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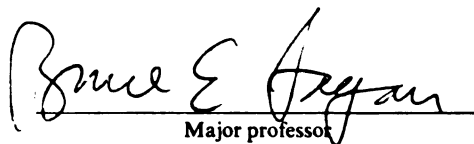
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**Lattice Path Proof of Some
Jacobi-Trudi Type Formulae**

presented by

Jose H. Giraldo

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LATTICE PATH PROOF OF SOME JACOBI-TRUDI TYPE FORMULAE

By

José H. Giraldo

A DISSERTATION

Submitted to
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ABSTRACT

LATTICE PATH PROOF OF SOME JACOBI-TRUDI TYPE FORMULAE

BY

José H. Giraldo

Ira Gessel gave a combinatorial proof of the Jacobi-Trudi identities using lattice paths. We use this lattice path technique to prove some Jacobi-Trudi type identities. These identities relate determinants, where each entry is the sum or difference of certain complete homogeneous symmetric functions, to sums of certain skew Schur functions. Similar determinantal identities are obtained when the entries of the determinant are elementary symmetric functions.

Contents

Introduction	1
1 Preliminaries	4
1.1 <i>Partitions and Tableaux</i>	4
1.2 <i>Symmetric Functions</i>	8
1.3 <i>Lattice Paths</i>	14
2 Main Results	22
2.1 <i>Jacobi-Trudi Identities</i>	22
2.2 <i>Some Jacobi-Trudi Type Identities</i>	30
2.3 <i>Symplectic and Orthogonal Analogs</i>	46
3 Open Problem	49
3.1 <i>Circulants</i>	49
Bibliography	58

Introduction

In the theory of group representations it is well-known that the irreducible polynomial representations of the general linear group $GL(n)$ of all n by n non-singular complex matrices are indexed by partitions λ of length at most n . The character of the irreducible representation indexed by λ is the Schur function s_λ which belongs to the ring Λ of symmetric functions. A nice combinatorial proof of the fact that s_λ is symmetric was given by Knuth [Knu 70]. Even more, the s_λ form a basis for the ring of symmetric functions.

There are two other interesting bases for Λ , formed by the elementary and complete homogeneous symmetric functions, denoted by e_λ and h_λ , respectively. Formulae expressing s_λ as the determinant of certain elementary or complete homogeneous symmetric function were found by Jacobi [Jac 41]. The proof of such formulae was later simplified by his student Trudi. These Jacobi-Trudi identities are stated as follows. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. Then

$$s_\lambda = |h_{\lambda_i - i + j}|$$

and

$$s_{\lambda'} = |e_{\lambda_i - i + j}|$$

where the determinants are n by n , and λ' is the conjugate of λ .

Ira Gessel [Ges um] gave a beautiful combinatorial proof of these results using lattice paths and tableaux. The same technique has been used by Gessel and Gerald

Viennot, [G-V 85,G-V ip], to prove other determinantal identities.

Bressoud and Wei [B-W um] in an attempt to find a lattice path proof of the Jacobi-Trudi analogs for symplectic and orthogonal groups, discovered a way to extend this method to prove the following result. For $t \geq -1$ an integer

$$\begin{aligned} & 2^{(t-|t|)/2} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_i-i+\sigma(i)} + (-1)^{(t+|t|)/2} h_{\lambda_i-i-\sigma(i)+1-t}) \\ &= \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{[|\mu|+\nu_\mu(|t|-1)]/2} s_{\lambda/\mu} \end{aligned}$$

where ν_μ is the length of the main diagonal of μ and P_t is the set of all $\mu = (\mu_1, \dots, \mu_n)$ such that $\mu_i = \mu'_i + t$, for $1 \leq i \leq \nu_\mu$. However their method lacked the elegance of the Gessel-Viennot proof.

In the present work we give a lattice path proof of some more general determinantal identities which imply the Bressoud-Wei result. These identities are stated as follows.

Theorem *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t \geq -1$, a fixed integer,*

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t+1)\nu_\mu)/2} s_{\lambda/\mu}$$

and

$$|h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t-1)\nu_\mu)/2} s_{\lambda/\mu}$$

In particular for $t = -1$ we have

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_{-1}}} (-1)^{\frac{|\mu|}{2}} s_{\lambda/\mu}$$

$$|h_{\lambda_i-i+j} - h_{\lambda_i-i-j+2}| = 0.$$

Similar formulae are obtained for elementary symmetric functions.

Theorem *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t \geq -1$, a fixed integer,*

$$|e_{\lambda_i-i+j} + e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t+1)\nu_\mu)/2} s_{\lambda'/\mu'}$$

and

$$|e_{\lambda_i-i+j} - e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t-1)\nu_\mu)/2} s_{\lambda'/\mu'}$$

In particular for $t = -1$ we have

$$|e_{\lambda_i-i+j} + e_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_{-1}}} (-1)^{\frac{|\mu|}{2}} s_{\lambda'/\mu'}$$

$$|e_{\lambda_i-i+j} - e_{\lambda_i-i-j+2}| = 0.$$

Chapter 1

Preliminaries

1.1 *Partitions and Tableaux*

A *partition* is any sequence (finite or infinite)

$$(1.1.1) \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$$

of non-negative integers in decreasing order, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots$, with only a finite number of non-zero terms. The partitions λ, μ are said to be equal if they have the same non-zero terms.

The *parts* of λ are the non-zero λ_i 's in (1.1.1). The number of parts is the *length* of λ , which is denoted by $l(\lambda)$. The *weight* of λ , denoted by $|\lambda|$, is the sum of the parts of λ ,

$$(1.1.2) \quad |\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \dots .$$

If $n = |\lambda|$ we say that λ is a *partition* of n . If the last non-zero entry of λ is λ_k , we will simply write

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) .$$

If λ is a partition of n , an alternate way of writing it is

$$\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$$

where m_k indicates the number of times the summand k appears in the partition. So $\sum_{i=1}^n m_i = l(\lambda)$ and $\sum_{i=1}^n im_i = n$. For instance, $\lambda = (3, 3, 1, 1, 1)$ and $\lambda = (1^3, 3^2)$ represent the same partition.

The *diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is an array of k left-aligned rows. The i^{th} row consists of λ_i dots (nodes) or boxes. The rows are numbered starting with the first row at the top and columns are numbered starting with the first column at the left. We use (i, j) to refer to a box located in the i^{th} row and j^{th} column. We write $(i, j) \in \lambda$, if the box in that position is in the diagram of λ .

For example if $\lambda = (4, 3, 2, 1)$, its diagram is given in Figure 1.1.1. In this case $(2, 3) \in \lambda$ but $(3, 3) \notin \lambda$.

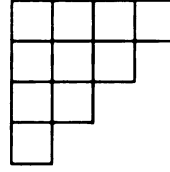


Fig. 1.1.1

The *main diagonal* of a diagram is the diagonal starting at the upper left corner of the diagram and moving southeast.

The *conjugate* of the partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \dots)$ whose diagram is the transpose of the diagram for λ . That is, the diagram obtained by reflecting the diagram of λ through the main diagonal. As an illustration, if $\lambda = (4, 3, 2)$, then $\lambda' = (3, 3, 2, 1)$. The diagrams for λ and λ' are given in Figure 1.1.2, where the main diagonal of λ is indicated.

A *semi-standard λ -tableau* is a filling of the diagram for λ with the elements from a totally ordered set $(A, \leq)$ where the rows are weakly increasing from left to right

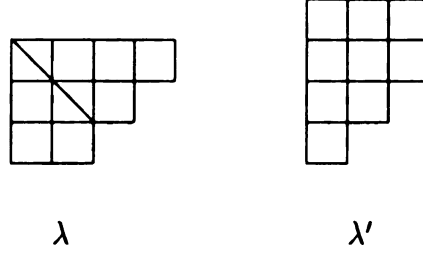


Fig. 1.1.2

and the columns are strictly increasing from top to bottom. As an example, if $\lambda = (4, 3, 2, 1)$, Figure 1.1.3 shows a semi-standard λ -tableau where A is the set of positive integers with the usual order.

1	2	5	8
4	4	6	
7	9		
8			

Fig. 1.1.3

Unless stated specifically, we fill the tableaux with positive integers. The entry in box (i, j) of a semi-standard λ -tableau T is denoted by $T_{(i,j)}$. To $T_{(i,j)}$ we assign a variable denoted by $x_{T_{(i,j)}}$. The *weight* x^T of a semi-standard λ -tableau T is defined by

$$(1.1.3) \quad x^T = \prod_{(i,j) \in \lambda} x_{T_{(i,j)}}.$$

If $\lambda = (3, 2, 2)$ some λ -tableaux are shown in Figure 1.1.4. These tableaux have weights: $x^{T_1} = x_1 x_1 x_1 x_2 x_3 x_3 x_3 = x_1^3 x_2 x_3^3$, $x^{T_2} = x_1 x_1 x_2 x_2 x_3 x_4 x_4 = x_1^2 x_2^2 x_3 x_4^2$, $x^{T_3} = x_1 x_2 x_3 x_3 x_4 x_4 x_5 = x_1 x_2 x_3^2 x_4^2 x_5$. The *degree* of a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} x_{i_3}^{\lambda_3} \cdots x_{i_q}^{\lambda_q}$ is defined as $\sum_{j=1}^q \lambda_j$. Then, the weight of a semi-standard λ -tableau is a monomial of degree $|\lambda|$.

1	1	1	1	1	2	1	3	4
T_1	2	3	T_2	2	3	T_3	2	4
	3	3		4	4		3	5

Fig. 1.1.4

The *length of the main diagonal* of the diagram for λ is denoted by ν_λ . When λ is clear from the context we will simply write ν . Alternatively, we can interpret ν_λ , as:

$$(1.1.4) \quad \nu_\lambda = |\{\lambda_i | \lambda_i \geq i\}|.$$

Observe that the diagram for λ is completely determined by the first ν rows and columns. We are interested in a particular set of partitions which is going to be denoted by P_t .

Definition 1.1.1 Let t be an integer. We define

$$P_t = \{\lambda | \lambda \text{ is a partition and } \lambda_i = \lambda'_i + t \text{ for all } 1 \leq i \leq \nu_\lambda\}.$$

For instance,

$$\begin{aligned} P_0 &= \{\lambda | \lambda_i = \lambda'_i, \text{ for } 1 \leq i \leq \nu_\lambda\} \\ &= \{\lambda | \lambda \text{ is self-conjugated}\}. \end{aligned}$$

Figure 1.1.5 shows an element of P_0 .

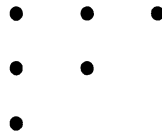


Fig. 1.1.5

Observe that the partition $\mu = 0$ vacuously belongs to P_t , since it has no non-zero parts. Note that if $\mu \in P_0$, and $t > 0$ then by adding t nodes to each of the first ν_μ rows of μ we obtain $\eta \in P_t$. We will say η is obtained by t -addition to μ . An example is shown in Figure 1.1.6.

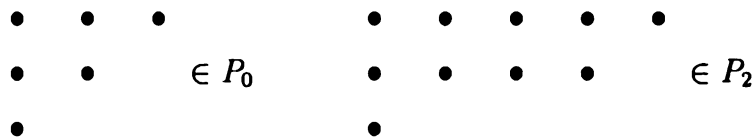


Fig. 1.1.6

Clearly, we can go backwards: given an element η of P_t , with $t > 0$, we can obtain an element of P_0 by deleting t nodes from each of the first ν_η rows.

Definition 1.1.2 For a diagram λ , the *hook* of the (i, j) node is

$$H_{i,j} = \{(i, j') | j' \geq j\} \cup \{(i', j) | i' > i\}.$$

The cardinality of $H_{i,j}$, denoted $h_{i,j}$, is called the *hooklength* of $H_{i,j}$. It follows that

$$(1.1.5) \quad h_{i,j} = \lambda_i + \lambda'_j - (i + j) + 1.$$

Note that if $\mu \in P_0$ then $h_{i,i}$ is odd. Thus $|\mu| - \nu_\mu \equiv 0 \pmod{2}$.

Lemma 1.1.3 Let t be a positive integer and $\lambda \in P_t$, then $|\lambda| - \nu_\lambda \equiv t\nu_\lambda \pmod{2}$.

Proof. Since $\lambda \in P_t$, λ can be obtained by t -addition to $\mu \in P_0$. Because $\nu_\mu = \nu_\lambda$, $|\lambda| = |\mu| + t\nu_\mu$ and $|\mu| \equiv \nu_\mu \pmod{2}$, we have

$$|\lambda| - \nu_\lambda = (|\mu| + t\nu_\mu) - \nu_\mu = (|\mu| - \nu_\mu) + t\nu_\mu \equiv t\nu_\mu \pmod{2}. \quad \square$$

Let μ and λ be partitions. The notation $\mu \subseteq \lambda$ indicates that $\mu_i \leq \lambda_i$ for all $i \geq 1$. In terms of diagrams, this means that the diagram of μ is contained in the diagram of λ . The boxes in λ which are not in μ determine a diagram, called a *skew-diagram* and denoted λ/μ . The i^{th} row of λ/μ has length $\lambda_i - \mu_i$.

The definitions of a semi-standard λ/μ tableau and its weight are analogous to the case $\mu = 0$.

1.2 Symmetric Functions

Let $C[[x]]$ be the ring of formal power series on the variables $x = \{x_1, x_2, x_3, \dots\}$. An element of $C[[x]]$ with all the monomials of degree n , is called a *homogeneous* formal power series of degree n .

For $k \geq 1$, let S_k be the symmetric group on k letters. For each k , consider the function

$$\phi : S_k \times C[[x]] \rightarrow C[[x]]$$

defined by

$$(1.2.1) \quad (\rho, f(x_1, x_2, x_3, \dots)) \xrightarrow{\phi} f(x_{\rho(1)}, x_{\rho(2)}, x_{\rho(3)}, \dots)$$

where $\rho(i) = i$ for $i > k$. The function ϕ defines an action of S_k on $C[[x]]$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. The *monomial symmetric function associated with λ* is

$$(1.2.2) \quad m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_n}^{\lambda_n}$$

where the sum is over all distinct monomials with exponents $\lambda_1, \lambda_2, \dots, \lambda_n$. For each λ , $m_\lambda \in C[[x]]$ is homogeneous of degree $|\lambda|$ and invariant under the action of S_k , for every k , as defined in (1.2.1).

Definition 1.2.1 The vector space spanned by the m_λ is denoted

$$\Lambda = \Lambda(x) = C[m_\lambda]$$

as λ varies over all partitions. This vector space is closed under product and hence is a ring. It is called the *ring of symmetric functions*.

It should be noticed that not all the elements of $C[[x]]$ which are invariant under the action of S_k as in (1.2.1), can be written as a finite linear combination of the m_λ . For example $\prod_{i \geq 1} (1 + x_i)$ is invariant under ϕ but is not in Λ . That is because the product contains monomials of every degree.

The space spanned by all the m_λ of degree n is denoted by Λ^n . Since the set $\{m_\lambda | \lambda \text{ is a partition of } n\}$ is linearly independent, it is a basis for Λ^n . So the dimension of Λ^n is the number of partitions of n , denoted $p(n)$. Note that if $f \in \Lambda^q$ and $g \in \Lambda^k$ then $fg \in \Lambda^{q+k}$ and thus Λ becomes a graded ring.

Definition 1.2.2 For $n \geq 0$, the n^{th} elementary symmetric function e_n is the sum of all products of n distinct variables x_i . That is, $e_0 = 1$, and for $n \geq 1$

$$(1.2.3) \quad e_n = m_{(1^n)} = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

As an example, for $n = 4$, $e_4 = x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_1 x_2 x_4 x_5 + \dots$.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ define

$$(1.2.3') \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}.$$

Definition 1.2.3 For $n \geq 0$, the n^{th} complete homogeneous symmetric function h_n is the sum of all monomials of degree n in the variables x_i . That is, $h_0 = 1$ and for $n \geq 1$

$$(1.2.4) \quad h_n = \sum_{|\lambda|=n} m_\lambda = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For instance, if $n = 4$

$$\begin{aligned} h_4 &= m_{(4)} + m_{(3,1)} + m_{(2^2)} + m_{(1^4)} \\ &= x_1^4 + x_2^4 + \cdots + x_1^3 x_2 + x_2^3 x_1 + \cdots + x_1^2 x_2^2 \\ &\quad + x_1^2 x_3^2 + \cdots + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \cdots \end{aligned}$$

Similarly, as in the case of the e 's, we define for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$,

$$(1.2.4') \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}.$$

Remark: For $n < 0$ we assume $h_n = e_n = 0$. The definitions for e_λ and h_λ make sense for any sequence of non-negative integers.

The proofs of the following results can be found in Macdonald [Mac 79] or Sagan [Sag 91].

Theorem 1.2.4 *The generating functions for e_n and h_n are respectively*

$$(1.2.5) \quad E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t),$$

$$(1.2.6) \quad H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}. \quad \square$$

This theorem immediately yields

$$H(t)E(-t) = 1.$$

Extracting the coefficient of t^n on both sides, results in the following corollary.

Corollary 1.2.5 *For each $n > 0$,*

$$(1.2.7) \quad \sum_{k=0}^n (-1)^k e_k h_{n-k} = 0.$$

In addition to the basis formed by the m_λ , we have two other bases for Λ .

Theorem 1.2.6 *The sets*

$$\{e_\lambda | \lambda \text{ is a partition of } n\}$$

and

$$\{h_\lambda | \lambda \text{ is a partition of } n\}$$

are bases for Λ^n . $\square$

Since the e_λ form a basis for Λ , the e_i are algebraically independent. Thus we can define a ring homomorphism

$$\omega : \Lambda \longrightarrow \Lambda$$

by

$$\omega(e_i) = h_i$$

for all $i \geq 0$. In fact ω is an isomorphism because of the next result.

Theorem 1.2.7 ω is an involution. That is ω^2 is the identity.

Proof: It suffices to show $\omega(h_n) = e_n$. If $n = 0$, $e_0 = h_0$, hence $\omega(h_0) = e_0$. Assume $\omega(h_i) = e_i$ for all $i \leq n - 1$. Because ω is a homomorphism, and by (1.2.7) each h_i can be written as a polynomial in terms of e_j 's for $j \leq i$, $\omega(e_k h_i) = \omega(e_k)\omega(h_i)$. Thus

$$\begin{aligned}\omega(h_n) &= \omega\left(\sum_{k=1}^n (-1)^{k+1} e_k h_{n-k}\right) \\ &= \sum_{k=1}^n (-1)^{k+1} \omega(e_k) \omega(h_{n-k}) \\ &= \sum_{k=1}^n (-1)^{k+1} h_k e_{n-k} \\ &= e_n.\end{aligned}$$

Definition 1.2.8 Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. The *Schur function* corresponding to λ is

$$(1.2.8) \quad s_\lambda = \sum_T x^T$$

where the sum is over all semi-standard λ -tableaux T , and x^T is as defined in (1.1.3).

In a similar way, the *skew Schur function* corresponding to λ/μ is

$$(1.2.8') \quad s_{\lambda/\mu} = \sum_T x^T$$

where the sum is over all skew semi-standard λ/μ -tableaux T .

To illustrate this, let $\lambda = (3, 2, 2)$ and $\mu = (3, 1)$. Some skew semi-standard λ/μ -tableaux are

	1	
1	2	

	1	
2	2	

	1	
2	3	

	2	
1	3	

whence the skew Schur function for λ/μ is

$$s_{\lambda/\mu} = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + \cdots .$$

Proposition 1.2.9 *The function $s_\lambda(x)$ is a symmetric function.*

Proof: The following is a combinatorial proof due to Knuth [Knu 70]. Since the function $s_\lambda(x)$ is homogeneous of degree $|\lambda|$, it suffices to show the function $s_\lambda(x)$ is invariant under ϕ for all k . Since any permutation is a product of transpositions of the form $(i, i+1)$ we need to show

$$((i, i+1), s_\lambda(x)) \xrightarrow{\phi} s_\lambda(x)$$

for all i .

Let $\mathcal{T}_\lambda$ be the collection of all semi-standard λ -tableaux. We want to find a bijection $f : \mathcal{T}_\lambda \rightarrow \mathcal{T}_\lambda$ such that $f(T) = T'$ implies $((i, i+1), x^T) \xrightarrow{\phi} x^{T'}$. We obtain T' from T as follows. Look at the entries of T equal to i or $i+1$. If an i and an $i+1$ are in the same column the entries are called fixed. If only one of i or $i+1$ appears in a column the entry is called free. In each row of T consider the free i 's and $i+1$'s. If a row contains r free i 's followed by t free $i+1$'s, then we change these into t i 's followed by r $i+1$'s. Call this new tableau T' . That the rows of T' are weakly increasing is clear. When i is free, the entry immediately below it is at least $i+2$, and when $i+1$ is free the entry immediately above it is at most $i-1$. So if a free i is changed into $i+1$ then the new column is strictly increasing. Similarly, if a free $i+1$ is changed into i , the new column is strictly increasing. Thus T' is a semi-standard λ -tableau. The map defined is an involution, since exchanging the number of i 's and $i+1$'s in each row twice maps T into itself. The fact that $((i, i+1), x^T) \xrightarrow{\phi} x^{T'}$ follows because the number of fixed i 's and $i+1$'s is the same in T and T' , while the number of free i 's

and $i+1$'s is switched under f . Therefore under the action of $(i, i+1)$, the exponents of x_i and x_{i+1} are exchanged. Thus, this involution guarantees that $s_\lambda(x)$ is invariant under the action of each transposition. $\square$

For instance, for the transposition $(3,4)$, Figure 1.2.1 shows a T and the corresponding T' . The fixed entries are enclosed by circles and the free entries are enclosed by squares.

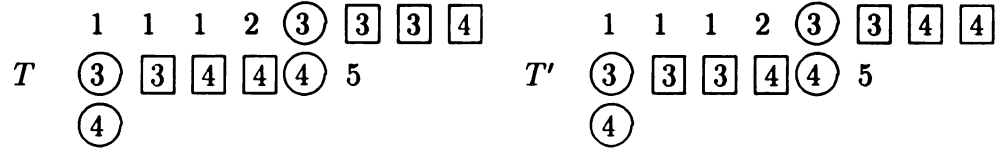


Fig. 1.2.1

Our final basis for Λ (see [Mac,70] or [Sag,91]) is as follows.

Theorem 1.2.10 *The set $\{s_\lambda | \lambda \text{ is a partition}\}$ forms a basis for Λ . $\square$*

1.3 Lattice Paths

Definition 1.3.1 a) Let $Z \times Z$ be the set of integer lattice points in $R \times R$. If $u = (a, b)$ and $v = (a+1, b)$ are in $Z \times Z$ then the line segment uv is called an *eastward* step. If $u = (a, b)$ and $v = (a, b+1)$ are in $Z \times Z$ then uv is called a *northward* step. We say that the step uv *starts* at u and *ends* at v . We call u and v the *initial* and *end* points of the step, respectively.

b) Suppose $u = (a, b)$ and $v = (c, d)$ are lattice points in $Z \times Z$ with $a \leq c$ and $b \leq d$. A *finite $u-v$ lattice path* is a sequence of northward and eastward steps $s_1, s_2, \dots, s_k$ such that the first step begins at u , the last step ends at v , and the end point of s_i is the initial point of s_{i+1} for $1 \leq i < k$.

c) We define $\tilde{Z} = Z \cup \{\infty\}$. Let $u = (a, b)$ and $v = (c, \infty)$ with $a \leq c$ be in $Z \times \tilde{Z}$. A $u - v$ path in $Z \times \tilde{Z}$ with initial point u and end point v , denoted $(a, b) \xrightarrow{p} (c, \infty)$, also written $p : (a, b) \rightarrow (c, \infty)$, is obtained by extending a finite $u - w$ path, where $w = (c, d)$ for some d , with an infinite number of northward steps along the line $x = c$. Note that in this case there are infinitely many $u - v$ paths but each one has only $c - a$ eastward steps. Figure 1.3.1 shows a path in $Z \times \tilde{Z}$.

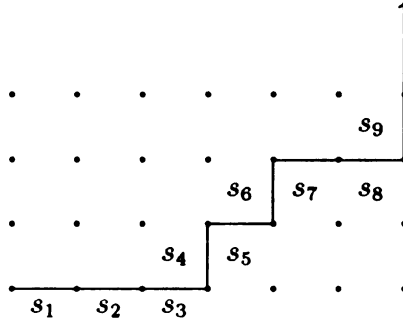


Fig. 1.3.1

Let p be a lattice path with steps $s_1, s_2, \dots$. To each eastward step we associate two labelings, the e -labeling and the h -labeling, as follows.

The e -labeling of the eastward step s_i is

$$L(s_i) = i.$$

The h -labeling of the eastward step s_i is

$$\hat{L}(s_i) = (\text{number of preceding northward steps}) + 1.$$

Figure 1.3.2 shows the e and h labelings for the path given in Figure 1.3.1

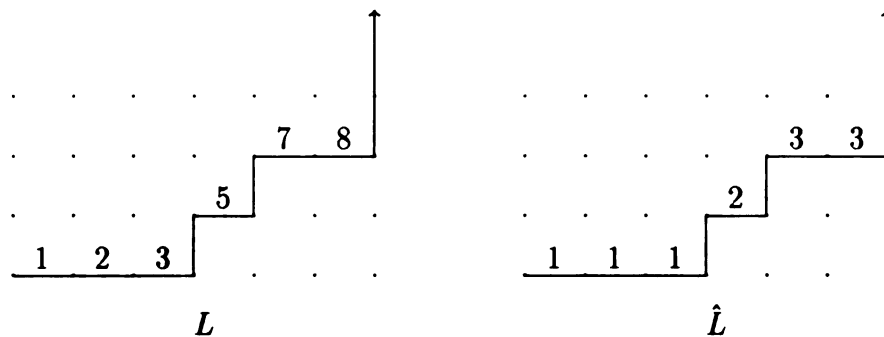


Fig. 1.3.2

Let p be a path. We assign to each of the eastward steps of p the weight $x_{L(s_i)}$ or $x_{\hat{L}(s_i)}$ depending upon the labeling we are considering. We now define two weights for a path p , depending on the labeling assigned to the eastward steps as:

$$(1.3.1) \quad x^p = \prod_{s_i} x_{L(s_i)}$$

$$(1.3.2) \quad \hat{x}^p = \prod_{s_i} x_{\hat{L}(s_i)}$$

where s_i varies over all the eastward steps in p .

The connection of these weights with elementary and complete homogeneous symmetric functions is as follows. For a fixed a, b and n , consider all paths $(a, b) \xrightarrow{p} (a+n, \infty)$. There is a one-to-one correspondence between square free monomials in n variables and the weights of these paths as determined by the e -labeling. Similarly, there is a one-to-one correspondence between all monomials of degree n and the weights of these paths as determined by the h -labeling. Hence,

$$(1.3.3) \quad e_n = \sum_{(a,b) \xrightarrow{p} (a+n,\infty)} x^p$$

and

$$(1.3.4) \quad h_n = \sum_{(a,b) \xrightarrow{p} (a+n,\infty)} \hat{x}^p.$$

Products of elementary and complete homogeneous symmetric functions can be obtained by considering weights of n -tuples of paths. Let $\mathcal{P}$ be the collection of all

n -tuples of paths $P = (p_1, p_2, \dots, p_n)$ with initial points $u_1, u_2, \dots, u_n$, and end points $v_1, \dots, v_n$. We insist that the initial point of p_i is u_i for all i . However the end points of $p_1, p_2, \dots, p_n$ can be $v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}$ (respectively) for any permutation $\sigma \in S_n$. If the n -tuple P corresponds to the permutation σ we will write $P = P_\sigma$. Each path in P_σ is written $p_i : u_i \rightarrow v_{\sigma(i)}$.

An n -tuple $P = (p_1, p_2, \dots, p_n)$ is called *intersecting* if it contains two paths with a common node, that is $p_i \cap p_j \neq \emptyset$ for some $i \neq j$. Otherwise, it is called *non-intersecting*. Count the number of steps from the initial point of a path to a node where intersection with other paths occurs. The node which corresponds to the minimum number of steps is called the *first intersection* of the path.

Definition 1.3.2 Let $P = (p_1, p_2, \dots, p_n)$ be an n -tuple of paths. If $P = P_\sigma$ we define its *sign* to be $(-1)^P = (-1)^\sigma$. Also, the *weight of P* corresponding to the e -labeling is

$$(1.3.5) \quad x^P = \prod_{i=1}^n x^{p_i},$$

while the weight corresponding to the h -labeling is

$$(1.3.6) \quad \hat{x}^P = \prod_{i=1}^n \hat{x}^{p_i}.$$

We define the *signed weights* of P to be $(-1)^P x^P$ and $(-1)^P \hat{x}^P$ respectively. For example, Figure 1.3.3 (a) - (b) shows a triple P with the e and h -labeling respectively. In this case $\sigma = (2, 3)$ and its sign is $(-1)^\sigma = -1$. The weights as defined in (1.3.5) and (1.3.6) are $x^P = x_1 x_2 x_3^2 x_4^2 x_5 x_6^2 x_9$ and $\hat{x}^P = x_1^6 x_3^2 x_4 x_5$.

Proposition 1.3.3 Let $\bar{\mathcal{P}}$ be the collection of all n -tuples of paths of the form $p_i : u_i \rightarrow v_i$, with $u_i = (a_i, b_i)$ and $v_i = (c_i, \infty)$. That is, for all n -tuples of paths in $\bar{\mathcal{P}}$, the initial and end point of the path p_i are fixed. Let $\lambda_i = c_i - a_i$ then, for

$\lambda = (\lambda_1, \dots, \lambda_n)$ we have

$$(1.3.7) \quad e_\lambda = \sum_P x^P$$

and

$$(1.3.8) \quad h_\lambda = \sum_P \hat{x}^P$$

where P varies over all n -tuples of paths in $\bar{\mathcal{P}}$.

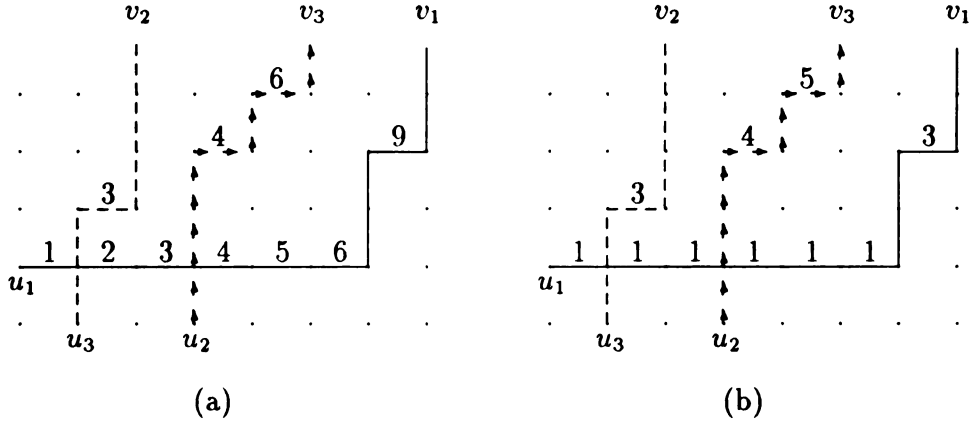


Fig. 1.3.3

Proof: From (1.2.3') $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$. By (1.3.3) $e_{\lambda_i} = \sum_{(a_i, b_i) \xrightarrow{p_i} (c_i, \infty)} x^{p_i}$. Hence,

$$\begin{aligned} e_\lambda &= \prod_{i=1}^n \left(\sum_{(a_i, b_i) \xrightarrow{p_i} (c_i, \infty)} x^{p_i} \right) \\ &= \sum_P x^P \end{aligned}$$

where P varies over all n -tuples in $\bar{\mathcal{P}}$.

Similarly, from (1.2.4') $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$. By (1.3.4) $h_{\lambda_i} = \sum_{(a_i, b_i) \xrightarrow{p_i} (c_i, \infty)} \hat{x}^{p_i}$.

Thus

$$\begin{aligned} h_\lambda &= \prod_{i=1}^n \left(\sum_{(a_i, b_i) \xrightarrow{p_i} (c_i, \infty)} \hat{x}^{p_i} \right) \\ &= \sum_P \hat{x}^P \end{aligned}$$

P varying over all n -tuples in $\bar{\mathcal{P}}$. $\square$

We define the x -coordinate function on $Z \times \tilde{Z}$ by $x((a, b)) = a$. For $u = (a, b)$ and $v = (c, d)$ in $Z \times \tilde{Z}$ we write $u < v$ if $x(u) < x(v)$ and say that u is *to the left* of v (or v is *to the right* of u). When we refer to a partial order between lattice points we assume it is the order induced by the function x , unless otherwise stated.

Given an n -tuple P of paths we will always write the initial points as $(u_1, \dots, u_{n-1}, u_n)$ where $u_k > u_{k+1}$. We also write an n -tuple of paths as $P = (p_1, \dots, p_{n-1}, p_n)$, where p_1 is the path with rightmost initial point, p_2 is the path with second rightmost initial point and so on.

Definition 1.3.4 We define a function $i : \mathcal{P} \rightarrow \mathcal{P}$ the following way:

- a) If P is a non-intersecting n -tuple of paths, then $i(P) = P$.
- b) If $P = (p_1, \dots, p_n)$ is an intersecting n -tuple of paths, choose the path with the smallest index, say p_k , such that $p_k \cap p_j \neq \emptyset$ for some $j > k$. Let v_0 be the lattice point in p_k where the first intersection occurs. Select the path with smallest index intersecting p_k at v_0 , call it p_q . Define $i(P) = P'$ with $P' = (p'_1, \dots, p'_n)$ where

$$\text{i) } p'_s = p_s \text{ if } s \neq k, q \text{ and}$$

$$\text{ii) } p'_k : u_{i_k} \xrightarrow{p_k} v_0 \xrightarrow{p_q} v_{\sigma(i_q)},$$

$$p'_q : u_{i_q} \xrightarrow{p_q} v_0 \xrightarrow{p_k} v_{\sigma(i_k)}.$$

Note that the permutations determined by P and P' differ by a transposition.

The next result follows immediately from the definition of i .

Lemma 1.3.5 *The function i as in definition 1.3.4 is an involution. Also, if P is an intersecting n -tuple with $i(P) = P'$ then $(-1)^P = -(-1)^{P'}$. $\square$*

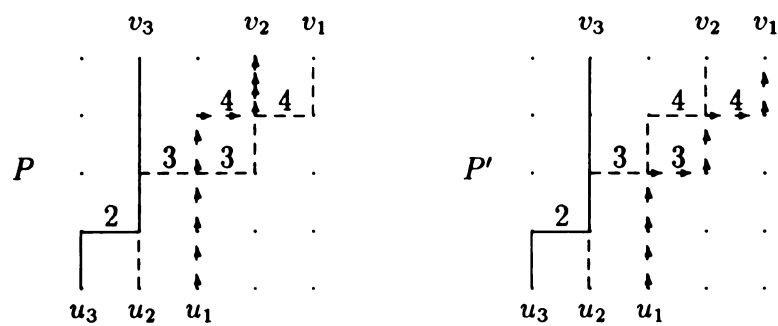
Lemma 1.3.6 *Let P be an intersecting n -tuple with $i(P) = P'$ as in definition 1.3.4.*

- a) *If the paths have the h -labeling and the initial points are on the line $y = c$, then $\hat{x}^{p_k} \hat{x}^{p_q} = \hat{x}^{p'_k} \hat{x}^{p'_q}$ and hence $(-1)^P \hat{x}^P + (-1)^{P'} \hat{x}^{P'} = 0$.*
- b) *If the paths have the e -labeling and the initial points are on the line $x + y = c$ then $x^{p_k} x^{p_q} = x^{p'_k} x^{p'_q}$ and hence $(-1)^P x^P + (-1)^{P'} x^{P'} = 0$.*

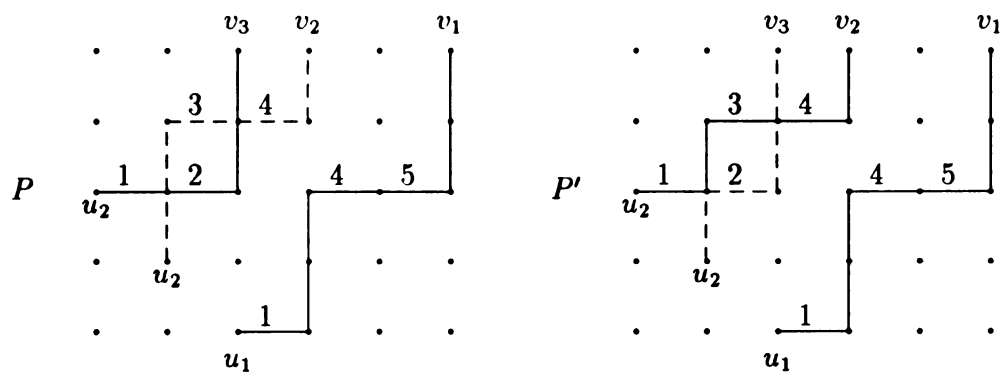
Proof: a) To reach v_0 we have to move the same number of northward steps from the initial points, since they are on the line $y = c$. Thus, after switching paths to obtain p'_k and p'_q the h -labeling remains the same and so $\hat{x}^{p_k} \hat{x}^{p_q} = \hat{x}^{p'_k} \hat{x}^{p'_q}$. Since all the other paths are unchanged, we have $\hat{x}^P = \hat{x}^{P'}$. But the permutations corresponding to P and P' differ by a transposition, so it follows that $(-1)^P \hat{x}^P + (-1)^{P'} \hat{x}^{P'} = 0$.

b) Say the point v_0 is on a line $x + y = a$, for some $a > c$. The number of steps on any lattice path between the lines $x + y = c$ and $x + y = a$ is $c - a$. So, after the first intersection on p_k the e -labeling for p'_k and p'_q is the same as the one for p_q and p_k , respectively. Therefore $x^{p_k} x^{p_q} = x^{p'_k} x^{p'_q}$ and because all the other paths in the n -tuple remain unchanged, $x^P = x^{P'}$. Since the signs of the permutations corresponding to P and P' differ by a transposition, it follows that $(-1)^P x^P + (-1)^{P'} x^{P'} = 0$.

Figures 1.3.4 (a)-(b) illustrates cases (a) and (b) of Lemma 1.3.6, respectively.



(a)



(b)

Fig. 1.3.4

Chapter 2

Main Results

2.1 *Jacobi-Trudi Identities*

By Theorems 1.2.6 and 1.2.10 the sets

$$\{e_\lambda | \lambda \text{ is a partition of } n\}$$

$$\{h_\lambda | \lambda \text{ is a partition of } n\}$$

$$\{s_\lambda | \lambda \text{ is a partition of } n\}$$

are bases for Λ^n .

Equation (1.2.7) can be used to express each h_λ as a linear combination of e_ρ and vice versa.

Jacobi [Jac 41] expressed each s_λ as the determinant of an array of certain symmetric functions. Later, his student Trudi [Tru 64] simplified the demonstration of these identities. In each determinant, all the entries are either complete homogeneous or elementary symmetric functions. Thus, these determinantal identities provide a way to write s_λ as a linear combination of h_ρ or e_ρ with $|\rho| = |\lambda|$. They can be proved combinatorially using a method of Gessel [Ges um] involving lattice paths. For completeness we present this proof here.

Theorem 2.1.1 (Jacobi-Trudi determinants) *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. Then*

$$(2.1.1) \quad s_\lambda = |h_{\lambda_i - i + j}|$$

and

$$(2.1.2) \quad s_{\lambda'} = |e_{\lambda_i - i + j}|.$$

Proof: To prove identity (2.1.1), we consider n -tuples of paths $P = (p_1, \dots, p_{n-1}, p_n)$ with initial points $u_i = (1 - i, 0)$ and end points $v_j = (\lambda_j - j + 1, \infty)$ for $1 \leq i, j \leq n$. We regard the weight determined by the h -labeling as in (1.3.6). Note that

$$u_1 > u_2 > \dots > u_n$$

and

$$v_1 > v_2 > \dots > v_n.$$

Due to the ordering of the initial and end points of the paths, the non-intersecting n -tuples correspond to the identity permutation id . Therefore the sign of non-intersecting n -tuples is $(-1)^{id} = 1$.

For an intersecting n -tuple P , take p_k and p_q as in Definition 1.3.4. Since the initial points are on the line $y = 0$, $\hat{x}^{p_k} \hat{x}^{p_q} = \hat{x}^{p'_k} \hat{x}^{p'_q}$ and so by Lemma 1.3.6 (a) $(-1)^P x^P + (-1)^{P'} x^{P'} = 0$. Hence, the signed weights of intersecting n -tuples of paths cancel in pairs.

Using the definition of determinant, identity (1.3.4), the definition in (1.3.6), Lemma 1.3.6 (a) and the previous discussion, we obtain

$$\begin{aligned} |h_{\lambda_i - i + j}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_{\sigma(i)} - \sigma(i) + i}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i, 0) \xrightarrow{p_i} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} \hat{x}^{p_i} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma} \\
&= \sum_P \hat{x}^P
\end{aligned}$$

where P varies over all non-intersecting n -tuples.

To complete the proof we establish a *weight preserving* bijection between non-intersecting n -tuples of paths and semi-standard λ -tableaux. For this part of the proof, weight preserving means that if the n -tuple P corresponds to the tableau T under a bijection, then $\hat{x}^P = x^T$. Let $P = (p_1, \dots, p_{n-1}, p_n)$ be a non-intersecting n -tuple. Since the path p_i has λ_i eastward steps, we can fill the diagram of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ as follows. Fill the i^{th} row of λ , from left to right, with the h -labeling of p_i , also read left to right, and call this tableau T . By the definition of the h -labeling of a path, the rows of T are weakly increasing. Because the paths do not intersect and the initial points are on the line $y = 0$, the k^{th} eastward step of p_i lies on a horizontal line with y coordinate at least one less than the y coordinate of the horizontal line containing the k^{th} eastward step of p_{i+1} . Therefore, $T_{(i,k)} < T_{(i+1,k)}$, and hence, the columns of T are strictly increasing. Thus, T is a semi-standard λ -tableau. From the definitions of $\hat{x}^P$ in (1.3.2) and x^T in (1.1.3), it follows $\hat{x}^P = x^T$.

Now given a semi-standard λ -tableau T , we map it to an n -tuple of paths as follows. Take the entries of the i^{th} row, weakly increasing, as the h -labeling of an infinite path p_i with initial point $u_i = (1 - i, 0)$. Note that the h -labeling uniquely determines p_i and that its end point must be $v_i = (\lambda_i - 1 + 1, 0)$. Because the columns of T are strictly increasing, that is $T_{(i,k)} < T_{(i+1,k)}$, and the initial points are on the line $y = 0$, we have that the k^{th} eastward step of p_i lies on a horizontal line which is below the horizontal line containing the k^{th} step of p_{i+1} . So, the k^{th} eastward steps do not intersect. Due to the choice of initial points, the k^{th} eastward step of p_{i+1} is one unit to the left of the k^{th} eastward step of p_i for $1 \leq k \leq \lambda_{i+1}$. Thus, the northward

steps between the k^{th} and $(k+1)^{st}$ eastward steps of the two paths do not intersect. Hence, the paths p_{i+1} and p_i are non-intersecting. From what we have shown and the relative placement of p_i and p_{i+1} , it follows that the semi-standard λ -tableau is mapped to an n -tuple of non-intersecting paths. Since this construction is a step by step reversal of the one given in the previous paragraph, we have a bijection. Because the bijection is weight preserving

$$(2.1.3) \quad \sum_P \hat{x}^P = \sum_T x^T$$

where P varies over all non-intersecting n -tuples of paths with initial points u_i and end points v_i , and T varies over all semi-standard λ -tableaux. But the right hand side of (2.1.3) is the definition of s_λ as in (1.2.8). Hence the identity (2.1.1) holds.

An example indicating the correspondence between non-intersecting n -tuples of paths and semi-standard λ -tableaux is given in Figure 2.1.1.

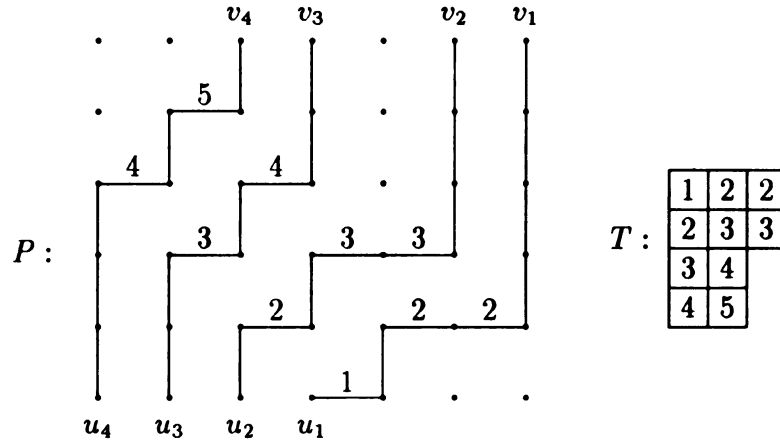


Fig. 2.1.1

To prove identity (2.1.2) we consider n -tuples of paths with weight corresponding to the e -labeling as defined in (1.3.5). We choose as initial points of the n -tuple

$u_i = (1 - i, i - 1)$ and end points $v_j = (\lambda_j - j + 1, \infty)$ for $1 \leq i, j \leq n$. Note that

$$u_1 > u_2 > \cdots > u_n$$

and

$$v_1 > v_2 > \cdots > v_n.$$

Due to the ordering of the initial and end points of the paths, the non-intersecting n -tuples correspond to the identity permutation id . Therefore the sign of non-intersecting n -tuples is $(-1)^{id} = 1$.

For an intersecting n -tuple of paths we take p_k and p_q as in Definition 1.3.4. Since the initial points are on the line $y + x = 0$ and we are considering the e -labeling, the weights of intersecting n -tuples of paths cancel in pairs by Lemma 1.3.6 (b).

The previous discussion together with the definition of determinant, the identity (1.3.3), the definition in (1.3.5) and Lemma 1.3.6 (b), yield

$$\begin{aligned} |e_{\lambda_i - i + j}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (e_{\lambda_{\sigma(i)} - \sigma(i) + i}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i, i-1) \xrightarrow{p_i} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} x^{p_i} \right) \\ &= \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} x^{P_\sigma} \\ &= \sum_P x^P \end{aligned}$$

where P varies over all n -tuples of non-intersecting paths.

We obtain a weight preserving bijection between non-intersecting n -tuples of paths and semi-standard λ' -tableaux as follows. Consider a non-intersecting n -tuple $P = (p_1, \dots, p_n)$. Since the path p_i has λ_i eastward steps we fill the i^{th} column of λ' from top to bottom with the e -labeling of p_i , read from left to right. Call this tableau T .

Columns are strictly increasing by definition of the e -labeling. Due to the choice of initial points, and since the paths p_i and p_{i+1} are non-intersecting, the k^{th} eastward step of the path p_{i+1} lies one unit to the left and at least one unit up from the k^{th} eastward step of the path p_i , for $1 \leq k \leq \lambda_{i+1}$. Since the difference in the e -labeling of these two steps is one less than the difference in their y coordinates, the rows of T are weakly increasing making it a semi-standard λ' -tableau. From definitions in (1.3.1) and (1.1.3) it follows $x^P = x^T$.

Conversely, given a semi-standard λ' -tableau T , let p_i be the infinite path with initial point $u_i = (1 - i, i - 1)$ and eastward steps with the e -labeling determined by the entries of the i^{th} column of λ' , from top to bottom. The path p_i is uniquely determined by the e -labeling and it must have end point $v_i = (\lambda_i - i + 1, 0)$. Since the rows of T are weakly increasing and the initial point of p_{i+1} is one unit to the left and one unit up from the initial point of p_i , the k^{th} eastward step of p_{i+1} is one unit to the left and at least one unit up from the k^{th} eastward step of p_i , for $1 \leq k \leq \lambda_{i+1}$. Thus the paths p_{i+1} and p_i do not intersect. Therefore the unique n -tuple $P = (p_1, \dots, p_n)$ so obtained is non-intersecting and $x^T = x^P$. Since the bijection is weight preserving

$$(2.1.3') \quad \sum_P x^P = \sum_T x^T$$

where P varies over all non-intersecting n -tuples of paths with initial points u_i and end points v_i and T varies over all semi-standard λ' -tableaux. The right hand side of (2.1.3') is the definition of $s_{\lambda'}$ as given in (1.2.8), hence the identity (2.1.2) follows.

□

There are Jacobi-Trudi identities corresponding to skew Schur functions.

Theorem 2.1.2 *Let μ and λ be partitions with $\mu \subseteq \lambda$. Then*

$$(2.1.4) \quad s_{\lambda/\mu} = |h_{\lambda_i - i + j - \mu_j}|$$

and

$$(2.1.4') \quad s_{\lambda'/\mu'} = |e_{\lambda_i - i + j - \mu_j}|.$$

Proof: To prove (2.1.4) we consider n -tuples of paths with the h -labeling. We take as initial points $u_i = (1 - i + \mu_i, 0)$ and end points $v_j = (\lambda_j - j + 1, \infty)$ for $1 \leq i, j \leq n$. Since $\mu_i \geq \mu_{i+1}$, we have $u_i > u_{i+1}$. Because the initial points are on the line $y = 0$, for two intersecting paths the first intersection occurs after an equal number of northward steps. Hence, by Lemma 1.3.6 (a) the signed weights of intersecting n -tuples of paths cancel in pairs. The non-intersecting n -tuples correspond to the identity permutation and so they have sign $(-1)^{id} = 1$. From these observations, it follows in the usual manner that

$$\begin{aligned} |h_{\lambda_i - i + j - \mu_j}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(h_{\lambda_{\sigma(i)} - \sigma(i) + i - \mu_i} \right) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i+\mu_i, 0) \xrightarrow{p_i} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} \hat{x}^{p_i} \right) \\ &= \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma} \\ &= \sum_P \hat{x}^P \end{aligned}$$

where the summation is over all non-intersecting n -tuples of paths P .

A weight preserving bijection between non-intersecting n -tuples of paths with the h -labeling and semi-standard λ/μ -tableaux T is obtained as follows. Let $P = (p_1, \dots, p_n)$ be a non-intersecting n -tuple. Since the path p_i has eastward steps, fill the i^{th} row of T , from left to right, with the h labeling of the path p_i . Rows are weakly increasing by the definition of h -labeling. Let's consider the paths p_i and p_{i+1} . Since $\mu_i - \mu_{i+1} = a \geq 0$, the $(a + k)^{\text{th}}$ eastward step of p_{i+1} , $1 \leq k \leq \lambda_{i+1} - a$, must be on a horizontal line with y coordinate at least one greater than the y coordinate of the

horizontal line containing the k^{th} eastward step of p_i . Since the initial points are on the line $y = 0$, the h -labeling of the $(a + k)^{\text{th}}$ eastward step of p_{i+1} is greater than the h -labeling of the k^{th} eastward step of p_i . Thus columns are strictly increasing and hence, T is a semi-standard λ/μ -tableau. Therefore, to each n -tuple of non-intersecting paths corresponds a unique semi-standard λ/μ -tableau T , with $\hat{x}^P = x^T$.

Conversely, given a semi-standard λ/μ -tableau T , a unique n -tuple of non-intersecting paths is obtained as follows. Construct the path p_i with initial point $(1 - i + \mu_i, 0)$ and take the entries of the i^{th} row of T as the h -labeling for its eastward steps. The paths are uniquely determined by the h -labeling and the end point of the path p_i is $v_i = (\lambda_i - i + 1, 0)$. Due to the choice of initial points and the fact that the difference in the h -labeling of the $(a + k)^{\text{th}}$ eastward step of p_{i+1} and the k^{th} eastward step of p_i is positive, the paths p_{i+1} and p_i are non-intersecting. Hence (2.1.4) follows from (1.2.8').

To prove (2.1.4'), we consider n -tuples of paths with the e -labeling. We choose as initial points $u_i = (1 - i + \mu_i, i - 1 - \mu_i)$ and end points $v_j = (\lambda_j - j + 1, \infty)$ for $1 \leq i, j \leq n$. Since $\mu_i \geq \mu_{i+1}$, we have $u_i > u_{i+1}$. Two intersecting paths have equal number of steps up to the first intersection (see Lemma 1.3.6.(b)). Thus, the weights of intersecting n -tuples of paths cancel in pairs by Lemma 1.3.6 (b). So,

$$\begin{aligned}
 |e_{\lambda_i - i + j - \mu_j}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(e_{\lambda_{\sigma(i)} - \sigma(i) + i - \mu_i} \right) \\
 &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i+\mu_i, 0) \xrightarrow{p_i} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} x^{p_i} \right) \\
 &= \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} x^{P_\sigma} \\
 &= \sum_P x^P
 \end{aligned}$$

where P varies over all non-intersecting n -tuples of paths.

A weight preserving bijection between n -tuples of non-intersecting paths with the e -labeling and semi-standard λ'/μ' tableaux T is obtained as follows. Since the path p_i has $\lambda_i - \mu_i$ eastward steps, fill the i^{th} column, from top to bottom, with the e -labeling of the path p_i , read from left to right. Columns are strictly increasing, by the definition of e -labeling. Let $\mu_i - \mu_{i+1} = a$, thus the $(a + k)^{\text{th}}$ eastward step of p_{i+1} must be at least one unit to the left and one unit up from the k^{th} eastward step of p_i . Also, due to the choice of initial points the difference in the e -labeling of these two steps is one less than the difference in the y coordinates of the horizontal lines containing them. Hence, the rows are weakly increasing and if the semi-standard tableau T corresponds to P then $x^T = x^P$.

Conversely, given a semi-standard λ'/μ' -tableau T , choose a path p_i with initial point $u_i = (1 - i + \mu_i, i - 1 - \mu_i)$ and eastward steps with e -labeling determined by the i^{th} column of λ'/μ' . The path p_i is uniquely determined by the e -labeling and its end point is $v_i = (\lambda_i - i + 1, \infty)$. Since the rows are weakly increasing, the e -labeling of the k^{th} eastward step of the path p_i is less than or equal to the e -labeling of the $(a + k)^{\text{th}}$ eastward step of the path p_{i+1} , where $\mu_i - \mu_{i+1} = a$ and $1 \leq k \leq \lambda_{i+1} - a$. Because of the choice of initial points, the $(a + k)^{\text{th}}$ eastward step of p_{i+1} is one unit to the left and at least one unit up from the k^{th} eastward step of the path p_i . Hence, p_i and p_{i+1} are non-intersecting and $x^T = x^P$. Thus we have a weight preserving bijection and (2.1.4') follows from (1.2.8'). $\square$

2.2 Some Jacobi-Trudi Type Identities

The identities we are about to prove relate certain determinants, where each entry is the sum or difference of certain complete homogeneous symmetric functions, to sums of certain skew Schur functions.

Theorem 2.2.1 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t \geq 0$, a fixed integer,*

$$(2.2.1) \quad |h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 1 - t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu| - (t+1)\nu_\mu)/2} s_{\lambda/\mu}$$

$$(2.2.2) \quad |h_{\lambda_i - i + j} - h_{\lambda_i - i - j + 1 - t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu| - (t-1)\nu_\mu)/2} s_{\lambda/\mu}.$$

We first prove some lemmas.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ a partition and $t \geq 0$ an integer we consider n -tuples of paths with initial points of the form

$$(2.2.3) \quad u_i = (1 - i, 0) \text{ or } w_j = (j + t, 0)$$

for $1 \leq i, j \leq n$, and end points

$$(2.2.3') \quad v_k = (\lambda_k - k + 1, \infty)$$

for $1 \leq k \leq n$. We write the initial points of an n -tuple of paths in decreasing order of their x coordinates as

$$(2.2.4) \quad (w_{j_1}, w_{j_2}, \dots, w_{j_m}, u_{j_{m+1}}, \dots, u_{j_n})$$

where $n \geq j_1 > j_2 > \dots > j_m$, $j_{m+1} < j_{m+2} < \dots < j_n \leq n$ and $j_r \neq j_t$ for $r \neq t$.

The set of indices of the initial points is $[n] = \{1, 2, \dots, n\}$. Thus, each n -tuple of paths corresponds to a $\sigma \in S_n$ as in Section 1.3. For $\lambda = (5, 4, 1)$ and $t = 2$ Figure 2.2.1 shows two triples of paths with initial points (w_1, u_2, u_3) and (w_3, w_1, u_2) , respectively. The permutations corresponding to these triples are $(2, 3)$ and $(1, 2, 3)$ in cycle notation.

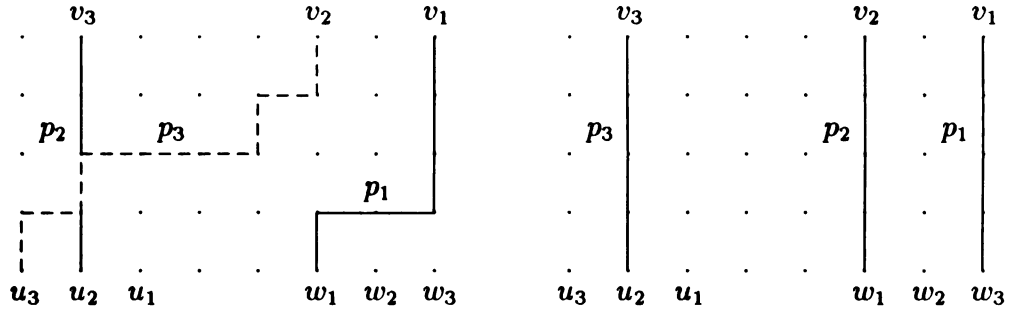


Figure 2.2.1

For $1 \leq k \leq m$ let $\mu_k = x(w_{j_k}) - x(u_k)$ and for $m < k \leq n$, let $\mu_k = x(u_{j_k}) - x(u_k)$. We say that the n -tuple $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ of non-negative integers, is *determined* by the initial points $(w_{j_1}, \dots, w_{j_m}, u_{j_{m+1}}, \dots, u_{j_n})$.

Lemma 2.2.2 *The n -tuple $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ determined by the initial points as in (2.2.4), is a partition. Furthermore, if there exists an n -tuple of non-intersecting paths with the given initial and end points then $\mu \subseteq \lambda$.*

Proof: Note that the points are in decreasing order with respect to the x coordinates and $x(u_i) - x(u_{i+1}) = 1$, for $1 \leq i < n$. We consider several cases.

For $1 \leq i < m$,

$$x(w_{j_i}) - x(w_{j_{i+1}}) \geq x(u_i) - x(u_{i+1}).$$

That is

$$x(w_{j_i}) - x(u_i) \geq x(w_{j_{i+1}}) - x(u_{i+1})$$

which by the definition of μ_i is $\mu_i \geq \mu_{i+1}$.

Similarly, for $i > m$, we have

$$x(u_{j_i}) - x(u_{j_{i+1}}) \geq x(u_i) - x(u_{i+1}).$$

So

$$x(u_{j_i}) - x(u_i) \geq x(u_{j_{i+1}}) - x(u_{i+1})$$

which is $\mu_i \geq \mu_{i+1}$, for $m < i < n$.

Finally,

$$x(w_{j_m}) - x(u_{j_{m+1}}) \geq x(u_m) - x(u_{m+1}).$$

Hence,

$$x(w_{j_m}) - x(u_m) \geq x(u_{j_{m+1}}) - x(u_{m+1}).$$

which is the same as $\mu_m \geq \mu_{m+1}$. All the cases together give $\mu_i \geq \mu_{i+1}$, for $1 \leq i \leq n$, so μ is a partition.

Now, let $P = (p_1, \dots, p_n)$ be a non-intersecting n -tuple of paths with the given initial points. Thus, for each i the path p_i has end point v_i , that is $p_i : (1 - i + \mu_i, 0) \rightarrow (\lambda_i - i + 1, \infty)$. So, the difference between the first coordinates of end and initial points of each path is greater than zero. It follows that $\mu_i \leq \lambda_i$, for all i , and so $\mu \subseteq \lambda$. $\square$

We now relate the length of the main diagonal of μ with the number of w_i and u_j occuring as initial points of the n -tuple of paths.

Lemma 2.2.3 *Let P be an n -tuple of paths with initial points as in (2.2.4). If μ is the partition determined by the initial points, then $v_\mu = m$.*

Proof: According to (1.1.4) it has to be shown that $\mu_i \geq i$ for $i \leq m$ and $\mu_i < i$ for $i > m$. Since $t \geq 0$, $x(w_{j_i}) > 0$. That is $1 - i + \mu_i > 0$ and so $\mu_i > i$ for $i \leq m$. Similarly $x(u_{j_i}) \leq 0$, that is $1 - i + \mu_i \leq 0$ for $m < i \leq n$. Hence, $\mu_i < i$ for $i > m$. $\square$

The next result explains the appearance of the partition from P_t in our main theorems.

Lemma 2.2.4 *Let $t \geq 0$ be a fixed integer. There is a bijection between n -tuples of initial points as in (2.2.4) and partitions in P_t .*

Proof: Note that if $j_r > j_1$ then $u_{j_r} = u_r$. Thus, $\mu_r = 0$ for $r > j_1$. On the other hand, if $j_r < j_1$ then $u_{j_r} \neq u_r$. So, $x(u_{j_r}) - x(u_r) \neq 0$, for $j_r < j_1$. Also $x(w_{j_r}) - x(u_r) \neq 0$, for $j_r \leq j_1$. It follows that $l(\mu) = j_1$.

Given an n -tuple of initial points we prove by induction on i that the partition $\mu = (\mu_1, \dots, \mu_i, \mu_{i+1}, \dots, \mu_n)$ determined by them is in P_t .

For $i = 1$, $\mu'_1 = l(\mu)$. Since $w_{j_1} = (j_1 + t, 0)$ and $\mu_1 = x(w_{j_1}) - x(u_1) = j_1 + t$, it follows that $\mu_1 = \mu'_1 + t$. For $1 \leq i < m$, we claim that $\mu_i - \mu_{i+1} = \mu'_i - \mu'_{i+1}$. By the definition of μ ,

$$\begin{aligned} \mu_i - \mu_{i+1} &= (x(w_{j_i}) - x(u_i)) - (x(w_{j_{i+1}}) - x(u_{i+1})) \\ &= x(w_{j_i}) - x(w_{j_{i+1}}) - 1 \\ &= j_i - j_{i+1} - 1. \end{aligned}$$

On the other hand

$$\begin{aligned} \mu'_i - \mu'_{i+1} &= \text{number of } \mu_k \text{ with } j_{i+1} < k < j_i \\ &= j_i - j_{i+1} - 1. \end{aligned}$$

Hence, the claim follows.

Since $\mu_i - \mu_{i+1} = \mu'_i - \mu'_{i+1}$, for $1 \leq i < m$, and $\mu_1 = \mu'_1 + t$, we have that $\mu_{i+1} = \mu'_{i+1} + t$, for $1 \leq i \leq m$. Thus, $\mu \in P_t$.

Conversely, let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be in P_t with $\nu_\mu = m$. Define

$$j_i = 1 - i + \mu'_i \text{ for } 1 \leq i \leq m$$

and

$$j_k = k - \mu_k \text{ for } m < k \leq n.$$

By definition of ν_μ , $j_i > 0$ for $1 \leq i \leq n$.

Since $\mu'_i \geq \mu'_{i+1}$ we have that $1 - i + \mu'_i > 1 - (1 + i) + \mu'_{i+1}$. That is $j_i > j_{i+1}$ for $1 \leq i < m$. Also, $\mu_k \geq \mu_{k+1}$ and so $k - \mu_k < (1 + k) - \mu_{k+1}$. Thus, $j_k < j_{k+1}$ for $m < k < n$. Thus we have that $j_1 > j_2 > \cdots > j_m$ and $j_{m+1} < \cdots < j_{n-1} < j_n$.

Because $l(\mu) \leq n$ it follows that $j_1 \leq n$. Since $\mu_n \geq 0$, $n - \mu_n \leq n$ and so $j_n \leq n$. Therefore, $1 \leq j_i \leq n$ for all i .

Now we claim that $j_r \neq j_i$. The proof of this claim follows the demonstration of Proposition (1.7) in Macdonald [Mac 70]. We make a diagram with $n \times \mu_1$ boxes which contains μ , as shown in Figure 2.2.2 for $\mu = (8, 7, 1, 1, 1, 1)$ and $n = 9$. In this diagram we consider the boundary path between λ and its complement as indicated by the thick line. This path has $n + \mu_1$ steps. Number its successive steps, starting at the bottom, with the numbers $0, 1, \dots, \mu_1 + n - 1$.

The i^{th} eastward step of the boundary path is in the i^{th} column. Up to and including that step the path has $n - \mu'_i$ northward steps and i eastward steps for a total of $n - \mu'_i + i$ steps. Since the labeling starts with zero, the number attached to the i^{th} eastward step counted from the bottom is

$$n - \mu'_i + i - 1 = n - (1 - i + \mu'_i) \text{ for } 1 \leq i \leq n.$$

X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	
X								
X								
X								
X								

Figure 2.2.2

The k^{th} northward step is in the $(n - k + 1)^{\text{th}}$ row of μ which has length μ_{n-k+1} . Up to and including that step, there are μ_{n-k+1} eastward steps and k northward steps, for a total of $\mu_{n-k+1} + k$ steps. Since the labeling starts with zero, the number attached to the k^{th} northward step from the bottom is

$$\mu_{n-k+1} + k - 1 \text{ for } 1 \leq k \leq n.$$

After we substitute $n - k + 1$ for k , this can be written as

$$\mu_k + (n - k) = n - (k - \mu_k) \text{ for } 1 \leq k \leq n.$$

Now consider the part of the boundary path starting at the bottom and ending at the main diagonal. The last northward step of this part of the boundary path is on the row μ_{m+1} . Thus the part of the path in consideration has m eastward steps and $n - m$ northward steps for a total of n steps. So, the numbers attached to these steps are $0, 1, \dots, n - 1$. Using the description of the labeling of northward and eastward steps in terms of μ as explained in the previous two paragraphs, we can write the numbers $0, 1, \dots, n - 1$ as

$$n - (1 - i + \mu'_i) = n - j_i \text{ for } 1 \leq i \leq m$$

and

$$n - (k - \mu_k) = n - j_k \text{ for } m < k \leq n.$$

This implies that $1 \leq j_i \neq j_k \leq n$, for $i \neq k$.

Finally, let

$$w_{j_i} = (1 - i + \mu_i, 0) \text{ for } 1 \leq i \leq m$$

and

$$u_{j_k} = (1 - k + \mu_k, 0) \text{ for } m < k \leq n.$$

Since $\mu_i = \mu'_i + t$, for $1 \leq i \leq m$, we have that $w_{j_i} = (1 - i + \mu'_i + t, 0)$. Using the definitions of j_i and j_k we obtain

$$w_{j_i} = (j_i + t, 0) \text{ for } 1 \leq i \leq m$$

and

$$u_{j_k} = (1 - j_k, 0) \text{ for } m < k \leq n$$

with $j_1, j_2, \dots, j_n$ satisfying the conditions in (2.2.4). $\square$

To tie in our paths with tableaux, we have the following

Lemma 2.2.5 *Let $\tilde{\mathcal{P}}$ be the collection of non-intersecting n -tuples of paths with fixed initial points $(w_{j_1}, \dots, w_{j_m}, u_{j_{m+1}}, \dots, u_{j_n})$ as in (2.2.4) and end points $v_j = (\lambda_j - j + 1, \infty)$, for $1 \leq j \leq n$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Let μ be the partition determined by the initial points. Then, there is a bijection between $\tilde{\mathcal{P}}$ and semi-standard λ/μ -tableaux.*

Proof: The coordinates of the initial and end points are the same as those of the initial and end points in the proof of (2.1.4). Thus the result has already been proved during the demonstration of Theorem 2.1.2. $\square$

Lemma 2.2.5 allows us to express $s_{\lambda/\mu}$ as

$$(2.2.5) \quad s_{\lambda/\mu} = \sum_P \hat{x}^P$$

where P varies over all n -tuples of non-intersecting paths with the given end points and whose initial points determine μ .

For the next result we write $\sigma \in S_n$ using one row notation as $\sigma = a_1 \cdots a_i a_{i+1} \cdots a_n$.

A pair (a_i, a_j) is called an *inversion* of the permutation if $i < j$ and $a_i > a_j$.

Lemma 2.2.6 *For $t \geq 0$ a fixed integer, let $P = P_\sigma$ be a non-intersecting n -tuple of paths with initial points as in (2.2.4) and μ the partition determined by the initial*

points of P . Then,

$$(-1)^P = (-1)^\sigma = (-1)^{\frac{|\mu| - (t+1)\nu_\mu}{2}}.$$

Note that since $\mu \in P_t$ by Lemma 1.1.3, $|\mu| - (t+1)\nu_\mu$ is even.

Proof: It suffices to show that the number of inversions of σ^{-1} is

$$\frac{|\mu| - (t+1)\nu}{2}.$$

Since $P_\sigma = (w_{j_1}, w_{j_2}, \dots, w_{j_m}, u_{j_{m+1}}, \dots, u_{j_n})$ is non-intersecting, using one row notation for permutations we have $\sigma^{-1} = j_1 \cdots j_m j_{m+1} \cdots j_n$ with $j_1 > \cdots > j_m$ and $j_{m+1} < \cdots < j_n$. For $1 \leq i \leq m$, all the numbers to the left of j_i are greater than it. So the numbers $1, 2, \dots, j_i - 1$ are all to the right of j_i and causing inversions with it. For $m < i \leq n$, all the numbers to the right of j_i are greater than it. So they do not generate any inversions. Thus, the number of inversions is

$$\text{inv } \sigma^{-1} = (j_1 - 1) + (j_2 - 1) + \cdots + (j_m - 1).$$

Now consider the diagram of $\mu \in P_t$. After we delete a strip with $(t+1)\nu_\mu$ boxes from the diagram of μ , as shown in Figure 2.2.3, we obtain an array with $|\mu| - (t+1)\nu_\mu$ boxes represented by the two unshaded regions. By the definition of P_t , the two unshaded parts have the same number of boxes. The size of the upper unshaded part is $(\mu_1 - (t+1)) + (\mu_2 - (t+2)) + \cdots + (\mu_m - (t+m))$. Since $\mu_k = j_k + t - (1 - k)$, for $1 \leq k \leq m$, we can write this as $(j_1 - 1) + (j_2 - 1) + \cdots + (j_m - 1)$. Therefore

$$\frac{|\mu| - (t+1)\nu}{2} = (j_1 - 1) + (j_2 - 1) + \cdots + (j_m - 1)$$

and the result follows. $\square$

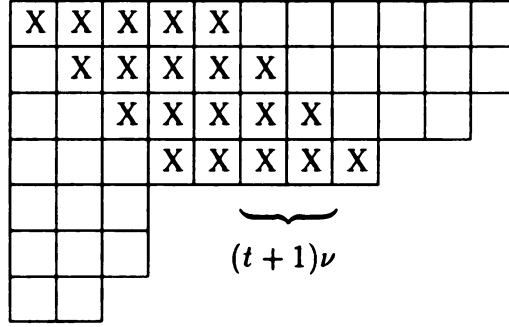


Fig. 2.2.3

Remark: If $\sigma \neq id$ is a permutation, there may be non-intersecting n -tuples of paths which correspond to σ . In Figure 2.2.1 we exhibited a non-intersecting triple of paths corresponding to $\sigma = (1, 2, 3)$.

Now, we put together these results to prove Theorem 2.2.1, which we restate here for easy reference

Theorem 2.2.1 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t \geq 0$, a fixed integer,*

$$(2.2.1) \quad |h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 1 - t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu| - (t+1)\nu_\mu)/2} s_{\lambda/\mu},$$

$$(2.2.2) \quad |h_{\lambda_i - i + j} - h_{\lambda_i - i - j + 1 - t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu| - (t-1)\nu_\mu)/2} s_{\lambda/\mu}.$$

Proof: We first prove identity (2.2.1). Consider initial points of the form (2.2.4).

Combining the definition of determinant and the identity (1.3.4) we have

$$\begin{aligned} |h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 1 - t}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_{\sigma(i)} - \sigma(i) + i} + h_{\lambda_{\sigma(i)} - \sigma(i) - i + 1 - t}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i, 0) \xrightarrow{P} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} \hat{x}^P + \sum_{(t+i, 0) \xrightarrow{P} (\lambda_{\sigma(i)} - \sigma(i) + 1, \infty)} \hat{x}^P \right) \\ &= \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma} \end{aligned}$$

where P_σ varies over all n -tuples of paths corresponding to σ . Since the initial points are along the line $y = 0$ and we are considering the h -labeling, by Lemma 1.3.6 (a) the signed weights of intersecting n -tuples cancel in pairs. Thus,

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1-t}| = \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma}$$

where now P_σ varies over all non-intersecting n -tuples of paths corresponding to σ .

Group the signed weights of non-intersecting paths that have the same initial points. By Lemma 2.2.4, identity (2.2.5) and Lemma 2.2.6 it follows that

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{\frac{|\mu|-(t+1)\nu_\mu}{2}} s_{\lambda/\mu}.$$

To prove the identity (2.2.2) we proceed as follows

$$\begin{aligned} |h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1-t}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_{\sigma(i)}-\sigma(i)+1} - h_{\lambda_{\sigma(i)}-\sigma(i)-i+1-t}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P - \sum_{(i+t,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P \right). \end{aligned}$$

In each factor, the second summation is over paths with w 's as initial points. When we multiply out the product indicated for each $\sigma \in S_n$, we obtain a sum of signed weights of n -tuples. Each n -tuple has as many factors with negative sign as paths with initial points w_i . Let ν_{P_σ} be the number of w which are initial points of the n -tuple P_σ . By Lemma 2.2.3 $\nu_\mu = \nu_{P_\sigma}$, where μ is the partition determined by the initial points of P_σ . Thus,

$$|h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1-t}| = \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma + \nu_{P_\sigma}} \hat{x}^{P_\sigma}.$$

By Lemma 1.3.6 (a), signed weights of intersecting n -tuples cancel in pairs. So, we can consider the last identity as summed over all non-intersecting n -tuples P_σ .

Group all the monomials for n -tuples with the same initial points. Then, by identity (2.2.5) and Lemma 2.2.6, we have

$$\begin{aligned} |h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1-t}| &= \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{\frac{|\mu|-(t+1)v_\mu+\nu_\mu}{2}} s_{\lambda/\mu} \\ &= \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{\frac{|\mu|-(t-1)v_\mu}{2}} s_{\lambda/\mu}. \quad \square \end{aligned}$$

Theorem 2.2.7 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t = -1$ we have*

$$(2.2.6) \quad |h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_{-1}}} (-1)^{\frac{|\mu|}{2}} s_{\lambda/\mu},$$

$$(2.2.7) \quad |h_{\lambda_i-i+j} - h_{\lambda_i-i-j+2}| = 0.$$

Proof: To prove (2.2.6) we consider initial points as in (2.2.3) and satisfying the condition established in (2.2.4). Since $t = -1$, we have $u_1 = w_1$. So we will always let u_1 be present as an initial point in the n -tuple; that is, we always have $j_{m+1} = 1$. Define μ as for the case $t \geq 0$. The proof of Lemma 2.2.2 is the same as before, so μ is a partition. Since each w that appears in the n -tuple still satisfies $x(w) > 0$, the demonstration of Lemma 2.2.3 holds for $t = -1$. The proofs of Lemmas 2.2.4 and 2.2.5 can be mimicked without change for $t = -1$. Thus, identity (2.2.5) is still valid. Finally, the proof of Lemma 2.2.6 proceeds similarly, except that since $t + 1 = 0$ we do not have to delete any strip. In this case the number of boxes to the right of the diagonal and including it is $|\mu|/2$. So we have

$$|\mu| = 2(\text{number of inversions of } \sigma).$$

Thus,

$$(-1)^\sigma = (-1)^{\frac{|\mu|}{2}}.$$

Therefore, we can apply all the results as for $t \geq 0$ to yield

$$\begin{aligned}
|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}| &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_{\sigma(i)}-\sigma(i)+i} + h_{\lambda_{\sigma(i)}-\sigma(i)-i+2}) \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \left(\sum_{(1-i,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P + \sum_{(i-1,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P \right) \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \left(2 \sum_{(0,0) \xrightarrow{P} ((\lambda_{\sigma(1)}-\sigma(1)+1,\infty))} \hat{x}^P \right) \\
&\quad \prod_{i=2}^n \left(\sum_{(1-i,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P + \sum_{(i-1,0) \xrightarrow{P} (\lambda_{\sigma(i)}-\sigma(i)+1,\infty)} \hat{x}^P \right) \\
&= 2 \sum_{\sigma \in S_n} \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma} \\
&= 2 \sum_{P_\sigma} (-1)^{P_\sigma} \hat{x}^{P_\sigma}
\end{aligned}$$

where, by Lemma 1.3.6 (a), P_σ varies over all non-intersecting n -tuples of paths with u_1 as one of the initial points. Group the signed weights of n -tuples of paths with the same initial points. Now, we use the bijection of Lemma 2.2.6, the identity (2.2.5) and the observation made at the beginning of the proof to obtain

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \in \mathcal{P}_{-1} \\ \mu \subseteq \lambda}} (-1)^{\frac{|\mu|}{2}} s_{\lambda/\mu}.$$

To prove (2.2.7), note that for $\sigma \in S_n$ the first factor of each of the summands participating in the determinant is

$$h_{\lambda_{\sigma(1)}-\sigma(1)+1} - h_{\lambda_{\sigma(1)}-\sigma(1)+1} = 0.$$

Thus, the result follows. $\square$

The next result was proved by Bressoud and Wei [B-W 92]. Here, we give a proof of it using Theorem 2.2.1.

Corollary 2.2.8 *Let $t \geq -1$ be an integer. Then*

$$\begin{aligned} & 2^{(t-|t|)/2} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n (h_{\lambda_i-i+\sigma(i)} + (-1)^{(t+|t|)/2} h_{\lambda_i-i-\sigma(i)+1-t}) \\ &= \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{[|\mu|+\nu_\mu(|t|-1)]/2} s_{\lambda/\mu}. \end{aligned}$$

Proof: We rewrite the statement of the corollary as:

(a) If $t = -1$,

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_{-1}}} (-1)^{|\mu|/2} s_{\lambda/\mu}.$$

(b) If t is even,

$$|h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{[|\mu|+\nu_\mu(t-1)]/2} s_{\lambda/\mu}.$$

(c) If t is a positive odd integer,

$$|h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{[|\mu|+\nu_\mu(t-1)]/2} s_{\lambda/\mu}.$$

Now we proceed to prove these three.

(a) This is the content of the identity (2.2.6) already proved in Theorem 2.2.7.

(b) We have that $\frac{|\mu|+(t-1)\nu}{2} - t\nu = \frac{|\mu|-(t+1)\nu}{2}$. Since t is even

$$(-1)^{(|\mu|+(t-1)\nu)/2} = (-1)^{(|\mu|-(t+1)\nu)/2}.$$

Hence, the identity in (b) is (2.2.1) for t even.

(c) We have that $\frac{|\mu|+(t-1)\nu}{2} - (t-1)\nu = \frac{|\mu|-(t-1)\nu}{2}$. Since $t-1$ is even

$$(-1)^{(|\mu|+(t-1)\nu)/2} = (-1)^{(|\mu|-(t-1)\nu)/2}.$$

Hence, the identity in (c) is (2.2.2) for t odd. $\square$

Using the ring homomorphism ω in Theorem 1.2.7, we have $\omega(h_\lambda) = e_\lambda$ and $\omega(s_\lambda) = s_{\lambda'}$. When we apply ω to the identities (2.2.1), (2.2.2), (2.2.6) and (2.2.7) we obtain, for $t \geq -1$,

$$|e_{\lambda_i-i+j} + e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t+1)\nu_\mu)/2} s_{\lambda'/\mu'}$$

and

$$|e_{\lambda_i-i+j} - e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t-1)\nu_\mu)/2} s_{\lambda'/\mu'}.$$

We can also give a combinatorial proof of these identities.

Theorem 2.2.9 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For $t \geq -1$, a fixed integer,*

$$(2.2.8) \quad |e_{\lambda_i-i+j} + e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t+1)\nu_\mu)/2} s_{\lambda'/\mu'}$$

and

$$(2.2.9) \quad |e_{\lambda_i-i+j} - e_{\lambda_i-i-j+1-t}| = \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_t}} (-1)^{(|\mu|-(t-1)\nu_\mu)/2} s_{\lambda'/\mu'}.$$

In particular for $t = -1$ we have

$$(2.2.10) \quad |e_{\lambda_i-i+j} + e_{\lambda_i-i-j+2}| = 2 \sum_{\substack{\mu \subseteq \lambda \\ \mu \in P_{-1}}} (-1)^{\frac{|\mu|}{2}} s_{\lambda'/\mu'},$$

$$(2.2.11) \quad |e_{\lambda_i-i+j} - e_{\lambda_i-i-j+2}| = 0.$$

First we make some remarks. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ a partition and $t \geq -1$ an integer we consider n -tuples of paths with initial points of the form

$$(2.2.12) \quad u'_i = (1-i, i-1) \text{ or } w'_j = (j+t, -j-t)$$

for $1 \leq i, j \leq n$, and end points

$$(2.2.12') \quad v_k = (\lambda_k - k + 1, \infty)$$

for $1 \leq k \leq n$. Observe that for $t = -1$, $u_1 = w_1$. In this case we always write the initial points of an n -tuple of paths as in (2.2.4) with $j_{m+1} = 1$ and u_1 in it.

The proofs of Lemmas 2.2.2, 2.2.3, 2.2.4 and 2.2.6 go through unchanged, since $x(u'_i) = x(u_i)$ and $x(w'_j) = x(w_j)$ for $1 \leq i, j \leq n$. Note that for Lemma 2.2.6 the strip of size $(1+t)v$ is removed from under the diagonal instead of from the right hand side.

For Lemma 2.2.5 we have the following analogue.

Lemma 2.2.10 *There is a bijection between n -tuples of non-intersecting paths as in (2.2.12) with fixed initial and end points, and semi-standard λ'/μ' tableaux, where μ is determined by the initial points and the end points are determined by λ .*

Proof: Consider non-intersecting n -tuples of paths with initial points as in (2.2.12) and satisfying the condition in (2.2.4). Assign the e -labeling to each path. Now the proof goes as for the skew case of the Jacobi-Trudi determinants involving elementary symmetric functions in Theorem 2.1.2. $\square$

By Lemma 2.2.10

$$\sum_T x^T = \sum_P x^P$$

where T varies over all semi-standard λ'/μ' tableaux and P varies over all non-intersecting n -tuples of paths with initial points as in (2.2.12) which determine μ and end points as in (2.2.12'). From these remarks and (1.2.8') we obtain

$$(2.2.13) \quad s_{\lambda'/\mu'} = \sum_P x^P.$$

Proof of Theorem 2.2.9: The proof of (2.2.8) goes exactly parallel to that of (2.2.1) in Theorem 2.2.1, using the corresponding lemmas about elementary symmetric functions and partitions determined by initial points as in (2.2.12). Since we are using the e -labeling, the cancellation of signed weights of intersecting paths is due to Lemma 1.3.6 (b).

Similarly, identity (2.2.9) is obtained by following the proof of (2.2.2) and using the elementary symmetric functions lemmas together with the ones about the partition determined by the initial points.

The identities (2.2.10) and (2.2.11) are obtained similarly as (2.2.6) and (2.2.7) using the parallel results for elementary symmetric functions. $\square$

2.3 Symplectic and Orthogonal Analogs

Definition 2.3.1 Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and $A = \{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}, \dots\}$ a totally ordered set with $1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n} < \dots$. A semi-standard λ -tableau T with entries from A is called a sp -tableau if all the entries in the row i are greater than or equal to i . In Figure 2.3.1 T_1 is a sp -tableau $(3, 2, 1)$ -tableau while T_2 is not.

1	1	2
$\bar{2}$	$\bar{2}$	
3		

T_1

1	2	$\bar{2}$
$\bar{1}$	$\bar{3}$	
3		

T_2

Figure 2.3.1

We assign *weights* to the entries of a sp - λ -tableau T as follows. If i appears as an entry then we give it weight $\omega(i) = x_i$, and if $\bar{i}$ is an entry then we give weight

$\omega(\bar{i}) = x_i^{-1}$. The *weight* of T is defined as

$$x^T = \prod_{(i,j) \in T} \omega_{(T(i,j))}.$$

For the sp -tableau T_1 in Figure 2.3.1 $x^{T_1} = x_1 x_1 x_2 x_2^{-1} x_2^{-1} x_3 = x_1^2 x_2^{-1} x_3$.

For the rest of the section we consider $x^{\pm 1} = (x_1, x_1^{-1}, x_2, x_2^{-1}, \dots)$.

Definition 2.3.2. Let λ be a partition. The sp Schur function corresponding to λ is

$$sp_{\lambda}(x^{\pm 1}) = \sum_T x^T$$

where T varies over all sp - λ -tableaux. The sp Schur functions are symmetric functions in the variables $x^{\pm 1}$.

Definition 2.3.3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition and

$$A = \{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}, \dots, \infty\}$$

a totally ordered set with $1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n} < \dots < \infty$. A semi-standard λ -tableau T with entries from A is called an so -tableau if

- a) all entries in row i are larger than or equal to i ;
- b) on any row, the symbol ∞ appears at most once.

The weight of an so -tableau T is defined as the weight of the sp -tableau obtained from T by deleting the symbol ∞ if it appears. In Figure 2.3.2 T is a so -(4, 2, 2)-tableau with weight $x^T = x_1^{-1} x_2^2 x_3$

$\bar{1}$	2	2	∞
$\bar{2}$	2		
3	∞		

Figure 2.3.2

Definition 2.3.4. Let λ be a partition. The so -Schur function corresponding to λ is

$$so_{\lambda}(\dot{x}^{\pm 1}) = \sum_T x^T$$

where T varies over all so - λ -tableaux and $\dot{x}^{\pm 1} = x^{\pm 1} \cup \{1\}$.

The Jacobi-Trudi identities are polynomial expressions for the character s_{λ} of the polynomial representation of the general linear group $GL(n)$. There are similar identities for the characters sp_{λ} and so_{λ} of the polynomial representations of the symplectic group $sp(2n)$ and orthogonal group $so(2n+1)$, respectively, which are due to Weyl [We 46]. These formulae are

$$(2.3.1) \quad sp_{\lambda}(x^{\pm 1}) = \frac{1}{2} \det(h_{\lambda_i - i - j + 2}(x^{\pm 1}) + h_{\lambda_i - i + j}(x^{\pm 1}))$$

and

$$(2.3.2) \quad so_{\lambda}(\dot{x}^{\pm 1}) = \det(h_{\lambda_i - i - j}(\dot{x}^{\pm 1}) + h_{\lambda_i - i + j}(\dot{x}^{\pm 1}))$$

where both are $l(\lambda)$ by $l(\lambda)$ determinants and $h_k(x^{\pm 1})$ is the complete homogeneous symmetric function on $x^{\pm 1}$.

Note the resemblance between the identity (2.2.6) in Theorem 2.2.7 and (2.3.1). We are attempting to prove (2.3.1) using the lattice path approach as in Theorem 2.2.1. In our attempt we have discovered that there may be a relationship between the Gessel-Viennot technique and Schützenberger’s jeu de taquin (or “teasing game”) [Scü 76]. An account of Schützenberger’s jeu de taquin can also be found in Sagan [Sa 90, Sa 91].

Okada [Ok um] recently gave a combinatorial proof of these formulae using lattice paths, but introducing some dummy variables. In his proof the relationship between lattice paths and tableaux is not as clear as it could be following our approach.

Chapter 3

Open Problem

3.1 *Circulants*

Definition 3.1.1 A *circulant determinant*, C , is an n by n determinant where the $(i + 1)^{st}$ row is obtained by rotating the i^{th} row one place to the right. Thus

$$C = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{vmatrix}$$

Let $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ be a polynomial having as coefficients the values from the entries of the first row of C and let r be a generator for the group of the n^{th} roots of unity. It is well known [Led 87] that

$$C = p(1)p(r) \cdots p(r^{n-1}).$$

That is

$$(3.1.1) \quad C = (a_0 + a_1 + \cdots + a_{n-1})(a_0 + a_1r + \cdots + a_{n-1}r^{n-1}) \cdots (a_0 + a_1(r^{n-1}) + \cdots + a_{n-1}(r^{n-1})^{n-1})$$

Consider the right hand side of (3.1.1). We call $p(r^{k-1})$ the k^{th} factor. From left to right, label the n summands in the k^{th} factor, $1 \leq k \leq n$, with the numbers

$k, k+1, \dots, n, 1, \dots, k-1$. With this labeling the coefficient of the term labeled j in the k^{th} factor is the entry of C in the k^{th} row and j^{th} column.

Let f be a function from $\{1, 2, \dots, n\}$ to itself. If $f(q) = s$, then from the q^{th} factor we choose the summand labeled with s . Thus, the product in (3.1.1) can be regarded as a sum of weights of functions f from a set with n elements to itself where the *weight of a function f* is the product of the summands picked by the function from each of the factors.

We will visualize functions and their weights using graphs as explained next.

Definition 3.1.2 Let V be a set and let A be a set of ordered pairs of elements from V . The set V will be called the set of *vertices* (or nodes) and A the set of arcs or *directed edges*, where loops are allowed. The pair (V, A) is called a *digraph* or *directed graph*.

Usually we represent a digraph by a diagram where the vertices are indicated by nodes and the directed edge (u, v) is represented by an arrow heading from u to v . We say that u is the *starting point* and v the *ending point* of the directed edge (u, v) . For the digraph shown in Figure 3.1.1 we have that $V = \{1, 2, 3, 4\}$ and $A = \{(1, 4), (2, 4), (3, 3), (4, 1)\}$.

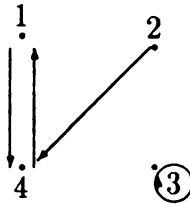


Figure 3.1.1

A digraph is of *outdegree one* if each node is the starting point of only one directed

edge. We are interested in digraphs of this type since they represent functions from a set with n elements to itself.

Given a digraph G with n vertices we label them $1, \dots, n$ in a clockwise manner after we select the vertex with the label zero. The graph in Figure 3.1.1 shows this type of labeling. With this assignment of labels we think of the i^{th} node of G as being associated with the root r^{i-1} .

Definition 3.1.3 Let G be an outdegree one digraph and denote the edge with starting point i and ending point j by e_{ij} . We define the *weight* of e_{ij} as

$$w(e_{ij}) = a_{j-i} r^{(i-1)(j-i)}$$

where $j - i$ is taken modulo n .

The *weight* of G is defined as

$$w(G) = \prod_{i=1}^n w(e_{ij})$$

which we also write as

$$w(G) = \prod_{(i,j)} a_{j-i} r^{(i-1)(j-i)}$$

where (i, j) indicates a directed edge.

As an example, Figure 3.1.1 shows a digraph G where $w(e_{14}) = a_3$, $w(e_{24}) = a_2 r^2$, $w(e_{33}) = a_0$ and $w(e_{41}) = a_1 r^3$. So, $w(G) = (a_3)(a_2 r^2)(a_0)(a_1 r^3) = a_0 a_1 a_2 a_3 r$.

Let G be an outdegree one digraph. With the edge $e_{ij} \in G$ we associate the number t_i , where $t_i \equiv j - i \pmod{n}$ and $0 \leq t_i < n$. Hence, with digraph G is associated a unique n -tuple $(t_1, \dots, t_n)$ where t_i is the number associated with $e_{ij} \in G$. Conversely, each n -tuple $(t_1, \dots, t_n)$ with $0 \leq t_i < n$ uniquely determines a digraph G . To see this we take as $e_{ij} \in G$ the edge with starting point i and ending point j , where $t_i + i \equiv j \pmod{n}$ with $1 \leq j \leq n$. This allows us to identify outdegree one digraphs

on n vertices with n -tuples of non-negative integers less than n . Note that if t_k is in the n -tuple of non-negative integers associated with the digraph G , then the weight of the edge e_{kj} has a_{t_k} as a factor on its weight. Thus, two edges with the same value t_k associated with them, contain the same factor of a_i . So, we have proved the next lemma.

Lemma 3.1.4 *Let G be a digraph associated with the n -tuple $(t_1, \dots, t_n)$. If G' is another digraph associated with an n -tuple which is a rearrangement of $(t_1, \dots, t_n)$ then the a 's participating in the weights of G and G' are the same. $\square$*

For instance, the digraph associated with $(1, 0, 0, 2)$ has $a_1 a_0 a_0 a_2$ as the a 's and $(r^0)^1 (r^1)^0 (r^2)^0 (r^3)^2 = r^2$ as the r 's in its weight. The digraph associated with $(0, 1, 2, 0)$ has $a_0 a_1 a_2 a_0$ as the a 's and has $(r^0)^0 (r^1)^1 (r^2)^2 (r^3)^0 = r$ as the r 's in its weight. Observe that the powers of r associated with two n -tuples which have the same set of values may be distinct.

Definition 3.1.5 Let $(t_1, \dots, t_n)$ be the n -tuple associated with the digraph G . We define the *ratio of G* or *ratio of the n -tuple* by

$$R_G = \sum_{i=1}^n t_i.$$

Let $(t_1, \dots, t_n)$ be the n -tuple associated with G and consider $\sigma = (1, 2, \dots, n) \in S_n$. We define an action of the cyclic group generated by σ on digraphs the following way

$$(3.1.2) \quad \sigma^k(G) = G' \quad 0 \leq k < n$$

where G' is the digraph associated with the n -tuple $(t_{n-k+1}, t_{n-k+2}, \dots, t_1, \dots, t_{n-k})$. Note that the k^{th} position in the n -tuple is occupied by t_1 . We say that G' is a k^{th} *rotation* of G .

The next lemma relates the weights of digraphs which are in the orbit of G under the action defined in (3.1.2).

Lemma 3.1.6 *If G' is the k^{th} rotation of G then*

$$w(G') = w(G)(r^R)^k$$

where $R = R_G$ is as in Definition 3.1.5.

Proof: Let $(t_1, \dots, t_n)$ be the n -tuple associated with G . Since the a_i 's in the weights of G and G' are the same, it suffices to concentrate on the powers of r . The power of r in the weight of G is $(r^0)^{t_1}(r^1)^{t_2} \dots (r^{n-1})^{t_n}$. Its exponent is $\sum_{i=1}^n (i-1)t_i$. On the other hand, the power of r in the weight of G' is

$$(r^0)^{t_{n-k+1}}(r^1)^{t_{n-k+2}} \dots (r^{k-1})^{t_n}(r^k)^{t_1}(r^{k+1})^{t_2} \dots (r^n)^{t_{n-k}}.$$

Its exponent is $\sum_{i=1}^n (k + (i-1))t_i$, where $k + (i-1)$ is taken modulo n . The relation between the exponents of r is

$$\sum_{i=1}^n (k + (i-1))t_i = kR + \sum_{i=1}^n (i-1)t_i$$

and the lemma follows. $\square$

Given an n -tuple $(t_1, \dots, t_n)$ associated with a digraph G we interpret the entry t_i as the number of nodes we have to move clockwise from the i^{th} node to reach the end point of the directed edge with initial point at that node. If the digraph G represents a bijection, then it is composed of cycles. Also, the sum of the t_i 's corresponding to the edges in a cycle is a multiple of n . So we have proved the following result.

Lemma 3.1.7 *If $(t_1, \dots, t_n)$ is the n -tuple associated with a digraph G representing a permutation then $n \mid \sum_{i=1}^n t_i = R$. $\square$*

By Lemma 3.1.7, if R is not a multiple of n then the digraph does not represent a bijection. However, if R is a multiple of n it does not imply that G represents a bijection. For instance, the graph corresponding to $(1, 4, 3, 1, 1)$, as shown in Figure 3.1.2, has ratio $R = 10$ and does not represent a permutation. In the graph, the numbers next to the edges are the entries of the n -tuple associated with the digraph.

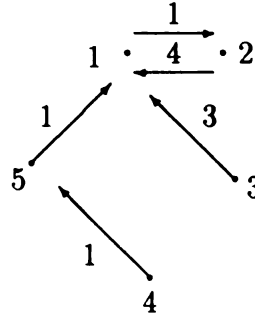


Figure 3.1.2

The next theorem allows us to reduce the right hand side of (3.1.1) to a sum associated with n -tuples whose ratio is a multiple of n . We consider the n -tuple $(t_1, \dots, t_n)$ as a circular word. The *degree* l of the n -tuple is the number of elements in the orbit of that n -tuple under circular permutation. In terms of graphs, the degree l represents the number of digraphs in the orbit of G under the action defined in (3.1.2).

Theorem 3.1.8 *Let G be an outdegree one digraph associated with the n -tuple $(t_1, \dots, t_n)$ of degree l . If $n \nmid R_G$ then*

$$\sum_{G'} w(G') = 0$$

where G' varies over all digraphs in the orbit of G under the action defined in (3.1.2).

Proof: Note that $(t_1, \dots, t_n)$ has $\frac{n}{l}$ identical blocks each of length l . If the sum of the entries in one of the blocks is t then $R_G = \frac{n}{l}t$. Thus $(r^R)^l = r^{nt} = 1$ and so

$1 - (r^{R_G})^l = 0$. Also, since $n \nmid R_G$ we have that $r^{R_G} \neq 1$. By Lemma 3.1.6, and the fact that the orbit of G has l digraphs we have that

$$\begin{aligned} \sum_{G'} w(G') &= w(G)(1 + (r^{R_G}) + (r^{R_G})^2 + \dots + (r^{R_G})^{l-1}) \\ &= w(G) \frac{1 - (r^{R_G})^l}{1 - r^{R_G}} \\ &= 0. \quad \square \end{aligned}$$

Lemma 3.1.9 *Let G be an outdegree one digraph associated with the n -tuple $(t_1, \dots, t_n)$ of degree l . If $n \mid R_G$ then*

$$\sum_{G'} w(G') = lw(G)$$

where G' varies over all digraphs in the orbit of G under the action defined in (3.1.2).

Proof: Since $r^{R_G} = 1$, the result follows from

$$\sum_{G'} w(G') = w(G)(1 + (r^{R_G}) + (r^{R_G})^2 + \dots + (r^{R_G})^{l-1}). \quad \square$$

Conjecture: We know each digraph corresponding to a permutation σ contributes $(-1)^\sigma a_{\sigma(1)-1} a_{\sigma(2)-2} \dots a_{\sigma(n)-n}$ to the determinant. Since on the product side the terms corresponding to digraphs with weight not a multiple of n cancel, we can partition the terms corresponding to digraphs with weight a multiple of n in a *natural manner* such that

- a) Each subset of the partition has the weight of a digraph corresponding to a unique permutation σ .
- b) The sum of the weights on the subset is $(-1)^\sigma a_{\sigma(1)-1} a_{\sigma(2)-2} \dots a_{\sigma(n)-n}$.

Note that the term in the determinant C corresponding to $\sigma \in S_n$ is

$$a_{\sigma(1)-1}a_{\sigma(2)-2} \cdots a_{\sigma(n)-n},$$

where $\sigma(i) - i$ is taken modulo n and $0 \leq \sigma(i) - i < n$.

Now we will show that the conjecture is true for the case of transpositions. Let $\sigma = (i, j)$, $i < j$, be a transposition on S_n , and let G be the outdegree one digraph representing σ . The n -tuple associated with G is $(0, \dots, j - i, 0, \dots, i - j, 0, \dots, 0)$, where $j - i$ and $i - j$ are in the i^{th} and j^{th} entries, respectively. We fix the i^{th} entry and make $i - j$ occupy each of the remaining $n - 1$ positions, to obtain $n - 1$ n -tuples which are associated with digraphs in different orbits. All these digraphs have the same factors of a_i in their weights, namely $a_0 a_0 \cdots a_{j-i} \cdots a_{i-j} \cdots a_0$ which can be written as $a_{\sigma(1)-1} a_{\sigma(2)-2} \cdots a_{\sigma(n)-n}$. If we call G_k the digraph associated with the n -tuple having $j - i$ in the i^{th} entry and $i - j$ in the k^{th} entry, $1 \leq k \leq n$, $k \neq i$, then the power of r in its weight is $(r^{i-1})^{j-i} (r^{k-1})^{i-j}$. With this setup we prove the next result.

Lemma 3.1.10 *If σ is a transposition in S_n and G is the outdegree one digraph representing it then*

$$\sum_{\substack{k=1 \\ k \neq i}}^n w(G_k) = -a_{\sigma(1)-1} a_{\sigma(2)-2} \cdots a_{\sigma(n)-n}$$

where the G_k is as defined previously.

Proof: Note that $r^{(j-i)+(i-j)} = 1$ and $\sum_{k=1, k \neq i}^n (r^{k-1})^{i-j} = 0$. Hence,

$$\begin{aligned} \sum_{\substack{k=1 \\ k \neq i}}^n w(G_k) &= a_{\sigma(1)-1} a_{\sigma(2)-2} \cdots a_{\sigma(n)-n} \left(\sum_{\substack{k=1 \\ k \neq i}}^n (r^{i-1})^{j-i} (r^{k-1})^{i-j} \right) \\ &= a_{\sigma(1)-1} a_{\sigma(2)-2} \cdots a_{\sigma(n)-n} \left(\sum_{k=1}^n (r^{i-1})^{j-i} (r^{k-1})^{i-j} - (r^{i-1})^{(j-i)+(i-j)} \right) \\ &= -a_{\sigma(1)-1} a_{\sigma(2)-2} \cdots a_{\sigma(n)-n}. \quad \square \end{aligned}$$

We expect to carry on this construction for all the subsets of digraphs where two of them are in different orbit and exactly one represents a permutation.

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