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Shangrong Deng

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Major professor

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VARIATIONAL PROBLEMS

ON

CONTACT MANIFOLDS

By

Shangrong Deng

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ABSTRACT

VARIATIONAL PROBLEMS ON CONTACT MANIFOLDS

Shangrong Deng

S.S.Chern and R.S.Hamilton in a paper of 1985 studied a kind of Dirichlet energy in terms of the torsion $\tau (\tau = \mathcal{L}_\xi g)$ of a 3-dimensional compact contact manifold and a problem analogous to the Yamabe problem. They raised the question of determining all 3-dimensional contact manifolds with $\tau = 0$ (i.e. K-contact). In a long paper of 1989 S.Tanno studied the Dirichlet energy and gauge transformations of contact manifolds. In 1984 D.E.Blair obtained the critical point condition of $I(g) = \int_M Ric(\xi) dV_g$ over $\mathcal{M}(\eta)$ (the space of all associated metrics), and proved that the regularity of the characteristic vector field ξ and the critical point condition force the metric to be K-contact. Since $Ric(\xi) = 2n - \frac{1}{4}|\tau|^2$, the study of $I(g)$ is the same as the study of the Dirichlet energy. In this thesis we investigate the second variation and prove the following results.

Theorem. Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy $L(g) = \int_M |\tau|^2 dV_g$, i.e. $\nabla_\xi \mathcal{L}_\xi g = 2(\mathcal{L}_\xi g)\phi$, then along any path $g_{ij}(t) = g_{ir}[\delta^r_j + tH^r_j + t^2 K^r_j + O(t^3)]$ in $\mathcal{M}(\eta)$

$$\frac{d^2 L}{dt^2}(0) = 2\int_M |\mathcal{L}_\xi H^i_j|^2 dV_g \geq 0,$$

and $L(g)$ has minimum at each critical metric.

Theorem. Let M^{2n+1} be a compact contact manifold, and suppose that ge^{Ht} is a geodesic with $g(0)$ K-contact, then ge^{Ht} is K-contact for each t if and only if $\mathcal{L}_\xi H^i_j = 0$. In general, $|\tau|$ is constant along any geodesic ge^{Ht} with $\mathcal{L}_\xi H^i_j = 0$.

In Chapter 3 we discuss almost Kähler manifold with Hermitian Ricci tensor and its relation to critical point conditions. It was conjectured that K-contact manifolds with $Q\phi = \phi Q$ are Sasakian. We give a negative answer to the conjecture; hence we have a new class of contact manifolds. We also give a variational characterization of this class. In Chapter 4 we study other functionals.

To my wife Lina,
son Peter.

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Contents

NOTES

Chapter 1

Preliminaries

In this chapter we review some formulas and results which we will need in this thesis. Section 1 is an introduction to contact manifolds; in this section we also present a new K-contact condition. In section 2 we describe the space of all Riemannian metrics on a Riemannian manifold and the space of all associated metrics on a symplectic or contact manifold. We follow basically the notations of [3], [18] and [20]. Differentiability always means differentiability of class C^∞. By a manifold or tensor field we mean a smooth one.

1.1 Contact Riemannian manifolds

A (2n+1)-dimensional manifold M^{2n+1} is a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. η is called the *contact form.* It follows that any contact manifold is orientable.

$\eta = 0$ defines a 2n-dimensional distribution $\mathcal{D}$ of the tangent bundle, i.e. for any $m \in M^{2n+1}, D_m = \{X \in T_mM | \eta(X) = 0\}$. Since $\eta \wedge (d\eta)^n \neq 0$, $\mathcal{D}$ is not integrable and $d\eta$ has rank 2n. The subspace $V_m = \{X | X \in T_mM, d\eta(X, T_mM) = 0\}$ of T_mM is of dimension 1. Let ξ_m be the element of V_m on which η has value 1. Then ξ is a vector field, which we call the *characteristic vector field*, defined on M^{2n+1} such that

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 1 \tag{1.1}$$

for any X.

Using (1.1) and the formula for Lie differentiation, $\mathcal{L}_\xi = d \cdot \iota(\xi) + \iota(\xi) \cdot d$, we have

$$\mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi d\eta = 0. \tag{1.2}$$

In this thesis we will also discuss symplectic manifolds. A 2n-dimensional manifold M^{2n} is called a *symplectic manifold* if it admits a global 2-form Ω such that $\Omega^n \neq 0$ and $d\Omega = 0$. On a symplectic manifold we have the following theorem.

Theorem 1.1. Let (M^{2n}, Ω) be a symplectic manifold. Then there exist a metric g and an almost complex structure J such that

$$\Omega(X, Y) = g(X, JY) \tag{1.3}$$

Outline of proof. Let k be any Riemannian metric on M^{2n} and $X_1 \cdots X_{2n}$ be a local k-orthonormal basis. We know that any non-singular matrix $A \in GL(n, R)$ can be written uniquely as FG with $F \in O(n)$ and G a positive definite symmetric matrix. Now consider $A = \Omega(X_i, X_j)$. Since A is non-singular, $A = FG$ by the polarization. Then G defines a new metric g and F defines an almost complex structure J locally. In fact this construction is independent of choice of k-o.n. basis. Therefore g and J are globally defined and $\Omega(X, Y) = g(X, JY)$. Q.E.D.

Such g and J are created simultaneously and g is called an *associated metric.* Thus the space of all associated metrics, denoted by $\mathcal{M}(\Omega)$, is the space of all almost Kähler metrics with Ω as their fundamental 2-form. It can be shown that all associated metrics have the same volume element $dV = \frac{1}{2^n n!}\Omega^n$.

On a contact manifold we have the following result .

Theorem 1.2. Let (M^{2n+1}, η) be a contact manifold. Then there exist a metric g and a type (1,1) tensor field ϕ such that

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi \\ d\eta(X, Y) &= g(X, \phi Y) \end{aligned} \tag{1.4}$$

$$\eta(X) = g(X, \xi)$$

proof. Let k' be any Riemannian metric. Then

$$k(X,Y) = k'(-X + \eta(X)\xi, -Y + \eta(Y)\xi) + \eta(X)\eta(Y)$$

is a new metric with $\eta(X) = k(X, \xi)$. Since $d\eta$ is a symplectic form on $\mathcal{D}$, we can polarize $d\eta$ on $\mathcal{D}$ as in Theorem 1.1. Therefore there exist g' and ϕ on $\mathcal{D}$ such that $g'(X, \phi Y) = d\eta(X,Y)$ and $\phi^2 = -I$. Extending g' to g agreeing with k in the direction of ξ and extending ϕ so that $\phi\xi = 0$, we have the theorem. Q.E.D.

Metrics constructed by polarization as above are called *associated metrics.* We refer to (ϕ, ξ, η, g) as a *contact metric structure.* A contact manifold with a contact metric structure is called a *contact metric manifold* (or simply a contact manifold in this thesis). It follows from Theorem 1.2 that

$$\phi\xi = 0, \quad \eta(\phi X) = 0 \tag{1.5}$$

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$$

$$d\eta(X, \phi Y) = -d\eta(\phi X, Y).$$

It is well known that all associated metrics have the same volume element $dV = \frac{1}{2^n n!}\eta \wedge (d\eta)^n$. We will discuss some properties of the space of all associated metrics in section 2.

Now we are ready to introduce the concept of a Sasakian manifold, which is the odd dimensional analogue of a Kähler manifold. We consider a product manifold $M^{2n+1} \times R$ of a contact manifold M^{2n+1} and the real line. A vector field on $M^{2n+1} \times R$ looks like $(X, f\frac{d}{dt})$, where $X \in TM^{2n+1}$ and t is the coordinate of R. We define a linear map on the tangent space of $M^{2n+1} \times R$ by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{1.6}$$

Then $J^2 = -I$, i.e. J is an almost complex structure on $M^{2n+1} \times R$.

Let $[J, J]$ be the Nijenhuis torsion of J, and similarly $[\phi, \phi]$ the torsion for ϕ. We have

$$\begin{aligned} [J,J](X,Y) &= J^2[X,Y] + [JX,JY] \\ &\quad - J[JX,Y] - J[X,JY] \end{aligned}$$

If J is integrable (or $[J, J] = 0$), then we call the contact metric manifold M^{2n+1} *Sasakian.*

Let $\Phi(X, Y) = d\eta(X, Y)$ and ∇ be the Riemannian connection. From the following 2 classical formulas

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} 3d\Phi(X,Y,Z) &= X\Phi(Y,Z) + Y\Phi(Z,X) + Z\Phi(X,Y) \\ &\quad - \Phi([X,Y],Z) - \Phi([Z,X],Y) - \Phi([Y,Z],X), \end{aligned}$$

we have ([3])

$$2g((\nabla_X \phi)Y, Z) = g(N^{(1)}(Y,Z), \phi X) + 2d\eta(\phi Y, X)\eta(Z) - 2d\eta(\phi Z, X)\eta(Y) \tag{1.8}$$

where $N^{(1)}(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi$.

Theorem 1.3. $[J, J] = 0$ if and only if $N^{(1)} = 0$.

proof. Enough to check the Nijenhuis torsion for all vector fields on $M^{2n+1} \times R$. See [3] pp.48-51 for details.

Let $h = \frac{1}{2}\mathcal{L}_\xi \phi$, $\tau = \mathcal{L}_\xi g$ on a contact manifold.

Proposition 1.4. On a contact manifold with contact metric structure (ϕ, ξ, η, g),

we have

(1) $\nabla_\xi \phi = 0$;

(2) $\nabla_\xi \xi = 0$;

(3) $\nabla_t \phi^t_i = -2n\eta_i$;

(4) $\nabla_X \xi = -\phi X - \phi h X$;

(5) $h\xi = 0, \phi h + h\phi = 0$, and hence $trh = 0$;

(6) $\tau_{ij} = -2\phi_{ir}h^r_j$, and $h = \tau = 0$ iff ξ is Killing.

Remarks: We define $g_{ir}\phi^r_j = \phi_{ij}$, hence ϕ^r_j means $\phi^r{}_j$. For differentiation we use the following notation

$$\nabla_r H^s_j \nabla^r H_{sk} = (\nabla_r H^s_j)(\nabla^r H_{sk})$$

$$\nabla_r \xi^j \xi^l = (\nabla_r \xi^j)\xi^l,$$

etc. Hence we differentiate only the first object which follows the derivative sign.

Proof. First we prove (4). By (1.8) we have

$$\begin{aligned} 2g((\nabla_X\phi)\xi, Z) &= g(\phi^2[\xi, Z] - \phi[\xi, \phi Z], X) - 2d\eta(\phi Z, X) \\ &= -2g(\phi h Z, \phi X) - 2g(\phi Z, \phi X) \\ &= -2g(hZ, X) - 2g(X, Z) + 2g(\eta(X)\xi, Z), \end{aligned}$$

that is

$$-\phi\nabla_X\xi = -hX - X + \eta(X)\xi.$$

Applying ϕ to both sides we have (4). (1), (2) and (3) follow from (1.8); (5) can be proved using (1), (2) and (3). see [3] pp.55.

Using (4) we have

$$\begin{aligned} \tau(X, Y) &= (\mathcal{L}_\xi g)(X, Y) \\ &= g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) \\ &= g(-\phi X - \phi h X, Y) + g(-\phi Y - \phi h Y, X) \end{aligned}$$

$$= -2g(\phi hX, Y).$$

This completes the proof. Q.E.D.

Let $l_{ij} = R_{iuvj}\xi^u\xi^v$. Then $l_{ij}\xi^j = 0$ and l is symmetric.

Proposition 1.5. On a contact metric manifold we have

(1) $\xi^r\nabla_r h_{ij} = \phi_{ij} - \phi_{ir}h^r_s h^s_j - \phi_{ir}l^r_j$;

(2) $l_{ir}\phi^r_j + \phi_{ir}l^r_j = 2\phi_{ij} - 2\phi_{ir}h^r_s h^s_j$,

and hence, $Ric(\xi) = 2n - trh^2 = 2n - \frac{1}{4}|\tau|^2$;

(3) $\nabla_t\nabla_k\phi^t_j + \nabla_t\nabla_j\phi^t_k = R_{kt}\phi^t_j + R_{jt}\phi^t_k - 2n(h_{kr}\phi^r_j + h_{jr}\phi^r_k)$.

Proof. Using Ricci identities and Proposition 1.4 we have

$$\begin{aligned}
\phi_{ir}l^r_j &= \phi_{ir}R_{juv}{}^r\xi^u\xi^v \\
&= \phi_{ir}(\nabla_j\nabla_u\xi^r - \nabla_u\nabla_j\xi^r)\xi^u \\
&= \phi_{ir}[\nabla_j(-\phi^r_u - \phi_{us}h^{us})\xi^u - \nabla_\xi(-\phi^r_j - \phi_{js}h^{sr})] \\
&= \phi_{ij} - \phi_{ir}h^r_s h^s_j - \nabla_\xi h_{ij}.
\end{aligned}$$

Then (1) and (2) follow from above.

By Proposition 1.4 (3) and the Ricci identitiy

$$\nabla_t\nabla_k\phi^s_j - \nabla_k\nabla_t\phi^s_j = R_{tkp}{}^s\phi^p_j - R_{tkj}{}^p\phi^s_p$$

from which

$$\nabla_t\nabla_k\phi^t_j = -R_{kp}\phi^p_j - R_{ktpj}\phi^{tp} - 2n\nabla_k\eta_j.$$

(3) then follows immediately. Q.E.D.

From Proposition 1.5 we have the following formulas

$$l_{ir}\phi^r_j - \phi_{ir}l^r_j = 2\nabla_\xi h_{ij} \tag{1.9}$$

$$lh - hl = \nabla_\xi h^2 \cdot \phi$$

$$|\nabla_\xi \tau|^2 = 4[|l|^2 + trh^4 + 2tr(h^2 l) - 2n].$$

If ξ is a Killing vector field, then we call the manifold M^{2n+1} *K-contact.* By Proposition 1.4, M^{2n+1} is K-contact if and only if $\tau = h = 0$; and $\nabla_X \xi = -\phi X$ on a K-contact manifold. From Proposition 1.5 we have another K-contact condition, namely, $Ric(\xi) = 2n$.

We can also characterize K-contact manifolds as the following.

Proposition 1.6. M^{2n+1} is K-contact if and only if $(\nabla_X \phi)Y = R_{\xi X} Y$.

Proof. (a) If ξ is Killing, then $\nabla_Y \xi = -\phi Y$. We have

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi = R_{X\xi} Y.$$

Therefore

$$(\nabla_X \phi) Y = R_{\xi X} Y.$$

(b) If $(\nabla_X \phi)Y = R_{\xi X} Y$, we set $Y = \xi$. Then

$$\begin{aligned} lX &= -R_{\xi X} \xi \\ &= -(\nabla_X \phi)\xi \\ &= \phi \nabla_X \xi \\ &= \phi(-\phi X - \phi h X) \\ &= X + hX - \eta(X)\xi. \end{aligned}$$

But from Proposition 1.5 (2) we have

$$R_{\xi X}\xi - \phi R_{\xi \phi X}\xi = 2h^2 X + 2\phi^2 X.$$

Therefore

$$(-X - hX + \eta(X)\xi) + (\phi^2 X - \phi^2 h X) = 2h^2 X + 2\phi^2 X,$$

from which we have

$$2h^2 X = 0.$$

But h is symmetric, and hence we have $h = 0$. Q.E.D.

Combining Proposition 1.6 and Proposition 1.4 (3), we have on K-contact manifolds that $Q\xi = 2n\xi$ with Q denoting the Ricci operator s and from Proposition 1.6 we also have that for any $X \perp \xi$

$$R_{X\xi}\xi = X$$

Now we consider the Sasakian condition.

Theorem 1.7. M^{2n+1} is Sasakian if and only if $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$.

Proof. (a) Combining (1.8) and Theorem 1.3 we have $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$. (b) If $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$, by (1.8) and a straightforward computation we have $N^{(1)} = 0$ (see [3] p73 for details). Q.E.D.

Theorem 1.8. M^{2n+1} is Sasakian if and only if $R_{XY}\xi = \eta(Y)X - \eta(X)Y$.

Proof. (a) If M^{2n+1} is Sasakian, from Proposition 1.6 and Theorem 1.7 we have $R_{XY}\xi = \eta(Y)X - \eta(X)Y$. (b) From $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ and Proposition 1.5 (2), it is easy to see that $h = 0$; hence by Proposition 1.6 and Theorem 1.7 M^{2n+1} is Sasakian. Q.E.D.

By Proposition 1.5 (3) and Theorem 1.7 we have $Q\phi = \phi Q$ on a Sasakian manifold.

1.2 The space of all associated metrics

Let M be a compact orientable manifold. The space of all Riemannian metrics on M, denoted by $\mathcal{M}$, has a Riemannian structure which was studied by Ebin [24] and others. $\mathcal{M}$ is infinite dimensional. The tangent space at a point g consists of symmetric (0,2)-type tensors on M. The inner product at g is defined by

$$< S, T >_g = \int_M S_{ij} T_{kl} g^{ik} g^{jl} dV_g.$$

Let $\mathcal{M}_1$ be the space of all Riemannian metrics on M with fixed total volume. Then $\mathcal{M}_1$ is a subspace of $\mathcal{M}$. By normalization we may assume

$$\mathcal{M}_1 = \{g | \int_M dV_g = 1\}.$$

We begin with any metric $g \in \mathcal{M}_1$. Let $g(t)$ be any curve in $\mathcal{M}_1$ with $g(0) = g$. On a coordinate neighborhood

$$dV_{g(t)} = \sqrt{det(g_{ij}(t))} dx^1 \wedge \cdots \wedge dx^n,$$

and

$$\begin{aligned} \frac{\partial}{\partial t} dV_{g(t)} &= \frac{\partial}{\partial t} \sqrt{det(g_{ij}(t))} dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2det(g_{ij}(t))} \frac{\partial}{\partial t}(det(g_{ij}(t))) \sqrt{det(g_{ij}(t))} dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2} g^{ij}(t) \frac{\partial}{\partial t} g_{ij}(t) dV_{g(t)}. \end{aligned}$$

Hence for any $g(t)$ in $\mathcal{M}_1$

$$\int_M g^{ij}(t) \frac{\partial}{\partial t} g_{ij}(t) dV_{g(t)} = 2 \frac{\partial}{\partial t} \int_M dV_{g(t)} = 0.$$

Now we put

$$g_{ij}(t) = g_{ij} + tH_{ij} + t^2 K_{ij} + O(t^3).$$

Then

$$g^{ij}(t) = g^{ij} - tH^{ij} + t^2(H^i_r H^{rj} - K^{ij}) + O(t^3)$$

where $H^i_j = g^{ir}H_{rj}$, *etc.*

To study variational problems over $\mathcal{M}_1$ we will need the following lemma (see [11]).

Lemma 1.9. Let T_{ij} be any symmetric 2 tensor. Then

$$\int_M T_{ij}H_{kl}g^{ik}g^{jl}dV_g = 0$$

for any symmetric 2 tensor H_{ij} satisfying $\int_M g^{ij}H_{ij}dV_g = 0$ if and only if $T_{ij} = ag_{ij}$ for some constant a.

Now we consider the space of all associated metrics $\mathcal{M}(\eta)$ of a contact manifold and $\mathcal{M}(\Omega)$ of a symplectic manifold.

Let $g(t)$ be any curve in $\mathcal{M}(\eta)$ with $g(0) = g$. Then the structure tensors $(\phi(t), \xi, \eta, g(t))$ corresponding to $g(t)$ satisfy the following:

$$\begin{gathered} g_{ir}(t)\xi^r = \eta_i \\ 2g_{ir}(t)\phi^r_j(t) = 2\phi_{ij} = \nabla_i\eta_j - \nabla_j\eta_i \qquad (1.10) \\ \phi^i_r(t)\phi^r_j(t) = -\delta^i_j + \xi^i\eta_j. \end{gathered}$$

Now we put

$$\begin{gathered} g_{ij}(t) = g_{ir}[\delta^r_j + tH^r_j + t^2K^r_j + O(t^3)] \\ \phi^i_j(t) = \phi^i_j + tS^i_j + t^2T^i_j + O(t^3). \end{gathered}$$

Then from the above conditions we have the following lemma.

Lemma 1.10.

$$\begin{gathered} H_{ir}\xi^r = K_{ir}\xi^r = S^i_r\xi^r = T^i_r\xi^r = 0 \\ H_{ij} + H_{rs}\phi^r_i\phi^s_j = 0, \quad \textit{hence} \quad H^i_i = 0 \\ S^i_j = \phi^i_rH^r_j, \quad S^i_rS^r_j = H^i_rH^r_j \end{gathered}$$

$$T^i_j = \phi^i_r K^r_j$$

$$K_{ij} + K_{rs}\phi^r_i\phi^s_j = H_{ir}H^r_j$$

$$2K^r_r = H^{rs}H_{rs}$$

and

$$tr(HHH) = tr(H^2h) = tr(h^2H) = 0.$$

The proof is straightforward but note that H and h anti-commute with ϕ.

It is easy to see that the tangent space of $\mathcal{M}(\eta)$ consists of all symmetric (0,2) tensor fields H satisfying

$$H_{ij} + H_{rs}\phi^r_i\phi^s_j = 0, \quad H_{ir}\xi^r = 0 \tag{1.11}$$

Similarly the tangent space of $\mathcal{M}(\Omega)$ consists of all symmetric (0,2) tensor fields H satisfying

$$H_{ij} + H_{rs}J^r_iJ^s_j = 0 \tag{1.12}$$

In fact $\mathcal{M}(\eta)$ and $\mathcal{M}(\Omega)$ are symmetric Hilbert manifolds. Geodesics in $\mathcal{M}(\eta)$ are of the form $g(t) = ge^{Ht}$ with $H\xi = 0$, and $H\phi = -\phi H$. For details see [5]. In [25] Freed and Groisser found the general formula for geodesics in $\mathcal{M}$ and computed the curvature of $\mathcal{M}$. $\mathcal{M}(\eta)$ and $\mathcal{M}(\Omega)$ are totally geodesic submanifolds of $\mathcal{M}$ and are path connected.

To study variational problems on $\mathcal{M}(\eta)$ or $\mathcal{M}(\Omega)$ we will need the following lemma([6], [12]).

Lemma 1.11. Let T_{ij} be any symmetric 2 tensor. Then

$$\int_M T_{ij}H_{kl}g^{ik}g^{jl}dV_g = 0$$

for any symmetric 2-tensor H_{ij} satisfying (1.12) in the symplectic case and (1.11) in the contact case if and only if $TJ = JT$ in the symplectic case and $T\phi = \phi T$ on $\mathcal{D}$ in the contact case.

Proof. We sketch the proof in the symplectic case; the proof in the contact case being similar. Let $X_1, \cdots, X_{2n}$ be a local J basis on a neighborhood U and note that X_1 can be any unit vector on U. Let f be a C^∞ function with compact support in U and define $g(t)$ by the change in the subspace spanned by X_1 and JX_1 given by the matrix

$$\begin{pmatrix} 1 + tf + \frac{1}{2}t^2f^2 & \frac{1}{2}t^2f^2 \\ \frac{1}{2}t^2f^2 & 1 - tf + \frac{1}{2}t^2f^2 \end{pmatrix}$$

with no change in other directions. Then $g(t) \in \mathcal{M}(\Omega)$ and $H_{11} = -H_{22} = f$. Therefore $\int_M T_{ij}H_{kl}g^{ik}g^{jl}dV_g = 0$ becomes

$$\int_M (T^{11} - T^{22})fdV_g = 0$$

Thus since X_1 was any unit vector field on U,

$$T(X, X) = T(JX, JX)$$

for any vector field X. Since T is symmetric, linearization gives $TJ = JT$. Conversely, if T commutes with J and H anti-commutes with J, then $trTH = trTJHJ = trJTHJ = -trTH$, giving $T^{ij}H_{ij} = 0$. Q.E.D.

Chapter 2

The Dirichlet Energy

S. S. Chern and R. S. Hamilton in a paper of 1985 [21] studied a kind of Dirichlet energy in terms of the torsion $\tau(\tau = \mathcal{L}_\xi g)$ of a 3-dimensional compact contact manifold and a problem analogous to the Yamabe problem. They raised the question of determining all 3-dimensional contact manifolds with $\tau = 0$ (i.e. K-contact). In a long paper of 1989 [43] S. Tanno studied the Dirichlet energy and gauge transformations of contact manifolds. In 1984 D. E. Blair [6] obtained the critical point condition of $I(g) = \int_M Ric(\xi) dV_g$ over $\mathcal{M}(\eta)$ (the space of all associated metrics), and proved that the regularity of the characteristic vector field ξ and the critical point condition force the metric to be K-contact. Since $Ric(\xi) = 2n - \frac{1}{4}|\tau|^2$, the study of $I(g)$ is the same as the study of the Dirichlet energy. In section 2 we investigate the second variation and prove the following result.

Theorem 2.2. Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy $L(g) = \int_M |\tau|^2 dV_g$, i.e. $\nabla_\xi \mathcal{L}_\xi g = 2(\mathcal{L}_\xi g)\phi$, then along any path $g_{ij}(t) = g_{ir}[\delta^r_j + tH^r_j + t^2 K^r_j + O(t^3)]$ in $\mathcal{M}(\eta)$

$$\frac{d^2 L}{dt^2}(0) = 2\int_M |\mathcal{L}_\xi H^i_j|^2 dV_g \geq 0,$$

and $L(g)$ has minimum at each critical metric.

In section 3 we show that the critical points of the Dirichlet energy are also critical

in $\mathcal{M}_1$. In section 4 we study the behavior of the Dirichlet energy at any associated metric. In section 5 we study the isolatedness of special metrics.

2.1 The critical point condition

For completeness we show how to obtain the critical point condition in this section. The computation will also be used later on.

Let M^{2n+1} be a compact contact manifold with contact metric structure (ϕ, ξ, η, g) and

$$g_{ij}(t) = g_{ij} + tH_{ij} + t^2K_{ij} + O(t^3)$$

be any curve in $\mathcal{M}(\eta)$ with $g(0) = g$. Let Γ^i_{jk} be the Christoffel symbols for the metric g and $\Gamma^i_{jk}(t)$ for $g(t)$. We assume that $\Gamma^i_{jk}(t) = \Gamma^i_{jk} + W^i_{jk}(t)$ and that $\nabla^{(t)}$ is the Riemannian connection for $g(t)$. We have

$$\begin{aligned} 0 &= \nabla^{(t)}_i g_{jk}(t) \\ &= \frac{\partial g_{jk}(t)}{\partial x^i} - \Gamma^r_{ij}(t)g_{rk}(t) - \Gamma^r_{ik}(t)g_{rj}(t) \\ &= \nabla_i g_{jk}(t) - W^r_{ij}(t)g_{rk}(t) - W^r_{ik}(t)g_{rj}(t) \end{aligned}$$

Rotating the indices $i \to j \to k \to i$, we have

$$\nabla_j g_{ki}(t) = W^r_{jk}(t)g_{ri}(t) + W^r_{ji}(t)g_{rk}(t)$$

and

$$-\nabla_k g_{ij}(t) = -W^r_{ki}(t)g_{rj}(t) - W^r_{kj}(t)g_{ri}(t)$$

Adding these we have

$$\begin{aligned} W^i_{jk}(t) &= \Gamma^i_{jk}(t) - \Gamma^i_{jk} \\ &= \frac{1}{2}g^{ir}(t)[\nabla_j g_{rk}(t) + \nabla_k g_{rj}(t) - \nabla_r g_{jk}(t)] \end{aligned}$$

Therefore

$$\Gamma^i_{jk}(t) = \Gamma^i_{jk} + W^i_{jk}(t) \tag{2.1}$$

$$
\begin{aligned}
&= \Gamma^i_{jk} + \frac{t}{2}(\nabla_j H^i_k + \nabla_k H^i_j - \nabla^i H_{jk}) \\
&\quad + \frac{t^2}{2}[(\nabla_j K^i_k + \nabla_k K^i_j - \nabla^i K_{jk}) - H^{ir}(\nabla_j H_{rk} + \nabla_k H_{rj} - \nabla_r H_{jk})] \\
&\quad + O(t^3) \\
&= \Gamma^i_{jk} + \frac{t}{2} D_{jk}{}^i + \frac{t^2}{2}(E_{jk}{}^i - H^i_r D_{jk}{}^r) + O(t^3)
\end{aligned}
$$

where $D_{jk}{}^i = \nabla_j H^i_k + \nabla_k H^i_j - \nabla^i H_{jk}$ and $E_{jk}{}^i = \nabla_j K^i_k + \nabla_k K^i_j - \nabla^i K_{jk}$.

For the curvature tensor we have

$$
\begin{aligned}
R_{ijk}{}^h(t) &= R_{ijk}{}^h + \nabla_i W^h_{jk}(t) - \nabla_j W^h_{ik}(t) + W^h_{ir}(t) W^r_{jk}(t) - W^h_{jr}(t) W^r_{ik}(t) \\
&= R_{ijk}{}^h + \frac{t}{2}(\nabla_i D_{jk}{}^h - \nabla_j D_{ik}{}^h) + \qquad (2.2) \\
&\quad + \frac{t^2}{2}[\nabla_i(E_{jk}{}^h - H^h_r D_{jk}{}^r) - \nabla_j(E_{ik}{}^h - H^h_r D_{ik}{}^r) + \\
&\quad + \frac{1}{2}(D_{ir}{}^h D_{jk}{}^r - D_{jr}{}^h D_{ik}{}^r)] + O(t^3)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
R_{jk}(t) &= R_{jk} + \frac{t}{2}(\nabla_r \nabla_j H^r_k + \nabla_r \nabla_k H^r_j - \nabla^r \nabla_r H_{jk}) + \qquad (2.3) \\
&\quad + \frac{t^2}{4}[2(\nabla_r \nabla_j K^r_k + \nabla_r \nabla_k K^r_j - \nabla^r \nabla_r K_{jk} - \nabla_j \nabla_k K^r_r) \\
&\quad - 2H^{rs}(\nabla_s \nabla_j H_{rk} + \nabla_s \nabla_k H_{rj} - \nabla_s \nabla_r H_{jk} - \nabla_j \nabla_k H_{rs}) \\
&\quad - 2\nabla_s H^{sr}(\nabla_j H_{rk} + \nabla_k H_{rj} - \nabla_r H_{jk}) \\
&\quad + \nabla_j H^{rs} \nabla_k H_{rs} - 2\nabla_r H^s_j \nabla_s H^r_k + 2\nabla_r H^s_j \nabla^r H_{sk}] + O(t^3)
\end{aligned}
$$

Let $I(g) = \int_M Ric(\xi) dV_g$ and the Dirichlet energy $L(g) = \int_M |\tau|^2 dV_g$. For any associated metric we have $Ric(\xi) = 2n - \frac{1}{4}|\tau|^2$ by Proposition 1.5, hence $I(g) = 2n\, vol(M) - \frac{1}{4}L(g)$. Therefore they have the same critical point condition.

Theorem 2.1.(Blair [6]) Let M^{2n+1} be a compact contact manifold. An associated metric $g \in \mathcal{M}(\eta)$ is critical with respect to the Dirichlet energy if and only if

$$
\nabla_\xi \tau = 2\tau\phi. \qquad (2.4)
$$

Remarks: Chern and Hamilton studied this over the set of all CR-structures. Strongly pseudo-convex CR-manifolds are contact manifolds satisfying an integrability condition, $\mathcal{Q} = 0$, where $\mathcal{Q}$ is a (1,2)-tensor field on M^{2n+1} defined by [43]

$$\mathcal{Q}(X,Y) = (\nabla_Y \phi)(X) + (\nabla_Y \eta)(\phi X)\xi + \eta(X)\phi\nabla_Y \xi, \tag{2.5}$$

in dimension 3, $\mathcal{Q} = 0$ trivially.

Proof. We consider $I(g) = \int_M Ric(\xi) dV_g$ here. Let $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ be any curve in $\mathcal{M}(\eta)$ with $g(0) = g$. Then

$$\frac{dI}{dt}(0) = \frac{1}{2}\int_M \xi^i\xi^j(\nabla_r\nabla_i H^r_j + \nabla_r\nabla_j H^r_i - \nabla^r\nabla_r H_{ij})dV_g$$

Using Green's Theorem we have

$$\begin{aligned}\int_M \xi^i\xi^j\nabla_r\nabla_i H^r_j dV_g &= \int_M \{\nabla_r(\xi^i\xi^j\nabla_i H^r_j) - \nabla_r\xi^i\xi^j\nabla_i H^r_j - \xi^i\nabla_r\xi^j\nabla_i H^r_j\}dV_g \\ &= \int_M (\nabla^r\xi^i\nabla_i\xi^s + \xi^i\nabla_i\nabla^r\xi^s)H_{rs}dV_g\end{aligned}$$

and

$$\int_M \xi^i\xi^j\nabla^r\nabla_r H_{ij}dV_g = 2\int_M \nabla^r\xi^i\nabla_r\xi^j H_{ij}dV_g.$$

Therefore

$$\frac{dI}{dt}(0) = \int_M (\nabla^r\xi^i\nabla_i\xi^s + \xi^i\nabla_i\nabla^r\xi^s - \nabla^i\xi^r\nabla_i\xi^s)H_{rs}dV_g.$$

Let

$$\begin{aligned}U^{rs} &= \frac{1}{2}\nabla^r\xi^i\nabla_i\xi^s + \frac{1}{2}\nabla^s\xi^i\nabla_i\xi^r \\ &\quad + \frac{1}{2}\xi^i\nabla_i\nabla^r\xi^s + \frac{1}{2}\xi^i\nabla_i\nabla^s\xi^r - \nabla^i\xi^r\nabla_i\xi^s.\end{aligned}$$

Then

$$U^{rs}\eta_r = 0.$$

By Lemma 1.11 we have that g is a critical metric if and only if $U\phi = \phi U$, namely,

$$\begin{aligned}&\nabla_r\xi^i\nabla_i\eta_s\phi^s_t + \nabla_s\xi^i\nabla_i\eta_r\phi^s_t + \xi^i\nabla_i\nabla_r\eta_s\phi^s_t + \xi^i\nabla_i\nabla_s\eta_r\phi^s_t - 2\nabla^i\eta_r\nabla_i\eta_s\phi^s_t \\ =\ &\phi_{rs}\nabla^s\xi^i\nabla_i\eta_t + \phi_{rs}\nabla_t\xi^i\nabla_i\xi^s + \phi_{rs}\xi^i\nabla_i\nabla^s\eta_t + \phi_{rs}\xi^i\nabla_i\nabla_t\xi^s - 2\phi_{rs}\nabla^i\xi^s\nabla_i\eta_t\end{aligned}$$

Using

$$\nabla^r \xi^i \nabla_i \xi^s = -g^{rs} + \xi^r \xi^s + h^r_j h^{js}$$

$$\nabla^i \xi^r \nabla_i \xi^s = g^{rs} - \xi^r \xi^s - 2h^{rs} + h^r_j h^{js}$$

we have

$$\nabla_\xi \tau = 2\tau\phi.$$

Q.E.D.

Example 2.1. Any K-contact metric g is critical since $\tau = 0$, and $L(g)$ has a minimum at g.

Example 2.2. Let $T_1M(-1)$ be the tangent sphere bundle of a compact Riemannian manifold of constant curvature (-1). The standard associated metric is a critical point of $L(g)$, but τ is not 0 (see [8]). In fact, non-trivial examples must be irregular (see [6]). A vector field X on M^{2n+1} is said to be *regular* if every point $p \in M^{2n+1}$ has a cubical coordinate neighborhood U such that the integral curves of X passing through U pass through U only once. If ξ is regular, then M^{2n+1} is called a regular contact manifold. We will study regular contact manifolds in Chapter 3.

2.2 The second variation

In this section we study the second variation of $L(g)$ and prove the following result.

Theorem 2.2. Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy $L(g) = \int_M |\tau|^2 dV_g$, i.e. $\nabla_\xi \mathcal{L}_\xi g = 2(\mathcal{L}_\xi g)\phi$, then along any path $g(t)$ in $\mathcal{M}(\eta)$ with $g(0) = g$

$$\frac{d^2L}{dt^2}(0) = 2\int_M |\mathcal{L}_\xi H^i_j|^2 dV_g \geq 0,$$

and $L(g)$ has minimum at each critical metric.

Remarks: On any contact manifold $Ric(\xi) = 2n - \frac{1}{4}|\tau|^2$; hence $I(g) = \int_M Ric(\xi)dV_g$ has maximum at each critical metric. Since in dimension 3, $\mathcal{Q} = 0$, the space of all CR structures and the space of all associated metrics are the same. Our theorem has extended the theory developed by Blair, Chern, Hamilton and Tanno to the second variation. $I(g) = \int_M Ric(\xi)dV_g$ and $L(g) = \int_M |\tau|^2 dV_g$ are nice functionals on the the space of all CR structures and the space of all associated metrics .

Proof. Let $g_{ij}(t) = g_{ij} + tH_{ij} + t^2K_{ij} + O(t^3)$ be any curve in $\mathcal{M}(\eta)$ with $g(0) = g$ critical. By Theorem 2.1 we have $\nabla_\xi \mathcal{L}_\xi g = 2(\mathcal{L}_\xi g)\phi$ or $\nabla_\xi h = 2h\phi$. Now we compute the second derivative. First we consider $I(g) = \int_M Ric(\xi)dV_g$; we know from section 2.1 that

$$\begin{aligned} R_{jk}(t) \quad = \quad & R_{jk} + \frac{t}{2}(\nabla_r\nabla_j H^r_k + \nabla_r\nabla_k H^r_j - \nabla^r\nabla_r H_{jk}) + \\ & + \frac{t^2}{4}[2(\nabla_r\nabla_j K^r_k + \nabla_r\nabla_k K^r_j - \nabla^r\nabla_r K_{jk} - \nabla_j\nabla_k K^r_r) \\ & - 2H^{rs}(\nabla_s\nabla_j H_{rk} + \nabla_s\nabla_k H_{rj} - \nabla_s\nabla_r H_{jk} - \nabla_j\nabla_k H_{rs}) \\ & - 2\nabla_s H^{sr}(\nabla_j H_{rk} + \nabla_k H_{rj} - \nabla_r H_{jk}) \\ & + \nabla_j H^{rs}\nabla_k H_{rs} - 2\nabla_r H^s_j \nabla_s H^r_k + 2\nabla_r H^s_j \nabla^r H_{sk}] + O(t^3). \end{aligned}$$

If we set

$$
\begin{aligned}
I_1 &= \int_M \xi^j \xi^l (\nabla_r \nabla_l K_j^r + \nabla_r \nabla_j K_l^r - \nabla_r \nabla^r K_{jl} - \nabla_l \nabla_j K_r^r) dV_g \\
I_2 &= \int_M \xi^j \xi^l [-H^{rs}(\nabla_r \nabla_l H_{sj} + \nabla_r \nabla_j H_{sl} - \nabla_r \nabla_s H_{jl} - \nabla_l \nabla_j H_{rs}) \\
&\quad - \nabla_s H^{sr}(\nabla_l H_{rj} + \nabla_j H_{rl} - \nabla_r H_{jl}) + \frac{1}{2} \nabla_l H^{rs} \nabla_j H_{rs} \\
&\quad + \nabla_r H_{sj} \nabla^r H_l^s - \nabla_r H_{sj} \nabla^s H_l^r] dV_g,
\end{aligned}
$$

then for the second derivative of $I(g)$ we have

$$
\frac{d^2 I}{dt^2}(0) = I_1 + I_2. \tag{2.6}
$$

Using Green's Theorem, the critical point condition and the facts that

$$
H_r^i H_s^r h_i^s = 0
$$

$$
\nabla_\xi H_s^r H_i^s h_j^i \phi_r^j = 0
$$

$$
\nabla^r \xi^i \nabla_i \xi^s = -g^{rs} + \xi^r \xi^s + h_j^r h^{js}
$$

$$
\nabla^i \xi^r \nabla_i \xi^s = g^{rs} - \xi^r \xi^s - 2h^{rs} + h_j^r h^{js}
$$

we compute as follows:

$$
\begin{aligned}
\int_M \xi^j \xi^l \nabla_r \nabla_l K_j^r dV_g &= \int_M (-\nabla_r \xi^j \xi^l \nabla_l K_j^r - \xi^j \nabla_r \xi^l \nabla_l K_j^r) dV_g \\
&= \int_M (\nabla_l \nabla_r \xi^j \xi^l K_j^r + \nabla_l \xi^j \nabla_r \xi^l K_j^r) dV_g \\
&= \int_M (\xi^l \nabla_l \nabla^r \xi^s + \nabla_l \xi^s \nabla^r \xi^l) K_{rs} dV_g,
\end{aligned}
$$

$$
\begin{aligned}
\int_M \xi^j \xi^l \nabla_r \nabla^r K_{jl} dV_g &= \int_M (-\nabla_r \xi^j \xi^l \nabla^r K_{jl} - \xi^j \nabla_r \xi^l \nabla^r K_{jl}) dV_g \\
&= 2 \int_M \nabla_r \xi^j \nabla^r \xi^l K_{jl} dV_g,
\end{aligned}
$$

$$
\int_M \xi^j \xi^l \nabla_l \nabla_j K_r^r dV_g = 0
$$

and hence

$$
\begin{aligned}
I_1 &= \int_M \xi^j \xi^l (\nabla_r \nabla_l K_j^r + \nabla_r \nabla_j K_l^r - \nabla_r \nabla^r K_{jl} - \nabla_l \nabla_j K_r^r) dV_g \\
&= 2\int_M (\xi^l \nabla_l \nabla^r \xi^s + \nabla_l \xi^s \nabla^r \xi^l - \nabla_i \xi^s \nabla^i \xi^r) K_{rs} dV_g \\
&= 2\int_M [\nabla_\xi (\phi^{rs} - \phi_i^r h^{is}) + (-g^{rs} + \xi^r \xi^s + h_j^r h^{js}) \\
&\quad - (g^{rs} - \xi^r \xi^s - 2h^{rs} + h_j^r h^{js})] K_{rs} dV_g \\
&= 2\int_M (-\phi_i^r \nabla_\xi h^{is} - 2g^{rs} + 2\xi^r \xi^s + 2h^{rs}) K_{rs} dV_g \\
&= 2\int_M (2\phi_i^r \phi_j^i h^{js} - 2g^{rs} + 2\xi^r \xi^s + 2h^{rs}) K_{rs} dV_g \\
&= -4\int_M K_r^r dV_g \\
&= -2\int_M |H|^2 dV_g.
\end{aligned}
$$

Now we consider I_2:

$$
\begin{aligned}
&\int_M \xi^j \xi^l H^{rs} \nabla_r \nabla_l H_{sj} dV_g \\
= &\int_M [-\nabla_r \xi^j \xi^l H^{rs} \nabla_l H_{sj} - \xi^j \nabla_r \xi^l H^{rs} \nabla_l H_{sj} - \xi^j \xi^l \nabla_r H^{rs} \nabla_l H_{sj}] dV_g \\
= &\int_M [\nabla_l \nabla_r \xi^j \xi^l H^{rs} H_{sj} + \nabla_r \xi^j \xi^l \nabla_l H^{rs} H_{sj} + \\
&+ \nabla_l \xi^j \nabla_r \xi^l H^{rs} H_{sj} - \xi^j \xi^l \nabla_r H^{rs} \nabla_l H_{sj}] dV_g,
\end{aligned}
$$

$$
\begin{aligned}
&\int_M \xi^j \xi^l H^{rs} \nabla_r \nabla_s H_{lj} dV_g \\
= &\int_M [-\nabla_r \xi^j \xi^l H^{rs} \nabla_s H_{jl} - \xi^j \nabla_r \xi^l H^{rs} \nabla_s H_{jl} - \xi^j \xi^l \nabla_r H^{rs} \nabla_s H_{jl}] dV_g \\
= &\int_M [\nabla_r \xi^j \nabla_s \xi^l H^{rs} H_{jl} + \nabla_s \xi^j \nabla_r \xi^l H^{rs} H_{jl} - \xi^j \xi^l \nabla_r H^{rs} \nabla_s H_{jl}] dV_g \\
= &\int_M [2\nabla_r \xi^j \nabla_s \xi^l H^{rs} H_{jl} - \xi^j \xi^l \nabla_r H^{rs} \nabla_s H_{jl}] dV_g,
\end{aligned}
$$

$$
\begin{aligned}
\int_M \xi^j \xi^l H^{rs} \nabla_l \nabla_j H_{rs} dV_g &= \int_M (-\xi^j \xi^l \nabla_l H^{rs} \nabla_j H_{rs}) dV_g \\
&= -\int_M |\nabla_\xi H|^2 dV_g,
\end{aligned}
$$

$$\int_M \xi^j\xi^l\nabla_r H_{sj}\nabla^r H_l^s dV_g = -\int_M \nabla_r\xi^j\xi^l H_{sj}\nabla^r H_l^s dV_g$$
$$= \int_M \nabla_r\xi^j\nabla^r\xi^l H_{sj}H_l^s dV_g,$$

$$\int_M \xi^j\xi^l\nabla_r H_{sj}\nabla^s H_l^r dV_g = -\int_M \nabla_r\xi^j\xi^l H_{sj}\nabla^s H_l^r dV_g$$
$$= \int_M \nabla_r\xi^j\nabla^s\xi^l H_{sj}H_l^r dV_g.$$

Therefore

$$\begin{aligned}
I_2 &= \frac{1}{2}\int_M \xi^j\xi^l[-2H^{rs}(\nabla_r\nabla_l H_{sj} + \nabla_r\nabla_j H_{sl} - \nabla_r\nabla_s H_{jl} - \nabla_l\nabla_j H_{rs}) \\
&\quad - 2\nabla_s H^{rs}(\nabla_l H_{rj} + \nabla_j H_{rl} - \nabla_r H_{jl}) \\
&\quad + \nabla_l H^{rs}\nabla_j H_{rs} + 2\nabla_r H_{sj}\nabla^r H_l^s - 2\nabla_r H_{sj}\nabla^s H_l^r]dV_g \\
&= \frac{1}{2}\int_M[-4\nabla_l\nabla_r\xi^j\xi^l H^{rs}H_{sj} - 4\nabla_r\xi^j\xi^l\nabla_l H^{rs}H_{sj} \\
&\quad - 4\nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj} + 4\xi^j\xi^l\nabla_r H^{rs}\nabla_l H_{sj} \\
&\quad + 4\nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl} - 2\xi^j\xi^l\nabla_r H^{rs}\nabla_s H_{jl} \\
&\quad - 2\xi^j\xi^l\nabla_s H^{rs}(\nabla_l H_{rj} + \nabla_j H_{rl} - \nabla_r H_{jl}) - \xi^j\xi^l\nabla_l H^{rs}\nabla_j H_{rs} \\
&\quad + 2\nabla_r\xi^j\nabla^r\xi^l H_{sj}H_l^s - 2\nabla_r\xi^j\nabla^s\xi^l H_{sj}H_l^r]dV_g \\
&= \frac{1}{2}\int_M[-4\nabla_l\nabla_r\xi^j\xi^l H^{rs}H_{sj} - 4\nabla_r\xi^j\xi^l\nabla_l H^{rs}H_{sj} \\
&\quad - 4\nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj} + 4\nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl} \\
&\quad - \xi^j\xi^l\nabla_l H^{rs}\nabla_j H_{rs} \\
&\quad + 2\nabla_r\xi^j\nabla^r\xi^l H_{sj}H_l^s - 2\nabla_r\xi^j\nabla^s\xi^l H_{sj}H_l^r]dV_g \\
&= -\frac{1}{2}\int_M |\nabla_\xi H|^2 dV_g \\
&\quad + \int_M[-2\nabla_l\nabla_r\xi^j\xi^l H^{rs}H_{sj} - 2\nabla_r\xi^j\xi^l\nabla_l H^{rs}H_{sj} \\
&\quad - 2\nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj} + 2\nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl} \\
&\quad + \nabla_r\xi^j\nabla^r\xi^l H_{sj}H_l^s - \nabla_r\xi^j\nabla^s\xi^l H_{sj}H_l^r]dV_g
\end{aligned}$$

but

$$\begin{aligned}\int_M \xi^l \nabla_l \nabla_r \xi^j H^{rs} H_{sj} dV_g &= \int_M \nabla_\xi (-\phi_r^j - \phi_{ri} h^{ij}) H_s^r H_j^s dV_g \\ &= -\int_M \phi_{ri} \nabla_\xi h^{ij} H_s^r H_j^s dV_g \\ &= -2\int_M h_r^j H_s^r H_j^s dV_g \\ &= 0,\end{aligned}$$

$$\begin{aligned}\int_M \nabla_l \xi^j \nabla_r \xi^l H^{rs} H_{sj} dV_g &= \int_M (-\delta_r^j + \xi^j \eta_r + h_k^j h_r^k) H_s^r H_j^s dV_g \\ &= \int_M (-|H|^2 + |hH|^2) dV_g,\end{aligned}$$

$$\begin{aligned}\int_M \nabla_r \xi^j \nabla_s \xi^l H^{rs} H_{jl} dV_g &= \int_M (-\phi_r^j - \phi_{ri} h^{ij})(-\phi_s^l - \phi_{sk} h^{kl}) H^{rs} H_{jl} dV_g \\ &= \int_M [-\phi_r^j H_s^r \phi_l^s H_j^l + \phi_r^j H_s^r \phi_k^s h_l^k H_j^l \\ &\quad + \phi_s^l H_r^s \phi_i^r h_j^i H_l^j - h_j^i H_l^j h_k^l \phi_s^k H_r^s \phi_i^r] dV_g \\ &= \int_M (-|H|^2 - tr(hH)^2) dV_g,\end{aligned}$$

$$\begin{aligned}\int_M \nabla_r \xi^j \nabla^r \xi^l H_{sj} H_l^s dV_g &= \int_M (g^{jl} - \xi^j \xi^l - 2h^{jl} + h_i^j h^{il}) H_{js} H_l^s dV_g \\ &= \int_M (|H|^2 + |hH|^2) dV_g,\end{aligned}$$

$$\begin{aligned}\int_M \nabla_r \xi^j \nabla^s \xi^l H_{sj} H_l^r dV_g &= \int_M (-\phi_r^j - \phi_{ri} h^{ij})(-\phi_s^l - \phi_{sk} h^{kl}) H_j^s H_l^r dV_g \\ &= \int_M (\phi_r^j H_l^r \phi_s^l H_j^s - h_k^l \phi_s^k H_j^s \phi_r^j H_l^r \\ &\quad - \phi_s^l H_j^s h_i^j \phi_r^i H_l^r + h_k^l \phi_s^k H_j^s h_i^j \phi_r^i H_l^r) dV_g \\ &= \int_M (|H|^2 - tr(hH)^2) dV_g\end{aligned}$$

and hence

$$\begin{aligned} I_2 &= \int_M [2(\phi_r^j + \phi_{ri}h^{ij})\nabla_\xi H_s^r H_j^s - tr(hH)^2 \\ &\quad - |hH|^2 - \frac{1}{2}|\nabla_\xi H|^2] dV_g. \end{aligned}$$

Since $\phi_{ri}h^{ij}\nabla_\xi H_s^r H_j^s = 0$, we have from (2.6) that

$$\begin{aligned} \frac{d^2 I}{dt^2}(0) &= I_1 + I_2 \\ &= \int_M [-2|H|^2 + 2\phi_r^j \nabla_\xi H_s^r H_j^s - \frac{1}{2}|\nabla_\xi H|^2 \\ &\quad - tr(hH)^2 - |hH|^2] dV_g \\ &= \int_M [-\frac{1}{2}|2H - \phi\nabla_\xi H|^2 - tr(hH)^2 - |hH|^2] dV_g. \end{aligned}$$

Now note that

$$\begin{aligned} \mathcal{L}_\xi H_j^i &= \xi^r \nabla_r H_j^i + H_r^i \nabla_j \xi^r - H_j^r \nabla_r \xi^i \\ &= \nabla_\xi H_j^i + H_r^i(-\phi_j^r - \phi_{js}h^{sr}) - H_j^r(-\phi_r^i - \phi_{rs}h^{si}) \\ &= (\nabla_\xi H_j^i - 2H_r^i \phi_j^r) + (H_r^i h_s^r + h_r^i H_s^r)\phi_j^s \end{aligned} \tag{2.7}$$

and hence

$$\begin{aligned} |\mathcal{L}_\xi H_j^i|^2 &= |\nabla_\xi H - 2H\phi|^2 + |Hh + hH|^2 \\ &= |\nabla_\xi H - 2H\phi|^2 + 2tr(hH)^2 + 2|hH|^2. \end{aligned} \tag{2.8}$$

Therefore

$$\frac{d^2 L}{dt^2}(0) = (-4)\frac{d^2 I}{dt^2}(0) = 2\int_M |\mathcal{L}_\xi H_j^i|^2 dV_g \geq 0.$$

We show in the next theorem that $|\tau(t)|^2$ is constant along any geodesic $g(t) = ge^{Ht}$ with $\mathcal{L}_\xi H_j^i = 0$; hence, $L(g)$ is constant along all such geodesics. $\mathcal{M}(\eta)$ is geodesically complete [5], therefore $L(g)$ has minimum at each critical metric. Q.E.D.

Theorem 2.3. $\tau^i_j(t) = \tau^i_j(0)$ **along any geodesic** $g(t) = ge^{Ht}$ **with** $\mathcal{L}_\xi H^i_j = 0$. **In particular,** $|\tau(t)|^2$ **is constant along such geodesics.**

Proof. Let $D^{(n)i}_{jk} = \nabla_j(H^n)^i_k + \nabla_k(H^n)^i_j - \nabla^i(H^n)_{jk}$. We first compute $\Gamma^i_{jk}(t)$ along the geodesic ge^{Ht},

$$\begin{aligned}
&\Gamma^i_{jk}(t)\\
= \; & \Gamma^i_{jk} + \frac{t}{2}D^i_{jk} + \frac{t^2}{2}(\frac{1}{2}D^{(2)i}_{jk} - H^i_r D^r_{jk})\\
& + \cdots +\\
& + \frac{t^n}{2}[\frac{1}{n!}D^{(n)i}_{jk} + \frac{1}{(n-1)!}(-1)H^i_r D^{(n-1)r}_{jk} + \frac{1}{(n-2)!2!}(H^2)^i_r D^{(n-2)r}_{jk} + \cdots +\\
& + \frac{1}{(n-l)!l!}(-1)^l(H^l)^i_r D^{(n-l)r}_{jk} + \cdots + \frac{1}{(n-1)!}(-1)^{n-1}(H^{n-1})^i_r D_{jk}{}^r] +\\
& + \cdots
\end{aligned}$$

If $\mathcal{L}_\xi H^i_j = 0$, we have $\nabla_\xi H = 2H\phi$ and $hH = -Hh$ from (2.8). We now show that $D^{(n)i}_{jk}\xi^k = 2(H^n)^i_r\phi^r_j$ for any n.

$$\begin{aligned}
D^{(n)i}_{jk}\xi^k &= [\nabla_j(H^n)^i_k + \nabla_k(H^n)^i_j - \nabla^i(H^n)_{jk}]\xi^k\\
&= \nabla_\xi(H^n)^i_j - (H^n)^i_k(-\phi^k_j - \phi^k_r h^r_j) + (H^n)^k_j(\phi^{ik} - \phi^i_r h^{rk})\\
&= \nabla_\xi(H^n)^i_j + (H^n)^i_k\phi^k_j + \phi^i_k(H^n)^k_j - (H^n)^i_r h^r_k\phi^k_j - \phi^i_r h^r_k(H^n)^k_j\\
&= 2(H^n)^i_r\phi^r_j
\end{aligned}$$

for any n. Thus along ge^{Ht} with $\mathcal{L}_\xi H^i_j = 0$,

$$\begin{aligned}
\nabla^{(t)}_j\xi^i = \; & \nabla_j\xi^i + \frac{t}{2}D_{jk}{}^i\xi^k + \frac{t^2}{2}(\frac{1}{2}D^{(2)i}_{jk}\xi^k - H^i_r D_{jk}{}^i\xi^k) + \cdots +\\
& + \frac{t^n}{2}[\frac{1}{n!}D^{(n)i}_{jk} + \frac{1}{(n-1)!}(-1)H^i_r D^{(n-1)r}_{jk} + \frac{1}{(n-2)!2!}(H^2)^i_r D^{(n-2)r}_{jk} + \cdots +
\end{aligned}$$

$$+\frac{1}{(n-l)!l!}(-1)^l(H^l)^i_r D^{(n-l)r}_{jk} + \cdots + \frac{1}{(n-1)!}(-1)^{n-1}(H^{n-1})^i_r D_{jk}{}^r]\xi^k +$$
$$+\cdots$$

and therefore

$$\begin{aligned}
&-\phi^i_j(t) + \frac{1}{2}\tau^i_j(t) \\
= \;& -\phi^i_j + \frac{1}{2}\tau^i_j - t\phi^i_r H^r_j \\
& -\frac{t^2}{2}(H^2)^i_r\phi^r_j - \cdots - \\
& -\frac{t^n}{n!}(H^n)^i_k\phi^k_j[1-n+(-1)^2\frac{n!}{(n-2)!2!} + \cdots + \frac{n!}{(n-1)!}(-1)^{n-1}] \\
& +\cdots \\
= \;& -\phi^i_j + \frac{1}{2}\tau^i_j - t\phi^i_r H^r_j - \cdots - \frac{t^n}{n!}\phi^i_r(H^n)^r_j - \cdots .
\end{aligned}$$

Note that $\phi H = -H\phi$; hence

$$\phi e^{Ht} = e^{-Ht}\phi.$$

Therefore we have

$$\begin{aligned}
g_t(X, \phi e^{Ht}Y) &= g(X, e^{Ht}\phi e^{Ht}Y) \\
&= g(X, \phi Y) \\
&= d\eta(X, Y)
\end{aligned}$$

and

$$\phi e^{Ht}\phi e^{Ht} = \phi^2 = -I + \xi \otimes \eta$$

from which $\phi(t) = \phi e^{Ht}$. Thus we have

$$\tau^i_j(t) = \tau^i_j(0)$$

along ge^{Ht} with $\mathcal{L}_\xi H^i_j = 0$. Q.E.D.

Remarks: From Example 2.2 we know that the standard associated metric is a critical point of $L(g)$, but τ is not 0. In fact, non-trivial examples must be irregular (see [6]). Theorem 2.2 says that $L(g)$ has local minimum at the standard metric. It seems that it is also a global minimum, or in other words, one can not deform the metric to have $\tau = 0$ (see also Example 2.3).

Recently Jack Lee and others studied the moduli space of all CR structures on a compact 3-dimensional CR-manifold. Since in 3-dimension $Q = 0$, 3-dimensional CR-manifolds are contact manifolds. Our theorem applies as a special case. But little is known about the differential structure of $\mathcal{M}(\eta)$. It seems to be difficult to determine whether we have Morse theory here, i.e. to verify the condition (C)(see [38] for details; it is a condition to have Morse theory of differentiable real functions on Hilbert manifolds).

2.3 Critical even in $\mathcal{M}_1$

In this section we prove an interesting result, namely, that the critical metrics of the Dirichlet energy in $\mathcal{M}(\eta)$ are also critical metrics of the same functional in $\mathcal{M}_1$.

Proposition 2.4. The critical metrics of the Dirichlet energy $L(g) = \int_M |\tau|^2 dV_g$ in $\mathcal{M}(\eta)$ are also critical metrics in $\mathcal{M}_1$.

Proof. We begin with a contact metric structure (ϕ, ξ, η, g) . For any path $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ in $\mathcal{M}_1$.

$$\begin{aligned}
L(g(t)) &= \int_M |\mathcal{L}_\xi g(t)|^2 dV_{g(t)} \\
&= \int_M [(g^{ir} - tH^{ir})(g^{js} - tH^{js})(\mathcal{L}_\xi g_{ij} + t\mathcal{L}_\xi H_{ij})(\mathcal{L}_\xi g_{rs} + t\mathcal{L}_\xi H_{rs}) \\
&\quad + O(t^2)] dV_{g(t)} \\
&= \int_M \{|\mathcal{L}_\xi g|^2 - 2t[H^{ir} g^{js} \mathcal{L}_\xi g_{ij} \mathcal{L}_\xi g_{rs} - (\mathcal{L}_\xi H; \mathcal{L}_\xi g)] + O(t^2)\} dV_{g(t)}
\end{aligned}$$

where we use notation $(T; S) = T^{ij} S_{ij}$, therefore

$$\frac{dL}{dt}(0) = \int_M [2(\mathcal{L}_\xi H; \mathcal{L}_\xi g) - 2H^{ir} g^{js} \mathcal{L}_\xi g_{ij} \mathcal{L}_\xi g_{rs} + \frac{1}{2} |\mathcal{L}_\xi g|^2 g^{ij} H_{ij}] dV_g.$$

Using Green's theorem we compute as follows:

$$\begin{aligned}
&\int_M (\mathcal{L}_\xi H; \mathcal{L}_\xi g) dV_g \\
&= \int_M [(\xi^k \nabla_k H_{ij} + \nabla_i \xi^k H_{kj} + \nabla_j \xi^k H_{ik}) \\
&\quad (\nabla_r \eta_s + \nabla_s \eta_r) g^{ir} g^{js}] dV_g \\
&= \int_M \{\nabla_k [\xi^k H_{ij} (\nabla^i \xi^j + \nabla^j \xi^i)] - \xi^k H_{ij} \nabla_k (\nabla^i \xi^j + \nabla^j \xi^i) \\
&\quad + (\nabla^i \xi^j + \nabla^j \xi^i) \nabla_i \xi^k H_{kj} + (\nabla^i \xi^j + \nabla^j \xi^i) \nabla_j \xi^k H_{ik}\} dV_g \\
&= \int_M H_{ij} [-\xi^k \nabla_k (\nabla^i \xi^j + \nabla^j \xi^i) + (\nabla^r \nabla^j + \nabla^j \xi^r) \nabla_r \xi^i \\
&\quad + (\nabla^i \xi^r + \nabla^r \xi^i) \nabla_r \xi^j] dV_g \\
&= \int_M [-\xi^k \nabla_k (\nabla^i \xi^j + \nabla^j \xi^i) \\
&\quad + 2\nabla_r \xi^i (\nabla^r \nabla^j + \nabla^j \xi^r)] H_{ij} dV_g
\end{aligned}$$

and

$$\int_M H^{ir} g^{js} \mathcal{L}_\xi g_{ij} \mathcal{L}_\xi g_{rs} dV_g = \int_M H^{ir} g^{js} (\nabla_i \eta_j + \nabla_j \eta_i)(\nabla_r \eta_s + \nabla_s \eta_r) dV_g$$
$$= \int_M (\nabla^i \xi^r + \nabla^r \xi^i)(\nabla^j \eta_r + \nabla_r \xi^j) H_{ij} dV_g$$

Let

$$T^{ij} = -\xi^k \nabla_k (\nabla^i \xi^j + \nabla^j \xi^i) + (\nabla^r \nabla^j + \nabla^j \xi^r) \nabla_r \xi^i + (\nabla^i \xi^r + \nabla^r \xi^i) \nabla_r \xi^j$$
$$- (\nabla^i \xi^r + \nabla^r \xi^i)(\nabla^j \eta_r + \nabla_r \xi^j) + \frac{1}{4} |\mathcal{L}_\xi g|^2 g^{ij}.$$

Then T is symmetric and

$$\frac{dL}{dt}(0) = \int_M 2T^{ij} H_{ij} dV_g.$$

We can simplify T^{ij} to

$$T^{ij} = -\nabla_\xi \tau^{ij} + \frac{1}{4} |\mathcal{L}_\xi g|^2 g^{ij} - 4h^{ij}.$$

By Lemma 1.9 in Chapter 1 , the critical point condition is $T^{ij} = ag^{ij}$. Therefore

$$-\nabla_\xi \tau^{ij} + \frac{1}{4} |\mathcal{L}_\xi g|^2 g^{ij} - 4h^{ij} = ag^{ij}.$$

By taking the trace we have $a = \frac{1}{4} |\mathcal{L}_\xi g|^2$. Therefore

$$\nabla_\xi \tau = -4h$$

or

$$\nabla_\xi \tau = 2\tau\phi.$$

2.4 Directions of most rapid change

In this section we study the behavior of the Dirichlet energy at any associated metric g. We find the direction in which the Dirichlet energy changes most rapidly.

First we have

$$\begin{aligned}
\tau_{ij}(t) &= \nabla_i^{(t)}\eta_j + \nabla_j^{(t)}\eta_i \\
&= \nabla_i\eta^j + \nabla_j\eta^i + \frac{t}{2}(-D_{ij}{}^r\eta_r - D_{ji}{}^r\eta_r) + O(t^2) \\
&= \tau_{ij} - tD_{ij}{}^r\eta_r + O(t^2)
\end{aligned}$$

and

$$\begin{aligned}
D_{ij}{}^r\eta_r &= (\nabla_i H_j^r + \nabla_j H_i^r - \nabla^r H_{ij})\eta_r \\
&= -H_j^r(\phi_{ir} - \phi_{is}h_r^s) - H_i^r(\phi_{jr} - \phi_{js}h_r^s) - \nabla_\xi H_{ij} \\
&= -2\phi_{ir}H_j^r - \nabla_\xi H_{ij} + \phi_{ir}(h_s^r H_j^s + H_s^r h_j^s).
\end{aligned}$$

Therefore

$$\begin{aligned}
\tau_i^j(t) &= [\tau_{is} - tD_{is}{}^r\eta_r + O(t^2)][g^{sj} - tH^{sj} + O(t^2)] \\
&= \tau_i^j - t(D_{is}{}^r\eta_r g^{sj} + \tau_{is}H^{sj}) + O(t^2)
\end{aligned}$$

and

$$\begin{aligned}
|\tau(t)|^2 &= [\tau_i^j - t(D_{is}{}^r\eta_r g^{sj} + \tau_{is}H^{sj}) + O(t^2)][\tau_j^i - t(D_{js}{}^r\eta_r g^{si} + \tau_{js}H^{si}) + O(t^2)] \\
&= |\tau|^2 - 2t(D_{is}{}^r\eta_r\tau^{sj} + \tau_s^i H_j^s\tau_i^j) + O(t^2) \\
&= |\tau|^2 - 2t(D_{is}{}^r\eta_r\tau^{sj}) + O(t^2);
\end{aligned}$$

then we have

$$\begin{aligned}
|\tau(t)|^2 &= |\tau|^2 - 2t[-2\phi_{ir}H_j^r - \nabla_\xi H_{ij} + \phi_{ir}(h_s^r H_j^s + H_s^r h_j^s)]\tau^{ij} + O(t^2) \\
&= |\tau|^2 - 2t[4\phi_t^s h_j^t \phi_k^j H_s^k + 2\phi_t^s h_j^t \nabla_\xi H_s^j - 2\phi_t^s h_j^t \phi_k^j(h_l^k H_s^l + H_l^k h_s^l)] + O(t^2) \\
&= |\tau|^2 - 4t(2h_i^j H_j^i + \phi_i^j h_t^i \nabla_\xi H_j^t) + O(t^2).
\end{aligned}$$

Therefore

$$
\begin{aligned}
L(g(t)) &= L(g) - 4t\int_M (2h_i^j H_j^i + \phi_i^j h_t^i \nabla_\xi H_j^t) dV_g + O(t^2) \\
&= L(g) - 4t\int_M (2h^{rs} - \phi_t^r \nabla_\xi h^{ts}) H_{rs} dV_g + O(t^2)
\end{aligned}
$$

and we have the following result.

Proposition 2.5. If $2h^{rs} - \phi_t^r \nabla_\xi h^{ts} \neq 0$, then $L(g)$ changes most rapidly in the direction $H^{rs} = 2h^{rs} - \phi_t^r \nabla_\xi h^{ts}$.

This is essentially Theorem 2.1 in Chapter 2. Here we compute $\nabla_j^{(t)}\eta_i$ first, hence the expression of $L(g(t))$ becomes nicer. We can study the equation of evolution; but at this time it seems to be difficult.

2.5 Isolatedness of special metrics

In Gauge theory people study "good" connections in the moduli space. We are in a similar situation. It is very natural to study "good" or special metrics in the space of Riemannian metrics. Our interest here will be Riemannian metrics associated to a contact structure or symplectic structure. We will discuss the isolatedness of K-contact metrics and Sasakian metrics in contact manifolds and Kähler metrics in symplectic manifolds.

First we consider the symplectic case. Let M^{2n} be a compact symplectic manifold with symplectic form Ω, i.e. $\Omega^n \neq 0$ and $d\Omega = 0$. By polarization, an associated metric g and an almost complex structure J can be created simultaneously such that

$$\Omega(X,Y) = g(X, JY)$$

as in Theorem 1.1. There we began with any Riemannian metric k; one would think that the space of all associated metrics $\mathcal{M}(\Omega)$ is a large set of metrics. In fact we can see that $\mathcal{M}(\Omega)$ is infinite dimensional through the following deformation of metrics ([6] , [12]).

Let $g \in \mathcal{M}(\Omega)$ and $X_1, \cdots, X_{2n}$ a local J basis on a neighborhood U. Let f be a C^∞ function with compact support in U. Define $g(t)$ by the change in the subspace spanned by X_1 and JX_1 given by the matrix

$$\begin{pmatrix} 1 + tf + \frac{1}{2}t^2f^2 & \frac{1}{2}t^2f^2 \\ \frac{1}{2}t^2f^2 & 1 - tf + \frac{1}{2}t^2f^2 \end{pmatrix}$$

with no change in other directions. Then $g(t)$ is also an associated metric for each t.

$\mathcal{M}(\Omega)$ can also be considered as the set of all almost Kähler metrics of M^{2n} which have Ω as their fundamental 2-form. Let's consider the following problem: how isolated are Kahler metrics in $\mathcal{M}(\Omega)$? This was first studied by Blair [10].

We begin with a Kähler structure (g, J) on M^{2n} . Let

$$g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$$

$$J^i_j(t) = J^i_j + tS^i_j + O(t^2)$$

be any curve in $\mathcal{M}(\Omega)$ with almost complex structure $J(t)$. Then we have $g_{ir}(t)J^r_j(t) = J_{ij}$. Therefore

$$JH + HJ = 0 \tag{2.9}$$

and

$$S = JH \tag{2.10}$$

Proposition 2.6. Let g be a Kähler metric and $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ any path in $\mathcal{M}(\Omega)$. If each metric on $g(t)$ is Kähler, then it is necessary that

$$\nabla^i H_{kl} - \nabla_l H^i_k = J^i_r \nabla^r H_{ks} J^s_l - J^i_r \nabla_s H^r_k J^s_l \tag{2.11}$$

Proof. Let $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ and $J^i_j(t) = J^i_j + tS^i_j + O(t^2) = J^i_j + tJ^i_r H^r_j + O(t^2)$. We have

$$\begin{aligned} \nabla^{(t)}_k J^i_j(t) &= \nabla^{(t)}_k (J^i_j + tJ^i_r H^r_j + O(t^2)) \\ &= \nabla_k J^i_j + \frac{t}{2}[D_{kr}{}^i J^r_j - D_{jk}{}^r J^i_r + 2\nabla_k (J^i_r H^r_j)] + O(t^2). \end{aligned} \tag{2.12}$$

It is well-known that g is a Kähler metric if and only if

$$\nabla J = 0.$$

If $g(t)$ is Kähler for each t, then

$$\begin{aligned} 2\nabla_k (H^i_r J^r_j) &= D_{kr}{}^i J^r_j - D_{jk}{}^r J^i_r \\ &= (\nabla_k H^i_r + \nabla_r H^i_k - \nabla^i H_{kr}) J^r_j \\ &\quad - (\nabla_k H^r_j + \nabla_j H^r_k - \nabla^r H_{kj}) J^i_r \end{aligned}$$

from which

$$\nabla^i H_{kl} - \nabla_l H^i_k = J^i_r \nabla^r H_{ks} J^s_l - J^i_r \nabla_s H^r_k J^s_l$$

This completes the proof. Q.E.D.

If H is parallel, then from (2.12) we can see that along ge^{Ht}, $\nabla_k^{(t)} J_j^i(t) = 0$. The conditions of $\nabla H = 0$ and $HJ + JH = 0$ are strong. If a Kähler manifold admits such symmetric (0,2) tensor, then it is locally a product of a Kähler manifold on which H is trivial and a flat Kähler manifold [10]. For example, $T^n \times T^n$ with H the difference of the projection map. Problems in the symplectic case are still being studied.

Naturally we are interested in the similar problems in contact geometry. We now consider the isolatedness of K-contact metrics.

Proposition 2.7. Let g be a K-contact metric. Then any metric on the path $g(t) = ge^{Ht}$ with $\mathcal{L}_\xi H_j^i = 0$ is K-contact.

Proof. Since $|\tau(t)| = 0$ along such path by Theorem 2.3 , we have this result immediately. Q.E.D.

Proposition 2.8. Let g be a K-contact metric. Then g is isolated in $\mathcal{M}(\eta)$ if and only if equation $\mathcal{L}_\xi H_j^i = 0$ has no non-trivial solution on M^{2n+1}.

Proof. (1) If $\mathcal{L}_\xi H_j^i = 0$ has no non-trivial solution on M^{2n+1} , then by Theorem 2.2.

we have

$$\frac{d^2 L}{dt^2}(0) = 2\int_M |\mathcal{L}_\xi H_j^i|^2 dV_g > 0$$

therefore g is isolated in $\mathcal{M}(\eta)$.

(2) If g is isolated in $\mathcal{M}(\eta)$, we know that $|\tau(t)| = 0$ along geodesic $g(t) = ge^{Ht}$ with $\mathcal{L}_\xi H_j^i = 0$, then $\mathcal{L}_\xi H_j^i = 0$ has no non-trivial solution. Q.E.D.

The equation $\mathcal{L}_\xi H_j^i = 0$ appears to be important in contact geometry. Let's look at the following example.

Example 2.3. Let M^{2n+1} be a regular contact manifold. Then M^{2n+1} fibers over an almost Kahler manifold M^{2n}, and equation $\mathcal{L}_\xi H^i_j = 0$ means that H^i_j is projectable. Let H' be any type (1,1) tensor field on M^{2n} such that $H'J = JH'$. Then the horizontal lift of H' denoted by H is a non-trivial solution of $\mathcal{L}_\xi H^i_j = 0$; therefore there are plenty of K-contact metrics on a regular contact manifold.

From the above example it seems that equation $\mathcal{L}_\xi H^i_j = 0$ may have no non-trivial solution on an irregular contact manifold . That is why we tend to believe that $L(g)$ has global minimum at the standard metric of the tangent sphere bundle of a compact Riemannian manifold of constant curvature (-1), i.e. $T_1M(-1)$.

Now we turn to Sasakian metrics. Let M^{2n+1} with contact metric structure (ϕ, ξ, η, g) be a Sasakian manifold. It is well known that M^{2n+1} is Sasakian if and only if $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ (Theorem 1.8).

Proposition 2.9. Let $g(t)$ be any curve in $\mathcal{M}(\eta)$ with $g(0) = g$ Sasakian. Then $g(t)$ is Sasakian for any t if and only if

$$\xi^r R_{ijr}{}^k(t) = \xi^r R_{ijr}{}^k$$

for any t. In particular, if $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$, then H satisfies the following equation

$$\nabla_i H^h_r \phi^r_j - \nabla_r H^h_i \phi^r_j - H^h_i \eta_j = \nabla_j H^h_r \phi^r_i - \nabla_r H^h_j \phi^r_i - H^h_j \eta_i$$

Proof. M^{2n+1} is Sasakian if and only if $R_{XY}\xi = \eta(Y)X - \eta(X)Y$. Since $g(0) = g$ is Sasakian, we have

$$\xi^r R_{ijr}{}^k = \eta_j \delta^k_i - \eta_i \delta^k_j$$

and that $g(t)$ is Sasakian for any t if and only if

$$\xi^r R_{ijr}{}^k(t) = \eta_j \delta^k_i - \eta_i \delta^k_j$$

Therefore $g(t)$ is Sasakian for any t if and only if

$$\xi^r R_{ijr}{}^k(t) = \xi^r R_{ijr}{}^k$$

or

$$\xi^r[\frac{t}{2}(\nabla_j D_{kr}{}^i - \nabla_k D_{jr}{}^i) + O(t^2)] = 0$$

From

$$\xi^r(\nabla_j D_{kr}{}^i - \nabla_k D_{jr}{}^i) = 0$$

we have

$$\nabla_i H_r^h \phi_j^r - \nabla_r H_i^h \phi_j^r - H_i^h \eta_j = \nabla_j H_r^h \phi_i^r - \nabla_r H_j^h \phi_i^r - H_j^h \eta_i$$

completing the proof.

Chapter 3

A New Class

3.1 Hermitian Ricci tensor and critical condition

It is well known that the critical metrics of $A(g) = \int_M R dV_g$ over the space of all Riemannian metrics with fixed total volume are Einstein metrics.

In [12] Blair and Ianus studied the same functional over the space of all associated metrics of a compact symplectic manifold. They found that $g \in \mathcal{M}(\Omega)$ is a critical point of $A(g) = \int_M R dV_g$ if and only if

$$QJ = JQ \tag{3.1}$$

where Q is the Ricci operator of g and J is the almost complex structure.

Recently Blair showed in [9] that on a compact symplectic manifold

$$\begin{aligned} E(g) &= \int_M (R + R^\star) dV_g \\ &= \int_M 4\pi \gamma_{ij} J^{ij} dV_g \\ &= \frac{\pi}{2^{n-3}(n-1)!} \int_M \gamma \wedge \Omega^{n-1} \end{aligned}$$

where $R^\star$ is the star scalar curvature and γ is the generalized Chern form. Therefore $E(g) = \int_M (R + R^\star) dV_g$ is a symplectic invariant.

On an almost Kähler manifold we have

$$R - R^\star = -\frac{1}{2}|\nabla J|^2.$$

Thus Kähler metrics are maxima of the functional $K(g) = \int_M (R - R^\star) dV_g$. By (3.1) and the symplectic invariance of $E(g)$, the critical point condition of $K(g)$ is also $QJ = JQ$. When Q satisfies (3.1), we say the Ricci tensor is Hermitian.

It was a long standing question whether or not almost Kähler manifolds with $QJ = JQ$ are Kähler manifolds . In 1990 Davidov and Muskarov [23] gave a counter-example by studying twistor spaces with Hermitian Ricci tensor. Before we can state their result, we need to review twistor spaces (see [23]).

Let (M^4, g) be an oriented Riemannian manifold. g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by $g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2} det(g(X_i, X_j)), i = 1, 2, j = 3, 4$. We define the curvature operator $\mathbf{R} : \Lambda^2 TM \longrightarrow \Lambda^2 TM$ by

$$g(\mathbf{R}(X \wedge Y), U \wedge V) = -g(R_{XY} U, V).$$

We now need the Hodge $\star$ - operator on an n-dimensional Riemannian manifold,

$$\star : \Lambda^p TM \longrightarrow \Lambda^{n-p} TM;$$

and it satisfies $\star^2 = (-1)^{p(n-p)}$. Since now $n = 4$ and $p = 2$, we have $\star^2 = id$. Thus $\star$ has eigenvalues 1 and -1 and we have the bundle decomposition $\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$. If $\{e_1, e_2, e_3, e_4\}$ is a local orthonormal oriented frame field on M^4, set

$$u_1 = e_1 \wedge e_2 - e_3 \wedge e_4$$

$$u_2 = e_1 \wedge e_3 - e_4 \wedge e_2$$

$$u_3 = e_1 \wedge e_4 - e_2 \wedge e_3$$

and

$$v_1 = e_1 \wedge e_2 + e_3 \wedge e_4$$

$$v_2 = e_1 \wedge e_3 + e_4 \wedge e_2$$

$$v_3 = e_1 \wedge e_4 + e_2 \wedge e_3$$

then $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are local oriented orthonormal frame fields for $\Lambda^2_+ TM$ and $\Lambda^2_- TM$ respectively. Let $W = W_+ + W_-$ be the decomposition of the Weyl conformal curvature tensor. If $W_- = 0$ (resp. $W_+ = 0$), then we say the Riemannian manifold M^4 is self-dual (resp. anti-self-dual).

The twistor space $\pi : Z \longrightarrow M^4$ is the sphere bundle of unit vectors in $\Lambda^2_- TM$. The Riemannian connection on M^4 gives rise to a splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ into horizontal and vertical parts. The vertical space $\mathcal{V}_\sigma$ at $\sigma \in Z$ is the orthogonal complement of σ in $\Lambda^2_- T_p M$, $p = \pi(\sigma)$.

Each point $\sigma \in Z$ defines an almost complex structure K_σ on $T_p M$ by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y)$$

$X, Y \in T_p M$.

For example, if $\sigma = e_1 \wedge e_2 - e_3 \wedge e_4$,

$$g(K_\sigma X, e_j) = 2g(e_1 \wedge e_2 - e_3 \wedge e_4, X^i e_i \wedge e_j)$$

$$= X^1 \delta_{2j} - X^2 \delta_{1j} - X^3 \delta_{4j} + X^4 \delta_{3j}$$

that is

$$K_\sigma \begin{pmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} = \begin{pmatrix} -X^2 \\ X^1 \\ X^4 \\ -X^3 \end{pmatrix}$$

Then we have $K_\sigma^2 = -I$ and $g(K_\sigma X, K_\sigma Y) = g(X, Y)$.

Let $\times$ be the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_- T_p M$. We define two almost complex structures on Z. The first one was introduced by Atiyah, Hitchin and Singer and is defined by

$$J_1 V = -\sigma \times V, V \in \mathcal{V}_\sigma$$

$$\pi_\star J_1 X = K_\sigma(\pi_\star X), X \in \mathcal{H}_\sigma$$

and the second one by Eells and Salamon and is defined by

$$J_2 V = \sigma \times V, V \in \mathcal{V}_\sigma$$

$$\pi_\star J_2 X = K_\sigma(\pi_\star X), X \in \mathcal{H}_\sigma.$$

J_1 is integrable if and only if M^4 is self-dual and J_2 is never integrable. Define a pseudo-Riemannian metric h_t on Z by $h_t = \pi^\star g + t g^v$, $t \neq 0$ where g^v is the restriction of the metric of $\Lambda^2 TM$ to $\mathcal{V}$. h_t is compatible with both J_1 and J_2 .

We now state the result of Davidov and Muskarov

Theorem 3.1. Let (M^4, g) be a connected oriented real analytic Riemannian manifold. If the Ricci tensor of $(Z, J_n, h_t), n = 1,2$ is J_n Hermitian (i.e. $QJ = JQ$) then either: (1) M^4 is self-dual and Einstein or (2) M^4 self-dual with $R = \frac{12}{t}$ and at each point of M^4 at least three eigenvalues of Q coincide. Conversely given (M^4, g), (1) or (2) imply that (Z, J_n, h_t) is J_n Hermitian .

Thus letting M^4 be a compact Einstein, self-dual manifold with negative scalar curvature R and $t = -\frac{12}{R}$, we see that there exist compact almost Kähler manifolds with $QJ = JQ$ which are not Kähler. The only known examples of such M^4 are compact quotients of the unit ball in $\mathbf{C}^2$ with the metric of constant negative curvature or the Bergman metric.

The famous conjecture of Goldberg is still open: Is a compact almost Kähler Einstein manifold Kähler?

3.2 A new class of contact manifolds

In section 1 we discussed almost Kähler manifold with Hermitian Ricci tensor and its relation to critical point conditions. Again we are interested in the analogue in contact geometry. We know that $Q\phi = \phi Q$ on any Sasakian manifold. It was conjectured that K-contact manifolds with $Q\phi = \phi Q$ are Sasakian. In the following we will present a negative answer to this conjecture; hence we have a new class of contact manifolds.

For compact regular contact manifold, we have the following celebrated theorem of Boothby and Wang [17].

Theorem 3.2. Let M^{2n+1} be a compact regular contact manifold with contact form $\acute{\eta}$. Then there exists a contact form $\eta = \rho\acute{\eta}$ for some non-vanishing function ρ whose characteristic vector field ξ generates a free effective S^1 action on M^{2n+1}. Moreover M^{2n+1} is a principal circle bundle over a symplectic manifold M^{2n} whose symplectic-form Ω determines an integral cocycle on M^{2n}; η is a connection form on the bundle with curvature form $d\eta = \pi^\star\Omega$.

Hereafter in this section we assume that M^{2n+1} is compact and regular. By the theorem of Boothby and Wang, M^{2n+1} is a principal circle bundle over a symplectic manifold M^{2n} and $d\eta = \pi^\star\Omega$ with $\pi : M^{2n+1} \longrightarrow M^{2n}$ the projection.

Since M^{2n} carries a symplectic form Ω, there exist a Riemannian metric g and an almost complex structure J such that (g, J) is an almost Kähler structure on M^{2n}.

Let X denote a vector field on M^{2n} and $\tilde{X}$ a vector field on M^{2n+1}. We define ϕ on M^{2n+1} by

$$\phi\tilde{X} = (J\pi_\star\tilde{X})^\star \tag{3.2}$$

for $\tilde{X} \in T_xM^{2n+1}$, where upper $\star$ denotes the horizontal lift with respect to η. Then

$$\phi^2\tilde{X} = (J\pi_\star(J\pi_\star\tilde{X})^\star)^\star = (J^2\pi_\star\tilde{X})^\star$$

$$= -(\pi_\star \tilde{X})^\star = -X + \eta(\tilde{X})\xi,$$

i.e. $\phi^2 = -I + \eta \otimes \xi$. We can also define $\tilde{g}$ on M^{2n+1} by

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(\pi_\star \tilde{X}, \pi_\star \tilde{Y}) + \eta(\tilde{X})\eta(\tilde{Y}) \tag{3.3}$$

i.e. $\tilde{g} = \pi^\star g + \eta \otimes \eta$. By (1.2) in Chapter 1 $\mathcal{L}_\xi \eta = 0$, we have $\mathcal{L}_\xi \tilde{g} = 0$; hence ξ is Killing. Moreover

$$\begin{aligned}
\tilde{g}(\tilde{X}, \phi\tilde{Y}) &= g(\pi_\star \tilde{X}, \pi_\star (J\pi_\star \tilde{Y})^\star) \\
&= g(\pi_\star \tilde{X}, J\pi_\star \tilde{Y}) \\
&= \Omega(\pi_\star \tilde{X}, \pi_\star \tilde{Y}) \\
&= \pi^\star \Omega(\tilde{X}, \tilde{Y}) \\
&= d\eta(\tilde{X}, \tilde{Y});
\end{aligned}$$

similarly

$$\begin{aligned}
\tilde{g}(\phi\tilde{X}, \phi\tilde{Y}) &= g(\pi_\star \tilde{X}, \pi_\star \tilde{Y}) \\
&= \tilde{g}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y}).
\end{aligned}$$

Therefore the contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is a K-contact structure.

Now since $\mathcal{L}_\xi \phi = 0$ we have

$$\begin{aligned}
N^{(1)}(\xi, \tilde{X}) &= [\phi, \phi](\xi, \tilde{X}) + 2d\eta(\xi, \tilde{X}) \\
&= \phi^2[\xi, \tilde{X}] - \phi[\xi, \phi\tilde{X}] \\
&= 0.
\end{aligned}$$

For projectable horizontal vector fields $\tilde{X}$ and $\tilde{Y}$ we have

$$\begin{aligned}
&[\phi, \phi](\tilde{X}, \tilde{Y}) + 2d\eta(\tilde{X}, \tilde{Y})\xi \\
= \; &(J^2\pi_\star[\tilde{X}, \tilde{Y}])^\star + [(J\pi_\star \tilde{X})^\star, (J\pi_\star \tilde{Y})^\star] - (J\pi_\star[(J\pi_\star \tilde{X})^\star, \tilde{Y}])^\star \\
&- (J\pi_\star[\tilde{X}, (J\pi_\star \tilde{Y})^\star])^\star + 2d\eta(\tilde{X}, \tilde{Y})\xi
\end{aligned}$$

$$
\begin{aligned}
&= (J^2[\pi_\star\tilde{X},\pi_\star\tilde{Y}])^\star + [J\pi_\star\tilde{X},J\pi_\star\tilde{Y}] + \eta([(J\pi_\star\tilde{X})^\star,(J\pi_\star\tilde{Y})^\star]) \\
&\quad - (J[J\pi_\star\tilde{X},\pi_\star\tilde{Y}])^\star - (J[\pi_\star\tilde{X},J\pi_\star\tilde{Y}])^\star + 2d\eta(\tilde{X},\tilde{Y})\xi \\
&= ([J,J](\pi_\star\tilde{X},\pi_\star\tilde{Y}))^\star.
\end{aligned}
$$

Thus from Theorem 1.3 we see that the K-contact structure $(\phi,\xi,\eta,\tilde{g})$ is Sasakian if and only if the base manifold M^{2n} is Kählerian(see e.g. [3]).

Given a symplectic manifold such that Ω determines an integral cohomology class, by a theorem of Kobayashi [28] we have

$$
\mathcal{P}(M^{2n},S^1) \approx H^2(M^{2n},Z)
$$

where $\mathcal{P}(M^{2n},S^1)$ is the set of all principal circle bundles over M^{2n}. Through $\mathcal{P}(M^{2n},S^1)$ the construction above is reversible.

Now we are ready to prove the following result.

Theorem 3.3. On a (compact) regular contact manifold M^{2n+1}, let $(\phi,\xi,\eta,\tilde{g})$ be the contact metric structure and (g,J) be the almost complex structure described above. Then $QJ = JQ$ if and only if $\tilde{Q}\phi = \phi\tilde{Q}$, where $\tilde{Q}$ is the Ricci operator on M^{2n+1}.

Proof. Let ∇ (resp. $\tilde{\nabla}$) denote covariant differentiation with respect to g (resp. $\tilde{g}$). From the classical formula (1.7) in Chapter 1 we have

$$
\begin{aligned}
&2g(\pi_\star\tilde{\nabla}_{X^\star}Y^\star,Z) \\
&= 2\tilde{g}(\tilde{\nabla}_{X^\star}Y^\star,Z^\star) \\
&= X^\star\tilde{g}(Y^\star,Z^\star) + Y^\star\tilde{g}(X^\star,Z^\star) - Z^\star\tilde{g}(X^\star,Y^\star) \\
&\quad + \tilde{g}([X^\star,Y^\star],Z^\star) + \tilde{g}([Z^\star,X^\star],Y^\star) - \tilde{g}(X^\star,[Y^\star,Z^\star]) \\
&= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
&\quad + g([X,Y],Z) + g([Z,X],Y) - g(X,[Y,Z]) \\
&= 2g(\nabla_X Y,Z)
\end{aligned}
$$

which shows that the horizontal component of $\tilde{\nabla}_{X^\star}Y^\star$ is $(\nabla_X Y)^\star$. On the other hand

$$\begin{aligned}
& 2\tilde{g}(\tilde{\nabla}_{X^\star}Y^\star, \xi) \\
= \;& X^\star\tilde{g}(Y^\star, \xi) + Y^\star\tilde{g}(X^\star, \xi) - \xi\tilde{g}(X^\star, Y^\star) \\
& + \tilde{g}([X^\star, Y^\star], \xi) + \tilde{g}([\xi, X^\star], Y^\star) - \tilde{g}(X^\star, [Y^\star, \xi]) \\
= \;& \tilde{g}([X^\star, Y^\star], \xi)
\end{aligned}$$

which means $2\eta(\tilde{\nabla}_{X^\star}Y^\star) = \eta([X^\star, Y^\star])$; hence the vertical component of $\tilde{\nabla}_{X^\star}Y^\star$ is given by $\frac{1}{2}\eta([X^\star, Y^\star])\xi$. Therefore

$$\tilde{\nabla}_{X^\star}Y^\star = (\nabla_X Y)^\star + \frac{1}{2}\eta([X^\star, Y^\star])\xi. \tag{3.4}$$

Now we can compute the curvature tensor [35]:

$$\begin{aligned}
& (\nabla_X \nabla_Y Z)^\star \\
= \;& -\phi^2 \tilde{\nabla}_{X^\star}(\nabla_Y Z)^\star \\
= \;& -\phi^2 \tilde{\nabla}_{X^\star}(\tilde{\nabla}_{Y^\star}Z^\star - \frac{1}{2}\eta([Y^\star, Z^\star])\xi) \\
= \;& -\phi^2 \{\tilde{\nabla}_{X^\star}\tilde{\nabla}_{Y^\star}Z^\star - \frac{1}{2}(X^\star\eta([Y^\star, Z^\star])\xi + \eta([Y^\star, Z^\star])\tilde{\nabla}_{X^\star}\xi)\} \\
= \;& -\phi^2 \{\tilde{\nabla}_{X^\star}\tilde{\nabla}_{Y^\star}Z^\star - \frac{1}{2}\eta([Y^\star, Z^\star])\tilde{\nabla}_{X^\star}\xi\}
\end{aligned}$$

moreover we have

$$\begin{aligned}
(\nabla_{[X,Y]}Z)^\star &= -\phi^2 \tilde{\nabla}_{[X,Y]^\star}Z^\star \\
&= -\phi^2 \{\tilde{\nabla}_{[X^\star,Y^\star]}Z^\star - \eta([X^\star, Y^\star])\tilde{\nabla}_\xi Z^\star\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& (R(X,Y)Z)^\star \\
= \;& -\phi^2 \{\tilde{R}(X^\star, Y^\star)Z^\star - \frac{1}{2}\eta([Y^\star, Z^\star])\tilde{\nabla}_{X^\star}\xi \\
& + \frac{1}{2}\eta([X^\star, Z^\star])\tilde{\nabla}_{Y^\star}\xi + \eta([X^\star, Y^\star])\tilde{\nabla}_\xi Z^\star\}.
\end{aligned} \tag{3.5}$$

Let $(E_1, \cdots, E_{2n})$ be an orthonormal J basis on M^{2n}. Then $(E_1{}^\star, \cdots, E_{2n}{}^\star, \xi)$ is an orthonormal basis on M^{2n+1} with $E_2{}^\star = (JE_1)^\star = \phi E_1{}^\star$ and so on.

Since M^{2n+1} is K-contact, $\tilde{Q}\xi = 2n\xi$ and $\tilde{R}(X^\star, \xi)\xi = X^\star$ (Proposition 1.6). We have

$$\begin{aligned} \tilde{Q}X^\star &= \sum_{i=1}^{2n} \tilde{R}(X^\star, E_i{}^\star)E_i{}^\star + \tilde{R}(X^\star, \xi)\xi \\ &= \sum_{i=1}^{2n} \tilde{R}(X^\star, E_i{}^\star)E_i{}^\star + X^\star \end{aligned}$$

and

$$\tilde{Q}\phi X^\star = \sum_{i=1}^{2n} \tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star + \phi X^\star.$$

Therefore $\tilde{Q}\phi = \phi\tilde{Q}$ if and only if

$$\sum_{i=1}^{2n} \tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star = \phi \sum_{i=1}^{2n} \tilde{R}(X^\star, E_i{}^\star)E_i{}^\star \tag{3.6}$$

Since $E_i{}^\star$ is invariant under ξ, $[E_i{}^\star, \xi] = 0$; hence $\tilde{\nabla}_\xi E_i{}^\star = \tilde{\nabla}_{E_i{}^\star}\xi$. Thus we have

$$\begin{aligned} &(R(X, E_i)E_i)^\star \qquad\qquad (3.7)\\ =\ & -\phi^2\{\tilde{R}(X^\star, E_i{}^\star)E_i{}^\star - \frac{1}{2}\eta([E_i{}^\star, E_i{}^\star])\tilde{\nabla}_{X^\star}\xi \\ & + \frac{1}{2}\eta([X^\star, E_i{}^\star])\tilde{\nabla}_{E_i{}^\star}\xi + \eta([X^\star, E_i{}^\star])\tilde{\nabla}_\xi E_i{}^\star\} \\ =\ & -\phi^2\{\tilde{R}(X^\star, E_i{}^\star)E_i{}^\star - \frac{3}{2}\eta([X^\star, E_i{}^\star])\phi E_i{}^\star\} \\ =\ & -\phi^2\tilde{R}(X^\star, E_i{}^\star)E_i{}^\star - \frac{3}{2}\eta([X^\star, E_i{}^\star])\phi E_i{}^\star \end{aligned}$$

hence

$$\begin{aligned} & -\phi^2\tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star \\ =\ & -\phi^2\tilde{R}((JX)^\star, E_i{}^\star)E_i{}^\star \\ =\ & (R(JX, E_i)E_i)^\star + \frac{3}{2}\eta([\phi X^\star, E_i{}^\star])\phi E_i{}^\star \end{aligned}$$

from which

$$\begin{aligned} & -\phi^2\sum_{i=1}^{2n}\tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star \\ =\ & \sum_{i=1}^{2n}(R(JX, E_i)E_i)^\star + \frac{3}{2}\sum_{i=1}^{2n}\eta([\phi X^\star, E_i{}^\star])\phi E_i{}^\star. \end{aligned}$$

Since

$$\eta([\phi X^\star, E_i{}^\star]) = -2d\eta(\phi X^\star, E_i{}^\star)$$

$$= 2d\eta(X^\star, \phi E_i{}^\star) = -\eta([X^\star, \phi E_i{}^\star]),$$

we have

$$\begin{aligned} & \sum_{i=1}^{2n} \eta([\phi X^\star, E_i{}^\star])\phi E_i{}^\star \\ = & -\sum_{i=1}^{2n} \eta([X^\star, \phi E_i{}^\star])\phi E_i{}^\star \\ = & -\phi \sum_{i=1}^{2n} \eta([X^\star, \phi E_i{}^\star]) E_i{}^\star \\ = & \ \phi \sum_{i=1}^{2n} \eta([X^\star, E_i{}^\star])\phi E_i{}^\star . \end{aligned}$$

Now if $JQ = QJ$, then

$$\begin{aligned} & -\phi^2 \sum_{i=1}^{2n} \tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star \\ = & \ (\sum_{i=1}^{2n} R(JX, E_i)E_i)^\star + \frac{3}{2}\sum_{i=1}^{2n} \eta([\phi X^\star, E_i{}^\star])\phi E_i{}^\star \\ = & \ (J\sum_{i=1}^{2n} R(X, E_i)E_i)^\star + \frac{3}{2}\phi \sum_{i=1}^{2n} \eta([X^\star, E_i{}^\star])\phi E_i{}^\star \\ = & \ \phi\{\sum_{i=1}^{2n} (R(X, E_i)E_i)^\star + \frac{3}{2}\sum_{i=1}^{2n} \eta([X^\star, E_i{}^\star])\phi E_i{}^\star\}. \end{aligned} \tag{3.8}$$

By (3.7) we have

$$-\phi^2 \sum_{i=1}^{2n} \tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star = \phi\{-\phi^2 \sum_{i=1}^{2n} \tilde{R}(X^\star, E_i{}^\star)E_i{}^\star\}. \tag{3.9}$$

Since M^{2n+1} is K-contact,

$$\tilde{g}(\tilde{Q}X^\star, \xi) = \tilde{g}(X^\star, \tilde{Q}\xi) = 0$$

and hence $\tilde{Q}X^\star \perp \xi$. Then

$$\sum_{i=1}^{2n} \tilde{R}(\phi X^\star, E_i{}^\star)E_i{}^\star = \phi \sum_{i=1}^{2n} \tilde{R}(X^\star, E_i{}^\star)E_i{}^\star$$

By (3.6) we have

$$\tilde{Q}\phi = \phi\tilde{Q}.$$

Conversely if $\tilde{Q}\phi = \phi\tilde{Q}$, then

$$\tilde{Q}\phi X^\star = \phi\tilde{Q}X^\star$$

and therefore (3.9) holds; and considering (3.8) we have

$$(\sum_{i=1}^{2n} R(JX, E_i)E_i)^\star = (J\sum_{i=1}^{2n} R(X, E_i)E_i)^\star .$$

Therefore

$$JQ = QJ$$

completing the proof. Q.E.D.

We have seen that there exist compact almost Kähler manifolds with $JQ = QJ$ which are not Kähler. By considering the circle bundles over such manifolds, from the construction and the theorem above we can see that there exist K-contact manifolds with $Q\phi = \phi Q$, but which are not Sasakian.

As in the symplectic case, the condition $Q\phi = \phi Q$ has been studied extensively ([13], [14] etc.).

Now we have a new class of contact manifolds. In the following we give a variational characterization of this class.

In a recent paper [16] Blair and Perrone found a scalar curvature which is much more natural than the generalized Tanaka-Webster scalar curvature. They studied the critical point conditions of the following functionals:

$$E_1(g) = \int_M W_1 dV_g = \int_M (R - Ric(\xi) + 4n)dV_g$$

$$E_2(g) = \int_M W_2 dV_g = \int_M (R^\star + Ric(\xi) + 4n^2)dV_g$$

$$E_3(g) = \int_M W_3 dV_g = \int_M [R + R^\star + 4n(n+1)]dV_g$$

Theorem 3.4. Let M^{2n+1} be a compact contact manifold. Then $g \in \mathcal{M}(\eta)$ is a critical point of $E_1(g) = \int_M W_1 dV_g$ if and only if

$$(Q\phi - \phi Q) - (l\phi - \phi l) = 4\phi h - \eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi$$

(see [16] [43]). In the 3-dimensional case the critical point condition becomes $h = 0$ or K-contact(which is proved by Chern and Hamilton [21]).

Theorem 3.5. Let M^{2n+1} be a compact contact manifold. Then $g \in \mathcal{M}(\eta)$ is a critical point of $E_2(g) = \int_M W_2 dV_g$ if and only if

$$(Q\phi - \phi Q) - (l\phi - \phi l) = -4(2n-1)\phi h - \eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi.$$

Theorem 3.6. Let M^{2n+1} be a compact contact manifold. Then $g \in \mathcal{M}(\eta)$ is a critical point of $E_3(g) = \int_M W_3 dV_g$ if and only if g is K-contact.

Therefore W_3 is a better scalar curvature in view of the results here.

Proposition 3.7. Let M^{2n+1} be a compact contact manifold. Then $g \in \mathcal{M}(\eta)$ is a critical point of $\int_M (aW_1 + bW_2) dV_g$ for any a and b if and only if

$$h = 0, \qquad Q\phi = \phi Q$$

Proof. By the results above we have that $g \in \mathcal{M}(\eta)$ is a critical point of $\int_M (aW_1 + bW_2) dV_g$ for any a and b if and only if

$$\begin{aligned} & (-8nb + 4(b-a))\phi h - (b-a)(Q\phi - \phi Q) \\ = \ & (b-a)[\eta \otimes \phi Q\xi - (\eta \circ Q\phi) \otimes \xi - (l\phi - \phi l)] \end{aligned}$$

is true for any a and b. By (1.9) and the fact that $Q\xi = 2n\xi$ on K-contact manifolds, the proposition follows immediately. Q.E.D.

Chapter 4

Other Functionals

4.1 Some classical functionals

In this section we study the critical point conditions of functionals $B(g) = \int_M R^2 dV_g$, $C(g) = \int_M |Ric|^2 dV_g$ and $D(g) = \int_M |Riem|^2 dV_g$. These functionals have been studied in the Riemannian context by Berger, Calabi, Muto and others([2], [33], [34], [46] etc.). Our interest here will again be critical point conditions of these functionals in the space of all associated metrics $\mathcal{M}(\eta)$ or $\mathcal{M}(\Omega)$.

First we consider $B(g) = \int_M R^2 dV_g$. Let $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ be any curve in $\mathcal{M}(\eta)$ or $\mathcal{M}(\Omega)$. From the expression of $R_{jk}(t)$ (2.3) we have, summarizing a lengthy computation,

$$\begin{aligned} & \frac{dB}{dt}(0) \\ = & \int_M (Rg^{ij}\nabla_r\nabla_i H_j^r + Rg^{ij}\nabla_r\nabla_j H_i^r \\ & - Rg^{ij}\nabla_r\nabla^r H_{ij} - 2RR_{ij}H^{ij})dV_g \\ = & \int_M 2(\nabla_i\nabla_j R - RR_{ij})H^{ij}dV_g. \end{aligned}$$

Let $T_{ij} = \nabla_i\nabla_j R - RR_{ij}$. Then T is symmetric and by Lemma 1.11 in Chapter 1 we have that g is critical in $\mathcal{M}(\Omega)$ (resp. in $\mathcal{M}(\eta)$) if and only if $TJ = JT$ (resp. $\phi T = T\phi$ on $\mathcal{D}$).

Now we consider $C(g(t)) = \int_M R_{ij}(t)R^{ij}(t)dV_g$, where $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$

is any curve in $\mathcal{M}(\eta)$ or $\mathcal{M}(\Omega)$. Then by similar computation

$$\begin{aligned}
&\frac{dC}{dt}(0) \\
= &\int_M (2\nabla_r\nabla_i H^r_j R^{ij} - \nabla_r\nabla^r H_{ij} R^{ij} \\
&- 2R_{ij}R_{lk}g^{il}g^{jk})dV_g \\
= &\int_M (2\nabla_r\nabla_i R^r_j - \nabla_r\nabla^r R_{ij} - 2R_{ir}R^r_j)H^{ij}dV_g.
\end{aligned}$$

Let

$$U_{ij} = \nabla_r\nabla_i R^r_j + \nabla_r\nabla_j R^r_i - \nabla_r\nabla^r R_{ij} - 2R_{ir}R^r_j.$$

Then g is critical in $\mathcal{M}(\Omega)$ (resp. in $\mathcal{M}(\eta)$) if and only if $UJ = JU$ (resp. $\phi U = U\phi$ on $\mathcal{D}$).

For $D(g) = \int_M |Riem|^2 dV_g$ we have from (2.2)

$$\begin{aligned}
&\frac{dD}{dt}(0) \\
= &\ 2\int_M (2\nabla_k\nabla_h H_{ij} R^{ikjh} - R_{klh}{}^i R^{klhj} H_{ij})dV_g \\
= &\ 2\int_M (2\nabla_h\nabla_k R^{ikjh} - R_{klh}{}^i R^{klhj})H_{ij}dV_g.
\end{aligned}$$

We assume that $V^{ij} = \nabla_h\nabla_k R^{ikjh} + \nabla_h\nabla_k R^{jkih} - R_{klh}{}^i R^{klhj}$. Then we can summarize the results in the following proposition.

Proposition 4.1. In symplectic case the critical point conditions are

(1) critical for $B(g)$ if and only if $TJ = JT$;

(2) critical for $C(g)$ if and only if $UJ = JU$;

(3) critical for $D(g)$ if and only if $VJ = JV$.

In contact case the critical point conditions are

(4) critical for $B(g)$ if and only if $\phi T = T\phi$ on $\mathcal{D}$;

(5) critical for $C(g)$ if and only if $\phi U = U\phi$ on $\mathcal{D}$;

(6) critical for $D(g)$ if and only if $\phi V = V\phi$ on $\mathcal{D}$.

4.2 Other functionals on $\mathcal{M}(\eta)$

On contact manifolds we have something like the Dirichlet energy which has no counterpart for symplectic manifolds. Consider the following functional

$$F_1(g) = \int_M |l|^2 dV_g = \int_M |R_{\cdot\xi}\xi|^2 dV_g.$$

Let $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ be any curve in $\mathcal{M}(\eta)$. Then

$$\begin{aligned} &\frac{dF_1}{dt}(0) \\ = &\int_M [\xi^k\xi^l\xi^u\xi^v(\nabla_k\nabla_i H^j_l + \nabla_k\nabla_l H^j_i \\ &- \nabla_k\nabla^j H_{il})R_{usvj}g^{si} - (\nabla_i\nabla_k H^j_l \\ &+ \nabla_i\nabla_l H^j_k - \nabla_i\nabla^j H_{kl})R_{usvj}g^{si}]dV_g. \end{aligned}$$

Using Green's theorem we have

$$\begin{aligned} &\frac{dF_1}{dt}(0) \\ = &\int_M (-l^{is}l^r_i - \nabla_\xi\nabla_\xi l^{rs} + l^{jr}\nabla_j\xi^i\nabla_i\xi^s \\ &+ 2\nabla_\xi l^{jr}\nabla_j\xi^s + l^{jr}\nabla_\xi\nabla_j\xi^s - 2l^{ij}\nabla_i\xi^r\nabla_j\xi^s)H_{rs}dV_g. \end{aligned}$$

Let

$$\begin{aligned} U^{rs} &= -l^{is}l^r_i - \nabla_\xi\nabla_\xi l^{rs} + l^{jr}\nabla_j\xi^i\nabla_i\xi^s \\ &\quad + 2\nabla_\xi l^{jr}\nabla_j\xi^s + l^{jr}\nabla_\xi\nabla_j\xi^s - 2l^{ij}\nabla_i\xi^r\nabla_j\xi^s. \end{aligned}$$

Since $U^{rs}\eta_s = 0$, by Lemma 1.11 the critical point condition is $\phi U = U\phi$; and it can be written as

$$\begin{aligned} &-2\nabla_\xi\nabla_\xi\nabla_\xi h + 2\nabla_\xi\nabla_\xi\tau + 4\nabla_\xi h + \nabla_\xi hhh \\ &+hh\nabla_\xi h - 2h\nabla_\xi hh + 8lh\phi + 8\nabla_\xi hh \\ &-4l\nabla_\xi h - 4\nabla_\xi hl \quad = \quad 0. \end{aligned}$$

Therefore we have the following.

Proposition 4.2. g is critical metric of $F_1(g) = \int_M |l|^2 dV_g = \int_M |R_{\cdot\xi}\xi|^2 dV_g$ in $\mathcal{M}(\eta)$ if and only if $\phi U = U\phi$. K-contact metrics are critical points. Furthermore, if $\nabla_\xi h = 0$ for a critical metric, then $h^3 = h$.

Let us also study a similar functional

$$F_2(g) = \int_M |R_{\xi\cdot} \cdot |^2 dV_g.$$

Let $g_{ij}(t) = g_{ij} + tH_{ij} + O(t^2)$ be any curve in $\mathcal{M}(\eta)$. Then

$$\begin{aligned}
&\frac{dF_2}{dt}(0) \\
= &\int_M (2\xi^r\xi^s\nabla_r\nabla_j H_{ik} R_s{}^{ijk} + 2\xi^r\xi^s\nabla_i\nabla_k H_{rj} R_s{}^{ijk} \\
&- \xi^r\xi^s R_{rijk} R_{st}{}^{jk} H^{it}) dV_g \\
= &\int_M (2\nabla^j\nabla_\xi R_{\xi rjs} + 2\nabla^k\nabla^i\eta_r R_{\xi isk} + 2\nabla^i\eta_r\nabla^k R_{\xi isk} \\
&+ 2\nabla^k\eta_r\nabla^i R_{\xi isk} - R_{\xi rjk} R_{\xi s}{}^{jk}) H^{rs} dV_g.
\end{aligned}$$

Now set

$$\begin{aligned}
V_{rs} &= \nabla^j\nabla_\xi R_{\xi rjs} + \nabla^j\nabla_\xi R_{\xi sjr} + \nabla^k\nabla^i\eta_r R_{\xi isk} \\
&+ \nabla^k\nabla^i\eta_s R_{\xi irk} + \nabla^i\eta_r\nabla^k R_{\xi isk} + \nabla^i\eta_s\nabla^k R_{\xi irk} \\
&+ \nabla^k\eta_r\nabla^i R_{\xi isk} + \nabla^k\eta_s\nabla^i R_{\xi irk} - R_{\xi rjk} R_{\xi s}{}^{jk}.
\end{aligned}$$

Then $V^{rs}\eta_s = 0$ and the critical condition is $\phi V = V\phi$. If g is Sasakian, then

$$R_{\xi ijk} = \eta_k g_{ij} - \eta_j g_{ik}$$

$$\nabla_l R_{\xi ijk} = \phi_{lk} g_{ij} - \phi_{lj} g_{ik}$$

etc.; it is easy to see that g is a critical metric. Thus we have the following result.

Proposition 4.3. g is a critical metric of $F_2(g) = \int_M |R_{\xi\cdot} \cdot |^2 dV_g$ in $\mathcal{M}(\eta)$ if and only if $\phi V = V\phi$. Moreover Sasakian metrics are critical points.

By Theorem 1.7 in Chapter 1, a contact manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X.$$

We define $P = (P_{ijk})$ on a contact manifold by [42]

$$P_{ijk} = \nabla_i \phi_{jk} - \eta_j g_{ik} + \eta_k g_{ij}.$$

Then $P = (P_{ijk})$ measures how far away a metric is from being Sasakian. It is natural to consider the functional

$$F_3(g) = \int_M |P|^2 dV_g.$$

Since $|P|^2 = |\nabla\phi|^2 - 4n$, we have

$$\begin{aligned} & |P(t)|^2 \\ = \; & |P|^2 + t[2\nabla^r \phi^{si} \nabla_r (\phi_{sj} H_i^j) - \nabla^r \phi^{si} D_{rs}{}^j \phi_{ji} - \nabla^r \phi^{si} D_{ri}{}^j \phi_{sj} \\ & - \nabla_r \phi_{si} \nabla_j \phi^{si} H^{jr} + \nabla_r \phi_{si} \nabla^r \phi_j^i H^{js} - \nabla_r \phi_{si} \nabla^r \phi_j^s H^{ji}] + O(t^2) \end{aligned}$$

hence

$$\begin{aligned} & \frac{dF_3}{dt}(0) \\ = \; & \int_M (2\phi_i^r \nabla_j \nabla^j \phi^{is} + 2\phi_i^s \nabla_j \nabla^r \phi^{ji} - 2\phi_i^j \nabla_j \nabla^r \phi^{si} \\ & - 4nh^{rs} - \nabla^r \phi_{ji} \nabla^s \phi^{ji}) H_{rs} dV_g. \end{aligned}$$

Taking the symmetric part we set

$$\begin{aligned} & W_{rs} \\ = \; & \phi_{ri} \nabla_j \nabla^j \phi_s^i + \phi_{si} \nabla_j \nabla^j \phi_r^i + \phi_{ri} \nabla_j \nabla_s \phi^{ji} + \phi_{si} \nabla_j \nabla_r \phi^{ji} \\ & + \phi_i^j \nabla_j \nabla_r \phi_s^i + \phi_i^j \nabla_j \nabla_s \phi_r^i - 4nh_{rs} - \nabla_r \phi_{ji} \nabla_s \phi^{ji} \end{aligned}$$

Since $W_{rs}\xi^s = 0$, we have the following proposition.

Proposition 4.4. g is critical metric of $F_3(g) = \int_M |P|^2 dV_g$ in $\mathcal{M}(\eta)$ if and only if $\phi W = W\phi$; and Sasakian metrics are minima.

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