GLS DETRENDING AND THE POWER OF UNIT ROOT AND STATIONARITY TESTS

By

Jaeyoun Hwang

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1993

ABSTRACT

GLS DETRENDING AND THE POWER OF UNIT ROOT AND STATIONARITY TESTS

By

Jaeyoun Hwang

This dissertation considers the problem of testing whether deviations of a time series from deterministic trend are stationary or contain a unit root. Common tests detrend the series either in levels, which is appropriate under stationarity, or in differences, which is appropriate given a unit root. This dissertation considers detrending by generalized least squares (GLS), based on an assumed value of the parameter of interest. This idea is closely related to King's theory of point optimal invariant (POI) tests.

We consider two tests based on GLS residuals: the Bhargava-Schmidt-Phillips (BSP) test of a unit root, and the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test of stationarity. We derive asymptotic distributions for these GLS-based tests and for the corresponding POI tests, and we compare their finite sample properties through detailed Monte Carlo simulations. Our results show that the power of the GLS-based BSP unit root test is comparable to that of the POI test. However, the GLS-based KPSS test of stationarity is not very powerful, and is dominated by the POI test. This supports the relevance of our theoretical result that the GLS-based KPSS test is inconsistent.

Dec

in

Dedicated to my parents from whom I have inherited health, intelligence, and especially the spirit of independence.

ACKNOWLEDGEMENTS

I am indebted to many people in different ways. I wish to thank them for their guidance, support, encouragement and patience. Especially, I wish to thank Professor Peter Schmidt, my dissertation chairman, for his considerate and effective guidance from the beginning to the end of this project. I appreciate his sharing of scarce Sabbatical time and his patience, particularly over the last five months. I also wish to thank my other committee members, Professor Robert H. Rasche and Jeffrey Wooldridge, for their invaluable comments.

I gratefully acknowledge the financial support I have received from the Ssangyong Cement Industrial Co. Had it not been for their support, I could not have even gotten started.

Foremost thanks must go to my wife, Younghee, and my little son and daughter, Innyoung and Sunny. They gratefully put up with my negligence of duties at home and impatience for the past five years. I also have to thank my mother who always has been praying for her children.

Finally, I wish to thank the Invisible Hand for my unexplained good fortune. Whenever I am in troubled waters, She comes to help me.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 2

CHAPTER 1

CHAPTER 1

INTRODUCTION

The finding of Nelson and Plosser (1982) that most U.S. macroeconomic data are nonstationary rather than stationary around a deterministic trend has had a huge impact on the character of empirical work in macroeconomics. It has become standard to test the hypothesis of a unit root in macroeconomic time series before proceeding with further analysis. This is so for the following two reasons. First, the presence (or absence) of a unit root in certain series is predicted by alternative economic theories; for example, the efficient market hypothesis, real business cycle theory, and the permanent income theory of consumption. Second, the presence of a unit root has strong implications for methods of statistical inference in regression. Regression with nonstationary data may produce spurious results, so that common statistics such as t-statistics and measures like R^2 are not correct even asymptotically (Granger and Newbold (1974) and Phillips (1986)).

One of the stylized facts in the unit root literature of the past decade is that standard unit root tests often fail to reject the null hypothesis of a unit root for many economic time series. The conclusion that can be drawn from this empirical evidence is that most economic time series do not show strong evidence against the unit root hypothesis. It is not clear whether this occurs because most series actually have a unit root, or because standard unit root tests have low power against relevant alternatives. Therefore Kiwatkowski, Phillips, Schmidt and Shin (1992), hereafter, KPSS, suggest that, in trying to decide whether economic time series are stationary or integrated, it would be useful to perform tests of the null hypothesis of stationarity as well as tests of the null hypothesis of a unit root.

To do so, we consider the Data Generating Process (DGP) to be of the following form:

(1A) $$y_t = \psi + \xi t + u_t,$$

(1B) $$u_t = \rho u_{t-1} + \omega_t - \theta\omega_{t-1}, \quad t = 1,\ldots,T.$$

Clearly u_t is the deviation of y_t from deterministic trend ($\psi + \xi t$). For the moment we assume that $\omega_t \sim NID(0, \sigma_\omega^2)$. In matrix form,

(2) $$y = Z\gamma + u,$$

where Z is a T×2 matrix with t^{th} row $z_t' = [1,t]$, $\gamma' = [\psi,\xi]$, and u is a T×1 vector of realizations of the error process. The point of this parameterization is that it allows for linear deterministic trend under the null and alternative hypotheses, and the interpretation of the parameters ψ (level) and ξ (trend) does not change whether the series is stationary or has a unit root. In addition, the distributions of all the unit root tests and stationarity tests considered in this thesis (except for the GLS-based KPSS tests in chapter 3) do not depend on the nuisance parameters ψ, ξ and σ_ω.

Though many testable hypotheses can be formulated in terms of this DGP, by selecting particular values of the parameters ρ and θ, we are interested in two specific cases which imply trend stationary and difference stationary processes under the null and alternative hypotheses. First, we will consider testing the null hypothesis $\rho = 1$ against the alternative hypothesis $\rho \in [0,1)$, assuming $\theta = 0$. Then u_t has a unit root so that y_t is difference stationary under the null hypothesis. All of the unit root tests that we will consider can be viewed as tests of the hypothesis $\rho = 1$ in this parameterization. Second, we will consider testing the null hypothesis $\theta = 1$ against the alternative $\theta \in [0,1)$, assuming $\rho = 1$. Then $u_t = \omega_t$ are iid errors so that y_t is trend stationary under the null hypothesis. We may note that even though the case of $\rho = 0$ and $\theta = 0$ constitutes the same null hypothesis of stationarity as the case of $\rho = 1$ and $\theta = 1$, the latter is more naturally related to the alternative hypothesis of a unit root, since y_t contains a unit root when $\rho = 1$ and $\theta \in [0,1)$.

This dissertation considers tests based on various types of residuals from equation (1A) for testing whether deviations of a time series from deterministic trend are stationary or contain a unit root. Obviously, different types of residuals correspond to different methods of detrending the series y_t. First, we will define $\hat{u}_t$, $t = 1,\ldots,T$, as the OLS residuals from (1A). That is, they are the residuals from an OLS regression of y on an intercept and time trend. The unit root tests of Dickey and Fuller, hereafter DF, and the KPSS stationarity test are based on these OLS residuals. Second, Bhargava (1986), Schmidt and Phillips (1992) and Schmidt and Lee (1991) consider tests based on detrending in differences. That is, their tests are based on the residuals

(3) $\tilde{u}_t = y_t - \tilde{\psi}_x - \tilde{\xi}t = [(T-1)y_t - (t-1)y_T - (T-t)y_1]/(T-1)$,

where $\tilde{\xi} = \overline{\Delta y} = (y_T-y_1)/(T-1)$ and $\tilde{\psi}_x = y_1 - \tilde{\xi}$ are the normal MLE's of the parameters $\psi_x = \psi + u_0$ and ξ when the restrictions $\rho = 1$ and $\theta = 0$ are imposed. Following the terminology in Schmidt and Phillips, we will refer to the $\tilde{u}_t$ as BSP residuals, and to their unit root tests as BSP tests.

The main contribution of this thesis is to consider tests based on generalized least squares (GLS) residuals from (1A). For the case of unit root testing, GLS would be based on an assumed value of ρ, say ρ_*, against which we wish to maximize power. The case of testing the null of stationarity is similar, except that GLS is based on an assumed value of θ, say θ_*. Tests based on the GLS residuals are closely related to the point optimal invariant (hereafter POI) tests proposed by King (1980) and developed in his later work (King and Hillier (1985), King (1988), and Dufour and King (1991)). King (1988) defines a point optimal test as a test that optimizes power at a predetermined point under the alternative hypothesis, and develops a theory of point optimal tests as a second best in cases in which a uniformly most powerful test does not exist. The theory of point optimal testing ensures that the test is most powerful among the set of invariant tests at a predetermined point in the alternative parameter space but one hopes that it also may have better power than other tests in a neighborhood of that point. In addition, point optimal tests can be used to find the power envelope for a given testing problem, which will be a benchmark for other tests.

Chapter 2 considers the problem of testing the null hypothesis of a unit root. Thus in equation (1B) we impose $\theta = 0$ and we wish to test the null hypothesis $\rho = 1$ against the alternative $\rho < 1$. Given a set of residuals, say $\hat{u}_t$, we will consider tests based on the artificial regression

$$\Delta\hat{u}_t = \phi\hat{u}_{t-1} + \text{error}\ , \quad t = 2,\ldots,T. \tag{4}$$

Let $\hat{\phi}$ be the OLS estimate of ϕ in (4). We will consider coefficient-based tests of the form $T\hat{\phi}$, and also tests based on the t-statistic for the hypothesis $\phi = 0$. These can be regarded as variants of the Dickey-Fuller tests. Specifically, if the $\hat{u}_t$ are OLS residuals from (1A) and $\hat{\phi}$ is the OLS estimate from (4), then the DF statistic $\hat{\rho}_\tau$ equals $T\hat{\phi}$ and the DF statistic $\hat{\tau}_\tau$ is the t-statistic for $\phi = 0$ in equation (4). The BSP tests are also of this general form. Consider the equivalent of equation (4), using $\tilde{u}_t$ in place of $\hat{u}_t$:

$$\Delta\tilde{u}_t = \phi\tilde{u}_{t-1} + \text{error}, \quad t = 2,\ldots,T, \tag{5}$$

and let $\bar{\phi}$ be the OLS estimate of ϕ in (5). Then Schmidt and Lee (1991) and Schmidt and Phillips (1992) consider the statistics $\bar{\rho} = T\bar{\phi}$ and $\bar{\tau}$ = t-statistic for the hypothesis $\phi = 0$. In the absence of corrections for autocorrelation, $\bar{\rho}$ and $\bar{\tau}$ are equivalent to each other and to Bhargava's statistic N_2.

From this perspective, the Dickey-Fuller tests and the BSP tests are of exactly the same form, except that $\hat{u}_t$ is used in Dickey-Fuller tests while $\tilde{u}_t$ is used in BSP tests. Both $\tilde{u}_t$ and $\hat{u}_t$ are residuals from the levels equation (1), but $\tilde{u}_t$ is based on parameters estimated using differences (i.e., GLS estimates under the null that $\rho = 1$) whereas $\hat{u}_t$ is based on the parameters estimated using levels. Since the regression in levels is spurious under the null, in the sense of Granger and Newbold (1974) and Phillips (1986), we might expect BSP tests to be more powerful than Dickey-Fuller tests against alternatives near the null. Conversely, we might expect the Dickey-Fuller tests to be more powerful than the BSP tests against alternatives far from the null. In fact, this pattern is exactly what Schmidt and Phillips (1992) and Schmidt and Lee (1991) find in their Monte Carlo experiments. This seems to be a dilemma from a practical point of view. However, we may ask a more fundamental question here; is there any other test which can dominate Dickey-Fuller and BSP tests?

In order to answer this question we consider test statistics based on the GLS residuals from (1A), where GLS is based on an assumed value of ρ, say ρ_*, against which we wish to maximize power. The Dickey-Fuller tests and BSP tests correspond to $\rho_* = 0$ and $\rho_* = 1$, respectively. In fact, a value like $\rho_* = 0.85$ might be reasonable in annual data, and the resulting tests might be expected to have better power than Dickey-Fuller and BSP tests not only against the specific alternative $\rho = \rho_*$, but also against alternatives in a (hopefully large) neighborhood of ρ_*. Dufour and King (1991) derive the point optimal invariant (POI) test of the hypothesis $\rho = \rho_0$ against the alternative $\rho = \rho_*$, so that the unit root case corresponds to $\rho_0 = 1$. Its calculation compares the unexplained sums of squares in GLS regressions based on ρ_0 and ρ_*, so that the POI unit root test statistic is also a function of GLS residuals.

In chapter 2 we present six unit root tests. We discuss coefficient-based and t-statistic tests based on GLS detrending, and a Dufour-King type POI test. However, there are two versions of each of these tests, depending on whether the alternative is taken to be a stationary AR(1) process or a particular type of nonstationary AR(1) process. This distinction occurs because we consider two of the several possible ways of treating the initial observation. According to our DGP as expressed in equation (1B), the initial "observation" u_1 is generated as

(6) $$u_1 = \rho u_0 + \epsilon_1.$$

We consider two different assumptions about u_0. First, we consider the case that u_0 is fixed. In this case the distribution of u_t is nonstationary, and the error covariance matrix used in GLS estimation is given in equation (7) of chapter 2. For a given value of ρ_*, we obtain GLS residuals which we denote by $\tilde{u}_{(N)t}(\rho_*)$; GLS-based tests $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$; and a Dufour-King type POI test $DK_N(\rho_*)$. Second, we consider the case that u_0 is random, with mean zero and variance $\sigma^2/(1-\rho^2)$. In this case the distribution of u_t is covariance stationary, and the error covariance matrix used in GLS estimation is given in equation (9) in chapter 2. For a given value of ρ_*, we obtain GLS residuals $\tilde{u}_{(s)t}(\rho_*)$; GLS-based tests $\tilde{\rho}_s(\rho_*)$ and $\tilde{\tau}_s(\rho_*)$; and a POI test $DK_s(\rho_*)$. The limits of these tests as $\rho_* \to 1$ are well defined.

In chapter 2, we derive the asymptotic distributions of these test statistics, and we show how to construct asymptotically valid tests in the presence of error autocorrelation. We tabulate critical values for our tests, and we investigate their power in a set of Monte Carlo experiments. Specifically, the value of ρ_* used in GLS detrending affects the size and power of the tests asymptotically and in finite samples. Let ρ_1 denote the true value of ρ in the DGP. Then power depends on T, ρ_*, ρ_1, and the treatment of the initial observation. We perform extensive Monte Carlo experiments to investigate the power of the tests as a function of these parameters. The GLS-based tests offer a clear gain in power relative to the Dickey-Fuller and BSP tests over an empirically relevant range of the parameter space. Their power is comparable to that of the POI test.

In chapter 3 we consider the problem of testing the null hypothesis of trend stationarity. Thus in equation (1B) we impose $\rho = 1$ and we wish to test the null hypothesis $\theta = 1$ against the alternative $\theta < 1$. Thus we are testing for a unit root in the moving-average representation of Δu_t (i.e., overdifferencing). Alternatively and equivalently, we can follow KPSS in expressing u_t in terms of a components representation:

(7) $$u_t = r_t + \epsilon_t, \quad r_t = r_{t-1} + v_t, \quad t = 1,\ldots,T,$$

where ϵ_t are iid(0, σ_ϵ^2) errors and v_t are iid(0, σ_v^2). Here λ ($\equiv \sigma_v^2/\sigma_\epsilon^2$, ≥ 0) is the signal to noise ratio, which measures the ratio of the changes in permanent versus transitory components (Shepard and Harvey (1990)). The signal to noise ratio λ is related to the moving average parameter θ in the following way:

(8) $$\theta = \{(\lambda + 2) - [\lambda(\lambda + 4)]^{1/2}\}/2, \quad \lambda = (\theta - 1)^2/\theta.$$

(9) $$\sigma_v^2 = \lambda\sigma_\epsilon^2.$$

Thus the null hypothesis of trend stationarity corresponds to $\lambda = 0$ (or $\sigma_v^2 = 0$ or $\theta = 1$) and the alternative hypothesis of difference stationarity corresponds to $\lambda > 0$ (or $\sigma_v^2 > 0$ or $\theta < 1$).

In this context, the one-sided LM test can be derived under the stronger assumption that the ϵ_t are iid N(0, σ_ϵ^2) and the v_t are iid N(0, σ_v^2). Let $\hat{e}_t$, t = 1,...,T, be the OLS residuals from the regression of y on intercept and trend; they correspond to $\hat{u}_t$ above. Define $\hat{\sigma}_\epsilon^2$ and $\hat{S}_t$ to be the estimate of the error variance from this regression and the partial sum process of the residuals, respectively:

(10) $$\hat{\sigma}_\epsilon^2 = T^{-1}\sum_{t=1}^{T}\hat{e}_t^2,$$

(11) $$\hat{S}_t = \sum_{j=1}^{t}\hat{e}_j, \quad t = 1,\ldots,T.$$

Then the LM statistic is given as follows:

(12) $$LM = \sum_{t=1}^{T}\hat{S}_t^2 \,/\, \hat{\sigma}_\epsilon^2.$$

KPSS (1992) consider the asymptotic distribution of the LM statistic under the null hypothesis with weaker assumptions about the errors. They modify the LM statistic to allow for autocorrelation in ϵ_t by replacing the denominator $\hat{\sigma}_\epsilon^2$ with a consistent estimate of the long run variance of ϵ_t. Define the estimated autocovariances $\hat{\gamma}(j) = T^{-1}\sum_{t=j+1}^{T}\hat{e}_t\hat{e}_{t-j}$, $j = 0, 1, \ldots, T-1$, and the long run variance estimator $\hat{\sigma}^2(\ell) = \hat{\gamma}(0) + 2\sum_{s=1}^{\ell} w(s,\ell)\hat{\gamma}(s)$. Here $w(s,\ell)$ is an optional weighting function, such as the Bartlett-window $w(s,\ell) = 1-s/(\ell+1)$, and ℓ is the number of lags used to estimate σ^2, satisfying $\ell \to \infty$ but $\ell/T \to 0$ as $T \to \infty$. Then the KPSS statistic is

$$\hat{\eta}_\tau = T^{-2}\sum_{t=1}^{T}\hat{S}_t^2 \;/\; \hat{\sigma}^2(\ell). \tag{13}$$

In chapter 3 we modify the KPSS statistic by basing it on GLS residuals instead of OLS residuals. GLS is based on an assumed value $\theta_* < 1$ in the MA representation (1B), or equivalently, on an assumed value $\lambda_* > 0$ in the components representation (7). A given value of θ_* implies the covariance matrix $\Omega_N(\theta_*)$ given by equation (10) in chapter 3, and a set of GLS residuals $\tilde{e}_t(\theta_*)$. Let $\tilde{S}_t(\theta_*)$ be the partial sum process of this residual process. Let $\tilde{\sigma}(\ell)^2$ be an estimator of the long run variance defined in the same way as $\hat{\sigma}(\ell)^2$ above except that $\tilde{e}_t(\theta_*)$ replaces $\hat{e}_t$. Then the GLS-based KPSS test can be defined as an upper tail test based on the statistic

$$\tilde{\eta}_\tau(\theta_*) = T^{-2}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \;/\; \tilde{\sigma}(\ell)^2. \tag{14}$$

Thus $\tilde{S}_t(\theta_*)$ and $\tilde{\sigma}(\ell)^2$ are used in the KPSS statistic instead of

$\hat{S}_t$ and $\hat{\sigma}(\ell)^2$.

We also consider the POI test of the stationarity hypothesis. Thus we consider the problem of testing the null $\theta = 1$ against the specific alternative $\theta = \theta_* < 1$. The POI test is a lower tail test based on the statistic $P_\tau(\theta_*)$, defined as the ratio of quadratic forms in GLS residuals:

$$(15) \qquad P_\tau(\theta_*) = \tilde{e}(\theta_*)'\Omega_N^{-1}(\theta_*)\tilde{e}(\theta_*) \ / \ \tilde{e}(1)'\Omega_N^{-1}(1)\tilde{e}(1),$$

where $\tilde{e}(\theta_*)$ and $\tilde{e}(1)$ are GLS residual vectors from (1A) under the alternative $\theta = \theta_*$ and under the null $\theta = 1$, respectively.

In chapter 3, we derive the asymptotic distributions of the GLS-based KPSS test and the POI test under the stationary null and under the unit root alternative. The GLS-based KPSS turns out to be inconsistent against unit root alternatives, so we do not expect it to have good power properties in finite samples. We tabulate critical values for our tests, and we investigate their power in a set of Monte Carlo experiments. As expected, the GLS-based KPSS test is not very powerful. However, the POI test based on a reasonable value for θ_* is considerably more powerful than the (standard, OLS-based) KPSS test over a wide range of θ. Thus for this problem, as for the unit root test problems, the POI approach offers the promise of substantial gains in power over other standard tests.

Finally, chapter 4 contains some concluding remarks.

CHAPTER 2

CHAPTER 2

ALTERNATIVE METHODS OF DETRENDING AND THE POWER OF UNIT ROOT TESTS

1. INTRODUCTION

The purpose of this chapter is to provide new tests of the null hypothesis of a unit root against the alternative of trend stationarity. These tests are based upon detrending the series by a generalized least squares (GLS) regression, using an empirically plausible value of the autoregressive root. These tests are related to the unit root tests of Bhargava (1986), Schmidt and Phillips (1992) and Schmidt and Lee (1991), and also to the point optimal tests of Dufour and King (1991). Elliott, Rothenberg and Stock (1992), in work done independently of ours, have recently proposed essentially the same tests.

Following Dickey (1984), Bhargava (1986), Schmidt and Phillips (1992) and others, we consider the data generating process (DGP) to be of the form:

(1) $y_t = \psi + \xi t + u_t, \quad u_t = \rho u_{t-1} + \epsilon_t, \quad t = 1,\ldots,T,$

where $\epsilon_t \sim NID(0, \sigma^2)$. In matrix form,

(1') $y = Z\gamma + u,$

where Z is a matrix of dimension T×2 with t^{th} row $z_t = [1,t]$, $\gamma' = [\psi,\xi]$, and u is a T×1 vector of realizations of the error process. The null hypothesis of a unit root corresponds to $\rho = 1$, and the alternative hypothesis to be considered in this chapter corresponds to $\rho \in [0,1)$. This parameterization is useful because it allows for linear deterministic trend under the null and alternative hypotheses, with the interpretation of the parameters ψ (level) and ξ (trend) being the same whether the null hypothesis holds or not. In addition, the distributions of most common unit root tests, and of all of the tests considered in this chapter, are independent of the nuisance parameters ψ, ξ, and σ under both the null and the alternative hypotheses.

In this chapter we will consider tests based on various types of residuals (OLS and GLS) from equation (1). Given a set of residuals, say $\hat{u}_t$, we will consider tests based on the artificial regression

(2) $$\Delta\hat{u}_t = \phi\hat{u}_{t-1} + \text{error} , \quad t = 2,\ldots,T.$$

Let $\hat{\phi}$ be the OLS estimate of ϕ in (2). We will consider coefficient-based tests of the form $T\hat{\phi}$, and also tests based on the t-statistic for the hypothesis $\phi = 0$. These can be regarded as variants of the Dickey-Fuller tests. Specifically, if the $\hat{u}_t$ are OLS residuals from (1) and $\hat{\phi}$ is the OLS estimate from (2), then the Dickey-Fuller statistic $\hat{\rho}_\tau$ equals $T\hat{\phi}$ and the Dickey-Fuller statistic $\hat{\tau}_\tau$ is the t-statistic for $\phi = 0$ in equation (2).

Bhargava (1986), Schmidt and Phillips (1992) and Schmidt and Lee (1991) consider tests based on detrending in differences. That is, their tests are based on the residuals

(3) $$\tilde{u}_t = y_t - \tilde{\psi}_x - \tilde{\xi}t = [(T-1)y_t - (t-1)y_T - (T-t)y_1]/(T-1),$$

where $\tilde{\xi} = \overline{\Delta y} = (y_T-y_1)/(T-1)$ and $\tilde{\psi}_x = y_1 - \tilde{\xi}$ are the normal

MLE's of the parameters $\psi_x = \psi + X_0$ and ξ when the restriction $\rho = 1$ is imposed. (Following the terminology in Schmidt and Phillips, we will refer to tests based on $\tilde{u}_t$ as BSP tests. Note that our $\tilde{u}_t$ is Schmidt and Phillips' $\tilde{S}_t$.) Consider the equivalent of equation (2), using $\tilde{u}_t$ in place of $\hat{u}_t$:

(4) $$\Delta\tilde{u}_t = \phi\tilde{u}_{t-1} + \text{error}, \quad t = 2,\ldots,T,$$

and let $\bar{\phi}$ be the OLS estimate of ϕ in (4). Then Schmidt and Lee (1991) and Schmidt and Phillips (1992) consider the statistics $\bar{\rho} = T\bar{\phi}$ and $\bar{\tau}$ = t-statistic for the hypothesis $\phi = 0$. In the absence of corrections for autocorrelation, $\bar{\rho}$ and $\bar{\tau}$ are equivalent to each other and to Bhargava's statistic N_2. In this chapter we will not consider the statistics $\tilde{\rho}$ and $\tilde{\tau}$ of Schmidt and Phillips (1992), or the closely related R_2 statistic of Bhargava (1986), which are based on an artificial regression like (4) above but with an intercept.

From this perspective, the Dickey-Fuller tests and the BSP tests are of exactly the same form, except that $\hat{u}_t$ is used in Dickey-Fuller tests while $\tilde{u}_t$ is used in BSP tests. Both $\tilde{u}_t$ and $\hat{u}_t$ are residuals from the levels equation (1), but $\tilde{u}_t$ is based on parameters estimated using differences (i.e., GLS estimates under the null that $\rho = 1$) whereas $\hat{u}_t$ is based on the parameters estimated using levels. Since the regression in levels is spurious under the null, in the sense of Granger and Newbold (1974) and Phillips (1986), we might expect BSP tests to be more powerful than Dickey-Fuller tests against alternatives near the null. Conversely, we might expect the Dickey-Fuller tests to be more powerful than the BSP tests against alternatives far from the null. In fact, this pattern is exactly what Schmidt and Phillips (1992) and Schmidt and Lee (1991) find in their Monte Carlo experiments.

In this chapter we construct test statistics based on the GLS residual from (1), where GLS is based on an assumed value of ρ, say ρ_*, against which we wish to maximize power. The Dickey-Fuller tests and BSP tests correspond to $\rho_* = 0$ and $\rho_* = 1$, respectively. In fact, a value like $\rho_* = 0.85$ might be reasonable in annual data, and the resulting tests might be expected to have better power than Dickey-Fuller and BSP tests not only against the specific alternative $\rho = \rho_*$, but also against alternatives in a (hopefully large) neighborhood of ρ_*.

This idea dates back at least to King (1980) and has been developed in his later work (King and Hillier (1985), King (1988), and Dufour and King (1991)). King (1988) defines a point optimal test as a test that optimizes power at a predetermined point under the alternative hypothesis, and develops a theory of point optimal tests as a second best in cases in which a UMP test does not exist. Dufour and King (1991) derive the point optimal invariant (POI) test of the hypothesis $\rho = \rho_0$ against the alternative $\rho = \rho_*$, so that the unit root case corresponds to $\rho_0 = 1$. Its calculation compares the unexplained sums of squares in GLS regressions based on ρ_0 and ρ_*, so that the POI unit root test statistic is also a function of GLS residuals. The Dufour-King POI test is based on a specific assumption about the generation of the initial value of the series, and it is not guaranteed to be point optimal under some initializations that we consider. Nevertheless, as we shall see, the POI test and the Dickey-Fuller type tests based on GLS residuals are not very different.

The value of ρ_* used in GLS detrending affects the size and power of the tests asymptotically and in finite samples. Let ρ_1 denote the true value of ρ in the DGP. Then power depends on T, ρ_*, ρ_1, and the treatment of the initial observation. We perform extensive Monte Carlo experiments to investigate the power of the tests as a function of these parameters. The new tests offer a clear gain in power relative to the Dickey-Fuller and BSP tests over an empirically relevant range of the parameter space. Their power is comparable to that of the POI test.

2. UNIT ROOT TESTS AGAINST STATIONARY AND NONSTATIONARY AR(1) PROCESSES: NEW TESTS AND POI TESTS

In this section we present six unit root tests. We discuss coefficient-based and t-statistic tests based on GLS detrending, and a Dufour-King type test. However, there are two versions of each of these tests, depending on whether the alternative is taken to be a stationary AR(1) process or a particular type of nonstationary AR(1) process. This distinction occurs because we consider two of the several possible ways of treating the initial observation.

According to our DGP given in equation (1), the initial "observation" u_1 is generated as

(5) $$u_1 = \rho u_0 + \epsilon_1 \quad .$$

We consider two different assumptions about u_0. First, we consider the case that u_0 is fixed. In this case the distribution of u_t is nonstationary. Second, we consider the case that u_0 is random, with mean zero and variance $\sigma^2/(1-\rho^2)$. In this case the distribution of u_t is covariance stationary. Neither of these assumptions generally corresponds to the Dufour-King treatment of the initial observation. They assume

(6) $$u_1 = d_1\epsilon_1$$

for some constant d_1. This is different from either of our assumptions, except in two special cases to be discussed below. We note in passing that Elliott, Rothenberg and Stock (1992) focus on asymptotics and therefore do not discuss the treatment of the initial observation in detail. However, even though it will not matter asymptotically, the treatment of the initial observation can be important in finite samples.

Consider first the case that u_0 is assumed to be fixed, so that u_t is a nonstationary AR(1) process. Then the covariance matrix of the T×1 vector u is $\sigma^2\Omega_N(\rho)$, where $\Omega_N(\rho)$ and its inverse are as follows:

(7) $$\Omega_N(\rho) = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1+\rho^2 & (1+\rho^2)\rho & \cdots & (1+\rho^2)\rho^{T-2} \\ \rho^2 & (1+\rho^2)\rho & 1+\rho^2+\rho^4 & \cdots & (1+\rho^2+\rho^4)\rho^{T-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho^{T-1} & \cdot & \cdot & \cdots & 1+\rho^2+\cdots+\rho^{2(T-1)} \end{bmatrix}.$$

$$(8) \quad \Omega_N^{-1}(\rho) = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}.$$

We may note that our $\Omega_N(\rho)$ is the same as Dufour and King's $\Omega(\rho,1)$, as defined on p. 123 of their article. This correspondence occurs because our DGP with $u_0 = 0$ is the same as the Dufour-King DGP with $d_1 = 1$; in each case $u_1 = \epsilon_1$. When $u_0 \neq 0$, our model is not the same as the Dufour-King model, even though the covariance matrix of u is the same under both models. All of the tests in this chapter have distributions that are invariant to the value of u_0 under the null hypothesis, but power depends on u_0, and the Dufour-King POI test has no known optimality properties for $u_0 \neq 0$.

Next consider the case that u_0 is assumed to be random, with mean zero and variance $\sigma^2/(1-\rho^2)$, so that u_t is a covariance stationary AR(1) process. Then the covariance matrix of the vector u is $\sigma^2\Omega_s(\rho)$, where $\Omega_s(\rho)$ and its inverse are as follows:

$$(9) \quad \Omega_s(\rho) = (1-\rho^2)^{-1} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho^{T-1} & \cdot & \cdot & \cdots & 1 \end{bmatrix}$$

$$(10)\quad \Omega_s^{-1}(\rho) \quad = \quad \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}.$$

We may note that our model with random u_0 is the same as the Dufour-King model with $d_1 = (1-\rho^2)^{-1/2}$.

We can now define our GLS-based tests. For a given ρ_* in the interval [0,1), let $\tilde{u}_{(s)t}(\rho_*)$, t = 1,...,T, be the residuals from the GLS regression of y_t on [1,t], using the (assumed) error covariance matrix $\Omega_s(\rho_*)$, and consider the regression

(11) $\Delta\tilde{u}_{(s)t}(\rho_*) = \phi\tilde{u}_{(s)t-1}(\rho_*)$ + error, t = 2,...,T,

similarly to equations (2) and (4) above. Define $\tilde{\phi}_s(\rho_*)$ as the OLS estimate of ϕ in this regression:

(12) $\tilde{\phi}_s(\rho_*) = \Sigma_{t=2}^T \Delta\tilde{u}_{(s)t}(\rho_*)\tilde{u}_{(s)t-1}(\rho_*)/\Sigma_{t=2}^T \tilde{u}_{(s)t-1}(\rho_*)^2$.

Then we define $\tilde{\rho}_s(\rho_*) = T\tilde{\phi}_s(\rho_*)$, and $\tilde{\tau}_s(\rho_*)$ = usual t-statistic for the hypothesis $\phi = 0$ in (11).

The tests $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$ are defined in exactly the same way, except that we use the residuals $\tilde{u}_{(N)t}(\rho_*)$ from the GLS regression of y_t on [1,t], using the (assumed) error covariance matrix $\Omega_N(\rho_*)$.

When $\rho_* = 0$, the GLS residuals $\tilde{u}_{(s)t}(0)$ and $\tilde{u}_{(N)t}(0)$ become the OLS residuals $\hat{u}_t$, and correspondingly our tests become the Dickey-Fuller tests: $\tilde{\rho}_s(0) = \tilde{\rho}_N(0) = \hat{\rho}_\tau$, and $\tilde{\tau}_s(0) = \tilde{\tau}_N(0) = \hat{\tau}_\tau$. Similarly, when $\rho_* = 1$, the GLS residuals $\tilde{u}_{(s)t}(1)$ and

$\tilde{u}_{(N)t}(1)$ become the BSP residuals $\tilde{u}_t$, and our tests become the BSP tests $\bar{\rho}$ and $\bar{\tau}$. More precisely, $\tilde{\rho}_N(1) = \lim_{r\to 1} \tilde{\rho}_s(r) = \bar{\rho}$ and $\tilde{\tau}_N(1) = \lim_{r\to 1} \tilde{\tau}_s(r) = \bar{\tau}$; the limits are taken in the stationary case because $\Omega_s(1)$ is singular. The mathematical details for $\rho_* = 1$ are given in Appendix 1.

The relationship between these tests and the Dufour-King POI test is slightly more complicated. We consider the statistic $S_2(1,\rho_*,d_1^*)$ as given by Dufour and King (Theorem 5, p. 127). Here d_1^* is an assumed value of d_1 in equation (6) above, and the statistic equals the ratio of quadratic forms in GLS residuals, using the covariance matrices $\Omega(\rho_*,d_1^*)$ and $\Omega(1,1)$ as defined on their p. 123. In order to make their treatment as comparable to ours as possible, we consider only the case that $d_1^* = 1$; as noted above, their model with $d_1 = 1$ corresponds to our nonstationary case with $u_0 = 0$. Then their $\Omega^{-1}(\rho_*,1)$ = our $\Omega_N^{-1}(\rho_*)$ and their $\Omega^{-1}(1,1)$ = our $\Omega_N^{-1}(1)$, where our notation $\Omega_N^{-1}(\cdot)$ is defined in equation (8) above. Thus we obtain their statistic in our notation as

(13) $$DK_N(\rho_*) = S_2(1,\rho_*,1) = \tilde{u}_{(N)}(\rho_*)'\Omega_N^{-1}(\rho_*)\tilde{u}_{(N)}(\rho_*)/\tilde{u}'\Omega_N^{-1}(1)\tilde{u},$$

where as before $\tilde{u} = \tilde{u}_{(N)}(1)$ are the BSP residuals defined in equation (3) above. The denominator of DK_N is proportional to the numerator of the $\bar{\rho}$ statistic. The numerator of DK_N does not have any clear relationship to tests of the type we propose in this paper, though its asymptotic distribution is proportional to the asymptotic distribution of the denominator of $\tilde{\rho}_N(\rho_*)$.

In their development of the POI test of the unit root hypothesis, Dufour and King consider only nonstationary AR(1) alternatives. The reason is that stationarity requires $d_1 = (1-\rho^2)^{-1/2}$ and cannot accommodate the null of $\rho = 1$. However, there is no reason that we should want to rule out the stationary AR(1) process as a plausible alternative hypothesis. It is sensible to consider the Dufour-King POI statistic for testing the null hypothesis that $\rho = \rho_0$ against the stationary alternative with $\rho = \rho_*$, and then to take the limit of this statistic as $\rho_0 \to 1$. By the same algebra as in Appendix 1, this yields the statistic

(14) $$DK_s(\rho_*) = \tilde{u}_{(s)}(\rho_*)'\Omega_s^{-1}(\rho_*)\tilde{u}_{(s)}(\rho_*)/\tilde{u}'\Omega_s^{-1}(1)\tilde{u} .$$

We may note that the matrix $\Omega_s^{-1}(1)$ is singular, but nevertheless well defined. In fact, the denominator of DK_s is exactly the same as the denominator of DK_N, because the only difference between $\Omega_s^{-1}(1)$ and $\Omega_N^{-1}(1)$ is in their (1,1) elements, and $\tilde{u}_1 = 0$.

3. DISTRIBUTION THEORY

In the previous section we considered three tests [$\tilde{\rho}_N(\rho_*)$, $\tilde{\tau}_N(\rho_*)$ and $DK_N(\rho_*)$] designed to be powerful against nonstationary AR(1) alternatives, and three tests [$\tilde{\rho}_s(\rho_*)$, $\tilde{\tau}_s(\rho_*)$ and $DK_s(\rho_*)$] designed to be powerful against stationary alternatives. In this section we discuss their distributional properties under the unit root null and under stationary and nonstationary alternatives.

The above six test statistics are all based on GLS residuals from the regression of y_t on [1,t], using different covariance matrices. It is easy to show, along the same lines as Schmidt and Phillips (1992, pp. 262-263), that under the null hypothesis of a unit root the residuals $\tilde{u}_t$, $\tilde{u}_{(N)t}(\rho_*)$ and $\tilde{u}_{(s)t}(\rho_*)$ are independent of the nuisance parameters ψ and ξ, and also of the initial value u_0 in the case that u_0 is fixed. Furthermore all six statistics are independent of the error variance σ^2, because it scales the numerator and the denominator of each statistic in the same way. Thus, under the null hypothesis, the distributions of the six statistics are independent of ψ, ξ, u_0 and σ^2. They obviously depend on ρ_* and the sample size T.

Under the alternative hypothesis, the distributions of the statistics do not depend on ψ, ξ and σ^2. They depend on ρ_*, T, and the true value of ρ, say ρ_1. In the case that u_0 is fixed, they also depend on u_0/σ.

All of the statements of the last two paragraphs are true for most common unit root tests, such as the Dickey-Fuller and BSP tests, as well for the tests discussed in this chapter.

We next consider the asymptotic distributions of our GLS-based tests as $T \to \infty$ with ρ_* fixed, under standard assumptions about the errors ϵ_t. Specifically, we assume the regularity conditions of Phillips and Perron (1988, p. 336), though other similar sets of conditions would yield the same results. Interestingly, the asymptotic distributions of the statistics $\tilde{\rho}_s(\rho_*)$, $\tilde{\tau}_s(\rho_*)$, $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$ do not depend on ρ_*, for any value of ρ_* in the interval $[0,1)$. Specifically, as we prove in Appendix 2, the asymptotic distributions of these statistics for any value of $\rho_* < 1$ are the same as for $\rho_* = 0$. That is, using any value of $\rho_* < 1$, the new tests have asymptotically the same distributions as the Dickey-Fuller test statistics $\hat{\rho}_\tau$ and $\hat{\tau}_\tau$. From this perspective, there is a discontinuity in the asymptotic distribution theory at $\rho_* = 1$, since choosing $\rho_* = 1$ yields the BSP statistics $\bar{\rho}$ and $\bar{\tau}$, which do not have the same asymptotic distributions as the Dickey-Fuller statistics.

One important implication of these results is that we can modify the $\tilde{\rho}_S(\rho_*)$, $\tilde{\tau}_S(\rho_*)$, $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$ statistics to allow for error autocorrelation in exactly the same ways as are currently done for the Dickey-Fuller tests. We can create augmented versions of these tests along the same lines as in Said and Dickey (1984), by adding lagged values of $\Delta\tilde{u}_{(S)t}(\rho_*)$ or $\Delta\tilde{u}_{(N)t}(\rho_*)$ to the regression that yields the test statistics, where the number of lagged values grows at a suitable rate with sample size. Alternatively, the corrections of Phillips and Perron (1988), based on consistent estimates of the innovation variance σ^2 and the long run variance ω^2, also lead to an asymptotically valid test.

The asymptotic distributions of the Dufour-King POI tests $DK_N(\rho_*)$ and $DK_S(\rho_*)$ are derived in Appendix 3. The two statistics have the same asymptotic distribution, which is given by $(\omega^2/\sigma^2)(1-\rho_*)^2$ times a functional of Brownian motion. Thus, in contrast to the GLS-based tests just discussed, the asymptotic distribution depends on ρ_*. Furthermore, to correct for error autocorrelation we need simply to multiply the statistic by a consistent estimate of (σ^2/ω^2). This is a correction of the same general type as in Phillips and Perron (1988), but the fact that the statistic is simply scaled by the ratio of nuisance parameters is very similar to the results in Schmidt and Phillips (1992). There is no obvious analogy to the augmented versions of the previous tests. This is a potential disadvantage of the POI tests, since in previous Monte Carlo studies of the Dickey-Fuller and BSP tests, the augmented versions have typically had smaller size distortions than the Phillips-Perron corrected versions.

We repeat that our asymptotics are done as $T \to \infty$ for fixed ρ_*. This is standard and perhaps natural, but it is not the only possibility. Elliott, Rothenberg and Stock (1992) consider asymptotics for the same statistics as $T \to \infty$ but with $\rho_* = 1-c_*/T$, for fixed c_*. Therefore they obtain different asymptotic distributions than we do. In particular, the asymptotic distributions of all of the test statistics then depend on c_*. Furthermore, the corrections that make the statistics asymptotically valid in the presence of error autocorrelation are also different under their type of asymptotics than under ours. Which type of asymptotic analysis leads to tests with better finite sample performance in the presence of error autocorrelation is an important topic for further research.

Despite our asymptotic results, for values of ρ_* close to one we would not expect the critical values for the Dickey-Fuller statistics to be very accurate for our GLS-based tests, for empirically relevant sample sizes. Therefore the finite sample distributions of the above six test statistics will be tabulated by a Monte Carlo simulation. Since the distributions of all of the test statistics under the null hypothesis depend only on the two parameters ρ_* and T, critical values can be tabulated through simulations using various values of these two parameters. We consider sample sizes T = 25, 50, 100, 200, and 500. We also consider values of ρ_* = 0.0, 0.5, 0.7, 0.8, 0.85, 0.9, 0.95, 0.99, and 1.0. The critical values are calculated by a direct simulation using 25,000 replications, with random normal deviates generated by the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986). Normality does not matter asymptotically, and from previous results for the Dickey-Fuller tests it seems unlikely to matter much here. The critical values are presented in Table 1.

The critical values in Table 1 look pretty much as one would expect. For our GLS-based tests [$\tilde{\rho}_s(\rho_*)$, $\tilde{\tau}_s(\rho_*)$, $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$], for each sample size and critical level, the critical values are monotonically increasing (i.e., monotonically decreasing in absolute value) as ρ_* increases from zero to one. This reflects a continuous movement from the Dickey-Fuller critical values toward the BSP critical values as ρ_* varies from zero to one. Furthermore, for each ρ_* between zero and one, as $T \to \infty$ the critical values should converge to the Dickey-Fuller asymptotic critical values. This convergence is apparent in Table 1, but it is relatively slow for ρ_* close to unity. This convergence of critical values as $T \rightarrow \infty$ is faster for the $\tilde{\rho}_s$ and $\tilde{\tau}_s$ tests than for the $\tilde{\rho}_N$ and $\tilde{\tau}_N$ tests. For ρ_* in the empirically relevant range between 0.8 and 0.99, use of the finite sample critical values instead of the asymptotic values will make a difference even for rather large sample sizes, such as T = 500.

For the Dufour-King POI tests DK_s and DK_N, for any T the critical values approach one as $\rho_* \rightarrow 1$. For a given value of ρ_*, the critical values are not very sensitive to T, except when ρ_* is small. When ρ_* is small, the critical values are roughly proportional to sample size, as we would expect from the asymptotic results in Appendix 3. The fact that this is true only for small values of ρ_* casts doubt on the accuracy of the asymptotic results for large values of ρ_*, for reasonable sample sizes, and suggests that it will be important to use the finite sample critical values.

4. SIMULATION RESULTS

In this section we consider the powers of the six tests described above. We will consider both the nonstationary case in which u_0 is fixed and the stationary case in which u_0 is drawn from the stationary distribution of u_t. As before, let ρ_* represent the value of ρ used in GLS detrending, and ρ_1 represent the true value of ρ in the DGP. Then the powers of the tests are independent of the parameters ψ, ξ and σ^2, but they depend on T, ρ_* and ρ_1. When u_0 is fixed, the powers also depend on u_0/σ. Without loss of generality we set $\psi = \xi = 0$

and $\sigma^2 = 1$. We consider sample sizes T = 25, 50, 100, 200 and 500; values of $\rho_* = 0.0, 0.5, 0.7, 0.8, 0.85, 0.9, 0.95, 0.99$ and 1.0; and values of $\rho_1 = 0.0, 0.5, 0.7, 0.8, 0.85, 0.9,$ 0.95 and 0.99. For experiments in which u_0 is fixed, we consider $u_0 = 0, -1, -2, -5$ and -10. (Because the distribution of our errors is symmetric, power depends only on $|u_0|$ and we do not need to consider positive values of u_0.)

We consider only 5% lower tail tests, and we consider only the case of iid errors ϵ_t. Power is calculated using a Monte Carlo simulation with 25,000 replications, and with normal random deviates generated as described in the previous section. We use the critical values presented in the previous section, so size should be exact apart from randomness; there are no size distortions to correct for, as there would be if we used the asymptotic critical values. We will present our experiments in three sets, according to what is assumed about the initial value u_0.

The first set of experiments corresponds to the case that u_0 is fixed at zero. The results for T = 50, 100 and 200 are given in Tables 2-4; the results for T = 25 and 500 are given in Tables 9 and 10.

Since our model with $u_0 = 0$ corresponds to Dufour and King's model with $d_1 = 1$, their POI test DK_N using $\rho_* = \rho_1$ should have maximum power against the specific alternative hypothesis $\rho = \rho_1$. Our simulation results generally support this expectation. That is, for each value (pair) of T and ρ_1, the $DK_N(\rho_*)$ test with $\rho_* = \rho_1$ generally has power at least as high as that of any of the other tests, apart from randomness. Exceptions to this statement are found mainly for small sample sizes (e.g., T = 25, 50), are only marginally larger than could be explained as randomness, and are not substantively significant. The gain to using a POI test can be substantial; for example, for T = 100 and ρ_1 = 0.85 (Table 3), compare the power of 0.580 for the POI test to 0.393 for $\hat{\tau}_\tau$ [i.e., $\tilde{\tau}_N(0)$ or $\tilde{\tau}_s(0)$] and 0.467 for $\hat{\rho}_\tau$ [i.e., $\tilde{\rho}_N(0)$ or $\tilde{\rho}_s(0)$], or to .526 and .524 for the BSP tests. Furthermore, these gains in power occur over an optimistically wide range of the parameter space. For example, the $DK_N(.85)$ test dominates the Dickey-Fuller tests for ρ_1 over at least the range from 0.7 to 0.95, and hence arguably over the empirically relevant range of ρ_1.

Our GLS-based tests $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$ are quite similar in performance to the POI test $DK_N(\rho_*)$. When $\rho_* = \rho_1$, they are generally slightly less powerful than the POI test. An interesting result is that, for a given value of ρ_1, the maximal power of our GLS-based tests is generally obtained at a value of ρ_* slightly larger than ρ_1. These values of maximal power are comparable to those of the POI test.

Finally, since the DGP in this set of experiments is nonstationary, we would expect the nonstationary variants of our tests (DK_N, $\tilde{\rho}_N$ and $\tilde{\tau}_N$) to be more powerful than their stationary counterparts (DK_s, $\tilde{\rho}_s$ and $\tilde{\tau}_s$). Our results generally support this expectation, though the differences in power are not large.

The second set of experiments considers a fixed <u>nonzero</u>

initial value u_0. Since this does not correspond to Dufour and King's setup, neither DK_S nor DK_N is a point optimal test in these experiments, though they may be approximately point optimal when u_0 is close to 0. Table 5 presents results for $T = 100$, $\rho_1 = 0.85$, and $u_0 = -1, -2, -5$ and -10. Some results for other values of ρ_1 are given in Tables 11-14. Only the absolute value of u_0 matters in this experiment because our errors have a symmetric (normal) distribution.

From Schmidt and Lee (1991) and Schmidt and Phillips (1992) it is known that the power functions of the Dickey-Fuller and BSP tests are monotonic in $|u_0|$, but in opposite directions; small $|u_0|$ favors the BSP tests while large $|u_0|$ favors the Dickey-Fuller $\hat{\tau}_\tau$ test. Our results in Table 5 show similar results for the tests proposed in this chapter. The power of the $\tilde{\rho}_S(\rho_*)$, $\tilde{\rho}_N(\rho_*)$, $DK_S(\rho_*)$ and $DK_N(\rho_*)$ tests decreases monotonically as $|u_0|$ increases, especially for large values of ρ_*. Their power becomes even less than nominal size under some alternatives. However, the power functions of the test statistics $\tilde{\tau}_N(\rho_*)$ and $\tilde{\tau}_S(\rho_*)$ depend on u_0 in more complicated ways. Power tends to increase with $|u_0|$ when ρ_* is close to zero and to decrease with $|u_0|$ when ρ_* is close to one, reflecting the differing behaviors of the Dickey-Fuller $\hat{\tau}_\tau$ test ($\rho_* = 0$) and the BSP $\bar{\tau}$ test ($\rho_* = 1$).

When $|u_0|$ is very large, for example $u_0 = -10$, all the tests have their maximum power at $\rho_* = 0$ and power decreases monotonically as ρ_* gets closer to one, so that the Dickey-Fuller tests $\hat{\tau}_\tau$ and $\hat{\rho}_\tau$ are most powerful. In fact, $\hat{\tau}_\tau$

dominates all of the other tests in every experiment with $u_0 = -10$.

The third set of experiments takes u_0 as random and drawn from the stationary distribution of u_t. Our results for T = 50, 100 and 200 are given in Tables 6-8.

The DGP for this set of experiments does not match the DGP assumed for Dufour and King's unit root test. Nevertheless, as argued in the previous section, the statisic $DK_s(\rho_*)$ should be expected to be most powerful against ρ_1 in the neighborhood of ρ_*. The results in Tables 6-8 generally support this expectation, although there is not much difference in power between $DK_s(\rho_*)$ and $DK_N(\rho_*)$. Also, the gain in power from using a POI test instead of the Dickey-Fuller tests is smaller than it was in the first set of experiments. For example, for T = 100 and $\rho_1 = 0.85$, the power of $DK_s(.85)$ is 0.509, compared to 0.411 and 0.468 for $\hat{\tau}_\tau$ and $\hat{\rho}_\tau$. Nevertheless, the POI test with a reasonable value of ρ_*, such as 0.85, still dominates the Dickey-Fuller tests over much or all of the empirically relevant range of ρ_1.

As in the previous experiments, our GLS-based tests are similar in performance to the POI test. Interestingly, despite the fact that the DGP for this set of experiments is a stationary AR(1) process, the $\tilde{\rho}_N$ and $\tilde{\tau}_N$ tests are generally more powerful than the $\tilde{\rho}_s$ and $\tilde{\tau}_s$ tests. The reason for this result is not clear. The loss in power from using the $\tilde{\rho}_N$ test rather than the DK_s test is generally negligible.

5. CONCLUDING REMARKS

We have proposed new unit root tests based on the residuals from a GLS regression of y_t on [1,t], using a value $\rho_* \in [0,1)$ against which maximal power is desired. These tests are constructed in the same way as the Dickey-Fuller tests and the tests of Bhargava (1986) and Schmidt and Phillips (1992), but they are based on detrending by GLS rather than in levels or differences. They are similar in spirit to the point optimal invariant test of Dufour and King (1991). The power of the tests depends on the true value of ρ (ρ_1), the value of ρ used in detrending (ρ_*), and the sample size (T). In finite samples power also depends on the way in which the initial observation is generated. Our results indicate that, for reasonable values of ρ_*, such as ρ_* in the range from 0.85 to 0.95, the new tests are more powerful than the Dickey-Fuller tests or the Bhargava-Schmidt-Phillips tests over the empirically relevant range of ρ_1. Furthermore, the new tests have power comparable to the power of Dufour-King's point optimal invariant tests. The new tests are perhaps easier to relate to existing tests than the point optimal invariant tests, and they can be modified to allow for error autocorrelation either by augmentation or by applying the corrections of Phillips and Perron (1988). Thus they would appear to be of practical importance.

TABLE 1a 1%, 5%, AND 10% CRITICAL VALUES OF $\tilde{\tau}_s(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	-4.53	-4.29	-4.11	-3.94	-3.84	-3.73	-3.58	-3.44	-3.37
	-3.74	-3.56	-3.42	-3.25	-3.18	-3.07	-2.91	-2.72	-2.70
	-3.36	-3.23	-3.10	-2.93	-2.84	-2.75	-2.58	-2.40	-2.38
50	-4.23	-4.11	-4.01	-3.95	-3.85	-3.77	-3.57	-3.33	-3.28
	-3.57	-3.51	-3.41	-3.33	-3.26	-3.15	-2.97	-2.74	-2.65
	-3.25	-3.19	-3.12	-3.03	-2.95	-2.84	-2.68	-2.43	-2.35
100	-4.10	-4.03	-4.04	-3.97	-3.91	-3.79	-3.62	-3.37	-3.20
	-3.48	-3.45	-3.41	-3.37	-3.32	-3.24	-3.07	-2.77	-2.63
	-3.19	-3.16	-3.13	-3.07	-3.03	-2.96	-2.81	-2.49	-2.34
200	-4.03	-4.00	-3.97	-3.93	-3.93	-3.86	-3.73	-3.44	-3.22
	-3.45	-3.43	-3.41	-3.39	-3.37	-3.33	-3.20	-2.87	-2.62
	-3.16	-3.14	-3.12	-3.10	-3.09	-3.05	-2.93	-2.61	-2.33
500	-4.03	-3.99	-3.96	-3.93	-3.93	-3.89	-3.82	-3.60	-3.15
	-3.45	-3.42	-3.42	-3.40	-3.39	-3.38	-3.32	-3.05	-2.61
	-3.15	-3.12	-3.13	-3.12	-3.11	-3.10	-3.04	-2.79	-2.33

TABLE 1b 1%, 5%, AND 10% CRITICAL VALUES OF $\tilde{\tau}_N(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	-4.53	-4.21	-3.93	-3.75	-3.66	-3.57	-3.46	-3.40	-3.37
	-3.74	-3.46	-3.23	-3.05	-2.97	-2.91	-2.79	-2.69	-2.70
	-3.36	-3.13	-2.89	-2.72	-2.65	-2.57	-2.45	-2.36	-2.38
50	-4.23	-4.03	-3.83	-3.70	-3.63	-3.55	-3.42	-3.28	-3.28
	-3.57	-3.45	-3.22	-3.08	-3.00	-2.92	-2.80	-2.67	-2.65
	-3.25	-3.13	-2.93	-2.77	-2.71	-2.61	-2.51	-2.36	-2.35
100	-4.10	-4.00	-3.89	-3.69	-3.63	-3.53	-3.42	-3.28	-3.20
	-3.48	-3.42	-3.28	-3.12	-3.05	-2.97	-2.85	-2.67	-2.63
	-3.19	-3.13	-3.00	-2.84	-2.75	-2.67	-2.57	-2.38	-2.34
200	-4.03	-3.98	-3.89	-3.75	-3.68	-3.58	-3.46	-3.31	-3.22
	-3.45	-3.41	-3.33	-3.20	-3.12	-3.02	-2.89	-2.74	-2.62
	-3.16	-3.12	-3.05	-2.93	-2.84	-2.73	-2.59	-2.46	-2.33
500	-4.03	-3.98	-3.92	-3.84	-3.75	-3.62	-3.47	-3.41	-3.15
	-3.45	-3.41	-3.38	-3.31	-3.23	-3.11	-2.95	-2.83	-2.61
	-3.15	-3.12	-3.10	-3.03	-2.96	-2.82	-2.66	-2.54	-2.33

TABLE 1c 1%, 5%, AND 10% CRITICAL VALUES OF $\tilde{\rho}_s(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	-22.65	-22.07	-21.12	-20.14	-19.53	-18.87	-17.89	-16.95	-16.52
	-18.04	-17.49	-16.74	-15.75	-15.24	-14.53	-13.48	-12.20	-12.00
	-15.66	-15.25	-14.61	-13.57	-13.01	-12.33	-11.21	-10.01	-9.87
50	-25.76	-25.43	-24.84	-24.32	-23.55	-22.83	-20.97	-18.80	-18.29
	-19.85	-19.67	-18.99	-18.55	-17.98	-17.06	-15.50	-13.52	-12.72
	-17.08	-16.74	-16.41	-15.78	-15.27	-14.37	-13.00	-10.95	-10.31
100	-27.54	-27.24	-27.52	-26.89	-26.42	-25.43	-23.54	-20.79	-18.92
	-20.81	-20.51	-20.50	-20.10	-19.79	-19.12	-17.49	-14.47	-13.22
	-17.68	-17.44	-17.42	-17.04	-16.74	-16.16	-14.83	-11.91	-10.57
200	-28.62	-28.14	-28.41	-27.75	-28.12	-27.51	-26.09	-22.61	-19.90
	-21.24	-21.20	-21.00	-20.83	-20.98	-20.59	-19.40	-16.02	-13.43
	-17.93	-17.86	-17.70	-17.70	-17.67	-17.41	-16.35	-13.25	-10.69
500	-29.53	-29.34	-28.81	-28.50	-28.55	-28.49	-27.51	-25.32	-19.54
	-21.78	-21.43	-21.48	-21.35	-21.47	-21.31	-20.99	-18.32	-13.49
	-18.29	-18.05	-18.00	-18.02	-18.04	-17.94	-17.70	-15.30	-10.73

TABLE 1d 1%, 5%, AND 10% CRITICAL VALUES OF $\tilde{\rho}_N(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	-22.65	-21.79	-20.17	-19.11	-18.45	-17.83	-17.14	-16.69	-16.52
	-18.04	-17.12	-15.80	-14.54	-14.00	-13.50	-12.62	-11.94	-12.00
	-15.66	-14.87	-13.58	-12.33	-11.84	-11.21	-10.38	-9.75	-9.87
50	-25.76	-25.17	-23.62	-22.43	-21.78	-21.00	-19.67	-18.31	-18.29
	-19.85	-19.40	-18.02	-16.78	-16.03	-15.21	-14.11	-12.95	-12.72
	-17.08	-16.51	-15.34	-14.16	-13.53	-12.60	-11.66	-10.41	-10.31
100	-27.54	-27.07	-26.64	-24.76	-24.08	-22.83	-21.45	-19.79	-18.92
	-20.81	-20.38	-19.72	-18.41	-17.63	-16.79	-15.42	-13.59	-13.22
	-17.68	-17.31	-16.68	-15.55	-14.76	-13.90	-12.84	-10.96	-10.57
200	-28.62	-28.05	-27.93	-26.48	-25.84	-24.79	-23.04	-20.96	-19.90
	-21.24	-21.14	-20.60	-19.71	-19.11	-18.06	-16.52	-14.66	-13.43
	-17.93	-17.81	-17.39	-16.64	-15.91	-14.93	-13.47	-11.85	-10.69
500	-29.53	-29.33	-28.70	-28.01	-27.17	-26.07	-23.97	-22.90	-19.54
	-21.78	-21.41	-21.34	-20.82	-20.43	-19.36	-17.59	-15.96	-13.49
	-18.29	-18.02	-17.88	-17.56	-17.16	-16.07	-14.50	-12.91	-10.73

TABLE 1e 1%, 5%, AND 10% CRITICAL VALUES OF $DK_s(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	0.5825	0.6538	0.7586	0.8268	0.8655	0.9070	0.9518	0.9900	0.9999
	0.7362	0.6993	0.7756	0.8356	0.8705	0.9093	0.9524	0.9901	0.9999
	0.8502	0.7319	0.7883	0.8420	0.8742	0.9111	0.9529	0.9901	0.9999
50	1.0065	0.7654	0.7995	0.8456	0.8762	0.9120	0.9531	0.9901	0.9999
	1.3110	0.8461	0.8314	0.8611	0.8855	0.9166	0.9543	0.9901	0.9999
	1.5322	0.9082	0.8553	0.8725	0.8924	0.9198	0.9553	0.9902	0.9999
100	1.8586	0.9882	0.8760	0.8828	0.8972	0.9220	0.9558	0.9902	0.9999
	2.4716	1.1407	0.9383	0.9107	0.9135	0.9294	0.9580	0.9903	0.9999
	2.9324	1.2628	0.9833	0.9308	0.9261	0.9353	0.9595	0.9904	0.9999
200	3.5681	1.4174	1.0327	0.9542	0.9370	0.9397	0.9606	0.9904	0.9999
	4.8139	1.7240	1.1537	1.0063	0.9673	0.9540	0.9646	0.9906	0.9999
	5.6932	1.9565	1.2367	1.0450	0.9901	0.9642	0.9676	0.9908	0.9999
500	8.5545	2.6778	1.5014	1.1638	1.0579	0.9940	0.9746	0.9910	0.9999
	11.7442	3.4956	1.7815	1.2918	1.1267	1.0254	0.9828	0.9915	0.9999
	13.9972	4.0596	1.9920	1.3812	1.1794	1.0497	0.9892	0.9918	0.9999

TABLE 1f 1%, 5%, AND 10% CRITICAL VALUES OF $DK_N(\rho_*)$

T	ρ_*=0.0	0.5	0.7	0.8	0.85	0.9	0.95	0.99	1.0
25	0.5825	0.6566	0.7602	0.8277	0.8660	0.9072	0.9518	0.9900	0.9999
	0.7362	0.7032	0.7789	0.8372	0.8714	0.9097	0.9525	0.9901	0.9999
	0.8052	0.7377	0.7925	0.8445	0.8757	0.9117	0.9530	0.9901	0.9999
50	1.0065	0.7692	0.8033	0.8477	0.8775	0.9126	0.9533	0.9901	0.9999
	1.3110	0.8529	0.8376	0.8651	0.8880	0.9177	0.9546	0.9901	0.9999
	1.5322	0.9165	0.8634	0.8784	0.8957	0.9215	0.9556	0.9902	0.9999
100	1.8586	0.9951	0.8829	0.8870	0.9003	0.9233	0.9561	0.9902	0.9999
	2.4716	1.1493	0.9477	0.9188	0.9190	0.9319	0.9586	0.9903	0.9999
	2.9324	1.2727	0.9964	0.9411	0.9335	0.9387	0.9604	0.9904	0.9999
200	3.5681	1.4229	1.0428	0.9606	0.9425	0.9429	0.9613	0.9904	0.9999
	4.8139	1.7323	1.1668	1.0194	0.9772	0.9593	0.9661	0.9907	0.9999
	5.6932	1.9672	1.2526	1.0617	1.0028	0.9715	0.9698	0.9908	0.9999
500	8.5545	2.6879	1.5150	1.1771	1.0690	1.0010	0.9771	0.9911	0.9999
	11.7442	3.5065	1.7966	1.3106	1.1440	1.0371	0.9871	0.9916	0.9999
	13.9972	4.0702	2.0145	1.4045	1.2027	1.0655	0.9949	0.9920	0.9999

TABLE 2

POWER, 5% LOWER TAIL TESTS, T = 50

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
3A	50	0.9	0.0	0	.087	.087	.100	.100	.103	.103
3A	50	0.9	0.5	0	.089	.093	.099	.102	.106	.109
3A	50	0.9	0.7	0	.097	.105	.103	.105	.104	.108
3A	50	0.9	0.8	0	.099	.112	.103	.112	.111	.114
3A	50	0.9	0.85	0	.106	.114	.109	.114	.112	.113
3A	50	0.9	0.9	0	.103	.109	.104	.109	.111	.111
3A	50	0.9	0.95	0	.109	.113	.110	.113	.102	.108
3A	50	0.9	1.0	0	.114	.114	.116	.116		
3B	50	0.85	0.0	0	.123	.123	.151	.151	.160	.160
3B	50	0.85	0.5	0	.139	.147	.156	.160	.173	.177
3B	50	0.85	0.7	0	.155	.182	.172	.179	.181	.184
3B	50	0.85	0.8	0	.164	.190	.171	.188	.188	.191
3B	50	0.85	0.85	0	.170	.196	.176	.194	.191	.195
3B	50	0.85	0.9	0	.176	.189	.178	.191	.191	.193
3B	50	0.85	0.95	0	.187	.193	.188	.192	.176	.186
3B	50	0.85	1.0	0	.183	.183	.186	.186		
3C	50	0.8	0.0	0	.198	.198	.244	.244	.264	.264
3C	50	0.8	0.5	0	.210	.226	.245	.251	.274	.280
3C	50	0.8	0.7	0	.240	.282	.265	.280	.285	.292
3C	50	0.8	0.8	0	.247	.292	.258	.291	.289	.295
3C	50	0.8	0.85	0	.263	.298	.271	.299	.293	.295
3C	50	0.8	0.9	0	.275	.295	.279	.296	.297	.298
3C	50	0.8	0.95	0	.293	.295	.295	.294	.272	.282
3C	50	0.8	1.0	0	.281	.281	.284	.284		
3D	50	0.7	0.0	0	.415	.415	.492	.492	.524	.524
3D	50	0.7	0.5	0	.440	.464	.493	.505	.543	.552
3D	50	0.7	0.7	0	.485	.554	.525	.554	.561	.568
3D	50	0.7	0.8	0	.512	.575	.532	.573	.570	.569
3D	50	0.7	0.85	0	.533	.579	.547	.579	.570	.565
3D	50	0.7	0.9	0	.557	.574	.565	.578	.577	.563
3D	50	0.7	0.95	0	.568	.554	.571	.554	.527	.527
3D	50	0.7	1.0	0	.511	.511	.514	.514		
3E	50	0.5	0.0	0	.891	.891	.933	.933	.945	.945
3E	50	0.5	0.5	0	.908	.921	.936	.940	.952	.954
3E	50	0.5	0.7	0	.930	.952	.946	.953	.951	.943
3E	50	0.5	0.8	0	.938	.948	.945	.947	.945	.923
3E	50	0.5	0.85	0	.940	.938	.946	.939	.939	.913
3E	50	0.5	0.9	0	.949	.918	.951	.921	.928	.894
3E	50	0.5	0.95	0	.934	.888	.935	.889	.883	.855
3E	50	0.5	1.0	0	.807	.807	.809	.809		

TABLE 3

SIZE AND POWER, 5% LOWER TAIL TESTS, T = 100

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
4	100	1	0.0	0	.049	.048	.048	.048	.048	.048
4	100	1	0.5	0	.051	.051	.052	.051	.048	.049
4	100	1	0.7	0	.052	.050	.051	.050	.050	.049
4	100	1	0.8	0	.051	.052	.051	.052	.052	.052
4	100	1	0.85	0	.050	.050	.050	.050	.049	.050
4	100	1	0.9	0	.051	.050	.052	.050	.049	.048
4	100	1	0.95	0	.054	.053	.054	.053	.053	.052
4	100	1	1.0	0	.051	.051	.051	.051		
4A	100	0.95	0.0	0	.082	.082	.094	.094	.098	.098
4A	100	0.95	0.5	0	.089	.091	.101	.102	.105	.106
4A	100	0.95	0.7	0	.091	.102	.099	.103	.104	.105
4A	100	0.95	0.8	0	.095	.112	.102	.110	.110	.114
4A	100	0.95	0.85	0	.099	.110	.103	.111	.107	.112
4A	100	0.95	0.9	0	.103	.110	.105	.110	.110	.110
4A	100	0.95	0.95	0	.117	.120	.119	.121	.119	.117
4A	100	0.95	1.0	0	.118	.118	.117	.117		
4B	100	0.9	0.0	0	.191	.191	.234	.234	.254	.254
4B	100	0.9	0.5	0	.201	.211	.247	.250	.266	.271
4B	100	0.9	0.7	0	.212	.251	.244	.262	.275	.283
4B	100	0.9	0.8	0	.221	.289	.251	.286	.288	.304
4B	100	0.9	0.85	0	.237	.289	.257	.291	.282	.295
4B	100	0.9	0.9	0	.256	.290	.268	.290	.285	.290
4B	100	0.9	0.95	0	.291	.306	.297	.310	.304	.304
4B	100	0.9	1.0	0	.291	.291	.289	.289		
4C	100	0.85	0.0	0	.393	.393	.467	.467	.503	.503
4C	100	0.85	0.5	0	.418	.433	.492	.498	.535	.543
4C	100	0.85	0.7	0	.431	.492	.479	.512	.537	.553
4C	100	0.85	0.8	0	.453	.564	.504	.563	.563	.591
4C	100	0.85	0.85	0	.477	.571	.510	.574	.559	.580
4C	100	0.85	0.9	0	.507	.571	.528	.572	.561	.568
4C	100	0.85	0.95	0	.566	.586	.573	.589	.580	.573
4C	100	0.85	1.0	0	.526	.526	.524	.524		
4D	100	0.8	0.0	0	.656	.656	.736	.736	.769	.769
4D	100	0.8	0.5	0	.676	.691	.750	.755	.788	.796
4D	100	0.8	0.7	0	.692	.755	.745	.774	.796	.811
4D	100	0.8	0.8	0	.717	.817	.765	.817	.816	.835
4D	100	0.8	0.85	0	.738	.824	.771	.827	.813	.824
4D	100	0.8	0.9	0	.771	.821	.791	.823	.814	.808
4D	100	0.8	0.95	0	.815	.822	.823	.826	.816	.799
4D	100	0.8	1.0	0	.725	.725	.723	.723		

TABLE 4

POWER, 5% LOWER TAIL TESTS, T = 200

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
5A	200	0.95	0.0	0	.185	.185	.236	.236	.257	.257
5A	200	0.95	0.5	0	.198	.206	.242	.243	.263	.265
5A	200	0.95	0.7	0	.200	.223	.243	.252	.273	.281
5A	200	0.95	0.8	0	.210	.269	.252	.278	.284	.303
5A	200	0.95	0.85	0	.212	.280	.240	.279	.277	.298
5A	200	0.95	0.9	0	.228	.291	.254	.288	.286	.299
5A	200	0.95	0.95	0	.265	.315	.280	.312	.306	.317
5A	200	0.95	1.0	0	.292	.292	.290	.290		
5B	200	0.9	0.0	0	.626	.626	.722	.722	.761	.761
5B	200	0.9	0.5	0	.639	.650	.719	.722	.762	.766
5B	200	0.9	0.7	0	.658	.703	.737	.754	.788	.803
5B	200	0.9	0.8	0	.669	.768	.742	.787	.797	.830
5B	200	0.9	0.85	0	.683	.799	.736	.801	.794	.829
5B	200	0.9	0.9	0	.705	.826	.750	.822	.810	.834
5B	200	0.9	0.95	0	.763	.840	.786	.841	.824	.835
5B	200	0.9	1.0	0	.747	.747	.746	.746		
5C	200	0.85	0.0	0	.952	.952	.977	.977	.984	.984
5C	200	0.85	0.5	0	.957	.961	.979	.979	.986	.987
5C	200	0.85	0.7	0	.961	.972	.980	.984	.989	.991
5C	200	0.85	0.8	0	.964	.985	.981	.989	.990	.993
5C	200	0.85	0.85	0	.968	.991	.981	.992	.991	.994
5C	200	0.85	0.9	0	.972	.991	.982	.992	.990	.991
5C	200	0.85	0.95	0	.984	.991	.988	.991	.990	.987
5C	200	0.85	1.0	0	.922	.922	.921	.921		

TABLE 5

POWER, 5% LOWER TAIL TESTS, T = 100

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
13C	100	0.85	0.0	-1	.396	.396	.467	.467	.498	.498
13C	100	0.85	0.5	-1	.409	.423	.481	.486	.514	.521
13C	100	0.85	0.7	-1	.438	.497	.487	.516	.536	.547
13C	100	0.85	0.8	-1	.447	.552	.495	.550	.550	.567
13C	100	0.85	0.85	-1	.478	.557	.510	.559	.549	.559
13C	100	0.85	0.9	-1	.509	.555	.527	.556	.552	.547
13C	100	0.85	0.95	-1	.552	.562	.560	.566	.561	.547
13C	100	0.85	1.0	-1	.499	.499	.496	.496		
12C	100	0.85	0.0	-2	.402	.402	.463	.463	.480	.480
12C	100	0.85	0.5	-2	.422	.434	.481	.486	.495	.499
12C	100	0.85	0.7	-2	.440	.485	.475	.494	.501	.494
12C	100	0.85	0.8	-2	.454	.521	.489	.515	.517	.504
12C	100	0.85	0.85	-2	.468	.500	.492	.501	.502	.482
12C	100	0.85	0.9	-2	.493	.472	.505	.474	.497	.457
12C	100	0.85	0.95	-2	.523	.471	.526	.475	.497	.458
12C	100	0.85	1.0	-2	.422	.422	.420	.420		
11C	100	0.85	0.0	-5	.471	.471	.438	.438	.365	.365
11C	100	0.85	0.5	-5	.470	.465	.440	.434	.343	.320
11C	100	0.85	0.7	-5	.470	.423	.426	.380	.320	.242
11C	100	0.85	0.8	-5	.458	.311	.419	.288	.300	.193
11C	100	0.85	0.85	-5	.449	.221	.406	.216	.269	.160
11C	100	0.85	0.9	-5	.415	.157	.378	.157	.236	.132
11C	100	0.85	0.95	-5	.321	.132	.305	.133	.197	.125
11C	100	0.85	1.0	-5	.113	.113	.112	.112		
10C	100	0.85	0.0	-10	.667	.667	.338	.338	.101	.101
10C	100	0.85	0.5	-10	.636	.591	.329	.297	.059	.039
10C	100	0.85	0.7	-10	.580	.270	.268	.110	.033	.007
10C	100	0.85	0.8	-10	.495	.026	.222	.017	.021	.003
10C	100	0.85	0.85	-10	.408	.006	.177	.005	.014	.002
10C	100	0.85	0.9	-10	.218	.002	.095	.002	.007	.001
10C	100	0.85	0.95	-10	.032	.001	.023	.001	.003	.001
10C	100	0.85	1.0	-10	.001	.001	.001	.001		

TABLE 6

POWER, 5% LOWER TAIL TESTS, T = 50

u_0 DRAWN FROM $N(0, 1/(1-\rho_1^2))$

Exp. No.	T	ρ_1	ρ_*	$\tilde{\tau}_S$	$\tilde{\tau}_N$	$\tilde{\rho}_S$	$\tilde{\rho}_N$	DK_S	DK_N
7A	50	0.95	0.0	.058	.058	.061	.061	.061	.061
7A	50	0.95	0.5	.061	.061	.062	.061	.062	.062
7A	50	0.95	0.7	.064	.066	.065	.064	.065	.064
7A	50	0.95	0.8	.059	.064	.060	.064	.063	.064
7A	50	0.95	0.85	.062	.065	.063	.065	.064	.064
7A	50	0.95	0.9	.064	.066	.064	.067	.068	.068
7A	50	0.95	0.95	.063	.065	.064	.064	.058	.062
7A	50	0.95	1.0	.063	.063	.064	.064		
7B	50	0.9	0.0	.085	.085	.095	.095	.096	.096
7B	50	0.9	0.5	.085	.087	.090	.092	.094	.095
7B	50	0.9	0.7	.095	.101	.100	.100	.099	.100
7B	50	0.9	0.8	.094	.101	.096	.101	.100	.100
7B	50	0.9	0.85	.093	.099	.095	.100	.098	.098
7B	50	0.9	0.9	.097	.100	.099	.101	.104	.104
7B	50	0.9	0.95	.102	.104	.102	.103	.093	.099
7B	50	0.9	1.0	.099	.099	.100	.100		
7C	50	0.85	0.0	.128	.128	.148	.148	.153	.153
7C	50	0.85	0.5	.137	.143	.149	.151	.158	.159
7C	50	0.85	0.7	.148	.164	.158	.161	.162	.160
7C	50	0.85	0.8	.153	.164	.155	.163	.164	.165
7C	50	0.85	0.85	.154	.165	.157	.164	.162	.162
7C	50	0.85	0.9	.160	.164	.162	.166	.168	.166
7C	50	0.85	0.95	.168	.167	.169	.166	.155	.161
7C	50	0.85	1.0	.162	.162	.164	.164		
7D	50	0.8	0.0	.201	.201	.234	.234	.244	.244
7D	50	0.8	0.5	.209	.220	.231	.237	.250	.252
7D	50	0.8	0.7	.234	.265	.253	.260	.262	.261
7D	50	0.8	0.8	.238	.260	.244	.260	.258	.256
7D	50	0.8	0.85	.243	.257	.247	.257	.259	.253
7D	50	0.8	0.9	.256	.258	.260	.259	.267	.258
7D	50	0.8	0.95	.260	.253	.261	.253	.238	.240
7D	50	0.8	1.0	.242	.242	.244	.244		

TABLE 7

POWER, 5% LOWER TAIL TESTS, T = 100

u_0 DRAWN FROM N(0, $1/(1-\rho_1^2)$)

Exp. No.	T	ρ_1	ρ_*	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
8A	100	0.95	0.0	.084	.084	.091	.091	.092	.092
8A	100	0.95	0.5	.086	.087	.095	.096	.095	.096
8A	100	0.95	0.7	.093	.099	.097	.100	.099	.100
8A	100	0.95	0.8	.089	.102	.096	.102	.101	.103
8A	100	0.95	0.85	.093	.096	.096	.098	.095	.097
8A	100	0.95	0.9	.100	.098	.101	.099	.100	.098
8A	100	0.95	0.95	.106	.106	.107	.108	.106	.105
8A	100	0.95	1.0	.103	.103	.102	.102		
8B	100	0.9	0.0	.199	.199	.230	.230	.238	.238
8B	100	0.9	0.5	.203	.209	.234	.236	.242	.244
8B	100	0.9	0.7	.214	.238	.233	.244	.248	.246
8B	100	0.9	0.8	.218	.260	.237	.257	.258	.263
8B	100	0.9	0.85	.228	.248	.240	.249	.248	.248
8B	100	0.9	0.9	.246	.249	.254	.251	.253	.245
8B	100	0.9	0.95	.260	.252	.262	.255	.258	.250
8B	100	0.9	1.0	.238	.238	.236	.236		
8C	100	0.85	0.0	.411	.411	.468	.468	.484	.484
8C	100	0.85	0.5	.420	.433	.482	.485	.500	.503
8C	100	0.85	0.7	.443	.489	.480	.498	.508	.506
8C	100	0.85	0.8	.452	.524	.488	.517	.521	.514
8C	100	0.85	0.85	.467	.510	.492	.511	.509	.501
8C	100	0.85	0.9	.494	.492	.507	.493	.506	.480
8C	100	0.85	0.95	.525	.494	.529	.497	.507	.480
8C	100	0.85	1.0	.445	.445	.443	.443		
8D	100	0.8	0.0	.665	.665	.733	.733	.752	.752
8D	100	0.8	0.5	.684	.697	.749	.753	.772	.774
8D	100	0.8	0.7	.702	.753	.743	.763	.772	.768
8D	100	0.8	0.8	.717	.787	.754	.782	.785	.770
8D	100	0.8	0.85	.735	.771	.760	.772	.775	.747
8D	100	0.8	0.9	.760	.745	.776	.747	.769	.716
8D	100	0.8	0.95	.790	.726	.793	.730	.754	.705
8D	100	0.8	1.0	.637	.637	.635	.635		

TABLE 8

POWER, 5% LOWER TAIL TESTS, T = 200

u_0 DRAWN FROM $N(0, 1/(1-\rho_1^2))$

Exp. No.	T	ρ_1	ρ_*	$\tilde{r}_s$	$\tilde{r}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
9A	200	0.95	0.0	.191	.191	.227	.227	.235	.235
9A	200	0.95	0.5	.203	.209	.233	.233	.241	.241
9A	200	0.95	0.7	.207	.224	.236	.242	.249	.250
9A	200	0.95	0.8	.206	.245	.239	.249	.250	.256
9A	200	0.95	0.85	.217	.255	.232	.252	.252	.254
9A	200	0.95	0.9	.220	.245	.237	.241	.250	.244
9A	200	0.95	0.95	.246	.255	.254	.255	.259	.253
9A	200	0.95	1.0	.235	.235	.233	.233		
9B	200	0.9	0.0	.637	.637	.718	.718	.739	.739
9B	200	0.9	0.5	.649	.661	.716	.718	.740	.742
9B	200	0.9	0.7	.668	.705	.731	.744	.764	.766
9B	200	0.9	0.8	.680	.757	.741	.768	.769	.768
9B	200	0.9	0.85	.683	.760	.723	.755	.756	.747
9B	200	0.9	0.9	.703	.745	.737	.740	.762	.725
9B	200	0.9	0.95	.749	.726	.765	.726	.765	.713
9B	200	0.9	1.0	.630	.630	.628	.628		
9C	200	0.85	0.0	.956	.956	.979	.979	.983	.983
9C	200	0.85	0.5	.960	.963	.978	.979	.984	.984
9C	200	0.85	0.7	.960	.971	.978	.981	.985	.986
9C	200	0.85	0.8	.966	.984	.980	.986	.987	.983
9C	200	0.85	0.85	.972	.987	.982	.987	.987	.974
9C	200	0.85	0.9	.974	.976	.982	.975	.985	.955
9C	200	0.85	0.95	.982	.948	.985	.949	.980	.936
9C	200	0.85	1.0	.851	.851	.850	.850		

TABLE 9

POWER, 5% LOWER TAIL TESTS, T = 25

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{r}_s$	$\tilde{r}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
2A	25	0.9	0.0	0	.060	.060	.064	.064	.064	.064
2A	25	0.9	0.5	0	.064	.065	.063	.063	.063	.063
2A	25	0.9	0.7	0	.063	.063	.063	.064	.064	.064
2A	25	0.9	0.8	0	.066	.066	.066	.066	.068	.066
2A	25	0.9	0.85	0	.064	.067	.065	.066	.067	.065
2A	25	0.9	0.9	0	.066	.065	.066	.065	.065	.065
2A	25	0.9	0.95	0	.065	.065	.064	.065	.060	.062
2A	25	0.9	1.0	0	.062	.062	.062	.062		
2B	25	0.85	0.0	0	.071	.071	.081	.081	.081	.081
2B	25	0.85	0.5	0	.079	.081	.081	.082	.084	.085
2B	25	0.85	0.7	0	.076	.077	.078	.077	.079	.080
2B	25	0.85	0.8	0	.084	.084	.083	.084	.085	.082
2B	25	0.85	0.85	0	.082	.088	.083	.087	.086	.085
2B	25	0.85	0.9	0	.077	.076	.077	.077	.077	.076
2B	25	0.85	0.95	0	.083	.082	.083	.083	.077	.079
2B	25	0.85	1.0	0	.083	.083	.084	.084		
2C	25	0.8	0.0	0	.087	.087	.098	.098	.102	.102
2C	25	0.8	0.5	0	.097	.102	.100	.102	.106	.106
2C	25	0.8	0.7	0	.098	.102	.100	.103	.105	.106
2C	25	0.8	0.8	0	.105	.108	.104	.108	.108	.106
2C	25	0.8	0.85	0	.101	.107	.102	.105	.104	.103
2C	25	0.8	0.9	0	.105	.105	.105	.105	.106	.106
2C	25	0.8	0.95	0	.107	.107	.106	.108	.099	.103
2C	25	0.8	1.0	0	.103	.103	.104	.104		
2D	25	0.7	0.0	0	.133	.133	.158	.158	.168	.168
2D	25	0.7	0.5	0	.157	.169	.167	.172	.181	.180
2D	25	0.7	0.7	0	.164	.174	.168	.174	.176	.179
2D	25	0.7	0.8	0	.182	.188	.183	.188	.189	.186
2D	25	0.7	0.85	0	.177	.183	.178	.181	.182	.178
2D	25	0.7	0.9	0	.176	.174	.176	.174	.174	.171
2D	25	0.7	0.95	0	.174	.174	.173	.175	.163	.165
2D	25	0.7	1.0	0	.176	.176	.178	.178		
2E	25	0.5	0.0	0	.328	.328	.386	.386	.409	.409
2E	25	0.5	0.5	0	.380	.404	.399	.410	.426	.425
2E	25	0.5	0.7	0	.397	.415	.405	.413	.415	.411
2E	25	0.5	0.8	0	.433	.437	.434	.439	.435	.417
2E	25	0.5	0.85	0	.418	.419	.421	.416	.416	.398
2E	25	0.5	0.9	0	.417	.398	.417	.398	.399	.380
2E	25	0.5	0.95	0	.408	.389	.407	.392	.374	.372
2E	25	0.5	1.0	0	.371	.371	.374	.374		

TABLE 10

POWER, 5% LOWER TAIL TESTS, T = 500

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
6A	500	0.99	0.0	0	.077	.077	.092	.092	.098	.097
6A	500	0.99	0.5	0	.085	.086	.101	.101	.107	.107
6A	500	0.99	0.7	0	.084	.089	.099	.100	.103	.104
6A	500	0.99	0.8	0	.085	.093	.099	.102	.107	.109
6A	500	0.99	0.85	0	.089	.103	.100	.106	.106	.110
6A	500	0.99	0.9	0	.088	.107	.099	.105	.105	.108
6A	500	0.99	0.95	0	.094	.112	.100	.112	.107	.113
6A	500	0.99	1.0	0	.116	.116	.116	.116		
6B	500	0.95	0.0	0	.808	.808	.882	.882	.909	.909
6B	500	0.95	0.5	0	.829	.834	.895	.895	.919	.920
6B	500	0.95	0.7	0	.830	.849	.896	.900	.921	.926
6B	500	0.95	0.8	0	.839	.877	.900	.915	.931	.945
6B	500	0.95	0.85	0	.842	.904	.893	.920	.927	.947
6B	500	0.95	0.9	0	.848	.936	.899	.938	.931	.956
6B	500	0.95	0.95	0	.879	.960	.908	.962	.942	.964
6B	500	0.95	1.0	0	.874	.874	.874	.874		

TABLE 11

POWER, 5% LOWER TAIL TESTS, T = 100, u_0 = -10

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
10	100	1	0.0	-10	.049	.049	.048	.048	.048	.048
10	100	1	0.5	-10	.051	.050	.052	.051	.049	.049
10	100	1	0.7	-10	.052	.051	.051	.050	.050	.049
10	100	1	0.8	-10	.051	.052	.051	.052	.052	.052
10	100	1	0.85	-10	.050	.050	.050	.050	.049	.050
10	100	1	0.9	-10	.051	.049	.052	.050	.049	.048
10	100	1	0.95	-10	.054	.052	.054	.053	.053	.052
10	100	1	1.0	-10	.051	.051	.051	.051		
10A	100	0.95	0.0	-10	.089	.089	.057	.057	.044	.044
10A	100	0.95	0.5	-10	.086	.079	.060	.059	.044	.042
10A	100	0.95	0.7	-10	.080	.055	.056	.047	.040	.035
10A	100	0.95	0.8	-10	.068	.040	.053	.039	.040	.034
10A	100	0.95	0.85	-10	.062	.035	.051	.035	.037	.031
10A	100	0.95	0.9	-10	.054	.032	.048	.032	.036	.031
10A	100	0.95	0.95	-10	.045	.032	.044	.033	.036	.032
10A	100	0.95	1.0	-10	.026	.026	.026	.026		
10B	100	0.9	0.0	-10	.304	.304	.121	.121	.045	.045
10B	100	0.9	0.5	-10	.277	.247	.116	.105	.033	.025
10B	100	0.9	0.7	-10	.246	.101	.100	.051	.024	.012
10B	100	0.9	0.8	-10	.189	.019	.084	.015	.017	.006
10B	100	0.9	0.85	-10	.147	.009	.066	.009	.014	.005
10B	100	0.9	0.9	-10	.081	.005	.047	.005	.010	.003
10B	100	0.9	0.95	-10	.025	.004	.021	.004	.007	.003
10B	100	0.9	1.0	-10	.003	.003	.003	.003		
10C	100	0.85	0.0	-10	.667	.667	.338	.338	.101	.101
10C	100	0.85	0.5	-10	.636	.591	.329	.297	.059	.039
10C	100	0.85	0.7	-10	.580	.270	.268	.110	.033	.007
10C	100	0.85	0.8	-10	.495	.026	.222	.017	.021	.003
10C	100	0.85	0.85	-10	.408	.006	.177	.005	.014	.002
10C	100	0.85	0.9	-10	.218	.002	.095	.002	.007	.001
10C	100	0.85	0.95	-10	.032	.001	.023	.001	.003	.001
10C	100	0.85	1.0	-10	.001	.001	.001	.001		
10D	100	0.8	0.0	-10	.912	.912	.677	.677	.265	.265
10D	100	0.8	0.5	-10	.894	.871	.666	.626	.163	.097
10D	100	0.8	0.7	-10	.868	.598	.599	.299	.081	.012
10D	100	0.8	0.8	-10	.816	.058	.527	.035	.047	.003
10D	100	0.8	0.85	-10	.742	.006	.428	.006	.024	.001
10D	100	0.8	0.9	-10	.513	.001	.252	.001	.012	.001
10D	100	0.8	0.95	-10	.068	.000	.045	.000	.003	.000
10D	100	0.8	1.0	-10	.000	.000	.000	.000		

TABLE 12

POWER, 5% LOWER TAIL TESTS, T = 100, u_0 = -5

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_S$	$\tilde{\tau}_N$	$\tilde{\rho}_S$	$\tilde{\rho}_N$	DK_S	DK_N
11A	100	0.95	0.0	-5	.085	.085	.084	.084	.082	.082
11A	100	0.95	0.5	-5	.088	.088	.088	.088	.085	.085
11A	100	0.95	0.7	-5	.087	.087	.086	.085	.084	.081
11A	100	0.95	0.8	-5	.086	.088	.088	.086	.087	.086
11A	100	0.95	0.85	-5	.090	.085	.088	.086	.085	.084
11A	100	0.95	0.9	-5	.088	.083	.088	.083	.084	.081
11A	100	0.95	0.95	-5	.091	.085	.090	.086	.087	.084
11A	100	0.95	1.0	-5	.079	.079	.079	.079		
11B	100	0.9	0.0	-5	.216	.216	.199	.199	.170	.170
11B	100	0.9	0.5	-5	.213	.212	.200	.197	.162	.153
11B	100	0.9	0.7	-5	.218	.198	.195	.179	.157	.132
11B	100	0.9	0.8	-5	.210	.156	.193	.148	.152	.120
11B	100	0.9	0.85	-5	.208	.130	.189	.130	.144	.112
11B	100	0.9	0.9	-5	.192	.110	.180	.111	.135	.102
11B	100	0.9	0.95	-5	.169	.106	.164	.107	.129	.102
11B	100	0.9	1.0	-5	.098	.098	.097	.097		
11C	100	0.85	0.0	-5	.471	.471	.438	.438	.365	.365
11C	100	0.85	0.5	-5	.470	.465	.440	.434	.343	.320
11C	100	0.85	0.7	-5	.470	.423	.426	.380	.320	.242
11C	100	0.85	0.8	-5	.458	.311	.419	.288	.300	.193
11C	100	0.85	0.85	-5	.449	.221	.406	.216	.269	.160
11C	100	0.85	0.9	-5	.415	.157	.378	.157	.236	.132
11C	100	0.85	0.95	-5	.321	.132	.305	.133	.197	.125
11C	100	0.85	1.0	-5	.113	.113	.112	.112		
11D	100	0.8	0.0	-5	.741	.741	.716	.716	.633	.633
11D	100	0.8	0.5	-5	.746	.745	.727	.721	.610	.572
11D	100	0.8	0.7	-5	.746	.708	.709	.655	.560	.414
11D	100	0.8	0.8	-5	.738	.548	.702	.508	.529	.316
11D	100	0.8	0.85	-5	.731	.373	.687	.363	.473	.247
11D	100	0.8	0.9	-5	.694	.245	.646	.244	.410	.198
11D	100	0.8	0.95	-5	.559	.191	.529	.192	.321	.178
11D	100	0.8	1.0	-5	.159	.159	.158	.158		

TABLE 13

POWER, 5% LOWER TAIL TESTS, T = 100, u_0 = -2

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_s$	$\tilde{\tau}_N$	$\tilde{\rho}_s$	$\tilde{\rho}_N$	DK_s	DK_N
12A	100	0.95	0.0	-2	.082	.082	.091	.091	.095	.095
12A	100	0.95	0.5	-2	.087	.089	.099	.099	.101	.102
12A	100	0.95	0.7	-2	.091	.098	.095	.099	.100	.101
12A	100	0.95	0.8	-2	.093	.110	.100	.108	.109	.112
12A	100	0.95	0.85	-2	.095	.104	.099	.105	.103	.105
12A	100	0.95	0.9	-2	.101	.102	.103	.104	.102	.103
12A	100	0.95	0.95	-2	.110	.113	.112	.114	.111	.111
12A	100	0.95	1.0	-2	.107	.107	.107	.107		
12B	100	0.9	0.0	-2	.192	.192	.225	.225	.237	.237
12B	100	0.9	0.5	-2	.203	.210	.238	.240	.249	.251
12B	100	0.9	0.7	-2	.212	.240	.234	.246	.252	.252
12B	100	0.9	0.8	-2	.221	.264	.242	.259	.261	.265
12B	100	0.9	0.85	-2	.225	.253	.240	.254	.250	.254
12B	100	0.9	0.9	-2	.247	.254	.254	.254	.258	.249
12B	100	0.9	0.95	-2	.265	.258	.267	.260	.264	.255
12B	100	0.9	1.0	-2	.246	.246	.245	.245		
12C	100	0.85	0.0	-2	.402	.402	.463	.463	.480	.480
12C	100	0.85	0.5	-2	.422	.434	.481	.486	.495	.499
12C	100	0.85	0.7	-2	.440	.485	.475	.494	.501	.494
12C	100	0.85	0.8	-2	.454	.521	.489	.515	.517	.504
12C	100	0.85	0.85	-2	.468	.500	.492	.501	.502	.482
12C	100	0.85	0.9	-2	.493	.472	.505	.474	.497	.457
12C	100	0.85	0.95	-2	.523	.471	.526	.475	.497	.458
12C	100	0.85	1.0	-2	.422	.422	.420	.420		
12D	100	0.8	0.0	-2	.675	.675	.737	.737	.750	.750
12D	100	0.8	0.5	-2	.687	.700	.746	.750	.761	.763
12D	100	0.8	0.7	-2	.702	.747	.739	.757	.761	.749
12D	100	0.8	0.8	-2	.721	.775	.755	.768	.771	.741
12D	100	0.8	0.85	-2	.734	.751	.758	.749	.763	.709
12D	100	0.8	0.9	-2	.759	.706	.768	.708	.752	.674
12D	100	0.8	0.95	-2	.772	.676	.775	.679	.725	.650
12D	100	0.8	1.0	-2	.589	.589	.588	.588		

TABLE 14

POWER, 5% LOWER TAIL TESTS, T = 100, u_0 = -1

Exp. No.	T	ρ_1	ρ_*	u_0	$\tilde{\tau}_S$	$\tilde{\tau}_N$	$\tilde{\rho}_S$	$\tilde{\rho}_N$	DK_S	DK_N
13A	100	0.95	0.0	-1	.083	.083	.095	.095	.101	.101
13A	100	0.95	0.5	-1	.088	.091	.101	.101	.103	.104
13A	100	0.95	0.7	-1	.089	.097	.095	.099	.101	.102
13A	100	0.95	0.8	-1	.092	.110	.100	.108	.109	.114
13A	100	0.95	0.85	-1	.097	.109	.102	.108	.106	.111
13A	100	0.95	0.9	-1	.104	.109	.108	.110	.109	.108
13A	100	0.95	0.95	-1	.112	.116	.113	.117	.115	.115
13A	100	0.95	1.0	-1	.113	.113	.112	.112		
13B	100	0.9	0.0	-1	.194	.194	.233	.233	.251	.251
13B	100	0.9	0.5	-1	.197	.205	.241	.244	.258	.262
13B	100	0.9	0.7	-1	.217	.248	.242	.256	.267	.274
13B	100	0.9	0.8	-1	.223	.286	.250	.283	.285	.296
13B	100	0.9	0.85	-1	.236	.281	.254	.282	.275	.288
13B	100	0.9	0.9	-1	.252	.284	.263	.285	.280	.283
13B	100	0.9	0.95	-1	.285	.294	.288	.296	.296	.290
13B	100	0.9	1.0	-1	.280	.280	.278	.278		
13C	100	0.85	0.0	-1	.396	.396	.467	.467	.498	.498
13C	100	0.85	0.5	-1	.409	.423	.481	.486	.514	.521
13C	100	0.85	0.7	-1	.438	.497	.487	.516	.536	.547
13C	100	0.85	0.8	-1	.447	.552	.495	.550	.550	.567
13C	100	0.85	0.85	-1	.478	.557	.510	.559	.549	.559
13C	100	0.85	0.9	-1	.509	.555	.527	.556	.552	.547
13C	100	0.85	0.95	-1	.552	.562	.560	.566	.561	.547
13C	100	0.85	1.0	-1	.499	.499	.496	.496		
13D	100	0.8	0.0	-1	.657	.657	.732	.732	.761	.761
13D	100	0.8	0.5	-1	.676	.691	.749	.754	.781	.788
13D	100	0.8	0.7	-1	.695	.753	.745	.773	.789	.796
13D	100	0.8	0.8	-1	.716	.805	.759	.804	.804	.811
13D	100	0.8	0.85	-1	.740	.805	.770	.809	.801	.796
13D	100	0.8	0.9	-1	.767	.793	.785	.794	.798	.776
13D	100	0.8	0.95	-1	.806	.783	.811	.789	.795	.759
13D	100	0.8	1.0	-1	.687	.687	.686	.686		

APPENDIX 1

In this Appendix, we show that $\tilde{\rho}_N(1) = \lim_{r\to 1} \tilde{\rho}_s(r) = \bar{\rho}$ and $\tilde{\tau}_N(1) = \lim_{r\to 1} \tilde{\tau}_s(r) = \bar{\tau}$. Let $\gamma = (\psi, \xi)'$ as in equation (1') of the main text. Let $\tilde{\gamma}$ be the restricted normal MLE's: $\tilde{\xi} = \overline{\Delta y} = (y_T - y_1)/(T-1)$ and $\tilde{\psi}_x = y_1 - \tilde{\xi}$, so that $\tilde{u}_t = y_t - z_t\tilde{\gamma}$ are the BSP residuals. Similarly let $\tilde{\gamma}_s(\rho_*)$ and $\tilde{\gamma}_N(\rho_*)$ be the GLS estimates using the covariance matrices $\Omega_s(\rho_*)$ and $\Omega_N(\rho_*)$, respectively, so that $\tilde{u}_{(s)t}(\rho_*) = y_t - z_t\tilde{\gamma}_s(\rho_*)$ and $\tilde{u}_{(N)t}(\rho_*) = y_t - z_t\tilde{\gamma}_N(\rho_*)$. Then it is sufficient to prove that $\tilde{\gamma}_N(1) = \lim_{r\to 1} \tilde{\gamma}_s(r) = \tilde{\gamma}$.

The nonstationary case is fairly straightforward. We have

$$\text{(A1.1)} \qquad \tilde{\gamma}_N(\rho_*) = [Z'\Omega_N^{-1}(\rho_*)Z]^{-1}Z'\Omega_N^{-1}(\rho_*)y$$

$$= \begin{bmatrix} B^2(T-1) + 1 & B^2\sum_{t=1}^{T} t + \rho_* BT + \rho_* \\ B^2\sum_{t=1}^{T} t + \rho_* BT + \rho_* & B^2\sum_{t=1}^{T} t^2 + \rho_* BT^2 + \rho_* T \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} B^2\sum_{t=1}^{T} y_t + \rho_* y_1 + \rho_* B y_T \\ B^2\sum_{t=1}^{T} t y_t + (\rho_* BT + \rho_*) y_T \end{bmatrix},$$

where $B = (1-\rho_*)$. When $\rho_* = 1$, so that $B = 0$,

$$\text{(A1.2)} \qquad \tilde{\gamma}_N(1) = \begin{bmatrix} 1 & 1 \\ 1 & T \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_T \end{bmatrix}$$

$$= (1-T)^{-1} \begin{bmatrix} Ty_1 - y_T \\ y_T - y_1 \end{bmatrix} = \begin{bmatrix} \tilde{\psi}_x \\ \tilde{\xi} \end{bmatrix} = \tilde{\gamma},$$

which is exactly the same as the restricted MLE. Therefore $\tilde{u}_{(N)t}(1) = y_t - z_t\tilde{\gamma} = \tilde{u}_t$ and the GLS-based tests are the same as the BSP tests.

The stationary case is more complicated because $\Omega_s(\rho_*)$ is singular for $\rho_* = 1$. However, we can evaluate the GLS estimator for $\rho_* \neq 1$ and take the limit as $\rho_* \to 1$. Thus, for $\rho_* \neq 1$ we have:

(A1.3) $$\tilde{\gamma}_s(\rho_*) = [Z'\Omega_s^{-1}(\rho_*)Z]^{-1}Z'\Omega_s^{-1}(\rho_*)y$$

$$= \begin{bmatrix} B^2T+2\rho_*B & B^2\sum_{t=1}^{T} t + \rho_*B(T+1) \\ B^2\sum_{t=1}^{T} t + \rho_*B(T+1) & B^2\sum_{t=1}^{T} t^2 + \rho_*BT^2 + \rho_*T - \rho_*^2 \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} B^2\sum_{t=1}^{T} y_t + \rho_*B(y_1+y_T) \\ B^2\sum_{t=1}^{T} ty_t + (\rho_*BT+\rho_*)y_T - \rho_*^2y_1 \end{bmatrix}$$

$$= \frac{1}{BD_*} \begin{bmatrix} B^2\sum_{t=1}^{T} t^2 + \rho_*BT^2 + \rho_*T - \rho_*^2 & -B(B\sum_{t=1}^{T} t + \rho_*(T+1)) \\ -B(B\sum_{t=1}^{T} t + \rho_*(T+1)) & B(BT+2\rho_*) \end{bmatrix}$$

$$\cdot \begin{bmatrix} B(B\sum_{t=1}^{T} y_t + \rho_*(y_1+y_T)) \\ B^2\sum_{t=1}^{T} ty_t + (\rho_*BT+\rho_*)y_T - \rho_*^2y_1 \end{bmatrix},$$

where $B = (1-\rho_*)$, BD_* is the determinant of $[Z'\Omega_s^{-1}(\rho_*)Z]$, and

$D_* = [(BT + 2\rho_*)(B^2 \sum_{t=1}^{T} t^2 + \rho_* BT^2 + \rho_* T - \rho_*^2) - B(B \sum_{t=1}^{T} t + \rho_*(T+1))^2]$. We now cancel B from the denominator, the second column of the first matrix, and the first element of the second matrix to obtain:

$$\tilde{\gamma}_s(\rho_*) = \frac{1}{D_*} \begin{bmatrix} B^2 \sum_{t=1}^{T} t^2 + \rho_* BT^2 + \rho_* T - \rho_*^2 & -(B \sum_{t=1}^{T} t + \rho_*(T+1)) \\ -B(B \sum_{t=1}^{T} t + \rho_*(T+1)) & BT+2\rho_* \end{bmatrix} \cdot \begin{bmatrix} B \sum_{t=1}^{T} y_t + \rho_*(y_1+y_T) \\ B^2 \sum_{t=1}^{T} t y_t + (\rho_* BT + \rho_*) y_T - \rho_*^2 y_1 \end{bmatrix}.$$

Now let $\tilde{\gamma}_s(1)$ denote $\lim_{r\to 1} \tilde{\gamma}_s(r)$. After some algebra,

$$\tilde{\gamma}_s(1) = (1-T)^{-1} \begin{bmatrix} Ty_1 - y_T \\ y_T - y_1 \end{bmatrix} = \begin{bmatrix} \tilde{\psi}_x \\ \tilde{\xi} \end{bmatrix} = \tilde{\gamma}$$

which is exactly the same as the restricted MLE. Therefore $\tilde{u}_{(s)t}(1) = y_t - z_t\tilde{\gamma} = \tilde{u}_t$ and the result follows.

APPENDIX 2

In this Appendix we show that the asymptotic distributions of the GLS-based statistics do not depend on ρ_*, for any ρ_* in the interval [0,1). We will give the proof for the $\tilde{\rho}_N(\rho_*)$ and $\tilde{\tau}_N(\rho_*)$ tests; the proof for the $\tilde{\rho}_S(\rho_*)$ and $\tilde{\tau}_S(\rho_*)$ tests is essentially the same.

Define the notation

$$\text{(A2.1)} \qquad D = \begin{bmatrix} T & 0 \\ 0 & T^3 \end{bmatrix}, \text{ so that } D^{-1/2} = \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{bmatrix}.$$

Our test statistics are functions of the normalized residual series $T^{-1/2}\,\tilde{u}_{(N)t}(\rho_*)$, and so we consider

$$\text{(A2.2)} \qquad T^{-1/2}\,\tilde{u}_{(N)t}(\rho_*) = T^{-1/2}\,u_t - T^{-1/2}\,z_t[\tilde{\gamma}_N(\rho_*)-\gamma]$$

$$= T^{-1/2}\,u_t - z_t(T^{1/2}D^{-1/2})[D^{-1/2}Z'\Omega_N^{-1}(\rho_*)ZD^{-1/2}]^{-1}(T^{-1}D^{-1/2})Z'\Omega_N^{-1}(\rho_*)u.$$

Now consider the terms on the right hand side of (A2.2). We have

$$\text{(A2.3)} \qquad z_t(T^{1/2}D^{-1/2}) = [1,\ \ t/T].$$

For the term $D^{-1/2}Z'\Omega_N^{-1}(\rho_*)ZD^{-1/2}$, note that $Z'\Omega_N^{-1}(\rho_*)Z$ is as given in the first matrix on the right hand side of equation (A1.1). Pre- and post-multiplication by $D^{-1/2}$ normalizes the 1,1 element by T^{-1}; the 1,2 element by T^{-2}; and the 2,2 element by T^{-3}. Taking probability limits, the first terms in each sum dominate, and thus

$$\text{(A2.4)} \quad \text{plim}[D^{-1/2}Z'\Omega_N^{-1}(\rho_*)ZD^{-1/2}]^{-1} = B^{-2}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} = B^{-2}\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

Finally, for the term $(T^{-1}D^{-1/2})Z'\Omega_N^{-1}(\rho_*)u$, note that $Z'\Omega_N^{-1}(\rho_*)u$ is the same as the vector on the right hand side of equation (A1.1), except that u replaces y. Pre-multiplication by $T^{-1}D^{-1/2}$ normalizes the first element by $T^{-3/2}$ and the second element by $T^{-5/2}$. Again the first terms in each sum dominate, and so

$$\text{(A2.5)} \qquad T^{-1}D^{-1/2}Z'\Omega_N^{-1}(\rho_*)u \quad = \quad B^2 \begin{bmatrix} T^{-3/2} \sum_{t=1}^{T} u_t \\ \\ T^{-5/2} \sum_{t=1}^{T} tu_t \end{bmatrix} \quad + \quad o_p(1) \quad .$$

We now substitute (A2.3), (A2.4) and (A2.5) into (A2.2). Note that the terms involving $B = (1-\rho_*)$ cancel. Doing a little algebra yields

$$\text{(A2.6)} \qquad T^{-1/2}\ \tilde{u}_{(N)t}(\rho_*) = T^{-1/2}\ u_t - (4T^{-3/2}\Sigma_t u_t - 6T^{-5/2}\Sigma_t tu_t)$$
$$- (t/T)(-6T^{-3/2}\Sigma_t u_t + 12T^{-5/2}\Sigma_t tu_t) + o_p(1).$$

Thus the asymptotic distribution of $T^{-1/2}\ \tilde{u}_{(N)t}(\rho_*)$ does not depend on ρ_*.

To be more precise, for any r between zero and one, define [rT] as the nearest lesser integer to rT; let W(r) be the Wiener process; and let ω^2 be the long-run variance of $\epsilon_t = \Delta u_t$. Then standard results applied to (A2.6) imply that

$$\text{(A2.7)} \qquad T^{-1/2}\ \tilde{u}_{(N)[rT]}(\rho_*) \rightarrow \omega W^*(r)$$

where $W^*(r) = [W(r) - (4-6r)\int_0^1 W(s)ds + (6-12r)\int_0^1 sW(s)ds]$ is a demeaned and detrended Wiener process, as defined by Park and Phillips (1988). This is exactly the same as the asymptotic distribution of $T^{-1/2}\ \hat{u}_{[rT]}$, where $\hat{u}_t$, t = 1,...,T are the residuals upon which the Dickey-Fuller tests are implicitly based. Thus our GLS-based tests based on any value of ρ_* in the interval [0,1) have the same asymptotic distributions as the corresponding Dickey-Fuller tests.

APPENDIX 3

In this Appendix we derive the asymptotic distribution of the Dufour-King POI statistic $DK_N(\rho_*)$. The statistic $DK_s(\rho_*)$ has the same asymptotic distribution.

Consider first the denominator of the statistic. We have

(A3.1) $$\begin{aligned}\tilde{u}'\Omega_N^{-1}(1)\tilde{u} &= \tilde{u}_T^2 + 2\Sigma_{t=1}^{T-1}\tilde{u}_t^2 - 2\Sigma_{t=2}^{T}\tilde{u}_t\tilde{u}_{t-1} \\ &= \Sigma_{t=2}^{T}\Delta\tilde{u}_t^2 \qquad \text{(using the fact that } \tilde{u}_1 = 0) \\ &= -2\Sigma_{t=2}^{T}\tilde{u}_{t-1}\Delta\tilde{u}_t ,\end{aligned}$$

where the last equality follows from Lemma 1 of Schmidt and Phillips (1992, p. 281). Schmidt and Phillips show that $T^{-1}\Sigma_{t=2}^{T}\tilde{u}_{t-1}\Delta\tilde{u}_t$ converges in probability to $-\sigma^2/2$, where σ^2 is the innovation variance (the variance of $\epsilon_t = \Delta u_t$). Therefore

(A3.2) $$T^{-1}\tilde{u}'\Omega_N^{-1}(1)\tilde{u} \rightarrow \sigma^2.$$

We next consider the numerator of the statistic. For typographical simplicity we will omit the subscript "N" from the residual vector $\tilde{u}_N(\rho_*)$ and from the individual residuals $\tilde{u}_{(N)t}(\rho_*)$. We have

(A3.3) $$\begin{aligned}\tilde{u}(\rho_*)'\Omega_N^{-1}(\rho_*)\tilde{u}(\rho_*) &= \tilde{u}_T^2(\rho_*) + (1+\rho_*^2)\Sigma_{t=1}^{T-1}\tilde{u}_t^2(\rho_*) \\ &\quad - 2\rho_*\Sigma_{t=2}^{T}\tilde{u}_t(\rho_*)\tilde{u}_{t-1}(\rho_*) \\ &= \Sigma_{t=2}^{T}[\tilde{u}_t(\rho_*) - \rho_*\tilde{u}_{t-1}(\rho_*)]^2 + \tilde{u}_1^2(\rho_*).\end{aligned}$$

Note that $[\tilde{u}_t(\rho_*) - \rho_*\tilde{u}_{t-1}(\rho_*)] = [\Delta\tilde{u}_t(\rho_*) + (1-\rho_*)\tilde{u}_{t-1}(\rho_*)]$ so that

(A3.4) $$\begin{aligned}T^{-2}\tilde{u}(\rho_*)'\Omega_N^{-1}(\rho_*)\tilde{u}(\rho_*) &= (1-\rho_*)^2T^{-2}\Sigma_{t=2}^{T}\tilde{u}_{t-1}^2(\rho_*) \\ &+ T^{-2}\Sigma_{t=2}^{T}\Delta\tilde{u}_t(\rho_*)^2 + 2(1-\rho_*)T^{-2}\Sigma_{t=2}^{T}\tilde{u}_{t-1}(\rho_*)\Delta\tilde{u}_t(\rho_*) + T^{-2}\tilde{u}_1^2(\rho_*).\end{aligned}$$

The last three terms on the right hand side of (A3.4) are

$o_p(1)$. Standard results applied to the first term imply that

(A3.5) $$T^{-2}\tilde{u}(\rho_*)'\Omega_N^{-1}(\rho_*)\tilde{u}(\rho_*) \to (1-\rho_*)^2\omega^2\int_0^1 W^*(r)^2dr \ ,$$

where W*(r) is a demeaned and detrended Wiener process and ω^2 is the long run variance of ϵ, as discussed in Appendix 2.

Combining (A3.2) and (A3.5), we obtain the asymptotic distribution of the statistic:

(A3.6) $$T^{-1}DK_N(\rho_*) \to (\omega^2/\sigma^2)\ (1-\rho_*)^2 \int_0^1 W^*(r)^2dr.$$

CHAPTER 3

CHAPTER 3

ALTERNATIVE METHODS OF DETRENDING THE POWER OF STATIONARITY TESTS

1. INTRODUCTION

The purpose of this chapter is to provide new tests of the null hypothesis of trend stationarity against the alternative hypothesis of a unit root. These tests are based upon detrending the series by a generalized least squares (GLS) regression, using various values of the moving average root. They are related to the stationarity tests of Kwiatkowski, Phillips, Schmidt and Shin (1992), hereafter KPSS, and Schmidt (1992), and also to the point optimal invariant (POI) tests of King (1980, 1988). Hence, in this chapter they are called GLS-based KPSS tests.

Following KPSS, consider the problem of testing the null hypothesis that an observable time series is stationary around a deterministic trend. They assume a components representation in which the series under study can be written as the sum of a deterministic trend, a random walk, and a stationary error:

(1) $$y_t = \xi t + r_t + \epsilon_t, \quad r_t = r_{t-1} + u_t, \quad t = 1,\ldots,T, \text{ or}$$

(1') $$y_t = r_0 + \xi t + \sum_{j=1}^{t} u_j + \epsilon_t, \quad t = 1,\ldots,T,$$

where ϵ_t are iid$(0,\sigma_\epsilon^2)$ errors and u_t are iid$(0, \sigma_u^2)$. Here λ ($= \sigma_u^2/\sigma_\epsilon^2$, ≥ 0) is the signal to noise ratio, which measures the ratio of the changes in permanent versus transitory components (Shepard and Harvey (1990)). The initial value r_0 is treated as fixed and plays the role of intercept.

The null hypothesis of trend stationarity corresponds to $\sigma_u^2 = 0$ (or $\lambda = 0$) and the alternative hypothesis of difference stationarity corresponds to $\sigma_u^2 > 0$ (or $\lambda > 0$). In this context, the one-sided LM test can be derived under the stronger assumption that the ϵ_t are iid $N(0, \sigma_\epsilon^2)$ and the u_t are iid $N(0, \sigma_u^2)$. Let $\hat{e}_t$, $t = 1,\ldots,T$, be the OLS residuals from the regression of y on intercept and trend. Define $\hat{\sigma}_\epsilon^2$ and $\hat{S}_t$ to be the estimate of the error variance from this regression and the partial sum process of the residuals, respectively:

(2) $$\hat{\sigma}_\epsilon^2 = T^{-1}\sum_{t=1}^{T}\hat{e}_t^2,$$

(3) $$\hat{S}_t = \sum_{j=1}^{t}\hat{e}_j, \quad t = 1,\ldots,T.$$

Then the LM statistic is given as follows:

(4) $$LM = \sum_{t=1}^{T}\hat{S}_t^2 \,/\, \hat{\sigma}_\epsilon^2.$$

Since the assumption of iid errors is restrictive and unrealistic, KPSS (1992) consider the asymptotic distribution of the LM statistic under the null hypothesis with weaker assumptions about the errors. See KPSS (1992) for more detailed discussion. Since the numerator normalized by T^{-2} converges to σ^2 (long run variance of the error) times a functional of a Brownian bridge, they modify the LM statistic by replacing the estimate of the error variance $\hat{\sigma}_\epsilon^2$ by a consistent estimate of the long run variance. Define the estimated autocovariances $\hat{\gamma}(j) = T^{-1}\sum_{t=j+1}^{T}\hat{e}_t\hat{e}_{t-j}$, $j = 0,1,\ldots,T-1$, and the long run variance estimator $\hat{\sigma}^2(\ell) = \hat{\gamma}(0) + 2 \sum_{s=1}^{\ell} w(s,\ell)\hat{\gamma}(s)$. Here $w(s,\ell)$ is an optional weighting function, such as the Bartlett-window $w(s,\ell) = 1-s/(\ell+1)$, and ℓ is the number of lags used to estimate σ^2, satisfying $\ell \rightarrow \infty$ but $\ell/T \rightarrow 0$ as $T \rightarrow \infty$. Then the KPSS statistic is

(5) $$\hat{\eta}_\tau = T^{-2}\sum_{t=1}^{T}\hat{S}_t^{\,2} \,/\, \hat{\sigma}^2(\ell).$$

Interestingly, the statistic (4) also may arise in the context of testing the hypothesis of a moving average unit root (or overdifferencing) using the ARIMA(0,1,1) parameterization:

(6) $$\Delta y_t = \xi + \omega_t - \theta\omega_{t-1}, \quad t = 1,\ldots,T,$$

where ω_t are iid$(0,\sigma_\omega^{\,2})$ and θ is a parameter which is assumed to be in the range [0,1]. In this model, difference stationarity corresponds to values of $\theta \in [0,1)$ and trend stationarity is the special case corresponding to $\theta = 1$. The null hypothesis of a moving average unit root, $\theta = 1$, implies overdifferencing in the ARIMA representation, while the alternative hypothesis of an invertible moving average process, $\theta \in [0,1)$, implies that y_t has an autoregressive unit root. However, we must note that while (1) and (6) are identical under the null of stationarity, they represent different processes under the alternative. Saikkonen and Luukkonen (1992a, b), in this context, derive a statistic (their R_2 statistic) of the same form as (4) as the locally best unbiased invariant (LBUI) test of the moving average unit root hypothesis. Campbell and Mankiw (1987, 1989) also use this parameterization to develop a method of measuring the long term effect of a current shock as a test to discriminate between trend stationary and difference stationary processes.

The relationship between the signal to noise ratio λ and the moving average parameter θ can be found without difficulty as follows:

(7) $$\theta = \{(\lambda + 2) - [\lambda(\lambda + 4)]^{1/2}\}/2, \quad \lambda = (\theta - 1)^2/\theta$$

(8) $$\sigma_\omega^2 = \sigma_\epsilon^2/\theta.$$

Thus $\lambda = 0$ corresponds to $\theta = 1$ (stationarity), while $\lambda = \infty$ corresponds to $\theta = 0$ (so y is a pure random walk). When λ is very small, or equivalently θ is very close to 1, y_t follows a nearly stationary process and standard unit root tests are expected to have low power.

Since the KPSS test is a modification of the LM test, it is therefore based on detrending under the null ($\lambda = 0$ or $\theta = 1$). Since the null is stationarity, this is the same type of detrending as in the Dickey-Fuller tests; an OLS regression of the variable y_t on intercept and trend. Another possibility is to detrend as Bhargava (1986) and Schmidt and Phillips (1992) do, using a regression in differences ($\lambda = \infty$ or $\theta = 0$). This leads to the residuals $(y_t-y_1) - (t-1)(y_T-y_1)/(T-1)$, which will be denoted $\tilde{e}_t(0)$ in the notation of the next section. Recall that in the case of testing the autoregressive unit root hypothesis, the Bhargava-Schmidt-Phillips (hereafter BSP) test detrends under the null, while the Dickey-Fuller tests detrend under the alternative. The result is that BSP tests are more powerful against alternatives close to the null (when power is low), while Dickey-Fuller tests are more powerful against alternatives far from the null (when power is high). See Schmidt and Lee (1991), Schmidt and Phillips (1992), and the previous chapter.

In the present context also, by analogy, we might expect the KPSS detrending method to maximize power against alternatives close to the null of stationarity, and this is consistent with the fact that it is the locally best invariant test. Conversely, we might expect KPSS test based on BSP residuals to give better power against alternatives far from the null. This is arguably important in the present context. As KPSS's simulations show, as $\lambda = \sigma_u^2/\sigma_\epsilon^2 \to \infty$, the power of the $\hat{\eta}_\tau$ test does not necessarily approach unity. For example, with T = 100, power as $\lambda \to \infty$ approaches 0.82 for $\ell = 4$ and approaches 0.41 for $\ell = 12$. Thus there is a clear need to increase power against alternatives far from the null. However, according to Schmidt (1992), the KPSS statistic using BSP residuals does not yield a satisfactory test. This is so for two reasons. First, its asymptotic distribution under the null of stationarity depends on the marginal distribution of ϵ. Second, the KPSS test based on BSP residuals is not consistent against unit root alternatives.

Another alternative, along the same lines as in the previous chapter, is to construct the KPSS test statistic with GLS residuals from (6), using an assumed value of θ, say θ_*, against which maximal power is desired. Let θ_1 denote the actual value of θ in the model (6). Then King's (1980, 1988) most powerful invariant test of the null of $\theta = 1$ against the alternative of a specific value, say θ_*, involves GLS regressions with $\theta = \theta_*$ and $\theta = 1$. The power of the POI test depends on θ_* as well as θ_1. Since the theory of point optimal testing ensures that the POI test will be at least as powerful as any other invariant test against $\theta = \theta_*$, we might expect that it is also more powerful against θ in a reasonable neighborhood of θ_*.

In the following section, we will derive the GLS-based KPSS test statistic and the point optimal invariant test statistic. In section 3, the asymptotic distributions of these tests statistics will be derived under the null and under the alternative hypothesis. In section 4, the finite sample size and power of the tests will be investigated using Monte Carlo simulation. Section 5 concludes.

2. STATIONARITY TEST: GLS-BASED KPSS TEST AND POI TEST

In this section we provide two tests of the hypothesis of trend stationarity. They consist of the GLS-based KPSS test and King's POI test. We assume the DGP:

(9A) $\qquad y_t = \psi + \xi t + X_t,$

(9B) $\qquad X_t = X_{t-1} + \omega_t - \theta\omega_{t-1},$

$t = 1,\ldots,T$, where ψ is r_0 in (1) and ω_t are iid $N(0, \sigma_\omega^2)$. The null hypothesis of stationarity corresponds to $\theta = 1$, so that X_t $(= \omega_t)$ is an iid process and the alternative hypothesis of unit root to be considered in this chapter corresponds to $\theta \in (0,1)$. Note that X_t can be expressed as a component representation of the form of equation (1); that is, $X_t = r_t + \epsilon_t$. This component representation and the ARIMA representation in (9) are identical under the null hypothesis. In matrix form,

(9') $$y = Z\gamma + x,$$

where Z is the T×2 matrix with t^{th} observation row $z_t' = [1,t]$, $\gamma' = [\psi,\xi]$, and x is a T×1 vector of realizations of the error process. Based on this specification, our GLS-based test and King's POI test are invariant under the transformation $y \rightarrow a_0 y + Za_1$, where a_0 and a_1 are a scalar and a vector of real constants, respectively.

In equation (9B), the initial value ω_0 is assumed to be fixed at zero, which implies that ΔX_t (or Δy_t) follows a (nonstationary) conditional MA(1) process under the alternative hypothesis. (If the initial value ω_0 were assumed to be a random variable having the same distribution as ω_t, ΔX_t would follow a stationary unconditional MA(1) process under the alternative hypothesis.) Thus we have $x \sim N(0, \sigma_\omega^2 \Omega_N(\theta))$, where $\Omega_N(\theta)$ and its component matrices are defined as follows:

(10) $$\Omega_N(\theta) = C^{-1}(1)C(\theta)C(\theta)'C^{-1}(1)',$$

(11) $$C(\theta) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\theta & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\theta & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\theta & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1 \end{bmatrix},$$

$$(12)\quad C^{-1}(\theta) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \theta & 1 & 0 & \cdots & 0 & 0 \\ \theta^2 & \theta & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \theta^{T-1} & \theta^{T-2} & \cdot & \cdots & \theta & 1 \end{bmatrix}.$$

From the above definition, C(1) is a differencing matrix so that premultiplying equation (9') by C(1) gives the following equivalent equation:

$$(13)\qquad \tilde{y} = \tilde{Z}\gamma + \tilde{x},$$

where $\tilde{y} \equiv C(1)y$, $\tilde{Z} \equiv C(1)Z$ and $\tilde{x} \equiv C(1)x$, so that $\tilde{x} = \Delta x$ follows a conditional MA(1) process. Thus, we have $\tilde{x} \sim (0,\sigma_\omega^2\tilde{\Omega}_N(\theta))$, where $\tilde{\Omega}_N(\theta) = C(\theta)C(\theta)'$ is defined as follows:

$$(14)\quad \tilde{\Omega}_N(\theta) = \begin{bmatrix} 1 & -\theta & 0 & \cdots & \cdot & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & \cdot & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1+\theta^2 & -\theta \\ 0 & \cdot & \cdot & \cdots & -\theta & 1+\theta^2 \end{bmatrix}.$$

We now introduce the GLS-based KPSS test. For a given θ_* ϵ [0,1], let $\tilde{e}_t(\theta_*)$, t = 1,...,T, be the residual series from the GLS regression of y_t on [1,t], using the assumed covariance matrix $\Omega_N(\theta_*)$. Let $\tilde{S}_t(\theta_*)$ be the partial sum process of this residual process. Let $\tilde{\sigma}(\ell)^2$ be an estimator of the long run variance defined in the same way as $\hat{\sigma}(\ell)^2$

above except that $\tilde{e}_t(\theta_*)$ replaces $\hat{e}_t$. Then the GLS-based KPSS test can be defined as an upper tail test based on the statistic

(15) $$\tilde{\eta}_\tau(\theta_*) = T^{-2}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \ / \ \tilde{\sigma}(\ell)^2.$$

Thus $\tilde{S}_t(\theta_*)$ and $\tilde{\sigma}(\ell)^2$ are used in the KPSS statistic instead of $\hat{S}_t$ and $\hat{\sigma}(\ell)^2$.

This statistic includes the KPSS statistic and the KPSS statistic based on BSP residuals (Schmidt (1992)) as special cases, corresponding to $\theta_* = 1$ and $\theta_* = 0$, respectively. Since GLS with $\theta_* = 1$ in (9) is just OLS in levels (because $\Omega_N(1)$ becomes the identity matrix in (10)), its residuals $\tilde{e}_t(1)$ are identical to the OLS residuals $\hat{e}_t$ and the new GLS-based KPSS test $\tilde{\eta}_\tau(1)$ becomes identical to the KPSS test $\hat{\eta}_\tau$. On the other hand, GLS with $\theta_* = 0$ is OLS in the differenced equation (9') ($\tilde{\Omega}_N(0)$ becomes the identity matrix in (14)) and its residuals $\tilde{e}_t(0)$ become identical to BSP residuals, so $\tilde{\eta}_\tau(0)$ becomes the same statistic as in Schmidt (1992). Mathematical details for GLS estimation using an assumed value of θ, (that is, θ_*) and its normalized residuals are discussed in Appendix 2.

In a series of papers, King developed the theory of point optimal testing in various contexts (King (1980, 1988), Dufour and King (1991)). Shively (1988) and Saikkonen and Luukkonen (1992b) derived the point optimal test based on King (1980) in the context of the current setting, but without deterministic trend. According to Theorem 3 in King (1980), the point optimal invariant test involves two quadratic forms in GLS residual vectors, corresponding to the null and the specific alternative hypothesis. Let $P_T(\theta_*)$ denote the point optimal invariant (POI) test of the null H_0: $\theta = 1$ against the specific alternative H_1: $\theta = \theta_*$. Then the POI test is a lower tail test based on the statistic $P_T(\theta_*)$, defined as the ratio of quadratic forms in GLS residuals:

$$(16) \qquad P_T(\theta_*) = \tilde{e}(\theta_*)'\Omega_N^{-1}(\theta_*)\tilde{e}(\theta_*) \;/\; \tilde{e}(1)'\Omega_N^{-1}(1)\tilde{e}(1),$$

where $\tilde{e}(\theta_*)$ and $\tilde{e}(1)$ are GLS residual vectors from (9) under the alternative $\theta = \theta_*$ and under the null $\theta = 1$, respectively. Since, as discussed above, $\tilde{e}(1)$ is just the OLS residual vector $\hat{e}$ and $\Omega_N(1)$ is an identity matrix, the denominator of $P_T(\theta_*)$ can be expressed simply as $\hat{e}'\hat{e}$. The numerator of $P_T(\theta_*)$ also can be expressed as the sum of squares of the OLS residual vector from the transformed regression equation; that is, as $\tilde{e}^*(\theta_*)'\tilde{e}^*(\theta_*)$, where $\tilde{e}^*(\theta_*)$ is the OLS residual vector from the following regression:

$$(17) \qquad \tilde{y}^* = \tilde{Z}^*\gamma + \tilde{x}^*,$$

where $\tilde{y}^* \equiv C^{-1}(\theta_*)C(1)y$, $\tilde{Z}^* \equiv C^{-1}(\theta_*)C(1)Z$, and $\tilde{x}^* \equiv C^{-1}(\theta_*)C(1)x$. The OLS residual vector from (17), $\tilde{e}^*(\theta_*)$, is related to the GLS residual vector $\tilde{e}(\theta_*)$ in the following way:

$$(18) \qquad \tilde{e}^*(\theta_*) = C^{-1}(\theta_*)C(1)\tilde{e}(\theta_*).$$

Therefore we have

$$(19) \qquad P_T(\theta_*) = \tilde{e}^*(\theta_*)'\tilde{e}^*(\theta_*) \;/\; \hat{e}'\hat{e}.$$

Since the residuals $\tilde{e}_t^*(\theta_*)$, $t = 1,\ldots,T$, are asymptotically equivalent to an exponentially weighted average process of x (see equation (A2.8) in Appendix 2), θ_* in the numerator can be seen as an optimal weight in the estimation of the permanent component r_t in (1). Also the denominator divided by T is simply an estimate of the variation of the transitory component and the numerator divided by T is asymptotically equivalent to the estimate of the variation of the permanent component (Muth (1960)).

3. DISTRIBUTION THEORY

In this section we consider the asymptotic distributions as $T \to \infty$ with θ_* fixed of the GLS-based KPSS test and the POI test. Since they are based on GLS residuals and these residuals can be expressed as functions of the error process X_t, we can analyze the properties of the statistics under the alternative assumptions that X_t is stationary (under H_0) and that it contains a unit root (under H_1).

Along the same lines as Schmidt (1992), we make the following simple alternative assumptions. In these assumptions and the rest of the chapter, $\Rightarrow$ denotes weak convergence, [rT] denotes the integer part of rT, σ^2 is the long run variance, W(r) is the Wiener process on [0,1], and integrals like $\int_0^1 W(r)dr$ and $\int_0^1 rW(r)dr$ will sometimes be denoted by simply as $\int W$ and $\int rW$.

ASSUMPTION A (Stationarity):

(i) Equation (9A) holds. (ii) For $r \in [0,1]$, the X_t satisfy the invariance principle $T^{-1/2}\sum_{j=1}^{[rT]} X_j \Rightarrow \sigma W(r)$, with $\sigma > 0$. (iii) $\sigma_x^2 = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E(X_t^2)$ exists.

ASSUMPTION B (Unit root):

(i) Equation (9A) holds. (ii) For $r \in [0,1]$, the X_t satisfy the invariance principle $T^{-1/2}X_{[rT]} \Rightarrow \sigma W(r)$, with $\sigma > 0$.

It is important to note that, in Assumptions A and B, X_t is simply the deviation of y_t from deterministic trend, as implied by equation (9A). However, for the purposes of our asymptotic distribution theory we do not assume equation (9B). Thus the assumption that X_t followed an ARIMA(0,1,1) process was used to derive the test statistics, but we now consider the asymptotic distributions of these statistics under more general assumptions on X_t.

As a preliminary step, we examine the order of probability of two exponentially weighted moving average processes under these assumptions. Let $\Theta_A^*(L)X_T$ and $\Theta_B^*(L)X_T$ be polynomials in the lag operator L, defined as follows:

(20) $$\Theta_A^*(L)X_T = \sum_{j=0}^{T-1} \theta_*^{\,j} L^j X_T.$$

(21) $$\Theta_B^*(L)X_T = \sum_{j=0}^{T-1} \theta_*^{\,T-1-j} L^j X_T.$$

These two polynomials produce absolutely summable series for any fixed value of $\theta_* \in [0,1)$, and so we claim the following two propositions as $T \to \infty$ with fixed θ_*.

LEMMA 1. Under Assumption A (stationarity),

i) $\Theta_A^*(L)X_T = O_p(1)$ and

ii) $\Theta_B^*(L)X_T = O_p(1)$.

Proof. The results are self-evident from the absolute summability of the series and the stationarity assumption.

LEMMA 2. Under Assumption B (unit root),

i) $\Theta_A^*(L)X_T = O_p(T^{1/2})$ and

ii) $\Theta_B^*(L)X_T = O_p(1)$.

Proof. See Appendix 1.

The asymptotic distribution of the GLS-based KPSS test is derived in Appendix 3. We summarize the main asymptotic results as Theorems 1 and 2. We deduce the important conclusions that the asymptotic distribution of the GLS-based KPSS test depends on the marginal distribution of x under the null hypothesis, and that it is not consistent against the alternative hypothesis of a unit root.

THEOREM 1. Denote the weak limit of X_T as $T \to \infty$ by X_∞ and let Assumption A (stationarity) hold. Then for any given $\theta_* \in [0,1)$,

(22) $\tilde{e}_t(\theta_*) = X_t-(1-\theta_*)\Theta_B^*(L)X_T-(t/T)(1-\theta_*)[\Theta_A^*(L)-\Theta_B^*(L)]X_T+o_p(1)$,

(23) $T^{-1}\tilde{S}_{[rT]}(\theta_*) \Rightarrow -(1-\theta_*)\{ r\Theta_B^*(L)X_\infty + (r^2/2)[\Theta_A^*(L)-\Theta_B^*(L)]X_\infty \}$,

(24) $$T^{-3}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \Rightarrow [(1-\theta_*)^2/60]\{8[\Theta_B^*(L)X_\infty]^2 + 9[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + 3[\Theta_A^*(L)X_\infty]^2\},$$

(25) $$\tilde{\sigma}^2(0) \Rightarrow \sigma_x^2 + [(1-\theta_*)^2/3]\{[\Theta_B^*(L)X_\infty]^2 + [\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + [\Theta_A^*(L)X_\infty]^2\},$$

(26) $$T^{-1}\tilde{\eta}_T(\theta_*) = T^{-3}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \,/\, \tilde{\sigma}(0)^2$$

$$\Rightarrow \frac{[(1-\theta_*)^2]\{8[\Theta_B^*(L)X_\infty]^2 + 9[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + 3[\Theta_A^*(L)X_\infty]^2\}}{60\sigma_X^2 + 20(1-\theta_*)^2\{[\Theta_B^*(L)X_\infty]^2+[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] +[\Theta_A^*(L)X_\infty]^2\}}.$$

Thus $\tilde{\eta}_\tau(\theta_*) = O_p(T)$.

THEOREM 2. Under Assumption B,

$$(27) \qquad T^{-1/2}\tilde{e}_t(\theta_*) = T^{-1/2}X_t - (t/T)X_T + o_p(1),$$

$$(28) \qquad T^{-1/2}\tilde{e}_{[rT]}(\theta_*) \Rightarrow \sigma B(s),$$

$$(29) \qquad T^{-3/2}\tilde{S}_{[rT]}(\theta_*) \Rightarrow \sigma\int_0^r B(s)ds,$$

$$(30) \qquad T^{-4}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \Rightarrow \sigma^2\int_0^1[\int_0^r B(s)ds]^2dr,$$

$$(31) \qquad T^{-1}\tilde{\sigma}^2(0) = T^{-2}\sum_{t=1}^{T}\tilde{e}_t(\theta_*)^2 \Rightarrow \sigma^2\int_0^1 B(s)^2ds,$$

$$(32) \qquad T^{-1}\tilde{\eta}_\tau(\theta_*) = T^{-4}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 / T^{-1}\tilde{\sigma}(0)^2 \Rightarrow \frac{\int_0^1[\int_0^r B(s)ds]^2dr}{\int_0^1 B(s)^2ds}$$

where $B(s) = W(r) - rW(1)$ is the Brownian bridge. Thus $\tilde{\eta}_\tau(\theta_*) = O_p(T)$.

Note that the polynomials $\Theta_A^*(L)$ and $\Theta_B^*(L)$ in equations (23) and below (also in Appendix 3) should be interpreted as $\sum_{j=1}^{\infty}\theta_*^{j-1}L^{j-1}$ and $\sum_{j=1}^{\infty}\theta_*^{T-j}L^{j-1}$, respectively.

Theorems 1 and 2 apply to the GLS-based KPSS tests for the case that $\ell = 0$, where ℓ is the number of covariance terms used in estimation of the long run variance. The analysis of the case that $\ell \to \infty$ but $\ell/T \to 0$ is more complicated. However, following the same lines as Schmidt (1992), it is possible to show that in this case $\tilde{\eta}_\tau(\theta_*)$ is $O_p(T/\ell)$ under both Assumption A (stationarity) and Assumption B (unit root). Thus the test is inconsistent in the case that $\ell \to \infty$, $\ell/T \to 0$ as well as in the case that $\ell = 0$.

Theorem 1 implies that the asymptotic distribution of the GLS-based KPSS test depends on the marginal distribution of x as well as $\theta_* \in [0,1)$. The basic problem here is that, even though the X_t process is stationary and ergodic, $\tilde{e}_t(\theta_*)$ is non-ergodic, and the usual central limit theorems do not apply because terms involving $\Theta_A^*(L)X_T$ and $\Theta_B^*(L)X_T$ do not average away. Furthermore, while $\tilde{e}_t(\theta_*) = O_p(1)$ from the Lemma 1, its cumulation is $O_p(T)$ rather than $O_p(T^{1/2})$. These are strong arguments against statistics, like $\tilde{\eta}_\tau(\theta_*)$, that depend on $\tilde{S}_t(\theta_*)$; such statistics have a limiting distribution which depends on the distribution of the data and they do not yield a consistent test.

Theorem 2 shows that under Assumption B, for any θ_* less than unity, the GLS-based KPSS test $\tilde{\eta}_\tau(\theta_*)$ has the same asymptotic distribution as the KPSS test based on BSP residuals, that is, $\theta_* = 0$. Recall that for $\theta_* = 1$ and under Assumption B,

(33) $$T^{-1/2}\tilde{e}_{[rT]}(1) \Rightarrow \sigma W^*(r),$$

where $W^*(r) = W(s) + (6s-4)\int W + (6-12s)\int rW$ is the demeaned and detrended Wiener process as in KPSS (1992). So we find that there is a discontinuity in the asymptotic distribution at $\theta_* = 1$, as there was in the previous chapter at $\rho_* = 1$. Finally, we note that we get the same asymptotic results as in Schmidt

(1992) when $\theta_* = 0$ is used in Theorems 1 and 2.

The asymptotic distribution of the POI statistic is derived in Appendix 4 based on the limiting distribution of sample autocorrelations as $T \to \infty$ with fixed θ_*. We summarize the main results under each assumption as Theorems 3 and 4.

THEOREM 3. Let $\rho_x(j)$ be the j^{th} population autocorrelation coefficient of X_t. Then under Assumption A,

(34) $$P \equiv \text{plim } P_\tau(\theta_*) = 2(1+\theta_*)^{-1}[1 - (1-\theta_*)\sum_{j=1}^{\infty} \theta_*^{j-1} \rho_x(j)],$$

(35) $$T^{1/2}[P_\tau(\theta_*) - P] \Rightarrow N(0, V),$$

where V is given by

(36) $$V \equiv [2(1-\theta_*)/(1+\theta_*)]^2 \sum_{i=1}^{T} \sum_{j=1}^{T} \theta_*^{i+j-2} w_{ij},$$

and w_{ij} is given by (A4.13) in Appendix 4. Thus $P_\tau(\theta_*)$ is $O_p(T^{-1/2})$.

Theorem 3 implies that the asymptotic null distribution of the POI test depends θ_*. If the X_t are not iid, it also depends on their covariance structure. Unfortunately, the way in which the asymptotic distribution of the POI test depends on the correlation structure of X_t is complicated, and does not suggest a simple Phillips-Perron type correction that would make the test robust to error autocorrelation. Recent papers by Saikkonen and Luukkonen (1992a, b) and Leybourne and McCabe (1992) suggest parametric corrections for autocorrelation. This would amount to assuming (9B) and also assuming an ARMA(p,q) model for ω_t, so that X_t is ARMA(p,q+1) with a unit moving average root under the null. The parametric model would be used to whiten ω_t and then the POI test would be applied to the whitened data. The finite sample properties of such corrected tests are an important topic for future research.

THEOREM 4. Let Assumption B hold and let $\hat{\gamma}(j)$ and $\gamma(j)$ be j^{th} sample and population autocovariance of ΔX_t, respectively. Then,

$$(37) \qquad T\,P_\tau(\theta_*) = \frac{T^{-1}\sum_{t=1}^{T} \tilde{e}_t^*(\theta_*)^2}{T^{-2}\sum_{t=1}^{T} \hat{e}_t^2} \Rightarrow \frac{[\gamma(0) - 2\sum_{j=1}^{\infty} \theta_*^j\,\gamma(j)]}{\sigma^2(1-\theta_*^2)\int_0^1 W^*(r)^2 dr}.$$

Thus $P_\tau(\theta_*)$ is $O_p(T^{-1})$. Hence, comparing Theorem 3 with Theorem 4 shows that the test is consistent.

Recent research by Saikkonen and Luukkonen (1992b) derives the asymptotic distribution of the POI test of level stationarity when $\theta_* = 1 - \delta_*/T$ with δ_* fixed. Hence their asymptotic distribution is quite different from ours both because they fix δ_* instead of θ_*, and because their model is level stationary under the null while ours is trend stationary.

Since the distributions of both the GLS-based KPSS and the POI test statistics depend only on the assumed value of θ (that is, θ_*) and the sample size T under the null hypothesis, the finite sample distributions can be tabulated by Monte Carlo simulation. We calculate the critical values of the tests through simulations using various values of these two parameters. For the purpose of comparison with the KPSS results, we consider the sample sizes T = 30, 50, 100, 200, and 500. We also consider the assumed values of $\theta_* = 1.0.,$ 0.99, 0.969, 0.905, 0.73, 0.382, 0.01, and 0.0001. These correspond to assumed values of the signal to noise ratio of $\lambda_* = 0.0$, 0.0001, 0.001, 0.01, 0.1, 1.0, 100, and 10000. The critical values are calculated by a direct simulation using 25,000 replications and normal random numbers are generated by the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986). These critical values are presented in Table 1.

The critical values in Table 1 reflect the analytical results given above. For our GLS-based KPSS test $\tilde{\eta}_\tau(\theta_*)$, the critical values for each sample size and critical level are monotonically increasing as θ_* decreases from one to zero. Also, for a given value of θ_* and a given critical level, the critical values of the statistic increase in proportion to the sample size, T. This reflects the fact that our GLS-based KPSS test $\tilde{\eta}_\tau(\theta_*)$ is $O_p(T)$ under the null hypothesis, as shown in Theorem 1. For the original KPSS test, which corresponds to our GLS-based test at $\theta_* = 1$, we see a very stable distribution with respect to the sample size under the null hypothesis, as expected.

As for the POI test $P_\tau(\theta_*)$, its critical values seem to depend on the values of θ_* and the sample size T in a very complicated way. However, we can see the convergence of the normalized POI test to the normal distribution in Table 2. Table 2 presents percentiles of the distributions of the POI tests at sample size T = 500. If $\theta_* = .730$, for example, we can find that the POI test $P_\tau(0.73)$ has an approximately normal distribution around the 50% critical value 1.1604, which is the approximate value of plim $P_\tau(0.73) = 2/1.73 = 1.156$. This corresponds to the result of Theorem 3.

4. SIMULATIONS RESULTS: SIZE AND POWER OF THE TESTS

In this section we present some limited evidence on the size and power of the $\tilde{\eta}_\tau(\theta_*)$ and $P_\tau(\theta_*)$ tests in finite samples. To do so, we perform Monte Carlo experiments which perform the 5% upper tail test for the GLS-based KPSS test with $\ell = 0$ and the 5% lower tail test for the POI test, using the critical values obtained in the above section. The results are generated using the same random number generator as in section 3 and using 25,000 replications in every experiment. Data are generated according to equations (9A) and (9B), with $\omega_0 = 0$.

We first consider the size of the tests in the presence of iid and AR(1) errors, ω_t. Under the null hypothesis that $\theta = 1$, the distributions of the POI test and the KPSS test do not depend on the nuisance parameters ψ, ξ and σ_x, because the GLS residuals upon which the tests are based do not depend on ψ and ξ and the scale factor σ_x appears in numerator and denominator and cancels. However, the null distribution of the GLS-based KPSS test does depend on σ_x. We assume that $\psi = \xi = 0$ and $\sigma_x^2 = 1$ in our experiments. As in KPSS (1992), we consider AR(1) errors $\omega_t = \phi\omega_{t-1} + v_t$, where v_t are iid N(0, 1) and $\phi = \pm.8$, $\pm.5$, $\pm.2$ and 0. Then the relevant parameters in this experiment are the sample size T, the chosen value θ_* used in detrending, and the AR(1) coefficient ϕ. We consider θ_* = 1.0, 0.99, 0.969, 0.905, 0.73, 0.382, 0.01 and 0.0001, and sample sizes T = 30, 50, 100, 200, and 500.

Tables 3-7 summarize the simulation results for the size of the tests in terms of T, ϕ and θ_*. The results for the cases of ϕ = -.5 and -.8 are not tabulated because all the numbers are very close to zero, except for the GLS-based KPSS test with very small values of θ_*. Under the null hypothesis of $\theta = 1$, the AR coefficient ϕ conveniently measures the distance of the null hypothesis from the alternative. When ϕ = 0 so that X_t are iid errors, the tests have size equal to their nominal level of 5% (the first block of each Table). When ϕ = .8, an overrejection problem can be predicted because X_t approaches a pure random walk process as $\phi \to 1$. For the KPSS (θ_* = 1) and POI tests, the results in the Tables correspond to our expectations. For a given T and θ_*, we have severe overrejection as $\phi \to 1$ and underrejection as $\phi \to -1$. For a given θ_* and $\phi > 0$, we have more rejections as T increases, and for $\phi < 0$ we have less rejections as T increases. Given T and ϕ, as $\theta_* \to 0$, the POI test shows more severe overrejections for positive ϕ and less severe underrejections for negative ϕ (but with very little difference for the negative values of ϕ). As for the GLS-based KPSS test (θ_* less than unity), size depends upon T, ϕ and θ_* in a very complicated way. When θ_* is closer to unity, it suffers from more overrejection as T increases and $\phi \to 1$;

when θ_* is closer to 0, it shows underrejection even for ϕ = .8, especially as T increases (see Table 7A, θ_* = 0.01 and 0.0001).

Next we consider the power of the tests in the presence of iid errors. The relevant parameters are the sample size T, the chosen value θ_* and the actual value θ_1. The main point in this chapter is to compare the power of the POI test and GLS-based KPSS test (including the KPSS test) with different possible values of θ_* under the alternative hypothesis of different values of θ_1. (As before, θ_1 represents the actual value of θ in the DGP while θ_* is the value of θ chosen to construct the test.) More specifically, we perform the experiments with the following values of the relevant parameters; θ_1 = 0.99, 0.969, 0.905, 0.73, 0.382, 0.01, 0.0001; θ_* = 1.0, 0.99, 0.969, 0.905, 0.73, 0.382, 0.01, 0.0001; sample size T = 30, 50, 100, 200, and 500.

The simulation results are summarized in Tables 8-12. As expected, power increases for the KPSS test $\tilde{\eta}_\tau(1)$ and the POI tests as T increases and as θ_1 decreases. We expect that the POI test $P_\tau(\theta_*)$ should have the maximum power against a specific alternative hypothesis. Our simulation results support this expectation. We can see that the $P_\tau(\theta_*)$ test with $\theta_* = \theta_1$ generally has higher power than any other tests within each experiment block (value of T and θ_1) and this pattern is quite clear except for a few values of θ_1 near the null. The gain to using a POI test can be substantial; for example, for T = 30 and θ_1 = 0.382, compare the power of 0.837 for the POI test $P_\tau(0.382)$ to 0.720 for KPSS test $\tilde{\eta}_\tau(1)$ in experiment 8E in Table 8. In addition, the gain from using the POI test is quite robust to the choice of assumed value of θ, that is, θ_*. In particular, $P_\tau(.730)$ generally dominates the KPSS test except when the power is very low.

As for the GLS-based KPSS tests $\tilde{\eta}_\tau(\theta_*)$, the KPSS test $\tilde{\eta}_\tau(1)$ dominates all of the GLS-based KPSS tests with θ_* less than unity, at all values of θ_1, apart from small differences due to randomness. While the power of the tests with values of θ_* close to unity improves as θ_1 decreases and as T increases, the power of the tests with small value of θ_* does not improve much as θ_1 decreases or as T increases. These simulation results reflect the fact that $\tilde{\eta}_\tau(\theta_*)$ is not consistent for $\theta_* \in [0,1)$.

5. CONCLUDING REMARKS

By analogy to the previous chapter, we have proposed GLS-based tests and POI tests in the context of testing the null of stationarity against the alternative of a unit root. These tests are based on the residuals from a GLS regression of y_t on [1,t], with the covariance matrix $\Omega_N(\theta)$ using a chosen value $\theta_* \in (0,1]$ of the moving average parameter against which maximal power is desired. Our GLS-based KPSS test statistic $\tilde{\eta}_\tau(\theta_*)$ includes the KPSS test and the KPSS test based on BSP residuals as special cases, corresponding to $\theta_* = 1$ and $\theta_* = 0$. For $\theta_* \in (0,1)$, its asymptotic behavior resembles that of the latter rather than that of former; its asymptotic distribution depends on the marginal distribution of x, and the test is not consistent, in the sense that it has the same order of probability under the null and under the alternative hypotheses. Our simulation results show that the GLS-based KPSS tests have low power. In sum, the GLS-based KPSS test seems to be a failure.

As for the POI test, we expect that it will be at least as powerful as any other invariant test against $\theta_1 = \theta_*$ and also might be more powerful against θ_1 in a reasonable neighborhood of θ_*. This expectation is also supported by our Monte Carlo experiments. However, the POI test depends on the assumption of iid errors and should not be used in the presence of more general stationary errors. So more research is needed to develop an autocorrelation-robust version of the POI test, either in a parametric fashion as in Saikkonen and Luukkonen (1992a, b) or in a nonparametric fashion as in Phillips and Perron (1988).

TABLE 1a

90%, 95%, 97.5%, AND 99% CRITICAL VALUES OF $\tilde{\eta}_T(\theta_*)$

T	%	θ_*=1.0	.990	.969	.905	.730	.382	.010	.0001
30	90	.122	.122	.130	.255	1.562	4.196	6.437	6.473
	95	.148	.150	.160	.329	2.054	5.136	7.439	7.477
	97.5	.174	.177	.189	.403	2.529	5.861	8.191	8.231
	99	.209	.211	.227	.495	3.083	6.713	8.875	8.934
50	90	.121	.122	.145	.575	2.904	6.768	10.45	10.44
	95	.148	.151	.180	.771	3.814	8.267	12.09	12.06
	97.5	.174	.180	.217	.959	4.585	9.470	13.31	13.30
	99	.210	.215	.269	1.195	5.644	10.82	14.48	14.50
100	90	.119	.127	.258	1.725	6.102	13.58	20.16	20.37
	95	.149	.157	.345	2.402	8.081	16.79	23.65	23.63
	97.5	.178	.188	.432	3.042	9.880	19.36	26.14	26.16
	99	.213	.233	.548	3.091	12.00	21.93	28.60	28.60
200	90	.118	.161	.853	4.115	12.29	26.66	40.22	40.29
	95	.147	.208	1.203	5.618	16.07	33.15	47.05	47.21
	97.5	.176	.256	1.548	7.108	19.58	38.36	52.25	52.40
	99	.218	.318	1.979	9.133	24.23	43.66	56.90	56.80
500	90	.119	.574	3.188	10.80	30.83	65.37	99.05	100.6
	95	.147	.793	4.550	14.96	41.28	81.52	116.1	116.7
	97.5	.176	1.020	5.935	19.29	51.08	94.65	128.8	128.4
	99	.215	1.302	7.879	24.57	61.88	107.4	141.1	140.4

TABLE 1b

1%, 2.5%, 5%, AND 10% CRITICAL VALUES OF $P_T(\theta_*)$

T	%	θ_*=1	.990	.969	.905	.730	.382	.010	.0001
30	1	-	1.0094	1.0256	1.0468	1.0314	1.0938	1.2601	1.2659
	2.5	-	1.0095	1.0266	1.0559	1.0700	1.1607	1.3813	1.3795
	5	-	1.0096	1.0274	1.0625	1.1001	1.2245	1.4844	1.4875
	10	-	1.0097	1.0282	1.0694	1.1321	1.2949	1.6024	1.6123
50	1	-	1.0090	1.0216	1.0267	1.0321	1.1477	1.3914	1.4076
	2.5	-	1.0091	1.0234	1.0384	1.0620	1.2101	1.4877	1.4979
	5	-	1.0093	1.0247	1.0471	1.0890	1.2598	1.5672	1.5823
	10	-	1.0094	1.0259	1.0561	1.1175	1.3126	1.6681	1.6849
100	1	-	1.0079	1.0136	1.0144	1.0536	1.2275	1.5504	1.5633
	2.5	-	1.0083	1.0167	1.0249	1.0787	1.2690	1.6199	1.6332
	5	-	1.0086	1.0191	1.0338	1.0974	1.3033	1.6791	1.6936
	10	-	1.0089	1.0213	1.0432	1.1170	1.3414	1.7508	1.7630
200	1	-	1.0060	1.0058	1.0159	1.0813	1.2892	1.6725	1.6810
	2.5	-	1.0067	1.0098	1.0245	1.0965	1.3197	1.7190	1.7340
	5	-	1.0072	1.0127	1.0314	1.1095	1.3427	1.7642	1.7800
	10	-	1.0078	1.0158	1.0386	1.1236	1.3692	1.8142	1.8311
500	1	-	1.0026	1.0045	1.0260	1.1086	1.3468	1.7788	1.7986
	2.5	-	1.0037	1.0074	1.0309	1.1176	1.3647	1.8107	1.8282
	5	-	1.0046	1.0100	1.0351	1.1253	1.3786	1.8377	1.8569
	10	-	1.0055	1.0125	1.0398	1.1336	1.3950	1.8711	1.8886

TABLE 2

PERCENTILES OF POINT OPTIMAL TESTS $P_T(\theta_*)$, T = 500

T	%	θ_* = .990	.969	.905	.730	.382	.010	.0001
500	1	1.0026	1.0045	1.0260	1.1086	1.3468	1.7788	1.7986
	2.5	1.0037	1.0074	1.0309	1.1176	1.3647	1.8107	1.8282
	5	1.0046	1.0100	1.0351	1.1253	1.3786	1.8377	1.8569
	10	1.0055	1.0125	1.0398	1.1336	1.3950	1.8711	1.8886
	20	1.0064	1.0152	1.0451	1.1431	1.4148	1.9109	1.9277
	30	1.0070	1.0169	1.0485	1.1496	1.4290	1.9391	1.9565
	40	1.0074	1.0182	1.0515	1.1552	1.4411	1.9626	1.9800
	50	1.0077	1.0194	1.0540	1.1604	1.4520	1.9845	2.0024
	60	1.0080	1.0204	1.0563	1.1653	1.4626	2.0066	2.0250
	70	1.0082	1.0214	1.0586	1.1704	1.4741	2.0302	2.0495
	80	1.0085	1.0224	1.0614	1.1764	1.4873	2.0580	2.0778
	90	1.0088	1.0237	1.0649	1.1842	1.5055	2.0963	2.1180
	95	1.0090	1.0246	1.0674	1.1905	1.5207	2.1284	2.1513
	97.5	1.0091	1.0253	1.0695	1.1959	1.5332	2.1555	2.1791
	99	1.0092	1.0260	1.0717	1.2018	1.5474	2.1854	2.2119

TABLE 3

SIZE OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 30

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
3	30	0	1	1.0	.051	-
3	30	0	1	.990	.050	.034
3	30	0	1	.969	.051	.047
3	30	0	1	.905	.050	.050
3	30	0	1	.730	.050	.050
3	30	0	1	.382	.050	.050
3	30	0	1	.010	.050	.050
3	30	0	1	.0001	.050	.050
3A	30	.8	1	1.0	.769	-
3A	30	.8	1	.990	.764	.713
3A	30	.8	1	.969	.745	.756
3A	30	.8	1	.905	.573	.792
3A	30	.8	1	.730	.403	.881
3A	30	.8	1	.382	.214	.954
3A	30	.8	1	.010	.093	.972
3A	30	.8	1	.0001	.090	.970
3B	30	.5	1	1.0	.419	-
3B	30	.5	1	.990	.413	.351
3B	30	.5	1	.969	.394	.402
3B	30	.5	1	.905	.289	.442
3B	30	.5	1	.730	.200	.552
3B	30	.5	1	.382	.110	.717
3B	30	.5	1	.010	.051	.770
3B	30	.5	1	.0001	.055	.765
3C	30	.2	1	1.0	.139	-
3C	30	.2	1	.990	.139	.105
3C	30	.2	1	.969	.131	.129
3C	30	.2	1	.905	.114	.143
3C	30	.2	1	.730	.096	.175
3C	30	.2	1	.382	.074	.219
3C	30	.2	1	.010	.051	.257
3C	30	.2	1	.0001	.053	.247
3D	30	-.2	1	1.0	.013	-
3D	30	-.2	1	.990	.013	.007
3D	30	-.2	1	.969	.013	.011
3D	30	-.2	1	.905	.018	.013
3D	30	-.2	1	.730	.022	.009
3D	30	-.2	1	.382	.032	.005
3D	30	-.2	1	.010	.043	.005
3D	30	-.2	1	.0001	.043	.005

TABLE 4

SIZE OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 50

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
4	50	0	1	1.0	.050	-
4	50	0	1	.990	.051	.040
4	50	0	1	.969	.051	.050
4	50	0	1	.905	.050	.050
4	50	0	1	.730	.050	.050
4	50	0	1	.382	.050	.050
4	50	0	1	.010	.050	.050
4	50	0	1	.0001	.050	.050
4A	50	.8	1	1.0	.879	-
4A	50	.8	1	.990	.873	.857
4A	50	.8	1	.969	.835	.883
4A	50	.8	1	.905	.534	.927
4A	50	.8	1	.730	.360	.988
4A	50	.8	1	.382	.169	.999
4A	50	.8	1	.010	.052	.999
4A	50	.8	1	.0001	.059	.999
4B	50	.5	1	1.0	.479	-
4B	50	.5	1	.990	.473	.438
4B	50	.5	1	.969	.426	.482
4B	50	.5	1	.905	.262	.558
4B	50	.5	1	.730	.189	.770
4B	50	.5	1	.382	.103	.919
4B	50	.5	1	.010	.044	.944
4B	50	.5	1	.0001	.044	.943
4C	50	.2	1	1.0	.149	-
4C	50	.2	1	.990	.148	.125
4C	50	.2	1	.969	.137	.147
4C	50	.2	1	.905	.105	.166
4C	50	.2	1	.730	.091	.235
4C	50	.2	1	.382	.074	.330
4C	50	.2	1	.010	.051	.368
4C	50	.2	1	.0001	.052	.371
4D	50	-.2	1	1.0	.012	-
4D	50	-.2	1	.990	.012	.009
4D	50	-.2	1	.969	.013	.011
4D	50	-.2	1	.905	.019	.011
4D	50	-.2	1	.730	.021	.005
4D	50	-.2	1	.382	.032	.003
4D	50	-.2	1	.010	.044	.002
4D	50	-.2	1	.0001	.046	.002

TABLE 5

SIZE OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 100

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
5	100	0	1	1.0	.051	-
5	100	0	1	.990	.050	.046
5	100	0	1	.969	.050	.050
5	100	0	1	.905	.050	.050
5	100	0	1	.730	.050	.050
5	100	0	1	.382	.050	.050
5	100	0	1	.010	.050	.050
5	100	0	1	.0001	.050	.050
5A	100	.8	1	1.0	.949	-
5A	100	.8	1	.990	.944	.951
5A	100	.8	1	.969	.760	.967
5A	100	.8	1	.905	.488	.996
5A	100	.8	1	.730	.305	1.00
5A	100	.8	1	.382	.099	1.00
5A	100	.8	1	.010	.026	1.00
5A	100	.8	1	.0001	.028	1.00
5B	100	.5	1	1.0	.518	-
5B	100	.5	1	.990	.512	.523
5B	100	.5	1	.969	.331	.573
5B	100	.5	1	.905	.244	.768
5B	100	.5	1	.730	.168	.964
5B	100	.5	1	.382	.083	.997
5B	100	.5	1	.010	.038	.999
5B	100	.5	1	.0001	.037	.999
5C	100	.2	1	1.0	.154	-
5C	100	.2	1	.990	.158	.154
5C	100	.2	1	.969	.119	.163
5C	100	.2	1	.905	.105	.227
5C	100	.2	1	.730	.086	.378
5C	100	.2	1	.382	.062	.554
5C	100	.2	1	.010	.049	.614
5C	100	.2	1	.0001	.053	.616
5D	100	-.2	1	1.0	.011	-
5D	100	-.2	1	.990	.012	.010
5D	100	-.2	1	.969	.015	.009
5D	100	-.2	1	.905	.018	.006
5D	100	-.2	1	.730	.021	.002
5D	100	-.2	1	.382	.032	.000
5D	100	-.2	1	.010	.044	.000
5D	100	-.2	1	.0001	.048	.000

TABLE 6

SIZE OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 200

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
6	200	0	1	1.0	.050	-
6	200	0	1	.990	.050	.045
6	200	0	1	.969	.050	.049
6	200	0	1	.905	.050	.050
6	200	0	1	.730	.050	.050
6	200	0	1	.382	.050	.050
6	200	0	1	.010	.050	.050
6	200	0	1	.0001	.050	.050
6A	200	.8	1	1.0	.976	-
6A	200	.8	1	.990	.932	.978
6A	200	.8	1	.969	.579	.995
6A	200	.8	1	.905	.448	1.00
6A	200	.8	1	.730	.259	1.00
6A	200	.8	1	.382	.068	1.00
6A	200	.8	1	.010	.011	1.00
6A	200	.8	1	.0001	.012	1.00
6B	200	.5	1	1.0	.563	-
6B	200	.5	1	.990	.453	.561
6B	200	.5	1	.969	.264	.688
6B	200	.5	1	.905	.234	.947
6B	200	.5	1	.730	.169	1.00
6B	200	.5	1	.382	.075	1.00
6B	200	.5	1	.010	.032	1.00
6B	200	.5	1	.0001	.030	1.00
6C	200	.2	1	1.0	.167	-
6C	200	.2	1	.990	.138	.154
6C	200	.2	1	.969	.108	.192
6C	200	.2	1	.905	.109	.340
6C	200	.2	1	.730	.093	.606
6C	200	.2	1	.382	.064	.823
6C	200	.2	1	.010	.045	.874
6C	200	.2	1	.0001	.047	.876
6D	200	-.2	1	1.0	.010	-
6D	200	-.2	1	.990	.013	.009
6D	200	-.2	1	.969	.014	.007
6D	200	-.2	1	.905	.018	.002
6D	200	-.2	1	.730	.022	.000
6D	200	-.2	1	.382	.033	.000
6D	200	-.2	1	.010	.044	.000
6D	200	-.2	1	.0001	.047	.000

TABLE 7

SIZE OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 500

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
7	500	0	1	1.0	.050	-
7	500	0	1	.990	.050	.048
7	500	0	1	.969	.050	.050
7	500	0	1	.905	.050	.050
7	500	0	1	.730	.050	.050
7	500	0	1	.382	.050	.050
7	500	0	1	.010	.050	.050
7	500	0	1	.0001	.050	.050
7A	500	.8	1	1.0	.987	-
7A	500	.8	1	.990	.655	.997
7A	500	.8	1	.969	.509	1.00
7A	500	.8	1	.905	.415	1.00
7A	500	.8	1	.730	.218	1.00
7A	500	.8	1	.382	.053	1.00
7A	500	.8	1	.010	.007	1.00
7A	500	.8	1	.0001	.006	1.00
7B	500	.5	1	1.0	.578	-
7B	500	.5	1	.990	.288	.670
7B	500	.5	1	.969	.248	.933
7B	500	.5	1	.905	.225	1.00
7B	500	.5	1	.730	.152	1.00
7B	500	.5	1	.382	.074	1.00
7B	500	.5	1	.010	.029	1.00
7B	500	.5	1	.0001	.029	1.00
7C	500	.2	1	1.0	.169	-
7C	500	.2	1	.990	.110	.185
7C	500	.2	1	.969	.103	.311
7C	500	.2	1	.905	.105	.601
7C	500	.2	1	.730	.085	.927
7C	500	.2	1	.382	.067	.994
7C	500	.2	1	.010	.048	.997
7C	500	.2	1	.0001	.050	.998
7D	500	-.2	1	1.0	.010	-
7D	500	-.2	1	.990	.014	.007
7D	500	-.2	1	.969	.016	.003
7D	500	-.2	1	.905	.019	.000
7D	500	-.2	1	.730	.021	.000
7D	500	-.2	1	.382	.032	.000
7D	500	-.2	1	.010	.047	.000
7D	500	-.2	1	.0001	.046	.000

TABLE 8

POWER OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 30

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
8C	30	0	.905	1.0	.074	-
8C	30	0	.905	.990	.076	.053
8C	30	0	.905	.969	.071	.068
8C	30	0	.905	.905	.073	.074
8C	30	0	.905	.730	.068	.076
8C	30	0	.905	.382	.060	.068
8C	30	0	.905	.010	.051	.065
8C	30	0	.905	.0001	.055	.064
8D	30	0	.730	1.0	.281	-
8D	30	0	.730	.990	.275	.231
8D	30	0	.730	.969	.273	.269
8D	30	0	.730	.905	.244	.276
8D	30	0	.730	.730	.199	.287
8D	30	0	.730	.382	.125	.257
8D	30	0	.730	.010	.081	.210
8D	30	0	.730	.0001	.080	.207
8E	30	0	.382	1.0	.720	-
8E	30	0	.382	.990	.715	.668
8E	30	0	.382	.969	.701	.707
8E	30	0	.382	.905	.591	.730
8E	30	0	.382	.730	.480	.790
8E	30	0	.382	.382	.314	.837
8E	30	0	.382	.010	.163	.809
8E	30	0	.382	.0001	.163	.799
8F	30	0	.010	1.0	.881	-
8F	30	0	.010	.990	.877	.845
8F	30	0	.010	.969	.867	.874
8F	30	0	.010	.905	.737	.895
8F	30	0	.010	.730	.592	.948
8F	30	0	.010	.382	.410	.985
8F	30	0	.010	.010	.221	.989
8F	30	0	.010	.0001	.221	.989
8G	30	0	.0001	1.0	.884	-
8G	30	0	.0001	.990	.876	.847
8G	30	0	.0001	.969	.870	.879
8G	30	0	.0001	.905	.741	.897
8G	30	0	.0001	.730	.596	.952
8G	30	0	.0001	.382	.407	.984
8G	30	0	.0001	.010	.220	.990
8G	30	0	.0001	.0001	.219	.990

TABLE 9

POWER OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 50

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
9C	50	0	.905	1.0	.129	-
9C	50	0	.905	.990	.126	.108
9C	50	0	.905	.969	.124	.125
9C	50	0	.905	.905	.110	.126
9C	50	0	.905	.730	.088	.117
9C	50	0	.905	.382	.069	.095
9C	50	0	.905	.010	.057	.079
9C	50	0	.905	.0001	.058	.083
9D	50	0	.730	1.0	.540	-
9D	50	0	.730	.990	.538	.508
9D	50	0	.730	.969	.523	.547
9D	50	0	.730	.905	.418	.570
9D	50	0	.730	.730	.316	.590
9D	50	0	.730	.382	.175	.552
9D	50	0	.730	.010	.098	.440
9D	50	0	.730	.0001	.099	.437
9E	50	0	.382	1.0	.911	-
9E	50	0	.382	.990	.908	.897
9E	50	0	.382	.969	.883	.909
9E	50	0	.382	.905	.703	.936
9E	50	0	.382	.730	.568	.973
9E	50	0	.382	.382	.368	.983
9E	50	0	.382	.010	.190	.977
9E	50	0	.382	.0001	.192	.979
9F	50	0	.010	1.0	.973	-
9F	50	0	.010	.990	.970	.965
9F	50	0	.010	.969	.959	.973
9F	50	0	.010	.905	.775	.985
9F	50	0	.010	.730	.639	.998
9F	50	0	.010	.382	.434	1.00
9F	50	0	.010	.010	.236	1.00
9F	50	0	.010	.0001	.234	1.00
9G	50	0	.0001	1.0	.972	-
9G	50	0	.0001	.990	.971	.966
9G	50	0	.0001	.969	.959	.974
9G	50	0	.0001	.905	.782	.985
9G	50	0	.0001	.730	.637	.999
9G	50	0	.0001	.382	.439	1.00
9G	50	0	.0001	.010	.236	1.00
9G	50	0	.0001	.0001	.238	1.00

TABLE 10

POWER, OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 100

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
10A	100	0	.990	1.0	.052	-
10A	100	0	.990	.990	.053	.048
10A	100	0	.990	.969	.053	.054
10A	100	0	.990	.905	.050	.051
10A	100	0	.990	.730	.051	.052
10A	100	0	.990	.382	.048	.050
10A	100	0	.990	.010	.050	.049
10A	100	0	.990	.0001	.052	.048
10B	100	0	.969	1.0	.077	-
10B	100	0	.969	.990	.083	.076
10B	100	0	.969	.969	.077	.080
10B	100	0	.969	.905	.069	.082
10B	100	0	.969	.730	.059	.071
10B	100	0	.969	.382	.051	.059
10B	100	0	.969	.010	.051	.057
10B	100	0	.969	.0001	.054	.058
10C	100	0	.905	1.0	.342	-
10C	100	0	.905	.990	.347	.339
10C	100	0	.905	.969	.300	.351
10C	100	0	.905	.905	.239	.366
10C	100	0	.905	.730	.140	.318
10C	100	0	.905	.382	.080	.231
10C	100	0	.905	.010	.064	.164
10C	100	0	.905	.0001	.069	.169
10D	100	0	.730	1.0	.877	-
10D	100	0	.730	.990	.872	.875
10D	100	0	.730	.969	.771	.890
10D	100	0	.730	.905	.625	.930
10D	100	0	.730	.730	.449	.950
10D	100	0	.730	.382	.245	.927
10D	100	0	.730	.010	.133	.872
10D	100	0	.730	.0001	.130	.865
10E	100	0	.382	1.0	.993	-
10E	100	0	.382	.990	.992	.993
10E	100	0	.382	.969	.956	.994
10E	100	0	.382	.905	.787	.999
10E	100	0	.382	.730	.631	1.00
10E	100	0	.382	.382	.404	1.00
10E	100	0	.382	.010	.217	1.00
10E	100	0	.382	.0001	.220	1.00

TABLE 11

POWER OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 200

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
11A	200	0	.990	1.0	.066	-
11A	200	0	.990	.990	.060	.055
11A	200	0	.990	.969	.059	.057
11A	200	0	.990	.905	.055	.061
11A	200	0	.990	.730	.055	.057
11A	200	0	.990	.382	.049	.054
11A	200	0	.990	.010	.046	.053
11A	200	0	.990	.0001	.048	.053
11B	200	0	.969	1.0	.194	-
11B	200	0	.969	.990	.178	.181
11B	200	0	.969	.969	.147	.183
11B	200	0	.969	.905	.107	.168
11B	200	0	.969	.730	.073	.123
11B	200	0	.969	.382	.056	.096
11B	200	0	.969	.010	.051	.076
11B	200	0	.969	.0001	.055	.079
11C	200	0	.905	1.0	.735	-
11C	200	0	.905	.990	.683	.722
11C	200	0	.905	.969	.534	.759
11C	200	0	.905	.905	.418	.798
11C	200	0	.905	.730	.240	.738
11C	200	0	.905	.382	.112	.602
11C	200	0	.905	.010	.073	.456
11C	200	0	.905	.0001	.075	.456
11D	200	0	.730	1.0	.990	-
11D	200	0	.730	.990	.981	.991
11D	200	0	.730	.969	.872	.996
11D	200	0	.730	.905	.742	.999
11D	200	0	.730	.730	.553	1.00
11D	200	0	.730	.382	.318	1.00
11D	200	0	.730	.010	.165	.998
11D	200	0	.730	.0001	.165	.998
11E	200	0	.382	1.0	1.00	-
11E	200	0	.382	.990	1.00	1.00
11E	200	0	.382	.969	.955	1.00
11E	200	0	.382	.905	.838	1.00
11E	200	0	.382	.730	.665	1.00
11E	200	0	.382	.382	.429	1.00
11E	200	0	.382	.010	.238	1.00
11E	200	0	.382	.0001	.233	1.00

TABLE 12

POWER OF $\tilde{\eta}_T(\theta_*)$ AND $P_T(\theta_*)$ TESTS, T = 500

Exp. No.	T	ϕ	θ_1	θ_*	$\tilde{\eta}_T(\theta_*)$	$P_T(\theta_*)$
12A	500	0	.990	1.0	.141	-
12A	500	0	.990	.990	.116	.133
12A	500	0	.990	.969	.093	.127
12A	500	0	.990	.905	.070	.102
12A	500	0	.990	.730	.051	.076
12A	500	0	.990	.382	.051	.062
12A	500	0	.990	.010	.052	.054
12A	500	0	.990	.0001	.052	.058
12B	500	0	.969	1.0	.615	-
12B	500	0	.969	.990	.465	.626
12B	500	0	.969	.969	.369	.669
12B	500	0	.969	.905	.221	.603
12B	500	0	.969	.730	.107	.454
12B	500	0	.969	.382	.068	.289
12B	500	0	.969	.010	.058	.190
12B	500	0	.969	.0001	.058	.192
12C	500	0	.905	1.0	.983	-
12C	500	0	.905	.990	.889	.989
12C	500	0	.905	.969	.772	.997
12C	500	0	.905	.905	.614	.999
12C	500	0	.905	.730	.375	.997
12C	500	0	.905	.382	.191	.986
12C	500	0	.905	.010	.103	.953
12C	500	0	.905	.0001	.102	.953
12D	500	0	.730	1.0	1.00	-
12D	500	0	.730	.990	.995	1.00
12D	500	0	.730	.969	.947	1.00
12D	500	0	.730	.905	.824	1.00
12D	500	0	.730	.730	.624	1.00
12D	500	0	.730	.382	.391	1.00
12D	500	0	.730	.010	.215	1.00
12D	500	0	.730	.0001	.203	1.00
12E	500	0	.382	1.0	1.00	-
12E	500	0	.382	.990	.999	1.00
12E	500	0	.382	.969	.972	1.00
12E	500	0	.382	.905	.869	1.00
12E	500	0	.382	.730	.671	1.00
12E	500	0	.382	.382	.451	1.00
12E	500	0	.382	.010	.253	1.00
12E	500	0	.382	.0001	.252	1.00

APPENDIX 1

LEMMA 2. Under Assumption B (unit root),

i) $\Theta_A^*(L)X_T = O_p(T^{1/2})$ and

ii) $\Theta_B^*(L)X_T = O_p(1)$.

Proof. The polynomial $\Theta_A^*(L)X_T$ can be expressed equivalently as (A1.2) through a well known polynomial decomposition, which decomposes a linear filter into long run and transitory elements.

(A1.1) $\Theta_A^*(L) \equiv \Theta_A^*(1) - (1-L)\tilde{\Theta}_A^*(L)$,

where $\tilde{\Theta}_A^*(L) = \sum_{j=0}^{T-2} \delta_j L^j$ and $\delta_j = \sum_{i=j+1}^{T-1} \theta_*^{\,i} = (\theta_*^{\,j+1} - \theta_*^{\,T})/(1-\theta_*)$. Thus we have

(A1.2) $\Theta_A^*(L)X_T = \Theta_A^*(1)X_T - \tilde{\Theta}_A^*(L)\Delta X_T$.

The first term equals $[(1-\theta_*^{\,T})/(1-\theta_*)]X_T$ because $\Theta_A^*(1) = \sum_{i=0}^{T-1} \theta_*^{\,i}$ $= (1-\theta_*^{\,T})/(1-\theta_*)$. For the second term, a little algebra shows that

(A1.3) $$\tilde{\Theta}_A^*(L)\Delta X_T = [\theta_*/(1-\theta_*)][\Delta X_T + \theta_*\Delta X_{T-1} + \cdots + \theta_*^{\,T-2}\Delta X_2 + \theta_*^{\,T-1}X_1] - [\theta_*^{\,T}/(1-\theta_*)]X_T$$
$$= [\theta_*/(1-\theta_*)]\Theta_A^*(L)\Delta X_T - [\theta_*^{\,T}/(1-\theta_*)]X_T.$$

Substituting $\Theta_A^*(1)X_T$ and (A1.3) into (A1.2) yields

(A1.4) $\Theta_A^*(L)X_T = [1/(1-\theta_*)]X_T - [\theta_*/(1-\theta_*)]\Theta_A^*(L)\Delta X_T$.

Under Assumption B, that is, X_t has a unit root, X_T is $O_p(T^{1/2})$ and ΔX_t and its absolutely summable series $\Theta_A^*(L)\Delta X_T$ are $O_p(1)$, so the result follows.

Similarly, $\Theta_B^*(L)X_T$ can be decomposed into as follows:

(A1.5) $$\Theta_B^*(L)X_T = \Theta_B^*(1)X_T - \tilde{\Theta}_B^*(L)\Delta X_T,$$

where $\tilde{\Theta}_B^*(L) = \sum_{j=0}^{T-2} d_j L^j$ and $d_j = \sum_{i=0}^{T-2-j} \theta_*^{\,i} = [(1-\theta_*^{\,T-1-j})/(1-\theta_*)]$. Since $\Theta_B^*(1) = \Theta_A^*(1) = \sum_{i=0}^{T-1} \theta_*^{\,i} = (1-\theta_*^{\,T})/(1-\theta_*)$, the first term equals $[(1-\theta_*^{\,T})/(1-\theta_*)]X_T$. The second term can be written as

(A1.6) $$\begin{aligned}\tilde{\Theta}_B^*(L)\Delta X_T &= [1/(1-\theta_*)]\{X_T - [\theta_*^{\,T-1}\Delta X_T + \cdots + \theta_*\Delta X_2 + X_1]\} \\ &= [1/(1-\theta_*)]X_T - [1/(1-\theta_*)]\Theta_B^*(L)\Delta X_T.\end{aligned}$$

Substituting $\Theta_B^*(1)X_T$ and (A1.6) into (A1.5) gives

(A1.7) $$\Theta_B^*(L)X_T = -[\theta_*^{\,T}/(1-\theta_*)]X_T + [1/(1-\theta_*)]\Theta_B^*(L)\Delta X_T.$$

Since $\lim_{T\to\infty}\theta_*^{\,T} \to 0$, the first term is $o_p(1)$ and ΔX_t and its absolutely summable series $\Theta_B^*(L)\Delta X_T$ are $O_p(1)$ under Assumption B. Hence, the result follows.

APPENDIX 2

In this Appendix, we derive GLS estimates and residuals, which will be used for constructing the GLS-based KPSS test, $\tilde{\eta}_\tau(\theta_*)$, and the POI test, $P_\tau(\theta_*)$. We show that $\tilde{\eta}_\tau(1) = \hat{\eta}_\tau$ (KPSS test statistic based on the OLS residuals) and $\tilde{\eta}_\tau(0) = \bar{\eta}_\tau$ (KPSS test based on BSP residuals). Let $\gamma' = [\psi, \xi]$ as in equation (9') of the main text, and define $\hat{\gamma}$ and $\bar{\gamma}$ to be the OLS and BSP estimates, respectively. Let $\tilde{\gamma}(\theta_*)$ be the GLS estimates using the covariance matrix $\Omega_N(\theta_*)$ so that $\tilde{e}_t(\theta_*) = y_t - z_t'\tilde{\gamma}(\theta_*)$. Then it is sufficient to show that $\tilde{\gamma}(1) = \hat{\gamma}$ and $\tilde{\gamma}(0) = \bar{\gamma}$.

We start with the derivation of the GLS estimates. Let $\tilde{Z}^* = C^{-1}(\theta_*)C(1)Z$ and $\tilde{y}^* = C^{-1}(\theta_*)C(1)y$ be the transformed variables as in equation (17) of the main text. They have the following form:

$$(A2.1) \qquad \tilde{Z}^{*\prime} = \begin{bmatrix} 1 & \theta_* & \theta_*^2 & \cdots & \theta_*^{T-1} \\ 1 & 1+\theta_* & 1+\theta_*+\theta_*^2 & \cdots & 1+\theta_*+\cdots\theta_*^{T-1} \end{bmatrix}$$

$$(A2.2) \quad \tilde{y}^{*\prime} = \begin{bmatrix} y_1 & \Delta y_2+\theta_* y_1 & \Delta y_3+\theta_*\Delta y_2+\theta_*^2 y_1 & \cdots & \Delta y_T+\theta_*\Delta y_{T-1}+\cdots+\theta_*^{T-1}y_1 \end{bmatrix}$$

$$\equiv \begin{bmatrix} \Theta_A^*(L)\Delta y_1 & \Theta_A^*(L)\Delta y_2 & \Theta_A^*(L)\Delta y_3 & \cdots & \Theta_A^*(L)\Delta y_T \end{bmatrix}.$$

Sinc

estimatio

(A2.3) be

$\theta_* = 1$, sc

so that $\bar{\gamma}$

(A2.3)

=

.

=

.

(A2.4) $\bar{\gamma}$(

Since GLS estimation from (9') is identical to OLS estimation from (17), the GLS estimates $\tilde{\gamma}(\theta_*)$ are defined as (A2.3) below. From (A2.1) and (A2.2), $\tilde{Z}^* = Z$ and $\tilde{y}^* = y$ for $\theta_* = 1$, so that $\tilde{\gamma}(1) = \hat{\gamma}$; and for $\theta_* = 0$, $\tilde{Z}^* = \Delta Z$ and $\tilde{y}^* = \Delta y$, so that $\tilde{\gamma}(0) = \bar{\gamma}$. For any $\theta_* \in (0,1)$,

$$\text{(A2.3)} \qquad \tilde{\gamma}(\theta_*) = [Z'\Omega_N^{-1}(\theta_*)Z]^{-1}Z'\Omega_N^{-1}(\theta_*)y$$
$$= [\tilde{Z}^{*\prime}\tilde{Z}^*]^{-1}\tilde{Z}^{*\prime}\tilde{y}^*$$

$$= \begin{bmatrix} (1-\theta_*^{2T})/(1-\theta_*^2) & \sum_{i=1}^{T}\theta_*^{i-1}(1-\theta_*^i)/(1-\theta_*) \\ \sum_{i=1}^{T}\theta_*^{i-1}(1-\theta_*^i)/(1-\theta_*) & \sum_{i=1}^{T}[(1-\theta_*^i)/(1-\theta_*)]^2 \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} \sum_{t=1}^{T}\theta_*^{t-1}\Theta_A^*(L)\Delta y_t \\ \sum_{t=1}^{T}[(1-\theta_*^t)/(1-\theta_*)]\Theta_A^*(L)\Delta y_t \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(1-\theta_*^{2T})}{(1-\theta_*^2)} & \dfrac{(1-\theta_*^T)(1-\theta_*^{T+1})}{(1-\theta_*)(1-\theta_*^2)} \\ \dfrac{(1-\theta_*^T)(1-\theta_*^{T+1})}{(1-\theta_*)(1-\theta_*^2)} & \dfrac{T(1-\theta_*^2)-2\theta_*(1-\theta_*^T)-\theta_*^2(1-\theta_*^T)^2}{(1-\theta_*)^2(1-\theta_*^2)} \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} (1+\theta_*)^{-1}\Theta_B^*(L)y_T + (1+\theta_*)^{-1}\theta_*^T\Theta_A^*(L)y_T \\ -(1-\theta_*^2)^{-1}\theta_*\Theta_B^*(L)y_T+(1-\theta_*)^{-1}[1-\theta_*^{T+1}/(1+\theta_*)]\Theta_A^*(L)y_T \end{bmatrix}.$$

$$\text{(A2.4)} \qquad \tilde{\gamma}(\theta_*) - \gamma = [Z'\Omega_N^{-1}(\theta_*)Z]^{-1}Z'\Omega_N^{-1}(\theta_*)x$$
$$= [\tilde{Z}^{*\prime}\tilde{Z}^*]^{-1}\tilde{Z}^{*\prime}\tilde{x}^*$$

=

.

Simplifyin

given θ_* ϵ

(A2.5) [

=

.

Note that

For

propertie

assumptio

these alt

Unde

we have (f

(A2.6) $\bar{e}_t$

$$
= \begin{bmatrix} \dfrac{(1-\theta_*^{2T})}{(1-\theta_*^2)} & \dfrac{(1-\theta_*^T)(1-\theta_*^{T+1})}{(1-\theta_*)(1-\theta_*^2)} \\ \dfrac{(1-\theta_*^T)(1-\theta_*^{T+1})}{(1-\theta_*)(1-\theta_*^2)} & \dfrac{T(1-\theta_*^2)-2\theta_*(1-\theta_*^T)-\theta_*^2(1-\theta_*^T)^2}{(1-\theta_*)^2(1-\theta_*^2)} \end{bmatrix}^{-1}
$$

$$
\cdot \begin{bmatrix} (1+\theta_*)^{-1}\Theta_B^*(L)X_T + (1+\theta_*)^{-1}\theta_*^T\Theta_A^*(L)X_T \\ -(1-\theta_*^2)^{-1}\theta_*\Theta_B^*(L)X_T + (1-\theta_*)^{-1}[1-\theta_*^{T+1}/(1+\theta_*)]\Theta_A^*(L)X_T \end{bmatrix}.
$$

Simplifying the elements of (A2.4) using $\lim_{T\to\infty}\theta_*^T \to 0$ for any given $\theta_* \in (0,1)$, we have

(A2.5) $[\tilde{\gamma}(\theta_*) - \gamma]$

$$
= \begin{bmatrix} \dfrac{1}{(1-\theta_*^2)} & \dfrac{1}{(1-\theta_*)(1-\theta_*^2)} \\ \dfrac{1}{(1-\theta_*)(1-\theta_*^2)} & \dfrac{T(1-\theta_*^2)-2\theta_*-\theta_*^2}{(1-\theta_*)^2(1-\theta_*^2)} \end{bmatrix}^{-1}
$$

$$
\cdot \begin{bmatrix} (1+\theta_*)^{-1}\Theta_B^*(L)X_T \\ -(1-\theta_*^2)^{-1}\theta_*\Theta_B^*(L)X_T + (1-\theta_*)^{-1}\Theta_A^*(L)X_T \end{bmatrix} + o_p(1).
$$

Note that this expression does not apply for $\theta_* = 1$.

For our asymptotic analysis we have to consider the properties of the GLS residuals under our alternative assumptions because they show quite different behavior under these alternative assumptions.

Under Assumption A, X_t, $\Theta_A^*(L)X_T$, and $\Theta_B^*(L)X_T$ are $O_p(1)$, so we have (for t = 1,...,T):

(A2.6) $\tilde{e}_t(\theta_*) = y_t - z_t'\tilde{\gamma}(\theta_*) = X_t - z_t'[\tilde{\gamma}(\theta_*) - \gamma]$

$= X_t$

When $\theta_* =$

which is a

$\theta_* = 1$, (A

(A2.7) $\hat{e}_t$

For

convenient

are relate

(A2.8) $\tilde{e}$

$=$

Note that

under the

residuals

overdiffer

Under

normalized

matrix

(A2.10)

Then

(A2.10) T

$= T^{-1/2}$

Now consid

$$= X_t - (1-\theta_*)\Theta_B^*(L)X_T - (t/T)(1-\theta_*)[\Theta_A^*(L)-\Theta_B^*(L)]X_T + o_p(1).$$

When $\theta_* = 0$, we can see that $\tilde{e}_t(0) = X_t - X_1 - (t/T)(X_T - X_1)$, which is asymptotically equivalent to the BSP residuals. For $\theta_* = 1$, (A2.5) and (A2.6) do not apply, but we have simply

$$\text{(A2.7)} \quad \hat{e}_t = \tilde{e}_t(1) = \text{OLS residuals from } (9').$$

For the construction of the POI test, it is more convenient to use the OLS residuals $\tilde{e}_t^*(\theta_*)$ from (17), which are related to $\tilde{e}_t(\theta_*)$ by (18) in the main text. Specifically,

$$\text{(A2.8)} \quad \tilde{e}_t^*(\theta_*) = \tilde{y}_t^* - \tilde{z}_t^{*\prime}\tilde{\gamma}(\theta_*) = \tilde{X}_t^* - \tilde{z}_t^{*\prime}[\tilde{\gamma}(\theta_*) - \gamma]$$
$$= \Theta_A^*(L)\Delta X_t - (1-\theta_*)\theta_*^{t-1}\Theta_1(L)X_T + o_p(1), \; t = 1,..,T.$$

Note that these residuals $\tilde{e}_t(\theta_*)$ are $O_p(1)$ under the null and under the alternative hypotheses. This is so because the residuals take the form of an exponentially weighted series of overdifferenced processes under the null.

Under Assumption B (unit root), we consider the normalized GLS residuals series $T^{-1/2}\tilde{e}(\theta_*)$. Define D as the matrix

$$\text{(A2.10)} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}, \text{ so that } D^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & T^{-1/2} \end{bmatrix}.$$

Then

$$\text{(A2.10)} \quad T^{-1/2}\tilde{e}_t(\theta_*) = T^{-1/2}X_t - T^{-1/2}z_t{}'D^{-1/2}D^{1/2}[\tilde{\gamma}(\theta_*) - \gamma]$$
$$= T^{-1/2}X_t - z_t{}'(T^{-1/2}D^{-1/2})[D^{-1/2}Z'\Omega_N^{-1}(\theta_*)ZD^{-1/2}]^{-1}D^{-1/2}Z'\Omega_N^{-1}(\theta_*)x.$$

Now consider the terms on the right hand side of equation

(A2.10). We have

$$\text{(A2.11)} \qquad z_t'(T^{-1/2}D^{-1/2}) = [\ T^{-1/2} \quad t/T\ ].$$

For the term $[D^{-1/2}Z'\Omega_N^{-1}(\theta_*)ZD^{-1/2}]^{-1}$, note that $Z'\Omega_N^{-1}(\theta_*)Z$ is as given in the first matrix on the right hand side of equation (A2.4). For any given $\theta_* \in (0,1)$, pre- and post-multiplying $Z'\Omega_N^{-1}(\theta_*)Z$ by $D^{-1/2}$ and taking probability limits of the elements (using the fact that $\lim_{T\to\infty}\theta_*^T \to 0$) yields

$$\text{(A2.12)} \qquad \text{plim } [D^{-1/2}Z'\Omega_N^{-1}(\theta_*)ZD^{-1/2}]^{-1} = \begin{bmatrix} 1-\theta_*^2 & 0 \\ 0 & (1-\theta_*)^2 \end{bmatrix}.$$

For the term $D^{-1/2}Z'\Omega_N^{-1}(\theta_*)x$, note that $Z'\Omega_N^{-1}(\theta_*)x$ is the same as the second matrix of the right hand side of equation (A2.4). Premultiplying it by $D^{-1/2}$ and taking probability limits of the elements yields (using Lemmas 1 and 2)

$$\text{(A2.13)} \quad \text{plim } D^{-1/2}Z'\Omega_N^{-1}(\theta_*)x = \begin{bmatrix} (1+\theta_*)^{-1}\Theta_B^*(L)X_T \\ T^{-1/2}(1-\theta_*)^{-1}\Theta_A^*(L)X_T \end{bmatrix} + o_p(1)$$

$$= \begin{bmatrix} (1+\theta_*)^{-1}\Theta_B^*(L)X_T \\ T^{-1/2}(1-\theta_*)^{-2}X_T \end{bmatrix} + o_p(1).$$

Note that the second equality follows from equation (A1.4).

Now substituting (A2.11), (A2.12) and (A2.13) into (A2.10) and doing some algebra yields

$$\text{(A2.14)} \quad T^{-1/2}\tilde{e}_t(\theta_*) = T^{-1/2}X_t - (t/T)X_T + o_p(1),\ t = 1,\ldots,T.$$

Note that these are asymptotically equal to the BSP residuals; asymptotically, they do not depend on the value of θ_*.

APPENDIX 3

In this Appendix we derive the asymptotic distribution of the new GLS-based KPSS test under our alternative assumptions. We show that its asymptotic distribution depends on the marginal distribution of x, and that the test is not consistent, because it has the same order of probability under the null and alternative hypotheses.

We consider the test with $\theta_* \in (0,1)$, because the asymptotics for $\theta_* = 1$ and $\theta_* = 0$ are given by KPSS (1992) and by Schmidt (1992), respectively. For the long run variance estimator, we consider only the case $\ell = 0$. For the case $\ell \neq 0$, the same results as in Schmidt (1992) can be derived without difficulty, just by applying the results of this Appendix and Schmidt (1992).

We first prove Theorem 1 under Assumption A. From Appendix 2, the GLS residuals are given as

$$\text{(A3.1)}\quad \tilde{e}_t(\theta_*) = X_t - (1-\theta_*)\Theta_B^*(L)X_T - (t/T)(1-\theta_*)[\Theta_A^*(L) - \Theta_B^*(L)]X_T + o_p(1).$$

Denote the weak limit of X_T as $T \to \infty$ by X_∞. Then under Assumption A (stationarity), for any $\theta_* \in [0,1)$,

$$\text{(A3.2)}\quad T^{-1}\tilde{S}_{[rT]}(\theta_*) \Rightarrow -(1-\theta_*)\{r\Theta_B^*(L)X_\infty + (r^2/2)[\Theta_A^*(L) - \Theta_B^*(L)]X_\infty\}.$$

Proof. $$T^{-1}\tilde{S}_{[rT]} = T^{-1}\sum_{j=1}^{[rT]}\tilde{e}_j(\theta_*)$$

$$= T^{-1}\sum_{j=1}^{[rT]}X_t - T^{-1}[rT](1-\theta_*)\Theta_B^*(L)X_T$$

$$- T^{-2}(1-\theta_*)[\Theta_A^*(L) - \Theta_B^*(L)]X_T\sum_{j=1}^{[rT]}j + o_p(1).$$

The first term converges in probability to zero and the second term converges to $r(1-\theta_*)\Theta_B^*(L)X_\infty$. For the third term, we use the fact that $\sum_{j=1}^{T} j = T(T+1)/2$. So replacing T with [rT] yields the result.

(A3.3) $T^{-3}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \Rightarrow [(1-\theta_*)^2/60]\{8[\Theta_B^*(L)X_\infty]^2 + 9[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + 3[\Theta_A^*(L)X_\infty]^2\}$

Proof.

From (A3.2), $T^{-3}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 = T^{-1}\sum_{t=1}^{T}[\tilde{S}_t(\theta_*)/T]^2$

$$\Rightarrow \int_0^r(1-\theta_*)^2\{\ r\Theta_B^*(L)X_\infty + (r^2/2)[\Theta_A^*(L)-\Theta_B^*(L)]X_\infty\ \}^2dr.$$

Then evaluating the integral yields the result.

(A3.4) $\tilde{\sigma}^2(0) \Rightarrow \sigma_x^2 + [(1-\theta_*)^2/3]\{[\Theta_B^*(L)X_\infty]^2 + [\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + [\Theta_A^*(L)X_\infty]^2\}.$

Proof.

(A3.5) $\tilde{\sigma}^2(0) = T^{-1}\sum_{t=1}^{T}\tilde{e}_t(\theta_*)^2$

$$= T^{-1}\sum_{t=1}^{T}\{X_t - (1-\theta_*)\Theta_B^*(L)X_T - (t/T)(1-\theta_*)[\Theta_A^*(L)-\Theta_B^*(L)]X_T\ \}^2$$

$$= T^{-1}\sum_{t=1}^{T}\{\ X_t - (1-\theta_*)\Theta_B^*(L)X_T\ \}^2$$

$$+ (1-\theta_*)^2\{[\Theta_A^*(L) - \Theta_B^*(L)]X_T\}^2\,T^{-3}\sum_{t=1}^{T}t^2$$

$$- 2(1-\theta_*)\{[\Theta_A^*(L) - \Theta_B^*(L)]X_T\}T^{-2}\sum_{t=1}^{T}t[X_t - (1-\theta_*)\Theta_B^*(L)X_T].$$

Since $T^{-1}\sum_{t=1}^{T}X_t = o_p(1)$ and $\Theta_1(L)X_T = O_p(1)$, the first term of the right hand side equals $T^{-1}\sum_{t=1}^{T}X_t^2 + (1-\theta_*)^2[\Theta_1(L)X_T]^2 + o_p(1)$ and converges to $\sigma_x^2 + (1-\theta_*)^2[\Theta_B^*(L)X_\infty]^2$. The second term converges to $(1-\theta_*)^2[\Theta_A^*(L)X_\infty - \Theta_B^*(L)X_\infty]^2/3$ (using $\sum_{t=1}^{T} t^2 = T(T+1)(2T+1)/6$, so that $T^{-3}\sum_{t=1}^{T} t^2$ has limit 1/3). In the third term, $T^{-2}\sum_{t=1}^{T} tX_t = o_p(1)$ and $[\Theta_A^*(L)X_T - \Theta_B^*(L)X_T] = O_p(1)$ so that the third term has the same asymptotic distribution as $2(1-\theta_*)^2[\Theta_A^*(L)X_T - \Theta_B^*(L)X_T][\Theta_B^*(L)X_T]T^{-2}\sum_{t=1}^{T} t$, which converges to $(1-\theta_*)^2[\Theta_A^*(L)X_\infty - \Theta_B^*(L)X_\infty][\Theta_B^*(L)X_\infty]$. Thus, collecting terms yields the result.

(A3.6) $T^{-1}\tilde{\eta}_\tau(\theta_*) = T^{-3}\sum_{t=1}^{T} \tilde{S}_t(\theta_*)^2 / \tilde{\sigma}(0)^2$

$$\Rightarrow \frac{[(1-\theta_*)^2]\{8[\Theta_B^*(L)X_\infty]^2 + 9[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] + 3[\Theta_A^*(L)X_\infty]^2\}}{60\sigma_x^2 + 20(1-\theta_*)^2\{[\Theta_B^*(L)X_\infty]^2+[\Theta_B^*(L)X_\infty][\Theta_A^*(L)X_\infty] +[\Theta_A^*(L)X_\infty]^2\}}$$

and $\tilde{\eta}_\tau(\theta_*) = O_p(T)$.

Proof. Simple substitution of (A3.3) and (A3.4) into the formula for the KPSS statistic yields the result.

Now we derive the asymptotic distribution of the GLS-based KPSS test under Assumption B. Under the nonstationarity assumption the test is regarded as a function of the normalized residuals

(A3.7) $\quad T^{-1/2}\tilde{e}_t(\theta_*) = T^{-1/2}X_t - (t/T)X_T + o_p(1),\ t = 1,\dots,T.$

Under Assumption B,

(A3.8) $\quad T^{-1/2}\tilde{e}_{[rT]}(\theta_*) \Rightarrow \sigma B(s),$

where $B(s) = W(r) - rW(1)$ is the Brownian bridge.

Proof. See Schmidt and Phillips (1992, Appendix 3).

Once we have the result (A3.8), exactly the same steps as in Schmidt (1992) apply. So we just state the main results.

(A3.9) $T^{-3/2}\tilde{S}_{[rT]}(\theta_*) \Rightarrow \sigma\int_0^r B(s)ds.$

(A3.10) $T^{-4}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \Rightarrow \sigma^2\int_0^1[\int_0^r B(s)ds]^2dr.$

(A3.11) $$T^{-1}\tilde{\sigma}^2(0) = T^{-2}\sum_{t=1}^{T}\tilde{e}_t(\theta_*)^2$$
$$\Rightarrow \sigma^2\int_0^1 B(s)^2ds.$$

(A3.12) $$T^{-1}\tilde{\eta}_\tau(\theta_*) = T^{-4}\sum_{t=1}^{T}\tilde{S}_t(\theta_*)^2 \ / \ T^{-1}\tilde{\sigma}(0)^2$$
$$\Rightarrow \frac{\int_0^1[\int_0^r B(s)ds]^2dr}{\int_0^1 B(s)^2ds}$$

and $\tilde{\eta}_\tau(\theta_*) = O_p(T)$.

Therefore, comparing (A3.6) and (A3.12) shows that the $\tilde{\eta}_\tau(\theta_*)$ test is not consistent, because the statistic is $O_p(T)$ under both the null and alternative hypotheses.

APPENDIX 4

In this Appendix we derive the asymptotic distribution of the POI statistic under our alternative assumptions.

We start with Assumption A (stationarity). First consider the denominator of the statistic. As we discussed in Appendix 2, for t = 1,...,T, $\tilde{e}_t(1)$ are identical to the OLS residuals $\hat{e}_t$ and $\Omega_N(1)$ becomes the identity matrix, so we have

(A4.1) $$\tilde{e}(1)'\Omega_N^{-1}(1)\tilde{e}(1) = \hat{e}'\hat{e} = \sum_{t=1}^{T}\hat{e}_t^2 \text{ and}$$

(A4.2) $$T^{-1}\sum_{t=1}^{T}\hat{e}_t^2 \Rightarrow \sigma_x^2 = \gamma_x(0).$$

Next consider the numerator of the statistic. From (A2.8), the residuals are given as

(A4.3) $$\tilde{e}_t^*(\theta_*) = \Theta_A^*(L)\Delta X_t - (1-\theta_*)\theta_*^{t-1}\Theta_B^*(L)X_T + o_p(1).$$

From (19) in the main text, we have

(A4.4) $$\tilde{e}(\theta_*)'\Omega_N^{-1}(\theta_*)\tilde{e}(\theta_*) = \tilde{e}^*(\theta_*)'\tilde{e}^*(\theta_*) = \sum_{t=1}^{T}\tilde{e}_t^*(\theta_*)^2.$$

(A4.5) $$\begin{aligned} T^{-1}\sum_{t=1}^{T}\tilde{e}_t^*(\theta_*)^2 &= T^{-1}\sum_{t=1}^{T}[\Theta_A^*(L)\Delta X_t - (1-\theta_*)\theta_*^{t-1}\Theta_B^*(L)X_T]^2 \\ &= T^{-1}\sum_{t=1}^{T}[\Theta_A^*(L)\Delta X_t]^2 + (1-\theta_*)^2[\Theta_B^*(L)X_T]^2T^{-1}\sum_{t=1}^{T}\theta_*^{2(t-1)} \\ &\quad - 2(1-\theta_*)[\Theta_B^*(L)X_T]T^{-1}\sum_{t=1}^{T}\theta_*^{t-1}[\Theta_A^*(L)\Delta X_t]. \end{aligned}$$

The second term converges to 0 because the limit of $\sum_{t=1}^{T}\theta_*^{2(t-1)}$ equals $1/(1-\theta_*^2)$ and $\Theta_B^*(L)X_T$ is $O_p(1)$ from Lemmas 1 and 2. After some algebra (using $\lim_{T\to\infty}\theta_*^T \to 0$) we can show that the third term asymptotically equals $2(1-\theta_*)(1+\theta_*)T^{-1}[\Theta_B^*(L)X_T]^2$, so the third term also converges to 0 under both assumptions.

(Recall that $\Theta_B^*(L)X_T$ is $O_p(1)$ under both the stationarity and unit root assumptions.) This implies that for our asymptotic analysis only the first term in (A4.5) matters under both assumptions about the errors.

Now we consider the first term in (A4.5) under Assumption A (stationarity). Since $\Theta_A^*(L)\Delta X_t = [X_t - (1-\theta_*)\sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}]$, we have

$$\text{(A4.6)}\quad T^{-1}\sum_{t=1}^{T}[\Theta_A^*(L)\Delta X_t]^2 = T^{-1}\sum_{t=1}^{T}[X_t - (1-\theta_*)\sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}]^2$$

$$= T^{-1}\sum_{t=1}^{T}X_t^2 - 2(1-\theta_*)T^{-1}\sum_{t=1}^{T}[X_t \sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}]$$

$$+ (1-\theta_*)^2 T^{-1}\sum_{t=1}^{T}[\sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}]^2.$$

Let $\hat{\gamma}_x(j) = T^{-1}\sum_{t=j+1}^{T}X_tX_{t-j}$ and $\gamma_x(j)$ be j^{th} sample and population autocovariance of X_t and let $\hat{\rho}_x(j)$ and $\rho_x(j)$ be the j^{th} sample and population autocorrelation coefficient of X_t, respectively. Then after a little algebra, we have

$$\text{(A4.7)}\quad T^{-1}\sum_{t=1}^{T}X_t^2 = \hat{\gamma}_x(0),$$

$$\text{(A4.8)}\quad T^{-1}\sum_{t=1}^{T}[X_t \sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}] = \sum_{j=1}^{T-1}\theta_*^{j-1}\hat{\gamma}_x(j) \text{ and}$$

$$\text{(A4.9)}\quad T^{-1}\sum_{t=1}^{T}[\sum_{j=1}^{t-1}\theta_*^{j-1}X_{t-j}]^2 = (1-\theta_*^2)^{-1}[\hat{\gamma}_x(0) + 2\sum_{j=1}^{T-1}\theta_*^{j}\hat{\gamma}_x(j)].$$

Substituting (A4.7), (A4.8) and (A4.9) into (A4.6) and collecting terms yields

$$\text{(A4.10)}\quad T^{-1}\sum_{t=1}^{T}[\Theta_A^*(L)\Delta X_t]^2 = 2(1+\theta_*)^{-1}[\hat{\gamma}_x(0) - (1-\theta_*)\sum_{j=1}^{T-1}\theta_*^{j-1}\hat{\gamma}_x(j)]$$

$$\rightarrow 2(1+\theta_*)^{-1}[\gamma_x(0)-(1-\theta_*)\sum_{j=1}^{\infty}\theta_*^{j-1}\gamma_x(j)].$$

Since plim $P_T(\theta_*) = \text{plim } T^{-1}\sum_{t=1}^{T} \tilde{e}_t^*(\theta_*)^2$ / plim $T^{-1}\sum_{t=1}^{T} \hat{e}_t^2$, from (A4.2) and (A4.10):

$$\text{(A4.11)} \quad P \equiv \text{plim } P_T(\theta_*) = 2(1+\theta_*)^{-1}[1 - (1-\theta_*)\sum_{j=1}^{\infty} \theta_*^{j-1} \rho_x(j)].$$

We know that under regularity conditions about the error process X_t, the joint distribution of $T^{1/2}[\hat{\rho}_x(i) - \rho_x(i)]$, $1 \leq i \leq p$, converges to the p-variate multivariate normal distribution with zero mean vector and covariance matrix $W = (w_{ij})$, that is,

$$\text{(A4.12)} \qquad T^{1/2}[\hat{\rho}_x(1) - \rho_x(1), \cdots, \hat{\rho}_x(p) - \rho_x(p)]' \Rightarrow N(\mathbf{0}, W),$$

$$\text{(A4.13)} \qquad w_{ij} = \sum_{k=1}^{\infty}\{\rho_x(k+i) + \rho_x(k-i) - 2\rho_x(i)\rho_x(k)\} \times \{\rho_x(k+j) + \rho_x(k-j) - 2\rho_x(j)\rho_x(k)\}$$

(Brockwell and Davis (1991), chapter 7). Note that when X_t are iid, $w_{ij} = 1$ for $i = j$ and $w_{ij} = 0$ for $i \neq j$, because $\rho_x(0) = 1$ and $\rho_x(j) = 0$ for $j \geq 1$.

Directly applying (A4.12) and (A4.13) to (A4.10) and (A4.11) gives the following result:

$$\text{(A4.14)} \qquad T^{1/2}[P_T(\theta_*) - P] \Rightarrow N(0, V),$$

where V is defined as

$$\text{(A4.15)} \qquad V \equiv [2(1-\theta_*)/(1+\theta_*)]^2 \sum_{i=1}^{T} \sum_{j=1}^{T} \theta_*^{i+j-2} w_{ij}.$$

Hence $P_T(\theta_*) = O_p(T^{-1/2})$ under Assumption A.

Next we derive the limiting distribution of POI statistic under Assumption B (unit root):

(A4.16) $$T^{-1/2}X_{[rT]} \Rightarrow \sigma W(r).$$

We know that the normalized OLS residuals converge to a function of the demeaned and detrended Wiener process $W*(r)$ (KPSS (1992), equation (26)), i.e.,

(A4.17) $$T^{-1/2}\hat{e}_{[rT]} \Rightarrow \sigma W*(r).$$

Hence, we have

(A4.18) $$T^{-2}\sum_{t=1}^{T}\hat{e}_t^2 \Rightarrow \sigma^2\int_0^1 W*(r)^2 dr.$$

Next consider the numerator. Assumption B implies that ΔX_t is a general stationary process. Let $\hat{\gamma}(j)$ and $\gamma(j)$ be its j^{th} sample and population autocovariance, respectively. Only the first term in the expression for the GLS residuals $\tilde{e}_t^*(\theta_*)$ in (A4.3) matters (see the discussion following equation (A4.5)). So we have

(A4.19) $$\begin{aligned} T^{-1}\sum_{t=1}^{T}\tilde{e}_t^*(\theta_*)^2 &= T^{-1}\sum_{t=1}^{T}[\Theta_A^*(L)\Delta X_t]^2 + o_p(1) \\ &= (1-\theta_*^2)^{-1}[\hat{\gamma}(0) - 2\sum_{j=1}^{T-1}\theta_*^j\,\hat{\gamma}(j)] + o_p(1) \\ &\rightarrow (1-\theta_*^2)^{-1}[\gamma(0) - 2\sum_{j=1}^{\infty}\theta_*^j\,\gamma(j)]. \end{aligned}$$

From (A4.18) and (A4.19),

(A4.20) $$T\,P_\tau(\theta_*) = \frac{T^{-1}\sum_{t=1}^{T}\tilde{e}_t^*(\theta_*)^2}{T^{-2}\sum_{t=1}^{T}\hat{e}_t^2} \Rightarrow \frac{[\gamma(0) - 2\sum_{j=1}^{\infty}\theta_*^j\,\gamma(j)]}{\sigma^2(1-\theta_*^2)\int_0^1 W*(r)^2 dr},$$

and $P_\tau(\theta_*)$ is $O_p(T^{-1})$ under Assumption B. Hence, comparing (A4.20) with (A4.15) shows that the $P_\tau(\theta_*)$ test is consistent.

CHAPTER 4

CHAPTER 4

CONCLUDING REMARKS

In this thesis, we have applied the theory of point optimal testing to the problem of testing whether a time series is trend stationary or whether it contains a unit root. We have considered the point optimal invariant (POI) tests of the unit root hypothesis and of the hypothesis of trend stationarity. Furthermore, we have stressed the connection of the POI tests to the detrending of the series by generalized least squares (GLS), based on an empirically plausible value of the relevant parameter under the alternative hypothesis. Our most important finding is that, compared to other standard tests, POI tests offer large enough gains in power over a wide enough range of the parameter space to make them potentially attractive.

For the unit root testing problem, our results are fairly complete. The POI test is very similar to a test of Dickey-Fuller type, but based on GLS detrending instead of OLS detrending. The asymptotic properties of these tests are straightforward, and they lead naturally to asymptotically valid corrections for error autocorrelation. The main question yet to be addressed is how well these autocorrelation-corrected tests work in finite samples. In particular, it is important to observe that, if ρ_* is the value of the autoregressive root assumed in the construction of the POI test (and used in GLS detrending), we have considered the asymptotic properties of our tests as $T \to \infty$ with ρ_* fixed. Elliott, Rothenberg and Stock (1992) have considered the asymptotic properties of the same statistics as $T \to \infty$, assuming that $\rho_* = 1 - c_*/T$ with c_* fixed, so that $\rho_* \to 1$ as $T \to \infty$. This results in very different asymptotics than ours, and it also results in different forms of corrections for error autocorrelation than we have. Which form of asymptotic analysis is more useful is basically a matter of which leads to autocorrelation-corrected statistics with better small sample properties; that is, with smaller size distortions and higher size-adjusted power. This is an important issue yet to be settled.

For the stationarity testing problem, our results are less complete. The POI test does offer a substantial gain in power relative to the KPSS test, which is an important and optimistic result. However, while the POI test depends on GLS residuals (that is, on the series detrended by GLS) and is consistent, the KPSS statistic based on GLS residuals does not yield a consistent test. More thought is needed to understand the reason for this result, and to see what forms of statistics based on GLS residuals lead to consistent tests. Furthermore, although we have derived the asymptotic distribution of the POI statistic under general forms of error autocorrelation, the asymptotic distribution depends on the covariance structure of the errors in a complicated way that does not lead to simple asymptotically-valid corrections for autocorrelation. The practical usefulness of the POI test is small unless a version that is asymptotically valid under autocorrelation is available. This is another important topic for further research.

LIST OF REFERENCES

LIST OF REFERENCES

Bhargava, A. (1986), "On the Theory of Testing for Unit Roots in Observed Time Series," Review of Economic Studies, 52 369-384.

Brockwell, P.J. and R.A. Davis (1991), Time Series: Theory and Methods, New York:Springer-Verlag.

Campbell, T.Y. and N.G. Mankiw (1987),"Are Output Fluctuations Transitory?," Quarterly Journal of Economics, 102, 857-880.

Campbell, T.Y. and N.G. Mankiw (1989),"International Evidence on the Persistence of Economic Fluctuations," Journal of Monetary Economics, 23, 319-333.

Dickey, D.A. (1984), "Powers of Unit Root Tests," Proceedings of the American Statistical Association, Business and Economic Statistics Section, 489-493.

Dickey, D.A. and W.A. Fuller (1979),"Distribution of the Estimators for Autoregressive Time Series with a Unit Root," Journal of the American Statistical Association, 74, 427-431.

Dickey, D.A. and W.A. Fuller (1981),"Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root," Econometrica, 49, 1057-1072.

Dufour, J-M and M.L. King (1991),"Optimal Invariant Tests for the Autocorrelation Coefficient in the Linear Regressions with Stationary and Nonstationary AR(1) Errors,"Journal of Econometrics, 47, 115-143.

Elliott, G, T.J. Rothenberg and J. H. Stock (1992),"Efficient Tests for an Autoregressive Unit Root," Unpublished Paper of Harvard University.

Fuller, W.A. (1976), Introduction to Statistical Time Series, New York:Wiley.

Granger, C.W.J. and P. Newbold (1974), "Spurious Regressions in Econometrics," Journal of Econometrics, 2, 121-130.

King, M.L. (1980),"Robust Tests for Spherical Symmetry and Their Application to Least Squares Regression," Annals of Statistics, 8, 1265-1271.

King, M.L. (1988),"Towards a Theory of Point Optimal Testing," Econometric Reviews, 6, 169-218.

King, M.L. and G.H. Hillier (1985),"Locally Best Invariant Tests of the Error Covariance Matrix of the Linear Regression Model," Journal of Royal Statistical Society, B47, 98-102.

Kiwatkowski, D., P.C.B. Phillips, P. Schmidt, and Y.C. Shin (1992),"Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root: How Sure Are We That Economic Time Series Have a Unit Root?," Journal of Econometrics, 54, 159-178.

Leybourne, S.J. and B.P.M. McCabe (1992),"An Alternative Test for a Unit Root," Unpublished Paper.

McCallum, B. (1992),"Unit Roots in Macroeconomic Time Series: A Critical Overview," Unpublished Paper of Carnegie Mellon University.

Muth, J.F. (1960),"Optimal Properties of Exponentially Weighted Forecasts," Journal of the American Statistical Association, 55, 299-306.

Nelson, C.R. and C.I. Plosser (1982),"Trend versus Random Walks in Macroeconomic Time Series: Some Evidence and Implications," Journal of Monetary Economics, 10, 136-162.

Nyblom, J. (1986),"Testing for Deterministic Linear Trend in Time Series," Journal of the American Statistical Association, 81, 545-549.

Park, J.Y. and P.C.B. Phillips (1988),"Statistical Inference in Regressions with Integrated Processes: Part I," Econometric Theory, 4, 468-498.

Phillips, P.C.B. (1986),"Understanding Spurious Regressions in Econometrics," Journal of Econometrics, 33, 311-340.

Phillips, P.C.B. (1987),"Time Series Regression with Unit Roots," Econometrica, 55, 277-301.

Phillips, P.C.B. and S. Ouliaris (1990),"Asymptotic Properties of residual Based Tests for Cointegration," Econometrica, 58, 165-194.

Phillips, P.C.B. and P. Perron (1988),"Testing for a Unit Root in Time Series Regression," Biometrika, 75, 335-346.

Press, W.H., B.P. Flannery, S.A. Teukolsky and W.T. Vetterling(1986), Numerical Recipes: The Art of Scientific Computing, Cambridge: Cambridge University Press.

Said, S.E. and D.A. Dickey (1984),"Testing for Unit Roots in Autoregressive - Moving Average Models of Unknown Order," Biometrika, 71, 599-608.

Saikkonen, P. and L. Luukkonen (1992a),"Testing for Moving Average Unit Root in Autoregressive Integrated Moving Average Models," Manuscript, University of Helsinki.

Saikkonen, P. and L. Luukkonen (1992b),"Point Optimal Tests for the Moving Average Unit Root Hypothesis," Manuscript, University of Helsinki.

Sargan, J.D. and A. Bhargava (1983),"Testing Residuals from Least Squares Regression for Being Generated by the Gaussian Random Walk," Econometrica, 51, 153-174.

Schmidt, P. and J. Lee (1991),"A Modification of the Schmidt-Phillips Unit Root Test," Economic Letters, 36, 285-293.

Schmidt, P. and P.C.B. Phillips (1992),"LM Tests for a Unit Root in the Presence of Deterministic Trends," Oxford Bulletin of Economics and Statistics, 54, 257-287.

Schmidt, P. (1992),"Some Results on Testing for Stationarity Using Data Detrended in Differences," Economic Letters, forthcoming.

Shepard N.G. and A.C. Harvey (1990),"On the Probability of Estimating a Deterministic Component in the Local Level Model," Journal of Time Series Analysis, 11, 339-347.

Shively, T.S. (1988),"An Exact Test for a Stochastic Coefficient in a Time Series Regression Model," Journal of Time Series Analysis, 9, 81-88.

Tanaka, K. (1990),"Testing for a Moving Average Unit Root," Econometric Theory, 6, 433-444.

Tanaka, K. and S.E. Satchell (1989),"Asymptotic Properties of the Maximum Likelihood and Nonlinear Least Squares Estimators for Noninvertible Moving Average Models," Econometric Theory, 5, 333-353.

White, H. (1984), Asymptotic Theory for Econometricians, New York: Academic Press.