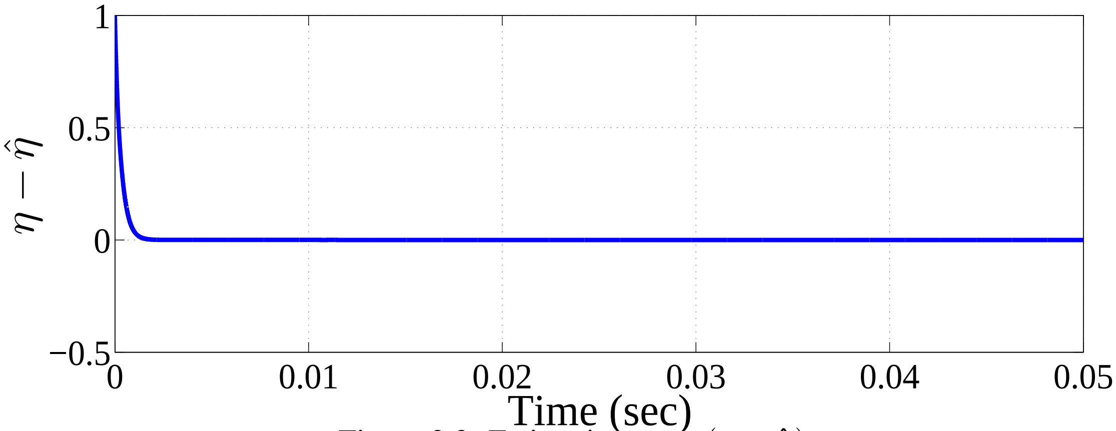

Figure 3.2: Estimation error $(\eta - \hat{\eta})$.

The parameters of the system are: $L = 1.36\,\text{mH}, C = 94\,\mu\text{F}, G = 1/120\,\Omega^{-1}$ and $E = 10V$. The parameters of the observer-based controller are chosen as: $\alpha_1 = 10, \alpha_2 = 3, V_* = 15, Q = 100000, R = 500$ and $\varepsilon = 0.00009$. The initial conditions are: $\eta(0) = 1, \xi(0) = 0.2, \hat{\eta}(0) = 0, \hat{\xi}(0) = 0, \hat{\sigma}(0) = 0$ and $P(0) = 1$. We first noticed that $\sigma = C_1\eta$ belongs to the range $[-7420, 7420]$. Therefore, to protect the system from peaking we saturated $\hat{\sigma}$ at the levels ± 7500. In this case, we did not saturate the control u, since it makes use of y and $\hat{\eta}$, which do not suffer from peaking.

Assumption 3.5 could be verified by showing that the observability Grammian, related to the pair $(A_1(t), C_1(t))$, is positive definite and bounded from above and below [65], which can be shown since $A_1(t) = 0$ and $C_1(t)$ is bounded for all t. We can first shape the transient response of the system (3.56) and $\dot{\tilde{\eta}} = -\frac{1}{R}PC_1^2\tilde{\eta}$, where $u = \gamma(\eta - \tilde{\eta}, \xi)$, by tuning the parameter λ, and then we can design the output feedback to recover this performance. Fig. 3.1 shows the response of the output $\bar{y}$ when the state feedback control is designed with $\lambda = 0.3$. This figure clearly shows that the output settles at the desired value. Fig. 3.2 shows the estimation error $\tilde{\eta} = \eta - \hat{\eta}$, while Fig. 3.3 shows the corresponding control effort. Fig. 3.4 shows the trajectory recovery feature of the output feedback control system as we decrease ε. It is clear from the figure that the response of the output feedback system almost identically matches that of the state feedback when $\varepsilon = 0.00001$.

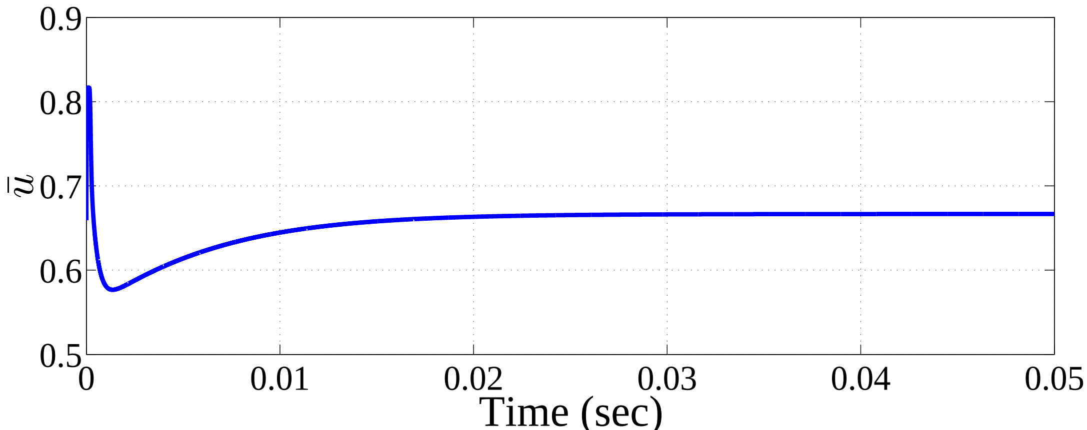

Figure 3.3: Control effort.

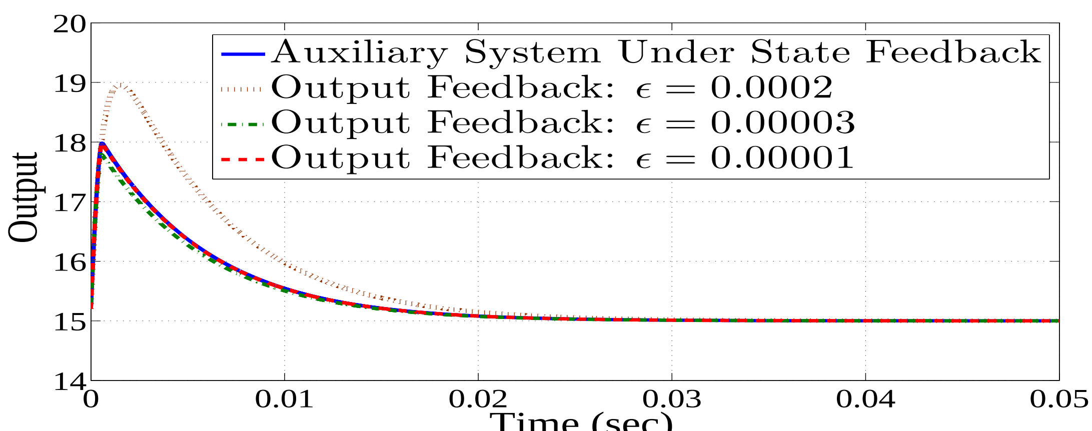

Figure 3.4: Recovery of the state of the auxiliary system.

3.6 Conclusions

We presented an output feedback control strategy that is based on the nonlinear observer proposed in Chapter 2. The strategy allows for the use of any globally asymptotically stabilizing state feedback controller. We proved semi-global stabilization of the origin of the closed-loop system of a class of nonlinear systems that could be non-minimum phase. We also proved recovery of the exponential stability of the origin of the closed-loop system under the same conditions. We finally demonstrated the effectiveness of the proposed control system in stabilizing the DC-to-DC boost converter system. We also showed that we can shape the transient performance using state feedback and then we can recover it using output feedback.

Chapter 4

Control of Nonlinear Systems Using Full-Order High Gain Observers: A Separation Principle Approach

4.1 Introduction

In the previous chapter, we proposed an output feedback control strategy that solves the problem of semi-global stabilization for a certain class of non-minimum phase nonlinear systems. The structure of this class is based on the normal form, where the system being linear in the internal state. The output feedback control is based on an Extended High Gain Observer (EHGO) augmented with an Extended Kalman Filter (EKF). The idea of this observer was first introduced in Chapter 2; it is based on using EHGO to provide estimates of the output and its derivatives plus a signal that is used as a virtual output for an auxiliary system comprised of the internal dynamics. The EKF, called the internal observer, was then used to estimate the states of this auxiliary system. This approach does not work in the case when the internal dynamics are nonlinear. Therefore, in this chapter, we propose an output feedback controller that is based on a different observer that is capable of dealing with nonlinear internal dynamics.

The proposed observer design in this chapter shares the same principle introduced in Chapter

2. The new technique used here, nonetheless, is in the design of the internal observer, where now we propose to use a high-gain observer to estimate the internal states. The challenge in this case, however, is to ensure that the observer for the output and its derivatives is faster than the internal observer. This can be realized when we consider the fact that both observers form a high gain observer that is going to be fast relative to the system dynamics. It turns out that by the use of this observer, we can achieve a number of properties for the considered class of nonlinear systems. In particular, it is possible to use any globally stabilizing full state feedback control that is designed independently of the observer. The output feedback control system can be designed to be robust to unmatched uncertainties and can totally recover the performance of the state feedback. This would lead to the assertion of the separation principle along the same lines as in [66].

In Chapter 3, we proposed a feedback control strategy, which is based on the observer proposed on Chapter 2, that solves the problem of semi-global stabilization for a certain class of non-minimum phase nonlinear systems. The structure of this class is based on the normal form, where the system is linear in the internal state. However, the approach of Chapter 3, in particular the observer, can not be adapted for the class of systems considered in this chapter, since this class of systems is nonlinear in the internal state.

The outline of this chapter is as follows. We start in Section 5.2 with a motivating example of the design of an output feedback control of flexible joint manipulators using only motor position feedback. Section 5.3 formulates the problem in a more general setting. Section 5.4 describes the state feedback stage while Section 5.5 describes the observer design. In Section 5.6, we describe the structure of the output feedback control scheme and present a theorem that describes the stability and trajectory recovery properties of the closed loop system. Finally, Section 5.7 includes concluding remarks.

4.2 Motivating Example: Tracking Control of Flexible Joint Manipulators Using Only Motor Position Feedback

We consider an n-link robot that is described by [67]

$$\begin{aligned} &D(q_1)\ddot{q}_1 + C_0(q_1,\dot{q}_1)\dot{q}_1 + g(q_1) = K(q_2 - q_1), \\ &J\ddot{q}_2 + K(q_2 - q_1) = u \end{aligned} \tag{4.1}$$

where $q_1 \in R^n$ and $q_2 \in R^n$ represent the link angles and the motor angles respectively, $D(q_1)$ is the $n \times n$ positive definite inertia matrix, J is a diagonal positive definite matrix of the actuator inertias, $C_0(q_1,\dot{q}_1)\dot{q}_1$ represents the Coriolis and centrifugal forces, $g(q_1)$ represents the gravitational terms, K is a diagonal positive definite matrix of the joint stiffness coefficients and $u \in R^n$ are the applied torques. Hereafter, we call the q_1 dynamics as the link dynamics and q_2 dynamics as the motor dynamics.

It is assumed that the angular part of the kinetic energy of each rotor is due to its own rotation and that the rotor/gear inertia is symmetric about the rotor axis of rotation. This model shares a lot of similarities with the rigid robot model, and in fact, reduces to one as K tends to infinity. The difference lies in the addition of elasticity, which typically is modeled by a linear torsional spring. For many manipulators, joint flexibility is significant. It is caused by effects such as shaft windup, bearing deformation, compressibility of the hydraulic fluid in hydraulic robots and torsional flexibility in the gears [68]. The challenges that are associated with these systems are due to a number of factors. These systems, first of all, are underactuated since the number of degrees of freedom is twice the number of control inputs. In addition, there is no matching between the system nonlinearities and the control input.

Motion control of robots with flexible joints has been extensively studied since the 1980s. Several surveys of the literature are available; see e.g. [69], [70]. Most of the results on output feedback control of flexible joint manipulators assume at least availability of measurement of either the link angle, see e.g. [71], [72], or both the link and rotor angles see, e.g. [73], [74]. Another recent example is the work in [75], which is based on the use of the discrete-time Extended Kalman Filter (EKF) and assumes availability of the measurements of both motor angle and velocity. In this chapter, on the other hand, we assume that the only measured variable is the motor angular position q_2. This choice is attractive when it is easier and/or cheaper to measure the motor angular position. However, this choice might make the system non-minimum phase, as for example in the case of the single-link manipulator shown in Section 4.2.4 of this chapter, and therefore, makes the design of the output feedback control scheme more challenging. In general, the motor position choice as the only measured output was considered in the set-point regulation problem [76], [77], [78], but, to the best of our knowledge, not in the tracking problem.

We consider the problem of designing an observer-based tracking controller for (4.1). The objective is to track the link angle q_1 using only the measurement of the motor angle q_2. We solve this problem by using ideas from high gain observer theory.

4.2.1 Problem Formulation

Using the change of variables $\eta_1 = q_1$, $\eta_2 = \dot{q}_1$, $\xi_1 = q_2$ and $\xi_2 = \dot{q}_2$, the system (4.1) takes the form

$$\dot{\eta}_1 = \eta_2, \tag{4.2}$$

$$\dot{\eta}_2 = \phi(\eta,\xi) \tag{4.3}$$

$$\dot{\xi}_1 = \xi_2, \tag{4.4}$$

$$\dot{\xi}_2 = J^{-1}[K(\eta_1 - \xi_1) + u] \tag{4.5}$$

$$y = \xi_1 \tag{4.6}$$

where the system states $\eta \in R^{2n}$ and $\xi \in R^{2n}$ are

$$\eta = [\eta_1^1, \eta_2^1, ..., \eta_1^n, \eta_2^n]^T, \quad \xi = [\xi_1^1, \xi_2^1, ..., \xi_1^n, \xi_2^n]^T,$$

and $\phi(\eta,\xi) = D^{-1}(\eta_1)[K(\xi_1 - \eta_1) - C(\eta_1,\eta_2)\eta_2 - g(\eta_1)]$. Equations (4.2)-(4.6) can be written in the compact form

$$\dot{\eta} = A\eta + B\phi(\eta,\xi) \tag{4.7}$$

$$\dot{\xi} = A\xi + B[J^{-1}[K(\eta_1 - \xi_1) + u]] \tag{4.8}$$

$$y = C\xi \tag{4.9}$$

where

$$
\begin{aligned}
&A = \text{blockdiag}\begin{bmatrix}A_1, ..., A_n\end{bmatrix}, A_i = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\\
&B = \text{blockdiag}\begin{bmatrix}B_1, ..., B_n\end{bmatrix}, B_i = \begin{bmatrix}0\\ 1\end{bmatrix},\\
&C = \text{blockdiag}\begin{bmatrix}C_1, ..., C_n\end{bmatrix}, C_i = \begin{bmatrix}1 & 0\end{bmatrix},\\
&\text{and} \quad i = 1, ..., n
\end{aligned}
$$

We assume that the function $\phi(\eta, \xi)$ is locally Lipschitz in its arguments.

The objective is to design an observer-based output feedback control such that for all sufficiently smooth η_{1d} with bounded derivatives and arbitrary initial conditions (in a known compact set) we have

$$\lim_{t\to\infty}(\eta_1 - \eta_{1d}) = 0. \tag{4.10}$$

4.2.2 Full-State Feedback Control

We first start by considering state feedback control to achieve the goal of tracking with the anticipation that we will be able to recover the performance of this control by a high gain observer. To this end, consider a full state feedback control for the system (4.2)-(4.6) that takes the form

$$u = \gamma(\eta, \xi, d(t)). \tag{4.11}$$

where $d(t) \in D$ and D is a compact subset of R^d that includes the reference signal and its derivatives. To solve the problem at hand, we follow the framework used in [79], which considers the change of variables

$$\bar{\eta}_1 = \eta_1 - \eta_{1d} \tag{4.12}$$

$$\bar{\eta}_2 = \dot{\bar{\eta}}_1 + \lambda_1 \bar{\eta}_1 \tag{4.13}$$

$$\bar{\xi}_1 = \xi_1 - \xi_{1d} \tag{4.14}$$

$$\bar{\xi}_2 = \dot{\bar{\xi}}_1 + \lambda_2 \bar{\xi}_1 \tag{4.15}$$

where ξ_{1d} needs to be designed and $\lambda_1 > 0, \lambda_2 > 0$ are diagonal design matrices.

For convenience, let $\vartheta = \begin{bmatrix} \bar{\eta} \\ \bar{\xi} \end{bmatrix}$. Accordingly, the closed-loop state feedback system can be written as

$$\dot{\vartheta} = f(\vartheta, \gamma(\eta, \xi, d(t)), d(t)). \tag{4.16}$$

We now require the state feedback control design procedure to satisfy:

1. γ is locally Lipschitz in η and ξ uniformly in d.

2. The closed loop system (4.44) is uniformly globally asymptotically stable with respect to the equilibrium point $(\bar{\eta} = 0, \bar{\xi} = 0)$.

3. The function $f(\vartheta, \gamma(\eta, \xi, d(t)), d(t))$ is locally Lipschitz in ϑ, η and ξ uniformly in d. Moreover, $f(\vartheta, \gamma(\eta, \xi, d(t)), d(t))$ is zero at $(\bar{\eta} = 0, \bar{\xi} = 0)$ uniformly in d.

The feedback control may also be required to satisfy extra requirements such as achieving particular transient response and/or provide robustness to uncertainties.

4.2.3 Output Feedback Control

Consider the system (4.2)-(4.6). We propose to use an extended high-gain observer to estimate the first two derivatives of the output. This would allow us to assume the availability of ξ_2 so that we can use it in estimating the η states. Furthermore, the second derivative of the output represents the right hand side of (4.5). As a result, as was shown in Chapter 2, it is possible to use this information to estimate the unknown state η_1. We now consider the auxiliary system (4.2)-(4.3) with the signal $\sigma = J^{-1}K\eta_1$ used as an output. This auxiliary system is in the standard normal form and has a full relative degree, viewing ξ as an input. Therefore, we can design a high-gain observer to estimate the η state vector. The crucial point is to design the extended high-gain observer for the output derivatives so that it is faster than the high-gain observer for the auxiliary system. Based on this concept, we propose the following full order observer

$$\dot{\hat{\eta}} = A\hat{\eta} + B\phi(\hat{\eta}, \hat{\xi}) + H_1(K^{-1}J\hat{\sigma} - \hat{\eta}_1) \tag{4.17}$$

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} - J^{-1}(K\hat{\xi}_1 - u)] + H_2(y - \hat{\xi}_1) \tag{4.18}$$

$$\dot{\hat{\sigma}} = J^{-1}K\hat{\eta}_2 + H_3(y - \hat{\xi}_1) \tag{4.19}$$

where

$$H_1 = \text{blockdiag}[H_{11}, ..., H_{1n}], H_{1i} = \begin{bmatrix} \alpha_1^i/\varepsilon \\ \alpha_2^i/\varepsilon^2 \end{bmatrix},$$

$$H_2 = \text{blockdiag}[H_{21}, ..., H_{2n}], H_{2i} = \begin{bmatrix} \beta_1^i/\varepsilon^2 \\ \beta_2^i/\varepsilon^4 \end{bmatrix},$$

$$H_3 = \text{diag}[\beta_3^i/\varepsilon^6]_{n\times n},$$

$i=1,...,n$, (α_1^i, α_2^i) and $(\beta_1^i, \beta_2^i, \beta_3^i)$ are chosen such that the polynomials $s^2+\alpha_1^i s+\alpha_2^i$ and $s^3+\beta_1^i s^2+\beta_2^i s+\beta_3^i$ are Hurwitz, and $\varepsilon>0$ is a small parameter.

We now replace the states in the state feedback control with their estimates, so that

$$u=\gamma(\hat{\eta}, \hat{\xi}, d(t)) \tag{4.20}$$

and use the full order observer (4.17)-(4.19) to provide these estimates.

Remark 4.1 *To protect against peaking, the auxiliary output $\hat{\sigma}$, and the nonlinear functions $\phi(\hat{\eta}, \hat{\xi})$ and $\gamma(\hat{\eta}, \hat{\xi}, d(t))$ should be saturated outside their domain under the state feedback control.*

It should be noted, however, that there would be no need to saturate $\phi(\hat{\eta}, \hat{\xi})$ and $\gamma(\hat{\eta}, \hat{\xi}, d(t))$ if they are originally bounded in $\hat{\eta}$ and $\hat{\xi}$.

In Section 4.3, we will prove, in a more general setting, that the output feedback control system recovers the stability properties of the state feedback system. Furthermore, we will show also that it is possible to recover the system trajectories under state feedback. In the next subsection, we will demonstrate by simulations these properties along with the behaviour of the system in the presence of external disturbances and uncertain parameters.

4.2.4 Simulation Example: Single-Link Flexible Joint Manipulators

Consider a single link manipulator with a flexible joint system, where an actuator is connected to a load through a torsional spring. The equations of motion are given by [67]

$$J_l\ddot{\theta}_l + Mgl\sin\theta_l + k(\theta_l - \theta_r) = 0 \tag{4.21}$$

$$J_r\ddot{\theta}_r - k(\theta_l - \theta_r) = u \tag{4.22}$$

where $\theta_l, \theta_r, J_l, J_r$ are the angular positions and the inertias of the link and rotor, respectively, M is the load mass, l is a distance, g is the gravity, k is the joint stiffness and u is a torque input.

The change of variables $\eta_1 = \theta_l,\ \eta_2 = \dot{\theta}_l,\ \xi_1 = \theta_r,\ \xi_2 = \dot{\theta}_r$, transforms the model (4.21) – (4.22) into the normal form

$$\dot{\eta}_1 = \eta_2 \tag{4.23}$$

$$\dot{\eta}_2 = -\frac{Mgl}{J_l}\sin\eta_1 - \frac{k}{J_l}(\eta_1 - \xi_1) \tag{4.24}$$

$$\dot{\xi}_1 = \xi_2 \tag{4.25}$$

$$\dot{\xi}_2 = \frac{k}{J_r}(\eta_1 - \xi_1) + \frac{1}{J_r}u \tag{4.26}$$

$$y = \xi_1 \tag{4.27}$$

The system (4.23) – (4.27) has a relative degree $\rho = 2$. The zero dynamics are

$$\dot{\eta}_1 = \eta_2 \tag{4.28}$$

$$\dot{\eta}_2 = -\frac{Mgl}{J_l}\sin\eta_1 - \frac{k}{J_l}\eta_1. \tag{4.29}$$

Using the Lyapunov function candidate

$$V = \int_0^{\eta_1} \frac{Mgl}{J_l}\sin z.dz + \frac{1}{2}\frac{k}{J_l}\eta_1^2 + \frac{1}{2}\eta_2^2,$$

it can be shown that $\dot{V} = 0$, i.e., the system is stable but not asymptotically stable. Thus, the system is not minimum phase. The objective is to design an output feedback control so that η_1 tracks a desired signal with arbitrarily large region of attraction.

For state feedback control, we use the passivity-based control scheme proposed in [79]. The control law is given by [79]

$$u = J_r \ddot{\xi}_{1r} + k(\xi_{1d} - \eta_{1d}) - B_2 \bar{\xi}_2, \tag{4.30}$$

with

$$\begin{aligned}
\xi_{1d} &= \eta_{1d} + \frac{1}{k} u_R \\
u_R &= J_l \ddot{\eta}_{1r} + Mgl \sin(\eta_1) - B_1 \bar{\eta}_2 \\
\ddot{\eta}_{1r} &= \ddot{\eta}_{1d} - \lambda_1 \dot{\bar{\eta}}_1 \\
\ddot{\xi}_{1r} &= \ddot{\xi}_{1d} - \lambda_2 \dot{\bar{\xi}}_1
\end{aligned}$$

where we used the definitions (4.12)-(4.15) and $B_1, B_2 > 0$ are design parameters. It is worth mentioning that the design of the control (4.30) is based on energy shaping technique, where ξ_{1d} is designed such that the closed loop energy function matches a desired function. Moreover, it can be shown that the equilibrium point of the closed loop system is globally exponentially stable [79]. We also notice that the control is independent of k. This can be seen by substituting ξ_{1d} back into (4.30).

Following the procedure proposed in Section 4.4, a full order observer can be designed as

$$\dot{\hat{\eta}}_1 = \hat{\eta}_2 + \frac{\alpha_1}{\varepsilon}\left(\frac{J_r}{k}\hat{\sigma} - \hat{\eta}_1\right) \tag{4.31}$$

$$\dot{\hat{\eta}}_2 = -\frac{Mgl}{J_l}\sin\hat{\eta}_1 - \frac{k}{J_l}(\hat{\eta}_1 - \hat{\xi}_1) + \frac{\alpha_2}{\varepsilon^2}\left(\frac{J_r}{k}\hat{\sigma} - \hat{\eta}_1\right) \tag{4.32}$$

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + \frac{\beta_1}{\varepsilon^2}(y - \hat{\xi}_1) \tag{4.33}$$

$$\dot{\hat{\xi}}_2 = \hat{\sigma} + \frac{k}{J_r}\hat{\xi}_1 + \frac{1}{J_r}u + \frac{\beta_2}{\varepsilon^4}(y - \hat{\xi}_1) \tag{4.34}$$

$$\dot{\hat{\sigma}} = \frac{k}{J_r}\hat{\eta}_2 + \frac{\beta_3}{\varepsilon^6}(y - \hat{\xi}_1) \tag{4.35}$$

The system parameters are given as [67]: $J_l = 1\,kg - m^2, J_r = 1\,kg - m^2, k = 100N - m/rad$, $Mgl = 9.8N - m$. The observer and controller constants are chosen to be: $\varepsilon = 0.06, \alpha_1 = 3, \alpha_2 = 1, \beta_1 = 5, \beta_2 = 3, \beta_3 = 1, B_1 = 7, B_2 = 5, \lambda_1 = \lambda_2 = 30$. $\eta_{1d} = \sin(\omega t)$ with $\omega = \frac{\pi}{3}$. The initial values of the states are chosen as: $\eta_1(0) = 0.1, \eta_2(0) = 0.05, \xi_1 = 0.1, \xi_2 = 0.05, \hat{\eta}_1(0) = 0, \hat{\eta}_2(0) = 0, \hat{\xi}_1(0) = 0, \hat{\xi}_2(0) = 0, \hat{\sigma}(0) = 0$.

4.2.4.1 Case 1: without parameter uncertainties

Fig. 4.1 shows the tracking performance of the link angle η_1 to a sinusoidal reference signal under output feedback control and Fig. 4.2 shows the control effort. It is clear that the control system achieved good tracking performance. Fig. 4.3 shows the trajectory recovery performance of the observer-based output feedback control system. More specifically, it shows the error $(\bar{\eta}_1)_{(output feedback)} - (\bar{\eta}_1)_{(state feedback)}$, where $\bar{\eta}_1$ is the tracking error, for different values of ε. As it can be seen from the figure, the smaller the value of ε, the closer the performance of the output feedback control gets to the state feedback control. It should be noted that the control u, $\hat{\eta}_1$, $\hat{\eta}_2$ and $\hat{\sigma}$ were saturated to prevent peaking. The saturation was done for the states that do not

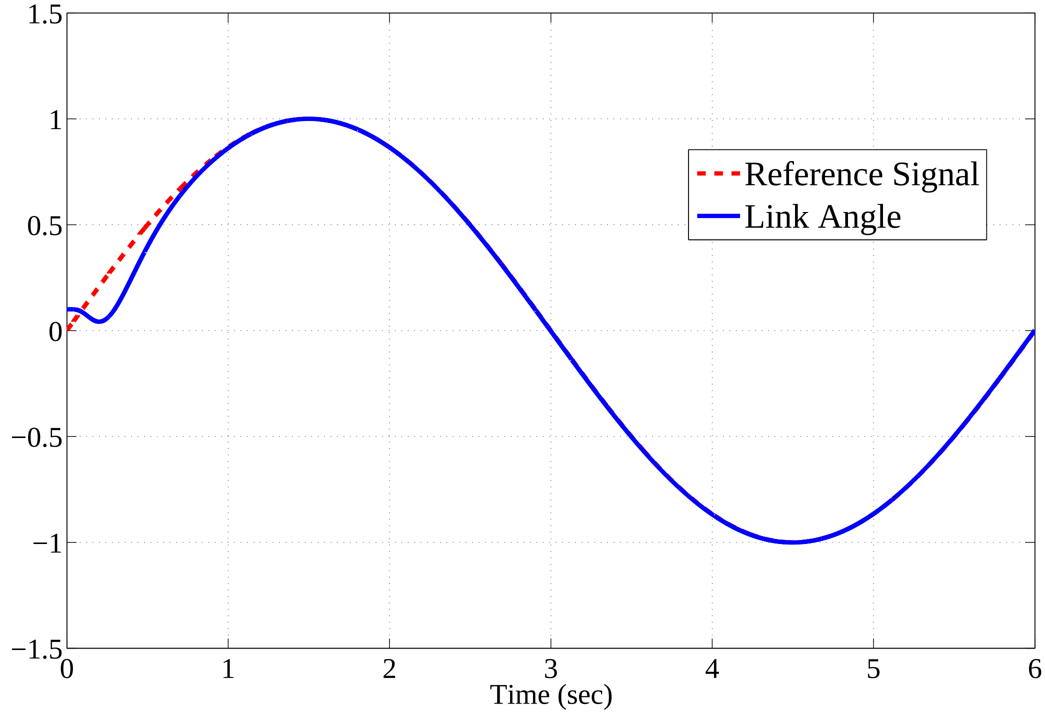

Figure 4.1: Tracking of the link angle to a reference signal.

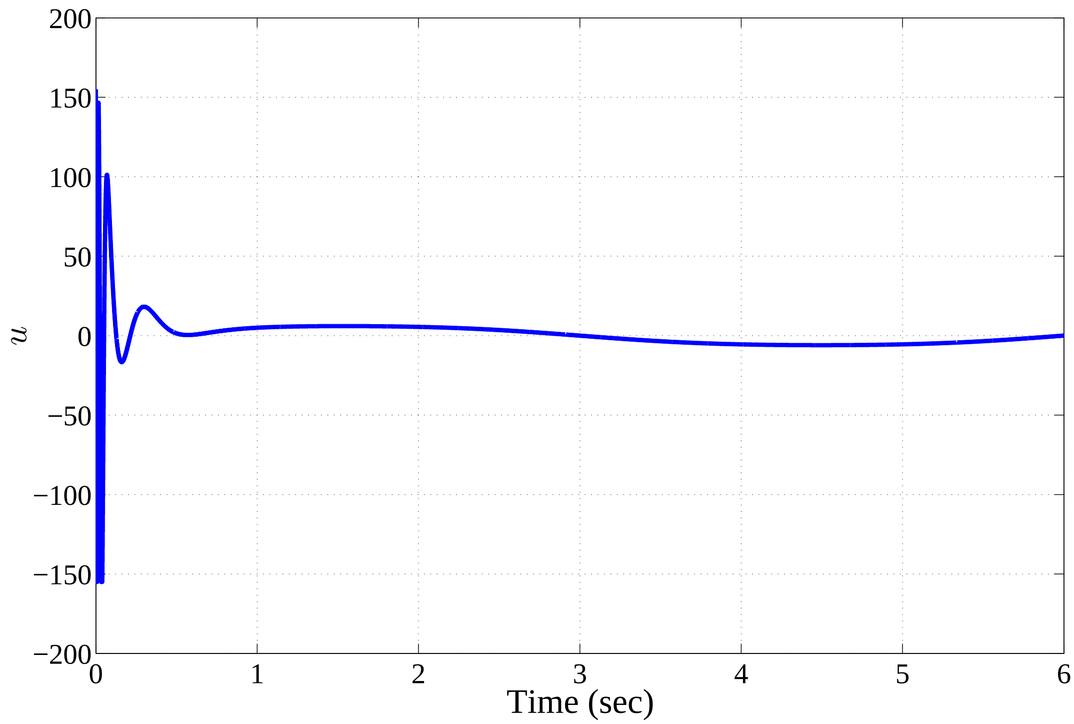

Figure 4.2: Control Effort.

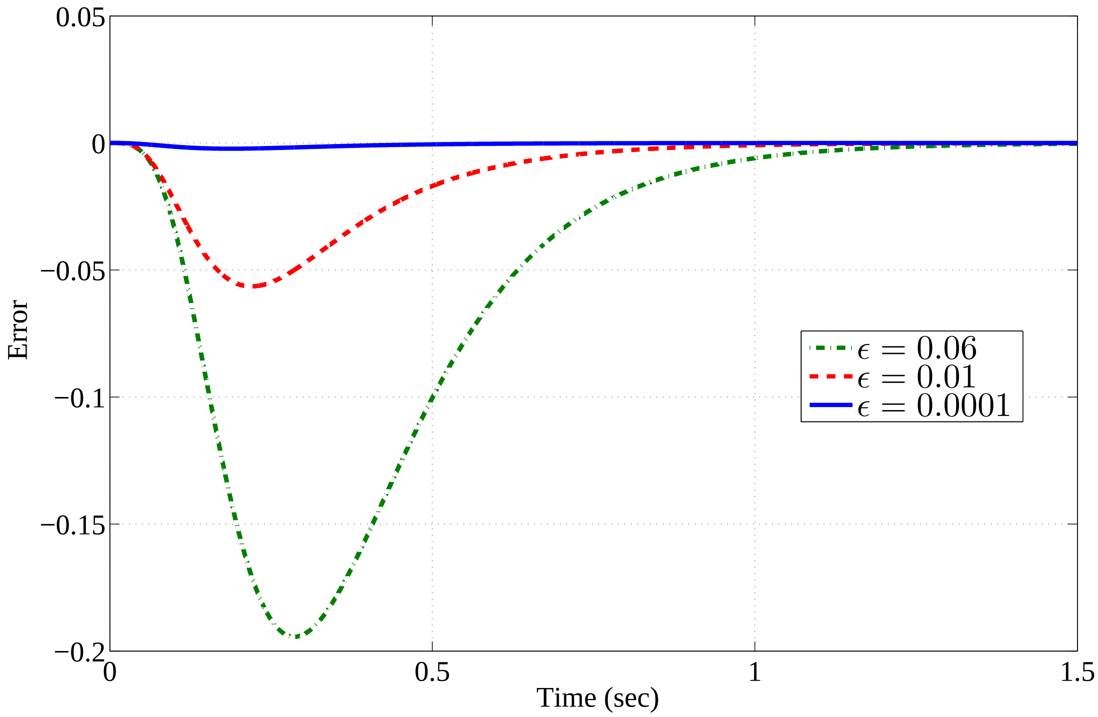

Figure 4.3: Recovery of the state feedback performance.

constitute the linear part of the observer error dynamics. Furthermore, the saturation was done after we observed that under state feedback, the control effort needed is in the range $[-152, 152]$, η_1 is in the range $[-1, 1]$, η_2 is in the range $[-1.5, 1.5]$ and σ is in the range $[-100, 100]$. Consequently, we used the saturation levels ±155, ±10, ±10, and ±110 for u $\hat{\eta}_1$, $\hat{\eta}_2$, and $\hat{\sigma}$, respectively. It is clear from Fig. 4.2 that the control passed the saturation period in a relatively quick time. We also tested the performance of the proposed output feedback control system in the presence of measurement noise. Fig. 4.4 shows the tracking error in this case. The noise signal is generated using the Simulink block "Uniform random Number" with a magnitude limit in the range $[-0.0016, 0.0016]$ and a sampling time of 0.0008 seconds. It is observed from the figure that despite the presence of noise, the tracking error can reach to a steady state level below 0.05. However, it is noticed, as it is well known for high gain observers, that there is a trade-off needed between how small can ε get (or how high can the observer gain get) and sensitivity to noise.

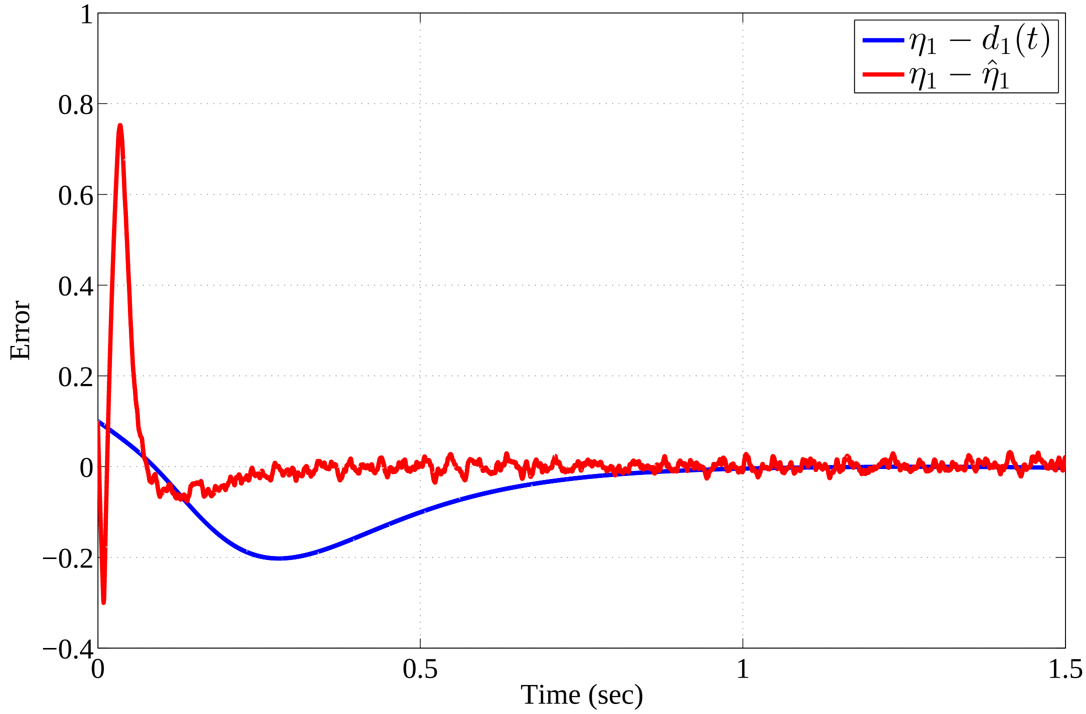

Figure 4.4: Tracking and estimation errors of the state η_1 in the presence of noise.

4.2.4.2 Case 2: with parameter uncertainties

In this subsection, we demonstrate the behaviour of the output feedback system in the presence of parameter uncertainties, specifically in the joint stiffness k and the load mass M. First of all, we should note that the state feedback control (4.30) does not provide robustness to variations in M but can be robust to variations in k. The main objective here is to examine how the observer (4.31)-(4.35) performs under these conditions. For the simulations presented in this section, we will use $\varepsilon = 0.001$.

We start by studying the effect of variations in M. In this case, we saturate $\hat{\sigma}$ at the levels ± 140 Fig. 4.5 shows the response of the system when the controller and observer use a value of $M = 1.5kg$, i.e. 1.5 times the real value. It is clear from the figure that there is a steady state tracking error of less than 0.13. However, it is also clear that the estimated state follows perfectly the real state. It is observed that as the difference between the nominal and the real values of M

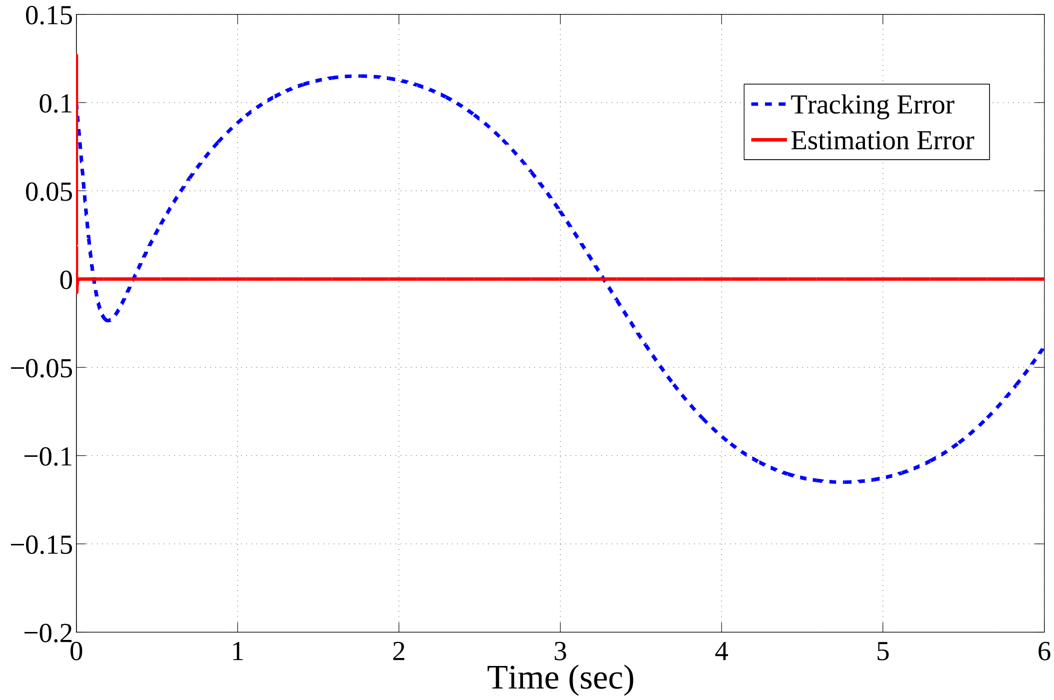

Figure 4.5: Tracking error and estimation error in the case of uncertain load mass M.

gets bigger the tracking error worsens, but the estimation error remains very close to zero. This robustness performance by the observer is expected and can be explained by noticing that M is only present in (4.24), thus, allowing the high gain observer, designed for the η dynamics, to dominate its uncertainty.

We now examine the performance of the system when there is uncertainty in the joint stiffness k. We observe that any uncertainty in the equation (4.27) will be regarded as a "noise" to the virtual output $\hat{\sigma}$. It is known that high-gain observers are sensitive to noise, therefore, we would expect a steady state error in this case [16]. Fig. 4.6 shows the response of the system when the controller and observer use a value of $k = 400N - m/rad$, i.e. 4 times the real value and saturation levels for $\hat{\sigma}$ at ± 600. It can be seen that, indeed, there is a steady state tracking and estimation errors of less than 0.06.

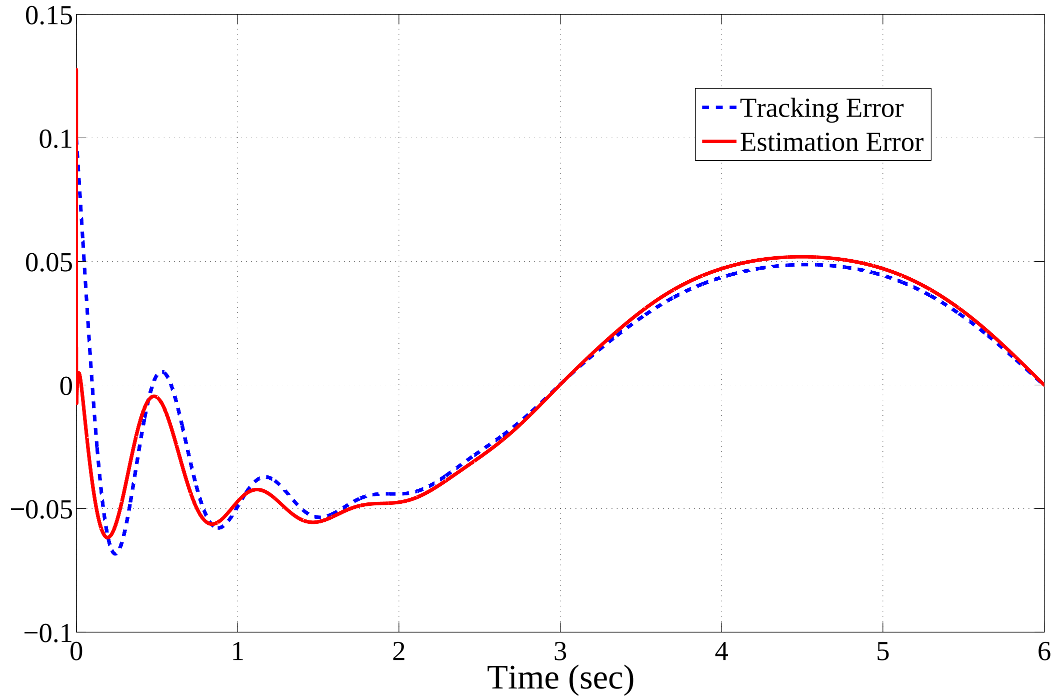

Figure 4.6: Tracking error and estimation error in the case of uncertain joint stiffness k.

4.2.4.3 Discussion

We demonstrated the effectiveness of the output feedback control system when used for the single-link flexible joint manipulator system. We showed by simulation that the output feedback control system can achieve the tracking goal when the proposed observer is used with a state feedback scheme that can achieve uniform global asymptotic stability of the origin of the tracking error dynamics. We also showed the capability of the scheme to recover the trajectories of the state feedback system. However, as it is well known for high gain observers, the observer gain is limited by the presence of measurement noise.

It is worthwhile to note that in this example it was assumed that there are no motor friction and Coriolis and centrifugal forces. Furthermore, the system would be minimum phase if there were positive Coriolis and centrifugal forces in (4.21). In any case, it can be shown that the proposed output feedback control system can handle the presence of these forces resulting in the implementation of the full information control law. In the case of minimum phase systems, high-gain observers have been used in the literature, in the context of output feedback control, in partial state estimation schemes or full state estimation schemes with the help of open loop observers for the internal dynamics. On the other hand, the proposed observer provides a contribution relative to these schemes, where now the observer is used to provide full state estimation and the internal observer in this case is a high-gain observer. The implication of this is that the internal observer is a closed-loop one and its speed can be controlled. It also raises the possibility to make the observer robust to modeling uncertainties and to have the ability to use the observer to recover the trajectories of the system under state feedback.

We also studied the performance of the system in the presence of parameter uncertainties. The state feedback control is independent of the joint stiffness constant. We observed that in the case of uncertain load mass, which appears in the system in an unmatched way, the proposed observer maintained a good performance in the presence of relatively large load uncertainty. The output feedback control system also shows a good performance in the case of a limited uncertainty in the joint stiffness constant. Based on these observations, we would expect the output feedback control system to provide reasonable robust performance if the state feedback controller is robust enough to uncertainties.

4.3 General Formulation

We consider the multi-input multi-output nonlinear system

$$
\begin{aligned}
\dot{\eta}_1^i &= \eta_2^i \\
\dot{\eta}_2^i &= \phi_i(\eta, \xi) \\
\dot{\xi}_1^i &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r-1}^i &= \xi_r^i \\
\dot{\xi}_r^i &= \eta_1^i + a_i(\xi, u) \\
y_i &= \xi_1^i \\
y_c^i &= \eta_1^i
\end{aligned}
\tag{4.36}
$$

for $1 \leq i \leq m$, where $y = \mathrm{col}(y_1, \ldots, y_m)$ is the measured output, $y_c = \mathrm{col}(y_c^1, \ldots, y_c^p)$ is the controlled output and $u \in R^l$ is the input. System (4.36) can be written in the compact form

$$\dot{\eta} = A_0\eta + B_0\phi(\eta, \xi) \tag{4.37}$$

$$\dot{\xi} = A_1\xi + B_1[C_0\eta + a(\xi, u)] \tag{4.38}$$

$$y = C_1\xi \tag{4.39}$$

$$y_c = C_0\eta \tag{4.40}$$

where

$$\eta = [\eta_1^1,\ \eta_2^1, \ldots, \eta_1^m,\ \eta_2^m]^T, \quad \xi = [\xi_1^1, \ldots, \xi_r^1, \ldots, \xi_1^m, \ldots, \xi_r^m]^T, \quad \phi = (\phi_1 \ldots \phi_m), \quad a = (a_1 \ldots a_m),$$

$$A_0 = \text{blockdiag}\begin{bmatrix}A_{01},\ldots,A_{0m}\end{bmatrix}, B_0 = \text{blockdiag}\begin{bmatrix}B_{01},\ldots,B_{0m}\end{bmatrix},$$
$$C_0 = \text{blockdiag}\begin{bmatrix}C_{01},\ldots,C_{0m}\end{bmatrix},$$
$$A_1 = \text{blockdiag}\begin{bmatrix}A_{11},\ldots,A_{1m}\end{bmatrix}, B_1 = \text{blockdiag}\begin{bmatrix}B_{11},\ldots,B_{1m}\end{bmatrix},$$
$$C_1 = \text{blockdiag}\begin{bmatrix}C_{11},\ldots,C_{1m}\end{bmatrix},$$

and , for $0 \le i \le m$, the 2×2 matrix A_{0i}, the 2×1 matrix B_{0i} and the 1×2 matrix C_{0i} represent a chain of 2 integrators and the $r\times r$ matrix A_{1i}, the $r\times 1$ matrix B_{1i} and the $1\times r$ matrix C_{1i} represent a chain of r integrators. Furthermore, $(\eta,\xi)\in R^n$, where $n = \bar{q}+\bar{r}$, $\bar{q} = m\times 2$ and $\bar{r} = m\times r$.

System (4.36) is motivated by the flexible joint manipulators system presented in Section 4.2. In general, (4.36) could represent a class of under-actuated mechanical systems, where the number of inputs is less than the degrees of freedom. These systems may have unstable zero dynamics with respect to the output y. In this case the zero dynamics are the η dynamics with $\xi = 0$. Another example of these systems is the two-mass-spring floating oscillator working under the influence of friction at both masses [80], [81]. It is also possible that the considered class of systems may include electro-mechanical systems where the mechanical part, for example, is comprised of the position and its derivatives and the electrical part is the current or voltage. On the other hand, a different source of (4.36) could be the normal form described in [56]. More specifically, system (4.36) can be derived from the normal form

$$\begin{aligned}\dot{x} &= n(x)+g(x)l(\xi)\\ \dot{\xi} &= A_1\xi + B_1[h(x)+a(\xi,u)]\\ y &= C_1\xi,\end{aligned} \tag{4.41}$$

where $x \in R^{\bar{q}}$ and $n(.), g(.)$ and $h(.)$ are sufficiently smooth, if the auxiliary system

$$\dot{x} = f(x) + g(x)l(\xi)$$
$$y_a = h(x)$$

with output y_a has a well defined ($m \times 1$ vector) relative degree $\{2, ..., 2\}$ and $2 + 2 + ... + 2 = \bar{q}$.[1]

Assumption 4.1 *The vector fields $\phi_i(.,.)$ and $a_i(.,.)$ are sufficiently smooth. Furthermore, $\phi_i(0,0) = 0$ and $a_i(0,0) = 0, \forall t \geq 0$.*

The objective is to design an observer-based output feedback controller so that all state variables are bounded and the controlled output y_c asymptotically tracks a reference signal $d_1(t)$, that is,

$$\lim_{t \to \infty} (y_c - d_1(t)) = 0,$$

for all initial states in a given compact set.

4.4 Full State Feedback Control

We first start by considering state feedback control to achieve the goal of tracking with the anticipation that we will be able to recover the performance of this control by the high-gain observer of Section 4.5. To this end, let $d(t) \in D$, where D is a known compact subset of R^n, a compact set that includes the desired reference signal $d_1(t)$ and its derivatives.

Typically the first step in designing a state feedback tracking controller is to determine a change of variables, which is dependent on the reference signal and its derivatives, that would transform

[1]The reference [56] states conditions for a system to have a well defined vector relative degree and provides the change of variables that can be used to transform the system to the normal form.

the original system into an error coordinate system with equilibrium at zero error. The task then is to design a state feedback control to make the origin of the transferred system asymptotically stable, and hence, achieving the desired tracking goal. More specifically, following the frame work used in Section 4.2, we propose a change of variables

$$e = \begin{bmatrix} \eta_1 - d_1 \\ \eta_2 - d_2 \\ \xi_1 - d_3 \\ \xi_2 - d_4 \\ \vdots \\ \xi_r - d_{2+r} \end{bmatrix}_{n\times 1} \tag{4.42}$$

where d_2 is composed of linear combination of d_1 and its derivative, d_3 is designed to achieve the overall tracking control goal and $d_4 - d_{2+r}$ are composed of linear combinations of $d_{\bar{q}+1}$ and its derivatives. Accordingly, consider a full state feedback control for the system (4.36) that takes the form

$$u = \gamma(\eta, \xi, d(t)). \tag{4.43}$$

Using the change of variables (4.42) and the control (4.43), the closed loop state feedback system can be written as

$$\dot{e} = f(e, \gamma(\eta, \xi, d(t)), d(t)). \tag{4.44}$$

Assumption 4.2 *The vector field $f(e, \gamma(\eta, \xi, d(t)), d(t))$ is locally Lipschitz in e, η and ξ and zero at $(e = 0)$ uniformly in $d(t)$.*

We now have the following assumption that pertains to the state feedback control.

Assumption 4.3 *1. γ is locally Lipschitz in η and ξ uniformly in d, and $\gamma(0,0,d(t)) = 0$.*

2. There exists a radially unbounded smooth Lyapunov function $V_1(t,e)$, for the closed loop system (4.44)*, and three positive definite and continuous functions $U_1(e), U_2(e)$ and $U_3(e)$, such that for all $t \geq 0$*

$$U_1(e) \leq V_1(t,e) \leq U_2(e) \tag{4.45}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial e} f(e, \gamma(\eta, \xi, d(t)), d(t)) \leq -U_3(e) \tag{4.46}$$

The feedback control may also be required to satisfy extra requirements such as achieving particular transient response performance.

4.5 Observer Design

Consider the system (4.36). We use an extended high-gain observer to estimate the first $r+1$ derivatives of the output y_i. The first r derivatives of the output comprise ξ^i, while the $(r+1)^{th}$ derivative is used to compute η_1^i. We consider an auxiliary system that is based on the η^i dynamics with η_1^i used as an output. This auxiliary system is in the standard normal form and has a full relative degree, viewing ξ^i as an input. Therefore, we can design a high-gain observer to estimate the η^i state vector. The crucial point is to design the extended high-gain observer for the output derivatives so that it is fast enough relative to the high-gain observer for the auxiliary system. This is achieved by choosing the eigenvalues of the high-gain observer to be of the order $O(1/\varepsilon)$ while the eigenvalues of the extended high-gain observer of the order $O(1/\varepsilon^2)$. Accordingly, the full order observer is given by

$$\dot{\hat{\eta}} = A_0\hat{\eta} + B_0\hat{\phi}(\hat{\eta}, \hat{\xi}) + H_0(\text{sat}(\hat{\sigma}) - C_0\hat{\eta}) \tag{4.47}$$

$$\dot{\hat{\xi}} = A_1\hat{\xi} + B_1[\hat{\sigma} + \hat{a}(\hat{\xi}, u)] + H_1(y - C_1\hat{\xi}) \tag{4.48}$$

$$\dot{\hat{\sigma}} = \text{sat}(\hat{\eta}_2) + H_2(y - C_1\hat{\xi}) \tag{4.49}$$

where

$$H_0 = \text{blockdiag}[H_{01}, ..., H_{0m}]_{q\times m}, \quad H_{0i} = \begin{bmatrix} \alpha_1^i/\varepsilon \\ \alpha_2^i/\varepsilon^2 \end{bmatrix}$$

$$H_1 = \text{blockdiag}[H_{11}, ..., H_{1m}]_{r\times m}, \quad H_{1i} = \begin{bmatrix} \beta_1^i/\varepsilon^2 \\ \beta_2^i/\varepsilon^4 \\ \vdots \\ \beta_r^i/\varepsilon^{2r} \end{bmatrix},$$

$$H_3 = \text{diag}[\beta_{r+1}^i/\varepsilon^{2(r+1)}]_{m\times m},$$

$i = 1, ..., n$, where ε is a positive constant to be specified. The nonlinear functions $\hat{\phi}(.,.)$ and $\hat{a}(.,.)$ are the same as $\phi(.,.)$ and $a(.,.)$ inside the working region of interest. The positive constants α_j^i and β_k^i are chosen such that the roots of $s^2 + \alpha_1^i s + \alpha_2^i = 0$ and $s^{r+1} + \beta_1^i s^r + ... + \beta_r^i s^1 + \beta_{r+1}^i = 0$ are in the open left-half plane. Notice the use of the function sat(.), which is the standard saturation function, to prevent peaking. The saturation of $\hat{\sigma}$ and $\hat{\eta}_2$, respectively, should be done outside the domain of $\sigma = \eta_1$ and η_2 under the state feedback control.

Assumption 4.4 *The nonlinear functions $\hat{\phi}(.,.), \hat{a}(.,.)$ and $\gamma(.,.,.)$ are bounded outside compact sets to which their ranges belong to under state feedback.*

Remark 4.2 *Assumption 4.4 is needed to protect against the peaking phenomenon and can always* *be achieved by using saturation outside the working region of interest.*

4.6 Output Feedback Control

We now replace the states in the state feedback control with their estimates, so that

$$u = \gamma(\hat{\eta}, \hat{\xi}, d(t)) \tag{4.50}$$

and use the full order observer (4.47)-(4.49) to provide these estimates. In what follows, we will prove that the output feedback control system recovers the stability properties of the state feedback system. Furthermore, we will show also that it is possible to recover the system trajectories under state feedback. To show these results, we combine the system dynamics under (4.50) with the estimation error dynamics of the observer. We use the scaled estimation error

$$\tilde{\eta}_j^i = \frac{\eta_j^i - \hat{\eta}_j^i}{\varepsilon^{2-j}}, \tag{4.51}$$

$$\chi_k^i = \frac{\xi_k^i - \hat{\xi}_k^i}{\varepsilon^{2r+3-2k}}, \tag{4.52}$$

$$\chi_{r+1}^i = \frac{\eta_1^i - \hat{\sigma}^i}{\varepsilon}, \tag{4.53}$$

for $i = 1, ..., m, j = 1, 2$ and $k = 1, ..., r$. It is worth mentioning that the scaling for $\tilde{\eta}$ is typical for high-gain observer results. However, the scaling for χ is chosen such that it is dependent of the dimension of the η dynamics, since there is coupling through $\hat{\sigma}$, and also to preserve the two time scale structure. Hence, we will be able to put the closed loop system in a multi-time scale structure.

To this end, let

$$\varphi_i = \begin{bmatrix} \chi_1^i \\ \vdots \\ \chi_r^i \end{bmatrix}, \quad \chi^i = \begin{bmatrix} \varphi_i \\ \chi_{r+1}^i \end{bmatrix}_{[(r+1)\times 1]}, \quad \chi = \begin{bmatrix} \chi^1 \\ \vdots \\ \chi^m \end{bmatrix}_{[(\bar{r}+m)\times 1]},$$

$$R(\varepsilon) = \text{blockdiag}[R_1, ..., R_m]_{\bar{q}\times\bar{q}}, \quad R_i = \text{diag}[\varepsilon, 1]_{2\times 2},$$

$$Q(\varepsilon) = \text{blockdiag}[Q_1, ..., Q_m]_{\bar{r}\times\bar{r}}, \quad Q_i = \text{diag}[\varepsilon^{2r+1}, ..., \varepsilon^3]_{r\times r},$$

$$S(\varepsilon) = \text{blockdiag}[S_1, ..., S_m]_{(\bar{r}+m)\times(\bar{r}+m)}, \quad S_i = \text{diag}[\varepsilon^{2r+1}, ..., \varepsilon^3, \varepsilon]_{(r+1)\times(r+1)},$$

then (4.52) becomes $Q(\varepsilon)\varphi = \xi - \hat{\xi}$ and equations (4.51)-(4.53) can be written in the compact form

$$R(\varepsilon)\tilde{\eta} = \eta - \hat{\eta} \tag{4.54}$$

$$S(\varepsilon)\chi = \begin{bmatrix} \xi - \hat{\xi} \\ \eta_1 - \hat{\sigma} \end{bmatrix} \tag{4.55}$$

As a result, the closed-loop system under output feedback takes the form

$$\dot{e} = f(e, \gamma(\eta - R\tilde{\eta}, \xi - Q\varphi, d(t)), d(t)) \triangleq f_1(e, R\tilde{\eta}, Q\varphi, d(t)) \tag{4.56}$$

$$\varepsilon\dot{\tilde{\eta}} = \Lambda_0\tilde{\eta} + \varepsilon B_0\Delta\phi + F\chi_{\bar{r}+1} \tag{4.57}$$

$$\varepsilon^2\dot{\chi} = \Lambda_1\chi + \varepsilon[\bar{B}_1\phi_1 + \varepsilon\bar{B}_2\bar{\Delta}a] \tag{4.58}$$

where

$$F = \text{blockdiag}\begin{bmatrix} F_1, ..., F_m \end{bmatrix}_{\bar{q}\times m}, \quad F_i = \begin{bmatrix} \alpha_1^i \\ \\ \alpha_2^i \end{bmatrix}_{2\times 1}, \quad \chi_{\bar{r}+1} = \begin{bmatrix} \chi_{r+1}^1 \\ \vdots \\ \chi_{r+1}^m \end{bmatrix}_{m\times 1},$$

$$\bar{B}_1 = \text{blockdiag}\begin{bmatrix} \bar{B}_{11}, ..., \bar{B}_{1m} \end{bmatrix}_{(\bar{r}+m)\times m}, \quad \bar{B}_{1i} = \begin{bmatrix} 0 \\ \\ B_{1i} \end{bmatrix}_{(r+1)\times 1}$$

$$\bar{B}_2 = \text{blockdiag}\begin{bmatrix} \bar{B}_{21}, ..., \bar{B}_{2m} \end{bmatrix}_{(\bar{r}+m)\times m}, \quad \bar{B}_{2i} = \begin{bmatrix} B_{1i} \\ \\ 0 \end{bmatrix}_{(r+1)\times 1},$$

$$\Delta\phi(\eta, \xi, R\tilde{\eta}, Q\varphi) = \phi(\eta, \xi) - \hat{\phi}(\hat{\eta}, \hat{\xi}),$$

$$\phi_1 = \eta_2 - \text{sat}(\hat{\eta}_2),$$

$$\bar{\Delta}a = N(\varepsilon)\Delta a/\varepsilon^2, \quad N(\varepsilon) = \text{diag}[\varepsilon^{-1}, ..., \varepsilon^{-1}]_{m\times m}$$

and

$$\Delta a(\eta, \xi, R\tilde{\eta}, Q\varphi, d(t)) = a(\xi, \gamma(\hat{\eta}, \hat{\xi}, d(t))) - \hat{a}(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi}, d(t))).$$

Because of the smoothness property of $a(.,.)$, we deduce that, for $(\eta, \xi, R\tilde{\eta}, Q\varphi) \in Z \subset R^{2n}$ where Z is a compact set, $\bar{\Delta}a$ is locally Lipschitz in its arguments, uniformly in ε and $d(t)$. In addition, for any $\tilde{\varepsilon} \leq 1$ and for all $(\eta, \xi, R\tilde{\eta}, Q\varphi) \in Z \subset R^{2n}$, there exists $0 \leq \varepsilon \leq \tilde{\varepsilon}$, such that

$$\begin{aligned} \left|\left|\frac{1}{\varepsilon^2}N(\varepsilon)[a(\xi, \gamma(\hat{\eta}, \hat{\xi}, d(t))) - \hat{a}(\hat{\xi}, \gamma(\hat{\eta}, \hat{\xi}, d(t)))]\right|\right| \leq \bar{L}\varepsilon^{-3}\left|\left|\xi - \hat{\xi}\right|\right| &\leq \bar{L}\varepsilon^{-3}\,||Q||\,||\chi|| \\ &\leq \bar{L}\,||\chi|| \end{aligned} \tag{4.59}$$

where $\bar{L}$ is the Lipschitz constant of $a(.,.)$ over Z. Notice that to get (4.59), we used the facts

$||Q|| \leq \varepsilon^3$ and $||\varphi|| \leq ||\chi||$. Henceforth, we will always consider $\varepsilon \leq \tilde{\varepsilon}$.

The matrices Λ_1 and Λ_2 are Hurwitz by design and given by

$$\Lambda_0 = \text{blockdiag}[\Lambda_{01}, ..., \Lambda_{0m}]_{\bar{q}\times\bar{q}}, \quad \Lambda_{0i} = \begin{bmatrix} -\alpha_1^i & 1 \\ -\alpha_2^i & 0 \end{bmatrix}_{2\times 2}$$

and

$$\Lambda_1 = \text{blockdiag}[\Lambda_{10}, ..., \Lambda_{1m}]_{(\bar{r}+m)\times(\bar{r}+m)}, \quad \Lambda_{1i} = \begin{bmatrix} -\beta_1^i & 1 & 0 & \dots & 0 \\ -\beta_2^i & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ -\beta_r^i & 0 & 0 & \ddots & 1 \\ -\beta_{r+1}^i & 0 & 0 & \dots & 0 \end{bmatrix}_{(r+1)\times(r+1)}.$$

Equations (4.56)-(4.58) are in the standard singularly perturbed form with three time scales structure. The slow variable of this structure is $e(t)$ and the fast variables are $(\tilde{\eta}(t), \chi(t))$, moreover, $\chi(t)$ is faster than $\tilde{\eta}(t)$. Notice that for ease of presentation, we dropped the explicit use of the function sat(.) for $\hat{\sigma}$. This will not cause a problem since will show later that the saturation will not affect the stability of the boundary layer system. We will also show that the trajectories of the slow system will remain bounded in the saturation period, which remains active only in the peaking period. This period can be made arbitrarily small by reducing ε. Setting $\varepsilon = 0$ in (4.56)-(4.58) we get $\tilde{\eta} = 0$ and $\chi = 0$, so that we have the reduced system

$$\dot{e} = f(e, \gamma(\eta, \xi, d(t)), d(t)) = f_1(e, 0, 0, d(t)). \tag{4.60}$$

Notice that system (4.60) is nothing but the state feedback system (4.44). Denote the solution of

(4.56)-(4.58) by $e(t,\varepsilon)$ and the solution of (4.60) by $e_r(t)$ starting from $e(0)$. Moreover, define the initial states as $e(0) \in \mathscr{M}$ and $(\hat{\eta}(0), \hat{\xi}(0), \hat{\sigma}(0)) \in \mathscr{N}$, where $\mathscr{M}$, containing the origin, and $\mathscr{N}$ are any compact subsets of R^n and R^{n+m}, respectively. We then have the following theorem.

Theorem 4.1 *Consider the closed-loop system* (4.56)-(4.58). *Let Assumptions 4.1-4.4 hold. Then we have the following*

- *there exists* $\varepsilon_1^* > 0$ *such that, for every* $0 < \varepsilon \leq \varepsilon_1^*$, *the solutions* $(e(t), \hat{\eta}(t), \hat{\xi}(t), \hat{\sigma}(t))$ *of the closed-loop system, starting in* $\mathscr{M} \times \mathscr{N}$, *are bounded for all* $t \geq 0$ *and* $d \in D$.

- *there exists* $\varepsilon_2^* > 0$ *such that, for every* $0 < \varepsilon \leq \varepsilon_2^*$, *the origin of the closed-loop system* (4.56)-(4.58) *is uniformly asymptotically stable and* $\mathscr{M} \times \mathscr{N}$ *is a subset of its region of attraction.*

- *for trajectories starting in* $\mathscr{M} \times \mathscr{N}$, *given any* $\mu > 0$, *there exists* $\varepsilon_3^* > 0$ *dependent on* μ, *such that, for every* $0 < \varepsilon \leq \varepsilon_3^*$, *we have*

$$||e(t,\varepsilon) - e_r(t)|| \leq \mu, \quad \forall t \geq 0, \quad \forall\, d \in D. \tag{4.61}$$

- *if the origin of* (4.44) *is exponentially stable and* $f(.,.,.)$ *is continuously differentiable in some neighborhood of* $(e = 0, \tilde{\eta} = 0, \chi = 0)$, *then there exists* $\varepsilon_4^* > 0$ *such that, for every* $0 < \varepsilon < \varepsilon_4^*$, *the origin of the closed-loop system* (4.56)-(4.58) *is exponentially stable and* $\mathscr{M} \times \mathscr{N}$ *is a subset of its region of attraction.*

Proof:

The proof follows the singular perturbation approach and uses similar arguments to the ones used in [35] and Chapters 2 and 3 of this dissertation.

We start by analyzing the boundary-layer model. For this purpose, use the new time scale $\tau = t/\varepsilon$ for the system (4.56)-(4.58) to get

$$\frac{de}{d\tau} = \varepsilon f(e, \gamma(\eta - R\tilde{\eta}, \xi - Q\varphi, d(t)), d(t)) \tag{4.62}$$

$$\frac{d\tilde{\eta}}{d\tau} = \Lambda_0 \tilde{\eta} + \varepsilon B_0 \Delta\phi + F\chi_{\bar{r}+1} \tag{4.63}$$

$$\varepsilon \frac{d\chi}{d\tau} = \Lambda_1 \chi + \varepsilon[\bar{B}_1 \tilde{\eta}_2 + \varepsilon \bar{B}_2 \bar{\Delta} a] \tag{4.64}$$

We notice that the subsystem (4.63)-(4.64) is the boundary layer subsystem to the original system (4.56)-(4.58). Moreover, subsystem (4.64) is the boundary layer subsystem to (4.63)-(4.64). Setting $\varepsilon = 0$ in this time scale yields $\chi = 0$ and

$$\frac{d\tilde{\eta}}{d\tau} = \Lambda_0 \tilde{\eta} \tag{4.65}$$

For (4.65), we define the Lyapunov function candidate $V_2(\tilde{\eta}) = \tilde{\eta}^T P_0 \tilde{\eta}$, where P_0 is the positive definite solution of $P_0 \Lambda_0 + \Lambda_0^T P_0 = -I$. This Lyapunov function satisfies

$$\lambda_{\min}(P_0)\,||\tilde{\eta}||^2 \leq V_2(\tilde{\eta}) \leq \lambda_{\max}(P_0)\,||\tilde{\eta}||^2 \tag{4.66}$$

$$\frac{\partial V_2}{\partial \tilde{\eta}} \Lambda_0 \tilde{\eta} \leq -\,||\tilde{\eta}||^2 \tag{4.67}$$

where $\lambda_{\min}(P_0)$ and $\lambda_{\max}(P_0)$ are the minimum and maximum eigenvalues of P_0, respectively.

Similarly, we analyze the stability of the boundary layer (4.64). To this end, we use the time scale $\rho = \frac{\tau}{\varepsilon}$. This leads to

$$\frac{de}{d\rho} = \varepsilon^2 f(e, \gamma(\eta - R\tilde{\eta}, \xi - Q\varphi, d(t)), d(t)) \tag{4.68}$$

$$\frac{d\tilde{\eta}}{d\rho} = \varepsilon[\Lambda_0\tilde{\eta} + \varepsilon B_0 \Delta\phi + F\chi_{\bar{r}+1}] \tag{4.69}$$

$$\frac{d\chi}{d\rho} = \Lambda_1\chi + \varepsilon[\bar{B}_1\tilde{\eta}_2 + \varepsilon\bar{B}_2\bar{\Delta}a] \tag{4.70}$$

Setting $\varepsilon = 0$, we get

$$\frac{d\chi}{d\rho} = \Lambda_2\chi \tag{4.71}$$

For this boundary layer subsystem we define the Lyapunov function candidate $V_3(\chi) = \chi^T P_1 \chi$, where P_1 is the positive definite solution of $P_1\Lambda_1 + \Lambda_1^T P_1 = -I$. It can be shown that this function satisfies

$$\lambda_{\min}(P_1)\,||\chi||^2 \leq V_3(\chi) \leq \lambda_{\max}(P_1)\,||\chi||^2 \tag{4.72}$$

$$\frac{\partial V_3}{\partial \chi}\Lambda_1\chi \leq -\,||\chi||^2 \tag{4.73}$$

where $\lambda_{\min}(P_1)$ and $\lambda_{\max}(P_1)$ are the minimum and maximum eigenvalues of P_1, respectively.

Using Assumption 4.3, we define a compact set $\Omega = \{e \in R^n : V_1(t,e) \leq c\}$, where $c > \max_{e\in\mathscr{M}} U_2(e)$. This implies

$$\mathscr{M} \subset \Omega \subseteq R^n.$$

Let $\mathscr{S} = \Omega \times \{V_2(\tilde{\eta}) \leq \rho_0\varepsilon^2\} \times \{V_3(\chi) \leq \rho_1\varepsilon^2\}$, where ρ_0 and ρ_1 are positive constants to be specified. We will show that $\mathscr{S}$ is positively invariant set for every $0 < \varepsilon \leq \varepsilon_1$, for some $\varepsilon_1 > 0$.

Using Assumption 4.44, we can show, for all $(e, \tilde{\eta}, \chi) \in \mathscr{S}$, that

$$||f_1(e, R\tilde{\eta}, Q\chi, d(t)) - f_1(e, 0, 0, d(t))|| \leq L_1 \, ||\tilde{\eta}|| + L_2 \, ||\chi|| \tag{4.74}$$

where L_1 and L_2 are positive constants independent of ε. Due to Assumptions 4.2 and 4.4, the fact that continuous functions are bounded over compact sets, and for all $e \in \Omega$ and $(\tilde{\eta}, \chi) \in R^{n+m}$, we have

$$||\Delta\phi(\eta, \xi, R\tilde{\eta}, Q\varphi)|| \leq k_1 \tag{4.75}$$

$$||\phi_1(\eta_2, \hat{\eta}_2)|| \leq k_2 \tag{4.76}$$

$$||f(e, \gamma(\eta - R\tilde{\eta}, \xi - Q\varphi, d(t)), d(t))|| \leq k_{10} \tag{4.77}$$

where k_1, k_2 and k_{10} are positive constants independent of ε. Using Assumption 4.43, inequalities (4.59) and (4.74) and the bounds (4.75) and (4.76), we can show that for all $(e, \tilde{\eta}, \chi) \in \mathscr{S}$ we have

$$\dot{V}_1 \leq -U_3(e) + (k_3 + k_4)\varepsilon \tag{4.78}$$

$$\dot{V}_2 \leq -\frac{1}{\varepsilon} \, ||\tilde{\eta}||^2 + 2k_1 \, ||\tilde{\eta}|| \, ||P_0|| + \frac{2k_6}{\varepsilon} \, ||\tilde{\eta}|| \, ||\chi|| \, ||P_0|| \tag{4.79}$$

$$\dot{V}_3 \leq -(\frac{1}{\varepsilon^2} - 2\bar{L} \, ||P_1||) \, ||\chi||^2 + \frac{2k_2}{\varepsilon} \, ||\chi|| \, ||P_1|| \tag{4.80}$$

where $k_3 = L_1 k_5 \sqrt{\rho_0/(\lambda_{\min}(P_0))}$, $k_4 = L_2 k_5 \sqrt{\rho_1/(\lambda_{\min}(P_1))}$, $||P_0|| = \lambda_{\max}(P_0)$, $||P_1|| = \lambda_{\max}(P_1)$, k_5 is an upper bound for $||\partial V_1/\partial e||$ over Ω, and $||F|| = k_6$, where k_6 is a positive constant. To get (4.78)-(4.80), we also used $||B_0|| = ||\bar{B}_1|| = ||\bar{B}_2|| = 1$.

Recall that we always consider $\varepsilon \leq 1$. Choose $\rho_1 = (64k_2^2 \lambda_{\max}^4(P_1))/(\lambda_{\min}(P_1))$ and $\varepsilon_a = (\lambda_{\min}(P_1))/(8\bar{L}\lambda_{\max}^2(P_1))$, then we can show that $\dot{V}_3 \leq 0$ for all $0 < \varepsilon \leq \varepsilon_a$ and $(e, \tilde{\eta}, \chi) \in \{V_1(t, e) \leq$

$c\} \times \{V_2(\tilde{\eta}) \leq \rho_0\varepsilon^2\} \times \{V_3(\chi) = \rho_1\varepsilon^2\}$.

By choosing $\rho_0 = (16\lambda_{\max}^4(P_0))/(\lambda_{\min}(P_0))[k_1 + k_6\sqrt{\rho_1/\lambda_{\min}(P_1)}]^2$, it can be shown that $\dot{V}_2 \leq 0$ for all $(e, \tilde{\eta}, \chi) \in \{V_1(t,e) \leq c\} \times \{V_2(\tilde{\eta}) = \rho_0\varepsilon^2\} \times \{V_3(\chi) \leq \rho_1\varepsilon^2\}$. Choosing $\varepsilon_b = \beta/(k_3+k_4)$, where $\beta = \min_{e \in \partial\Omega} U_3(e)$, it can be shown that, for every $0 < \varepsilon \leq \varepsilon_b$, we have $\dot{V}_1 \leq 0$ for all $(e, \tilde{\eta}, \chi) \in \{V_1(t,e) = c\} \times \{V_2(\tilde{\eta}) \leq \rho_0\varepsilon^2\} \times \{V_3(\chi) \leq \rho_1\varepsilon^2\}$. Choose $\varepsilon_1 = \min\{\varepsilon_a, \varepsilon_b\}$, then it can be seen that, for all $0 < \varepsilon \leq \varepsilon_1$, $\mathscr{S}$ is positively invariant.

Consider now the initial state $(e(0), \hat{\eta}(0), \hat{\xi}(0), \hat{\sigma}(0)) \in \mathscr{M} \times \mathscr{N}$. It can be checked that the corresponding initial errors $\tilde{\eta}(0)$ and $\chi(0)$ satisfy $||\tilde{\eta}(0)|| \leq k_7/\varepsilon$ and $||\chi(0)|| \leq k_8/\varepsilon^{2r+1}$, respectively, for some nonnegative constants k_7 and k_8 dependent on $\mathscr{M}$ and $\mathscr{N}$. It can also be shown that, since $e(0)$ is in the interior of ω and as long as $e(t,\varepsilon) \in \omega$, we have

$$||e(t,\varepsilon) - e(0)|| \leq k_{10}t. \tag{4.81}$$

Therefore, there exists a finite time T_0, independent of ε, such that $e(t,\varepsilon) \in \omega$ for all $t \in [0, T_0]$. During this time interval, we can show first that

$$\dot{V}_3 \leq -\frac{1}{2\varepsilon^2}||\chi||^2 \tag{4.82}$$

for $V_3 \geq \rho_1\varepsilon^2$ and $\varepsilon \leq \varepsilon_a$ Based on this fact, we can show that

$$\dot{V}_2 \leq -\frac{1}{2\varepsilon}||\tilde{\eta}||^2 \tag{4.83}$$

for $V_2 \geq \rho_0\varepsilon^2$ and $V_3 \leq \rho_1\varepsilon^2$. Inequalities (4.82) and (4.83) indicate that the variable χ first becomes $O(\varepsilon)$. This in turn allows the variable $\tilde{\eta}$ to become $O(\varepsilon)$.

As a result, it can be verified that

$$V_2(\tilde{\eta}(t)) \leq \frac{\sigma_2}{\varepsilon^2}\exp(-\sigma_1 t/\varepsilon) \tag{4.84}$$

$$V_3(\chi(t)) \leq \frac{\sigma_4}{\varepsilon^{2(2r+1)}}\exp(-\sigma_3 t/\varepsilon^2) \tag{4.85}$$

where $\sigma_1 = 1/2\,||P_0||$, $\sigma_2 = ||P_0||\,k_7^2$, $\sigma_3 = 1/2\,||P_1||$ and $\sigma_4 = ||P_1||\,k_8^2$. Now choose ε_2 small enough such that, for all $0 < \varepsilon \leq \varepsilon_2$, we have $T_1(\varepsilon) \triangleq T_2(\varepsilon) + T_3(\varepsilon) \leq \frac{1}{2}T_0$, where

$$T_2(\varepsilon) \triangleq \frac{\varepsilon}{\sigma_1}\ln\left(\frac{\sigma_2}{\rho_0\varepsilon^4}\right) \tag{4.86}$$

$$T_3(\varepsilon) \triangleq \frac{\varepsilon^2}{\sigma_3}\ln\left(\frac{\sigma_4}{\rho_1\varepsilon^{4r+4}}\right) \tag{4.87}$$

We note that ε_2 exists, since $T_2(\varepsilon)$ and $T_3(\varepsilon)$ tend to zero as ε tends to zero. It follows that $V_2(\tilde{\eta}(T_1)) \leq \rho_0\varepsilon^2$ and $V_3(\chi(T_1)) \leq \rho_1\varepsilon^2$ for every $0 < \varepsilon \leq \varepsilon_2$. Taking $\varepsilon_1^* = \min\{\tilde{\varepsilon}, \varepsilon_1, \varepsilon_2\}$ guarantees that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the trajectory $(e(t), \tilde{\eta}(t), \chi(t))$ enters $\mathscr{S}$ during the time interval $[0, T_1(\varepsilon)]$ and remains there for all $t \geq T_1(\varepsilon)$. We also note that, for $t \in [0, T_1(\varepsilon)]$, the trajectory is bounded by virtue of inequalities (4.81)-(4.85).

We assume that the trajectories are now inside the set $\mathscr{S}$, where we now prove asymptotic stability of the origin. For this purpose, consider the composite Lyapunov function candidate $V_4 = V_1 + c_1\sqrt{V_2} + \sqrt{V_3}$, where c_1 is positive constant to be determined. Using (4.46), smoothness properties of V_1 and Lipschitz properties of f_1, $\Delta\phi$ and Δa, it can be shown that, for all $(e, \tilde{\eta}, \chi) \in$

$\mathscr{S}$, we have

$$
\begin{aligned}
\dot{V}_4 =& \dot{V}_1 + \frac{c_1 \dot{V}_2}{2\sqrt{V_2}} + \frac{\dot{V}_3}{2\sqrt{V_3}} \\
\leq& -U_3(e) + \left[d_1 - \frac{c_1}{4\varepsilon\sqrt{\lambda_{\min}(P_0)}} + \frac{c_1 d_3}{2\sqrt{\lambda_{\min}(P_0)}} \right] ||\tilde{\eta}|| \\
&+ \left[-\frac{c_1}{4\varepsilon\sqrt{\lambda_{\min}(P_0)}} + \frac{d_7}{2\varepsilon\sqrt{\lambda_{\min}(P_1)}} \right] ||\tilde{\eta}|| + \left[d_2 + \frac{c_1(d_4/\varepsilon + d_5)}{2\sqrt{\lambda_{\min}(P_0)}} - \frac{c_2(1/\varepsilon^2 - d_6)}{2\sqrt{\lambda_{\min}(P_1)}} \right] ||\chi|| \\
\triangleq& -U_3(e) + a||\tilde{\eta}|| + b||\tilde{\eta}|| + c||\chi||
\end{aligned}
$$

where d_1 to d_7 are positive constants. It can be shown that we can choose c_1 large enough to make b negative and $0 < \varepsilon_2^* \leq \varepsilon_1^*$ small enough such that, for all $0 < \varepsilon \leq \varepsilon_2^*$, a and c are negative. This makes $\dot{V}_4$ negative definite, and hence, proves the second bullet.

To prove the third bullet, we divide the interval $[0,\infty]$ into three intervals $[0,T_1(\varepsilon)]$, $[T_1(\varepsilon),T_3]$ and $[T_3,\infty]$, where $T_3 > 0$ is to be determined, and show (4.61) for each interval.

1- The interval $[0,T_1(\varepsilon)]$.

Using similar arguments to the ones leading to the inequality (4.81), we can argue that

$$||e_r(t) - e(0)|| \leq k_{10} t.$$

Hence,

$$||e(t,\varepsilon) - e_r(t)|| \leq 2k_{10} T_1(\varepsilon), \quad \forall t \in [0,T_1(\varepsilon)]. \tag{4.88}$$

Since $T_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, given any $\mu > 0$ there exists $0 < \varepsilon_3 \leq \varepsilon_1^*$ such that, for every $0 < \varepsilon \leq \varepsilon_3$, we have

$$||e(t,\varepsilon) - e_r(t)|| \leq \mu, \quad \forall t \in [0,T(\varepsilon)]. \tag{4.89}$$

2- The interval $[T_1(\varepsilon), T_3]$.

During this interval $e(t,\varepsilon)$ satisfies $\dot{e} = f_1(e, R(\varepsilon)\tilde{\eta}(t), Q(\varepsilon)\varphi(t), d(t))$ with initial condition $e(T_1(\varepsilon), \varepsilon)$ and $R(\varepsilon)\tilde{\eta}$ and $Q(\varepsilon)\varphi$ are $O(\varepsilon)$ and $e_r(t)$ satisfies $\dot{e} = f_1(e, 0, 0, d(t))$, with initial condition $e_r(T_1(\varepsilon))$. From (4.88), we know that $||e(t,\varepsilon) - e_r(t)|| \leq 2k_{10}T_1(\varepsilon) \triangleq \delta(\varepsilon)$, where $\delta \to 0$ as $\varepsilon \to 0$. Therefore, by [Theorem 3.5,[61]], we conclude that, for any $\mu > 0$, there exists $0 < \varepsilon_4 \leq \varepsilon_1^*$ such that for every $0 < \varepsilon \leq \varepsilon_4$, we have

$$||e(t,\varepsilon) - e_r(t)|| \leq \mu, \quad \forall t \in [T_1(\varepsilon), T_3]. \tag{4.90}$$

3- The interval $[T_3, \infty)$.

From the second bullet, we know that for any $\mu > 0$, there exists $\varepsilon_5 > 0$ and $\tilde{T}_3 \geq T_1(\varepsilon) > 0$, both dependent on μ, such that, for every $0 < \varepsilon \leq \varepsilon_5$, we have

$$||e(t,\varepsilon)|| \leq \mu/2, \quad \forall t \geq \tilde{T}_3. \tag{4.91}$$

From the asymptotic stability of the origin of the reduced system (deduced from (4.46)), we know that there exits a finite time $\bar{T}_2$, independent of ε, such that

$$||e_r(t)|| \leq \mu/2, \quad \forall t \geq \bar{T}_3. \tag{4.92}$$

Take $T_3 = \max\{\tilde{T}_3, \bar{T}_3\}$. Then, using the triangular inequality, and from (4.91) and (4.92), we conclude that for every $0 < \varepsilon \leq \varepsilon_5$, we have

$$||e(t,\varepsilon) - e_r(t)|| \leq \mu, \quad \forall t \geq T_3. \tag{4.93}$$

Take $\varepsilon_3^* = \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}$, then using (4.89), (4.90) and (4.93) we conclude (4.61).

To prove the last bullet, we define a ball $B(0,r_1)$, for some radius $r_1 > 0$ inside the set $\mathscr{S}$ and around the origin $(e,\tilde{\eta},\chi) = (0,0,0)$. Since the origin of the closed loop system (4.44) is exponentially stable, there exists a smooth Lyapunov function V_5 that satisfies the following inequalities for all $e \in B(0,r_1)$ [[61], Th. 4.14]

$$a_1\,||e||^2 \leq V_5(t,e) \leq a_2\,||e||^2 \tag{4.94}$$

$$\frac{\partial V_5}{\partial t} + \frac{\partial V_5}{\partial e} f(e,\gamma(\eta,\xi,d(t)),d(t)) \leq -a_3\,||e||^2 \tag{4.95}$$

$$\left|\left|\frac{\partial V_5}{\partial e}\right|\right| \leq a_4\,||e|| \tag{4.96}$$

where a_1, a_2, a_3 and a_4 are positive constants. Consider now the composite Lyapunov function $V_6(t,e,\tilde{\eta},\chi) = \theta_1 V_5(t,e) + V_2(\tilde{\eta}) + V_3(\chi)$ with $\theta_1 > 0$ to be determined. Choose $r_2 < r_1$, then it can be shown, using (4.94)-(4.96), Lipschitz properties of f_1, $\Delta\phi$ and Δa and for all $(e,\tilde{\eta},\chi) \in B(0,r_2) \times \{||\tilde{\eta}|| \leq r_2\} \times \{||\chi|| \leq r_2\}$, that

$$\begin{aligned}
\dot{V}_6 \leq & -\theta_1 a_3\,||e||^2 + \theta_1 b_1\,||e||\,||\tilde{\eta}|| + \theta_1 b_2\,||e||\,||\chi|| - [\frac{1}{\varepsilon} - b_3]\,||\tilde{\eta}||^2 + [b_4 + \frac{b_5}{\varepsilon}]\,||\tilde{\eta}||\,||\chi|| \\
& + \frac{b_6}{\varepsilon}\,||\tilde{\eta}||\,||\chi|| - [\frac{1}{\varepsilon^2} - b_7]\,||\chi||^2 \\
= & - \begin{bmatrix} ||e|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}^T \begin{bmatrix} \theta_1 a_3 & -\theta_1 b_1/2 & -\theta_1 b_2/2 \\ -\theta_1 b_1/2 & [1/\varepsilon - b_3] & -(1/2)[b_4 + (b_5+b_6)/\varepsilon] \\ -\theta_1 b_2/2 & -(1/2)[b_4 + (b_5+b_6)/\varepsilon] & [1/\varepsilon^2 - b_7] \end{bmatrix} \begin{bmatrix} ||e|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix} \\
\triangleq & - \begin{bmatrix} ||e|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}^T \Gamma \begin{bmatrix} ||e|| \\ ||\tilde{\eta}|| \\ ||\chi|| \end{bmatrix}
\end{aligned}$$

where b_1 to b_7 are positive constants. It can be shown that we can choose $0 < \varepsilon_4^* \leq \varepsilon_1^*$ small enough such that, for all $0 < \varepsilon \leq \varepsilon_4^*$, the second principle minor of Γ is positive and, with θ_1 chosen to be small enough, the determinant of Γ is positive. This makes Γ a positive definite constant matrix, and hence, makes $\dot{V}_6$ negative definite. This proves the last bullet. $\square$

4.7 Conclusions

We considered the problem of output feedback tracking of a class of nonlinear systems. This class is characterized by having internal dynamics that have a full relative degree. This makes it possible to represent both the dynamics of the output and its derivatives and the internal dynamics in the chain-of-integrators form. This in turn allows the design of high-gain observers, based on the concept presented in Chapter 2, to estimate all the states of the system. The observer in this case can handle systems that may be non-minimum phase. We allow the use of any state feedback scheme that can achieve uniform global asymptotic stability of the origin of the tracking error dynamics. We showed that the proposed scheme can recover the uniform asymptotic stability, and exponential stability, with respect to the origin of the tracking error dynamics. We also showed the capability of the scheme to recover the trajectories of the state feedback system. However, as it is well known for high-gain observers, the observer gain is limited by the presence of measurement noise. Finally, we solved the problem of output feedback tracking control of n-link flexible joint manipulators. We demonstrated the effectiveness of the output feedback control system when used for the single-link flexible joint manipulator system. In this case, the output to be tracked is the link angle and the measured output is the motor position. This configuration makes the system non-minimum phase with respect to the measured output. This problem, to the best of our knowledge, has not been considered before.

Chapter 5

Conclusions and Future Work

5.1 Concluding Remarks

The focus of this dissertation is on estimation and control of nonlinear systems. The main objective is to estimate all the states of the system using only information from the measured output and to use these estimates in solving different control problems. This is certainly desired when it is hard or expensive to measure all the states of the system. We solved this problem for a large class of nonlinear systems, which are represented in the normal form. An important part of the solution is the use of high-gain observers. In the literature, the use of these observers, in feedback control applications, has been limited to partial state estimation and for minimum phase systems. In this dissertation, we solved the estimation problem by using the high-gain observer to provide estimates of the derivatives of the output. We also extended the derivatives of the output by one in the observer dynamics and used this information to provide an estimate of a state that is used as a virtual output to the remaining (not-estimated) system dynamics. This way if the remaining system dynamics are observable with respect to this virtual output we can use any suitable observer, we call it internal observer, to provide estimates of the remaining states. This is indeed possible because of the relative speed by which the high-gain observer can estimate the virtual output. To solve the estimation problem for nonlinear systems represented in the general normal form, we used an extended Kalman filter as an internal observer. The initial states of the internal dynamics, in this case, can not be arbitrarily large. This is a direct consequence of using linearization in the extended Kalman filter scheme. Nevertheless, thanks to the high-gain observer, the initial states of the other dynamics can belong to any known compact set. We demonstrated the effectiveness of the observer by using it to estimate all the states of a synchronous generator on infinite bus system.

It turns out that the initial states of the internal dynamics can be made as large as desired if the system is linear in the internal state. An example of this type of systems is the Translating Oscillator with a Rotating Actuator (TORA) system. For this particular class of systems, we solved in Chapter 3 the problem of output feedback stabilization, achieving a semi-global stability result. We showed that the output feedback controller can recover the stability properties of any globally stabilizing state feedback controller. We also showed that the output feedback controller can recover the performance of an auxiliary system comprised of the system under state feedback augmented with a systems that represents the estimation error of the internal dynamics. We showed the efficacy of the output feedback scheme when used for the control of a DC-DC boost converter system.

Results for arbitrarily large compact set of initial conditions can also be achieved if the internal dynamics have a full relative degree with respect to the virtual output. This was realised when we solved the problem of output feedback tracking of flexible joint manipulators in Chapter 4. The internal dynamics in this case can be represented by a chain of integrators, forming a double chain of integrators when augmented with the other dynamics. This allows the design of a high-gain observer to be augmented with the extended high-gain observer for the output and its derivatives. The result is a high-gain observer that is capable of estimating the full state vector of the system. Both the extended high-gain observer for the output and its dynamics and the internal observer form with the original closed loop system a three-time-scale structure. The two high-gain observers that make-up the overall observer are designed for two subsystems with each having a full relative degree, i.e. it does not have zero dynamics. This makes the overall observer capable of handling systems with unstable zero dynamics. This feature can be very beneficial in the context of designing output feedback control of non-minimum phase systems. As was shown in Chapter 4, output feedback control systems that use the proposed observer can recover the stability properties of the closed loop system under state feedback. It can also recover the trajectory performance of all the states, satisfying the separation principle along the same lines as in (Atassi and Khalil, 1999) [35]. We demonstrated the effectiveness of the output feedback schemes when used for the tracking control of single-link flexible joint manipulator system. In this example, the observer-based control system showed a reasonable performance in the presence of noise for relatively low value of the observer gain. On the other hand, for relatively high value of the observer gain, the output feedback control system showed a robust performance for unmatched uncertainty. For matched uncertainty, however, we believe that for relatively small uncertainty and when using a robust state feedback controller, the output feedback scheme can provide acceptable robust performance. This feature needs to be further verified and proven and is left as a subject of future work.

In summary, the main contributions of this work are:

- We propose a full order observer for a wide class of nonlinear systems that could include non-minimum phase nonlinear systems.
- The observer design procedure is relatively simple and constructive.
- The proposed observer gives a degree of freedom in designing the observer for the internal dynamics.
- We solve the problem of semi-global output feedback stabilization for systems that are linear in the internal state.
- We solve the problem of semi-global output feedback tracking of systems with internal dy-

namics having full relative degree with respect to a virtual output. The output feedback controller is capable of totally recovering the performance of the state feedback controller.

5.2 Future Work

The work that has been conducted in this dissertation can be extended in different directions.

- The first venue of possible future research is to investigate further the robustness properties of the proposed observer and, hence, the output feedback control system. The robustness properties of the proposed observer are clearly dependent on the robustness properties of the two observers that it is composed of, namely, the extended high-gain observer and the internal observer. It is well known that high-gain observers are robust to modeling uncertainties and external disturbances only if the uncertainties appear in the last system equation. Furthermore, one of the original uses of extended high-gain observers is to estimate modeling uncertainties. Therefore, as part of the observer proposed in Chapter 2, the extended high-gain observer will provide estimate of the virtual output corrupted with these uncertainties. This means that these uncertainties will act as ”noise” to the extended Kalman filter. Consequently, it is interesting to see how the EKF can handle the virtual output noise and the model uncertainties, and how this would be reflected on the performance of the overall observer.

  With respect to the observer proposed in Chapter 4, we showed by simulation that this observer could be robust to any unmatched uncertainty in the internal dynamics. We also believe that it is not hard to prove this property since any uncertainty that appear in the internal dynamics can be dominated by increasing the observer gain. However, as it is well known for high-gain observers, this observer is sensitive to any output noise. In general, we think that it might be possible to filter-output noise from the virtual output before it is used in the internal observer, hence making the observer fully robust to modeling uncertainties and external disturbances.

- It is important to note that the observer framework proposed in this dissertation is flexible enough to allow the design of any suitable observer for the auxiliary system that is dependent on the internal dynamics. Clearly, the main assumption that is needed in this case is the observability of this system. In this work, we only pursued two schemes, namely, the extended Kalman filter and high-gain observers. It would be beneficial, however, to determine a more general framework that could possibly include the two mentioned schemes as special cases. This framework might also provide conditions to achieve the desired requirements of having arbitrarily large initial conditions and robustness to modeling uncertainties and external disturbances.

- Another possible direction of future research is to extend the techniques proposed in this dissertation to solve control problems where the auxiliary system includes totally autonomous dynamics that could be viewed as an ”exogenous” signal generator. Depending on the control scenario, this generator could represent exogenous disturbances to be rejected and/or reference model to be tracked. In this regard, it might be possible to use the proposed observer to estimate the states of the system along with the states of the signal generator, which might in turn simplify the design of the output feedback control system.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] M. Zeitz, "The extended luenberger observer for nonlinear systems," *Systems & Control Letters*, vol. 9, no. 2, pp. 149–156, 1987.

[2] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Journal of Basic Engineering*, vol. 83, no. 3, pp. 95–108, 1961.

[3] J. Baras, A. Bensoussan, and M. James, "Dynamic observers as asymptotic limits of recursive filters: Special cases," *SIAM Journal on Applied Mathematics*, vol. 48, no. 5, pp. 1147–1158, 1988.

[4] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Systems & Control Letters*, vol. 3, no. 1, pp. 47–52, 1983.

[5] I. Souleiman, A. Glumineau, and G. Schreier, "Direct transformation of nonlinear systems into state affine miso form for observer design," *Automatic Control, IEEE Transactions on*, vol. 48, no. 12, pp. 2191–2196, 2003.

[6] Z. Ding, "Semi-global stabilisation of a class of non-minimum phase non-linear output-feedback systems," in *Control Theory and Applications, IEE Proceedings-*, vol. 152, no. 4. IET, 2005, pp. 460–464.

[7] X. Fan and M. Arcak, "Observer design for systems with multivariable monotone nonlinearities," *Systems & Control Letters*, vol. 50, no. 4, pp. 319–330, 2003.

[8] A. Zemouche and M. Boutayeb, "A unified h adaptive observer synthesis method for a class of systems with both lipschitz and monotone nonlinearities," *Systems & Control Letters*, vol. 58, no. 4, pp. 282–288, 2009.

[9] B. Açıkmeşe and M. Corless, "Observers for systems with nonlinearities satisfying incremental quadratic constraints," *Automatica*, vol. 47, no. 7, pp. 1339–1348, 2011.

[10] H. Khalil, "High-gain observers in nonlinear feedback control," in *Control, Automation and Systems, 2008. ICCAS 2008. International Conference on*. IEEE, 2008, pp. xlvii–lvii.

[11] S. K. Spurgeon, "Sliding mode observers: a survey," *International Journal of Systems Science*, vol. 39, no. 8, pp. 751–764, 2008.

[12] J. H. Ahrens and H. K. Khalil, “Closed-loop behavior of a class of nonlinear systems under ekf-based control,” *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 536–540, 2007.

[13] M. Fliess, C. Join, and H. Sira-Ramirez, “Non-linear estimation is easy,” *International Journal of Modelling, Identification and Control*, vol. 4, no. 1, pp. 12–27, 2008.

[14] D. Karagiannis, D. Carnevale, and A. Astolfi, “Invariant manifold based reduced-order observer design for nonlinear systems,” *Automatic Control, IEEE Transactions on*, vol. 53, no. 11, pp. 2602–2614, 2008.

[15] R. G. Sanfelice and L. Praly, “Convergence of nonlinear observers on rn with a riemannian metric (part i),” *IEEE Transactions on Automatic Control*, vol. 57, no. 7, p. 1709, 2012.

[16] L. Vasiljevic and H. Khalil, “Error bounds in differentiation of noisy signals by high-gain observers,” *Systems & Control Letters*, vol. 57, no. 10, pp. 856–862, 2008.

[17] H. K. Khalil and L. Praly, “High-gain observers in nonlinear feedback control,” *International Journal of Robust and Nonlinear Control*, pp. n/a–n/a, 2013. [Online]. Available: http://dx.doi.org/10.1002/rnc.3051

[18] K. Busawon, M. Farza, and H. Hammouri, “Observer design for a special class of nonlinear systems,” *International Journal of Control*, vol. 71, no. 3, pp. 405–418, 1998.

[19] F. Deza, E. Busvelle, J. Gauthier, and D. Rakotopara, “High gain estimation for nonlinear systems,” *Systems & control letters*, vol. 18, no. 4, pp. 295–299, 1992.

[20] J. Gauthier and I. Kupka, *Deterministic observation theory and applications*. Cambridge University Press, 2001.

[21] J. Gauthier, H. Hammouri, and S. Othman, “A simple observer for nonlinear systems applications to bioreactors,” *Automatic Control, IEEE Transactions on*, vol. 37, no. 6, pp. 875–880, 1992.

[22] F. Esfandiari and H. Khalil, “Output feedback stabilization of fully linearizable systems,” *International Journal of Control*, vol. 56, no. 5, pp. 1007–1037, 1992.

[23] H. Khalil and F. Esfandiari, “Semiglobal stabilization of a class of nonlinear systems using output feedback,” in *Decision and Control, 1992., Proceedings of the 31st IEEE Conference on*. IEEE, 1992, pp. 3423–3428.

[24] S. Oh and H. Khalil, "Nonlinear output-feedback tracking using high-gain observer and variable structure control," *Automatica*, vol. 33, no. 10, pp. 1845–1856, 1997.

[25] A. Memon and H. Khalil, "Full-order high-gain observers for minimum phase nonlinear systems," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*. IEEE, 2009, pp. 6538–6543.

[26] L. Freidovich and H. Khalil, "Performance recovery of feedback-linearization-based designs," *Automatic Control, IEEE Transactions on*, vol. 53, no. 10, pp. 2324–2334, 2008.

[27] A. Chakrabortty and M. Arcak, "Time-scale separation redesigns for stabilization and performance recovery of uncertain nonlinear systems," *Automatica*, vol. 45, no. 1, pp. 34–44, 2009.

[28] J. Back and H. Shim, "Adding robustness to nominal output-feedback controllers for uncertain nonlinear systems: A nonlinear version of disturbance observer," *Automatica*, vol. 44, no. 10, pp. 2528–2537, 2008.

[29] L. Freidovich and H. Khalil, "Logic-based switching for robust control of minimum-phase nonlinear systems," *Systems & control letters*, vol. 54, no. 8, pp. 713–727, 2005.

[30] ——, "Lyapunov-based switching control of nonlinear systems using high-gain observers," *Automatica*, vol. 43, no. 1, pp. 150–157, 2007.

[31] S. Nazrulla and H. Khalil, "Robust stabilization of non-minimum phase nonlinear systems using extended high-gain observers," *Automatic Control, IEEE Transactions on*, vol. 56, no. 4, pp. 802–813, 2011.

[32] D. Karagiannis, A. Astolfi, and R. Ortega, "Two results for adaptive output feedback stabilization of nonlinear systems," *Automatica*, vol. 39, no. 5, pp. 857–866, 2003.

[33] Y. Shtessel, S. Baev, C. Edwards, and S. Spurgeon, "Hosm observer for a class of non-minimum phase causal nonlinear mimo systems," *Automatic Control, IEEE Transactions on*, vol. 55, no. 2, pp. 543–548, 2010.

[34] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Systems & Control Letters*, vol. 22, no. 5, pp. 313–325, 1994.

[35] A. Atassi and H. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *Automatic Control, IEEE Transactions on*, vol. 44, no. 9, pp. 1672–1687, 1999.

[36] L. Praly and M. Arcak, "A relaxed condition for stability of nonlinear observer-based controllers," *Systems & control letters*, vol. 53, no. 3, pp. 311–320, 2004.

[37] A. Isidori, "A tool for semi-global stabilization of uncertain non-minimum-phase nonlinear systems via output feedback," *Automatic Control, IEEE Transactions on*, vol. 45, no. 10, pp. 1817–1827, 2000.

[38] V. Andrieu and L. Praly, "Global asymptotic stabilization for nonminimum phase nonlinear systems admitting a strict normal form," *Automatic Control, IEEE Transactions on*, vol. 53, no. 5, pp. 1120–1132, 2008.

[39] D. Karagiannis, Z. Jiang, R. Ortega, and A. Astolfi, "Output-feedback stabilization of a class of uncertain non-minimum-phase nonlinear systems," *Automatica*, vol. 41, no. 9, pp. 1609–1615, 2005.

[40] R. Marino and P. Tomei, "A class of globally output feedback stabilizable nonlinear non-minimum phase systems," *Automatic Control, IEEE Transactions on*, vol. 50, no. 12, pp. 2097–2101, 2005.

[41] S. Hoseini, M. Farrokhi, and A. Koshkouei, "Adaptive neural network output feedback stabilization of nonlinear non-minimum phase systems," *International journal of adaptive control and signal processing*, vol. 24, no. 1, pp. 65–82, 2010.

[42] A. Gelb, *Applied Optimal Estimation*. MIT Press, 1974.

[43] H. Soresnson, "Guest editorial: On applications of kalman filtering," *IEEE Transactions on Automatic Control*, vol. AC-28, no. 3, pp. 254–255, 1983.

[44] K. Reif, F. Sonnemann, and R. Unbehauen, "An ekf-based nonlinear observer with a prescribed degree of stability," *Automatica*, vol. 34, no. 9, pp. 1119–1123, 1998.

[45] K. Reif and R. Unbehauen, "The extended kalman filter as an exponential observer for nonlinear systems," *Signal Processing, IEEE Transactions on*, vol. 47, no. 8, pp. 2324–2328, 1999.

[46] M. Boutayeb and D. Aubry, "A strong tracking extended kalman observer for nonlinear discrete-time systems," *Automatic Control, IEEE Transactions on*, vol. 44, no. 8, pp. 1550–1556, 1999.

[47] M. Boutayeb, H. Rafaralahy, and M. Darouach, "Convergence analysis of the extended kalman filter used as an observer for nonlinear deterministic discrete-time systems," *Automatic Control, IEEE Transactions on*, vol. 42, no. 4, pp. 581–586, 1997.

[48] P. Moraal and J. Grizzle, “Observer design for nonlinear systems with discrete-time measurements,” *Automatic Control, IEEE Transactions on*, vol. 40, no. 3, pp. 395–404, 1995.

[49] H. Hammouri and J. de Leon Morales, “Observer synthesis for state-affine systems,” in *Decision and Control, 1990., Proceedings of the 29th IEEE Conference on*. IEEE, 1990, pp. 784–785.

[50] L. Praly and I. Kanellakopoulos, “Output feedback asymptotic stabilization for triangular systems linear in the unmeasured state components,” in *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, vol. 3. IEEE, 2000, pp. 2466–2471.

[51] G. Besancon, J. De León-Morales, and O. Huerta-Guevara, “On adaptive observers for state affine systems,” *International journal of Control*, vol. 79, no. 06, pp. 581–591, 2006.

[52] R. Bucy and P. Joseph, *Filtering for stochastic processes with applications to guidance*. American Mathematical Society, 1987, vol. 326.

[53] L. Torres, G. Besançon, and D. Georges, “Ekf-like observer with stability for a class of nonlinear systems,” *Automatic Control, IEEE Transactions on*, vol. 57, no. 6, pp. 1570–1574, 2012.

[54] N. Boizot, E. Busvelle, and J. Gauthier, “An adaptive high-gain observer for nonlinear systems,” *Automatica*, vol. 46, no. 9, pp. 1483–1488, 2010.

[55] A. Krener, “The convergence of the extended kalman filter,” *Directions in mathematical systems theory and optimization*, pp. 173–182, 2003.

[56] A. Isidori, *Nonlinear control systems*. Springer-Verlag New York, Inc., 1997.

[57] ——, *Nonlinear control systems II*. Springer Verlag, 1999, vol. 2.

[58] J. Baras, A. Bensoussan, and M. James, “Dynamic observers as asymptotic limits of recursive filters: Special cases,” *SIAM Journal on Applied Mathematics*, vol. 48, no. 5, pp. 1147–1158, 1988.

[59] M. Pai, *Power system stability: Analysis by the direct method of Lyapunov*. North-Holland Publishing Company, 1981, vol. 3.

[60] C. Wan, D. Bernstein, and V. Coppola, “Global stabilization of the oscillating eccentric rotor,” *Nonlinear Dynamics*, vol. 10, no. 1, pp. 49–62, 1996.

[61] H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, 2002, vol. 3.

[62] J. G. Kassakian, M. F. Schlecht, and G. C. Verghese, *Principles of power electronics*. Addison-Wesley Reading, USA, 1991, vol. 46.

[63] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications*. Springer, 2008.

[64] H. Rodriguez, R. Ortega, G. Escobar, and N. Barabanov, "A robustly stable output feedback saturated controller for the boost dc-to-dc converter," *Systems & control letters*, vol. 40, no. 1, pp. 1–8, 2000.

[65] R. S. Bucy and P. D. Joseph, "Filtering for stochastic processes with applications to guidance," 1968.

[66] A. N. Atassi and H. Khalil, "A separation principle for the control of a class of nonlinear systems," *Automatic Control, IEEE Transactions on*, vol. 46, no. 5, pp. 742–746, 2001.

[67] M. W. Spong, "Modeling and control of elastic joint robots," *ASME J. Dyn. Syst. Meas. Control*, vol. 109, pp. 310–319, 1990.

[68] M. W. Spong and M. Vidyasagar, *Robot dynamics and control*. Wiley.com, 2008.

[69] M. Benosman and G. Le Vey, "Control of flexible manipulators: A survey," *Robotica*, vol. 22, no. 5, pp. 533–545, 2004.

[70] S. K. Dwivedy and P. Eberhard, "Dynamic analysis of flexible manipulators, a literature review," *Mechanism and Machine Theory*, vol. 41, no. 7, pp. 749–777, 2006.

[71] S. Nicosia and P. Tomei, "A tracking controller for flexible joint robots using only link position feedback," *Automatic Control, IEEE Transactions on*, vol. 40, no. 5, pp. 885–890, 1995.

[72] W. Chatlatanagulchai, H. C. Nho, and P. H. Meckl, "Robust observer backstepping neural network control of flexible-joint manipulator," in *American Control Conference, 2004. Proceedings of the 2004*, vol. 6. IEEE, 2004, pp. 5250–5255.

[73] K. Melhem and W. Wang, "Global output tracking control of flexible joint robots via factorization of the manipulator mass matrix," *Robotics, IEEE Transactions on*, vol. 25, no. 2, pp. 428–437, 2009.

[74] S. J. Yoo, J. B. Park, and Y. H. Choi, “Output feedback dynamic surface control of flexible-joint robots,” *International Journal of Control Automation and Systems*, vol. 6, no. 2, p. 223, 2008.

[75] S. Ulrich and J. Z. Sasiadek, “Extended kalman filtering for flexible joint space robot control,” in *American Control Conference (ACC), 2011*. IEEE, 2011, pp. 1021–1026.

[76] R. Ortega, R. Kelly, and A. Loria, “A class of output feedback globally stabilizing controllers for flexible joints robots,” *Robotics and Automation, IEEE Transactions on*, vol. 11, no. 5, pp. 766–770, 1995.

[77] I. Burkov and L. B. Freidovich, “Stabilization of the position of a lagrangian system with elastic elements and bounded control, with and without measurement of velocities,” *Journal of applied mathematics and mechanics*, vol. 61, no. 3, pp. 433–441, 1997.

[78] A. Ailon, “Output controllers based on iterative schemes for set-point regulation of uncertain flexible-joint robot models,” *Automatica*, vol. 32, no. 10, pp. 1455–1461, 1996.

[79] B. Brogliato, R. Ortega, and R. Lozano, “Global tracking controllers for flexible-joint manipulators: a comparative study,” *Automatica*, vol. 31, no. 7, pp. 941–956, 1995.

[80] S. Liu, P. Garimella, and B. Yao, “Adaptive robust precision motion control of a flexible system with unmatched model uncertainties,” in *American Control Conference, 2005. Proceedings of the 2005*. IEEE, 2005, pp. 750–755.

[81] B. Wie and D. S. Bernstein, “Benchmark problems for robust control design,” *Journal of Guidance, Control, and Dynamics*, vol. 15, no. 5, pp. 1057–1059, 1992.