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Kimberly Kay Johannes Kinateder

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STRONG MARKOV PROPERTIES FOR MARKOV RANDOM FIELDS

by

Kimberly Kay Johannes Kinateder

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Abstract

Strong Markov Properties for Markov Random Fields

by

Kimberly Kay Johannes Kinateder

Markov properties and strong Markov properties for random fields are defined and discussed. Special attention is given to those defined by I. V. Evstigneev.

Various definitions of measurability for set-valued functions have been defined. These definitions are shown to be equivalent to each other for compact domain-valued functions, called random domains. The strong Markov nature of Markov random fields with respect to random domains such as $[0, \tau_1]$ and $[\tau_1, \tau_2]$ are explored, where τ_1 and τ_2 are stopping times. This concept is extended to higher dimensions by introducing an extension of stopping times called membranes. A special case of this extension is shown to generalize a recent work of Merzbach and Nualart.

Finally, the so-called corner Markov and strong corner Markov properties are introduced, and the strong corner Markov property is proven to hold under some conditions which include a Cairoli-Walsh (F4) type of condition. The strong Markov nature of reciprocal Markov processes is explored using techniques of Stroock and Varadahn.

This thesis is dedicated to the glory of God,

in Whom all things are possible. Philippians 4:13.

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Chapter 1

Introduction

The study of Markov properties for random fields was initiated by Lévy [Lev48]. McKean [McK63], Molchan [Mol71], Pitt [Pit71], Kallianpur and Mandrekar [KM74], and Kunch [Kun79] studied necessary and sufficient conditions for Markov properties of Gaussian random fields. In [Man83], Markov properties for general random fields were studied. Evstigneev initiated the study of strong Markov properties in multi-dimensions by introducing Markov times in [Evs77] and proposed a necessary and sufficient condition (splitting) for this strong Markov property in [Evs82]. In [Evs88], Evstigneev presented a nonanticipating sufficient condition for a Markov random field to have his strong Markov property with respect to a set-valued random function (specifically, random domains). We shall refer to this condition as (2.6). Rozanov [Roz82] also explored set-valued random functions and strong Markov properties for multi-dimensions. This study has found applications to various fields ([Dud82], [Nel73], [Sim74]).

For $\mathcal{R}_+^1$, our purpose is to systematically study Evstigneev's strong Markov property for random domains related to stopping times. In order to extend these results to $\mathcal{R}_+^d$ with $d > 1$, we introduce random membranes as an analogue of stopping times and study Evstigneev's strong Markov property with respect to random domains related to random membranes. As an example in $\mathcal{R}_+^2$, we show that the so-called decreasing stopping lines occuring in a recent work of Merzbach and Nualart [MN90] on point processes are a special type of random membrane that satisfies condition (2.6). [MN90] presents another strong Markov property and we show that, under some natural assumptions, condition (2.6) is sufficient for a point process to have this strong Markov property with respect to a decreasing stopping line.

In Cairoli and Walsh [CW78], a one-dimensional property (F4) is used to investigate two-dimensional Markov properties. In Chapter 4, we study one-dimensional strong Markov properties and relate them to two-dimensional strong Markov properties. Finally, the study of strong Markov properties for reciprocal Gaussian processes is undertaken. The results obtained present a good beginning to this study, and some methods from [Str87] are clarified. Since reciprocal processes play a major role in various applied problems, it appears that the study of strong Markov properties for reciprocal processes may have a significant impact on applications.

Chapter 2

Definitions and Preliminary Theorems

2.1 Markov properties

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Throughout this thesis, we assume that all sub-σ-algebras of $\mathcal{F}$ introduced will contain all sets of measure 0 from $\mathcal{F}$. Since our goal is to determine when certain random fields have a strong Markov property, we need to understand the simple notion of a random field.

Definition 2.1 Let $\xi = \{\xi_t\}_{t \in \mathcal{R}_+^d}$ be a family of random variables defined on $(\Omega, \mathcal{F}, P)$. We call ξ a random field.

Example 2.1 One of the special cases of random fields that we study is the point process. In [MN90], the point process for $d = 2$ is defined as follows. Let N be a random measure on $\mathcal{R}_+^2$ such that $N(\omega)$ is a finite or countable sum of Dirac measures on random and different points $Z_i(\omega)$, $i = 0, 1, \ldots$. We also assume that

$$N(\{z = (z_1, z_2) \in \mathcal{R}_+^2 : 0 \le z_i \le t_i,\ i = 1, 2\}) < \infty,$$

for all $t = (t_1, t_2) \in \mathcal{R}_+^2$ and that the measure of the axes is zero. The point process ξ is then defined for $t \in \mathcal{R}_+^2$ by

$$\xi_t = N(\{z = (z_1, z_2) \in \mathcal{R}_+^2 : 0 \leq z_i \leq t_i, i = 1, 2\}).$$

Associated with a random field ξ we define σ-algebras

$$\mathcal{G}_A \equiv \sigma(\xi_t, t \in A)$$

for $A \subseteq \mathcal{R}_+^d$ and germ-field σ-algebras

$$\mathcal{F}_A \equiv \bigcap_{\epsilon>0} \mathcal{G}_{A^\epsilon}$$

for $A \subseteq \mathcal{R}_+^d$, where

$$A^\epsilon = \{t \in \mathcal{R}_+^d : d(t, A) < \epsilon\}.$$

Throughout we assume $\mathcal{G}_{\mathcal{R}_+^d} = \mathcal{F}$.

Using the above σ-algebras, we can express several different Markov properties for a random field ξ. For this we need the concept of conditional independence.

Definition 2.2 Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{G}$ be sub σ-fields of $\mathcal{F}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are <u>conditionally independent</u> given $\mathcal{G}$, if

$$P(A \cap B \mid \mathcal{G}) = P(A \mid \mathcal{G}) P(B \mid \mathcal{G}), \quad \text{for all } A \in \mathcal{A} \text{ and } B \in \mathcal{B},$$

where $P(\cdot \mid \mathcal{G})$ is the conditional probability given $\mathcal{G}$. We denote this by $\mathcal{A} \perp\!\!\!\perp \mathcal{B} \mid \mathcal{G}$. In our case, $P(\cdot \mid \mathcal{G})$ will be an equivalence class.

The first Markov property that we shall introduced was proposed by Evstigneev [Evs88]. A subset $A \subseteq \mathcal{R}_+^d$ is called a <u>domain</u> if $A \subseteq \overline{A^o}$. Let T denote the set of all compact domains in $\mathcal{R}_+^d$. Assume $\emptyset \in T$. A random field ξ is said to be Markov with respect to $B \in T$ if for all $A, C \in T$ with $A \subseteq B \subseteq C$,

$$\mathcal{F}_B \perp\!\!\!\perp \mathcal{F}_{\overline{A^c}} \mid \mathcal{F}_{\overline{A^c \cap B}}.$$

Definition 2.3 We shall say that a random field ξ is Markov if ξ is Markov with respect to $B \in T$, for all $B \in T$.

Given this definition, a natural question arises. When $d = 1$, how does this Markov property relate to the Germ-field Markov Property (denoted GFMP),

$$(\mathcal{F}_{[0,t]} \perp\!\!\!\perp \mathcal{F}_{[t,\infty)} \mid \mathcal{F}_{\{t\}}, \text{ for all } t \geq 0)$$

and the classical Markov property

$$(\mathcal{G}_{[0,t]} \perp\!\!\!\perp \mathcal{G}_{[t,\infty)} \mid \mathcal{G}_{\{t\}}, \text{ for all } t \geq 0) \qquad ?$$

The answer is given below.

Lemma 2.1 *If ξ has the classical Markov property, then ξ is Markov.*

Before proving the claim, we will state a few results dealing with conditional independence. Proofs for Propositions 2.1 through 2.4 and Theorem 2.1 can be found in [Man83], pp.163-167. Proof of the Proposition 2.5 is given in [Roz82], p. 58.

Proposition 2.1 *If $\mathcal{A}, \mathcal{B}, \mathcal{G}'$, and $\mathcal{G}$ are sub-σ-algebras of $\mathcal{F}$ with $\mathcal{A} \perp\!\!\!\perp \mathcal{B} \mid \mathcal{G}$ and $\mathcal{G}' \subseteq \mathcal{G} \vee \mathcal{B}$, then $\mathcal{A} \perp\!\!\!\perp \mathcal{G}' \mid \mathcal{G}$.*

Proposition 2.2 *Let $\{O_i, i \in I\}$ be disjoint open subsets of X and $U = \bigcup_{i \in I} O_i$. If*

$$\mathcal{G}_{\overline{O}_i} \perp\!\!\!\perp \mathcal{G}_{\overline{O}_i^c} \mid \mathcal{G}_{\partial O_i},$$

for all $i \in I$, then

$$\mathcal{G}_{\overline{U}} \perp\!\!\!\perp \mathcal{G}_{\overline{U}^c} \,|\, \mathcal{G}_{\partial U}.$$

We call the latter condition the simple Markov property on U.

Proposition 2.3 *ξ has the simple Markov property on all open subsets of X if and only if ξ has the simple Markov property on all open intervals in X.*

Theorem 2.1 *If ξ has the simple Markov property with respect to a set A, then*

$$\bigcap_{\text{open } O \supseteq \overline{A}} \mathcal{G}_O \perp\!\!\!\perp \bigcap_{\text{open } O \supseteq \overline{A^c}} \mathcal{G}_O \,\Big|\, \bigcap_{\text{open } O \supseteq \overline{\partial A}} \mathcal{G}_O.$$

That is, ξ has the the germ-field Markov property (GMFP) on A.

Proposition 2.4 *ξ has the GFMP on a set A if and only if for every open set $\tilde{O} \supseteq \partial A$,*

$$\mathcal{G}_A \perp\!\!\!\perp \mathcal{G}_{A^c} \,|\, \mathcal{G}_{\tilde{O}}.$$

Proposition 2.5 *If $\{\mathcal{G}_n\}_n$ are monotonically decreasing sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{A} \perp\!\!\!\perp \mathcal{B} \,|\, \mathcal{G}_n$ for all $n \in \mathcal{N}$, then*

$$\mathcal{A} \perp\!\!\!\perp \mathcal{B} \,\Big|\, \bigcap_{n=1}^{\infty} \mathcal{G}_n.$$

Proof of Lemma 2.1. Assume ξ has the classical Markov property. Then

$$\mathcal{G}_{[0,t]} \perp\!\!\!\perp \mathcal{G}_{(t,\infty)} \,|\, \sigma(\xi_t), \text{ for all } t \geq 0.$$

From $\mathcal{G}_{[0,t]} = \mathcal{G}_{[0,t)} \vee \sigma(\xi_t)$, $\mathcal{G}_{[t,\infty)} = \mathcal{G}_{(t,\infty)} \vee \sigma(\xi_t)$, and Proposition 2.1, we get that ξ has the simple Markov property on sets of the form $[0,t)$ and (t,∞). By Proposition 2.2, ξ has the simple Markov property on sets of the form $[0,s) \cup (t,\infty)$, $s < t$. By Proposition 2.1,

$$\mathcal{G}_{[0,s]\cup[t,\infty)} \perp\!\!\!\perp \mathcal{G}_{(s,t)} \,|\, \mathcal{G}_{\{s,t\}}, \text{ for all } t > s;$$

that is, ξ has the simple Markov property on all open intervals in $(0,\infty)$. Proposition 2.3 now gives us the simple Markov property on all open sets in $(0,\infty)$. The GFMP on all open sets follows from Theorem 2.1. Now let $a, b \in T$ with $a \subseteq b$. Our goal is to show $\mathcal{F}_{\overline{a^c}} \perp\!\!\!\perp \mathcal{F}_b \,|\, \mathcal{F}_{\overline{a^c \cap b}}$. Let $\epsilon > 0$, $A = b^o$, $\tilde{O} = (\overline{a^c \cap b})^\epsilon$, and apply Proposition 2.4 to get $\mathcal{G}_{b^o} \perp\!\!\!\perp \mathcal{G}_{(b^o)^c} \,|\, \mathcal{G}_{(\overline{a^c \cap b})^\epsilon}$. Note that $\mathcal{G}_{b^\epsilon} = \mathcal{G}_{b^o} \vee \mathcal{G}_{(\overline{a^c \cap b})^\epsilon}$ and $\mathcal{G}_{\overline{(a^c)}^\epsilon} = \mathcal{G}_{(b^o)^c} \vee \mathcal{G}_{(\overline{a^c \cap b})^\epsilon}$. Hence $\mathcal{G}_{\overline{(a^c)}^\epsilon} \perp\!\!\!\perp \mathcal{G}_{b^\epsilon} \,|\, \mathcal{G}_{(\overline{a^c \cap b})^\epsilon}$ by Proposition 2.1. Finally, we apply Proposition 2.5 and the facts that $\mathcal{F}_{\overline{a^c}} \subseteq \mathcal{G}_{\overline{(a^c)^\epsilon}}$, for all $\epsilon > 0$ and $\mathcal{F}_b \subseteq \mathcal{G}_{b^\epsilon}$, for all $\epsilon > 0$ to get

$$\mathcal{F}_{\overline{a^c}} \perp\!\!\!\perp \mathcal{F}_b \,|\, \mathcal{F}_{\overline{a^c \cap b}}.$$

Therefore, ξ is Markov with respect to all $a, b \in T$ with $a \subseteq b$. □

The converse of the above claim is not true. A counterexample for this comes from $A = (0,t)$ and $\ddot{X}(t) + X(t) = B(dt)$, $t \geq 0$, $X(0) = \dot{X}(0) = 0$, where B is Brownian motion. By [Doo44], $\{X_t, t > 0\}$ has the GFMP for all A, but it does not have the simple Markov property for all A.

Another Markov property for $\mathcal{R}^1_+$ that we need is the reciprocal Markov property.

Definition 2.4. ξ has the reciprocal Markov property if

$$\mathcal{G}_{[s,t]} \perp\!\!\!\perp \mathcal{G}_{[0,s]\cup[t,\infty)} \,|\, \mathcal{G}_{\{s,t\}}$$

for all $0 \leq s < t < \infty$.

We now turn our attention toward strong Markov properties. We start by introducing stopping times.

Definition 2.5. A nonnegative random variable τ is called a stopping time if

$$\{\tau \leq u\} \in \mathcal{G}_{[0,u]}, \quad \text{for all } u \geq 0$$

and a two-sided stopping time if

$$\{u_1 \leq \tau \leq u_2\} \in \mathcal{G}_{[u_1,u_2]}, \quad \text{for all } u_2 \geq u_1 \geq 0.$$

Observe that τ is a stopping time if and only if $\{\omega : [0, \tau(\omega)] \subseteq [0, u]\} \in \mathcal{G}_{[0,u]}$ for all $u \geq 0$.

In order to generalize the concept of a stopping time to higher dimensions, we need to define measurability of set-valued functions ([Evs77], [Evs88], [MN90], [Roz82]). Define a σ- algebra $\mathcal{T}$ on T by letting

$$\mathcal{T} = \sigma(\{A \in T : A \subseteq U\} \ : U \text{ is an open subset of } \mathcal{R}^d_+).$$

It is easy to prove that $\mathcal{T}$ can also be generated by sets of the form $\{A \in T : A \subseteq U\}$ where U ranges over all Borel subsets of $\mathcal{R}^d_+$. For further information, see [Evs88].

Definition 2.6. A measurable map from $(\Omega, \mathcal{F})$ to $(T, \mathcal{T})$ is called a random domain [Evs88].

That is, a random domain is a compact domain-valued function on Ω such that

$$\{Q \in D\} \in \mathcal{F}, \quad \text{for all } D \in \mathcal{T} \tag{2.1}$$

It is worth noting that there are other ways to express this measurability (2.1).

Lemma 2.2 *If $Q(\omega) \in T$ for all $\omega \in \Omega$, then the following conditions are equivalent to* (2.1)

$$\{Q \cap A \neq \emptyset\} \in \mathcal{F}, \quad \textit{for all } A \in T \tag{2.2}$$

$$\{Q \subseteq A\} \in \mathcal{F} \quad \textit{for all } A \in T. \tag{2.3}$$

$$\{Q \subseteq B\} \in \mathcal{F}, \quad \textit{for all open subsets } B \in \mathcal{R}^d_+. \tag{2.4}$$

Proof. (2.1), (2.2), and (2.3) are equivalent using [Evs88], p. 31. The equivalence of (2.3) and (2.4) follows from the relations

$$\{Q \subseteq B\} = \bigcup_{n=1}^{\infty} \{Q \subseteq \overline{B^{-1/n}}\}$$

for any open domain B in $\mathcal{R}^d_+$ where $B^{-1/n} = \{t \in \mathcal{R}^d_+ : d(t, B^c) > \frac{1}{n}\}$ and

$$\{Q \subseteq A\} = \bigcap_{n=1}^{\infty} \{Q \subseteq A^{1/n}\} \quad \text{for any } A \text{ in } T.$$

□

Condition (2.4) is the definition of measurability used in [Roz82]. Rozanov also noted ([Roz82], p. 80) that the measurability of a T-valued map on Ω can be expressed in terms of the measurability of random variables, as described in the next lemma.

Lemma 2.3 *Suppose $Q(\omega) \in T$ for all $\omega \in \Omega$. Then $I_Q(z)$ is a measurable random variable for each z in $\mathcal{R}^d_+$ if and only if Q is measurable.*

We shall also study the strong Markov property for point processes. In order to relate this work to that in [MN90], we need the definition of measurability from [MN90]. In [MN90], a random set Q is said to be measurable if

$$\{Q \cap A \neq \emptyset\} \in \mathcal{F}, \quad \text{for all open } A \subseteq \mathcal{R}^d_+. \tag{2.5}$$

Once again, this definition of measurability is equivalent to Evstigneev's definition of measurability when Q takes values in T.

Lemma 2.4 *Assume* $Q(\omega) \in T$ *for each* $\omega \in \Omega$. *Then* (2.5) *holds if and only if* (2.1) *holds.*

Proof. (2.1) implies (2.5) by the relation

$$\{Q \cap A \neq \phi\} = \{Q \subseteq \overline{(A^c)^o}\}^c \quad \text{for open } A,$$

the equivalence of (2.1) and (2.3), and the fact that $\overline{(A^c)^o} \in T$. The reverse implication follows from the same relation after noting that any $B \in T$ can be written as $\overline{((B^c)^c)^o}$ and that B^c is open. □

Let $\mathcal{C}$ be a collection of sets C. A random set-valued function D is said to be <u>compatible</u> with $\mathcal{C}$ if

$$\{D \subseteq C\} \in \mathcal{F}_C, \text{ for all } C \in \mathcal{C}.$$

A random set-valued function D is said to be <u>co-compatible</u> with a collection of sets $\mathcal{C}$ if

$$\{D \supseteq C\} \in \mathcal{F}_C, \text{ for all } C \in \mathcal{C}.$$

As mentioned earlier in this section, an example of a random domain when $d = 1$ is the random interval $Q = [0, \tau]$ where τ is a positive random variable. We provide an extension of this concept to $\mathcal{R}^d_+$.

Definition 2.7. A $\mathcal{R}^d_+$-<u>membrane</u> is a subset $M \subseteq \mathcal{R}^d_+$ such that

if $d = 1$, then $M = \{m\}$ for some $m \in (0, \infty)$, and

if $d \geq 2$, then M satisfies (1) and (2) below:

(1) $M \cap \{u \in \mathcal{R}^d_+ : u_i = 0\}$ is a $\mathcal{R}^{d-1}_+$-membrane in $\{u \in \mathcal{R}^d_+ : u_i = 0\}$ for all $i = 1, \ldots, d$.

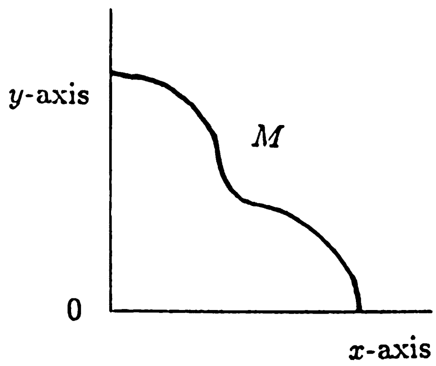
(2) there exists a continuous one-to-one map

$$\gamma : \overline{B_{d-1}(0,1)} \cap \mathcal{R}_+^{d-1} \longrightarrow \mathcal{R}_+^d$$

such that $\gamma(\overline{B_{d-1}(0,1)} \cap \mathcal{R}_+^{d-1}) = M$.

We call it a membrane because the first exit set τ 'attaches' itself to each axis ($d = 2$) or axis plane ($d = 3$) and 'stretches' itself between the axes ($d = 2$) or axis planes ($d \geq 3$).

Figure 2.1 membrane M $(d = 2)$

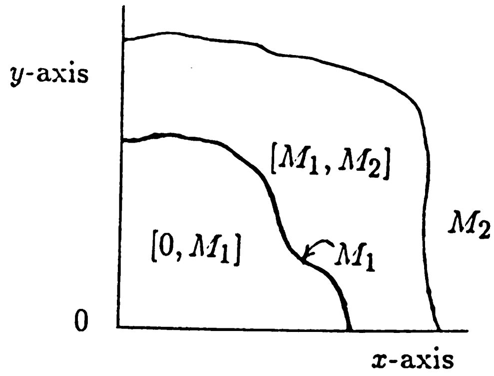


We can define sets of the form $[0, M]$ and $[M_1, M_2]$ for $d > 1$ dimensions in a natural way using membrane theory as described below. Let $\mathcal{M}$ denote the class of all membranes. If $M \in \mathcal{M}$, let

$$[0, M] \equiv \{u \in \mathcal{R}_+^d : u \text{ lies on some polygon } p \text{ in } \mathcal{R}_+^d \text{ which connects } 0 \text{ to an element } v \in M \text{ and such that } p \cap M = \{v\}\}.$$

Define a partial ordering $\leq$ on $\mathcal{M}$ by $M_1 \leq M_2$ if and only if $[0, M_1] \subseteq [0, M_2]$. Also define $[M_1, M_2] = [0, M_2] \cap [0, M_1]^c$ for $M_1, M_2 \in \mathcal{M}$ such that $M_1 \leq M_2$.

Figure 2.2 $[0, M_1]$ and $[M_1, M_2]$ for $M_1, M_2 \in \mathcal{M}$ such that $M_1 \leq M_2$



2.2 Strong Markov properties

Now let us recall the classical 1-dimensional strong Markov property. As mentioned in Section 2.1, a nonnegative random variable τ is a stopping time if $\{\tau \leq u\} \in \mathcal{G}_{[0,u]}$, for all $u \geq 0$. Define the stopped σ-algebra $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq u\} \in \mathcal{G}_{[0,u]}, \text{ for all } u \geq 0\}.$$

Definition 2.8. ξ is said to be strong Markov if for every stopping time τ,

$$P(\xi_t \in C \| \mathcal{F}_\tau) = P(\xi_t \in C \| \sigma(\xi_\tau)) \text{ a.s. on } \{t > \tau\}$$

for all $t \geq 0$ and $C \in B(\mathcal{R})$.

Intuitively, this says that the information from the process after the random time τ is conditionally independent of the future information after the random time τ, given the information from the process at the random time τ. As the extension of the Markov property to $\mathcal{R}_+^d$, $d > 1$, corresponds to the conditional independence of the information of the process in the 'interior' and 'exterior' regions of a set given the information on the boundary of the set, the extension of the strong Markov property to $\mathcal{R}_+^d$, $d > 1$, corresponds to the conditional independence of the information of the process in the 'interior' and 'exterior' regions of a random domain given the information on the boundary of the random domain.

Strong Markov properties of this type were proposed by Evstigneev ([Evs88] and [Evs77]) and Rozanov [Roz 82]. Exploring Evstigneev's most recent strong Markov property is the main purpose of this thesis, and it is defined below.

For notational convenience, define

$$\mathcal{A}_1(A,B) = \mathcal{F}_B$$

$$\mathcal{A}_2(A,B) = \mathcal{F}_{\overline{A^c}}$$

$$\mathcal{A}_3(A,B) = \mathcal{F}_{\overline{A^c \cap B}}$$

for $A, B \subseteq \mathcal{R}_+^d$ with $A \subseteq B$.

Let $\epsilon > 0$ and α, β be random domains such that $\alpha(\omega) \subseteq \beta(\omega)$ for all $\omega \in \Omega$. Let $i \in \{1,2,3\}$. Define $\mathcal{A}_i^\epsilon(\alpha,\beta)$ to be the σ-algebra generated by α, β and sets of the form

$$\{\overline{\alpha^{-\epsilon}} \subseteq A\} \bigcap \{\overline{\beta^\epsilon} \supseteq B\} \bigcap \Gamma$$

where $\Gamma \in \mathcal{A}_i(A,B)$ and

$$\alpha^{-\epsilon}(\omega) = \{t \geq 0 : d(t, \alpha(\omega)^c > \epsilon\}.$$

Let $\mathcal{A}_i(\alpha,\beta) \equiv \bigcap_{\epsilon>0} \mathcal{A}_i^\epsilon(\alpha,\beta)$.

Definition 2.9. A random field ξ is <u>strong Markov with respect to a random domain Q</u> if, for every two $\sigma(Q)$-measurable random domains α, β with $\alpha(\omega) \subseteq Q(\omega) \subseteq \beta(\omega)$ for all $\omega \in \Omega$,

$$\mathcal{A}_1(\alpha,\beta) \perp\!\!\!\perp \mathcal{A}_2(\alpha,\beta) \,|\, \mathcal{A}_3(\alpha,\beta).$$

Evstigneev [Evs88] proved that each of the following conditions on a random domain Q is sufficient for a Markov random field ξ to be strong Markov with respect to a random domain Q:

$$\{Q \subseteq B\} \in \mathcal{F}_B, \text{ for all } B \in T \tag{2.6}$$

$$\{Q \supseteq B\} \in \mathcal{F}_B, \text{ for all } B \in T \tag{2.7}$$

The following lemma is helpful in relating condition (2.6) to some more common conditions.

Lemma 2.5 *Let Q be a random domain.*

(i) $\{Q \subseteq A\} \in \mathcal{F}_A$, *for all open subsets A of $\mathcal{R}_+^d$ is equivalent to condition* (2.6).

(ii) (2.6) *implies* $\{Q \subseteq V\} \in \mathcal{F}_V$, *for all compact intervals V of $\mathcal{R}_+^d$.*

(iii) If $Q = [\tau_1, \tau_2]$ for some random variables τ_1, τ_2 such that $0 \le \tau_1(\omega) < \tau_2(\omega) < \infty$, for all $\omega \in \Omega$, then $\{Q \subseteq V\} \in \mathcal{F}_V$, for all compact intervals V of $\mathcal{R}_+^d$ implies (2.6).

Proof.

(i) We shall first show that $\{Q \subseteq A\} \in \mathcal{F}_A$ for all open subsets A of $\mathcal{R}_+^d$ implies condition (2.6). Let $V \in T$. Since $\bigcap_{\epsilon>0} \mathcal{F}_{V^\epsilon} = \mathcal{F}_V$, it is enough to show that $\{Q \subseteq V\} \in \mathcal{F}_{V^\epsilon}$ for every $\epsilon > 0$. Let $\epsilon > 0$. Then $\{Q \subseteq V\} = \bigcap_{n=[\frac{1}{\epsilon}]+1} \{Q \subseteq V^{\frac{1}{n}}\} \in \mathcal{F}_{V^\epsilon}$ and we are done. Now assume condition (2.6) and let A be an open subset of $\mathcal{R}_+^d$. Using that A is open and $Q(\omega)$ is closed for each $\omega \in \Omega$, $\{Q \subseteq A\} = \bigcup_{n=1} \{Q \subseteq \overline{A^{-\frac{1}{n}}}\} \in \mathcal{F}_A$.

(ii) Consider the compact interval $V = [a, b]$. If $a = b$, then $V = \{a\}$ and $\{Q \subseteq V\} = \phi \in \mathcal{F}_V$, because $Q(\omega)$ is a domain for each $\omega \in \Omega$. If $a < b$, then $V \in T$ and $\{Q \subseteq V\} \in \mathcal{F}_V$ by condition (2.6).

(iii) In order to prove (2.6), we shall show that $\{Q \subseteq V\} \in \mathcal{F}_{V^\epsilon}$ for every $V \in T$ and $\epsilon > 0$. Let $V \in T$ and $\epsilon > 0$. Then

$$\{Q \subseteq V\} = \bigcap_{n=[\frac{1}{\epsilon}]+1}^{\infty} \quad \bigcup_{s,t \in V^{\frac{1}{n}} \cap \mathbb{Q};\ [s,t] \subseteq V^{\frac{1}{n}}} \{Q \subseteq [s,t]\}$$

is an element of $\mathcal{F}_{V^{\frac{1}{n}}} \subseteq \mathcal{F}_{V^\epsilon}$ and the proof is complete. □

We will later show that condition (2.7) is not equivalent to ξ having the strong Markov property with respect to Q. The next theorem describes the relationship of a random domain Q with the random set $\overline{Q^c}$.

Theorem 2.2 *Consider the random field $\xi = \{\xi_t\}_{t \in D}$ where D is some compact subset of $\mathcal{R}^d_+$.*

(i) Q is a random domain if and only if $\overline{Q^c}$ is a random domain.

(ii) ξ is strong Markov with respect to Q if and only if ξ is strong Markov with respect to $\overline{Q^c}$.

Proof.

(i) Let Q be a random domain. Since Q is closed, we have that Q^c is open and thus $\overline{Q^c}$ is a domain. Hence $\overline{Q^c(\omega)} \in T$, for all $\omega \in \Omega$. Using results (2.10) and (2.15) on pp. 79,80 of [Roz82], we can get the measurability of Q. Therefore $\overline{Q^c}$ is a random domain.

The reverse implication follows by the above and the fact that $\overline{(\overline{Q^c})^c} = Q$.

(ii) Assume ξ is strong Markov with respect to Q. Let α and β be $\sigma(\overline{Q^c})$-measurable random domains such that $\alpha \subseteq \overline{Q^c} \subseteq \beta$. Note that for an arbitrary $A \in T$, it holds that $\overline{A^o} = A$, $\partial A^o = \partial A$, and $(\overline{A^c})^c = A^o$ ([Roz82, pp. 80-81, (2.17), (2.18)]). Thus

$$\{Q \subseteq A\} = \{Q^o \subseteq A^o\} = \{(Q^o)^c \supseteq (A^o)^c\} = \{\overline{Q^c} \supseteq \overline{A^c}\}.$$

Since $\overline{A^c} \in T$ and T is generated by sets of the form

$$\{B \in T : B \supseteq C\}$$

where $C \in T$ ([Evs88], Lemma 6.2(3)), we get that $\sigma(Q) = \sigma(\overline{Q^c})$. Combining this, $\sigma(\alpha) = \sigma(\overline{\alpha^c})$, and $\sigma(\beta) = \sigma(\overline{\beta^c})$, it follows that $\overline{\alpha^c}$ and $\overline{\beta^c}$ are $\sigma(Q)$-measurable. By the assumed strong Markov property,

$$\mathcal{A}_1(\overline{\beta^c}, \overline{\alpha^c}) \perp\!\!\!\perp \mathcal{A}_2(\overline{\beta^c}, \overline{\alpha^c}) \mid \mathcal{A}_3(\overline{\beta^c}, \overline{\alpha^c}).$$

In order to prove the desired strong Markov property, it is enough to show

$$\mathcal{A}_1(\overline{\beta^c}, \overline{\alpha^c}) = \mathcal{A}_2(\alpha, \beta), \tag{2.8}$$

$$\mathcal{A}_2(\overline{\beta^c}, \overline{\alpha^c}) = \mathcal{A}_1(\alpha, \beta), \text{ and} \tag{2.9}$$

$$\mathcal{A}_3(\overline{\beta^c}, \overline{\alpha^c}) = \mathcal{A}_3(\alpha, \beta). \tag{2.10}$$

For $\epsilon > 0$, $A, B \in T$ with $A \subseteq B$ and $\Gamma \in \mathcal{A}_i(A, B)$, $i \in \{1, 2, 3\}$,

$$\begin{aligned}\{\overline{\alpha^{-\epsilon}} \subseteq A\} \cap \{\overline{\beta^{\epsilon}} \supseteq B\} \cap \Gamma &= \{\overline{\overline{(\alpha^{-\epsilon})^c}} \supseteq \overline{A^c}\} \cap \{\overline{(\overline{\beta^{\epsilon}})^c} \subseteq \overline{B^c}\} \cap \Gamma \\ &= \{\overline{(\overline{\alpha^c})^{-\epsilon}} \supseteq \overline{A^c}\} \cap \{\overline{(\overline{\beta^c})^{\epsilon}} \subseteq \overline{B^c}\} \cap \Gamma.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{A}_1(A, B) &= \mathcal{F}_B = \mathcal{F}_{\overline{(\overline{B^c})^c}} = \mathcal{A}_2(\overline{B^c}, \overline{A^c}), \\ \mathcal{A}_2(A, B) &= \mathcal{F}_{\overline{A^c}} = \mathcal{A}_1(\overline{B^c}, \overline{A^c}), \text{ and} \\ \mathcal{A}_3(A, B) &= \mathcal{F}_{\overline{A^c} \cap B} = \mathcal{F}_{\overline{(\overline{B^c})^c} \cap \overline{A^c}} = \mathcal{A}_3(\overline{B^c}, \overline{A^c}).\end{aligned}$$

Hence $\mathcal{A}_1^{\epsilon}(\alpha, \beta) = \mathcal{A}_2^{\epsilon}(\overline{\beta^c}, \overline{\alpha^c})$, $\mathcal{A}_2^{\epsilon}(\alpha, \beta) = \mathcal{A}_1^{\epsilon}(\overline{\beta^c}, \overline{\alpha^c})$, and $\mathcal{A}_3^{\epsilon}(\alpha, \beta) = \mathcal{A}_3^{\epsilon}(\overline{\beta^c}, \overline{\alpha^c})$. Therefore (2.8) holds. (2.9) and (2.10) can be proven in a similar fashion.

The reverse implication follows from $\overline{(\overline{\alpha^c})^c} = \alpha$ and $\overline{(\overline{\beta^c})^c} = \beta$ and the same techniques used above. $\square$

Chapter 3

Special random domains

We will now look at some specific random fields ξ and random domains Q such that ξ has the strong Markov property with respect to Q.

3.1 Random domains of the form $Q = [0, \tau]$ when $d = 1$.

In this section, two theorems with different hypotheses and identical conclusions will be stated for $d = 1$. The second theorem is decidedly stronger than the first theorem. However, the proofs of the two theorems use different techniques. The proof of the first theorem uses an interesting result of Rozanov and condition (2.6). The second proof uses only (2.6) and is most similar to the rest of the proofs in Sections 3.3 and 3.5.

Theorem 3.1 *Let* $\xi = \{\xi_t\}_{t \geq 0}$ *be a Markov random field with continuous sample paths. Assume there exists some open set* $A_o \subseteq \mathcal{R}$ *such that* $\xi_o(\omega) \in A_o$*, for all*

$\omega \in \Omega$. Define the random set

$$\lambda(\omega) = \{u \geq 0 : \xi_u(\omega) \in A_o\}$$

and assume that $\lambda(\omega)$ is bounded for all $\omega \in \Omega$.

Then ξ is strong Markov with respect to the random domain $Q = [0, \tau]$, with

$$\tau(\omega) = \inf\{u \geq 0 : \xi_u(\omega) \notin A_o\}.$$

We call τ the <u>first exit time from A_o.</u>

Before we prove this theorem, we need to state a result of Rozanov ([Roz82], p. 82).

Theorem 3.2 *Let $\mathcal{O}$ denote the collection of all open subsets of $\mathcal{R}_d^+$, and let u_o be a fixed point in $\mathcal{R}_d^+$. If D is a random domain, co-compatible with $\mathcal{O}$, then the connected component D_o of D containing u_o is compatible with $\mathcal{O}$.*

Proof of Theorem 3.1. We will first show that $\overline{\lambda}$ is a random domain. Note that the set $\lambda(\omega)$ is open in X since A_o is open and $\xi(\omega)$ is continuous for all $\omega \in \Omega$. Thus $\overline{(\overline{\lambda})^o}(\omega) = \overline{\lambda}(\omega)$, for all $\omega \in \Omega$; that is, $\overline{\lambda}(\omega)$ is a domain for all $\omega \in \Omega$. By assumption that $\overline{\lambda}(\omega)$ is bounded, for all $\omega \in \Omega$, and since $\overline{\lambda}(\omega)$ is closed, for all $\omega \in \Omega$, we have that $\overline{\lambda}(\omega)$ is compact, for all $\omega \in \Omega$. Thus $\overline{\lambda}(\omega) \in T$, for all $\omega \in \Omega$.

Next we must check the measurability of λ. Since λ is open, it is enough to check if $\{\lambda \supseteq A\} \in \mathcal{F}$, for all $A \in \mathcal{O}$ ([Roz82], pp. 79-80). Now

$$\{\lambda \supseteq A\} = \{\overline{\lambda} \supseteq A\} = \bigcap_{\epsilon > 0} \bigcap_{u \in A \cap \mathbb{Q}} \bigcup_{\nu \in B(u,\epsilon) \cap \mathbb{Q}} \{\xi_\nu \in A_o\},$$

which is an element of $\mathcal{F}_A$, for all $A \in \mathcal{O}$ using that A is open. Hence $\overline{\lambda}$ is measurable, and thus $\overline{\lambda}$ is a random domain. From $\{\overline{\lambda} \supseteq A\} \in \mathcal{F}_A$, for all $A \in \mathcal{O}$, we also get that $\overline{\lambda}$ is co-compatible with the family $\{\sigma(\xi_u, u \in A)\}_{A \in \mathcal{O}}$.

Applying Theorem 3.2 with $D = \overline{\lambda}$ and $u_o = 0$, yields that $Q = [0,\tau]$ is compatible with the family $\{\mathcal{G}_A\}_{A\in\mathcal{O}}$, where $\tau = \inf\{u \geq 0 : \xi_u \notin A_o\}$. Let $B \in T$. Then $\{Q \subseteq B\} = \bigcap_{n=1}^{\infty}\{Q \subseteq B^{1/n}\}$ since B is closed. Also, $\{Q \subseteq B^{1/n}\} \in \mathcal{G}_{B^{\frac{1}{n}}}$, since $B^{\frac{1}{n}}$ is open and Q is compatible, for all $n \in \mathbb{N}$. So $\{Q \subseteq B\} \in \bigcap_{n=1}^{\infty}\mathcal{G}_{B^{\frac{2}{n}}} = \mathcal{F}_B$ since $\mathcal{G}_{B^{\frac{2}{n}}} \subseteq \bigcap_{\frac{2}{n}\leq\epsilon<\frac{2}{n-1}}\mathcal{G}_{B^\epsilon}$, for all n. This inclusion and condition (2.6) yield that ξ has the strong Markov property with respect to Q. □

It turns out that Theorem 3.1 holds for any positive stopping time τ, not just exit times. The path continuity and initial conditions on ξ can be removed to yield the result given below.

Theorem 3.3 *If ξ is a Markov random field and τ is a positive finite stopping time, then ξ is strong Markov with respect to the random domain $Q = [0,\tau]$.*

Proof. We must first show that Q is a random domain. Certainly, using that $0 < \tau(\omega) < \infty$, for all $\omega \in \Omega$, we get that $Q(\omega) \in T$, for all $\omega \in \Omega$. Moreover, given $V \in T$ and $\epsilon > 0$,

$$\begin{aligned}\{Q \subseteq V\} &= \bigcap_{n=[\frac{1}{\epsilon}]+1}^{\infty} \{[0,\tau] \subseteq V^{\frac{1}{n}}\} \\ &= \bigcap_{n=[\frac{1}{\epsilon}]+1}^{\infty} \bigcup_{s\in V^{\frac{1}{n}}\cap\mathbb{Q};\ [0,s]\subseteq V^{\frac{1}{n}}} \{\tau \leq s\}.\end{aligned}$$

The latter equality comes from the proof of the following lemma.

Lemma 3.1

$$\{[0,\tau] \subseteq V^{\frac{1}{n}}\} = \bigcup_{s \in V^{\frac{1}{n}} \cap \mathbb{Q};\ [0,s] \subseteq V^{\frac{1}{n}}} \{\tau \leq s\}$$

for arbitrary $n \in \mathcal{N}$.

Proof. If $u \in [0, \tau(\omega)]$ and there is an $s \in V^{\frac{1}{n}} \cap \mathbb{Q}$ such that $[0, s] \subseteq V^{\frac{1}{n}}$ and $\tau(\omega) \leq s$, then $u \in [0, s] \subseteq V^{\frac{1}{n}}$. Hence

$$\bigcup_{s \in V^{\frac{1}{n}} \cap \mathbb{Q};\ [0,s] \subseteq V^{\frac{1}{n}}} \{\tau \leq s\} \subseteq \{[0,\tau] \subseteq V^{\frac{1}{n}}\}.$$

Furthermore, if $[0, \tau(\omega)] \subseteq V^{\frac{1}{n}}$, then by $V^{\frac{1}{n}} \cap (\tau(\omega), \infty)$ being open, there exists some rational $s \in V^{\frac{1}{n}}$ such that $[0, \tau(\omega)] \subseteq [0, s] \subseteq V^{\frac{1}{n}}$. In particular, $\tau(\omega) \leq s$. Thus

$$\bigcup_{s \in V^{\frac{1}{n}} \cap \mathbb{Q};\ [0,s] \subseteq V^{\frac{1}{n}}} \{\tau \leq s\} \supseteq \{[0,\tau] \subseteq V^{\frac{1}{n}}\},$$

and the lemma is proven.

Continuing with the proof of Theorem 3.3, we have $\{Q \subseteq V\} \in \mathcal{F}_{V^\epsilon}$ for each $\epsilon > 0$ and $\{Q \subseteq V\} \in \mathcal{F}_V$. Hence Q is a random domain and ξ is strong Markov with respect to Q using condition (2.6). □

3.2 A counterexample

We are now ready to construct an example of a process ξ and a random domain D such that ξ is strong Markov with respect to D but D does not satisfy (2.7).

Let $\tilde{\xi}_0$, $\tilde{\xi}_1$, and $\tilde{\xi}_2$ be independent random variables. Define

$$\xi_t = \sum_{n=1}^{3} \tilde{\xi}_{n-1} 1_{(n-1,n]}(t) + \tilde{\xi}_0 1_{\{0\}}(t) \quad \text{for } t \in [0,3].$$

Suppose that $\tilde{\xi}_0$ lies in some open set A_0 and let $\tau = \inf\{t \geq 0 : \xi_t \notin A_0\}$. Furthermore, assume that $\tau(\omega) < \infty$ for all $\omega \in \Omega$ and $0 < P(\tau \geq 1) < 1$. Note that ξ is classical Markov since $\mathcal{G}_{[0,1]} = \sigma(\xi_0)$ is independent of $\sigma(\xi_1, \xi_2) = \mathcal{G}_{[1,3]}$. Thus ξ is Markov and has the strong Markov property with respect to the domain $Q = [0, \tau]$ by Theorem 3.3. By Theorem 2.2, ξ also is strong Markov with respect to the random domain $\overline{Q^c} = [\tau, 3]$.

However,

$$\begin{aligned}
\{\overline{Q^c} \supseteq [1,3]\} &= \{[\tau, 3] \supseteq [1,3]\} \\
&= \{\tau \geq 1\} \\
&= \{\tau < 1\}^c \\
&\in \mathcal{G}_{[0,1]}
\end{aligned}$$

since $\{\tau < 1\} = \bigcup_{n=1}^{\infty} \{\tau \leq 1 - \frac{1}{n}\}$ and $\{\tau \leq 1 - \frac{1}{n}\} \in \mathcal{G}_{[0,1-\frac{1}{n}]} \subseteq \mathcal{G}_{[0,1]}$ for each n. Suppose $\{\overline{Q^c} \supseteq [1,3]\} \in \mathcal{G}_{[1,3]}$. Then $\{\overline{Q^c} \supseteq [1,3]\} \in \mathcal{G}_{[0,1]} \cap \mathcal{G}_{[1,3]}$ and, since $\mathcal{G}_{[0,1]}$ is independent of $\mathcal{G}_{[1,3]}$, $P(\tau \geq 1) = P(\overline{Q^c} \supseteq [1,3]) = 0$ or 1. This contradicts our assumption that $0 < P(\tau \geq 1\} < 1$. Hence $\{\overline{Q^c} \supseteq [1,3]\} \notin \mathcal{G}_{[1,3]}$ and condition (2.7) does not hold.

3.3 Random domains of the form $Q = [0, \tau]$ when $d > 1$

Recall that membrane theory was discussed in Section 2.1. The extension of Theorem 3.3 now follows naturally.

Theorem 3.4 *Let ξ be a Markov random field and let $\tau : \Omega \longrightarrow \mathcal{M}$ be a map such that*

$$\{\tau \leq M\} \in \mathcal{G}_{[0,M]}, \quad \textit{for all } M \in \mathcal{M}. \tag{3.1}$$

Then ξ has the strong Markov property with respect to the random domain $Q = [0,\tau]$.

Proof. We need only to show that Q is a random domain and that (2.6) holds. Since $Q(\omega) \in T$, for all $\omega \in \Omega$, we may conclude that Q is a random domain once we have proven (2.3), as was done in the proof of Theorem 3.3. For arbitrary $V \in T$ and $\epsilon > 0$, define

$$\mathcal{H}_V^n = \{M \in \mathcal{M} : [0,M] \subseteq V^{1/n},\ M \cap (\bigcup_{i=1}^{d} \{u \in \mathcal{R}_+^d : u_j = 0 \text{ for } j \neq i\}) \subseteq \mathcal{Q}_+^d\}$$

Then given any $V \in T$ and $\epsilon > 0$,

$$\begin{aligned}\{Q \subseteq V\} &= \bigcap_{n=[\frac{1}{\epsilon}]+1} \bigcup_{M \in \mathcal{H}_V^n} \{Q \subseteq M\} \\ &\in \bigcap_{\epsilon>0} \mathcal{G}_{V^\epsilon}\end{aligned}$$

which equals $\mathcal{F}_V$ using the hypothesis. □

3.4 Stopping lines

In [MN90] a special type of membrane is defined for $d = 2$. Let M be a $\mathcal{R}_+^2$-membrane. Define $Q(u) = M \cap \{x \in \mathcal{R}_+^2 : x_1 = u\}$ for $u \in [0,1]$. We shall say that M is a <u>decreasing line</u> if $Q(u)$ is a singleton for each $u \in [0,1]$ and the map $\tilde{Q}$ on $[0,1]$ is non-increasing, where $\tilde{Q}(u)$ is defined to be the y-coordinate of $Q(u)$. Let $L(\omega)$ be a decreasing line for each $\omega \in \Omega$. Then L is said to be a <u>stopping line</u> if

$$\{z : C_z \leq L\} \in \mathcal{G}_{[0,C_z]},$$

for all $z \in \mathcal{R}_+^2$ where

$$C_z \equiv \{x \in \mathcal{R}_+^2 : x_1 = z_1,\ 0 \leq x_2 \leq z_2\} \cup \{x \in \mathcal{R}_+^2 : x_2 = z_2,\ 0 \leq x_1 \leq z_1\}.$$

Let N be a point process and L be a random membrane. Upon application of Theorem 3.4, if N is Markov and satisfies

$$\{L \leq M\} \in \mathcal{G}_{[0,M]} \text{ for } M \in \mathcal{M} \tag{3.2}$$

then N is strong Markov with respect to the random domain $Q = [0, L]$.

Evstigneev showed that there is another strong Markov property such that N is strong Markov with respect to Q which also follows from condition (3.1) for N and Q. This strong Markov property is

$$\mathcal{A}(Q) \perp\!\!\!\perp \mathcal{A}(\overline{Q^c}) \mid \mathcal{A}(\partial Q)$$

where $\mathcal{A}^\varepsilon(D) \equiv \sigma(D, \{\overline{D^\varepsilon} \supseteq B\} \cap \Gamma,\ \ B \in T,\ \ \Gamma \in \mathcal{G}_B)$ for $\varepsilon > 0$ and $\mathcal{A}(D) = \bigcap_{\varepsilon>0} \mathcal{A}^\varepsilon(D)$ for any random domain D ([Evs77]).

Merzbach & Nualart define the following strong Markov property:

For a set-valued random function D, let $\mathcal{H}_D$ be defined to be the σ-algebra generated by $D^{-1}(\mathcal{T})$ and the random variables $1_D(z)N_z$, $z \in \mathcal{R}^2_+$. We say that N has Merzbach & Nualart's strong Markov property with respect to D if

$$\mathcal{H}_D \perp\!\!\!\perp \mathcal{H}_{\overline{D^c}} \mid \mathcal{H}_{\partial D}.$$

Merzbach & Nualart's strong Markov result is stated below.

Theorem 3.5 *[MN90] If L is a decreasing stopping line and N has the simple Markov property with respect to sets of the form $[0, \ell]$, ℓ a decreasing line, then*

$$\mathcal{H}_Q \perp\!\!\!\perp \mathcal{H}_{\overline{Q^c}} \mid \mathcal{H}_L.$$

Next, we compare sufficient conditions on L.

Lemma 3.2 *If L is a decreasing stopping line, then L satisfies condition* (3.1).

Proof. Let L be a decreasing stopping line. Since L is decreasing, it is enough to show that $\{L \leq M\} \in \mathcal{G}_{[0,M]}$ for arbitrary decreasing lines $M \in \mathcal{M}$. Choose an M as described. Then

$$\begin{aligned}\{L \leq M\} &= \{L \not\leq M\}^c \\ &= \Big(\bigcup_{z \in M \cap \mathbb{Q}^2} \{C_z \leq L\}\Big)^c\end{aligned}$$

Now $\{C_z \leq L\} \in \mathcal{G}_{[0,C_z]}$ by L a stopping line. Moreover, $\mathcal{G}_{[0,C_z]} \subseteq \mathcal{G}_{[0,M]}$ for $z \in M \cap \mathbb{Q}_+^2$ since $[0,C_z] \subseteq [0,M]$ and M is decreasing. Thus $\{L \leq M\} \in \mathcal{G}_{[0,M]}$, and we have that L satisfies (3.1).

□

The converse of this proposition is not true, as the following example shows.

Counterexample 3.1. Here we present an exmaple of a point process N and random membrane L such that L is decreasing and satisfies (3.1), but L is not a stopping line. Let N be Poisson. Recall that N is a.s. boundedly finite and without fixed atoms. We can apply Theorem 2.4 VII of [DV88] (p. 35) to get that for every finite family of bounded disjoint Borel sets $\{A_i, i = 1, \ldots, k\}$, the random variables $N(A_1), \ldots, N(A_k)$ are mutually independent. Define $Mu = \{(x,y) : y = u - x\} \cap \mathcal{R}_+^2$ for $u > 0$ and

$$L(\omega) = \inf\{Mu : \int \xi_s(\omega) 1_{[0,Mu] \cap \xi_\cdot(u)^{-1}([3,\infty))}(s) \lambda_2(ds) \geq 6\}.$$

Assume that $z \in \mathcal{R}_+^2$ and $\{C_z \leq L\} \notin \{\emptyset, \Omega\}$. Then given any Mu,

$$\{L \leq Mu\} = \{\omega : \int \xi_s(\omega) 1_{[0,Mu]\cap\xi_\cdot(\omega)^{-1}([3,\infty))}(s)\lambda_2(ds) \geq 6\}$$
$$\in \mathcal{G}_{[0,Mu]}$$

Thus L satisfies (3.1). However, for $z \in \mathcal{R}_+^2$,

$$\begin{aligned}\{C_z \leq L\} &= \{L < M_{z_1+z_2}\}^c \\ &= \left(\bigcup_{\nu\in\mathbb{Q}_+^2\cap(0,z_1+z_2)} \{L \leq M_\nu\}\right)^c \\ &\in \mathcal{G}_{[0,M_{z_1+z_2}]}.\end{aligned}$$

By independence of $N([0, M_{z_1+z_2}])$ and $N([0, Cz] \cap [0, M_{z_1+z_2}]^c)$, it does not hold that $\{C_z \leq L\} \in \mathcal{G}_{[0,C_z]}$. Thus L is not a stopping line.

The above lemma and counterexample show that (3.1) is a weaker assumption on L than the assumption that L is a stopping line. However, condition (3.1) is sufficient for N to have Merzbach & Nualart's strong Markov property. We shall now state and prove our version of Theorem 3.5.

Theorem 3.6 *Assume $\mathcal{A}(Q) \vee \mathcal{A}(\overline{Q^c}) = \mathcal{F}$ and $\bigcap_{n=1}^{\infty} \mathcal{H}_{[L-\frac{1}{n},L+\frac{1}{n}]} = \mathcal{H}_L$. If N has the simple Markov property with respect to sets of the form $[0,\ell]$, ℓ a decreasing line, and L satisfies (3.1) then*

$$\mathcal{H}_Q \perp\!\!\!\perp \mathcal{H}_{\overline{Q^c}} \mid \mathcal{H}_L.$$

Remark. The condition $\mathcal{A}(Q) \vee \mathcal{A}(\overline{Q^c}) = \mathcal{F}$ is clearly satisfied if Q is a deterministic compact domain. Also, the condition $\bigcap_{n=1}^{\infty} \mathcal{H}_{[L-\frac{1}{n},L+\frac{1}{n}]} = \mathcal{H}_L$ is similar to the condition $\bigcap_{n=1}^{\infty} \mathcal{H}_{[L,L_n]} = \mathcal{H}_L$ which is proved in [MN90] for L a decreasing stopping line and $\{L_n\}$ a sequence of decreasing stopping lines which converge to L.

In order to prove this result, we will need a few lemmas.

Lemma 3.3 *If N has the simple Markov property with respect to set of the form*

$$[0,\ell], \quad \ell \text{ a decreasing line,}$$

then N is Markov.

Proof. Let $B \in T$. We must show that $\mathcal{F}_{\overline{A^c}} \perp\!\!\!\perp \mathcal{F}_C \mid \mathcal{F}_{\overline{A^c \cap C}}$ for all $A, C \in T$ with $A \subseteq B \subseteq C$. Certainly $\mathcal{G}_{[0,\ell)} \perp\!\!\!\perp \mathcal{G}_{[0,\ell]^c} \mid \mathcal{G}_\ell$ for any decreasing line ℓ. Moreover, $\mathcal{G}_{[0,\ell)} \perp\!\!\!\perp \mathcal{G}_{[0,\ell)^c} \mid \mathcal{G}_\ell$ using Proposition 2.1 and $\mathcal{G}_{[0,\ell)^c} \subseteq \mathcal{G}_{[0,\ell]^c} \vee \mathcal{G}_\ell$. Thus, by Proposition 2.2, we know that

$$\mathcal{G}_D \perp\!\!\!\perp \mathcal{G}_{\overline{D^c}} \mid \mathcal{G}_{\partial D}$$

for all sets D of the form

$$D = [0,\ell) \cup [0,s]^c$$

where $\ell \leq s$ and l, s are decreasing lines. This, of course, is equivalent to $\mathcal{G}_D \perp\!\!\!\perp \mathcal{G}_{\overline{D}^c} \mid \mathcal{G}_{\partial D}$ for $D = (\ell, s)$.

Given any open convex set $\theta \subseteq \mathcal{R}_+^2$, there exists a collection $\{(\ell_i, s_i)\}_{i \in \mathbb{N}}$ of pairs of decreasing stopping lines such that

$$\ell_i \leq s_i \text{ for each } i \in \mathbb{N},$$

$$\overline{\bigcup_{i \in \mathbb{N}} (\ell_i, s_i - i)} = \overline{\theta},$$

$$\text{and} \quad (\ell_i, s_i) \cap (\ell_j, s_j) = \phi \text{ for } i \neq j.$$

Thus, by Proposition 22, N has the simple Markov property on θ; that is,

$$\mathcal{G}_{\overline{\theta}} \perp\!\!\!\perp \mathcal{G}_{\overline{\theta^c}} \mid \mathcal{G}_{\partial\theta}.$$

According to Corollary 3.1 of [Man83], this is equivalent to N having the simple Markov property on all open subsets of $\mathcal{R}_+^2$. Hence, N has the GFMP on all open sets by Theorem 2.1, and the rest of the proof is identical to the last part of the proof of Lemma 2.1 with $a = A$ and $b = B$. □

Lemma 3.4

(i) $\mathcal{H}_{\overline{Q^\varepsilon}} \subseteq \mathcal{A}(Q) \vee \mathcal{A}^\varepsilon(L)$ *and* $\mathcal{H}_{\overline{(Q^c)^\varepsilon}} \subseteq \mathcal{A}(\overline{Q^c}) \vee \mathcal{A}^\varepsilon(L)$ *for all* $\varepsilon > 0$

(ii) $\mathcal{A}(Q) \subseteq \mathcal{H}_Q$ *and* $\mathcal{A}(\overline{Q^c}) \subseteq \mathcal{H}_{\overline{Q^c}}$

(iii) $\mathcal{A}^\varepsilon(L) = \mathcal{H}_{[L-\varepsilon, L+\varepsilon]}$

Proof.

(i) We shall first show $\mathcal{H}_{\overline{Q^\varepsilon}} \subseteq \mathcal{A}(Q) \vee \mathcal{A}^\varepsilon(L)$ for arbitrary $\varepsilon > 0$. Consider $C \in \mathcal{B}(\mathcal{R}^+)$, $z \in \mathcal{R}_+^2$, and note that $1_{\{1_{\overline{Q^\varepsilon}}(z) N_z \in C\}}$ is equal to

$$1_{\{1_{\overline{Q^\varepsilon}}(z)\}} 1_{\{N_z \in C\}} + 1_{(\overline{Q^\varepsilon})^c}(z) \qquad \text{if } 0 \in C$$

or

$$1_{\{1_{\overline{Q^\varepsilon}}(z)\}} 1_{\{N_z \in C\}} \qquad \text{if } 0 \notin C.$$

Furthermore, note that $(\overline{Q^\varepsilon})^c$ is $\sigma(Q)$-measurable and hence $\mathcal{A}(Q)$-measurable. Thus, it is enough to show $1_{\{z \in \overline{Q^\varepsilon}\}} 1_{\{N_z \in C\}}$ is $\mathcal{A}(Q) \vee \mathcal{A}^\varepsilon(L)$-measurable. Now

$$\begin{aligned} &1_{\{z \in \overline{Q^\varepsilon}\} \cap \{N_z \in C\}} \\ &= 1_{\{z \in Q\} \cap \{N_z \in C\}} + 1_{\{z \in [L-\varepsilon, L+\varepsilon]\} \cap \{N_z \in C\}} - 1_{\{z \in Q\}^c} 1_{\{z \in [L-\varepsilon, L+\varepsilon]\} \cap \{N_z \in C\}}. \end{aligned}$$

In addition, since $Q(\omega) \in T$ for each $\omega \in \Omega$,

$$\{z \in Q\} \cap \{N_z \in C\} = \{\overline{Q^\varepsilon} \supseteq B(z, \varepsilon)\} \cap \{N_z \in C\} \in \mathcal{A}^\varepsilon(Q).$$

Thus $\{z \in Q\} \cap \{N_z \in C\} \in \mathcal{A}(Q)$. Because L is decreasing, $\{z \in [L-\varepsilon, L+\varepsilon]\} = \{[L-\varepsilon, L+\varepsilon] \supseteq \overline{B^{++}(z,2\varepsilon)}\} \cup \{[L-\varepsilon, L+\varepsilon] \supseteq \overline{B^{--}(z,2\varepsilon)}\} \cup (\bigcup_{0<\delta<\varepsilon} \{[L-\varepsilon, L+\varepsilon] \supseteq B(z,\delta)\})$ where

$$B^{++}(z,2\varepsilon) = B(z,2\varepsilon) \cap \{(s,t) : s \geq z_1,\ t \geq z_2\}$$

$$B^{--}(z,2\varepsilon) = B(z,2\varepsilon) \cap \{(s,t) : s \leq z_1,\ t \leq z_2\}.$$

Since the three summands are disjoint, $\{z \in [L-\varepsilon, L+\varepsilon]\} \cap \{N_z \in C\} \in \mathcal{A}^{\varepsilon}(L)$. Therefore, $1_{\{z\in\overline{Q^{\varepsilon}}\}} 1_{\{N_z \in C\}}$ is $\mathcal{A}(Q) \vee \mathcal{A}^{\varepsilon}(L)$-measurable. The proof of $\mathcal{H}_{\overline{(Q^c)^{\varepsilon}}} \subseteq \mathcal{A}(\overline{Q^c}) \vee \mathcal{A}^{\varepsilon}(L)$ is proven in a similar fashion.

(ii) We show $\mathcal{A}(Q) \subseteq \mathcal{H}_Q$ by proving $\mathcal{A}^{\varepsilon}(Q) \subseteq \mathcal{H}_{\overline{Q^{\varepsilon}}}$ for all $\varepsilon > 0$. Consider a set $\{\overline{Q^{\varepsilon}} \supseteq \Gamma\} \cap \Delta$ where $\Gamma \in T$ and $\Delta \in \mathcal{G}_{\Gamma}$. Now $\Delta \in \mathcal{G}_{\Gamma}$ implies the existence of a countable collection $Z \subseteq \Gamma$ and a measurable function Φ such that $1_{\Delta} = \Phi(N_z, z \in Z)$. Moreover, $\{\overline{Q^{\varepsilon}} \supseteq \Gamma\} = \bigcap_{x\in\Gamma\cap\mathrm{supp}(N)} \{x \in \overline{Q^{\varepsilon}}\}$ for the countable collection $X = \Gamma \cap \mathrm{supp}(N) \subseteq \mathcal{R}^2_+$ since $\mathrm{supp}(N)$ is countable. Therefore

$$\begin{aligned} 1_{\{\overline{Q^{\varepsilon}}\supseteq\Gamma\}\cap\Delta} &= \prod_{x\in X} 1_{\{x\in\overline{Q^{\varepsilon}}\}} \Phi(N_z, z \in Z) \\ &= \prod_{x\in X\cup Z} 1_{\{x\in\overline{Q^{\varepsilon}}\}} \Phi(1_{\{z\in\overline{Q^{\varepsilon}}\}} N_z, z \in Z) \end{aligned}$$

is $\mathcal{H}_{\overline{Q^{\varepsilon}}}$-measurable. So $\mathcal{A}^{\varepsilon}(Q) \subseteq \mathcal{H}_{\overline{Q^{\varepsilon}}}$.

The proof of $\mathcal{A}^{\varepsilon}(\overline{Q^c}) \subseteq \mathcal{H}_{\overline{(Q^c)^{\varepsilon}}}$ is handled in a similar fashion.

(iii) Proof of $\mathcal{A}^{\varepsilon}(L) \subseteq \mathcal{H}_{[L-\varepsilon,L+\varepsilon]}$ uses the same technique as in (ii) and will not be shown here. In order to show that reverse inclusion, consider $\varepsilon > 0$, $z \in \mathcal{R}^2_+$, and $C \in \mathcal{B}(\mathcal{R})$. Then

$$1_{\{1_{[L-\varepsilon,L+\varepsilon]}(z)N_z\in C\}} = \begin{cases} 1_{\{[L-\varepsilon,L+\varepsilon]\supseteq\{z\}\}\cap\{N_z\in C\}} + 1_{\{[L-\varepsilon,L+\varepsilon]\supseteq\{z\}\}} & \text{if } 0 \in C \\ 1_{\{[L-\varepsilon,L+\varepsilon]\supseteq\{z\}\}\cap\{N_z\in C\}} & \text{if } 0 \notin C \end{cases}$$

Now $\{[L-\varepsilon, L+\varepsilon] \supseteq \{z\}\} \in \mathcal{A}^{\varepsilon}(L)$ and so it is enough to show that $\{[L-\varepsilon, L+\varepsilon] \supseteq \{z\}\} \cap \{N_z \in C\} \in \mathcal{A}^{\varepsilon}(L)$. Now $\{[L-\varepsilon, L+\varepsilon] \supseteq \{z\}\} = (\bigcup_{\delta>0}\{[L-\varepsilon, L+\varepsilon] \supseteq \overline{B(z,\delta)}\}) \bigcup \{[L-\varepsilon, L+\varepsilon] \supseteq \overline{B^{++}(z,2\varepsilon)}\} \bigcup \{[L-\varepsilon, L+\varepsilon] \supseteq \overline{B^{--}(z,2\varepsilon)}\}$ and so $\{[L-\varepsilon, L+\varepsilon] \supseteq \{z\}\} \bigcap \{N_z \in C\} \in \mathcal{A}^{\varepsilon}(L)$. Thus $\mathcal{H}_{[L-\varepsilon, L+\varepsilon]} \subseteq \mathcal{A}(L)$. □

Proof of Thm 3.6. By Lemma 3.3 and the fact that (3.1) holds, we get

$$\mathcal{A}(Q) \perp\!\!\!\perp \mathcal{A}(\overline{Q^c}) \mid \mathcal{A}(L).$$

Since $\mathcal{A}(Q) \vee \mathcal{A}(\overline{Q^c}) = \mathcal{F}$ and $\mathcal{A}(L) \subseteq \mathcal{A}^{\varepsilon}(L) \subseteq \mathcal{F}$ for any $\varepsilon > 0$,

$$\mathcal{A}(Q) \perp\!\!\!\perp \mathcal{A}(\overline{Q^c}) \mid \mathcal{A}^{\varepsilon}(L).$$

Now $\mathcal{A}(Q) \vee \mathcal{A}^{\varepsilon}(L) \perp\!\!\!\perp \mathcal{A}(\overline{Q^c}) \vee \mathcal{A}^{\varepsilon}(L) \mid \mathcal{A}^{\varepsilon}(L)$ by applying Prop. 2.1. Applying Lemma 3.4(i) now gives us $\mathcal{H}_{\overline{Q^\varepsilon}} \perp\!\!\!\perp \mathcal{H}_{\overline{(Q^c)^\varepsilon}} \mid \mathcal{A}^{\varepsilon}(L)$, and Lemma 3.4 (iii) yields $\mathcal{F}_{\overline{Q^\varepsilon}} \perp\!\!\!\perp \mathcal{F}_{\overline{(Q^c)^\varepsilon}} \mid \mathcal{H}_{[L-\varepsilon, L+\varepsilon]}$. By the Martingale convergence theorem, $\bigcap_{n=1}^{\infty} \mathcal{H}_{[L-\frac{1}{n}, L+\frac{1}{n}]} = \mathcal{H}_L$. Since $\mathcal{H}_D \subseteq \mathcal{H}_{D'}$ for all set-valued random functions $D \subseteq D'$, we conclude that $\mathcal{H}_Q \perp\!\!\!\perp \mathcal{H}_{\overline{Q^c}} \mid \mathcal{H}_L$. □

3.5 Random domains of the form $Q = [\tau_1, \tau_2]$

Another type of set-valued random function that may be of interest, particularly when considering reciprocal Markov processes, are of the form

$$Q = [\tau_1, \tau_2]$$

and τ_1 and τ_2 are positive random variables that satisfy

$$0 \leq \tau_1(\omega) < \tau_2(\omega) < \infty, \qquad \text{for all } \omega \in \Omega.$$

For τ_1 and τ_2 membrane-valued random functions satisfying the above inequalities, we have the following result for $d \geq 1$:

Theorem 3.7 *If ξ is a Markov random field and τ_1 and τ_2 satisfy*

$$\{M_1 \leq \tau_1 < \tau_2 \leq M_2\} \in \mathcal{G}_{[M_1,M_2]}, \tag{3.3}$$

for all $M_1 \in \mathcal{M} \cup \{0\}$ and $M_2 \in \mathcal{M}$ with $M_1 \leq M_2$ then ξ is strong Markov with respect to the random domain $Q = [\tau_1, \tau_2]$.

Note: If τ_1 and τ_2 are two-sided stopping membranes; that is, if for $i = 1,2$

$$\{M_1 \leq \tau_i \leq M_2\} \in \mathcal{G}_{[M_1,M_2]},$$

for all $M_1 \in \mathcal{M} \cup \{0\}$ and $M_2 \in \mathcal{M}$ with $M_1 \leq M_2$ then the sufficient condition (3.3) holds. This follows from the relation

$$\{M_1 \leq \tau_1 < \tau_2 \leq M_2\} = \{M_1 \leq \tau_1 \leq M_2\} \cap \{M_1 \leq \tau_2 \leq M_2\}.$$

Proof of Theorem 3.7. It is enough to show (2.6). Let $V \in T$ and $n \in \mathbb{N}$ be arbitrary and define

$$\begin{aligned} \mathcal{S}_V^n = \{(M_1, M_2) \in (\mathcal{M} \cup \{0\}) \times \mathcal{M} \ : M_1 < M_2,\ [M_1, M_2] \subseteq V^{\frac{1}{n}}, \\ M_i \cap (\bigcup_{j=0}^{d} \{u \in \mathcal{R}_+^d \ : \ u_k = 0,\ k \neq j\}) \subseteq \mathbb{Q}_+^d \text{ for } i = 1, 2\}. \end{aligned}$$

Then for $\epsilon > 0$,

$$\{Q \subseteq V\} = \bigcap_{n=[\frac{1}{\epsilon}]+1}^{\infty} \ \bigcup_{(M_1,M_2)\in\mathcal{S}_V^n} \{M_1 \leq \tau_1 < \tau_2 \leq M_2\} \in \mathcal{G}_{V^\epsilon}.$$

Hence $\{Q \subseteq V\} \in \bigcap_{\epsilon>0} \mathcal{G}_{V^\epsilon} = \mathcal{F}_V$, and we have that ξ is strong Markov with respect to Q. $\square$

Chapter 4

Corner Markov and Reciprocal processes

Definition 4.1 A random field ξ for $d = 1$ has the reciprocal Markov property if

$$\mathcal{G}_{[s,t]} \perp\!\!\!\perp \mathcal{G}_{[0,s]\cup[t,\infty)} \,|\, \mathcal{G}_{\{s,t\}}, \qquad \text{for all } s \geq t \geq 0.$$

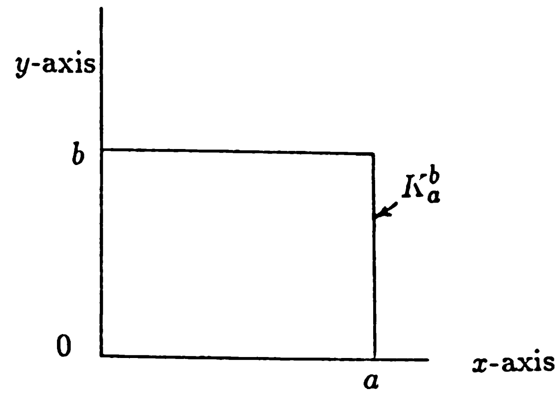
The type of strong Markov property that a reciprocal Markov process has and the extension of this reciprocal Markov property are two topics that are treated in this chapter.

4.1 The corner Markov property

One way that the reciprocal Markov process can be extended to $\mathcal{R}_+^d$ utilizes "corners". Define a corner K_a^b for $a, b \geq 0$ by

$$K_a^b = \{u \in \mathcal{R}_+^2 : u_1 = a, 0 \leq u_2 \leq b\} \cup \{u \in \mathcal{R}_+^2 : u_2 = b, 0 \leq u_1 \leq a\}.$$

Figure 4.1 corner K_a^b



The following result puts one-dimensional conditions on a process that in turn yields a "corner Markov" type of property; that is

$$\mathcal{G}_{[0,K_a^b]} \perp\!\!\!\perp \mathcal{G}_{\overline{[0,K_a^b]^c}} \,|\, \mathcal{G}_{K_a^b} \quad \text{for all } a, b \geq 0.$$

Theorem 4.1 *Assume the following:*

$$\mathcal{A}_i : \mathcal{G}^i_{[0,u]} \perp\!\!\!\perp \mathcal{G}^i_{[u,\infty)} \,|\, \mathcal{G}^i_{\{u\}}, \ \textit{for all } u \geq 0 \qquad i = 1,2$$

$$\mathcal{A}_3 : \mathcal{G}^1_{[0,u_1]} \perp\!\!\!\perp \mathcal{G}^2_{[0,u_2]} \,|\, \mathcal{G}_{[0,K_{u_1}^{u_2}]}, \ \textit{for all } u_1, u_2 \geq 0$$

$$\mathcal{B}_i : M_{\mathcal{G}^i_{\{u_i\}}} \bigcap M_{\mathcal{G}_{[0,K_{u_1}^{u_2}]}} = M_{\mathcal{G}i_{u_1}^{u_2}},$$

where $M_{\mathcal{A}} \equiv \{f \in L^2(\Omega, \mathcal{F}, P) : f$ *is* $\mathcal{A}$*-measurable*$\}$ *for any sub-*σ*-algebra* $\mathcal{A}$ *of* $\mathcal{F}$ *and* $\mathcal{G}i_{u_1}^{u_2} \equiv \sigma(X_{(x_1,s_2)} : s_i = u_i, 0 \leq s_{3-i} \leq u_{3-i})$ $i = 1,2$

$$\mathcal{C}_i : E(P(B|\mathcal{G}^i_{\{u_i\}})|\mathcal{G}_{[0,K_{u_1}^{u_2}]}) = E(P(B|\mathcal{G}_{[0,K_{u_1}^{u_2}]})|\mathcal{G}^i_{\{u_i\}}), \qquad i = 1,2$$

for all $B \in \mathcal{G}_{\{(s,t)\}}$ *such that* $s \geq u_1, t \geq u_2$ *for all* $u_1, u_2 \geq 0$.

Note that $\mathcal{A}_3$ is a Cairoli-Walsh [CW78] (F4)-type of condition.

Then $\mathcal{G}_{[0,K_{u_1}^{u_2}]} \perp\!\!\!\perp \mathcal{G}_{\overline{[0,K_{u_1}^{u_2}]^c}} \,|\, \mathcal{G}_{K_{u_1}^{u_2}}$ for all $u_1, u_2 \geq 0$.

Proof. Given any $u = (u_1, u_2) \in \mathcal{R}_+^2$, define the following regions.

$$[1] = \{(s,t) : s \geq u_1, \quad 0 \leq t \leq u_2\}$$

$$[2] = \{(s,t) : 0 \leq s \leq u_1, \quad t \geq u_2\}$$

$$[3] = \{(s,t) : s \geq u_1, \quad t \geq u_2\}.$$

For notation convenience, let $K = K_{u_1}^{u_2}$.

Part I

Let $B_i \in \mathcal{G}_{[i]}$ $i \in \{1,2,3\}$. If $i \in \{1,2\}$, then $P(B_i|\mathcal{G}_{[0,K]}) = P(B_i|\mathcal{G}^i_{[0,u_i]})$ using $\mathcal{A}_3$ and $B_i \in \mathcal{G}^{3-i}_{[0,u_{3-i}]}$. Furthermore, $P(B_i|\mathcal{G}^i_{[0,u_i]}) = P(B_i|\mathcal{G}^i_{\{u_i\}})$ by $\mathcal{A}_i$ and $B_i \in \mathcal{G}_{[u_i,\infty)}$. Thus $P(B_i|\mathcal{G}_{[0,K]}) = P(B_i|\mathcal{G}^i_{\{u_i\}})$. Otherwise, $P(B_3 \,|\, \mathcal{G}_{[0,K]})$ $= E(P(B_3 \,|\, \mathcal{G}^i_{[0,u_i]}) \,|\, \mathcal{G}_{[0,K]})$ by $\mathcal{G}_{[0,K]} \subseteq \mathcal{G}^i_{[0,u_i]}$, which in turn equals $E(P(B_3 \,|\, \mathcal{G}^i_{\{u_i\}}) \,|\, \mathcal{G}_{[0,K]})$ using $\mathcal{A}_i$ and $B_3 \in \mathcal{G}^i_{[u_i,\infty)}$. Thus

$$P(Bi \,|\, \mathcal{G}_{[0,K]}) = P(Bi \,|\, \mathcal{G}^i_{\{u_i\}}), \quad i = 1,2 \qquad *(i)$$

and

$$P(B_3 \,|\, \mathcal{G}_{[0,K]}) = E(P(B_3 \,|\, \mathcal{G}^i_{\{u_i\}}) \,|\, \mathcal{G}_{[0,K]}). \qquad *(3i)$$

Notation. Given a sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$, let $\mathbb{P}_{\mathcal{A}}$ denote the $L^2(\Omega)$-projection onto $M_{\mathcal{A}}$. Of course, $\mathbb{P}_{\mathcal{A}} f = E(f \,|\, \mathcal{A})$, for all $f \in L^2(\Omega)$.

Part II

Now we prove $P(B_i \,|\, \mathcal{G}_{[0,K]}) = P(B_i \,|\, \mathcal{G}i^{u_2}_{u_1})$, and $P(B_3 \,|\, \mathcal{G}_{[0,K]}) = P(B_3 \,|\, \mathcal{G}i^{u_2}_{u_1})$ for $i = 1,2$. First, let $i \in \{1,2\}$. Note that $\mathbb{P}_{\mathcal{G}_{[0,K^{u_1}_{u_2}]}} 1_{B_i} \in M_{\mathcal{G}^i_{\{u_i\}}} \cap M_{\mathcal{G}_{[0,K]}}$ by $*(i)$ and that $M_{\mathcal{G}^i_{\{u_i\}}} \cap M_{\mathcal{G}_{[0,K]}} = M_{\mathcal{G}i^{u_2}_{u_1}}$ by $\mathcal{B}_i$. Moreover,

$$\min_{X \in M_{\mathcal{G}i^{u_2}_{u_1}}} \|X - 1_{B_i}\| \geq \min_{X \in M_{\mathcal{G}^i_{\{u_i\}}}} \|X - 1_{B_i}\|$$

since $M_{\mathcal{G}i_{u_1}^{u_2}} \subseteq M_{\mathcal{G}i_{\{u_i\}}}$ and $\min_{X \in M_{\mathcal{G}^i_{\{u_i\}}}} \|X - 1_{B_i}\| = \|\mathbb{P}_{\mathcal{G}^i_{\{u_i\}}} 1_{B_i} - 1_{B_i}\| = \|\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} - 1_{B_i}\|$ by $*(i)$. But $\|\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} - 1_{B_i}\| \le \min_{X \in M_{\mathcal{G}i_{u_1}^{u_2}}} \|X - 1_{B_i}\|$ and $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} \in M_{\mathcal{G}_{u_1}^{u_2}}$ together imply $\|\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} - 1_{B_i}\| = \min_{X \in M_{\mathcal{G}i_{u_1}^{u_2}}} \|X - 1_{B_i}\|$. Hence $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} = \mathbb{P}_{\mathcal{G}i_{u_1}^{u_2}} 1_{B_i}$; that is, $P(B_i \,|\, \mathcal{G}_{[0,K]}) = P(B_i \,|\, \mathcal{G}i_{u_1}^{u_2})$.

Next we consider B_3.

Lemma 4.1 $(\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^n 1_{B_3} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3}$ *for all* $n \in \mathbb{N}$.

Proof. We shall prove this by induction. The proof for $n = 1$ follows from $*(3)$.

Next, assume the result holds for n; show the results holds for $n+1$: Now

$$\begin{aligned}(\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^{n+1} 1_{B_3} &= (\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})(\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^n 1_{B_3} \\ &= (\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})(\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3})\end{aligned}$$

by assumption. Moreover,

$$\begin{aligned}(\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})(\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3}) &= \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}} 1_{B_3} \\ &= \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}} \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3}\end{aligned}$$

by applying $*(3)$ and $\mathcal{C}_i$. One more application of $*(3)$ allows us to conclude $(\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^{n+1} 1_{B_3} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3}$.

Thus

$$P(B_3 \,|\, \mathcal{G}i_{u_1}^{u_2}) = P(B_3 \,|\, \mathcal{G}^i_{\{u_i\}} \cap \mathcal{G}_{[0,K]}) = \lim_{n \to \infty} (\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^n 1_{B_3}$$

using $\mathcal{B}_i$, and

$$\lim_{n \to \infty} (\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^i_{\{u_i\}}})^n 1_{B_i} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3} = P(B_3 \,|\, \mathbb{P}_{\mathcal{G}_{[0,K]}})$$

by Lemma 4.1. Hence $P(B_3 \,|\, \mathcal{G}i_{u_1}^{u_2}) = P(B_3 \,|\, \mathbb{P}_{\mathcal{G}_{[0,K]}})$.

Part III

Next, we show $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} = \mathbb{P}_{\mathcal{G}_K} 1_{B_i}$ for $i = 1,2,3$. Now $\mathbb{P}_{\mathcal{G}_K} 1_{B_i} = \mathbb{P}_{\mathcal{G}_K}(\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i})$ and $\mathbb{P}_{\mathcal{G}_K}(\mathbb{P}_{\mathcal{G}_{(2-|i-2|)_{u_1}^{u_2}}} 1_{B_i}) = \mathbb{P}_{\mathcal{G}_{(2-|i-2|)_{u_1}^{u_2}}} 1_{B_i}$ since $M_{\mathcal{G}i_{u_1}^{u_2}} \subseteq M_{\mathcal{G}_{K_{u_1}^{u_2}}}$ for $i = 1,2$. By applying $\mathbb{P}_{\mathcal{G}_{(2-|i-2|)_{u_1}^{u_2}}} 1_{B_i} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i}$ twice (by Part II), we get the desired result.

Part IV

In order to complete the proof, we must show that Part III holds for sets of the form $B_1 \cap B_2$, $B_1 \cap B_3$, $B_2 \cap B_3$, and $B_1 \cap B_2 \cap B_3$. We shall first consider $B_1 \cap B_2$. Observe that

$$
\begin{aligned}
\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2} &= \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2} && \text{by } \mathcal{A}_3 \\
&= \mathbb{P}_{\mathcal{G}_K} 1_{B_1} \cdot \mathbb{P}_{\mathcal{G}_K} 1_{B_2} && \text{by Part III} \\
&= \mathbb{P}_{\mathcal{G}_K}(1_{B_1} \cdot \mathbb{P}_{\mathcal{G}_K} 1_{B_2}) && \text{by } \mathbb{P}_{\mathcal{G}_K} 1_{B_2} \in M_{\mathcal{G}_K} \\
&= \mathbb{P}_{\mathcal{G}_K}(1_{B_1} \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2}) && \\
&= \mathbb{P}_{\mathcal{G}_K}(\mathbb{P}_{\mathcal{G}_{[0,K]}}(1_{B_i} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2})) && \text{by } M_{\mathcal{G}_K} \subseteq M_{\mathcal{G}_{[0,K]}} \\
&= \mathbb{P}_{\mathcal{G}_K}(\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1}) && \text{by } \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2} \in M_{\mathcal{G}_{[0,K]}} \\
&= \mathbb{P}_{\mathcal{G}_K}(\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2}) && \text{by } \mathcal{A}_3 \\
&= \mathbb{P}_{\mathcal{G}_K} 1_{B_1 \cap B_2} && \text{by } M_{\mathcal{G}_K} \subseteq M_{\mathcal{G}_{[0,K]}}.
\end{aligned}
$$

Note that the above implies

$$
\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i} f = \mathbb{P}_{\mathcal{G}_K} 1_{B_i} f, \text{ for all } f \in M_{\mathcal{G}_{[3-i]}}, \tag{4.1}
$$

where $i = 1,2$. Next we shall consider $B_i \cap B_3$ for $i \in \{1,2\}$. Recall that

$$\begin{aligned}
\mathbb{P}_{M_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}}} 1_{B_3} &= \mathbb{P}_{M_{\mathcal{G}_{[0,K]}} \bigvee M_{\mathcal{G}_{[i]}}} 1_{B_3} \\
&= \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3} + \mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3} - \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3} \\
&\quad + \mathbb{P}_{\mathcal{G}_{[i]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3} - \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[i]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3} + \cdots (4.2)
\end{aligned}$$

We shall show that each summand is in $M_{\mathcal{G}_{[i]}}$. First, $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_3} = \mathbb{P}_{\mathcal{G}_{i u_1}^{u_2}} 1_{B_3} \in M_{\mathcal{G}_{[i]}}$ using Part II and $M_{\mathcal{G}i_{u_1}^{u_2}} \subseteq M_{\mathcal{G}_{[i]}}$.

From Part II, we see that $E(f \,|\, \mathcal{G}_{[0,K]}) = E(f \,|\, \mathcal{G}i_{u_1}^{u_2})$, for all $f \in M_{\mathcal{G}_{[i]}}$. Combining this and the relation $M_{\mathcal{G}_{u_2}^{u_1}} \subseteq M_{\mathcal{G}_{[i]}}$, we conclude that

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3} = \mathbb{P}_{\mathcal{G}i_{u_1}^{u_2}} (\mathbb{P}_{\mathcal{G}_{[i]}} 1_{B_3}) = \mathbb{P}_{\mathcal{G}i_{u_1}^{u_2}} 1_{B_3} \in M_{\mathcal{G}_{[i]}}.$$

Using the same techniques as above, we see that each term in (4.2) is an element of $M_{\mathcal{G}_{[i]}}$, and so we can conclude that $\mathbb{P}_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}} 1_{B_3} \in M_{\mathcal{G}_{[i]}}$. We will use this fact below.

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i \cap B_3} = \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_i} \mathbb{P}_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}} 1_{B_3})$$

by $M_{\mathcal{G}_{[0,K]}} \subseteq M_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}}$ and $B_i \in \mathcal{G}^{3-i}_{[0,u_{3-i}]}$. Using (4.1) and $\mathbb{P}_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}} 1_{B_3} \in M_{\mathcal{G}_{[i]}}$, we have

$$\mathbb{P}_{\mathcal{G}_K} (1_{B_i} \cdot \mathbb{P}_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}} 1_{B_3}) = \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_i} \mathbb{P}_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}} 1_{B_3}).$$

We eventually conclude that $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_i \cap B_3} = \mathbb{P}_{\mathcal{G}_K} 1_{B_i \cap B_3}$ by recognizing that $M_{\mathcal{G}_K} \subseteq M_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}}$ and $B_i \in M_{\mathcal{G}^{3-i}_{[0,u_{3-i}]}}$.

Finally, we must prove $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3} = \mathbb{P}_{\mathcal{G}_K} 1_{B_1 \cap B_2 \cap B_3}$, for each $B_i \in \mathcal{G}_{[i]}$.

Lemma 4.2 $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3} \in M_{\mathcal{G}_K}$.

Proof. Recall that

$$\begin{aligned}\mathbb{P}_{\mathcal{G}_{[0,u_i]}} 1_{B_1 \cap B_3} &= \mathbb{P}_{M_{\mathcal{G}_{[0,K]}} \vee M_{\mathcal{G}_{[2]}}} 1_{B_1 \cap B_3} \\ &= \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_3} + \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} \\ &\quad - \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} + \mathbb{P}_{\mathcal{G}_{[2]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} \\ &\quad - \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} + \cdots\end{aligned}$$

Thus $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3}$

$$\begin{aligned}&= \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}^1_{[0,u_1]}} (1_{B_1 \cap B_2 \cap B_3}) && \text{by } M_{\mathcal{G}_{[0,K]}} \subseteq M_{\mathcal{G}^1_{[0,u_1]}} \\ &= \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}^1_{[0,u_1]}} 1_{B_1 \cap B_3}) && \text{by } B_2 \in M_{\mathcal{G}^1_{[0,u_1]}} \\ &= \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_3}) \\ &\quad + \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) - \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} - \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) \\ &\quad + \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[2]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) \\ &\quad - \mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) + \cdots\end{aligned}$$

We must consider each term separately and show it to be in $M_{\mathcal{G}_K}$. First,

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_3}) = \mathbb{P}_{\mathcal{G}_K} 1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_K} 1_{B_1 \cap B_3}$$

using $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_3} \in M_{\mathcal{G}_{[0,K]}}$ and Part III. Next,

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) = \mathbb{P}_{\mathcal{G}_{[0,K]}} (\mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_2 \cap B_3}) = \mathbb{P}_{\mathcal{G}_K} (\mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_2 \cap B_3})$$

by (4.1) and $B_2 \in \mathcal{G}_{[2]}$. $\mathbb{P}_{\mathcal{G}_{[0,K]}} \mathcal{G}_{[2]} 1_{B_1 \cap B_3} \in M_{\mathcal{G}_{[0,K]}}$ and (4.1) together imply

$$\begin{aligned}\mathbb{P}_{\mathcal{G}_{[0,K]}} (1_{B_2} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) &= \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_2} \\ &= \mathbb{P}_{\mathcal{G}_K} (B_2 \cdot \mathbb{P}_{\mathcal{G}_{[0,K]}} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3}) \\ &= \mathbb{P}_{\mathcal{G}_K} \mathbb{P}_{\mathcal{G}_{[2]}} 1_{B_1 \cap B_3} \cdot \mathbb{P}_{\mathcal{G}_K} 1_{B_2}.\end{aligned}$$

These same techniques will yield that every other term in the series is indeed in $M_{\mathcal{G}_K}$. Hence $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3} \in M_{\mathcal{G}_K}$.

This claim and the fact that

$$\begin{aligned} \min_{x \in M_{\mathcal{G}_K}} \|X - 1_{B_1 \cap B_2 \cap B_3}\| &\geq \min_{X \in M_{\mathcal{G}_{[0,K]}}} \|X - 1_{B_1 \cap B_2 \cap B_3}\| \\ &= \|\mathbb{P}_{\mathcal{G}_{[0,K]}} - 1_{B_1 \cap B_2 \cap B_3}\| \end{aligned}$$

will give us $\mathbb{P}_{\mathcal{G}_K} 1_{B_1 \cap B_2 \cap B_3} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3}$.

The last step is using the $\pi - \lambda$ Theorem to show that $\mathbb{P}_{\mathcal{G}_K} 1_{B_1 \cap B_2 \cap B_3} = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{B_1 \cap B_2 \cap B_3}$ for all $B_i \in \mathcal{G}_{[i]}$ implies $\mathbb{P}_{\mathcal{G}_K} 1_B = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_B$, all $B \in \mathcal{G}_{[0,K]^c}$.

$\mathcal{L} \equiv \{A \in \mathcal{F} : \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_A = \mathbb{P}_{\mathcal{G}_K} 1_A\}$ is a λ-system as follows:

i) $\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_\Omega = 1 = \mathbb{P}_{\mathcal{G}_K} 1_\Omega$ implies that $\Omega \in \mathcal{L}$.

ii) Let $A \in \mathcal{L}$. Then

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{A^c} = 1 - \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_A = 1 - \mathbb{P}_{\mathcal{G}_K} 1_A = \mathbb{P}_{\mathcal{G}_K} 1_{A^c}$$

since $A \in \mathcal{L}$.

iii) Let $\{A_n\} \subseteq \mathcal{L}$ be pairwise disjoint. Then

$$\mathbb{P}_{\mathcal{G}_{[0,K]}} 1_{\cup A_n} = \mathbb{P}(1_{\cup A_n} \mid \mathcal{G}_{[0,K]}) = \mathbb{P}(\sum_n 1_{A_n} \mid \mathcal{G}_{[0,K]}) = \sum_n \mathbb{P}(1_{A_n} \mid \mathcal{G}_{[0,K]}) A_n \in \mathcal{L}$$

by pairwise disjointness. Furthermore,

$$\sum_n \mathbb{P}(1_{A_n} \mid \mathcal{G}_{[0,K]}) = \sum_n \mathbb{P}(1_{A_n} \mid \mathcal{G}_K) = \mathbb{P}(\sum_n 1_{A_n} \mid \mathcal{G}_K) = \mathbb{P}_{\mathcal{G}_K} 1_{\cup A_n}.$$

Therefore $\bigcup_n A_n \in \mathcal{L}$, and we conclude that $\mathcal{L}$ is a λ-system.

Now apply the $\pi - \lambda$ theorem to get that $\mathbb{P}_{\mathcal{G}_K} 1_B = \mathbb{P}_{\mathcal{G}_{[0,K]}} 1_B$, for all $B \in \mathcal{G}_{[0,K]}$. Therefore $\mathcal{G}_{[0,K]} \perp\!\!\!\perp \mathcal{G}_{\overline{[0,K]^c}} \mid \mathcal{G}_K$.

The proof is now complete. □

Of particular interest are random domains Q which take on the form $Q = [0, K_{\tau_1}^{\tau_2}]$ for τ_1, τ_2 nonnegative random variables. That is, Q takes values in the class $\{[0, K_a^b] : a, b > 0\}$. Note that K_a^b is a membrane if $a, b > 0$.

It is desirable to determine what (2.6) means when $Q = [0, K_{\tau_1}^{\tau_2}]$.

Lemma 4.2 (2.6) *holds if and only if*

$$\{K_{\tau_1}^{\tau_2} \leq K_a^b\} \in \mathcal{F}_{[0,K_a^b]}, \; \textit{for all } a, b > 0. \tag{4.3}$$

Proof. Certainly, (2.6) implies (4.3), since $[0, K_a^b] \in T$ when $a, b > 0$. Moreover, (2.6) follows from (4.3) upon noting that, given $V \in T$ and $\epsilon > 0$,

$$\{Q \subseteq V\} = \bigcap_{n=[\frac{1}{\epsilon}]+1} \quad \bigcup_{a,b \in \mathbb{Q}^+, [0,K_a^b] \subseteq V^{\frac{1}{n}}} \{K_{\tau_1}^{\tau_2} \leq K_a^b\} \in \mathcal{F}_{V^\epsilon}.$$

Since $\bigcap_{\epsilon>0} \mathcal{F}_{V^\epsilon}$, we are done. □

When we write (4.3) as

$$\{\tau_1 \leq a, \tau_2 \leq b\} \mathcal{F}_{[0,K_a^b]}, \quad \forall\, a, b > 0,$$

we are reminded of a stopping time-type of condition.

Theorem 4.2 *Let ξ be a Markov random field and τ_1 and τ_2 be random variables which satisfy*

$$0 \leq \tau_1(\omega) \leq \tau_2(\omega), \quad \textit{for all } \omega \in \Omega.$$

If $\{K_{\tau_1}^{\tau_2} \leq K_a^b\} \in \mathcal{F}_{[0,K_a^b]}$, for all $a, b > 0$, then ξ is strong Markov with respect to the random domain $Q = [0, K_{\tau_1}^{\tau_2}]$.

The random analogue to Theorem 4.1 also arrives at a strong corner Markov property and is stated below.

Theorem 4.3. *Let τ_1, τ_2 be countably-valued stopping times, and let $K \equiv K_{\tau_1}^{\tau_2}$ for notational convenience.*

Let

$$\mathcal{F}_{\tau_i}^- = \sigma(A \cap \{\tau i = a\} : \quad a \in \tau i(\Omega), \quad A \in \mathcal{G}^i_{[0,a]}),$$
$$\mathcal{F}_{\tau i}^+ = \sigma(A \cap \{\tau i = a\} : \quad a \in \tau i(\Omega), \quad A \in \mathcal{G}^i_{[0,a]}),$$

and

$$\mathcal{F}_{\tau i} = \sigma(A \cap \{\tau i = a\} : \quad a \in \tau i(\Omega), \quad \mathcal{G}^i_{\{a\}}) \qquad \text{for } i = 1, 2.$$

Also define

$$\mathcal{F}_K^- = \sigma(A \cap \{\tau_1 = a\} \cap \{\tau_2 = b\} : \quad (a,b) \in \mathcal{R}^2, \quad A \in \mathcal{G}_{[0,K_a^b]}),$$
$$\mathcal{F}_K^+ = \sigma(A \cap \{\tau_1 = a\} \cap \{\tau_2 = b\} : \quad (a,b) \in \mathcal{R}^2, \quad A \in G_{[K_a^b,\infty)}),$$

and

$$\mathcal{F}_K = \sigma(A \cap \{\tau_1 = a\} \cap \{\tau_2 = b\} : \quad (a,b) \in \mathcal{R}^2, \quad A \in \mathcal{G}_{K_a^b}).$$

Assume the following for $i = 1, 2$.

$\mathcal{B}_i : M_{\mathcal{F}_K^-} \cap M_{\mathcal{F}_{\tau_i}} = M_{\mathcal{H}_i}$, where

$$\mathcal{H}_i \equiv \sigma(A \cap \{\tau_1 = u_1\} \cap \{\tau_2 = u_2\} : (u_1, u_2) \in \mathcal{R}^2, \quad A \in \mathcal{G}_{iu_1^{u_2}}).$$

$\mathcal{C}_i : \mathbb{P}(P(B_3 \,|\, \mathcal{F}_{K^-}) \,|\, \mathcal{F}_{\tau i}) = \mathbb{P}(P(B_3 \,|\, \mathcal{F}_{\tau i}) \,|\, \mathcal{F}_{K^-})$, for all $B_3 \in \mathcal{F}_3$, where

$$\mathcal{F}_3 \equiv \sigma(A \cap \{\tau_1 = a\} \cap \{\tau_2 = b\} : (a,b) \in \mathcal{R}^2,$$
$$A \in \mathcal{G}_{III_{a,b}} \equiv \sigma(\xi_{(s,t)} : s \geq 0, t \geq b))$$

$\mathcal{A}i : \mathcal{F}_{\tau_i}^- \perp\!\!\!\perp \mathcal{F}_{\tau_i}^+ \,|\, \mathcal{F}_{\tau_i}$ for $i = 1, 2$

$\mathcal{A}_3 : \mathcal{F}_{\tau_1}^- \perp\!\!\!\perp \mathcal{F}_{\tau_2}^- \,|\, \mathcal{F}_K^-$.

Then

$$\mathcal{F}_K^- \perp\!\!\!\perp \mathcal{F}_K^+ \,|\, \mathcal{F}_K \,.$$

Proof. The techniques used in the proof of theorem 4.1 are the techniques used to prove this theorem with the obvious substitutions. $\mathcal{F}_{T_i}^-$, $\mathcal{F}_{T_i}^+$, and $\mathcal{F}_{T_i}$ play the roles that $\mathcal{G}^i_{[0,u]}, \mathcal{G}^i_{[u,\infty)}$, and $\mathcal{G}^i_{\{u_i\}}$ played in the proof of Theorem 4.1, respectively. We use $\mathcal{F}_K^-, \mathcal{F}_K^+$, and $\mathcal{F}_K$ in this proof for the $\mathcal{F}_{[0,K^{u_2}_{u_1}]}$, $\mathcal{F}_{\overline{[0,K^{u_2}_{u_1}]^c}}$, and $\mathcal{F}_K$ used in the proof of Theorem 4.1, respectively. Also, $\mathcal{F}_3$ replaces $\mathcal{G}_{[3]}$ and $M_{\mathcal{H}_i}$ replaces $M_{\mathcal{G}^i{}^{u_2}_{u_1}}$ in this proof. □

4.2 A Martingale Approach

The following question arises naturally upon considering reciprocal processes $(d = 1)$:

What kind of strong Markov property, if any, does a given reciprocal process have?

Pasha [Pas82] showed that every Gaussian reciprocal process ξ on a compact interval $[a, b]$ can be expressed as

$$\xi_t = Y_t + A_t\xi_a + B_t\xi_b,$$

where Y_t is Markov with trivial tails, A and B are real functions on $[a, b]$, and $\sigma(\xi_a, \xi_b)$ is independent of $\sigma(Y_t)$ for each $t \in [a, b]$. If $Y_t \not\equiv 0$ for every t and if ξ is continuous in quadratic mean, then $Y_t = \Phi_t U_t$ for all t, where Φ_t is a real function on $[a, b]$ and U is a martingale. One can look at a stochastic differential equation for the process $Z_t = \langle Y_t, A_t\xi_a, B_t\xi_b\rangle^{\mathrm{T}}$. Z_t satisfies the stochastic differential equation

$$dZ_t = \begin{bmatrix} \Phi_t'\Phi_t^{-1} & 0 & 0 \\ 0 & A_t'A_t^{-1} & 0 \\ 0 & 0 & B_t'B_t^{-1} \end{bmatrix} Z_t dt + \begin{bmatrix} \Phi_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dU_t \\ 0 \\ 0 \end{bmatrix}$$

under appropriate differentiability assumptions on Φ, A, and B. By the nature of this differential equation, one needs techniques from [SV79] to study the strong Markov properties of its solution. In the special case of $A_t\xi_a + B_t\xi_b = 0$ for every t (that is, ξ is independent of its boundaries), one can use techniques exhibited in [Str87] and [SV79] to consider ξ as a solution of a stochastic differential equation, and then apply Theorem 6.6.2 of [SV79] to get a strong Markov property on ξ. Because the equation involved in the general case is a singular differential equation, one must define the strong Markov property delicately. This work is currently under investigation.

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