This is to certify that the

dissertation entitled

COMMUTANTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

presented by

Željko Čučković

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Date February 8, 1991

COMMUTANTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

By

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A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1991

ABSTRACT

COMMUTANTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

By

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In the first chapter we show that on the Bergman space L_a^2, two Toeplitz operators with harmonic symbols commute only in the obvious cases. The main tool is a characterization of harmonic functions by a conformally invariant mean value property.

The next chapter describes the commutants of certain analytic Toeplitz operators. To underline the difference between the Bergman and Hardy spaces, we first prove that on the Bergman space the only isometric Toeplitz operators with harmonic symbols are scalar multiples of the identity. If $\mathcal{T}$ denotes the closed subalgebra of $\mathcal{L}(L_a^2)$ generated by all Toeplitz operators, we show that for each positive integer n, $\{T_{z^n}\}' \cap \mathcal{T}$ is the set of all analytic Toeplitz operators. Here $\{T_{z^n}\}'$ denotes the commutant of T_{z^n}. If $n > 1$, then $\{T_{z^n}\}'$ is strictly larger than the set of all analytic Toeplitz operators, but we show that $\{T_{z^n+az}\}'$ is precisely the set of all analytic Toeplitz operators if $a \neq 0$. Using the same technique, we prove that if $p(z) = z + a_2 z^2 + ... + a_n z^n$, where $a_i \geq 0$ for $i = 2, ..., n$, and if $p(z) - p(1)$ has n distinct zeros, then $\{T_p\}' = \{T_\psi : \psi \in H^\infty\}$. These results are valid for both the Bergman and Hardy space.

The dissertation concludes with descriptions of the von Neumann algebras generated by operators T_{z^n} on the Bergman and Hardy space. For fixed $n > 1$, these von Neumann algebras have a different structure on the Bergman space than on the Hardy space. This problem is related to the problem of finding commutants of T_{z^n} and its adjoint, and we use some results proved in the second chapter.

THES

To my parents

HES

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Sheldon Axler, for all his help, encouragement and advice. His knowledge and enthusiasm were invaluable.

TABLE OF CONTENTS

INTRODUCTION

The purpose of this introductory chapter is to supply basic definitions, establish most of the notation, and list the major results of this thesis. The main goal of our study is to investigate some commutativity properties of Toeplitz operators acting on the Bergman space. Since this necessarily leads to a comparison with classical Toeplitz operators on the Hardy space, we will deal with both spaces throughout the thesis. Much more is known about the Hardy space, but techniques used there do not apply to the Bergman space. Therefore we had to find new approaches and techniques to obtain most of our results.

We start with definitions of these function spaces. Let $D = \{ z \in \mathbf{C} : |z| < 1 \}$. Let dA denote the usual area measure on D. The complex space $L^2(D,dA)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_D f \bar{g} \, dA.$$

The *Bergman space* L^2_a is the set of those functions in $L^2(D,dA)$ that are analytic on D (the a in L^2_a stands for "analytic"). The Bergman space L^2_a is a closed subspace of $L^2(D,dA)$, and so there is an orthogonal projection P from $L^2(D,dA)$ onto L^2_a. Fix $w \in D$. Point evaluation $f \to f(w)$ is a bounded linear functional on L^2_a (see J. B. Conway [11], Chapter III, Corollary 10.3) so there exists a function $k_w \in L^2_a$ such that

$$f(w) = \langle f, k_w \rangle$$

for all $f \in L^2_a$. This function is called the reproducing kernel for the point w. In [5], Proposition 1.7 it is shown that

$$k_w(z) = \pi^{-1}(1 - \overline{w}z)^{-2}.$$

Using this, we get an explicit formula for Pf:

$$Pf(w) = \langle Pf, k_w \rangle = \langle f, k_w \rangle = \pi^{-1}\int_D f(z)\,(1 - w\overline{z})^{-2} dA(z) \qquad \textbf{(1)}$$

For $\varphi \in L^\infty(D,dA)$, the *Toeplitz operator with symbol* φ, denoted T_φ, is the operator from L^2_a to L^2_a defined by $T_\varphi f = P(\varphi f)$.

Let m be normalized Lebesgue measure on the circle ∂D, i.e., $dm = \frac{dt}{2\pi}$. The complex space $L^2(\partial D)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\partial D} f\overline{g}\,\frac{dt}{2\pi}\,.$$

For each integer n, let e_n denote the function $e_n(z) = z^n$ for $|z| = 1$. Then $\{e_n\}$ is an orthonormal basis for $L^2(\partial D)$ (see, for example [25], Theorem 4.25) and the Hardy space $H^2(\partial D)$ is, by definition, the subspace span $\{e_n : n \geq 0\}^-$. For $\varphi \in L^\infty(\partial D)$, the *Toeplitz operator with symbol* φ, denoted again by T_φ, on $H^2(\partial D)$ is defined by $T_\varphi f = Q(\varphi f)$, where Q is the orthogonal projection from $L^2(\partial D)$ onto $H^2(\partial D)$. It will be clear from the context which Toeplitz operator we consider.

Although these two Toeplitz operators on different spaces differ in many ways, they do have the same basic algebraic properties: For $\varphi,\psi \in L^\infty(D)$ or $L^\infty(\partial D)$,

$$T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$$

$$(T_\varphi)^* = T_{\overline{\varphi}}$$

$$T_\varphi T_\psi = T_{\varphi\psi} \qquad \text{if } \psi \in H^\infty$$

$$T_{\bar\psi} T_\varphi = T_{\bar\psi\varphi} \qquad \text{if } \psi \in H^\infty,$$

where H^∞ denotes the set of all bounded analytic functions on D, or the set of all their boundary values. Standard references for the classical Toeplitz operators are [16] and [18].

The question of when two Toeplitz operators on $H^2(\partial D)$ commute was answered by Arlen Brown and Paul Halmos ([9], Theorem 9), where they proved the following result:

Theorem 1. *Suppose that* φ *and* ψ *belong to* $L^\infty(\partial D)$. *Then*

$$T_\varphi T_\psi = T_\psi T_\varphi$$

if and only if

(2) φ *and* ψ *are in* H^∞,

or (3) $\bar\varphi$ *and* $\bar\psi$ *are in* H^∞,

or (4) *there exist constants* $a, b \in \mathbf{C}$, *not both* 0, *such that* $a\varphi + b\psi$ *is constant on* ∂D.

Brown and Halmos proved their result by examining the matrix (with respect to the orthonormal basis $\{e_n\}$) of products of Toeplitz operators. It is well known that Toeplitz operators on $H^2(\partial D)$ have matrices that are constant on diagonals. It is natural to ask if Theorem 1 holds for the Bergman space Toeplitz operators. On L^2_a, Toeplitz operators do not have nice matrices in the standard orthonormal basis $e_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n$, for $|z| < 1$ and $n \in \mathbf{N}$, and the techniques used by Brown and Halmos do not seem to work in this context. In Chapter 1 we prove the analogous theorem, using function theory, rather than matrix manipulations, assuming that symbols are harmonic functions. By a harmonic function we mean a complex-valued function on D whose Laplacian is identically 0. Here is our theorem:

Theorem 2. *Suppose that φ and ψ are bounded harmonic functions on D. Then*

$$T_\varphi T_\psi = T_\psi T_\varphi$$

if and only if

(5) *φ and ψ are both analytic on D,*

or (6) *$\bar{\varphi}$ and $\bar{\psi}$ are both analytic on D,*

or (7) *there exist constants a, b* $\in$ **C**, *not both* 0, *such that $a\varphi + b\psi$ is constant on D.*

A special case of Theorem 1 was proved by Sheldon Axler and Pamela Gorkin ([7], Theorem 7), using function theory techniques quite different from those that we use here. Dechao Zheng ([33], Theorem 5) also proved a special case of Theorem 1; our proof makes use of some of his ideas.

There is a strong relationship between symbols of Toeplitz operators in these theorems. Of basic importance is the Poisson integral.

Definition 3. For $0 \le r < 1$ and θ real, the function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

is called the *Poisson kernel.* If $L^1(\partial D)$ denotes the space of Lebesgue integrable functions on ∂D, and $f \in L^1(\partial D)$, then the function

$$F(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{i\theta})\frac{dt}{2\pi}$$

is called the Poisson integral of f. We shall write $F = P[f]$. It is well known that $P[f]$ is a harmonic function on D for every $f \in L^1(\partial D)$.

Functions in $L^\infty(\partial D)$ correspond, via the Poisson integral, to bounded harmonic functions on D, so perhaps the restriction in Theorem 1 to consideration only of Toeplitz operators with harmonic symbols (as opposed to Toeplitz operators with symbol in $L^\infty(D,dA)$) is natural. More importantly, Theorem 1 does not hold if "harmonic" is replaced by "measurable". Paul Bourdon pointed out the following example: A function f is called *radial* if $f(z) = f(|z|)$. Suppose that φ and ψ are any two radial functions in $L^\infty(D,dA)$. Then,using (1), we obtain (for fixed $n \in$ N)

$$(T_\varphi z^n)(w) = P(\varphi\, z^n)(w) = \int_D \varphi(r)\, z^n \frac{1}{(1 - w\bar{z}^2)}\, dA(z) =$$

$$= \int_0^1 \int_{-\pi}^{\pi} \varphi(r)\, r^n e^{in\theta} \frac{1}{(1 - wre^{-i\theta})^2}\, r dr\, d\theta = \frac{1}{i} \int_0^1 \varphi(r)\, r^{n+1} \int_{-\pi}^{\pi} \frac{e^{i(n+1)\theta}}{(e^{i\theta} - wr)^2} d(e^{i\theta})\, dr =$$

$$= 2\pi(n+1)\, w^n \int_0^1 \varphi(r)\, r^{2n+1}\, dr.$$

Now, by the same reasoning as above, we obtain

$$T_\psi T_\varphi\, z^n(w) = 4\pi^2(n+1)^2 \left(\int_0^1 \varphi(r)\, r^{2n+1}\, dr \right) \cdot \left(\int_0^1 \psi(r)\, r^{2n+1}\, dr.\right) w^n.$$

HESIS

From this, it is clear that $T_\varphi T_\psi = T_\psi T_\varphi$ on the set of polynomials, which is dense in L_a^2, so that $T_\varphi T_\psi = T_\psi T_\varphi$. This example suggests that the following open problem may be hard: Find conditions on functions φ and ψ in $L^\infty(D,dA)$ that are necessary and sufficient for T_φ to commute with T_ψ.

Chapter 1 concludes with a corollary that describes the normal Toeplitz operators with harmonic symbol. For a Hilbert space H, $\mathcal{L}(H)$ denotes the algebra of all bounded linear operators on H. An operator $T \in \mathcal{L}(H)$ is called *normal* if $T^* T = T T^*$. An operator $T \in \mathcal{L}(H)$ *is* called *hyponormal* if $T^* T \geq T T^*$. Further research could go in the direction of describing the hyponormal Toeplitz operators with harmonic symbol. C. Cowen [14] characterized hyponormal Toeplitz operators on the Hardy space, and it would be interesting to do so for the Bergman space Toeplitz operators.

If $S \subset \mathcal{L}(H)$ then $S' = \{B \in \mathcal{L}(H): AB = BA \text{ for all } A \in S\}$ is the *commutant* of S. There is quite a bit of literature about commutants of different operators. In Chapter 2 we will study commutants of certain analytic Toeplitz operators on the Bergman space. Not much is known about this. It is well known that $\{T_z\}' = \{T_\varphi : \varphi \in H^\infty\}$. Much work has been done in studying commutants of analytic Toeplitz operators on $H^2(\partial D)$. In their paper [28], Shields and Wallen studied commutants of weighted shift operators. Deddens and Wong [15] studied the problem using operator theory and raised six questions that stimulated much of the further work on the problem. Abrahamse [1] answered some of Deddens-Wong questions negatively. Baker, Deddens and Ullman [8] found $\{T_f\}'$ in case that f is an entire function. In a series of papers [29] - [32], Thomson used function theoretic methods to find commutants or intersection of commutants of certain analytic Toeplitz operators. Finally, C. Cowen continued their work in [13] and found the commutant of Toeplitz operators whose symbol is a finite Blaschke product or a covering map.

Our approach to the Bergman space problem will be function theoretic, based on methods introduced by Shields and Wallen. We will not try to describe commutants of different analytic Toeplitz operators. Instead, we will try to find those Toeplitz operators whose commutant is equal to the set of all analytic Toeplitz operators. Methods used by Deddens and Wong cannot be applied for the Bergman Toeplitz operators, so that we will not pursue the course of study they suggested.

Function $f \in H^\infty$ is said to be an *inner function* if $|f^*(e^{i\theta})| = 1$ almost everywhere on the unit circle. Here $f^*(e^{i\theta}) = \lim f(re^{i\theta})$ as $r \to 1$. One of the basic facts about the Hardy space is that T_f^* is an isometry if and only if f is inner. To underline the difference between the Hardy and Bergman space Toeplitz operators, we first prove the following theorem:

Theorem 4. *Suppose that $h \in L^\infty(D,dA)$ is harmonic and that T_h is an isometry. Then h is a constant function of modulus* 1.

The proof of this theorem uses, among other results, some basic aspects of commutative Banach algebras. If A is a commutative Banach algebra with identity, the set of all multiplicative linear functionals on A is called the *maximal ideal space* and is usually denoted by M. It is easily seen that each multiplicative linear functional is continuous so that $M \subset A^*$, where A^* denotes the dual space of A. Given the relative weak-* topology of A, M is a compact Hausdorff space.

For each $a \in A$, the function $\hat{a} : M \to \mathbf{C}$ defined by $\hat{a}(\varphi) = \varphi(a)$, for $\varphi \in M$, is called the *Gelfand transform of a* and it is a continuous function. The map $\wedge : A \to C(M)$ defined by $a \to \hat{a}$ is a homomorphism and is called the *Gelfand transform of* A. We will be particularly interested in the Banach algebras H^∞ and $L^\infty(\partial D)$, which are crucial for

understanding the McDonald-Sundberg symbol calculus for the Bergman space Toeplitz operators (see [23]).

After this, we will prove the first result about commutants. Let $\mathcal{T}$ be the closed subalgebra of $\mathcal{L}(L_a^2)$ generated by all Toeplitz operators. We have the following theorem:

Theorem 5. *Let $S \in \mathcal{T}$ commute with T_{z^n}, $n \in \mathbb{N}$. Then $S = T_\psi$, for some $\psi \in H^\infty$.*

From the proof it is clear that operators in $\{T_{z^n}\}'$ have complicated structure. On the other hand, it turns out that $S \in \{T_{z^n+az}\}'$ must be an analytic Toeplitz operator in the case $a \neq 0$. This is our next result:

Theorem 6. *Suppose $h(z) = z^n + az$, $a \neq 0$, $n \in \mathbb{N}$, $n > 1$ and let $S \in \mathcal{L}(L_a^2)$ commutes with T_h. Then $S = T_\psi$ for some $\psi \in H^\infty$.*

At this point we use the Shields-Wallen ideas introduced in [28]. They proved that if S commutes with T_h, where $h \in H^\infty$, then $S^* k_\lambda$ is an eigenvector for T_h^*, for every $\lambda \in D$. Because

$$\text{Ker } T_{\overline{h}-\overline{h(\lambda)}} = (\text{Range } T_{h-h(\lambda)})^\perp$$

we try to characterize the range of $T_{h-h(\lambda)}$. The following theorem is based on the same technique:

Theorem 7. *Suppose $p(z) = z + a_2z^2 + \cdots + a_nz^n$, where $a_i \geq 0$, for $i = 2, \ldots, n$. If $p(z) - p(1)$ has n distinct zeros, then $\{T_p\}' = \{T_\psi : \psi \in H^\infty\}$.*

For Toeplitz operators whose symbols are polynomials with complex coefficients, we cannot say much about their commutants, except in a very special case.

Chapter concludes with several examples.

In Chapter 3 we study von Neumann algebras generated by T_{z^n}. Their definition is related to the double commutants. If $S \subset \mathcal{L}(H)$, then the *double commutant of S* is the set $S'' = \{S'\}'$. A *von Neumann algebra A* is a C*-subalgebra of $\mathcal{L}(H)$ such that $A = A''$. If $T \in \mathcal{L}(H)$, the von Neuman algebra generated by T will be denoted by $W^*(T)$. From Corollary 7.6, p. 119 in [24] it follows that $W^*(T) = \{T, T^*\}''$. Therefore, in order to determine $W^*(T_{z^n})$ for some fixed $n \in \mathbf{N}$, we have to determine $\{T_{z^n}, T_{z^n}^*\}''$. We will study the operator T_{z^n} acting on the Bergman space and determine $W^*(T_{z^n})$ in this case first and then we will consider T_{z^n} as an operator on the Hardy space and find the von Neumann algebra generated by this operator. We will show that they have different structures if $n > 1$. The difference comes from coefficients in the Bergman space basis vectors. In the case $n = 1$ the situation is simple: $W^*(T_z)$ is the set of all bounded linear operators, either on L_a^2 or $H^2(\partial D)$. The proof uses the injectivity of the map $\varphi \to T_\varphi$. In the Hardy space case, $T_\varphi = 0$ means that all matrix elements of $T_\varphi(e_n)$ are zero. Since matrix elements are Fourier coefficients of φ, it follows that $\varphi = 0$. For the Bergman space, the proof can be found in [6]. Now, if $S \in \{T_z, T_z^*\}'$, then S and S^* commute with T_z. Therefore $S = T_\varphi$ and $S^* = T_\psi$. By the previous comment, $\varphi = \bar{\psi}$, so that φ is a constant function. This means that $\{T_z, T_z^*\}' = \{cI : c \in \mathbf{C}\}$, so that $\{T_z, T_z^*\}''$ is equal to the set of all bounded linear operators.

The case $n > 1$ will use results in Chapter 2 about the commutant of T_{z^n}. For fixed $n > 1$ we'll show the following:

HESIS

Bergman space:

$$W^*(T_{z^n}) = \{\ T \in \mathcal{L}(L_a^2) : T = \begin{pmatrix} T_0 & & & \\ & T_1 & & \\ & & \ddots & \\ & & & T_{n-1} \end{pmatrix},\ where\ L_a^2 = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1},$$

$$T_i : X_i \rightarrow X_i\ for\ i = 0,1,\cdots, n-1\ \}.$$

Hardy space:

$$W^*(T_{z^n}) = \{\ T \in \mathcal{L}(H^2(\partial D)) : T = \begin{pmatrix} T_0 & & & \\ & M_z T_0 M_{\bar{z}} & & \\ & & \ddots & \\ & & & M_{z^{n-1}} T_0 M_{\bar{z}^{n-1}} \end{pmatrix},\ where$$

$H^2(\partial D) = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$, $T_0 : X_0 \rightarrow X_0$, M_z *and* $M_{\bar{z}}$, *are multiplication operators*$\}$.

CHAPTER 1

COMMUTING TOEPLITZ OPERATORS WITH HARMONIC SYMBOLS

Studying special classes of operators, like the class of Toeplitz operators, usually involves the question of when two arbitrary operators from this class commute. In this chapter we restrict our study to the class of Bergman space Toeplitz operators with symbols that are harmonic functions. In the first half of the chapter we examine harmonic functions more closely from the perspective of an invariant mean value property. Then, in the second half, we apply these results to our operator theory problem. This chapter, once again, shows close relationship between function theory and operator theory. It also shows how much harder the Bergman space case is than the Hardy space case.

1. The Invariant Mean Value Property

A continuous function on the disk D is harmonic if and only if it has the mean value property (see for example [10], Chapter X, Mean Value Theorem 1.4 and Theorem 2.11). In this section we characterize harmonic functions in terms of an invariant mean value property.

Let $Aut(D)$ denote the set of analytic, one-to-one maps of D onto D ("Aut" stands for "Automorphism"). A function h on D is in Aut(D) if and only if there exist $\alpha \in \partial D$ and $\beta \in D$ such that

$$h(z) = \alpha \frac{\beta - z}{1 - \bar{\beta} z}$$

for all $z \in D$.

A function $u \in C(D)$ is said to have the *invariant mean value property* if

$$\int_0^{2\pi} u(h(re^{i\theta}))\, \frac{d\theta}{2\pi} = u(h(0))$$

for every $h \in \mathrm{Aut}(D)$ and every $r \in [0,1)$. Here "invariant" refers to conformal invariance, meaning invariance under composition with elements of $\mathrm{Aut}(D)$. If u is harmonic on D, then so is $u \circ h$ for every $h \in \mathrm{Aut}(D)$; thus harmonic functions have the invariant mean value property. The converse is also true (see [26], Corollary 2 to Theorem 4.2.4): if a function $u \in C(D)$ has the invariant mean value property, then u is harmonic on D.

The invariant mean value property concerns averages over circles with respect to arc length measure. Because we are dealing with the Bergman space L^2_a, we need an invariant condition stated in terms of an area average over D. Thus we say that a function $u \in C(D) \cap L^1(D,dA)$ has the *area version of the invariant mean value property* if

$$\int_D u \circ h\, \frac{dA}{\pi} = u(h(0))$$

for every $h \in \mathrm{Aut}(D)$. If u is in $C(D) \cap L^1(D,dA)$, then so is $u \circ h$ for every $h \in \mathrm{Aut}(D)$, so the left-hand side of the above equation makes sense. Note that the area version of the invariant mean value property deals with integrals over all of D, as opposed to integrals over rD for $r \in (0,1)$.

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If u is harmonic on D and in $L^1(D,dA)$, then so is $u \circ h$ for every $h \in \text{Aut}(D)$. Thus, by the mean value property, harmonic functions have the area version of the invariant mean value property. Whether or not the converse is true is an open question. In other words, if $u \in C(D) \cap L^1(D,dA)$ has the area version of the invariant mean value property, must u be harmonic? This question has an affirmative answer if we replace the hypothesis that u is in $C(D) \cap L^1(D,dA)$ with the stronger hypothesis that u is in $C(\bar{D})$; see [26], Proposition 13.4.4 or [3], Proposition 10.2.

We need to consider functions that are not necessarily continuous on the closed disk $\bar{D}$, so the result mentioned in the last sentence will not suffice. However, our functions do have the property that their radializations (defined below) are continuous on the closed disk, and we will prove that this property, along with the area version of the invariant mean value property, is enough to imply harmonicity; see Lemma 1.1.

If $u \in C(D)$, then the *radialization of u*, denoted $\mathcal{R}(u)$, is the function on D defined by

$$\mathcal{R}(u)(w) = \int_0^{2\pi} u(we^{i\theta}) \frac{d\theta}{2\pi} .$$

In the following lemma, the statement $\mathcal{R}(u \circ h) \in C(\bar{D})$ means that $\mathcal{R}(u \circ h)$ can be extended to a continuous complex-valued function on $\bar{D}$.

The following lemma will be a key tool in what follows.

Lemma 1.1. *Suppose that* $u \in C(D) \cap L^1(D, dA)$. *Then* u *is harmonic on* D *if and only if*

$$\int_D u \circ h \, \frac{dA}{\pi} = u(h(0)) \tag{1.1}$$

and

$$\mathcal{R}(u \circ h) \in C(\bar{D}) \tag{1.2}$$

for every $h \in Aut(D)$.

PROOF: We first prove the easy direction (the other direction will be the one that we need in the proof of Theorem 1). Suppose that u is harmonic on D. Let $h \in \mathrm{Aut}(D)$. As we discussed earlier, $u \circ h$ is harmonic and so (1.1) holds. The mean value property implies that $\mathcal{R}(u \circ h)$ is a constant function on D, with value $u(h(0))$, so (1.2) also holds.

To prove the other direction, suppose that (1.1) and (1.2) hold. Let $h \in \mathrm{Aut}(D)$, and let

$$v = \mathcal{R}(u \circ h).$$

By (1.2), $v \in C(\bar{D})$.

We want to show that v has the area version of the invariant mean value property. To do this, fix $g \in \mathrm{Aut}(D)$. Then

$$\begin{aligned}\int_D v \circ g \, \frac{dA}{\pi} &= \int_D \mathcal{R}(u \circ h)(g(w)) \, \frac{dA(w)}{\pi} \\ &= \int_D \int_0^{2\pi} u(h(g(w)e^{i\theta})) \, \frac{d\theta}{2\pi} \, \frac{dA(w)}{\pi}. \end{aligned} \tag{1.3}$$

To check that interchanging the order of integration in the last integral is valid, for each $\theta \in [0,2\pi]$ define $f_\theta \in \mathrm{Aut}(D)$ by

$$f_\theta(w) = h(g(w)e^{i\theta}).$$

The inverse (under composition) f_θ^{-1} of f_θ is also an analytic automorphism of D, so there exist $\alpha \in \partial D$ and $\beta \in D$ such that

$$f_\theta^{-1}(z) = \alpha \frac{\beta - z}{1 - \bar{\beta} z} \qquad \text{for all } z \in D.$$

Thus

$$|(f_\theta^{-1})'(z)| = \frac{1 - |\beta|^2}{|1 - \bar{\beta} z|^2}$$

$$\leq \frac{1 + |\beta|}{1 - |\beta|} \qquad \text{for all } z \in D.$$

Note that $\beta = f_\theta(0) = h(g(0)e^{i\theta})$; then $|\beta| \leq \sup\{|h(z)|: |z| \leq |g(0)|\}$. We are thinking of h and g as fixed, so the above inequality shows there is a constant K (depending only on h and g) such that

$$|(f_\theta^{-1})'(z)| \leq K \qquad \text{for all } z \in D \text{ and all } \theta \in [0,2\pi].$$

Now

$$\int_0^{2\pi} \int_D |u(h(g(w)e^{i\theta}))| \frac{dA(w)}{\pi} \frac{d\theta}{2\pi} = \int_0^{2\pi} \int_D |u(z)| \, |(f_\theta^{-1})'(z)|^2 \frac{dA(z)}{\pi} \frac{d\theta}{2\pi}$$

THES

$$\leq K^2 \int_D |u(z)| \frac{dA(z)}{\pi}$$

$$< \infty.$$

Thus we can apply Fubini's Theorem to (1.3), getting

$$\int_D v \circ g \frac{dA}{\pi} = \int_0^{2\pi} \int_D u(h(g(w)e^{i\theta})) \frac{dA(w)}{\pi} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \int_D (u \circ f_\theta)(w) \frac{dA(w)}{\pi} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} u(f_\theta(0)) \frac{d\theta}{2\pi} \qquad \text{(by (1.1))}$$

$$= \int_0^{2\pi} u(h(g(0)e^{i\theta}) \frac{d\theta}{2\pi} = \mathcal{R}(u \circ h)(g(0))$$

$$= v(g(0)).$$

Thus v is a continuous function on $\bar{D}$ that has the area version of the invariant mean value property. Hence (see [26], Proposition 13.4.4 and Remark 4.1.4 or [3], Proposition 10.2, combined with a change of variables) v is harmonic on D. Because v is also a radial function, the mean value property implies that v is a constant function on D, with value $v(0)$.

Recall that $v = \mathcal{R}(u \circ h)$, so

THES

$$\int_0^{2\pi} (u \circ h)(re^{i\theta}) \frac{d\theta}{2\pi} = u(h(0))$$

for every $r \in [0,1)$ and for each $h \in \mathrm{Aut}(D)$. In other words, u has the invariant mean value property (the usual version, not the area version). Thus (see [26], Corollary 2 to Theorem 4.2.4 and Remark 4.1.4) u is harmonic on D, and the proof of Lemma 2 is complete. ■

2. Toeplitz Operators

In the previous section we obtained an invariant mean value property of harmonic functions. In addition to this, we need some other facts about these functions. For a real-valued harmonic function u on D, any v defined on D such that $u + iv$ is analytic is called a *harmonic conjugate of u*. Given the real harmonic function u, there is a unique real harmonic function $\tilde{u}$ that is conjugate to u with $\tilde{u}(0) = 0$. If $f = u + i\tilde{u}$, then $u = \frac{1}{2}(f + \overline{f})$. In case that u is a complex-valued harmonic function, the above procedure gives $u = f_1 + \overline{f_2}$ where f_1 and f_2 are analytic.

Suppose now that u is a bounded harmonic complex-valued function on D. What more can we say about f_1 and f_2 ? To answer this question we have to introduce another space of functions on ∂D. For a function $f \in L^1(\partial D)$ and an arc I in ∂D, let f_I denote the average of f over I:

$$f_I = \frac{1}{m(I)} \int_I f \; dm$$

For a function $f \in L^1(\partial D)$, let

$$\| f \|_{BMO} = sup \; \{ \frac{1}{m(I)} \int_I | f - f_I | \, dm : I \text{ is an arc in } \partial D \}.$$

A function $f \in L^1(\partial D)$ for which $\| f \|_{BMO} < \infty$ is said to be of *bounded mean oscillation*. The set of all these functions is denoted by BMO. The class BMO was first introduced in [21]. It satisfies

$$L^2(\partial D) \supset BMO \supset L^\infty(\partial D) \tag{1.4}$$

By assumption on u, Re $u = u_1$ is bounded and Fatou's theorem guarantees the existence of the radial limit of u_1 almost everywhere on ∂D. Also $u_1^* \in L^\infty(\partial D)$. The function $\tilde{u_1}$ has radial limits almost everywhere on ∂D (see [22], Chapter III, Lemma 1.3) and by [17], Chapter VI, Theorem 1.5, $\tilde{u_1}^*$ is in BMO. Formula (1.4) implies that $u_1^* + i\tilde{u_1}^*$ is in $L^2(\partial D)$ so that $u_1 + i\tilde{u_1} = P[u_1^* + i\tilde{u_1}^*]$ is an analytic function on D whose Taylor coefficients are square-summable. The set of all analytic functions on D with square-summable Taylor coefficients is denoted by $H^2(D)$ and is called the Hardy space, since it is isometrically isomorphic to $H^2(\partial D)$. Hence, if u_1 is bounded and harmonic, it is the real part of a $H^2(D)$ function and therefore $u = f_1 + \overline{f_2}$, where $f_1, f_2 \in H^2(D)$.

Now, we are ready to state the main theorem of this chapter.

Theorem 1.2. *Suppose that φ and ψ are bounded harmonic functions on D. Then*

$$T_\varphi T_\psi = T_\psi T_\varphi$$

if and only if

(1.5) *φ and ψ are both analytic on D,*

or (1.6) *$\overline{\varphi}$ and $\overline{\psi}$ are both analytic on D,*

or (1.7) *there exist constants $a, b \in \mathbf{C}$, not both 0, such that $a\varphi + b\psi$ is constant on D.*

Before beginning the proof of Theorem 1.2, we need the following two lemmas.

For $h \in \mathrm{Aut}(D)$, define an operator U_h on L_a^2 by

$$U_h f = (f \circ h) h'.$$

Lemma 1.3 is certainly well known.

Lemma 1.3. *Let $h \in Aut(D)$. Then U_h is a unitary operator from L^2_a onto L^2_a.*

PROOF: When making a change of variables $z = h(w)$ in an area integral, we must make the substitution $dA(z) = |h'(w)|^2\, dA(w)$. Thus

$$\int_D |f(h(w))|^2\, |h'(w)|^2\, dA(w) = \int_D |f(z)|^2\, dA(z)$$

for every $f \in L^2_a$, and hence U_h is an isometry of L^2_a into L^2_a. Clearly $U_{h^{-1}}$ is an inverse for U_h. An invertible isometry is a unitary operator, so we are done. ■

Lemma 1.4. *Let $h \in Aut(D)$ and let $\varphi \in L^\infty(D,dA)$. Then*

$$U_h T_\varphi U_h^* = T_{\varphi\cdot h}\,.$$

PROOF: Define $V_h: L^2(D,dA) \to L^2(D,dA)$ by $V_h f = (f\cdot h)h'$. As in the proof of Lemma 6, V_h is a unitary operator from $L^2(D,dA)$ onto $L^2(D,dA)$. Obviously $V_h|L^2_a = U_h$. Thus V_h maps L^2_a onto L^2_a, so

$$PV_h = V_h P. \tag{1.8}$$

If $f \in L^2_a$, then

$$T_{\varphi\cdot h} U_h f = T_{\varphi\cdot h}((f\cdot h)h') = P((\varphi\cdot h)(f\cdot h)h') = P(V_h(\varphi f))$$

$$= V_h(P(\varphi f)) \qquad \text{(by (1.8))}$$

$$= U_h T_\varphi f.$$

Thus $T_{\varphi \circ h} U_h = U_h T_\varphi$, and because U_h is unitary (Lemma 1.3), this implies the desired result. ■

We have now assembled all the ingredients needed to prove Theorem 1.2.

PROOF: We begin with the easy direction. First suppose that (1.5) holds, so that φ and ψ are analytic on D, which means that T_φ and T_ψ are, respectively, the operators on L_a^2 of multiplication by φ and ψ. Thus $T_\varphi T_\psi = T_\psi T_\varphi$.

Now suppose that (1.6) holds, so that $\overline{\varphi}$ and $\overline{\psi}$ are analytic on D. By the paragraph above, $T_{\overline{\varphi}} T_{\overline{\psi}} = T_{\overline{\psi}} T_{\overline{\varphi}}$. Take the adjoint of both sides of this equation, and then use the identities $T_{\overline{\varphi}}^* = T_\varphi$ and $T_{\overline{\psi}}^* = T_\psi$ to conclude that $T_\varphi T_\psi = T_\psi T_\varphi$.

Finally (for the easy direction) suppose that (1.7) holds, so there exist constants a, $b \in \mathbf{C}$, not both 0, such that $a\varphi + b\psi$ is constant on D. If $a \neq 0$, then there exist constants $\beta, \gamma \in \mathbf{C}$ such that $\varphi = \beta\psi + \gamma$ on D, which means that $T_\varphi = \beta T_\psi + \gamma I$ (here I denotes the identity operator on L_a^2), which clearly implies that $T_\varphi T_\psi = T_\psi T_\varphi$. If $b \neq 0$, we conclude in a similar fashion that T_φ and T_ψ commute.

To prove the other direction of Theorem 1.2, suppose now that φ and ψ are bounded harmonic functions on D such that $T_\varphi T_\psi = T_\psi T_\varphi$. Then there exist functions f_1, f_2, g_1, and g_2 in $H^2(D)$ such that

$$\varphi = f_1 + \overline{f_2} \quad \text{and} \quad \psi = g_1 + \overline{g_2} \qquad \text{on } D. \tag{1.9}$$

The Hardy space $H^2(D)$ is a subspace of L_a^2. In particular, f_1, f_2, g_1, and g_2 are all in L_a^2.

Let 1 denote the constant function 1 on D. Then

$$T_\varphi T_\psi 1 = T_\varphi(P\psi) = T_\varphi(P(g_1 + \overline{g_2})) = T_\varphi(g_1 + \overline{g_2(0)})$$

$$= P([f_1 + \overline{f_2}][g_1 + \overline{g_2(0)}])$$

$$= f_1 g_1 + \overline{g_2(0)} f_1 + P(\overline{f_2} g_1) + \overline{f_2(0)}\,\overline{g_2(0)}.$$

Thus

$$<T_\varphi T_\psi 1,\ 1> = <f_1 g_1 + \overline{g_2(0)} f_1 + \overline{f_2} g_1 + \overline{f_2(0)}\,\overline{g_2(0)},\ 1>$$

$$= \int_D f_1 g_1 + \overline{g_2(0)} f_1 + \overline{f_2} g_1 + \overline{f_2(0)}\,\overline{g_2(0)}\ dA$$

$$= \pi[f_1(0) g_1(0) + f_1(0)\overline{g_2(0)} + \overline{f_2(0)}\,\overline{g_2(0)}] + \int_D \overline{f_2} g_1\ dA. \qquad \textbf{(1.10)}$$

A similar formula (interchanging the f's and the g's) can be obtained for $<T_\psi T_\varphi 1, 1>$.

Because $T_\varphi T_\psi = T_\psi T_\varphi$, we can set the right-hand side of (1.10) equal to the corresponding formula for $<T_\psi T_\varphi 1, 1>$, getting

$$\int_D \overline{f_2} g_1 - f_1 \overline{g_2}\, \frac{dA}{\pi} = \overline{f_2(0)} g_1(0) - f_1(0)\overline{g_2(0)}. \qquad \textbf{(1.11)}$$

Let $h \in \mathrm{Aut}(D)$. Multiplying both sides of the equation $T_\varphi T_\psi = T_\psi T_\varphi$ by U_h on the left and by U_h^* on the right, and recalling (Lemma 1.3) that U_h is unitary (so $U_h^* U_h = I$), we get

$$U_h T_\varphi U_h^* U_h T_\psi U_h^* = U_h T_\psi U_h^* U_h T_\varphi U_h^*.$$

Lemma 1.4 now shows that

$$T_{\varphi\circ h}T_{\psi\circ h} = T_{\psi\circ h}T_{\varphi\circ h}. \tag{1.12}$$

Composing both sides of the equations in (1.9) with h expresses each of the bounded harmonic functions $\varphi\circ h$ and $\psi\circ h$ as the sum of an analytic function and a conjugate analytic function:

$$\varphi\circ h = f_1\circ h + \overline{f_2}\circ h \quad \text{and} \quad \psi\circ h = g_1\circ h + \overline{g_2}\circ h \qquad \text{on } D. \tag{1.13}$$

Equation (1.11) was derived under the assumption that $T_\varphi T_\psi = T_\psi T_\varphi$; thus (1.12), combined with (1.13), says that (1.11) is still valid when we replace each function in it by its composition with h. In other words,

$$\int_D (\overline{f_2}g_1 - f_1\overline{g_2})\circ h \,\frac{dA}{\pi} = \overline{f_2}(h(0))\, g_1(h(0)) - f_1(h(0))\, \overline{g_2}(h(0)).$$

Letting

$$u = \overline{f_2}g_1 - f_1\overline{g_2},$$

the above equation becomes

$$\int_D u\circ h \,\frac{dA}{\pi} = u(h(0)).$$

In other words, u has the area version of the invariant mean value property.

We want to show that u is harmonic on D. By the above equation and Lemma 1.1, we need only show that $\mathcal{R}(u \cdot h) \in C(\bar{D})$. To do this, represent the analytic functions $f_2 \cdot h$ and $g_1 \cdot h$ as Taylor series:

$$(f_2 \cdot h)(z) = \sum_{n=0}^{\infty} \alpha_n z^n \quad \text{and} \quad (g_1 \cdot h)(z) = \sum_{n=0}^{\infty} \beta_n z^n \quad \text{for all } z \in D.$$

Because $\varphi \cdot h$ and $\psi \cdot h$ are bounded harmonic functions on D, (13) implies that $f_2 \cdot h$ and $g_1 \cdot h$ are in $H^2(D)$; in other words,

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\beta_n|^2 < \infty. \tag{1.14}$$

Now for $z \in D$ we have

$$\mathcal{R}((\overline{f_2} g_1) \cdot h)(z) = \int_0^{2\pi} (\overline{f_2} \cdot h)(ze^{i\theta})\, (g_1 \cdot h)(ze^{i\theta})\, \frac{d\theta}{2\pi}$$

$$= \sum_{n=0}^{\infty} \overline{\alpha_n} \beta_n\, |z|^{2n}.$$

Inequalities (1.14) imply that $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$, so the above formula for $\mathcal{R}((\overline{f_2} g_1) \cdot h)$ shows that $\mathcal{R}((\overline{f_2} g_1) \cdot h) \in C(\bar{D})$; similarly, we get $\mathcal{R}((f_1 \overline{g_2}) \cdot h) \in C(\bar{D})$. Thus $\mathcal{R}(u \cdot h) \in C(\bar{D})$, as desired. Thus at this stage of the proof we know that u is harmonic.

Let $\partial/\partial z$ and $\partial/\partial \bar{z}$ denote the usual operators defined (on smooth functions on D) by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If f is analytic, then the Cauchy-Riemann equations show that

$$\frac{\partial f}{\partial z} = f', \quad \frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0, \quad \text{and} \quad \frac{\partial \bar{f}}{\partial \bar{z}} = \overline{f'}.$$

It is easy to check that $\partial/\partial z$ and $\partial/\partial \bar{z}$ obey the usual addition and multiplication formulas for derivatives and that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$

Thus, because u is harmonic, we have

$$0 = 4 \frac{\partial}{\partial \bar{z}} \left(\frac{\partial u}{\partial z}\right) = 4 \frac{\partial}{\partial \bar{z}} \left(\frac{\partial(\overline{f_2} g_1 - f_1 \overline{g_2})}{\partial z}\right) = 4 \frac{\partial}{\partial \bar{z}} \left(\overline{f_2} g_1' - f_1' \overline{g_2}\right)$$

$$= 4 \left(\overline{f_2'} g_1' - f_1' \overline{g_2'}\right) \quad \text{on } D.$$

Hence

$$f_1' \overline{g_2'} = \overline{f_2'} g_1' \quad \text{on } D. \tag{1.15}$$

We finish the proof by showing that the above equation implies that (1.5), (1.6), or (1.7) holds. If g_1' is identically 0 on D, then (1.15) shows that either g_2' is identically 0 on D (so ψ would be constant on D and (1.7) would hold) or f_1' is identically 0 on D (so both $\bar{\varphi}$ and $\bar{\psi}$ would be analytic on D and (1.6) would hold). Similarly, if g_2' is identically 0 on D, then (1.15) shows that either (1.7) or (1.5) would hold. Thus we may assume that neither g_1' nor g_2' is identically 0 on D, and so (1.15) shows that

$$\frac{f_1'}{g_1'} = \overline{\left(\frac{f_2'}{g_2'}\right)}$$

at all points of D except the countable set consisting of the zeroes of $g_1'g_2'$. The left-hand side of the above equation is an analytic function (on D with the zeroes of $g_1'g_2'$ deleted), and the right-hand side is the complex conjugate of an analytic function on the same domain, and so both sides must equal a constant $c \in \mathbf{C}$. Thus $f_1' = cg_1'$ and $f_2' = \bar{c}g_2'$ on D. Hence $f_1 - cg_1$ and $\overline{f_2} - c\overline{g_2}$ are constant on D, and so their sum, which equals $\varphi - c\psi$, is constant on D; in other words, (1.7) holds and the proof of Theorem 1.2 is complete. ■

Recall that an operator is called *normal* if it commutes with its adjoint. If $\varphi \in L^\infty(D, dA)$, then $T_\varphi^* = T_{\bar{\varphi}}$, so we can use Theorem 1.2 to prove the following corollary, which states that for φ a bounded harmonic function on D, the Toeplitz operator T_φ is normal only in the obvious case.

Corollary 1.5. *Suppose that φ is a bounded harmonic function on D. Then T_φ is a normal operator if and only if $\varphi(D)$ lies on some line in* $\mathbf{C}$.

PROOF: First suppose that $\varphi(D)$ lies on some line in $\mathbf{C}$. Then there exist constants $\alpha, \beta \in \mathbf{C}$, with $\alpha \neq 0$, such that $\alpha\varphi + \beta$ is real valued on D. Thus $T_{\alpha\varphi + \beta}$ is a self-adjoint operator, and hence T_φ, which equals $\alpha^{-1}(T_{\alpha\varphi + \beta} - \beta I)$, is a normal operator.

To prove the other direction, suppose now that T_φ is a normal operator. Thus $T_\varphi T_{\bar{\varphi}} = T_\varphi T_{\bar{\varphi}}$, and so Theorem 1.2 implies that φ and $\bar{\varphi}$ are both analytic on D (in which case φ is constant, so we are done) or there are constants $a, b \in \mathbf{C}$, not both 0, such that $a\varphi + b\bar{\varphi}$ is constant on D. The latter condition implies that $\varphi(D)$ lies on a line, completing the proof. ■

We conclude with a question. Does Theorem 1.2 remain true if we replace the disk by an arbitrary connected region in the plane? (So the Toeplitz operators now act on the Bergman space of this new region.) The statement of Theorem 1.2 certainly makes sense in this context and is plausible. However, the proof of Theorem 1.2 made extensive use of the set of analytic automorphisms of the disk. This approach will not work in general, because nonsimply-connected regions have too few analytic automorphisms to provide any useful information.

CHAPTER 2

THE COMMUTANTS OF CERTAIN TOEPLITZ OPERATORS

In this chapter we will be mostly interested in studying the commutants of some Toeplitz operators on the Bergman space whose symbols are polynomials. Only a little is known about this subject. The Bergman shift T_z has commutant that is equal to the set of all analytic Toeplitz operators. How about T_{z^n}?

It is well known that the Hardy space Toeplitz operator T_f is an isometry if and only if f is inner. If f is a nonconstant inner function then T_f is a pure isometry and is unitarily equivalent to a unilateral shift, whose commutant can be characterized matricially. Much work about the commutants of the Hardy space Toeplitz operators is based on this fact. On the Bergman space, a Toeplitz operator whose symbol is a nonconstant inner function is not an isometry. We will prove even more: The only Toeplitz operator with harmonic symbol that is an isometry is a scalar multiple of the identity. Before we prove this, we need some facts about bounded analytic functions. Good references are Hoffman [19] and Garnett [17].

By H^∞ we denote the Banach algebra of bounded analytic functions on D, with the sup norm

$$\|f\| = \sup \{ |f(z)| : z \in D \}$$

The set of all multiplicative linear functionals on H^∞ is denoted by M. The obvious elements of M are the point evaluations

$$\varphi_\lambda(f) = f(\lambda)$$

where λ is a point in D. Map $\lambda \to \varphi_\lambda$ is a homeomorphism from D into M, so that we regard D as a subset of M. D is an open subset of M and by the Corona Theorem, D is dense in M.

The Gelfand transform $\hat{}: H^\infty \to C(M)$ is defined by $\hat{f}(\varphi) = \varphi(f)$, for $\varphi \in M$. The Gelfand transform is an isometry from $H^\infty \to C(M)$, so that we can identify H^∞ with the uniformly closed subalgebra of $C(M)$. Hoffman ([20], Lema 4.4) has proved that the following algebras are identical:

(i) $C(M)$;

(ii) The sup norm closure of the algebra generated by H^∞ and $\overline{H^\infty}$ (the latter set denotes the complex conjugates of functions in H^∞);

(iii) The sup norm closure of the algebra generated by the bounded harmonic functions.

If m_1 and m_2 are in M, the pseudohyperbolic distance between m_1 and m_2 is

$$\rho(m_1, m_2) = sup \{ \, | \hat{f}(m_2) | : f \in H^\infty, \|f\| \leq 1, \hat{f}(m_1) = 0 \, \}$$

Clearly $\rho(m_1, m_2) \leq 1$. The relation $m_1 \sim m_2$ if and only if $\rho(m_1, m_2) < 1$ is an equivalence relation on M. The corresponding equivalence classes are called the Gleason parts of M. An element $m \in M$ is said to be an one-point part if its equivalence class consists of itself alone. The set of one-point parts in M will be denoted by M_1. Let

$$J = \{ \, \varphi \in C(M) : \varphi = 0 \; on \, M_1 \, \}.$$

Let $\mathcal{T}(C(M))$ be the closed subalgebra of $\mathcal{L}(L_a^2)$ generated by $\{T_\varphi : \varphi \in C(M)\}$ and let C be the commutator ideal of $\mathcal{T}(C(M))$. McDonald and Sundberg [23] has proved that the sequence

$$0 \to J \to C(M) \overset{\Phi}{\to} \mathcal{T}(C(M))/_C \to 0$$

is exact, where

$$\Phi(\varphi) = T_\varphi + C$$

for $\varphi \in C(M)$. This implies that $C(M)/_J$ is isomorphic to $\mathcal{T}(C(M))/_C$ with isomorphism

$$\Psi(\varphi + J) = T_\varphi + C$$

for $\varphi \in C(M)$. Let $\Pi : \mathcal{T}(C(M)) \to \mathcal{T}(C(M))/_C$ be the quotient map.

Another Banach algebra we need is $L^\infty(\partial D)$. The maximal ideal space of $L^\infty(\partial D)$, denoted by $M(L^\infty)$, plays an important role here. If we identify each function $f \in H^\infty$ with its boundary values, we may regard H^∞ as a closed subalgebra of $L^\infty(\partial D)$, which gives a natural map $\tau : M(L^\infty) \to M$. The map τ is defined by restricting each complex homomorphism of $L^\infty(\partial D)$ to H^∞. It is, in fact, a homeomorphism of $M(L^\infty)$ into M, so we may think of $M(L^\infty)$ as a subset of M. It turns out that $M(L^\infty)$ is a subset of M_1.

Now, we can prove our theorem.

Theorem 2.1. *Suppose that $h \in L^\infty(D,dA)$ is harmonic and that T_h is an isometry. Then h is a constant function of modulus* 1.

[illegible]

PROOF: Suppose that T_h is an isometry, i.e., $T^*_h T_h = I$. Then

$$\Pi(T^*_h)\Pi(T_h) = \Pi(I) \tag{2.1}$$

Since h is harmonic, Hoffman's result shows that h and $\overline{h} \in C(M)$. Applying Ψ^{-1} to both sides of (2.1), we obtain

$$(\overline{h} + J) \cdot (h + J) = 1 + J.$$

This shows that

$$h \cdot \overline{h} - 1 \in J.$$

By the definition of J,

$$h \cdot \overline{h} - 1 = 0 \quad \text{on } M_1.$$

As we said before, $M(L^\infty)$ is a subset of M_1, so that above equality means

$$\varphi(h \cdot \overline{h}) = 1$$

for every $\varphi \in M(L^\infty)$. Since $\varphi(h \cdot \overline{h}) = | \varphi(h) |^2$ we can conclude that

$$| \varphi(h) | = 1 \tag{2.2}$$

for every $\varphi \in M(L^\infty)$. Since $h \in L^\infty(D,dA)$ is harmonic, we can identify it with its boundary value function, denoted again by h. The Gelfand transform of $L^\infty(\partial D)$ maps $L^\infty(\partial D)$ isometrically and isomorphically onto $C(M(L^\infty))$ (see Hoffman, [19], p. 170), so that we have

$$sup\ \{\ |\ h(z)\ |: z \in D\ \} = \|\ h\ \|_{L^\infty(\partial D)} = \|\ \hat{h}\ \|_{C(M(L^\infty))} = sup\ \{\ |\ \hat{h}(\varphi)\ |: \varphi \in M(L^\infty)\ \} =$$

$$= sup\ \{\ |\ \varphi(h)\ |: \varphi \in M(L^\infty)\ \}.$$

Hence (2.2) implies $\sup\ \{\ |\ h(z)\ |: z \in D\ \} = 1$. If $|\ h(z)\ | = 1$ for some $z \in D$, then h is constant by the Maximum Principle. If $|\ h(z)\ | < 1$ for all $z \in D$, then

$$\pi = \|\ 1\ \|^2 = \|\ T_h\ 1\ \|^2 = \|\ Ph\ \|^2 \leq \|\ h\ \|^2 = \int_D |\ h(z)\ |^2\, dA < \int_D dA = \pi,$$

a contradiction. Here $\|\cdot\|$ denotes the $L^2(D,dA)$ - norm. Therefore h is a constant function, and since T_h is an isometry, $|\ h\ | = 1$. ■

We can slightly generalize this result and get the following:

Corollary 2.2. *Suppose that T_h is an isometry, where $h = f \cdot g^n$, where f is inner, $g \in L^\infty(D,dA)$ is harmonic, and $n \in$* **N**. *Then h is a constant function of modulus* 1.

PROOF: Since $f, g \in C(M)$, it is clear that $h \in C(M)$ so that we can apply the same reasoning as in Theorem 2.2 and thus obtain $|\varphi(h)| = 1$ for every $\varphi \in M(L^\infty)$. Since $|\varphi(f)| = 1$ if $\varphi \in M(L^\infty)$, it follows that

$$1 = |\varphi(g^n)| = |\varphi(g)|^n$$

what gives $|\varphi(g)| = 1$ for every $\varphi \in M(L^\infty)$. Because g is bounded harmonic,

$$sup \{ |g(z)| : z \in D \} = 1$$

As before, if $|g(z)| < 1$ for every $z \in D$, then $|h(z)| < 1$ for every $z \in D$, and we obtain the same contradiction. Thus $|g(z)| = 1$ for some $z \in D$, and then g = constant, i.e., $h = cf,\ c \in \mathbf{C}$. Then $T_h^* T_h = I$ implies $T_{|h|^2} = I$, i.e., $|h| = 1$ and we are done. ■

J. Thukral asked for which harmonic h is T_h a partial isometry. If T_h is a partial isometry, then by Halmos [18], Problem 98 , $T_h = T_h T_h^* T_h$. Applying $\Psi^{-1}\Pi$ on both sides gives

$$h - h^2\overline{h} \in J.$$

As before this means

$$\varphi(h)[1 - |\varphi(h)|^2] = 0$$

for every $\varphi \in M(L^\infty)$. Then, either $\varphi(h) = 0$ or $|\varphi(h)| = 1$, so that $\sup\{ |h(z)| : z \in D \} = 0$ or 1. If $\varphi(h) = 0$ for every $\varphi \in M(L^\infty)$, then $h = 0$. If $\varphi(h) \neq 0$ for some $\varphi \in M(L^\infty)$,

then $\sup \{ \, | h(z) | : z \in D \, \} = 1$. If $| h(z) | < 1$ for each $z \in D$, then taking any u in $(\operatorname{Ker} T_h)^{\perp}$ we get

$$\| u \|^2 = \| T_h u \|^2 = \| P(hu) \|^2 \le \| hu \|^2 = \int_D | h(z) |^2 | u(z) |^2 \, dA < \int_D | u(z) |^2 \, dA = \| u \|^2.$$

This is again a contradiction. Hence h must be a constant function. Thus we have proved the following theorem:

Theorem 2.3. *Suppose that $h \in L^{\infty}(D,dA)$ is harmonic and that T_h is a partial isometry. Then h is either a constant function of modulus* 1 *or h is identically* 0.

Now, we are going to consider our main problem - the commutants of some analytic Toeplitz operators. At first we will be interested in finding the commutant of T_{z^n}, for arbitrary positive integer n.

Let $\mathcal{T}$ be the closed subalgebra of $\mathcal{L}(L^2_a)$ generated by all Toeplitz operators. For $\varphi \in L^{\infty}(D,dA)$, the *Hankel operator with symbol φ*, denoted H_{φ}, is the operator from L^2_a to $(L^2_a)^{\perp}$ defined by $H_{\varphi} f = (I - P)(\varphi f)$. $\mathcal{K}$ will denote the ideal of all compact opertors acting on L^2_a. Before we state our next result, we need to define some function spaces.

For an analytic function f on D we set

$$\| f \|_B = \sup \{ \, (1 - | z |^2) | f'(z) | : z \in D \, \}$$

The *Bloch space B* is the set of all analytic functions f on D for which $\| f \|_B < \infty$. The quantity $| f(0) | + \| f \|_B$ defines a norm on B, and B equipped with this norm is a Banach

space. Contained in the Bloch space is the *little Bloch space* B_0, which is by definition the set of all analytic functions f on D for which

$$(1 - |z|^2) f'(z) \to 0 \text{ as } |z| \to 1.$$

For the basic properties of the Bloch space see [2].

Let $n \in \mathbf{N}$ be fixed.

Theorem 2.4. *Let $S \in \mathcal{T}$ commute with T_{z^n}. Then $S = T_\psi$ for some $\psi \in H^\infty$.*

PROOF: The equation $S T_{z^n} = T_{z^n} S$ gives us the following:

Let $g_0 = S\,1$. Then

$S z^n = S T_{z^n} 1 = T_{z^n} S\, 1 = z^n g_0.$

$S z^{2n} = S T_{z^n} z^n = T_{z^n} S z^n = z^{2n} g_0$

...

$S z^{kn} = z^{kn} g_0$, $k = 0,1,2 \ldots$.

Let $g_1 = S z$. Then

$S z^{n+1} = S T_{z^n} z = T_{z^n} S z = z^n g_1$

$S z^{2n+1} = S T_{z^n} z^{n+1} = T_{z^n} S z^{n+1} = z^{2n} g_1$

...

$S z^{kn+1} = z^{kn} g_1$, $k = 0,1,2 \ldots$.

Continuing in this way, we finally get

$g_{n-1} = S\, z^{n-1}$

$S\, z^{2n-1} = S\, T_{z^n}\, z^{n-1} = T_{z^n} S\, z^{n-1} = z^n g_{n-1}$

...

$S\, z^{kn+n-1} = z^{kn} g_{n-1}$, $k = 0,1,2 \ldots$.

Let $X_0 = \overline{\text{span}\ \{ e_{kn}: k = 0,1,2,\ldots \}}$,

$X_1 = \overline{\text{span}\ \{ e_{kn+1}: k = 0,1,2, \ldots \}}$,

...

$X_{n-1} = \overline{\text{span}\ \{ e_{kn+(n-1)}: k = 0,1,2, \ldots \}}$.

Then $L_a^2 = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$, i.e., each $f \in L_a^2$ can be written as

$$f = f_0 + f_1 + \cdots + f_{n-1},$$

$f_i \in X_i$, $i = 0,1,2, \cdots ,n-1$. Each $f_0 \in X_0$ has its Fourier series expansion

$$f_0 = \sum_{k=0}^{\infty} (f_0, e_{kn}) e_{kn}$$

i.e., $f_0 = \lim s_m$, where $s_m = \sum_{k=0}^{m} (f_0, e_{kn}) e_{kn}$. Since the point evaluations are bounded on L_a^2, we have $f_0(z) = \lim s_m(z)$, for each $z \in D$, so that

$$(f_0 \cdot g_0)(z) = lim\ (s_m \cdot g_0)(z) \tag{2.3}$$

for each $z \in D$. Since $S\, z^{kn} = z^{kn} g_0$, $k = 0,1,2 \ldots$, it follows that $S\, s_m = s_m \cdot g_0$, for every $m \in$ N. By continuity of S, we have

$$(S f_0) = \lim S s_m = \lim s_m \cdot g_0$$

so that

$$(S f_0)(z) = \lim (s_m \cdot g_0)(z) \tag{2.4}$$

for each $z \in D$. Comparing (2.3) and (2.4) we conclude that

$$S f_0 = g_0 f_0,$$

for each $f_0 \in X_0$. Repeating the above reasoning, we get that $S f_1 = \frac{g_1}{z} f_1$, for $f_1 \in X_1$ and so on. Thus operator S can be described as

$$S f = g_0 f_0 + \frac{g_1}{z} f_1 + \frac{g_2}{z^2} f_2 + \cdots + \frac{g_{n-1}}{z^{n-1}} f_{n-1}. \tag{2.5}$$

Let's observe another property of S, being an element of $\mathcal{T}$.

<u>Claim</u>: $S T_z - T_z S \in \mathcal{K}$.

At first, assume $S = T_\varphi$, $\varphi \in L^\infty(D, dA)$. Then

$$T_\varphi T_z - T_z T_\varphi = T_{z\varphi} - T_z T_\varphi = H_{\bar{z}}^* H_\varphi .$$

Because $z \in B_0$, the operator $H_{\bar{z}}^*$ is compact (see [4]), so that $T_\varphi T_z - T_z T_\varphi$ is compact. If $\Pi : \mathcal{L}(L_a^2) \to \mathcal{L}(L_a^2)/\mathcal{K}$ denotes the natural projection, then

$$\Pi(T_\varphi)\, \Pi(T_z) = \Pi(T_z)\, \Pi(T_\varphi) \tag{2.6}$$

for every $\varphi \in L^{\infty}(D,dA)$. If $S = T_{\varphi_1} \cdots T_{\varphi_n}$ then, because of (2.6), $\Pi(S\,T_z - T_z\,S) = 0$ so that $S\,T_z = T_z\,S \in \mathcal{K}$. An arbitrary operator S in $\mathcal{T}$ is the limit of sums of operators of the form $T_{\varphi_1} \cdots T_{\varphi_n}$. In that case $S = \lim S_n$, each S_n is of the form $\Sigma\, T_{\varphi_1} \cdots T_{\varphi_n}$, and so $S\,T_z - T_z\,S = \lim\,(S_n\,T_z - T_z\,S_n) \in \mathcal{K}$. Hence claim is proved.

Now, let's express $S\,T_z - T_z\,S$ in terms of (2.5).

$S\,T_z f_0 = S(z f_0) = \frac{g_1}{z}(z f_0) = g_1 f_0$, because $z f_0 \in X_1$.

$S\,T_z f_1 = S(z f_1) = \frac{g_2}{z^2}(z f_1) = \frac{g_2}{z} f_1$, because $z f_1 \in X_2$, and so on. Finally we get

$$S\,T_z f = g_1 f_0 + \frac{g_2}{z} f_1 + \frac{g_3}{z^2} f_2 + \cdots + z\, g_0 f_{n-1}.$$

From this and (2.5) we obtain

$$(S\,T_z - T_z\,S) f = f_0\,(g_1 - z g_0) + f_1\left(\frac{g_2}{z} - g_1\right) + f_2\left(\frac{g_3}{z^2} - \frac{g_2}{z}\right) + \cdots + f_{n-1}\left(z g_0 - \frac{g_{n-1}}{z^{n-2}}\right).$$

By the claim, $(S\,T_z - T_z\,S)_{|X_0} = M_{g_1 - z g_0} : X_0 \to L^2_a$ is compact ($M_{g_1 - z g_0}$ is a multiplication operator). Let $\varphi = g_1 - z g_0$.

<u>Claim</u> : $\varphi = 0$.

Let $w \in D$ be fixed and let $R_w \in X_0$ be the reproducing kernel for X_0. Recall that $M^*_{\varphi} : L^2_a \to X_0$ and take any $g \in X_0$ to obtain the following:

$$\langle g, M^*_{\varphi} k_w \rangle = \langle M_{\varphi} g, k_w \rangle = \varphi(w) \cdot g(w) = \varphi(w) \langle g, R_w \rangle = \langle g, \overline{\varphi(w)} R_w \rangle .$$

It follows that $M^*_{\varphi} k_w = \overline{\varphi(w)} R_w$, so that $M^*_{\varphi} \frac{k_w}{\| k_w \|} = \overline{\varphi(w)} \frac{R_w}{\| k_w \|}$. It is well known that $\frac{k_w}{\| k_w \|}$ converges weakly to 0 as $|w| \to 1$. Since M^*_{φ} is compact, $M^*_{\varphi} \frac{k_w}{\| k_w \|} \to 0$ in norm as $|w| \to 1$. Thus

$$|\varphi(w)|\frac{\| R_w \|}{\| k_w \|}\to 0 \text{ as } |w|\to 1 \tag{2.7}$$

We'll calculate the numerator first.

$$R_w(z)=\sum_{k=0}^{\infty}\frac{kn+1}{\pi}(R_w,z^{kn})z^{kn}=\frac{1}{\pi}\sum_{k=0}^{\infty}(kn+1)\,\overline{w}^{kn}\,z^{kn},$$

so that

$$\| R_w \|^2=\frac{1}{\pi}\sum_{k=0}^{\infty}(kn+1)\,|w|^{2kn}.$$

Let $\lambda=|w|^{2n}$. Then we have the following:

$$\| R_w \|^2=\frac{1}{\pi}\sum_{k=0}^{\infty}(kn+1)\lambda^k=\frac{n}{\pi}\sum_{k=1}^{\infty}k\lambda^k+\frac{1}{\pi}\sum_{k=0}^{\infty}\lambda^k=\frac{n\lambda}{\pi}(\sum_{k=1}^{\infty}\lambda^k)'+\frac{1}{\pi(1-\lambda)}=$$

$$=\frac{n\lambda}{\pi(1-\lambda)^2}+\frac{1}{\pi(1-\lambda)}=\frac{1+(n-1)\lambda}{\pi(1-\lambda)^2}.$$

Therefore

$$\frac{\| R_w \|^2}{\| k_w \|^2}=\frac{1+(n-1)|w|^{2n}}{(1-|w|^{2n})^2}\cdot(1-|w|^2)^2=[\frac{1}{1+|w|^2+|w|^4+\cdots+|w|^{2(n-1)}}]^2\,[n|w|^{2n}+1-|w|^{2n}]$$

$$\geq\frac{1}{n}|w|^{2n}.$$

Thus $\lim \frac{\| R_w \|}{\| k_w \|} \geq \frac{1}{\sqrt{n}} > 0$, as $|w| \to 1$. Therefore by (2.7), $\lim | \varphi(w) | = 0$, as $|w| \to 1$.

Since φ is analytic, $\varphi = 0$, and claim is proved. This means that $g_0 = \frac{g_1}{z}$.

Similarly, $(S\,T_z - T_z\,S)|_{X_1} = M_{g_2/z - g_1} : X_1 \to L_a^2$ is compact. Thus $M_{g_2/z - g_1} M_z|_{X_0} = M_{g_2 - zg_1} : X_0 \to L_a^2$ is compact. By the previous claim, $g_2 - zg_1 = 0$ so that $g_1 = \frac{g_2}{z}$ and therefore $g_0 = \frac{g_2}{z^2}$. If we continue this way, (2.5) shows that

$$S f = g_0 \cdot f = T_{g_0} f$$

for every $f \in L_a^2$. The function g_0 must be an H^∞ function as a multiplier of L_a^2 (see [28]). If we let $\psi = g_0$, the theorem is proved. ■

We can extend this result. Let $u \in Aut(D)$ and define an operator $V : L_a^2 \to L_a^2$ by $V f = f \circ u^{-1}$. Clearly, V is a bounded linear operator, with the inverse operator $V^{-1} f = f \circ u$. Observe that $T_z V f = z \cdot V f = z \cdot (f \circ u^{-1})$ and $V T_u f = (u \cdot f) \circ u^{-1} = z \cdot (f \circ u^{-1})$ which means that

$$T_z\, V = V\, T_u\,,$$

i.e., T_u is similar to T_z. Thus

$$T_{z^n} V = V\, T_{u^n} \tag{2.8}$$

for every $n \in \mathbf{N}$.

Suppose now that $S \in \mathcal{T}$ and $S\, T_{u^n} = T_{u^n} S$ for some n. Formula (2.8) implies

$$S\,V^{-1}\,T_{z^n}\,V = V^{-1}\,T_{z^n}\,V\,S\,.$$

After multiplying both sides of this equation by V from left and by V^{-1} from right, we conclude that

$$V\,S\,V^{-1} \in \{\,T_{z^n}\,\}'.$$

Claim: The operator $B = V\,S\,V^{-1}$ has the property that $B\,T_z - T_z\,B$ is in $\mathcal{K}$.

$B\,T_z - T_z B = V\,S\,V^{-1}T_z - T_z\,V\,S\,V^{-1} = V(S\,V^{-1}T_zV - V^{-1}\,T_z\,V\,S)V^{-1}$

$= V(S\,T_u - T_u\,S)V^{-1}.$

The operator $S\,T_u - T_u\,S$ is compact because $u \in B_0$ and the claim is proved.

Now, by the proof of Theorem 2.4, $B = T_\varphi$, for some $\varphi \in H^\infty$. This means that $S = V^{-1}T_\varphi\,V$, i.e., $S\,f = V^{-1}T_\varphi\,V\,f = V^{-1}(\varphi \cdot (f{\cdot}u^{-1})) = (\varphi{\cdot}u) \cdot f$, for $f \in L^2_a$. If we let $\psi = \varphi{\cdot}u$, we have proved the following corollary:

Corollary 2.5. *Let $S \in \mathcal{T}$ commute with T_{u^n}, for some $n \in$ N. Then $S = T_\psi$ for some $\psi \in H^\infty$.*

Now, knowing that, for example, $\{\,T_{z^2}\,\}' \cap \mathcal{T} = \{\,T_\psi : \psi \in H^\infty\,\}$ leads us to the question of finding $\{\,T_{z^2+z}\,\}' \cap \mathcal{T}$. The answer is rather surprising: the commutant $\{\,T_{z^2+z}\,\}'$ itself is equal to $\{\,T_\psi : \psi \in H^\infty\,\}$, as we will see. One of the crucial tools in what follows is this well-known lemma:

Lemma 2.6. *If $h \in H^\infty$ and $S \in \mathcal{L}(L^2_a)$ commutes with T_h, then $S^*\,k_\lambda$ is an eigenvector for $T_h{}^*$ for every $\lambda \in D$, with eigenvalue $\overline{h(\lambda)}$.*

For the sake of simplicity, let $h(z) = z^2 + z$. By the Lemma 2.6,

$$S^* k_\lambda \in \text{ Ker } T_{\bar{h}-\overline{h(\lambda)}} = (\text{Range } T_{h-h(\lambda)})^\perp$$

so that we have to find Range $T_{h-h(\lambda)}$.

Suppose that $g \in$ Range $T_{h-h(\lambda)}$. Then $g = [\, h - h(\lambda)]f$ for some $f \in L^2_a$. This implies

$$g(z) = (\, z^2 + z - \lambda^2 - \lambda \,)f = (\, z - \lambda \,) \cdot (\, z + \lambda + 1 \,)f.$$

Let

$$D_0 = \{\, \lambda \in D : -\lambda - 1 \in \overline{D} \,\}$$

Suppose that $\lambda \in D \setminus D_0$; then $z = -\lambda - 1$ is not a zero of g.

Claim : Range $T_{h-h(\lambda)} = \{\, g \in L^2_a : g(\lambda) = 0 \,\}$.

Clearly, Range $T_{h-h(\lambda)}$ is a subset of $\{\, g \in L^2_a : g(\lambda) = 0 \,\}$. Conversely, suppose that $v \in \{\, g \in L^2_a : g(\lambda) = 0 \,\}$. Then $v(z) = (\, z - \lambda \,)u(z)$, for some analytic function u. It is easy to see that $u \in L^2_a$: First choose $0 < \varepsilon < 1$ big enough so that $\lambda \in D_\varepsilon$, where $D_\varepsilon = \{\, z \in D : |z| < \varepsilon \,\}$. Since u is analytic on D, it is bounded on $\overline{D_\varepsilon}$, let's say, by a constant C. For $z \in D \setminus D_\varepsilon$, $|z - \lambda| > k = \text{dist}\,(\, \lambda, \partial D_\varepsilon \,)$ so that we have the following:

$$\int_D |\, u(z) \,|^2 \, dA = \int_{D_\varepsilon} |\, u(z) \,|^2 \, dA + \int_{D \setminus D_\varepsilon} |\, u(z) \,|^2 \, dA \le C^2\pi + \int_{D \setminus D_\varepsilon} \frac{|\, v(z) \,|^2}{|\, z - \lambda \,|^2} \, dA$$

$$\le C^2\pi + \frac{\|\, v \,\|^2}{k^2}$$

so that $u \in L_a^2$. We can write $v(z) = (z - \lambda) \cdot (z + \lambda + 1)f$, where $f(z) = \dfrac{u(z)}{z + \lambda + 1}$.
Recall that $\lambda \in D \setminus D_0$, so that $z + \lambda + 1$ is bounded away from zero. Then $f \in L_a^2$, because $u \in L_a^2$, and we proved that $v \in$ Range $T_{h-h(\lambda)}$ and the claim is proved.

This claim shows that Range $T_{h-h(\lambda)} = \{k_\lambda\}^\perp$ for $\lambda \in D \setminus D_0$. Then for these λ's, $S^* k_\lambda = \overline{f(\lambda)} k_\lambda$, where f is some function defined on $D \setminus D_0$. For fixed $u \in L_a^2$ we have

$$(S\,u)(\lambda) = \langle S\,u, k_\lambda\rangle = \langle u, S^* k_\lambda\rangle = \langle u, \overline{f(\lambda)} k_\lambda\rangle = f(\lambda) \cdot u(\lambda)$$

for each $\lambda \in D \setminus D_0$. For such λ, $[(ST_z - T_zS)(u)](\lambda) = [S(zu)](\lambda) - [zSu](\lambda) = 0$. Because $(ST_z - T_zS)(u)$ is an analytic function, that is equal to 0 on an open set, it must be identically 0 on D. The function u was arbitrary, so that $ST_z = T_zS$ and therefore $S = T_\psi$, $\psi \in H^\infty$. Hence we showed that $\{T_{z^2+z}\}' = \{T_\psi : \psi \in H^\infty\}$.

The same reasoning would show that $\{T_h\}' = \{T_\psi : \psi \in H^\infty\}$, in the case that $h(z) = (z - a)(z - b)$, where $a, b \in \mathbf{C}$ and $b \neq -a$. Of course, when $b = -a$, $h(z) = z^2 - a^2$ and $\{T_h\}' = \{T_{z^2}\}'$. The difference between $\{T_{z^2}\}'$ and $\{T_{z^2+z}\}'$ leads us to the question of characterizing the commutant of T_{z^n+az}, $a \neq 0$, and $n \in \mathbf{N}$. For that reason, let's fix $n \in \mathbf{N}$. The case $|a| > 1$ is easy:

Suppose $S \in \{T_{z^n+az}\}'$. Denote $h(z) = z^n + az$ and we will find Range $T_{h-h(\lambda)}$. If $g \in$ Range $T_{h-h(\lambda)}$, then

$$g(z) = [(z^n - \lambda^n) + a(z - \lambda)]f = (z - \lambda)[\,z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1} + a\,]f. \qquad (2.9)$$

Let $p(z) = z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1}$. Since $|a| > 1$, there exists $\varepsilon > 0$ such that $1 - \dfrac{1}{|a|} > \varepsilon$, i.e., $\dfrac{1}{1 - \varepsilon} < |a|$. For this ε, let $D_\varepsilon = \{z \in D : |z| < \varepsilon\}$, and suppose $\lambda \in D_\varepsilon$. Then

$$|p(z)| < 1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-1} < \frac{1}{1-\varepsilon} < |a|$$

so that $p(z) + a \neq 0$ for every $z \in D$. In fact $p(z) + a$ is bounded away from 0 if $\lambda \in D_\varepsilon$:

$$|p(z) + a| \geq |a| - |p(z)| > |a| - \frac{1}{1-\varepsilon} > 0.$$

From (2.9) we conclude that Range $T_{h-h(\lambda)} = \{ g \in L_a^2 : g(\lambda) = 0 \}$ if $\lambda \in D_\varepsilon$. As in the example above, it will follow that $(S\, u)(\lambda) = f(\lambda) \cdot u(\lambda)$, for some function f and $\lambda \in D_\varepsilon$. This will again give

$$(ST_z - T_z S)(u) = 0$$

for $u \in L_a^2$, since $(ST_z - T_z S)(u)$ is an analytic function that is equal to 0 on an open set D_ε. Therefore $S \in \{T_z\}'$, so that $S = T_\psi$, for some $\psi \in H^\infty$.

Now we want to show that the same holds for any $a \in \mathbb{C} \setminus \{0\}$. This is the content of the next theorem:

Theorem 2.7. *Suppose* $h(z) = z^n + az$, *where* $a \neq 0$, $n \in \mathbb{N}$, and $n > 1$. *If* $S \in \mathcal{L}(L_a^2)$ commutes with T_h, *then* $S = T_\psi$ *for some* $\psi \in H^\infty$.

PROOF: The case $|a| > 1$ has already been discussed. Then we may assume $0 < |a| \leq 1$. By Lemma 2.6, $S^* k_\lambda$ is an eigenvector for $T_h{}^*$, for every $\lambda \in D$. Again, suppose $g \in$ Range $T_{h-h(\lambda)}$. Then $g(z) = (z^n + az - \lambda^n - a\lambda)f$.

<u>GOAL</u>: We want to find an open subset U of D such that $z = \lambda$ is the only solution of $z^n + az - \lambda^n - a\lambda = 0$ in $\overline{D}$, if $\lambda \in U$, and furthermore $z = \lambda$ is simple.

So, we shall look at the equation

$$z^n + az = \lambda^n + a\lambda \tag{2.10}$$

For explanatory reasons, we'll consider two cases.

Case 1 : $a > 0$.

For such a, choose $\lambda = 1$ and look at the equation

$$z^n + az = 1 + a \tag{2.11}$$

For $z \in D$, $| z^n + az | \leq | z^n | + a | z | < 1 + a$, so there is no solution of (2.11) in D. If $z \in \partial D$ satisfies (2.11), we have

$$1 + a = | z^n + az | \leq | z^n | + a | z | = 1 + a .$$

Equality in the triangle inequality occurs only in the case that all terms are linearly dependent. Thus $arg\ (z^n + az) = arg\ az + 2k\pi$, $k \in \mathbf{Z}$. But $z^n + az \in \mathbf{R}$ so that $arg\ z = arg\ az = 2m\pi$, for some $m \in \mathbf{Z}$, i.e., $z = 1$. Thus, (2.11) has only one root in $\overline{D}$: $z = 1 = \lambda$.

Since $(z^n + az - 1 - a)'(1) = (nz^{n-1} + a)(1) = n + a \neq 0$ (because $a > 0$), $z = 1$ is a simple zero.

Let $F : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ be defined by $F(\lambda,z) = z^n + az - \lambda^n - a\lambda$. By the previous arguments, $F(1,z)$ has $n - 1$ zeros outside $\overline{D}$: $z_1, z_2, \cdots, z_{n-1}$. The other zero is $z = 1$. Each zero z_i is simple, because $(z^n + az -1 - a)'(z_i) = n\, z_i^{n-1} + a \neq 0$, for $a \leq 1$.

Let's choose $\varepsilon > 0$ small enough such that $K(z_i, \varepsilon) \cap \overline{D} = \emptyset$ for every i, and $K(z_i, \varepsilon) \cap K(z_j, \varepsilon) = \emptyset$, for $i \neq j$. Here $K(z_i, \varepsilon) = \{ z \in D : | z - z_i | < \varepsilon \}$.

We know that $F(1,z_1) = 0$. Taking partial derivatives with respect to z gives $\frac{\partial F}{\partial z} = n z^{n-1} + a$ and $\frac{\partial F}{\partial z}(1,z_1) = nz_1^{n-1} + a \neq 0$, because $0 < |a| \leq 1$. By the Implicit Function Theorem, there exists W_1, an open neighborhood of 1, and $\varphi_1 : W_1 \to \mathbf{C}$ continuous such that $\varphi_1(1) = z_1$ and $F(\lambda,\varphi_1(\lambda)) = 0$ for all $\lambda \in W_1$. By continuity of φ_1, there exists V_1, an open subset of W_1, such that $1 \in V_1$ and $K(z_1, \varepsilon) \supset \varphi_1(V_1)$.

Similarly, $F(1,z_2) = 0$ and $\frac{\partial F}{\partial z}(1,z_2) \neq 0$. By the Implicit Function Theorem, there exists W_2, an open neighborhood of 1, and $\varphi_2 : W_2 \to \mathbf{C}$ continuous such that $\varphi_2(1) = z_2$ and $F(\lambda,\varphi_2(\lambda)) = 0$ for all $\lambda \in W_2$. By continuity of φ_2, there exists V_2, an open subset of W_2, such that $1 \in V_2$ and $K(z_2, \varepsilon) \supset \varphi_2(V_2)$. Let's continue in this way. Then we get $V_1, V_2, \ldots, V_{n-1}$ as open neighborhoods of 1. Let $V = \cap V_i$. Then V is also an open neighborhood of 1 and $U = V \cap D$ is a nonempty open subset of D. For $\lambda \in U$ fixed, $\lambda \in V_i$ for every i, and thus $(\lambda,\varphi_1(\lambda))$, $(\lambda,\varphi_2(\lambda))$, ..., $(\lambda,\varphi_{n-1}(\lambda))$ are zeros of F. This means that $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_{n-1}(\lambda)$ are roots of (2.10). Since $\varphi_i(\lambda) \in K(z_i, \varepsilon)$, we have $\varphi_i(\lambda) \notin \overline{D}$. In addition to that, $\varphi_i(\lambda) \neq \varphi_j(\lambda)$, if $i \neq j$, so that we have exactly $n - 1$ roots outside $\overline{D}$. The only other root is $z = \lambda$. Thus we found a nonempty open set U of D such that for each $\lambda \in U$, $z^n + az - \lambda^n - a\lambda = 0$ has exactly one root in $\overline{D}$: $z = \lambda$.

General case : $a = re^{i\theta}, 0 < r < 1$.

For such a, choose $\lambda_0 = exp(i\frac{\theta}{n-1})$. Then $\lambda_0^{n-1} = e^{i\theta}$, and hence

$$|\lambda_0^{n-1} + a| = 1 + |a|$$

so that $|\lambda_0^n + a\lambda_0| = |\lambda_0(\lambda_0^{n-1} + a)| = 1 + |a|$.

We will investigate the equation

$$z^n + az = \lambda_0^n + a\lambda_0 \tag{2.12}$$

24

If $z \in D$, then $|z^n + az| < 1 + |a| = |\lambda_0^n + a\lambda_0|$, so there is no solution of (2.12) in D.

If $z \in \partial D$ satisfies the above equation, then

$$1 + |a| = |\lambda_0^n + a\lambda_0| = |z^n + az| \leq |z^n| + |a||z| = 1 + |a|.$$

This means that equality holds in the triangle inequality so that

$$arg\ (z^n + az) = arg\ az + 2k\pi = \theta + arg\ z + 2k\pi\ ,\ k \in \mathbf{Z}.$$

On the other hand,

$$arg\ (z^n + az) = arg\ (\lambda_0^n + a\lambda_0) = arg\ (exp(i\frac{n\theta}{n-1}) + r\ exp(i\frac{\theta}{n-1} + i\theta)) = \frac{n\theta}{n-1}.$$

Comparing these two quantities, we get

$$\theta + arg\ z + 2k\pi = \frac{n\theta}{n-1}$$

and therefore $arg\ z = \frac{\theta}{n-1} - 2k\pi$, which means that $z = \lambda_0$. Hence, we have proved that $z = \lambda_0$ is the only solution of (2.12) in $\overline{D}$.

Since $(z^n + az - \lambda_0^n - a\lambda_0)'(\lambda_0) = n\lambda_0^{n-1} + a$, and $|a| \leq 1$ it follows that $(z^n + az - \lambda_0^n - a\lambda_0)'(\lambda_0) \neq 0$, so that $z = \lambda_0$ is a simple zero. Again, by the same reasoning as above, the Implicit Function Theorem gives a nonempty open set U in D such that for each $\lambda \in U$, $z^n + az - \lambda^n - a\lambda = 0$ has exactly one root in $\overline{D}$.

Let's fix one $\lambda \in U$. If $g \in$ Range $T_{h-h(\lambda)}$ it is clear that $g(\lambda) = 0$. On the other hand, if $g \in L^2_a$ and $g(\lambda) = 0$, then $g(z) = (z - \lambda)u(z)$, where $u \in L^2_a$. We can write g as

$$g(z) = (z^n + az - \lambda^n - a\lambda) \cdot \frac{u(z)}{z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1} + a}.$$

By the previous discussion, the polynomial $z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1} + a$ has no zeros in $\overline{D}$, so it is bounded away from 0 in D. Hence, $\dfrac{u(z)}{z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1} + a} \in L^2_a$, so that $g \in$ Range $T_{h-h(\lambda)}$. In other words, we have proved that Range $T_{h-h(\lambda)} = \{ g \in L^2_a : g(\lambda) = 0 \}$ if $\lambda \in U$. Thus for $\lambda \in U$, $S^* k_\lambda = \overline{f(\lambda)} k_\lambda$, where f is some function defined on U. This leads to the conclusion that $S \in \{ T_z \}'$ so that $S = T_\psi$, for some $\psi \in H^\infty$. ■

After proving this theorem one may ask if this method can help us describing the commutant of a Toeplitz operator whose symbol is a polynomial. It turns out that the same method works in this situation, under some additional conditions. The nicest result can be obtained under the assumption that all polynomial coefficients are nonnegative. In order to apply the same reasoning, we should look at the proof of the previous theorem more carefully. The crucial point in the proof was observing that the equation $z^n + az = 1 + a$, with $0 < a < 1$, has n distinct zeros. If we let $p(z) = z^n + az$, we needed that $p(z) - p(1)$ has n distinct zeros. Then we could apply the Implicit Function Theorem. In light of this, we can understand the following theorem:

Theorem 2.8. *Suppose* $p(z) = z + a_2 z^2 + \cdots + a_n z^n$, *where* $a_i \geq 0$, for $i = 2,\ldots,n$. *If* $p(z) - p(1)$ *has* n *distinct zeros, then* $\{T_p\}' = \{T_\psi : \psi \in H^\infty\}$.

PROOF: Consider the equation $p(z) - p(1) = 0$, i.e.,

$$z + a_2 z^2 + \cdots + a_n z^n = 1 + a_2 + \cdots + a_n. \tag{2.13}$$

For $z \in D$, $| z + a_2z^2 + \cdots + a_nz | < 1 + a_2 + \cdots + a_n$. and consequently there is no solution of (2.13) in D.

If $z \in \partial D$ satisfies (2.13), then

$$1 + a_2 + \cdots + a_n = | z + a_2z^2 + \cdots + a_nz^n | \leq | z | + | a_2 | | z^2 | + \cdots + | a_n | | z^n |$$

$$\leq 1 + a_2 + \cdots + a_n.$$

Therefore, $z, z^2, \cdots, z^n$ are linearly dependent. Hence $arg\ (z + a_2z^2 + \cdots + a_nz^n) =$ $= arg\ z + 2k\pi$.

On the other hand, $z + a_2z^2 + \cdots + a_nz$ is positive so that $arg\ z = 2m\pi$, showing that $z = 1$. Therefore $z = 1$ is the only solution of 2.13 in $\overline{D}$. Other zeros $z_1, z_2, \cdots, z_{n-1}$ are outside $\overline{D}$. Let's choose $\varepsilon > 0$ small enough such that $K(z_i, \varepsilon) \cap \overline{D} = \emptyset$ for every i, and $K(z_i, \varepsilon) \cap K(z_j, \varepsilon) = \emptyset$, for $i \neq j$. Define a function $F : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ by

$$F(\lambda,z) = z + a_2z^2 + \cdots + a_nz - \lambda - a_2\lambda^2 - \cdots - a_n\lambda^n.$$

Then $z \rightarrow F(1,z)$ has $n - 1$ zeros outside $\overline{D}$. After taking partial derivatives of F with respect to z, we get $\frac{\partial F}{\partial z} = p'(z)$, so that for every i

$$\frac{\partial F}{\partial z}(1,z_i) = p'(z_i) = [\ p(z) - p(1)\ ]'(z_i) \neq 0,$$

because of the assumption that all zeros of $p(z) - p(1)$ are simple. By the Implicit Function Theorem, we can find a nonempty open set U in D such that for each $\lambda \in U$, $p(z) - p(\lambda) = 0$ has exactly one root in $\overline{D}$. By the same argument as before, $S = T_\psi$, for some $\psi \in H^\infty$. ■

Corollary 2.9. *Suppose* $p(z) = z + a_2 z^2 + \cdots + a_n z^n$, *where* $a_i \geq 0$, for $i = 2,\ldots,n$. *If* $p'(z)$ *has no zeros outside* D, *then* $\{T_p\}' = \{T_\psi : \psi \in H^\infty\}$.

If $p(z)$ is a polynomial with complex coefficients, then we need very strong restrictions on coefficients , in order to obtain similar result. We'll omit that case.

Remark: A reader should realize that proofs of Theorem 2.8, Corollary 2.9, and Theorem 2.10 are valid for both the Bergman and Hardy spaces.

We'll conclude this chapter with several examples, where we use the same technique.

Example: Suppose $h(z) = z \cdot \dfrac{a - z}{1 - \bar{a}z}$, where $a \in D$ is fixed and $z \in D$. We want to find the commutant of T_h. The function $\varphi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ is an analytic automorphism of D that is inverse to itself, i.e.,

$$\varphi_a(\varphi_a(z)) = z \tag{2.14}$$

for $z \in D$. Suppose $g \in$ Range $T_{h-h(\lambda)}$, for some $\lambda \in D$. Then

$$g(z) = [\, z \cdot \frac{a - z}{1 - \bar{a}z} - \lambda \cdot \frac{a - \lambda}{1 - \bar{a}\lambda}\,] f = \frac{(z - \lambda)[\, a - z - \lambda + \bar{a}z\lambda \,]}{(1 - \bar{a}z)(1 - \bar{a}\lambda)} f$$

for some $f \in L^2_a$. This fraction is equal 0 if $z = \lambda$ or $z = \dfrac{a - \lambda}{1 - \bar{a}\lambda}$, so that we conclude

$$\text{Range } T_{h-h(\lambda)} = \{ g \in L_a^2 : g(\lambda) = g(\frac{a - \lambda}{1 - \bar{a}\lambda}) = 0 \}.$$

In other words Ker $T_{\bar{h}-\overline{h(\lambda)}}$ = span $\{k_\lambda, k_{\varphi_a(\lambda)}\}^-$. If S commutes with T_h, then $S^* k_\lambda \in$ Ker $T_{\bar{h}-\overline{h(\lambda)}}$ for every $\lambda \in D$. Thus

$$S^* k_\lambda = \overline{f(\lambda)} k_\lambda + \overline{g(\lambda)} k_{\varphi_a(\lambda)},$$

for some functions f and g defined on D. Now

$$(S\, u)(\lambda) = f(\lambda)\, u(\lambda) + g(\lambda)\, u(\varphi_a(\lambda)).$$

Therefore,

$$S\, u = f \cdot u + g \cdot (u \circ \varphi_a) = (T_f + T_g\, C_{\varphi_a}))(u)\,, \tag{2.15}$$

where $C_{\varphi_a} : L_a^2 \to L_a^2$ is the composition operator defined by $C_{\varphi_a} = u \circ \varphi_a$. This shows what an operator in $\{ T_h \}'$ looks like. Suppose, in addition to this, that $S \in \mathcal{T}$. Let's calculate $S\, T_z - T_z\, S$:

$$(S\, T_z - T_z\, S)(u) = S(z \cdot u) - z \cdot (Su) = z \cdot f \cdot u + g \cdot [(z \cdot u) \circ \varphi_a] - z \cdot f \cdot u - z \cdot g \cdot (u \circ \varphi_a) =$$

$$= g \cdot (\varphi_a - z) \cdot (u \circ \varphi_a).$$

In other words, $S\, T_z - T_z\, S = T_{g(\varphi_a - z)}\, C_{\varphi_a}$. In Theorem 2.4, we proved that $S\, T_z - T_z\, S$ must be compact, so that $T_{g(\varphi_a - z)}\, C_{\varphi_a}$ is compact. Note that $C_{\varphi_a}{}^{-1} = C_{\varphi_a}$ (because of

(2.14)) and that implies that $T_{g(\varphi_a - z)}$ is compact. A multiplication operator T_f (i.e., $f \in H^\infty$) has spectrum equal to $\overline{f(D)}$ and can be compact only if $f = 0$, because the spectrum of a compact operator must be countable. Hence $g \cdot (\varphi_a - z) = 0$. If $a = 0$, then $h(z) = -z^2$ and we already proved that S must be an analytic Toeplitz operator. If $a \neq 0$, $\varphi_a(z) \neq z$, so that $g = 0$. Then (2.15) gives that

$$S\,u = f \cdot u$$

for $u \in L^2_a$. In particular, $f = S\,1 \in L^2_a$ so that S must be an analytic Toeplitz operator.

<u>Example</u>: For fixed $w \in D$, let $h(z) = k_w(z)$. Then $g \in$ Range $T_{k_w - k_w(\lambda)}$ implies

$$g(z) = \frac{(z-\lambda)(2\overline{w} - \overline{w}^2 z - \overline{w}^2\lambda)}{(1-\overline{w}z)^2(1-\overline{w}\lambda)^2}) f.$$

The numerator is equal to 0 if $z = \lambda$ or $z = \frac{2 - \overline{w}\lambda}{\overline{w}}$. In the latter case

$$|z| \geq \frac{(2 - |w\lambda|)}{|w|} > \frac{1}{|w|} > 1$$

so that Range $T_{k_w - k_w(\lambda)} = \{ g \in L^2_a : g(\lambda) = 0 \}$, and as before, if S commutes with T_{k_w}, it must be an analytic Toeplitz operator.

<u>Example</u>: Let $h(z) = \frac{z-a}{z-b}$, $a \neq b$, $|b| > 1$. We want to find the commutant of T_h. If $g \in$ Range $T_{h-h(\lambda)}$ then

$$g(z) = \frac{(z - \lambda)(a - b)}{(z - b)(\lambda - b)} f.$$

Hence, Range $T_{h-h(\lambda)} = \{ g \in L_a^2 : g(\lambda) = 0 \}$ and $\{ T_h \}' = \{ T_\psi : \psi \in H^\infty \}$.

We conclude this chapter with a question related to Theorem 2.4. Suppose $S \in \mathcal{L}(L_a^2)$ is such that $S\, T_z - T_z\, S \in \mathcal{K}$. Then $S\, T_{z^n} - T_{z^n} S \in \mathcal{K}$, for every n. If $S\, T_{z^n} - T_{z^n} S = 0$ for some n, then Theorem 2.4 shows that S must be an analytic Toeplitz operator. Now, the same assumption $S\, T_z - T_z\, S \in \mathcal{K}$ implies that $S\, T_p - T_p\, S \in \mathcal{K}$ for every polynomial p. If $S\, T_p - T_p\, S = 0$ for some polynomial p (other than z^n), must S be an analytic Toeplitz operator ?

CHAPTER 3

VON NEUMANN ALGEBRAS GENERATED BY T_{z^n}

In the previous chapter we studied the commutant of an operator T_{z^n} acting on L_a^2 and we found its intersection with the algebra $\mathcal{T}$. Now we are interested in characterizing the von Neumann algebra generated by T_{z^n}, denoted by $W^*(T_{z^n})$. Because $W^*(T_{z^n}) = \{ T_{z^n}, T_{z^n}^* \}''$, our problem is again related to the problem of finding commutants of certain operators. In this chapter we will study operators T_{z^n} acting on the Bergman and Hardy spaces and show that von Neumann algebras generated by them have different structure if $n > 1$.

1. T_{z^n} acts on the Bergman space

We will prove the following theorem:

Theorem 3.1. *Let $n \in \mathbb{N}$ be fixed. Then*

$$W^*(T_{z^n}) = \{ T \in \mathcal{L}(L_a^2): T = \begin{pmatrix} T_0 & & & \\ & T_1 & & \\ & & \cdot & \\ & & & T_{n-1} \end{pmatrix}, \textit{ where } L_a^2 = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1},$$

$$T_i : X_i \to X_i \textit{ for } i = 0,1, \ldots , n\text{-}1 \}.$$

This theorem follows easily from the following theorem:

Theorem 3.2. *Let $S \in \mathcal{L}(L_a^2)$, and let $n \in \mathbb{N}$ be fixed. Then $S \in \{ T_{z^n}, T_{z^n}^* \}'$ if and only if*

$$S f = a_0 f_0 + a_1 f_1 + \cdots + a_{n-1} f_{n-1} , a_i \in \mathbb{C}, f = \sum_{k=0}^{n-1} f_i, \; f_i \in X_i .$$

PROOF: Let S be in $\{ T_{z^n}, T^*_{z^n} \}'$. Then $S\, T_{z^n} = T_{z^n} S$ and $S\, T^*_{z^n} = T^*_{z^n} S$. This means that $S, S^* \in \{ T_{z^n} \}'$. From Chapter 2, Theorem 2.4 we know that

$$S f = g_0 f_0 + \frac{g_1}{z} f_1 + \frac{g_2}{z^2} f_2 + \cdots + \frac{g_{n-1}}{z^{n-1}} f_{n-1}$$

and

$$S^* f = \hat{g}_0 f_0 + \frac{\hat{g}_1}{z} f_1 + \frac{\hat{g}_2}{z^2} f_2 + \cdots + \frac{\hat{g}_{n-1}}{z^{n-1}} f_{n-1}$$

for $f \in L^2_a$, where $g_j = S\, z^j$ and $\hat{g}_j = S^* z^j$, $j = 0, 1, \cdots, n$-1. We want to show that all g_j and $\hat{g}_j$ are polynomials.

Let's start with g_0.

$$<g_0, z^{kn}> = <S\ 1, z^{kn}> = <1, S^* z^{kn}> = <1, \hat{g}_0\, z^{kn}>$$

so that $<g_0, z^{kn}> = 0$ if $k > 0$, i. e., $g_0 \perp \{ e_{kn} : k > 0 \}$. Similarly

$$<g_0, z^{kn+1}> = <S\ 1, z^{kn+1}> = <1, S^* z^{kn+1}> = <1, \hat{g}_1\, z^{kn}>$$

so that $<g_0, z^{kn+1}> = 0$ if $k > 0$, i. e. $g_0 \perp \{ e_{kn+1} : k > 0 \}$. Continuing this procedure gives that $g_0 \perp \{ e_{kn+j} : k > 0 \}$ for $j = 0, 1, \cdots, n$-1. Therefore

$$g_0 = \sum_{k=0}^{n-1} \alpha_k e_k$$

i.e., g_0 is a polynomial of degree at most n - 1.

By the exactly same argument, $\hat{g}_0$ is a polynomial of degree at most n - 1.

Now look at g_1. Similarly as before we have

$$<g_1, z^{kn}> = <S\, z, z^{kn}> = <z, S^* z^{kn}> = <z, \hat{g}_0\, z^{kn}>$$

so that $<g_1, z^{kn}> = 0$ if $k > 0$, i. e., $g_1 \perp \{ e_{kn} : k > 0 \}$. Also

$$<g_1, z^{kn+1}> = <S\, z, z^{kn+1}> = <z, S^* z^{kn+1}> = <z, \hat{g}_1\, z^{kn}>$$

so that $<g_1, z^{kn+1}> = 0$ if $k > 0$, i. e., $g_1 \perp \{ e_{kn+1} : k > 0 \}$. Continuing this procedure gives that $g_1 \perp \{ e_{kn+j} : k > 0 \}$ for $j = 0, 1, \cdots, n\text{-}1$. Therefore

$$g_1 = \sum_{k=0}^{n-1} \alpha_k e_k$$

i.e., g_1 is a polynomial of degree at most n - 1. By the same argument, $\hat{g}_1$ is a polynomial of degree $\leq n$ - 1. In this way we can show that all g_j and $\hat{g}_j$ are polynomials of degree at most n -1. Hence,

$$S f = p_0 f_0 + \frac{p_1}{z} f_1 + \frac{p_2}{z^2} f_2 + \cdots + \frac{p_{n-1}}{z^{n-1}} f_{n-1}$$

and

$$S^* f = \hat{p}_0 f_0 + \frac{\hat{p}_1}{z} f_1 + \frac{\hat{p}_2}{z^2} f_2 + \cdots + \frac{\hat{p}_{n-1}}{z^{n-1}} f_{n-1}\ . \qquad (3.1)$$

We will observe these polynomials more closely. Let

$$p_i = a_0^i + a_1^i z + a_2^i z^2 + \ldots + a_{n-1}^i z^{n-1}$$

$$\hat{p}_i = b_0^i + b_1^i z + b_2^i z^2 + \ldots + b_{n-1}^i z^{n-1} \,,$$

for $i = 0, 1, \ldots, n-1$. Let $f_0 = \alpha_0 + \alpha_n z^n + \alpha_{2n} z^{2n} + \ldots$ be an arbitrary function in X_0. Look at the following scalar products:

$$<S f_0, z^{kn}> = <f_0, S^* z^{kn}> = <f_0, \hat{p}_0 \, z^{kn}> = <f_0, b_0^0 \, z^{kn}> = \bar{b}_0^0 \, \alpha_{kn} \frac{\pi}{kn+1}$$

$$<S f_0, z^{kn+1}> = <f_0, S^* z^{kn+1}> = <f_0, \hat{p}_1 \, z^{kn}> = <f_0, b_0^1 \, z^{kn}> = \bar{b}_0^1 \, \alpha_{kn} \frac{\pi}{kn+1}$$

$$\cdots$$

$$<S f_0, z^{kn+n-1}> = <f_0, S^* z^{kn+n-1}> = <f_0, \hat{p}_{n-1} \, z^{kn}> = \bar{b}_0^{n-1} \, \alpha_{kn} \frac{\pi}{kn+1} \quad .$$

Thus

$$S f_0 = \sum_{k=0}^{\infty} (S f_0, e_k) e_k = \sum_{k=0}^{\infty} (S f_0, e_{kn}) e_{kn} +$$

$$\sum_{k=0}^{\infty} (S f_0, e_{kn+1}) e_{kn+1} + \ldots + \sum_{k=0}^{\infty} (S f_0, e_{kn+n-1}) e_{kn+n-1} =$$

$$= \bar{b}_0^0 \sum_{k=0}^{\infty} \alpha_{kn} z^{kn} + \bar{b}_0^1 \sum_{k=0}^{\infty} \frac{kn+2}{kn+1} \alpha_{kn} z^{kn+1} + \bar{b}_0^2 \sum_{k=0}^{\infty} \frac{kn+3}{kn+1} \alpha_{kn} z^{kn+2} + \ldots +$$

$$+ \bar{b}_0^{n-1} \sum_{k=0}^{\infty} \frac{(k+1)n}{kn+1} \alpha_{kn} z^{kn+n-1} \qquad (3.2)$$

On the other hand, $S f_0 = p_0 f_0$, as was shown in (3.1). Now

$$p_0 = \sum_{k=0}^{n-1} (p_0, e_k)e_k = \sum_{k=0}^{n-1} (S\,1, e_k)e_k = \sum_{k=0}^{n-1} \frac{k+1}{\pi}(1, S^* z^k)z^k = \sum_{k=0}^{n-1} \frac{k+1}{\pi}(1, \hat{p}_k)z^k =$$

$$= \sum_{k=0}^{n-1} (k+1)\bar{b}_0^k z^k.$$

Thus

$$S f_0 = (\bar{b}_0^0 + 2\bar{b}_0^1 z + 3\bar{b}_0^2 z^2 + \ldots + n\bar{b}_0^{n-1} z^{n-1}) f_0 =$$

$$= \bar{b}_0^0 \sum_{k=0}^{\infty} \alpha_{kn} z^{kn} + \bar{b}_0^1 \sum_{k=0}^{\infty} 2\alpha_{kn} z^{kn+1} + \bar{b}_0^2 \sum_{k=0}^{\infty} 3\alpha_{kn} z^{kn+2} + \ldots + \bar{b}_0^{n-1} \sum_{k=0}^{\infty} n\alpha_{kn} z^{kn+n-1} \quad (3.3)$$

Equating (3.2) and (3.3) , we get

$$\bar{b}_0^1 \frac{kn+2}{kn+1} \alpha_{kn} = \bar{b}_0^1\, 2\alpha_{kn}, \quad \bar{b}_0^2 \frac{kn+3}{kn+1} \alpha_{kn} = \bar{b}_0^2\, 3\alpha_{kn}, \quad \bar{b}_0^{n-1} \frac{(k+1)n}{kn+1} \alpha_{kn} = \bar{b}_0^{n-1}\, n\alpha_{kn}$$

for all k. The function f_0 was arbitrary, so that the above equalities are true for all α's. This forces

$$b_0^1 = b_0^2 = \cdots = b_0^{n-1} = 0$$

so that $p_0 = \bar{b}_0^0 =$ constant. Therefore $S f_0 = \text{constant} \cdot f_0$.

Let $f_1 = \alpha_1 z + \alpha_{n+1} z^{n+1} + \alpha_{2n+1} z^{2n+1} + \ldots$ be an arbitrary function in X_1. Look at the following scalar products:

$$\langle S f_1, z^{kn} \rangle = \langle f_1, S^* z^{kn} \rangle = \langle f_1, \hat{p}_0\, z^{kn} \rangle = \langle f_1, b_1^0\, z^{kn+1} \rangle = \bar{b}_1^0 \alpha_{kn+1} \frac{\pi}{kn+2}$$

$$\langle S f_1, z^{kn+1} \rangle = \langle f_1, S^* z^{kn+1} \rangle = \langle f_1, \hat{p}_1\, z^{kn} \rangle = \langle f_1, b_1^1\, z^{kn+1} \rangle = \bar{b}_1^1\, \alpha_{kn+1} \frac{\pi}{kn+2}$$

...

$$<Sf_1, z^{kn+n-1}> = <f_1, S^* z^{kn+n-1}> = <f_1, \hat{p}_{n-1} z^{kn}> = \bar{b}_1^{n-1} \alpha_{kn+1} \frac{\pi}{kn+2} \quad .$$

Then

$$Sf_1 = \sum_{k=0}^{\infty} (Sf_1, e_{kn})e_{kn} + \sum_{k=0}^{\infty} (Sf_1, e_{kn+1})e_{kn+1} + \ldots + \sum_{k=0}^{\infty} (Sf_1, e_{kn+n-1})\, e_{kn+n-1} =$$

$$= \bar{b}_1^0 \sum_{k=0}^{\infty} \frac{kn+1}{kn+2} \alpha_{kn+1} z^{kn} + \bar{b}_1^1 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} + \bar{b}_1^2 \sum_{k=0}^{\infty} \frac{kn+3}{kn+2} \alpha_{kn+1} z^{kn+2} + \ldots +$$

$$+ \bar{b}_1^{n-1} \sum_{k=0}^{\infty} \frac{(k+1)n}{kn+2} \alpha_{kn+1} z^{kn+n-1}. \qquad (3.4)$$

On the other hand $Sf_1 = \frac{p_1}{z} f_1$, so that

$$p_1 = \sum_{k=0}^{n-1} (p_1, e_k)e_k = \sum_{k=0}^{n-1} (S\, z, e_k)e_k$$

$$= \sum_{k=0}^{n-1} \frac{k+1}{\pi} (z, S^* z^k) z^k = \sum_{k=0}^{n-1} \frac{k+1}{\pi} (z, \hat{p}_k) z^k = \sum_{k=0}^{n-1} \frac{k+1}{2} \bar{b}_1^k z^k.$$

Thus

$$Sf_1 = \frac{p_1}{z} f_1 = \left(\frac{\bar{b}_1^0}{2} + \bar{b}_1^1 z + 3\frac{\bar{b}_1^2}{2} z^2 + \ldots + n\frac{\bar{b}_1^{n-1}}{2} z^{n-1}\right)\frac{f_1}{z} =$$

$$= \bar{b}_1^0 \sum_{k=0}^{\infty} \tfrac{1}{2} \alpha_{kn+1} z^{kn} + \bar{b}_1^1 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} + \bar{b}_1^2 \sum_{k=0}^{\infty} \tfrac{3}{2} \alpha_{kn+1} z^{kn+2} + \ldots$$

$$+ \bar{b}_1^{n-1} \sum_{k=0}^{\infty} \tfrac{n}{2} \alpha_{kn+1} z^{kn+n-1} \,. \tag{3.5}$$

Comparing (3.4) and (3.5) we get (similarly as above)

$$b_1^0 = b_1^2 = \cdots = b_1^{n-1} = 0$$

so that $p_1 = \bar{b}_1^1 z = \text{constant} \cdot z$. Therefore $S f_1 = \text{constant} \cdot f_1$.

Contining in this way, we get $p_k = \text{constant} \cdot z^k$, so that $S f_k = \text{constant} \cdot f_k$, where

$k = 0, 1, \ldots, n-1$. Hence

$$S f = a_0 f_0 + a_1 f_1 + \ldots + a_{n-1} f_{n-1} \ , \quad a_i \in \mathbf{C}.$$

Thus, we have proved that $\{ T_{z^n}, T_{z^n}^* \}'$ is a subset of

$$\{ S \in \mathcal{L}(L_a^2) : S f = a_0 f_0 + a_1 f_1 + \ldots + a_{n-1} f_{n-1} \ , \quad a_i \in \mathbf{C}, \ f = \sum_{k=0}^{n-1} f_i \ \}.$$

On the other hand, if S belongs to this set, then

$$S\, T_{z^n} f = S\, T_{z^n} (f_0 + f_1 + \ldots + f_{n-1}) = S\, (z^n f_0 + z^n f_1 + \ldots + z^n f_{n-1}) =$$

$$= a_0 z^n f_0 + a_1 z^n f_1 + \ldots + a_{n-1} z^n f_{n-1} \ (\textit{since } z^n f_i \in X_i) = z^n S f = T_{z^n} S f$$

for each $f \in L^2_a$. Since $S^* f = \bar{a}_0 f_0 + \bar{a}_1 f_1 + \ldots + \bar{a}_{n-1} f_{n-1}$, then $S^* T_{z^n} = T_{z^n} S^*$ so that $S \in \{ T_{z^n}, T^*_{z^n} \}'$. Hence, we proved Theorem 3.2.

The second step in our study is to find the commutant of

$$\{ S \in \mathcal{L}(L^2_a) : S f = a_0 f_0 + a_1 f_1 + \ldots + a_{n-1} f_{n-1} \ , \ a_i \in \mathbf{C}, \ f = \sum_{k=0}^{n-1} f_i \}, \text{ i.e., } W^*(T_{z^n}).$$

It is readily seen that

$$W^*(T_{z^n}) = \{ T \in \mathcal{L}(L^2_a) : T = \begin{pmatrix} T_0 & & & \\ & T_1 & & \\ & & \cdot & \\ & & & T_{n-1} \end{pmatrix}, \text{ where } L^2_a = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1},$$

$$T_i : X_i \to X_i \text{ for } i = 0,1, \ldots, n-1 \}$$

and Theorem 3.1 is proved.

2. T_{z^n} acts on the Hardy space

We will prove a theorem that is analogous to the Theorem 3.1 in the Hardy space case. Here is our result. We don't know if this theorem is already known.

Theorem 3.3. *Let $n \in \mathbb{N}$ be fixed. Then*

$$W^*(T_{z^n}) = \{\, T \in \mathcal{L}(H^2(\partial D)) : T = \begin{pmatrix} T_0 & & & \\ & M_z T_0 M_{\bar z} & & \\ & & \cdot & \\ & & & M_{z^{n-1}} T_0 M_{\bar z^{n-1}} \end{pmatrix}, \text{ where}$$

$H^2(\partial D) = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$, $T_0 : X_0 \to X_0$, M_z and $M_{\bar z}$ are multiplication operators$\}$.

This theorem will follow from the following theorem:

Theorem 3.4. *Let $S \in \mathcal{L}(H^2(\partial D))$, and let $n \in \mathbb{N}$ be fixed. Then $S \in \{\, T_{z^n}, T_{z^n}^* \,\}'$ if and only if*

$$Sf = p_0 f_0 + \frac{p_1}{z} f_1 + \frac{p_2}{z^2} f_2 + \cdots + \frac{p_{n-1}}{z^{n-1}} f_{n-1},$$

where $p_0, p_1, \ldots, p_{n-1}$ are polynomials of degree at most $n - 1$, where $f = \sum_{k=0}^{n-1} f_i$, with $f_i \in X_i$.

PROOF: Suppose $S \in \{\, T_{z^n}, T_{z^n}^* \,\}'$. By the same argument as for the Bergman space,

$$Sf = p_0 f_0 + \frac{p_1}{z} f_1 + \frac{p_2}{z^2} f_2 + \cdots + \frac{p_{n-1}}{z^{n-1}} f_{n-1} \tag{3.6}$$

where $p_0, p_1, \dots, p_{n-1}$ are polynomials of degree at most $n - 1$.

Suppose, conversely, that S is an operator of the form (3.6).

Claim: $S^* f = \hat{p}_0 f_0 + \frac{\hat{p}_1}{z} f_1 + \frac{\hat{p}_2}{z^2} f_2 + \dots + \frac{\hat{p}_{n-1}}{z^{n-1}} f_{n-1}$, $\hat{p}_i$ are polynomials.

Let $p_i = a_0^i + a_1^i z + a_2^i z^2 + \dots + a_{n-1}^i z^{n-1}$, and let $f_0 = \alpha_0 + \alpha_n z^n + \alpha_{2n} z^{2n} + \dots$ be an arbitrary function in X_0. Then

$$<S^* f_0, z^{kn}> = <f_0, S\, z^{kn}> = <f_0, p_0 z^{kn}> = \bar{a}_0^0\, \alpha_{kn}$$

...

$$<S^* f_0, z^{kn+n-1}> = <f_0, S\, z^{kn+n-1}> = <f_0, p_{n-1} z^{kn}> = \bar{a}_0^{n-1}\, \alpha_{kn}$$

Then

$$S^* f_0 = \sum_{k=0}^{\infty} (S^* f_0, e_{kn}) e_{kn} + \sum_{k=0}^{\infty} (S^* f_0, e_{kn+1}) e_{kn+1} + \dots + \sum_{k=0}^{\infty} (S^* f_0, e_{kn+n-1}) e_{kn+n-1} =$$

$$= \bar{a}_0^0 \sum_{k=0}^{\infty} \alpha_{kn} z^{kn} + \bar{a}_0^1 \sum_{k=0}^{\infty} \alpha_{kn} z^{kn+1} + \bar{a}_0^2 \sum_{k=0}^{\infty} \alpha_{kn} z^{kn+2} + \dots + \bar{a}_0^{n-1} \sum_{k=0}^{\infty} \alpha_{kn} z^{kn+n-1}$$

$$= (\bar{a}_0^0 + \bar{a}_0^1 z + \bar{a}_0^2 z^2 + \dots + \bar{a}_0^{n-1} z^{n-1}) f_0 = \hat{p}_0 f_0 .$$

Let $f_1 = \alpha_1 z + \alpha_{n+1} z^{n+1} + \alpha_{2n+1} z^{2n+1} + \dots$ be an arbitrary function in X_1. Look at the following scalar products:

$$<S^* f_1, z^{kn}> = <f_1, S\, z^{kn}> = <f_1, p_0 z^{kn}> = \bar{a}_1^0\, \alpha_{kn+1} .$$

...

$$<S^* f_1, z^{kn+n-1}> = <f_1, S z^{kn+n-1}> = <f_1, p_{n-1} z^{kn}> = \bar{a}_1^{n-1} \alpha_{kn+1}$$

Thus

$$S^* f_1 = \sum_{k=0}^{\infty} (S^* f_1, e_{kn}) e_{kn} + \sum_{k=0}^{\infty} (S^* f_1, e_{kn+1}) e_{kn+1} + \ldots + \sum_{k=0}^{\infty} (S^* f_1, e_{kn+n-1}) e_{kn+n-1} =$$

$$= \bar{a}_1^0 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn} + \bar{a}_1^1 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} + \bar{a}_1^2 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+2} + \ldots + \bar{a}_1^{n-1} \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+n-1}$$

$$= \frac{\bar{a}_1^0}{z} \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} + \bar{a}_1^1 \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1}$$

$$+ \bar{a}_1^2 z \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} + \ldots + \bar{a}_1^{n-1} z^{n-2} \sum_{k=0}^{\infty} \alpha_{kn+1} z^{kn+1} =$$

$$= \left(\frac{\bar{a}_1^0}{z} + \bar{a}_1^1 + \bar{a}_1^2 z + \ldots + \bar{a}_1^{n-1} z^{n-2} \right) f_1 = \frac{\hat{p}_1}{z} f_1 .$$

If we continue in this way, we'll get $S^* f_i = \frac{\hat{p}_i}{z^i} f_i$, for $i = 0, 1, \ldots, n-1$, where $\hat{p}_i$ are polynomials of degree at most $n - 1$. Hence we have proved the claim.

Now, it is easy to see that S and S^* both commute with T_{z^n} and we proved the theorem.

Our goal is to determine $\{ T_{z^n}, T_{z^n}^* \}''$. Suppose that $T \in \{ T_{z^n}, T_{z^n}^* \}''$. Then T commutes with all operators S of the form

$$S f = p_0 f_0 + \frac{p_1}{z} f_1 + \frac{p_2}{z^2} f_2 + \cdots + \frac{p_{n-1}}{z^{n-1}} f_{n-1} ,$$

where $p_0, p_1, \dots, p_{n-1}$ are polynomials, by Theorem 3.4. In particular, T commutes with operators of the form

$$Sf = a_0 f_0 + a_1 f_1 + \dots + a_{n-1} f_{n-1}, \quad a_i \in \mathbf{C}.$$

Thus $T = \begin{pmatrix} T_0 & & & \\ & T_1 & & \\ & & \cdot & \\ & & & T_{n-1} \end{pmatrix}$, where $H^2(\partial D) = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$ and $T_i : X_i \to X_i$.

Choose S such that $Sf = p_0 f_0$, for any $f \in H^2(\partial D)$ and such that

$$p_0 = a_0^0 + a_1^0 z + a_2^0 z^2 + \ \dots + a_{n-1}^0 z^{n-1}.$$

Thus

$$STf = p_0\,(T_0 f_0) = a_0^0 (T_0 f_0) + a_1^0 z (T_0 f_0) + a_2^0 z^2 (T_0 f_0) + \dots + a_{n-1}^0 z^{n-1} (T_0 f_0).$$

On the other hand, this is equal to

$$TSf = T\,(p_0 f_0) = a_0^0\,(T_0 f_0) + a_1^0\,T_1\,(z f_0) + a_2^0\,T_2\,(z^2 f_0) + \dots + a_{n-1}^0\,T_{n-1}\,(z^{n-1} f_0).$$

After comparing these two expressions, we get

$$z(T_0 f_0) = T_1\,(z f_0),$$
$$z^2(T_0 f_0) = T_2\,(z^2 f_0),$$
$$\dots$$
$$z^{n-1}(T_0 f_0) = T_{n-1}\,(z^{n-1} f_0),$$

or, using multiplication operators, we obtain

$$M_z T_0 = T_1 M_z /X_0$$
$$M_{z^2} T_0 = T_2 M_{z^2}/X_0$$
$$\dots$$
$$M_{z^{n-1}} T_0 = T_{n-1} M_{z^{n-1}}/X_0 .$$

Let $f_1 = \alpha_1 z + \alpha_{n+1}z^{n+1} + \alpha_{2n+1}z^{2n+1} + \dots = z(\alpha_1 + \alpha_{n+1}z^n + \alpha_{2n+1}z^{2n} + \dots)$ be an arbitrary function in X_1. Denote $\hat{f_1} = \alpha_1 + \alpha_{n+1}z^n + \alpha_{2n+1}z^{2n} + \dots \in X_0$. Hence,

$$T_1 f_1 = T_1 (z\hat{f_1}) = T_1 M_z /X_0\hat{f_1} = M_z T_0 \hat{f_1} = M_z T_0 M_{\bar{z}} f_1$$

so that

$$T_1 = M_z T_0 M_{\bar{z}}.$$

Let $f_2 = \alpha_2 z^2 + \alpha_{n+2}z^{n+2} + \alpha_{2n+2}z^{2n+2} + \dots = z^2(\alpha_2 + \alpha_{n+2}z^n + \alpha_{2n+2}z^{2n} + \dots)$ be an arbitrary function in X_2. Denote $\hat{f_2} = \alpha_1 + \alpha_{n+1}z^n + \alpha_{2n+1}z^{2n} + \dots \in X_0$. As before, we have

$$T_2 f_2 = T_2 (z^2\hat{f_2}) = T_2 M_{z^2}/X_0\hat{f_2} = M_{z^2} T_0 \hat{f_2} = M_{z^2} T_0 M_{\bar{z}^2} f_2,$$

and therefore

$$T_2 = M_{z^2} T_0 M_{\bar{z}^2}.$$

Continuing this way, we finally get

$$T_{n-1} = M_{z^{n-1}}\, T_0\, M_{\bar{z}^{n-1}}.$$

Thus

$$T = \begin{pmatrix} T_0 & & & \\ & M_z T_0 M_{\bar{z}} & & \\ & & \cdot & \\ & & & M_{z^{n-1}} T_0 M_{\bar{z}^{n-1}} \end{pmatrix}$$

Conversely, suppose T is of the above form. Let S be as in (3.6). We want to show that T commutes with S. First,

$$S\,T\,f = ST\,(f_0 + f_1 + \cdots + f_{n-1}) = S\,(T_0 f_0 + M_z\, T_0\, M_{\bar{z}} f_1 + \cdots + M_{z^{n-1}}\, T_0\, M_{\bar{z}^{n-1}} f_{n-1}) =$$

$$p_0(T_0 f_0) + p_1(T_0\, M_{\bar{z}} f_1) + \cdots + p_{n-1}(T_0\, M_{\bar{z}} f_{n-1}\,). \tag{3.7}$$

On the other hand,

$$TS\,f = T\,(p_0 f_0 + \frac{p_1}{z} f_1 + \frac{p_2}{z^2} f_2 + \cdots + \frac{p_{n-1}}{z^{n-1}} f_{n-1}) =$$

$$= T_0\,(a_0^0 f_0 + a_0^1\, \bar{z} f_1 + a_0^2\, \bar{z}^2 f_2 + \ldots + a^{n-1}_0\; \bar{z}^{n-1} f_{n-1}) +$$

$$+ M_z\, T_0\, M_{\bar{z}}\,(a_1^0\, z f_0 + a_1^1 f_1 + a_1^2\, \bar{z} f_2 + \ldots + a^{n-1}_1\, \bar{z}^{n-2} f_{n-1}) +$$

$$\ldots$$

$$+ M_{z^{n-1}}\, T_0\, M_{\bar{z}^{n-1}}(a_{n-1}^0\, z^{n-1} f_0 + a_{n-1}^1\, z^{n-2} f_1 + a_{n-1}^2\, z^{n-3} f_2 + \ldots + a_{n-1}^{n-1} f_{n-1}) =$$

$$a_0^0(T_0 f_0) + a_0^1(T_0\, M_{\bar{z}} f_1) + \cdots + a^{n-1}_0(T_0\, M_{\bar{z}^{n-1}} f_{n-1}) +$$

$$a_1^0\, z(T_0 f_0) + a_1^1 z(T_0\, M_{\bar{z}} f_1) + \cdots + a^{n-1}_1 z(T_0\, M_{\bar{z}^{n-1}} f_{n-1}) +$$

$$\ldots$$

$$a_{n-1}^{0}\, z^{n-1}(T_0 f_0) + a_{n-1}^{1} z^{n-1}(T_0\, M_{\bar{z}} f_1) + \cdots + a_{n-1}^{n-1} z^{n-1}(T_0\, M_{\bar{z}^{n-1}} f_{n-1}) =$$

$$p_0(T_0 f_0) + p_1(T_0\, M_{\bar{z}} f_1) + \cdots + p_{n-1}(T_0\, M_{\bar{z}} f_{n-1}\,). \tag{3.8}$$

Comparing (3.7) and (3.8), we get that

$$S\,T = T\,S.$$

Hence $T \in \{\ T_{z^n}, T_{z^n}^{*}\ \}''$. Thus, we proved the following:

$$T \in \{\ T_{z^n}, T_{z^n}^{*}\ \}'' \text{ if and only if } T = \begin{pmatrix} T_0 & & & \\ & M_z T_0 M_{\bar{z}} & & \\ & & \cdot & \\ & & & M_{z^{n-1}} T_0 M_{\bar{z}^{n-1}} \end{pmatrix}.$$

Therefore, we proved Theorem 3.3.

If we compare theorems 3.1 and 3.3 we see that in both cases operators in $W^*(T_{z^n})$ have diagonal matrices. In the Bergman space case, each diagonal entry is an arbitrary operator T_i, while on the Hardy space each diagonal element is of the form $M_{z^{i-1}} T_0 M_{\bar{z}^{i-1}}$, for some T_0. The difference comes from the coefficients of the orthonormal basis in the Bergman space.

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