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ABSTRACT

DIRICHLET SPACES ON FINITELY CONNECTED DOMAINS

By

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Suppose Ω is a finitely connected nonempty domain in $\mathbf{C}$ such that no connected component of $\partial\Omega$ is equal to a point.

In the second chapter we show that $\sigma_e(M_\varphi)$, the essential spectrum of a multiplication operator M_φ on the Dirichlet space $D(\Omega)$, is equal to $\mathrm{cl}(\varphi; \partial\Omega)$, the cluster set of φ on $\partial\Omega$. In order to prove this equality, we first show that, if Ω has an analytic boundary, then the set of rational functions in $D(\Omega)$ whose poles are off $\overline{\Omega}$ is dense in $D(\Omega)$ and $D(\Omega)$ is contained in $B(\Omega)$, the Bergman space on Ω. And then we prove $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi; \partial\Omega)$ when Ω has an analytic boundary. By the conformal invariance of $\sigma_e(M_\varphi)$ and $\mathrm{cl}(\varphi; \partial\Omega)$, we have the desired equality.

The next chapter characterizes the finite codimensional closed invariant subspaces of $D(\Omega)$ under any multiplication operator when Ω has an analytic boundary. We show that those subspaces are of the form $qD(\Omega)$ where q is a polynomial with all its zeros in Ω. To prove this we show that $(z - \lambda)D(\Omega)$ is dense in $D(\Omega)$ for any λ in $\partial\Omega$ when Ω has an analytic boundary. Here we use the result from Chapter 2; namely $R(\Omega)\cap D(\Omega, z_0)$ is a dense subset of $D(\Omega, z_0)$ where $R(\Omega)$ is the set of rational functions whose poles are off $\overline{\Omega}$.

To my parents and my wife Ae-Young

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CHAPTER 1

PRELIMINARY

In this chapter, we will introduce some definitions, notations, and basic facts about the Dirichlet space. Throughout this thesis, Ω will denote a domain, namely a nonempty open connected set in the complex plane $\mathbf{C}$, such that no connected component of $\partial\Omega$ is equal to a point.

The *Bergman space* $B(\Omega)$ is the Hilbert space of analytic functions f on Ω such that $\int_\Omega |f|^2 dA < \infty$, with the inner product

$$<f, g>_{B(\Omega)} = \int_\Omega f \bar{g} \, dA \tag{1.1}$$

where dA denotes the usual area measure on Ω.

Let z_0 be in Ω. The *Dirichlet space* $D(\Omega, z_0)$ is the Hilbert space of analytic functions f on Ω such that $\int_\Omega |f'|^2 dA < \infty$ and $f(z_0) = 0$, with the inner product

$$<f, g>_{D(\Omega)} = \int_\Omega f' \, \overline{g'} \, dA. \tag{1.2}$$

Changing the distinguished point z_0 gives a space that is obtained from the original by subtracting a suitable constant from each function. We will use $D(\Omega)$ instead of $D(\Omega, z_0)$ if the distinguished point is irrelevant.

The square of the Dirichlet norm of f is just the area of the image of Ω under f, counting multiplicity.

We will use $\|f\|^2_{D(\Omega)}$ to denote $\int_{\Omega} |f'|^2 \, dA$ even when $f \in H(\Omega) \setminus D(\Omega, z_0)$ where $H(\Omega)$ is the set of analytic functions on Ω.

It is well known that point evaluation maps on $B(\Omega)$ (see Conway [8], Chapter III, Corollary 10.3) and $D(\Omega)$ (see Taylor [14]) are bounded. Here we will prove the boundedness of point evaluation of each derivative on $D(\Omega)$.

Lemma 1.3: *Let $z \in \Omega$ and let $n \in \mathrm{N} \cup \{0\}$. Then the map $\lambda_{z,n} : D(\Omega, z_0) \to \mathrm{C}$ defined by $\lambda_{z,n}(f) = f^{(n)}(z)$ is a bounded linear functional.*

Proof: Let $z \in \Omega$. First assume $n = 0$. We will use λ_z instead of $\lambda_{z,0}$. Let Γ be a rectifiable path in Ω from z_0 to z. Then

$$|\lambda_z(f)| = |f(z)| = \left| \int_{\Gamma} f'(w) \, dw \right| \leq (\text{length of } \Gamma) \ \sup\{ |f'(w)| : w \in \Gamma \}. \qquad (1.4)$$

Since Γ is a compact subset of Ω, there exists $r > 0$ such that the distance between Γ and $\partial\Omega$ is bigger than r. Let $g \in B(\Omega)$ and let $B(w,r)$ be the open disk in C centered at the point w with radius r. For each $w \in \Gamma$,

$$|g(w)| \leq \frac{1}{\pi r^2} \int_{B(w,r)} |g(t)| \, dA(t) \qquad \text{by the mean value property}$$

$$\leq \frac{1}{\sqrt{\pi}\, r} \|g\|_{B(\Omega)}. \qquad \text{by Hölder's inequality}$$

Since $f' \in B(\Omega)$, there is a constant K, which depends only on Γ and Ω, such that

$$\text{the right hand side of (1.4)} \leq K \|f'\|_{B(\Omega)} = K \|f\|_{D(\Omega)}.$$

Therefore the point evaluation map λ_z is bounded.

Now let $n \geq 1$. Choose $\delta > 0$ such that $\overline{B(z,\delta)} \subset \Omega$. Here $\overline{B(z,\delta)}$ denotes the closure of $B(z,\delta)$ in C. Since $\{\lambda_w: w \in \partial B(z,\delta)\}$ is a subset of the dual space $D(\Omega,z_0)^*$ and $\sup\{\,|\lambda_w(f)|: w \in \partial B(z,\delta)\} < \infty$ for all $f \in D(\Omega,z_0)$, by the Uniform Boundedness Principle, there is a constant K such that $\sup\{\,\|\lambda_w\|: w \in \partial B(z,\delta)\} \leq K$.
Hence $|f(w)| \leq \|\lambda_w\|\,\|f\|_{D(\Omega)} \leq K\,\|f\|_{D(\Omega)}$ for all $f \in D(\Omega,z_0)$ and for all $w \in \partial B(z,\delta)$. By the Cauchy Formula,

$$|\lambda_{z,n}(f)| = |f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{\partial B} \frac{|f(w)|}{|w-z|^{n+1}}\,|dw| \leq \frac{n!}{\delta^n} K \|f\|_{D(\Omega)}$$

for all $f \in D(\Omega,z_0)$. Q.E.D.

Using the same argument as in the proof of the above lemma, we can prove that every norm bounded subset of $D(\Omega,z_0)$ is uniformly bounded on each compact subset of Ω. In a normed vector space, every weakly convergent sequence is norm bounded. Hence we get the following lemma by the normal family argument.

Lemma 1.5: *Let $n \in \mathrm{N}\cup\{0\}$. If $\{f_m\}$ is a sequence in $D(\Omega,z_0)$ converging to f weakly, then $f_m^{(n)} \to f^{(n)}$ uniformly on compact subsets of Ω as $m \to \infty$.*

Remarks: (*a*) If $\{f_\alpha\}_{\alpha\in A}$ is a bounded net in $D(\Omega)$ such that f_α converges to f weakly, then we can still apply the normal family argument to prove that $f_\alpha^{(n)} \to f^{(n)}$ uniformly on compact subsets of Ω for all $n \in \mathbf{N}$.

(*b*) A weaker version of the converse of the above lemma will be discussed in Lemma 3.3.

An analytic function φ on Ω is called a *multiplier* of $D(\Omega,z_0)$ if $\varphi D(\Omega,z_0) \subset D(\Omega,z_0)$. We denote by $M(D(\Omega,z_0))$ the set of all multipliers of $D(\Omega,z_0)$. For any multiplier φ, the linear transformation $M_\varphi : D(\Omega,z_0) \to D(\Omega,z_0)$ defined by $M_\varphi f = \varphi f$ is bounded; this follows from the Closed Graph Theorem and the boundedness of point evaluation maps. M_φ is called a *multiplication operator*. Giving each function in $M(D(\Omega,z_0))$ the operator norm of the corresponding multiplication operator makes $M(D(\Omega,z_0))$ into a normed space. Standard references for $M(D(U))$ are [13] and [14]; here U denotes the open unit disk in $\mathbf{C}$.

If $\varphi \in M(D(\Omega,z_0))$, then φ is in the set of bounded analytic functions $H^\infty(\Omega)$ with $\| \varphi \|_\infty \leq \| M_\varphi \|$ (see [10], Lemma 11). But the converse is not true. Actually $M(D(U))$ is not even closed under $\| \cdot \|_\infty$ norm (see Axler and Shields [5], Theorem 10).

Lemma 1.6: *If* Ω *is bounded, then* $\varphi' \in B(\Omega)$ *and* $\varphi - \varphi(z_0) \in D(\Omega,z_0)$ *for all* $\varphi \in M(D(\Omega))$.

Proof: Let $\varphi \in M(D(\Omega))$. Since Ω is bounded, $z - z_0 \in D(\Omega)$. Thus $[(z - z_0)\varphi]' = \varphi + (z - z_0)\varphi'$ is in $B(\Omega)$. Since φ is bounded, $(z - z_0)\varphi'$ is in $B(\Omega)$. Choose $r > 0$ such that $B(z_0, r) \subset \Omega$. Then

$$\infty > \int_{\Omega} |z - z_0|^2 |\varphi'|^2 \, dA > \int_{B(z_0, r)} |z - z_0|^2 |\varphi'|^2 \, dA + r^2 \int_{\Omega \setminus B(z_0, r)} |\varphi'|^2 \, dA.$$

Hence $\int_{\Omega \setminus B(z_0, r)} |\varphi'|^2 \, dA < \infty$. Since φ' is bounded on $B(z_0, r)$, φ' is in $B(\Omega)$. Now the second assertion follows immediately. Q.E.D.

The following lemma can be proved using change-of-variables.

Lemma 1.7: *Let* Ω_1 *and* Ω_2 *be two domains in* $\mathbf{C}$ *and let* $z_0 \in \Omega_1$ *and* $w_0 \in \Omega_2$. *Suppose* ψ *is a conformal mapping from* Ω_2 *onto* Ω_1 *such that* $\psi(w_0) = z_0$. *Then*

(1) *The composition map* $C_\psi: D(\Omega_1, z_0) \to D(\Omega_2, w_0)$ *defined by* $C_\psi(f) = f \circ \psi$ *is a unitary map.*

(2) *The composition map* $C_\psi: M(D(\Omega_1, z_0)) \to M(D(\Omega_2, w_0))$ *defined by* $C_\psi(\varphi) = \varphi \circ \psi$ *is an onto isometry.*

For the open unit disk U in $\mathbf{C}$, the spaces $D(U,0)$ and $B(U)$ can be described in terms of Taylor coefficients using (1.1) and (1.2); namely

$$\| f \|^2_{D(U,0)} = \pi \sum_{n=1}^{\infty} n |a_n|^2 , \tag{1.8}$$

$$\| f \|^2_{B(U)} = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} . \tag{1.9}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Hence we have $D(U,0) \subset B(U)$. For a simply connected domain Ω, there are some equivalent conditions to $D(\Omega) \subset B(\Omega)$ (see Axler and Shields [5], Theorem 1).

Denote the annular region in $\mathbf{C}$ centered at 0 with the inner radius $r > 0$ and outer radius 1 by A_r. Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Then $\|f\|^2_{D(A_r)}$, $\|f\|^2_{B(A_r)}$ can be written in terms of the Laurent series coefficients similar to those which are in (1.8) and (1.9). The formulae are given in the following lemma, which can be proved by direct calculation.

Lemma 1.10: *Suppose* $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

(1) *If* $f \in D(A_r, \sqrt{r})$, *then*

$$\|f\|^2_{D(A_r)} = \pi \sum_{n \neq 0} n\, |a_n|^2 (1 - r^{2n}). \tag{1.11}$$

(2) *If* $f \in B(A_r)$, *then*

$$\|f\|^2_{B(A_r)} = \pi \sum_{n \neq -1} \frac{|a_n|^2}{n+1} (1 - r^{2n+2}) - 2\pi\, |a_{-1}|^2 \log r. \tag{1.12}$$

If the infinite series (1.11) converges, then so does the series (1.12). Hence we have $D(A_r, \sqrt{r}) \subset B(A_r)$. Now, by the Closed Graph Theorem, the inclusion map $I : D(A_r) \to B(A_r)$ is bounded.

The next lemma establishes some equivalent conditions for $D(\Omega) \subset B(\Omega)$ when Ω is a bounded doubly connected domain in $\mathbf{C}$. These results can be proved as Theorem 1 in [5]. Recall that any doubly connected domain is conformally equivalent to some annulus (see, for example, Axler [1], Doubly Connected Mapping Theorem on page 255).

Lemma 1.13: *Let* Ω *be a bounded doubly connected domain in* $\mathbb{C}$, *and let* $z_0 \in \Omega$. *Suppose* ψ *is a conformal mapping from* A_r *onto* Ω *for some* $r > 0$. *Then the following are equivalent.*

(1) $D(\Omega,z_0) \subset B(\Omega)$

(2) $z\, D(\Omega,z_0) \subset D(\Omega,z_0)$

(3) $\psi\, D(A_r, \psi^{-1}(z_0)) \subset D(A_r, \psi^{-1}(z_0))$

(4) $\psi' D(A_r, \psi^{-1}(z_0)) \subset B(A_r)$

Remarks: (a) (1) and (2) are equivalent for any bounded domain Ω.

(b) (3) and (4) are equivalent for any bounded domain Ω; namely if $\psi \in H(\Omega)$ where Ω is a bounded domain, the two statements $\psi \in M(D(\Omega))$ and $\psi' D(\Omega) \subset B(\Omega)$ are equivalent.

(c) Suppose Ω in the previous lemma has an analytic boundary; namely Ω has two analytic curves as a boundary. Then ψ can be extended analytically up to ∂A_r by Schwarz Reflection Principle. (For the definition of an analytic curve and the analytic extension of ψ, see page 12 of this thesis.) Hence $|\psi'|$ is bounded on A_r and so condition (4) in the above lemma is satisfied. Here we used the fact that $D(A_r,\sqrt{r}) \subset B(A_r)$. Therefore we have $D(\Omega) \subset B(\Omega)$, z is in $M(D(\Omega,z_0))$, and especially $\psi \in M(D(A_r))$. In Chapter 2, we will see that $D(\Omega) \subset B(\Omega)$ when Ω is a finitely connected domain with an analytic boundary.

(d) For any domain Ω, if $D(\Omega) \subset B(\Omega)$, then the inclusion map $I : D(\Omega) \to B(\Omega)$ is bounded by the Closed Graph Theorem.

We do not know exactly when the inclusion map $I : D(\Omega) \subset B(\Omega)$ is compact. For simply connected domains, Axler and Shields got some results (see Axler and Shields [5]). For doubly connected domains, we have the following lemma.

Lemma 1.14: *Let* Ω *be a doubly connected domain and let* ψ *be a conformal mapping from* A_r *onto* Ω *for some* $r > 0$. *Then the following are equivalent.*

(1) *The inclusion map* $I : D(\Omega) \to B(\Omega)$ *is compact.*

(2) *The multiplication operator* $M_{\psi}' : D(A_r) \to B(A_r)$ *defined by* $M_{\psi}'(f) = \psi' f$ *is compact.*

Proof: Define an operator $T : B(\Omega) \to B(A_r)$ by $T(g) = \psi'(g \circ \psi)$. Then, by change-of-variables, T is isometry . Let $h \in B(A_r)$. It is easy to see that $(\frac{1}{\psi'} h) \circ \psi^{-1}$ is the preimage of h under T, again by change-of-variables. Hence T is a unitary map. Note that $M_{\psi}' \circ C_{\psi} = T \circ I$ on $D(\Omega)$ where C_{ψ} is the composition map as in Lemma 1.7. Since C_{ψ} and T are both unitary, (1) is equivalent to (2). Q.E.D.

We proved that a point evaluation map on the Dirichlet space is bounded in Lemma 1.3. When Ω is either U or A_r, we can find λ_z explicitly by direct calculation.

Suppose $\Omega = U$ and $z \in U$. Then λ_z defined by

$$\lambda_z(w) = \frac{1}{\pi} \log \frac{1}{1-\bar{z}w} \tag{1.15}$$

is the point evaluation map at z on $D(U,0)$.

When $\Omega = A_r$ and $z \in A_r$, λ_z defined by

$$\lambda_z(w) = \frac{1}{\pi} \sum_{n \neq 0} \frac{\bar{z}^n - (\sqrt{r})^n}{n\,(1 - r^{2n})} \, [w^n - (\sqrt{r})^n] \tag{1.16}$$

is the point evaluation map at z on $D(A_r, \sqrt{r}\,)$. It does not seem possible to express the infinite sum in (1.16) in closed form.

CHAPTER 2

ESSENTIAL SPECTRUM OF MULTIPLICATION OPERATORS

Recall that an operator T on a Hilbert space H is called *Fredholm* if the kernel of T and H/TH are both finite dimensional vector spaces. These conditions imply that T has closed range (see [6], Cor 3.2.5).

Suppose T is an operator on a Hilbert space H. The *essential spectrum* of T, denoted $\sigma_e(T)$, is defined to be the set of complex numbers c such that $T - c$ is not Fredholm. $\sigma_e(T)$ is precisely the spectrum of T in the Calkin algebra $L(H)/K(H)$ where $L(H)$ denotes the set of all bounded operators on H, and $K(H)$ denotes the set of all compact operators on H (see Douglas [9]).

If φ is an analytic function on Ω, then the *cluster set* of φ on $\partial\Omega$, denoted $\mathrm{cl}(\varphi;\partial\Omega)$, is the set of complex numbers c such that there exists a sequence $\{z_n\}$ in Ω such that z_n tends to $\partial\Omega$ and $f(z_n) \to c$ as $n \to \infty$.

Suppose G is any open set in the complex plane $\mathbf{C}$ such that no connected component of ∂G is equal to a point. On the Bergman space $B(G)$, Sheldon Axler showed that $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi;\partial G)$ when $\varphi \in M(B(G))$ (see Axler [2], Theorem 23). No result of this generality is known for the Dirichlet space. If Ω is a bounded simply connected domain, and if φ is a multiplier of $D(\Omega)$, then $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi;\partial G)$ (see Axler and Shields [5], Theorem 11). In this chapter, we will prove that the same conclusion holds when Ω is a bounded finitely connected domain (with an analytic boundary) and φ is a multiplier of $D(\Omega)$.

Suppose Ω is bounded. If $M_\varphi : D(\Omega,z_0) \to D(\Omega,z_0)$ is a multiplication operator and $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi;\partial\Omega)$, then M_φ is also a multiplication operator on $D(\Omega,z_1)$ for any $z_1 \in \Omega$ and has the same essential spectrum; namely the essential spectrum of a multiplication operator on the Dirichlet space does not depend on the choice of the distinguished point. Furthermore the essential spectrum of a multiplication operator is conformally invariant in the following sense.

Lemma 2.1: *Suppose Ω_1 and Ω_2 are domains in $\mathbf{C}$ and ψ is a conformal mapping from Ω_1 onto Ω_2 such that $\psi(z_0) = w_0$. Suppose $\varphi \in M(D(\Omega_2,w_0))$ and $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi;\partial\Omega_2)$. Then $\varphi\circ\psi \in M(D(\Omega_1,z_0))$ and $\sigma_e(M_{\varphi\circ\psi}) = \mathrm{cl}(\varphi\circ\psi;\partial\Omega_1)$.*

Proof: Define composition operators C_ψ as in Lemma 1.7. Even though we are using the same notation for two different composition operators, which one we mean will be clear by the context. The fact that $\varphi\circ\psi \in M(D(\Omega_1,z_0))$ is the result of Lemma 1.7. We will prove that

$$\sigma_e(M_\varphi) = \sigma_e(M_{\varphi\circ\psi}) \tag{2.2}$$

and

$$\mathrm{cl}\,(\varphi;\partial\Omega_2) = \mathrm{cl}\,(\varphi\circ\psi;\partial\Omega_1). \tag{2.3}$$

To prove (2.2), let $\lambda \in \mathbf{C}$. For all $g \in D(\Omega_2,w_0)$,

$$(C_\psi\circ(M_\varphi-\lambda))\,(g) = ((M_{\varphi\circ\psi}-\lambda)\circ C_\psi)\,(g). \tag{2.4}$$

Since C_ψ is a unitary map, (2.4) shows that $\sigma_e(M_\varphi) = \sigma_e(M_{\varphi\circ\psi})$.

Now, to prove (2.3), let $\lambda \in \mathrm{cl}\,(\varphi\circ\psi;\partial\Omega_1)$. Then there is a sequence $\{z_n\}$ in Ω_1 such that $z_n \to \partial\Omega_1$ and $\varphi\circ\psi(z_n) \to \lambda$ as $n \to \infty$. Since $\{\psi(z_n)\}$ is a sequence in Ω_2, and $\psi(z_n) \to \partial\Omega_2$ (maybe some subsequence of $\{\psi(z_n)\}$), $\lambda \in \mathrm{cl}(\varphi;\partial\Omega_2)$. Hence $\mathrm{cl}(\varphi\circ\psi;\partial\Omega_1) \subset \mathrm{cl}(\varphi;\partial\Omega_2)$. The other inclusion can be proved similarly. Q.E.D.

By an *analytic curve*, we mean the image of the unit circle in **C** under a one-to-one function analytic on a neighbourhood of the unit circle.

Lemma 2.5: *Suppose Ω is a bounded doubly connected domain with an analytic boundary. Let ψ be a conformal mapping from Ω onto A_r for some r. Then ψ and ψ^{-1} can be extended analytically up to $\partial\Omega$ and ∂A_r respectively.*

Proof: A well known extension of Jordan Curve Theorem (see, for example, Koosis [11], page 53) says that ψ has a continuous and one-to-one extension up to $\partial\Omega$. Suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where Γ_0 is the boundary of the unbounded component of $S^2 \setminus \Omega$. Since Γ_1 is an analytic curve, there exists a neighbourhood N_1 of ∂U and a one-to-one analytic function φ_1 on N_1 such that $\varphi_1(\partial U) = \Gamma_1$. We may assume that $\varphi_1(U \cap N_1)$ is in Ω. Note that $\psi\circ\varphi_1$ is analytic on $U \cap N_1$ and continuous on $\overline{U} \cap N_1$. Since $|\psi\circ\varphi_1(z)| \to r$ as $z \to \partial U$, by the Schwarz Reflection Principle, $\psi\circ\varphi_1$ has an analytic extension up to ∂U. Suppose N_1' is a neighbourhood of ∂U on which $\psi\circ\varphi_1$ is analytic. Then $\psi\circ\varphi_1\circ\varphi_1^{-1}$ is an analytic extension of ψ on $\varphi(N_1 \cap N_1')$. Similarly we can extend ψ analytically up to Γ_0.

Note that $\varphi_1^{-1}\circ\psi^{-1}$ is analytic on a neighbourhood in A_r of $\{z \in \mathbf{C}: |z| = r\}$ and continuous up to $\{z \in \mathbf{C}: |z| = r\}$. Since $|\varphi_1^{-1}\circ\psi^{-1}(z)| \to 1$ as $|z| \to r$, by the Schwarz Reflection Principle, $\varphi_1^{-1}\circ\psi^{-1}$ has an analytic extension up to $\{z \in \mathbf{C}: |z| = r\}$. Suppose N_2 is a neighbourhood of $\{z \in \mathbf{C}: |z| = r\}$ on which $\varphi_1^{-1}\circ\psi^{-1}$ is analytic. Choose a neighbourhood N_2' of $\{z \in \mathbf{C}: |z| = r\}$ such that $N_2' \subset N_2$ and $\varphi_1^{-1}\circ\psi^{-1}(N_2') \subset N_1$.

Then $\varphi_1 \circ \varphi_1^{-1} \circ \psi^{-1}$ is an analytic extension of ψ^{-1} on N_2'. Similarly we can extend ψ^{-1} analytically up to $\{z \in \mathbf{C}: |z| = 1\}$. Q.E.D.

Suppose Ω is a bounded simply connected domain in $\mathbf{C}$ (or an interior of the complement in $\mathbf{S}^2$ of a bounded simply connected domain in $\mathbf{C}$) with an analytic boundary. We claim that each ζ in $\partial\Omega$ belongs to a closed ball contained in the complement of Ω. (This condition is called the external ball condition.) Let ψ be a conformal mapping from Ω onto U. Then, by the above lemma, ψ can be extended analytically to some neighbourhood G of $\overline{\Omega}$. Define a function φ on $\psi(G)$ by $\varphi(z) = |z|^2 - 1$. Then $\varphi \circ \psi$ is called a C^∞ defining function of Ω; namely $\varphi \circ \psi$ is a real-valued C^∞ function on some neighbourhood O of $\partial\Omega$ satisfying following three conditions: (i) $\Omega \cap O = \{ z \in \Omega : \varphi \circ \psi(z) < 0\}$; (ii) $\partial\Omega \cap O = \{ z \in \Omega: \varphi \circ \psi(z) = 0\}$; (iii) the gradient vector of $\varphi \circ \psi$ on $\partial\Omega$ is never 0. Note that the tangent line of $\partial\Omega$ at ζ is perpendicular to the gradient vector $\nabla \varphi \circ \psi(\zeta)$. Now, to prove the claim, we may assume that $0 \in \partial\Omega$ and the tangent line of $\partial\Omega$ at 0 is the real axis. Then near 0 in $\mathbf{R}$, $\partial\Omega$ is the graph $(x, \Lambda(x))$ of a C^∞ function Λ defined on a neighbourhood of 0 in $\mathbf{R}$, where $\Lambda'(0) = 0$; this follows from the implicit function theorem. Since $\Lambda'(0) = 0$, $|\Lambda(x)| = O(|x|^2)$ as $x \to 0$ by Taylor's Theorem. Hence the external ball condition at 0 is satisfied. Actually the external ball condition is satisfied when Ω has a C^2 boundary (see, for example, [4], Chapter 10).

Let Ω be a bounded domain in $\mathbf{C}$ whose complement (in $\mathbf{S}^2$) consists of exactly $m+1$ nontrivial components where m is a positive integer. Then $m+1$ applications of the Riemann Mapping Theorem produce a one-to-one holomorphic mapping of Ω onto a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves. Hence, as far as the essential spectrum is concerned, by Lemma 2.1 we may assume that

Ω is a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves.

Let Ω be a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves. The following notation is used throughout this thesis. The $m+1$ mutually disjoint analytic curves consisting of $\partial\Omega$ will be denoted by $\Gamma_0, \Gamma_1, \cdots, \Gamma_m$, where Γ_0 is the boundary of the unbounded component of $S^2 \setminus \Omega$. Ω_0, or sometimes U_0, will be used to denote the bounded component of $S^2 \setminus \Gamma_0$, and U_j will be used to denote the unbounded component of $S^2 \setminus \Gamma_j$ for each $j = 1, \ldots, m$. And we will also denote $\Omega_0 \cap U_j$ by Ω_j for each $j = 1, \cdots, m$.

Lemma 2.6: *Suppose Ω is a bounded domain in $\mathbf{C}$ whose boundary consists of $m+1$ mutually disjoint analytic curves. Let $z_0 \in \Omega$. If $f \in D(\Omega, z_0)$, then there is a function f_j in $D(\Omega_j, z_0)$ for each $j = 1, 2, \cdots, m$, such that $f = f_0 + f_1 + \cdots + f_m$ on Ω.*

Proof: For simplicity, we will prove this lemma when $m = 2$. Let $z \in \Omega$, and let γ_0, γ_1, and γ_2 be mutually disjoint smooth simple closed curves in Ω so near Γ_0, Γ_1, and Γ_2 respectively that z is interior to γ_j for each $j = 0, 1, 2$. (γ_0 is oriented counterclockwise, γ_1 and γ_2 are oriented clockwise.) By the Cauchy Formula,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_0 \cup \gamma_1 \cup \gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (2.7)$$

We will denote the $(j+1)^{th}$ integral in (2.7) by $g_j(z)$ for each $j = 0, 1, 2$. Then $g_j(z)$ is independent of the choice of γ_j and is in $H(U_j)$. Let $f_j(z) = g_j(z) - g_j(z_0)$. Then f_j is in $H(\Omega_j)$ and $f_j(z_0) = 0$ for each j. Let A_0, A_1, and A_2 be mutually disjoint connected neighbourhoods of Γ_0, Γ_1, and Γ_2 in Ω respectively. Now we will prove that $|f_0'|$ is square integrable on Ω_0 with respect to the usual area measure dA.
On $\Omega_0 \setminus A_0$, f_0' is bounded. On A_0, $|f'|, |f_1'|$, and $|f_2'|$ are square integrable and $f_0' = f' - f_1' - f_2'$. Hence $|f_0'|$ is square integrable on Ω_0 and so f_0 is in $D(\Omega_0,z_0)$. Similarly we can prove that $f_1 \in D(\Omega_1,z_0)$ and $f_2 \in D(\Omega_2,z_0)$. Q.E.D.

Remark: Since $D(\Omega)$ is conformally invariant, Lemma 2.6 is true when Ω is any finitely connected domain.

Corollary 2.8: *Suppose Ω is a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves. Then $D(\Omega) \subset B(\Omega)$ and $z \in M(D(\Omega))$.*

Proof: Let $f \in D(\Omega)$. By Lemma 2.6, $f = f_0 + f_1 + \cdots + f_m$ where $f_j \in D(\Omega_j)$. Since each Ω_j is either a simply connected domain with an analytic boundary or a doubly connected domain with an analytic boundary, by remark (c) following Lemma 1.13, f_j is in $B(\Omega_j)$ for all j. Hence $f \in B(\Omega)$. Now $z \in M(D(\Omega))$ follows from remark (a) following Lemma 1.13. Q.E.D.

Let $P(\Omega)$ be the set of polynomials. Then (1.8) shows that $P(U) \cap D(U,0)$ is a dense subset of $D(U,0)$. Also, from (1.11), we can see that

$$\left\{ \sum_{j=-n}^{n} a_j z^j : n \in \mathbf{N} \cup \{0\}, a_j \in \mathbf{C} \text{ for all } j = 0, \pm 1, \ldots, \pm n, \sum_{j=-n}^{n} a_j z_0^j = 0 \right\} \quad (2.9)$$

is a dense subset $D(A_r,z_0)$. For a finitely connected domain Ω with an analytic boundary, we will prove, in the following theorem, that the set of rational functions in $D(\Omega,z_0)$ whose poles are off $\overline{\Omega}$ is a dense subset of $D(\Omega,z_0)$. Let $R(\Omega)$ denote the set of rational functions whose poles are in $S^2\setminus\overline{\Omega}$.

Theorem 2.10: *Suppose Ω is a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves. Let z_0 be* in *Ω_0. Then $R(\Omega)\cap D(\Omega,z_0)$ is a dense subset of $D(\Omega,z_0)$. If Ω is simply connected, then $P(\Omega)\cap D(\Omega,z_0)$ is dense in $D(\Omega,z_0)$.*

Proof: Let $f\in D(\Omega,z_0)$. By Lemma 2.6, there is $f_j\in D(\Omega_j,z_0)$ for each $j=0, 1,\cdots,m$ such that $f=f_0+f_1+\cdots+f_m$ on Ω. Let $\varepsilon>0$. For Ω_0, there is a conformal mapping ψ_0 from Ω_0 onto U such that $\psi(z_0)=0$. By Lemma 1.7,

$$f_0\circ\psi_0^{-1}\in D(U,0).$$

Since $P(U)\cap D(U,0)$ is dense in $D(U,0)$, there is a polynomial $p\in P(U)\cap D(U,0)$ such that

$$\| f_0\circ\psi_0^{-1}-p\|_{D(U)}<\frac{\varepsilon}{2(m+1)}.$$

By Lemma 1.7, the composition map C_{ψ_0} is a unitary map from $D(U,0)$ onto $D(\Omega_0,z_0)$. Hence

$$\|f_0-p\circ\psi_0\|_{D(\Omega_0)}=\|(f_0\circ\psi_0^{-1}-p)\circ\psi_0\|_{D(\Omega_0)}=\|f_0\circ\psi_0^{-1}-p\|_{D(U)}<\frac{\varepsilon}{2(m+1)}. \quad (2.11)$$

Since Γ_0 is an analytic curve, ψ_0 extends to be analytic to a simply connected neighbourhood G of the closure of Ω_0 by Lemma 2.5. Hence $p \circ \psi_0$ is analytic on G. By Runge's Theorem, there is a sequence of polynomials $\{p_n\}$ that converge to $p \circ \psi_0$ uniformly on compact subsets of G. Therefore p_n' converges to $(p \circ \psi_0)'$ uniformly on the closure of Ω_0, and so p_n converges to $p \circ \psi_0$ in $D(\Omega_0, z_0)$. We may assume $p_n(z_0) = 0$ for all n, by replacing $p_n(z)$ by $p_n(z) - p_n(z_0)$ if necessary, because the constant term does not contribute to the Dirichlet norm. Hence there is a polynomial p_0 in $D(\Omega_0, z_0)$ such that

$$\| p \circ \psi_0 - p_0 \|_{D(\Omega_0)} < \frac{\varepsilon}{2(m+1)} .$$

By the triangle inequality and (2.11), $\| f_0 - p_0 \|_{D(\Omega_0)} < \frac{\varepsilon}{m+1}$. Note that this proves the second part of theorem.

Now fix $j = 1, 2, \cdots, m$. Since Ω_j is doubly connected, there is a conformal mapping ψ_j from Ω_j onto A_r for some r. Let $w_0 = \psi_j(z_0)$. By Lemma 1.7,

$$f_j \circ \psi_j^{-1} \in D(A_r, w_0).$$

Denote the set in (2.9) by $R(A_r, z_0)$. Since $R(A_r, w_0)$ is dense in $D(A_r, w_0)$, there is $h_j \in R(A_r, w_0)$ that is analytic on a neighbourhood of the closure of A_r such that

$$\| f_j \circ \psi_j^{-1} - h_j \|_{D(A_r)} < \frac{\varepsilon}{2(m+1)} .$$

By Lemma 1.7,

$$\| f_j - h_j \circ \psi_j \|_{D(\Omega_j)} = \| (f_j \circ \psi_j^{-1} - h_j) \circ \psi_j \|_{D(\Omega_j)} = \| f_j \circ \psi_j^{-1} - h_j \|_{D(A_r)} < \frac{\varepsilon}{2(m+1)} . \quad (2.12)$$

Since $\partial\Omega_j = \Gamma_0 \cup \Gamma_j$ and Γ_0, Γ_j are analytic curves, ψ_j extends to be analytic to some neighbourhood G of the closure of Ω_j by Lemma 2.5. Choose a compact subset K_j of G such that $S^2 \backslash K_j$ has two connected components and

$$\text{the closure of } \Omega_j \subset \text{the interior of } K_j \subset K_j.$$

Then, by Runge's Theorem, there is a sequence $\{r_n\}$ of rational functions whose poles are off K_j such that r_n converges to $h_j \cdot \psi_j$ uniformly on K_j. Hence r_n' converges to $(h_j \cdot \psi_j)'$ uniformly on the closure of Ω_j and so r_n converges to $h_j \cdot \psi_j$ in $D(\Omega_j)$. Again $r_n(z_0)$ may be assumed to be 0 for all n. Therefore we can choose a rational function r_j whose poles are off K_j such that

$$\| h_j \cdot \psi_j - r_j \|_{D(\Omega_j)} < \frac{\varepsilon}{2(m+1)} .$$

Hence, by the triangle inequality and (2.12), $\| f_j - r_j \|_{D(\Omega_j)} < \frac{\varepsilon}{m+1}$.

After choosing r_j for each j, let $r = p_0 + r_1 + \cdots + r_m$. Then r is in $R(\Omega) \cap D(\Omega, z_0)$ and

$$\| f - r \|_{D(\Omega)} < \| f_0 - p_0 \|_{D(\Omega_0)} + \| f_1 - r_1 \|_{D(\Omega_1)} + \cdots + \| f_m - r_m \|_{D(\Omega_m)} < \varepsilon.$$

Hence $R(\Omega) \cap D(\Omega, z_0)$ is a dense subset of $D(\Omega, z_0)$. Q.E.D.

Now we are ready to prove our main theorem of this chapter.

Theorem 2.13: *Suppose Ω is a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves and let $z_0 \in \Omega$. Let $\varphi \in M(D(\Omega, z_0))$. Then* $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi; \partial\Omega)$.

Proof: By Lemma 2.1, we may assume that $\Gamma_0 = \partial U$. We first will show that $cl(\varphi;\partial\Omega) \subset \sigma_e(M_\varphi)$. It suffices to show that if $0 \in cl(\varphi;\partial\Omega)$, then M_φ is not a Fredholm operator. Suppose $0 \in cl(\varphi;\partial\Omega)$ and M_φ is Fredholm. Let $\{z_n\}$ be a sequence in Ω such that $z_n \to \partial\Omega$ and $\varphi(z_n) \to 0$. Then there is a j such that $z_n \to \Gamma_j$. (Use a subsequence of $\{z_n\}$, if necessary.) Suppose ψ_j is a conformal mapping from U onto U_j. Let $\alpha_n = \psi_j^{-1}(z_n)$. Then $\alpha_n \to \partial U$ and $\varphi(\psi_j(\alpha_n)) = \varphi(z_n) \to 0$ as $n \to \infty$. Let λ_z be the point evaluation map at z on $D(U,0)$. Then, by (1.15),

$$\| \lambda_z \|^2_{D(U)} = \langle \lambda_z, \lambda_z \rangle_{D(U)} = \lambda_z(z) = \frac{1}{\pi} \log \frac{1}{1-|z|^2} .$$

Hence

$$\| \lambda_{\alpha_n} \|_{D(U,0)} \to \infty \text{ as } n \to \infty.$$

By the Uniform Boundedness Principle, there is a function $f \in D(U,0)$ such that

$$\sup_n | \langle f, \lambda_{\alpha_n} \rangle_{D(U,0)} | = \infty .$$

Therefore there is a subsequence of $\{\alpha_n\}$, for which we will use the same notation $\{\alpha_n\}$, such that

$$\lim_{n\to\infty} | \langle f, \lambda_{\alpha_n} \rangle_{D(U,0)} | = \lim_{n\to\infty} | f(\alpha_n) | = \infty. \tag{2.14}$$

Let $\Omega^* = \psi_j^{-1}(\Omega)$. Let k_z be the point evaluation mapping at $z \in \Omega^*$ on $D(\Omega^*,w_0)$ where $w_0 = \psi_j^{-1}(z_0)$. Since $f|_{\Omega^*} - f(w_0) \in D(\Omega^*,w_0)$,

$$|f(\alpha_n) - f(w_0)| = | \langle f|_{\Omega^*} - f(w_0), k_{\alpha_n} \rangle_{D(\Omega^*)} | \le \|f\|_{D(U)} \| k_{\alpha_n} \|_{D(\Omega^*)} . \tag{2.15}$$

By (2.14) and (2.15), $\| k_{\alpha_n} \|_{D(\Omega^*)} \to \infty$ as $n \to \infty$. Define a function f_n by

$$f_n = \frac{k_{\alpha_n}}{\| k_{\alpha_n} \|_{D(\Omega^*)}}$$

for each $n \in \mathbf{N}$. We claim that $f_n \to 0$ weakly in $D(\Omega^*, w_0)$. We must show that $< f_n , g >_{D(\Omega^*)} \to 0$ for all $g \in D(\Omega^*, w_0)$ as $n \to \infty$. Let $g \in D(\Omega^*, w_0)$. By Theorem 2.10, there is a sequence $\{r_m\}$ of rational functions in $R(\Omega^*) \cap D(\Omega^*, w_0)$ such that $r_m \to g$ in $D(\Omega^*, w_0)$. For all $n \in \mathbf{N}$ and $m \in \mathbf{N}$,

$$| < f_n , g >_{D(\Omega^*)} | \leq | < f_n , g - r_m >_{D(\Omega^*)} | + | < f_n , r_m >_{D(\Omega^*)} |$$

$$\leq \| f_n \|_{D(\Omega^*)} \| g - r_m \|_{D(\Omega^*)} + | < f_n , r_m >_{D(\Omega^*)} | .$$

Let $\varepsilon > 0$. Choose a positive integer K_1 such that $\| g - r_m \|_{D(\Omega^*)} < \frac{\varepsilon}{2}$ if $m \geq K_1$. Fix $m \geq K_1$. Note that

$$| < f_n , r_m >_{D(\Omega^*)} | = \frac{| r_m (\alpha_n) |}{\| k_{\alpha_n} \|_{D(\Omega^*)}} .$$

Since r_m is bounded on Ω and $\| k_{\alpha_n} \|_{D(\Omega^*)} \to \infty$ as $n \to \infty$, there is a positive integer K_2 such that

$$| < f_n , r_m >_{D(\Omega^*)} | < \frac{\varepsilon}{2}$$

if $n \geq K_2$. Hence $< f_n , g >_{D(\Omega^*)} \to 0$ and so we have $f_n \to 0$ weakly in $D(\Omega^*, w_0)$ as $n \to \infty$. Since we assumed that M_φ is a Fredholm operator on $D(\Omega, z_0)$, $M_{\varphi \circ \psi_j}$ is a Fredholm operator on $D(\Omega^*, w_0)$ by Lemma 2.1. Hence there is a compact operator T

on $D(\Omega^*,w_0)$ such that $1 - M_{\varphi \cdot \psi_j} T$ is compact. Since $f_n \to 0$ weakly in $D(\Omega^*,w_0)$, we have $\| (1 - M_{\varphi \cdot \psi_j} T)(f_n) \|_{D(\Omega^*)} \to 0$. Therefore

$$1 - (\varphi \cdot \psi_j)(\alpha_n) < T(f_n), f_n >_{D(\Omega^*)} = < f_n - (\varphi \cdot \psi_j)\, T(f_n)\, ,\, f_n >_{D(\Omega^*)}$$

$$= < (1 - M_{\varphi \cdot \psi_j} T)(f_n)\, ,\, f_n >_{D(\Omega^*)} \to 0. \qquad (2.16)$$

But, since $(\varphi \cdot \psi_j)(\alpha_n) \to 0$ and $| < T(f_n), f_n >_{D(\Omega^*)} | \leq \| T \| \| f_n \|_{D(\Omega^*)} = \| T \|$, the left hand side of (2.16) approaches 1. This contradiction shows that M_φ is not Fredholm. Hence $\mathrm{cl}(\varphi;\partial\Omega) \subset \sigma_e(M_\varphi)$.

To prove the converse inclusion, suppose that $0 \notin \mathrm{cl}(\varphi;\partial\Omega)$, i.e. φ is bounded away from 0 near $\partial\Omega$. Let $z_1, \cdots, z_n$ be the distinct zeros of φ in Ω. Assume first that $z_j \neq z_0$ for all j. Let $m(z_j)$ be the multiplicity of the zero of φ at z_j. Let E be the subspace of $D(\Omega,z_0)$ consisting of all functions f in $D(\Omega,z_0)$ such that f vanishes on $\{ z_1, \ldots ,z_n \}$ with multiplicity bigger than or equal to $m(z_j)$ at each z_j. Let $f \in E$. Then

$$\frac{f}{\varphi} \in H(\Omega) \text{ and } \frac{f}{\varphi}(z_0) = 0.$$

To see $\frac{f}{\varphi}$ is in $D(\Omega,z_0)$, observe that

$$\left(\frac{f}{\varphi}\right)' = (f'\varphi - f\varphi') / \varphi^2.$$

By the remark *(b)* following Lemma 1.13, the numerator is square integrable on Ω. Since φ is bounded away from 0 near $\partial\Omega$,

$$\frac{f}{\varphi} \in D(\Omega,z_0).$$

Hence f is in the range of M_φ and so E is contained in the range of M_φ. Note that

$$E = \cap \{ \text{Ker } \lambda_{z_j, k} : j = 1, \cdots, n \text{ and } k = 0, \cdots, m(z_j) - 1 \}.$$

Being an intersection of the kernels of finitely many linear functionals, E has a finite codimension. Since $\text{Ker}\, M_\varphi = \{0\}$, M_φ is Fredholm.

Now suppose there is $j = 1, \cdots, n$, such that $z_j = z_0$, say $z_1 = z_0$. Then redefine $m(z_1)$ to be (the multiplicity of the zero of φ at z_1) + 1, and define E as before. Then, by the same argument, we can conclude that M_φ is Fredholm. Thus $\sigma_e(M_\varphi) \subset \text{cl}(\varphi;\partial\Omega)$. Q.E.D.

Remark: Note that to prove $\sigma_e(M_\varphi) \subset \text{cl}(\varphi;\partial\Omega)$, we used neither the analytic boundary condition nor the finite connectedness of Ω, but the fact that $D(\Omega) \subset B(\Omega)$. Hence $\sigma_e(M_\varphi) \subset \text{cl}(\varphi;\partial\Omega)$ is true when φ is a multiplier of $D(\Omega)$ on bounded domains Ω in $\mathbf{C}$ such that $D(\Omega) \subset B(\Omega)$.

By Lemma 2.1 and Theorem 2.13, we have the following corollary.

Corollary 2.17: *Suppose* Ω *is a finitely connected bounded domain in* $\mathbf{C}$. *Let* $\varphi \in M(D(\Omega))$. *Then* $\sigma_e(M_\varphi) = \text{cl}(\varphi;\partial\Omega)$.

CHAPTER 3

CLOSED FINITE CODIMENSIONAL INVARIANT SUBSPACES

In this chapter we will study finite codimensional invariant closed subspaces of the Dirichlet space of a finitely connected domain with an analytic boundary. A characterization of those subspaces of the Bergman spaces defined on a large class of bounded domains in **C** was obtained by Axler and Bourdon in [3]. Also Chan characterized those spaces on $D(\Omega)$ when Ω is a circular domain; see [7]. In his paper, Chan used a Laurent series expansion to prove his characterization, which cannot be applied on noncircular domains. In this chapter, we will establish the same characterization of finite codimensional invariant closed subspaces of $D(\Omega)$ when Ω is a finitely connected domain with an analytic boundary. Recall we assumed that no component of $\partial\Omega$ is equal to a point.

We start this chapter with the Bergman norm estimation of certain class of functions on U that will be used repeatedly throughout this chapter.

Lemma 3.1: *For* $r > 1$, *define a function* g_r *on* U *by* $g_r = \frac{r-1}{(z-r)^2}$. *Then* $sup\ \{ \| g_r \|_{B(U)} : r > 1 \} < \infty$.

Proof: Let $r > 1$. Note that

$$\frac{1}{(z-r)^2} = \sum_{n=0}^{\infty} \frac{n+1}{r^{n+2}} z^n$$

where the series converges uniformly and absolutely on U.

Also

$$\frac{1}{(1-\frac{1}{z})^2} = \sum_{n=0}^{\infty} \frac{n+1}{z^n} \quad \text{for } |z| > 1. \tag{3.2}$$

Hence

$$\|g_r\|^2_{B(U)} = \pi\,(r-1)^2 \sum_{n=0}^{\infty} \frac{n+1}{r^{2(n+2)}} \qquad \text{by (1.9)}$$

$$= \pi r^{-4}\,(r-1)^2 \sum_{n=0}^{\infty} \frac{n+1}{r^{2n}} = \pi r^{-4}\,(r-1)^2 \frac{r^4}{(r^2-1)^2} \qquad \text{by (3.2)}$$

$$= \frac{\pi}{(r+1)^2} < \frac{\pi}{4} \qquad \text{Q.E.D.}$$

The following lemma is well known in general function spaces. For later use, we state it explicitly.

Lemma 3.3: *Suppose that* $\{f_\alpha\}_{\alpha\in A}$ *is a norm bounded net in a closed subspace* H *of* $D(\Omega)$*, which converges to* f *pointwise on* Ω*. Then* $f \in H$*.*

Proof: By the Banach-Alaoglu Theorem, any closed ball of $D(\Omega)$ is weak* (hence weak) compact. Therefore $\{f_\alpha\}_{\alpha\in A}$ has a weak convergent subnet $\{f_{\alpha_\beta}\}_{\beta\in B}$, say $f_{\alpha_\beta} \to g$ weakly in $D(\Omega)$. Hence, by remark (*a*) following Lemma 1.5, $f_{\alpha_\beta}(z) \to g(z)$ pointwise on Ω and so $f = g$ and $f_{\alpha_\beta} \to f$ weakly in $D(\Omega)$. But the norm topology and the weak topology have the same closed convex sets (see, for example, Rudin [12], Theorem 3.12). Since H is a (norm) closed convex set, $f \in H$. Q.E.D.

Remark: A sequence version of the above lemma is still true. We only need to prove that a bounded sequence in $D(\Omega)$ has a weak convergent subsequence. Define an operator $T: D(\Omega) \to B(\Omega)$ by $T(f) = f'$. Then T is an isometry. Now, since $B(\Omega)$ is separable, so is $D(\Omega)$. Hence any closed ball of $D(\Omega)$ with the weak topology is a metrizable compact set (see, for example, Rudin [12], Theorem 3.16) and so any bounded sequence in $D(\Omega)$ has a weak convergent subsequence.

Suppose Ω is a bounded domain such that $D(\Omega) \subset B(\Omega)$. If λ is in Ω, then $(z - \lambda)D(\Omega,z_0)$ is a closed proper subspace of $D(\Omega,z_0)$ that is invariant under multiplication by z. If q is a polynomial that has all of its zeros in Ω, then we will see in Proposition 3.12 that $qD(\Omega)$ is a finite codimensional closed subspace of $D(\Omega)$ that is invariant under multiplication by z. If $\lambda \in \mathbf{C} \setminus \overline{\Omega}$, then $(z - \lambda)D(\Omega,z_0) = D(\Omega,z_0)$. We will prove in the following two theorems that $(z - \lambda)D(\Omega,z_0)$ is dense in $D(\Omega,z_0)$ if Ω is a finitely connected domain with an analytic boundary and $\lambda \in \partial\Omega$. These theorems are key steps toward obtaining a characterization of finite codimensional invariant closed subspaces of $D(\Omega)$ on finitely connected domains with an analytic boundary.

By a *wedge* W_λ in $\mathbf{C}$, we mean the convex hull of a point λ (called the *vertex* of the wedge) and an arc of a circle centered at λ. We mentioned in Chapter 2 that, if Ω has an analytic boundary, then each boundary point satisfies the external ball condition. Hence, for each $\lambda \in \partial\Omega$, there is a wedge W_λ in $\mathbf{C} \setminus \Omega$ with vertex at λ. Actually, in order to satisfy this "wedge condition", $\partial\Omega$ need only be a C^1 boundary by the implicit function theorem and Taylor's Theorem.

Theorem 3.4: *Let Ω be a simply connected domain with an analytic boundary. Then $(z - \lambda)D(\Omega,z_0)$ is dense in $D(\Omega,z_0)$ for every $\lambda \in \partial\Omega$.*

Proof: Let $\lambda \in \partial\Omega$. We know that $P(\Omega) \cap D(\Omega,z_0)$ is dense in $D(\Omega,z_0)$ by Theorem 2.10. Hence it suffices to show that $P(\Omega) \cap D(\Omega,z_0) \subset \overline{(z - \lambda)D(\Omega,z_0)}$, the closure of $(z - \lambda)D(\Omega,z_0)$ in $D(\Omega,z_0)$.

Assume that $z - z_0 \in \overline{(z-\lambda)D(\Omega,z_0)}$. Then, since $\overline{(z-\lambda)D(\Omega,z_0)}$ is invariant under multiplication by a polynomial, and each $p \in P(\Omega)\cap D(\Omega,z_0)$ is of the form $q(z-z_0)$ where $q \in P(\Omega)$, we would have $P(\Omega)\cap D(\Omega,z_0) \subset \overline{(z-\lambda)D(\Omega,z)}$ as desired.

To prove $z - z_0 \in \overline{(z-\lambda)D(\Omega,z_0)}$, we first assume that $\lambda=1\in\partial\Omega$. Since $\partial\Omega$ is an analytic curve, there exists a wedge W_1 in $\mathbf{C}\setminus\Omega$ with vertex at 1. Assume that there exist $a\in(0, 1)$ and $\delta\in(0, \frac{\pi}{4})$ such that

$$W_1 = \{z\in \mathbf{C} : |z-1| \leq 2a, -2\delta \leq \arg(z-1) \leq 2\delta\}.$$

Let
$$G_1 = \{ z\in \mathbf{C} : |z-(1-a)| < a \},$$
$$G_2 = \{ z\in \mathbf{C} : |z-b| < a \}$$
where b is the point which is obtained from $1-a$ rotating by $\delta - \frac{\pi}{2}$ around 1,
and let
$$G_3 = \{ z\in \mathbf{C} : \bar{z} \in G_2 \}.$$

Let $r \in (1, 1+a)$. Define a function g_r by $g_r(z) = (z-1)\frac{z-z_0}{z-r}$. Then $g_r(z)$ is in $(z-1)D(\Omega,z_0)$. Note that

$$g_r' = 1 + \frac{r(1-r)}{(z-r)^2} + z_0\frac{r-1}{(z-r)^2}. \tag{3.5}$$

Hence, by Lemma 3.1, $\sup\{\| g_r \|_{D(G_1)} : r\in(1, 1+a)\} < \infty$ since $G_1 \subset U$. For $z \in G_2$, let w be the point obtained from z rotating by $\frac{\pi}{2} - \delta$ around 1. Then $w \in G_1$. Let $\theta = \arg(z-1)$. Then

$$\frac{|w-r|^2}{|z-r|^2} = \frac{|w-1|^2 + (r-1)^2 - 2|w-1|(r-1)\cos(\theta+\frac{\pi}{2}-\delta)}{|z-1|^2 + (r-1)^2 - 2|z-1|(r-1)\cos\theta} \text{ by Law of Cosines}$$

$$\leq \frac{(|z-1| + (r-1))^2}{|z-1|^2 + (r-1)^2 - 2|z-1|(r-1)\cos\delta} \quad \text{since } |w-1|=|z-1| \text{ and } \cos\delta>\cos\theta$$

$$\leq \frac{(\,|z-1|+(r-1)\,)^2}{\mathrm{Max}\ \{\ |z-1|^2 \sin^2\delta,\ (r-1)^2 \sin^2\delta\ \}}\ . \tag{3.6}$$

If $|z-1| \geq r-1$, then the numerator of the right hand side of (3.6) is less than or equal to $4\,|z-1|^2$. When $r-1 \geq |z-1|$, the numerator of the right hand side of (3.6) is less than or equal to $4\,(r-1)^2$. In any cases, we have

$$\text{the right hand side of (3.6)} \leq \frac{4}{\sin^2\delta}\ . \tag{3.7}$$

Hence there is a constant K such that

$$\|\, g_r \,\|_{D(G_2)} \;=\; \|\, g_r{}' \,\|_{B(G_2)}$$

$$\leq \sqrt{\pi} + \|\,(r^2-r)/(z-r)^2\|_{B(G_2)} + \|\, z_0\,(r-1)/(z-r)^2\|_{B(G_2)} \qquad \text{by (3.5)}$$

$$\leq \sqrt{\pi} + \frac{4}{\sin^2\delta}\,[\,\|\,(r^2-r)/(z-r)^2\,\|_{B(G_1)} + \|\, z_0\,(r-1)/(z-r)^2\,\|_{B(G_1)}\,]$$

by change-of-variables and (3.7)

$$< K. \qquad \text{by Lemma (3.1)}$$

Therefore $\sup\{\|\, g_r \,\|_{D(G_2)} : r \in (1,\ 1+a\,)\} < \infty$. $\mathrm{Sup}\{\|\, g_r \,\|_{D(G_3)} : r \in (1,\ 1+a\,)\} < \infty$ can be proved similarly.

On the other hand, note that $|z-r|$ is bounded away from 0 for all $r \in (1,\ 1+a\,)$ and for all $z \in \Omega \setminus (G_1 \cup G_2 \cup G_3)$. Hence $\sup\{\|\, g_r \,\|_{D(\Omega)} : r \in (1,\ 1+a)\ \} < \infty$.

Consider $\{g_r\}_{r \in (1,1+a)}$ as a net in $D(\Omega)$. By the Banach-Alaoglu Theorem, $\{g_r\}_{r \in (1,1+a)}$ has a weak convergent subnet $\{g_{r_\alpha}\}_{\alpha \in A}$ where A is some index set. Note that g_{r_α} converges to $z - z_0$ pointwise on Ω. Since $\{g_{r_\alpha}\}_{\alpha \in A} \in \overline{(z-1)D(\Omega, z_0)}$,

$g_{r_\alpha}(z) \to (z - z_0)$ pointwise, and $\overline{(z-1)D(\Omega,z_0)}$ is a closed subspace of $D(\Omega,z_0)$, $z - z_0$ is in $\overline{(z-1)D(\Omega,z_0)}$ by Lemma 3.3.

For general $\lambda \in \partial\Omega$, suppose there exist $t_0 \in \mathbf{C}$, $a \in (0,1)$, and $\delta \in (0, \frac{\pi}{4})$ such that $W_\lambda = \{z \in \mathbf{C} : |z-1| \le 2a,\ \arg(t_0 - \lambda) - 2\delta \le \arg(z-1) \le \arg(t_0 - \lambda) + 2\delta\}$. Let $L = \{ t \in \mathbf{C} : |t-1| < a,\ \arg(t-\lambda) = \arg(t_0 - \lambda) \}$. Define a function g_t by

$$g_t(z) = (z - \lambda)\frac{z - z_0}{z - t}$$

for each $t \in L$. Then $\{g_t\} \subset (z-\lambda)D(\Omega,z_0)$, and

$$g_t' = 1 + \frac{t(\lambda - t)}{(z-t)^2} + \frac{z_0(t-\lambda)}{(z-t)^2}. \tag{3.8}$$

Let G be the region obtained from Ω by rotating and translating so that $1 \in \partial G$ corresponds to λ and W_1 corresponds to W_λ. For each t in L, there is the unique r in $(1, 1+a)$ such that $\| (\lambda - t)/(z-t)^2 \|_{B(\Omega)} = \| (1-r)/(z-r)^2 \|_{B(G)}$. By (3.8) and by the same argument as in case of $\lambda = 1$, $\sup\{ \| g_t \|_{D(\Omega)} : t \in L\} < \infty$. Again, by the same argument as in case of $\lambda = 1$, $(z - z_0) \in \overline{(z-\lambda)D(\Omega,z_0)}$. Q.E.D.

In the following theorem, we will generalize Theorem 3.4 to the case where Ω is a bounded finitely connected domain with an analytic boundary. The fact that $R(\Omega) \cap D(\Omega, z_0)$ is dense in $D(\Omega, z_0)$ plays a crucial role.

Theorem 3.9: *Suppose that Ω be a bounded domain in $\mathbf{C}$ whose boundary consists of $m + 1$ mutually disjoint analytic curves and let $z_0 \in \Omega$. Then $(z-\lambda)D(\Omega,z_0)$ is dense in $D(\Omega,z_0)$ for all $\lambda \in \partial\Omega$.*

Proof: For simplicity, assume $m = 2$. Let $\lambda \in \partial\Omega$. By Theorem 2.10, it suffices to show that

$$R(\Omega) \cap D(\Omega, z_0) \subset \overline{(z - \lambda)D(\Omega, z_0)} \tag{3.10}$$

The proof of the previous theorem shows that $P(\Omega) \cap D(\Omega, z_0) \in \overline{(z - \lambda)D(\Omega, z_0)}$. Hence, in order to prove (3.10), it suffices to show that

$$\frac{1}{(z - w)^n} - \frac{1}{(z_0 - w)^n} \in \overline{(z - \lambda)D(\Omega, z_0)} \tag{3.11}$$

where $w \in \mathbf{C} \setminus \overline{\Omega}$ and $n \in \mathbf{N}$.

Since $\partial\Omega$ is an analytic curve, there exists a wedge W_λ in $\mathbf{C} \setminus \Omega$ with vertex at λ. To prove (3.11), without loss of generality, assume that $\lambda = 1$ and the wedge W_1 is of the form $\{z \in \mathbf{C} : |z - 1| < 2a, -2\delta < \arg(z - 1) < 2\delta\}$ for some a in $(0, 1)$ and δ in $(0, \frac{\pi}{4})$. And we also may assume that $w \in \mathbf{C} \setminus \overline{\Omega}_1$.

We will show (3.11) by induction. Let $n = 1$. Define a function h on Ω by

$$h(z) = \frac{1}{(z - w)} - \frac{1}{(z_0 - w)}.$$

Then $h \in D(\Omega, z_0)$. Let r be in $(1, 1 + a)$. Since $D(\Omega) \subset B(\Omega)$, $h / (z - r) \in D(\Omega, z_0)$. Note that $\{(z - 1)h / (z - r)\}$ converges to h pointwise on Ω as $r \to 1$. Hence, in order to prove that h is in $\overline{(z - 1)D(\Omega, z_0)}$, by Lemma 3.3, it suffices to show that

$$\sup \{ \| (z - 1)\frac{h}{z - r} \|_{D(\Omega)} : r \in (1, 1 + a) \} < \infty.$$

Note that

$$[\,(z-1)\frac{h}{z-r}\,]' = \frac{h(1-r)}{(z-r)^2} + \frac{(z-1)h'}{z-r}$$

$$= \frac{1}{(z-w)}\frac{(1-r)}{(z-r)^2} - \frac{1}{(z_0-w)}\frac{(1-r)}{(z-r)^2} - \frac{z-1}{(z-r)(z-w)^2}$$

$$= \frac{1}{(z-w)^2}\,[\,-1+(2z-r-w)\frac{1-r}{(z-r)^2}\,] + \frac{1}{(z_0-w)}\,\frac{r-1}{(z-r)^2}\,.$$

Hence $\sup\{\, \| (z-1)\frac{h}{z-r} \|_{D(\Omega)} : r \in (1, 1+a)\,\} < \infty$ by Lemma 3.1.

Therefore $h = \frac{1}{(z-w)} - \frac{1}{(z_0-w)} \in \overline{(z-1)D(\Omega,z_0)}$ as desired.

Now assume that (3.11) is true when $n = k$ with $w \in \mathbf{C}\setminus\overline{\Omega}_1$ and $\lambda = 1$. Let $\{\zeta_j\}$ be a sequence in $\mathbf{C}\setminus\overline{\Omega}_1$ such that ζ_j converges to w. For each j, define a function g_j on Ω by

$$g_j\,(z) = \frac{1}{w-\zeta_j}\,[\,(\frac{1}{(z-w)^k} - \frac{1}{(z_0-w)^k}) - (\frac{1}{(z-\zeta_j)^k} - \frac{1}{(z_0-\zeta_j)^k})\,]$$

Then , by induction hypothesis, g_j is in $\overline{(z-1)D(\Omega,z_0)}$ for all j. Note that

$$\sup\{\, \| g_j' \|_\infty : j \in \mathbf{N}\,\} < \infty$$

since $|z-w|$ is bounded away from zero and $\inf\{|z-\zeta_j| : j \in \mathbf{N}\} > 0$. Hence

$$\sup\{\, \| g_j \|_{D(\Omega)} : j \in \mathbf{N}\,\} < \infty.$$

Note that $g_j(z)$ converges to

$$k\left[\frac{1}{(z-w)^{k+1}} - \frac{1}{(z_0-w)^{k+1}}\right]$$

pointwise as $j \to \infty$ since

$$g_j(z) = \frac{(z-\zeta_j)^{k-1} + (z-\zeta_j)^{k-2}(z-w) + \ldots + (z-w)^{k-1}}{(z-w)^k (z-\zeta_j)^k}$$

$$- \frac{(z_0-\zeta_j)^{k-1} + (z_0-\zeta_j)^{k-2}(z_0-w) + \ldots + (z_0-w)^{k-1}}{(z_0-w)^k (z_0-\zeta_j)^k}.$$

Therefore, by the remark following Lemma 3.3, (3.11) holds when $n = k + 1$. By induction we are done. Q.E.D.

Proposition 3.12: *Suppose Ω is a bounded domain such that $D(\Omega) \subset B(\Omega)$. Let q be a polynomial that has all its zeros in Ω. Then $qD(\Omega,z_0)$ is a finite codimensional closed subspace of $D(\Omega,z_0)$ that is invariant under multiplication by z. Furthermore the codimension of $qD(\Omega,z_0)$ in $D(\Omega,z_0)$ is the degree of q.*

Proof: Suppose q is a polynomial that has all its zeros in Ω. Let the degree of q be n and let $z_1, \cdots, z_m$ be the distinct zeros of q. Denote the multiplicity of the zero of q at z_j by k_j for each j.

Let $E = \bigcap_{j=1}^{m} \{ \text{Ker } \lambda_{z_j,k} : k = 0, \cdots, k_j - 1 \text{ if } z_j \neq z_0;\ k = 1, \cdots, k_j \text{ if } z_j = z_0 \}$.

Then, obviously, $qD(\Omega,z_0) \subset E$. To see the other inclusion, assume that $f \in E$. Then

$$\frac{f}{q} \in H(\Omega) \text{ and } \frac{f}{q}(z_0) = 0.$$

Since $D(\Omega) \subset B(\Omega)$ and q is bounded away from 0 near $\partial\Omega$, $\frac{f}{q} \in D(\Omega,z_0)$. Hence $f \in qD(\Omega,z_0)$ and so $qD(\Omega,z_0) = E$. Being an intersection of n closed subspaces whose codimension is 1, $qD(\Omega,z_0)$ is a closed subspace of $D(\Omega,z_0)$ with codimension $\leq n$. Since $qD(\Omega,z_0)$ is invariant under multiplication by z, we proved the first part of this proposition.

In order to prove that $qD(\Omega,z_0)$ has a codimension n in $D(\Omega,z_0)$, we must show that the set of n bounded linear functionals H defined by

$$H = \bigcup_{j=1}^{m} \{ \lambda_{z_j,k} : k = 0, \cdots, k_j - 1 \text{ if } z_j \neq z_0;\ k = 1, \cdots, k_j \text{ if } z_j = z_0 \}$$

is linear independent in $D(\Omega,z_0)^*$. To prove that H is a linearly independent subset of $D(\Omega,z_0)^*$, it suffices to show that, for each T in H, there is a function f in $D(\Omega,z_0)$ such that $T(f) = 1$ and $S(f) = 0$ for all S in $H \setminus \{T\}$.

Let $T \in H$. Without loss of generality, we may assume that $T = \lambda_{z_1,k}$ where k is either a fixed element in $\{1, \cdots, k_1\}$ (if $z_1 = z_0$), or a fixed element in $\{0, \cdots, k_1 - 1\}$ (if $z_1 \neq z_0$). We first assume that $z_1 = z_0$. Let $f = g\,p$ where

$$g(z) = [\, (z - z_2)(z - z_3)\cdots(z - z_m) \,]^n$$

and

$$p(z) = \alpha_1(z - z_1) + \cdots + \alpha_{k_1}(z - z_1)^{k_1}$$

where $\alpha_1, \cdots, \alpha_{k_1}$ are constants to be determined so that $S(f) = 0$ for all S in $H \setminus \{T\}$ and $T(f) = 1$. From the definition of f, it is easy to see that $\lambda_{z_j,l}(f) = 0$ for all $j = 2, \cdots, m$ and corresponding l's. So we want to find $\alpha_1, \cdots, \alpha_{k_1}$ satisfying $f^{(l)}(z_1) = 0$ for all l in $\{1, 2, \cdots, k_1\} \setminus \{k\}$ and $f^{(k)}(z_1) = 1$. By direct calculation, we can see that this problem is equivalent to solving a k_1 by k_1 linear system of the form

$$M\ [\alpha_1, \alpha_2, \cdots, \alpha_{k_1}] = [\, 0, \cdots, 0, 1, 0, \cdots, 0] \tag{3.13}$$

where [⋯] is a column vector in $\mathbf{C}^{k_1}$, the 1 in the right hand side vector of (3.13) is in the k^{th} slot, and M is a k_1 by k_1 matrix with

$$|\det M| = k_1! \cdot (k_1 - 1)! \cdots 2! \cdot (g(z_1))^{k_1} \neq 0.$$

Hence the linear system in (3.13) has the unique solution and so H is linear independent in $D(\Omega, z_0)^*$.

Now assume $z_1 \neq z_0$. Let $f = g\,p$ where

$$g(z) = [\,(z - z_2)(z - z_3)\cdots(z - z_m)\,]^n$$

and

$$p(z) = \alpha_1(z - z_1) + \cdots + \alpha_{k_1}(z - z_2)^{k_1} - \alpha_1(z_0 - z_1) - \cdots - \alpha_{k_1}(z_0 - z_2)^{k_1}$$

where $\alpha_1, \cdots, \alpha_{k_1}$ are constants to be determined so that $S(f) = 0$ for all S in $H \setminus \{T\}$ and $T(f) = 1$. As before, this problem is equivalent to solving a k_1 by k_1 linear system of the form

$$M\ [\alpha_1, \alpha_2, \cdots, \alpha_{k_1}] = [\,0, \cdots, 0, 1, 0, \cdots, 0] \qquad \textbf{(3.14)}$$

where [⋯] is a column vector in $\mathbf{C}^{k_1}$, the 1 in the right hand side vector of (3.14) is in the $(k+1)^{\text{th}}$ slot, and M is a k_1 by k_1 matrix with

$$|\det M| = (k_1 - 1)! \cdot (k_1 - 2)! \cdots 2! \cdot (g(z_1))^{k_1} \cdot (z_0 - z_1)^{k_1} \neq 0.$$

Thus the linear system in (3.14) has the unique solution and so the linear independency of H in $D(\Omega)^*$ is proved. Q.E.D.

Proposition 3.12 is true when Ω is a bounded domain with an analytic boundary by Corollary 2.8. Furthermore we will prove in next theorem that all of the finite codimensional closed subspaces of $D(\Omega)$ that are invariant under multiplication by z are of the form $qD(\Omega)$, where q is a polynomial with all of its roots in Ω.

Theorem 3.15: *Suppose Ω is a bounded domain whose boundary consists of $m+1$ mutually disjoint analytic curves. Let E be a finite codimensional closed subspace of $D(\Omega)$ that is invariant under multiplication by z. Then there is a polynomial q that has all of its zeros in Ω such that $E = qD(\Omega)$.*

Proof: Define an operator $T : D(\Omega) / E \to D(\Omega) / E$ by $T(f + E) = zf + E$. The invariance of E implies that T is well defined. Since T is an operator on a finite dimensional space, there is a nonzero polynomial h, with degree at most $\dim (D(\Omega) / E)$, such that $h(T) = 0$. Since $h(T)(f + E) = hf + E$ for all f in $D(\Omega)$, $hD(\Omega) \subset E$. Factor h as $h = qk$ where q is a polynomial that has all of its zeros in Ω and k is a polynomial that has all of its zeros in $\mathbf{C} \setminus \Omega$.

We claim that $kD(\Omega)$ is dense in $D(\Omega)$. Note that $(z - \lambda)D(\Omega)$ is $D(\Omega)$ if $\lambda \in \mathbf{C} \setminus \overline{\Omega}$, and $(z - \lambda)D(\Omega)$ is dense in $D(\Omega)$, by Theorem 3.9, if $\lambda \in \partial\Omega$. Let λ_1 and λ_2 be two roots of k. We will show that

$$\overline{(z - \lambda_1)(z - \lambda_2)D(\Omega)} = \overline{(z - \lambda_1)D(\Omega)}. \tag{3.16}$$

The inclusion $\overline{(z - \lambda_1)(z - \lambda_2)D(\Omega)} \subset \overline{(z - \lambda_1)D(\Omega)}$ is obvious since $z \in M(D(\Omega))$.

In order to prove the other inclusion, let $\varepsilon > 0$ and let $\sup\{|z - \lambda_1| : z \in \Omega\} = K$. Let $f \in D(\Omega) = \overline{(z - \lambda_1)D(\Omega)}$. Then there is a function $g \in D(\Omega)$ such that

$\| (z - \lambda_1)g - f \|_{D(\Omega)} < \frac{\varepsilon}{3}$. For $g \in D(\Omega) = \overline{(z - \lambda_2)D(\Omega)}$, there is a sequence of functions $\{ g_n \}$ in $D(\Omega)$ such that $\| (z - \lambda_2)g_n - g \|_{D(\Omega)} \to 0$ as $n \to \infty$. Since the inclusion map from $D(\Omega)$ into $B(\Omega)$ is bounded, we have $\| (z - \lambda_2)g_n - g \|_{B(\Omega)} \to 0$ as $n \to \infty$. Hence there is a function g_0 in $D(\Omega)$ such that

$$\| (z - \lambda_2)g_0 - g \|_{D(\Omega)} < \frac{\varepsilon}{3K} \quad \text{and} \quad \| (z - \lambda_2)g_0 - g \|_{B(\Omega)} < \frac{\varepsilon}{3}.$$

Therefore
$$\| (z - \lambda_1)(z - \lambda_2)g_0 - f \|_{D(\Omega)} \le$$

$$\| (z - \lambda_1) [(z - \lambda_2)g_0 - g] \|_{D(\Omega)} + \| (z - \lambda_1)g - f \|_{D(\Omega)} \le$$

$$\| (z - \lambda_2)g_0 - g \|_{B(\Omega)} + K \| (z - \lambda_2)g_0 - g \|_{D(\Omega)} + \| (z - \lambda_1)g - f \|_{D(\Omega)} < \varepsilon.$$

Thus f is in $\overline{(z - \lambda_1)(z - \lambda_2)D(\Omega)}$ and so we proved (3.16). Since k has only finitely many zeros, we can conclude that $\overline{kD(\Omega)} = D(\Omega)$ by repeating a similar argument.

Suppose f is in $qD(\Omega) = q\,(\overline{kD(\Omega)})$. Then $f = qg$ for some g in $D(\Omega)$. For g, there is a sequence of functions $\{g_n\}$ in $D(\Omega)$ such that $\| k\, g_n - g \|_{D(\Omega)} \to 0$ as $n \to 0$. Hence

$$\| qk\, g_n - f \|_{D(\Omega)} = \| qk\, g_n - qg \|_{D(\Omega)} = \| q(k\, g_n - g) \|_{D(\Omega)}$$

$$\le \| q' (k\, g_n - g) + q\, (k\, g_n - g)' \|_{B(\Omega)}. \tag{3.17}$$

Since q and q' are bounded on Ω, and the inclusion map from $D(\Omega)$ into $B(\Omega)$ is bounded, the right hand side of (3.17) approaches 0 as $n \to 0$. Hence $q(\overline{kD(\Omega)}) \subset \overline{qkD(\Omega)}$. Therefore

$$qD(\Omega) = q\,(\overline{kD(\Omega)}) \subset \overline{qkD(\Omega)} = \overline{hD(\Omega)} \subset E. \tag{3.18}$$

Hence $\dim(D(\Omega)/qD(\Omega)) = \text{degree of } q$ by Proposition 3.12

$\leq \text{degree of } h \leq \dim(D(\Omega)/E)$ by the choice of h

$\leq \dim(D(\Omega)/qD(\Omega))$. by (3.18)

Hence $\dim(D(\Omega)/qD(\Omega)) = \dim(D(\Omega)/E)$ and so, by (3.18), $E = qD(\Omega)$. Q.E.D.

Cor 3.19: *Let E be a finite codimensional closed subspace of $D(\Omega)$ where Ω is a finitely connected bounded domain with an analytic boundary. Then the following are equivalent.*

(1) $zE \subset E$

(2) $\varphi E \subset E$ *for all* $\varphi \in M(D(\Omega))$

(3) $E = qD(\Omega)$ *where* q *is a polynomial with all of its zeros in* Ω.

Proof: (1) implies (3) by Theorem 3.15. (3) implies (2) since $\varphi E = \varphi qD(\Omega) = q\varphi D(\Omega) \subset qD(\Omega) = E$. And (2) implies (1) trivially. Q.E.D.

Hence we can conclude this chapter as follows: Suppose Ω is a finitely connected bounded domain with an analytic boundary. If E is a finite codimensional closed subspace of $D(\Omega)$ that is invariant under any multiplication operator M_φ, then E is of the form $qD(\Omega)$ where q is a polynomial with all its zeros in Ω.

We conclude this thesis by raising a few questions. For which domain Ω is $D(\Omega)\subset B(\Omega)$? Is Corollary 2.17 true for any domain Ω in **C**? Suppose Ω is a bounded domain in **C** such that no connected component of $\partial\Omega$ is equal to a point. Then the finite codimensional invariant closed subspaces $B(\Omega)$ are of the form $qD(\Omega)$ where q is a polynomial with all of its zeros in Ω; see [4], Theorem 5. Can Theorem 3.15 be generalized as in case of Bergman spaces?

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