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DETERMINING THE STEADY STATE SOLUTIONS OF
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Homotopy Methods and Computer Implementation

By

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ABSTRACT

DETERMINING THE STEADY STATE SOLUTIONS OF NONLINEAR MODELS OF POWER SYSTEMS

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Determining the steady state solutions (equilibria) of models of interconnected power systems, known as the load flow problem, has become increasingly important, and is presenting challenging problems, facing theoretical as well as applied researchers. The load flow problem has continuously received the attention of researchers due to its essential role in the planning and operation of power systems. It is the core problem in the studies focusing on the stability and bifurcations of (models) of power systems.

In this thesis, we use powerful analytical tools and modern techniques from algebraic geometry to infer the number of the steady state (equilibrium) solutions of nonlinear models of power systems. We develop the theorems and the methods to predict and determine the steady state (equilibrium) solutions for various levels of detailed models of power systems. Sufficient conditions are provided which guarantee the precise number of solutions to the full-fledged load flow. The sufficient conditions are cast in terms of properties of the physical admittance matrix of the power grid. Consequently, these sufficient conditions are placed on the structure or the topology of the given power network. When the sufficient conditions are not satisfied, we develop the cluster method to provide a "tighter" upper bound on the load flow solutions for special power grid structures. Consequently, our results lower the upper bound for the

many practical power grids which are normally sparsely connected.

We also present the special homotopy method, due to Li et al, to reduce the computational complexity, and to "guarantee" the finding of all possible solutions of the load flow equations of power systems. We then develop the imbedding-based and the homotopy-based heuristic methods to simplify the computations in finding the solutions of the so-called *deficient* systems (with particular interest to power systems). Moreover, the methods render procedures which are directly implementable on digital serial and parallel processors.

We specialize some of our results to prototype models and numerical examples of power systems to illustrate as well as demonstrate the procedures capabilities. The algorithmic techniques are then implemented to obtain the steady state (equilibrium) solutions of various models of power systems.

To

my wife, Xuemei; my son, Zheng; and my Motherland.

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Nomenclature

M	:	moment of inertia,
D	:	damping coefficient,
δ	:	angle at generator buses,
Θ	:	angle at buses excluding the slack bus,
Φ	:	angle at load buses,
P^m	:	mechanical power,
P^G	:	active power injection at generator buses,
P^L	:	active power injection at load buses,
Q^L	:	reactive power injection at load buses,
S	:	complex power injection at buses,
$T_{do}^{'}$	:	direct axis transient open-circuit time constant ,
x_d	:	direct axis synchronous reactance,
E_F	:	the exciter output voltage,
$E_q^{'}$	:	quadrature axis magnitude of voltage behind transient reactance,
E_q	:	quadrature axis magnitude of voltage behind synchronous reactance,

λ_{Fi} : the flux linkage of the field winding,

V : voltage magnitude at buses,

V_{ref} : the reference voltage,

V_C : a voltage error,

U_1 : the measured terminal voltage,

U_3 : the stabilizer output voltage,

V_R : the amplifier output limit voltage,

S_E : the saturation function,

K_A : the amplifier gain,

T_A : the amplifier time constant,

T_R : the time constant measured through a potential transformer, rectified and filtered,

K_F : the stabilizer gain,

T_F : the stabilizer time constant,

K_E : the exciter gain,

T_E : the exciter time constant,

$H(.,t)$: the homotopy function with homotopy parameter $0 \le t \le 1$,

$\hat{S}(.)$: the "initial" starting system in the homotopy function, i.e., $H(.,0):=\hat{S}(.)$,

$\hat{T}(.)$: the target system in the homotopy function, i.e., $H(.,1):=\hat{T}(.)$,

CP^n : the projective space.

Chapter 1

INTRODUCTION

Determining the steady state constant solutions (i.e., equilibria) for any dynamic system is the most basic and fundamental problem in the quest for analyzing and investigating a system's behavior. Once the equilibria are determined, one may calculate the linearization of the vector field about a chosen equilibrium point, i.e. the Jacobian matrix, and then calculate the eigenvalues of the Jacobian matrix. By Lyapunov's direct method, the eigenvalues generically infer the local stability or instability of the chosen equilibrium point. These types of calculations have generally been programmed onto digital computers. Alternatively, one may use the Lyapunov function techniques within a neighborhood of the equilibrium point to analytically infer the local stability or instability of the equilibrium point.

For many systems, an equilibrium point represents the operating point of that system if it is (asymptotically) stable. It may otherwise belong to the boundary of the basin of the stability (or attraction) for a certain desired operating point; in this case, the equilibrium point may characterize a part of the boundary within its vicinity. (The described view was specialized for the particular case of power systems in [7,8,10].)

The question is then: how to determine the equilibria?

Various numerical procedures have been used over the years with limited degree of success. Examples of these numerical procedures are Newton-Raphson, Gauss, Seidel, Poncel Fletcher etc.. All these procedures, however, suffer from a "poor" choice of initial conditions. Variations of Newton-Raphson procedures are common and are believed to be frequent in use. Indeed instability or non-convergence of these procedures have been reported sporadically in the literature. Yet, for a lack of better alternatives, the use of these procedures have continued to dominate.

Recently, a study [11] by Thorp and Naqavi has articulated a form of instability associated with the use of the Newton-Raphson method in determining equilibria for the polar-coordinate models of the swing equations of power systems. The conclusion of [11] illustrates via numerical computations of some power system examples that each isolated equilibrium point attracts basins of initial guesses. However, the basins of attraction for some equilibrium points can be very irregular, and/or extremely small. The assertion in [11] is that fractal structure are present in their power system examples.

It would be a welcome relief therefore to develop numerical procedures or algorithms that are independent from the choice of initial conditions of the algorithms and would succeed despite a "poor" choice of the initial conditions. It would even be more welcome if such numerical procedures are guaranteed via mathematical foundations to successfully find all (or some of) solutions.

1.1. The Load Flow Problem: basic issues

The load flow (or power flow) problem is the calculation of line loading given the generation and demand levels for the normal balanced three-phase steady-state operating conditions of an electric power system ([1]-[5]). Basically, the problem can be intuitively described as follows: given the forecast (real and reactive) load demands, the secure operation of power systems entails determining the required power generation so that a known set of inequality constraints are satisfied. The load flow equations are a system of nonlinear models which relate the real and reactive power, or the real power and voltage magnitudes at each node or bus in an electrical network operating in steady state. In general, load flow calculations are performed routinely for power system planning, and in connection with system operation and control. The data obtained from the load flow are used for the study of normal operating mode, contingency analysis, outage security assessment, as well as optimal dispatching, stability and bifurcations.

Although the full load flow equations have extensively been used for a long time, there remain a number of very basic open questions.

(a) What is the least upper bound on the number of (complex) solutions of the full fledged polynomial load flow?

(b) The more important but difficult problem is how to solve and obtain all (or some of) the equilibria of the nonlinear models of power systems.

(c) What are the number of real system solutions of the load flow equations for a given N-node power system?

These challenging questions have fascinated and puzzled mathematicians and engineers for many years. Many researchers have devoted large part of their professional lives to such problems. This thesis will address and provide answers to the first two significant questions.

1.2. Scanning the Literature

Because of its difficulty, few theoretical investigations of the load flow equations have been reported and with rather limited practical results ([12]-[17], [28], [30], [31]). For instance, it is not yet possible to infer the number of solutions of the full-fledged PV- and PQ-bus load flow equations.

The possible existence of multiple (stable) solutions to the load flow equations has been realized both analytically as well as via simulation of realistic power networks (see, e.g., [55], [57], [58], [74]). Related works using computational methods have appeared in [47], [52], and [56] though they emphasize the security aspects of power systems. Works that emphasize the optimal load flow computations are available in [42], [53], [61], [62], [68], and [71]. Works that emphasize the stability and bifurcations of power systems have appeared in [6]-[10], [16], [17], [26], [29], [33], [41], [49], [59], [60], [65], and [70]. Some theoretical analysis of various simple models of the load flow have been initiated by Galiana ([50], [51]). Analysis on the real power of the load flow equations began with the work of Tavora and Smith [31]. Recent investigations made by Arapostathis, Sastry and Varaiya [27] have utilized tools from bifurcation theory. The most recent work on the real power flow equations have been reported by Baillieul and Byrnes ([15]-[17]). As for the full fledged power flow equations, the work by Wu and Kumagai, namely [72], [73], represents the only analytical treatment. Yet, none of these theoretical works determines upper bounds on the number of solutions for a general N-node (PQ and PV buses) power system. None addresses the dependence of the number of solutions on the network structure in addition to the number of nodes. More importantly, none of them addresses guaranteeing the finding of all possible solutions of the load flow equations. The reason stems from the difficulty of the problem analytically even though the mathematical tools employed in the works ([15]-[17], [27], [46]) are sophisticated.

In the early work on multiple equilibria of swing equations, i.e. on multiple solutions of the load flow equations restricted to be lossless PV buses, for an N-node (excluding the slack bus) lossless power system model, Prabhakara et al [12] have stated that the number of equilibria of the swing equations for an N-node (excluding the slack bus) lossless power system model is 2^N. The upper bound 2^N for an N-node power system (excluding the slack bus) has been rationalized as follows. Consider N decoupled nodes connected only to the reference or infinity node. If one assumes that each node connected to an infinity node produces 2 possible solutions, then the N decoupled nodes will give rise to 2^N possible solutions. It is also assumed that the coupling would not increase the possible number of solutions. This is the best rational for an otherwise a completely heuristic justification.

In the following extension on the number of solutions of the load flow equations, Tamura et al ([13], [14]) have concluded that the number of solutions of the load flow equations for an N-node power system (excluding the slack bus) is 2^N. Tamura et al have developed both basic and simplified algorithms to compute multiple solutions to the load flow equations. These algorithms are based on the Newton-Raphson method (with some optimal multiplier which is assumed to prevent divergence and occurrence of oscillation) to compute the multiple solutions to the load flow equations. Even so, there is no guarantee that these algorithms would find all possible solutions to the load flow problem since the algorithms are based on the incorrect conclusion that the upper bound on the number of solutions of the full load flow equations for an N-node power system (excluding the slack bus) is 2^N.

The subsequent works on the number of solutions of the load flow equations by Baillieul and Byrnes ([15]-[17]) have used some powerful results from Algebraic Geometry, the Morse Theory and the Intersection Theory to study the load flow equations of the lossless PV-bus power system models. These theories were specialized to the lossless PV-bus models of power systems cast as a special quadratic system of polynomials. Their work has extensively exploited the particular features of the lossless PV-bus model. They have finally concluded that the number of complex solutions of the load flow equations for an N-node (excluding the slack bus) power system (with all buses (i.e. nodes) restricted to be PV buses) is bounded above by

$$N_n = \binom{2N}{N}.$$

The most recent works by Li, Sauer, and Yorke ([18], [19]) have used different tools from Intersection Theory, and the Homotopy Continuation methods to develop Homotopies to find all the solutions of general classes of polynomial systems. They have developed some theorems that can be used to determine the upper bound on the number of solutions for general classes of (generic) polynomial systems. Some of their applied results have stated that the number of solutions of the load flow equations for an N-node (excluding the slack bus) power system (with all buses (i.e. nodes) restricted to be PQ buses) is bounded above by

$$N_n = \binom{2N}{N}.$$

It appears that the framework of Li, Sauer, and Yorke is more general and is systematic. It only applies to general (generic) polynomial systems however. It should be emphasized that neither work is applicable to a general power system which includes both PV and PQ buses. It would be of interest, however, to prove that this bound applies to general PV- and PQ-bus power system. Moreover, for more detailed models that include the excitation system, e.g., one has yet to develop representations that describe the overall model in terms of systems of polynomials. Only through the general polynomial systems approach of [18], [19] as well as our newly developed representations have we been able to extend, then apply, the Homotopy method techniques to general power system models.

Because of the nonlinearity of the models of power systems, numerical procedures have continued to be the only possible avenue for obtaining all *(real)* solutions of the load flow problem [25]. However, traditional methods, such as the Newton method and its variations, are not capable of solving for all the roots. The popular IMSL package, for instance, can only find one root based on the MINPACK implementation of M.J.D. Powell's hybrid algorithm and P. Wolfe's secant method [54]. The Newton-Raphson method is also known to fail in many case studies. It is well known that Newton-Raphson method can be unpredictable when the initial guess is poor, and it breaks down when the Jacobian at any stage of the iteration becomes singular. Thence this method does not always converge to a solution, and if it does, there is no guarantee that it would find all possible solutions to the load flow problem. The authoritative review on the subject of computational load flow equations is still perhaps [25]. Various subsequent works have employed variations of the Newton method and have used parallel processing concepts ([48], [66], [67], [69]).

Solving for all the roots of any system of polynomial equations has been almost impossible until the advent of the *globally convergent probability-one basic homotopy method* (subsequently, referred to as the homotopy method) ([20]-[24], [35]-[39], [43]-[45]). The method is globally convergent in the sense that it will converge to solutions of the problem from an arbitrary set of starting initial points ([2]-[5]).

Homotopy continuation methods have been proven to be superior to the quasi-Newton methods ([76]). A recent survey of this method can be found in [77]. Here we briefly describe the homotopy method as applied to solving systems of polynomial equations and underline some of its properties. First, the numerical computation of the homotopy method can be systematically implemented in parallel processors. Second, the homotopy method is globally convergent, i.e., one may choose any initial guesses and the homotopy method is *guaranteed* to converge to all solutions with probability one.

Since each homotopy curve only depends on the "initial" starting point, we can trace the homotopy curves separately. This makes it possible to exploit the inherent parallelism in the (polynomial) load flow model to take advantage of massively parallel computers ([45], [63]). Within the last five years, various types of parallel machines have been commercially produced [64]. Large-scale scientific computing is one of the major application domains which demands the huge computing power of parallel machines. However, the move into parallel territory requires new conceptual strategies in formulating a problem, and new algorithms to shape the problem for parallel computers. This implies that a brute-force approach to solve a large-scale problem, such as the load flow equations of power systems, on parallel computers does not render the problem tractable.

1.3. The Contributions of This Thesis

In this thesis, we use some powerful analytical tools and modern techniques from algebraic geometry and the homotopy continuation methods (with their algorithmic implementation onto computer) to investigate and calculate the steady state solutions of various nonlinear models of power systems. In particular,

(A) We extend the model of the (equilibrium) steady-state equations in the appropriate (complex) polynomial representations in chapter 2. This representation form is convenient for the homotopy approach we pursue.

(B) We develop theorems in chapter 3 to investigate and predict the number of solutions of the full-fledged (equilibrium) steady-state equations for various levels of detailed models of power systems. Sufficient conditions are provided which guarantee the precise number of solutions to the load flow. The sufficient conditions are cast in terms of properties of the physical admittance matrix of the power grid. Consequently, these sufficient conditions are placed on the structure or the topology of the given power network. When the sufficient conditions are not satisfied, we describe the cluster method to provide a "tighter" upper bound on the load-flow solutions for special power grid structures. The upper bound, therefore, depends on the topology or the structure of the power grid in addition to the number of nodes. Consequently, our results reduce the upper bound for the many practical power grids which are normally sparsely connected.

(C) We present the special homotopy method in chapter 4 to reduce the computational complexity, and to *guarantee* the finding of all possible solutions of the load flow equations of the polynomial power systems with probability one. The amount of computation depends on the size of the system.

(D) We develop the "imbedding-based" method in chapter 5 to simplify the computer computations of finding the solutions of the load flow equations, and to make the algorithm capable of handling relatively large-sized power system networks. This method, largely heuristic, draws from experience with the Homotopy procedures and from properties of power systems.

(E) We develop the "homotopy-based" method in chapter 6 to further reduce the computational complexity in the finding of the geometrically isolated roots of deficient systems, such as power systems.

The final conclusions and suggestions are given in chapter 7.

Chapter 2

(NONLINEAR) MODELS OF POWER SYSTEMS

The mathematical model of representations of a power system should be chosen or developed to accommodate, and be accommodated by, the techniques and the computational facilities available. In the following, we present various representations of power systems that can be used as per convenience. From the view point of the homotopy-based or the imbedding-based methods, the representations that are in the form of systems of polynomials are of interest.

2.1. The Classical Model

The well-known classical model is formulated by assuming that the flux linkage, λ_{Fi}, of the field winding is constant, and a voltage regulator holds the magnitude of its terminal voltage fixed by automatically varying the generator field excitation. We have listed the nomenclature at the beginning of this thesis (see page xi). The model of a power system consists of three main components (see Figure 2.1): generators, loads, and a transmission network that connects generators and loads. Let the model of the power system consist of $N+1$ buses or nodes. Let the generator buses be subscripted from 1 to N_g, and let the load buses be subscripted from N_g+1 to N. We choose to subscript the slack bus by $N+1$. We denote the node or bus complex admittance matrix by $[Y]$ where its ki-th component is $y_{ki} = G_{ki} + jB_{ki}$ ($j = \sqrt{-1}$). The term G_{ki} is the conductance of the line connecting buses k and i, and B_{ki} is the susceptance of the line connecting buses k and i. Each load is represented as a constant real (P^d) and reactive (Q^d) power demand. Therefore, we refer to a load bus as a PQ bus.

In the general case, a change in input, or load, or structure, or a sequence of such changes, causes dynamic motion. If we ignore the system direct axis synchronous reactances x_d, then the motion of the i-th generator with round rotor is governed by the following swing equations

$$0 = \dot{\theta}_i - \omega_i, \quad i = 1, 2, \cdots, N_g, \tag{2.1.1a}$$

$$0 = M_i \dot{\omega}_i + D_i \omega_i + F_i^G(\Theta, V) - P_i^m, \tag{2.1.1b}$$

$$i = 1, 2, \cdots, N_g,$$

with constraints

$$0 = F_k^L(\Theta, V) - P_k^L, \quad k = N_g+1, \cdots, N, \tag{2.1.1c}$$

$$0 = T_k^L(\Theta, V) - Q_k^L, \quad k = N_g+1, \cdots, N. \tag{2.1.1d}$$

The load flow equations or the equilibrium equations are obtained by setting the derivative terms to zero, that is,

$$0 = F_i^G(\Theta, V) - P_i^m, \quad i = 1, 2, \cdots, N_g, \tag{2.1.2a}$$

$$0 = F_k^L(\Theta, V) - P_k^L, \quad k = N_g+1, \cdots, N, \tag{2.1.2b}$$

$$0 = T_k^L(\Theta, V) - Q_k^L, \quad k = N_g+1, \cdots, N. \tag{2.1.2c}$$

The power flow, or load flow, problem has been formulated based on sinusoidal steady state nodal analysis of circuit theory. Load flow calculations are performed in

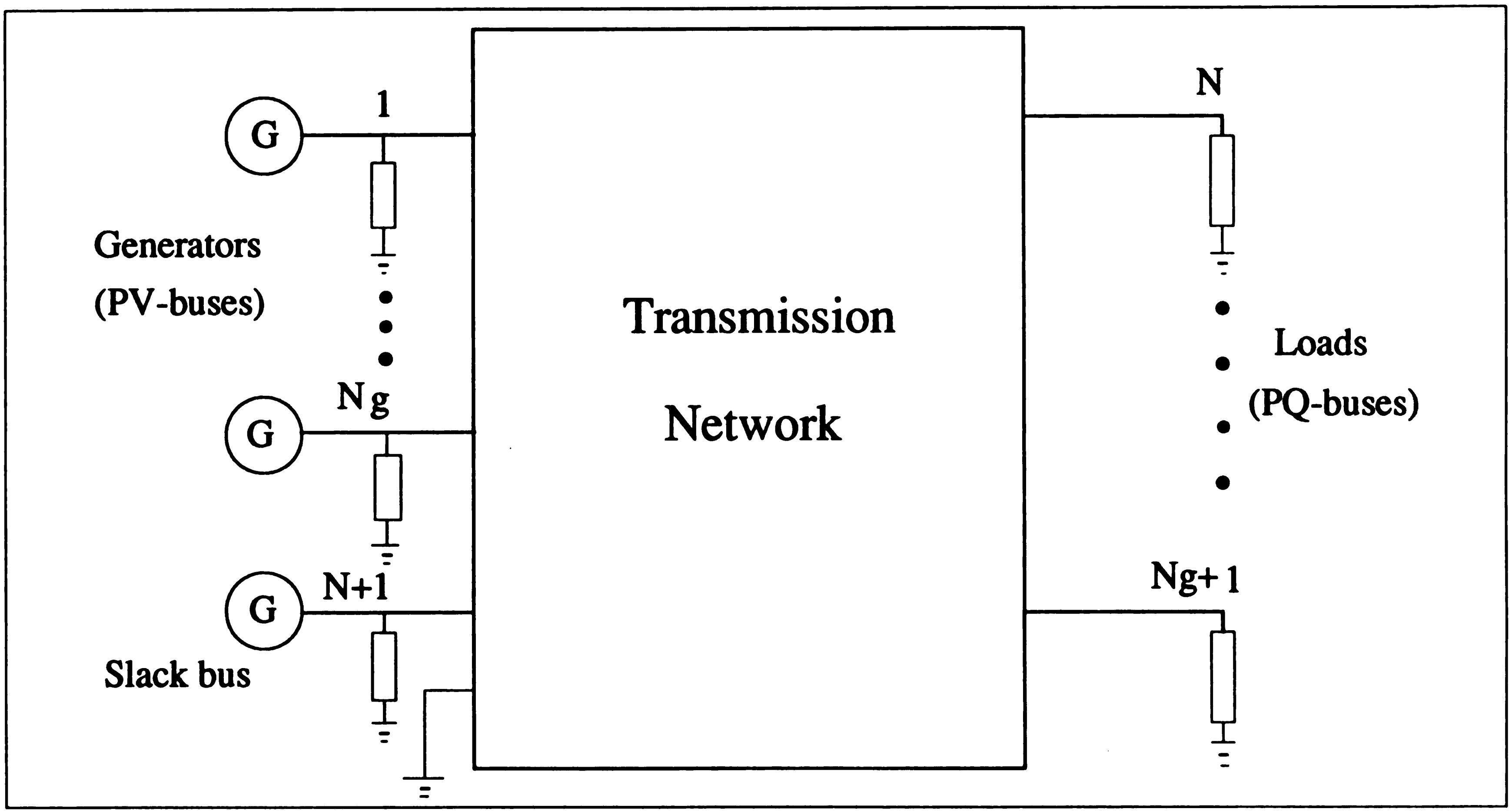


Figure 2.1. The main components of a power system.

power system planning, operation and control. They are increasingly and extensively being used to solve very large systems, to solve multiple cases for purposes such as outage security assessment and voltage collapse, and as part of more involved calculations such as optimization, stability, and bifurcations of (models of) power systems.

There are three types of buses in a power system network:

(i) PQ bus: a bus where the real and reactive powers are specified.

(ii) PV bus: a bus where the real power and the voltage amplitude are specified.

(iii) A slack bus: a fictitious concept whereby one of the generator buses has only its complex voltage specified. One purpose of this bus is to guarantee that the total power injection into the network equals the total power consumed. Consequently, ensuring the existence of steady state solutions (i.e., equilibria).

It is conventional to model loads as PQ-buses, one generator as a slack bus, and the rest of the generators as PV buses. All phase angles are measured relative to the angle of the slack bus which is set to be $\theta_{N+1} = 0$; furthermore the slack bus voltage is set to be $V_{N+1} = 1$ per unit. In the following, we derive various forms of the power system (equilibrium) steady-state models that are convenient representations for the analytical development treated later.

2.1.1. The polar-coordinate form

In the polar coordinate representation, the complex voltage at the k-th bus can be expressed as:

$$V_k\, e^{j\Theta_k} \;=\; V_k \cos\Theta_k + j\; V_k \sin\Theta_k ,$$

where V_k represents its voltage amplitude and Θ_k represents its phase angle. Let Θ_{ki} denote $\Theta_k - \Theta_i$ $(1 \le k, i \le N+1)$.

At the k-th generator node, we have [40]

$$0 = \sum_{i=1}^{N+1} V_k V_i (G_{ki}\cos\Theta_{ki} + B_{ki}\sin\Theta_{ki}) - P_k^m, \tag{2.1.3a}$$

$$k = 1, 2, \cdots, N_g.$$

At the k-th load node, we obtain

$$0 = \sum_{i=1}^{N+1} V_k V_i (G_{ki}\cos\Theta_{ki} + B_{ki}\sin\Theta_{ki}) - P_k^L, \tag{2.1.3b}$$

$$0 = \sum_{i=1}^{N+1} V_k V_i (G_{ki}\sin\Theta_{ki} - B_{ki}\cos\Theta_{ki}) - Q_k^L, \tag{2.1.3c}$$

$$k = N_g+1, \cdots, N.$$

2.1.2. The rectangular-coordinate form

In the rectangular coordinate representation, we have

$$V_k \cos\Theta_k + j\, V_k \sin\Theta_k =: \hat{X}_k + j\, \hat{Y}_k, \quad k = 1, 2, \cdots, N.$$

For convenience, we use the notation P_k to denote the real power injected at the k-th bus including both the generator bus and the load bus. similarly, we use Q_k for the "imaginary" power at the k-th bus. Therefore, the load flow equations can be as:

$$0 = \sum_{i=1}^{N+1} \left[G_{ki}(\hat{X}_k\hat{X}_i + \hat{Y}_k\hat{Y}_i) + B_{ki}(\hat{X}_i\hat{Y}_k - \hat{X}_k\hat{Y}_i) \right] - P_k, \tag{2.1.4a}$$

$$k = 1, 2, \cdots, N,$$

$$0 = \hat{X}_k^2 + \hat{Y}_k^2 - V_k^2, \quad k = 1, 2, \cdots, N_g, \tag{2.1.4b}$$

$$0 = \sum_{i=1}^{N+1} \left[G_{ki}(\hat{X}_i\hat{Y}_k - \hat{X}_k\hat{Y}_i) - B_{ki}(\hat{X}_k\hat{X}_i + \hat{Y}_k\hat{Y}_i) \right] - Q_k, \tag{2.1.4c}$$

$$k = N_g+1, \cdots, N,$$

or

$$0 = \hat{X}_k \sum_{i=1}^{N+1}(G_{ki}\hat{X}_i - B_{ki}\hat{Y}_i) + \hat{Y}_k \sum_{i=1}^{N+1}(B_{ki}\hat{X}_i + G_{ki}\hat{Y}_i) - P_k, \qquad (2.1.4a')$$

$$k = 1, 2, \cdots, N,$$

$$0 = \hat{X}_k^2 + \hat{Y}_k^2 - V_k^2, \quad k = 1, 2, \cdots, N_g, \qquad (2.1.4b')$$

$$0 = \hat{Y}_k \sum_{i=1}^{N+1}(G_{ki}\hat{X}_i - B_{ki}\hat{Y}_i) - \hat{X}_k \sum_{i=1}^{N+1}(B_{ki}\hat{X}_i + G_{ki}\hat{Y}_i) - Q_k, \qquad (2.1.4c')$$

$$k = N_g+1, \cdots, N,$$

where $\hat{X}_k$ represents the real part of the complex voltage E_k, while $\hat{Y}_k$ represents its imaginary part. The equations (2.1.4b) and (2.1.4b') represent the constraints on the voltage amplitude of the PV-buses.

2.1.3. The zero-conductance form

In this form, the transmission line conductances are assumed negligible. Consequently, we set $G_{ki} \approx 0$ in (2.1.4) to obtain the simplified model as follows:

$$0 = \sum_{i=1}^{N+1}\left[B_{ki}(\hat{X}_i\hat{Y}_k - \hat{X}_k\hat{Y}_i) \right] - P_k, \qquad k = 1, 2, \cdots, N, \qquad (2.1.5a)$$

$$0 = \hat{X}_k^2 + \hat{Y}_k^2 - V_k^2, \qquad k = 1, 2, \cdots, N_g, \qquad (2.1.5b)$$

$$0 = \sum_{i=1}^{N+1}\left[B_{ki}(\hat{X}_k\hat{X}_i + \hat{Y}_k\hat{Y}_i) \right] + Q_k, \qquad k = N_g+1, \cdots, N, \qquad (2.1.5c)$$

or

$$0 = \hat{Y}_k \sum_{i=1}^{N+1} B_{ki}\hat{X}_i - \hat{X}_k \sum_{i=1}^{N+1} B_{ki}\hat{Y}_i - P_k, \qquad k = 1, 2, \cdots, N, \qquad (2.1.5a')$$

$$0 = \hat{X}_k^2 + \hat{Y}_k^2 - V_k^2, \qquad k = 1, 2, \cdots, N_g, \qquad (2.1.5b')$$

$$0 = \hat{X}_k \sum_{i=1}^{N+1} B_{ki}\hat{X}_i + \hat{Y}_k \sum_{i=1}^{N+1} B_{ki}\hat{Y}_i + Q_k, \qquad k = N_g+1, \cdots, N. \qquad (2.1.5c')$$

2.1.4. The decoupled load flow equations

In addition to assuming that $G_{ki} = 0$, it is often assumed that the phase angle difference between nodes (or buses) are small. Consequently, one may write $\sin\Theta_{ki} \approx \Theta_{ki}$ and $\cos\Theta_{ki} \approx 1$. These two assumptions now reduce the polar representation (2.1.3) to the so called *decoupled load flow equations*, namely

$$0 = V_k \sum_{i=1}^{N+1} V_i B_{ki} \, \Theta_{ki} - P_k, \qquad k = 1, 2, \cdots, N, \qquad (2.1.6a)$$

$$0 = V_k \, [\, \sum_{i=1}^{N+1} V_i B_{ki} \,] + Q_k, \qquad k = N_g + 1, \cdots, N. \qquad (2.1.6b)$$

We remark that the simplified models, namely, Models (2.1.5) and (2.1.6), can be used to obtain approximate solutions for the full-fledged model. The motivation for doing so is three-fold:

(a) The solutions of the approximate models can be computed efficiently (through parallel processing which will be tailored for each of the simplified models).

(b) A simplified model, with its solutions, may be used as the "initial configuration" solution-set required in a given homotopy method.

(c) The (parallel processing) techniques developed for the simplified models are important in their own right, since the simplified models are employed in various applications ranging from planning to (transient) stability.

The next model is a new model that was introduced in [3, 4]. The model uses the complex space directly and hence one does not need to go though the necessary steps of complexifying the space to permit (complex) analysis and/or facilitate the computational procedures for solving the roots of systems of equations. We call this form "the complex form".

2.1.5. The complex form

Let E_k denote the complex voltage of bus k. It is of the form $E_k = V_k \cos\Theta_k + j\ V_k \sin\Theta_k$, where V_k represents the amplitude of the voltage at bus k. Let $S_k := P_k + jQ_k$ denote the complex injected power at node k, then the injected complex power balance equation can be expressed as $[E^*]\ [Y]\ E\ - S^* = 0$, where [E] $= diag\,[E_1, \cdots, E_N]$, E = $[E_1, \cdots, E_N]^T$, S = $[S_1, \cdots, S_N]^T$, and the superscript * denotes the complex conjugate.

We will give the 2N equations that govern the load flow equations in each of the following cases [3,4]:

(a) The load flow equations of the PQ-bus Network:

$$E_k \sum_{i=1}^{N+1} y_{ki}^* E_i^* - S_k = 0, \qquad k = 1, 2, \cdots, N, \qquad (2.1.7a)$$

$$E_k^* \sum_{i=1}^{N+1} y_{ki} E_i - S_k^* = 0, \qquad k = 1, 2, \cdots, N. \qquad (2.1.7b)$$

(b) The load flow equations of the PV-bus network:

$$E_k \sum_{i=1}^{N+1} y_{ki}^* E_i^* + E_k^* \sum_{i=1}^{N+1} y_{ki} E_i - 2P_k = 0, \qquad k = 1, 2, \cdots, N, \qquad (2.1.8a)$$

$$E_k E_k^* - V_k^2 = 0, \qquad k = 1, 2, \cdots, N. \qquad (2.1.8b)$$

(c) The load flow equations for the general power network:

$$E_k \sum_{i=1}^{N+1} y_{ki}^* E_i^* + E_k^* \sum_{i=1}^{N+1} y_{ki} E_i - 2P_k = 0, \qquad k = 1, 2, \cdots, N_g, \qquad (2.1.9a)$$

$$E_k E_k^* - V_k^2 = 0, \qquad k = 1, 2, \cdots, N_g, \qquad (2.1.9b)$$

$$E_k \sum_{i=1}^{N+1} y_{ki}^* E_i^* - S_k = 0, \qquad k = N_g+1, \cdots, N, \qquad (2.1.9c)$$

$$E_k^* \sum_{i=1}^{N+1} y_{ki} E_i - S_k^* = 0, \qquad k = N_g+1, \cdots, N. \qquad (2.1.9d)$$

2.2. The Model with Internal and Terminal Buses of a Generator

This model includes one circuit for the field winding of the round-rotor machine. Let the i-th ($1 \le i \le n$) generator's terminal voltage be denoted by $V_i e^{j\Theta_i}$, and its internal stator voltage be $E'_{qi} e^{j\delta_i}$, or its internal stator-based voltage be $E_{qi} e^{j\delta_i}$ [40]. Let the k-th ($n+2 \le k \le n+m+1$) load-bus voltage be denoted by $V_k e^{j\Phi_k}$. The motion of the i-th generator with round rotor is governed by the following swing equations [40].

$$0 = \dot{\delta}_i - \omega_i, \qquad i = 1, 2, \cdots, n, \tag{2.2.1a}$$

$$0 = M_i \dot{\omega}_i - P_i^m + D_i \omega_i + P_i^e, \quad i = 1, 2, \cdots, n, \tag{2.2.1b}$$

$$0 = T'_{doi} \dot{E}'_{qi} + E_{qi} - E_{Fi}, \qquad i = 1, 2, \cdots, n, \tag{2.2.1c}$$

with the algebraic constraints

$$0 = P_k^d - g_k(\delta, E_q, \Theta, V, \Phi), \tag{2.2.1d}$$

$$0 = Q_k^d - h_k(\delta, E_q, \Theta, V, \Phi), \tag{2.2.1e}$$

$$k = 1, \cdots, n, n+2, \cdots, n+m+1,$$

where

$$P_i^e = f_i(\delta, E_q, \Theta, V, \Phi) = \frac{E_{qi} V_i \sin(\delta_i - \Theta_i)}{x_{di}} \tag{2.2.2a}$$

$$= b_{di} E_{qi} V_i \sin(\delta_i - \Theta_i), \qquad (b_{di} = 1/x_{di}),$$

$$i = 1, 2, \cdots, n.$$

At the i-th ($1 \le i \le n$) terminal generator bus, we have

$$0 = g_i(\delta, E_q, \Theta, V, \Phi) \tag{2.2.2b}$$

$$= b_{di} E_{qi} V_i \sin(\Theta_i - \delta_i) +$$

$$\sum_{j=1}^{n+1} [g_{ij} V_i V_j \cos(\Theta_i - \Theta_j) + b_{ij} V_i V_j \sin(\Theta_i - \Theta_j)] +$$

$$\sum_{k=n+2}^{n+m+1} [g_{ik} V_i V_k \cos(\Theta_i - \Phi_k) + b_{ik} V_i V_k \sin(\Theta_i - \Phi_k)],$$

and

$$0 = h_i(\delta, E_q, \Theta, V, \Phi) \tag{2.2.2c}$$

$$= b_{di} V_i [V_i - E_{qi} \cos(\Theta_i - \delta_i)] +$$

$$\sum_{j=1}^{n+1} [g_{ij} V_i V_j \sin(\Theta_i - \Theta_j) - b_{ij} V_i V_j \cos(\Theta_i - \Theta_j)] +$$

$$\sum_{k=n+2}^{n+m+1} [g_{ik} V_i V_k \sin(\Theta_i - \Phi_k) - b_{ik} V_i V_k \cos(\Theta_i - \Phi_k)].$$

At the k-th ($n+2 \le k \le n+m+1$) load bus, we obtain

$$P_k^d = g_k(\delta, E_q, \Theta, V, \Phi)$$

$$= \sum_{i=1}^{n+1} [g_{ki} V_k V_i \cos(\Phi_k - \Theta_i) + b_{ki} V_k V_i \sin(\Phi_k - \Theta_i)] +$$

$$\sum_{l=n+2}^{n+m+1} [g_{kl} V_k V_l \cos(\Phi_k - \Phi_l) + b_{kl} V_k V_l \sin(\Phi_k - \Phi_l)], \tag{2.2.2d}$$

and

$$Q_k^d = h_k(\delta, E_q, \Theta, V, \Phi)$$

$$= \sum_{i=1}^{n+1}[g_{ki}V_kV_i\sin(\Phi_k-\Theta_i) - b_{ki}V_kV_i\cos(\Phi_k-\Theta_i)] +$$

$$\sum_{l=n+2}^{n+m+1}[g_{kl}V_kV_l\sin(\Phi_k-\Phi_l) - b_{kl}V_kV_l\cos(\Phi_k-\Phi_l)]. \quad (2.2.2e)$$

Let the complex variables $\bar{E}_i$, $\hat{E}_i$ and $\tilde{E}_k$ be defined by

$$\bar{E}_i := E_{qi}\cos\delta_i + j\ E_{qi}\sin\delta_i, \quad (j = \sqrt{-1})$$

$$\hat{E}_i := V_i\cos\Theta_i + j\ V_i\sin\Theta_i,$$

$$i = 1, 2, \cdots, n,$$

and

$$\tilde{E}_k := V_k\cos\Phi_k + j\ V_k\sin\Phi_k,$$

$$k = n+2, \cdots, n+m+1.$$

The static equations of the model (see the depicted equivalent schematic diagram in Figure 2.2) in polynomial representations are obtained by taking all time-derivatives to be zero. Therefore

$$0 = P_i^m - P_i^e \quad (2.2.3a)$$

$$= P_i^m - \frac{1}{2}\bar{E}_i(\bar{E}_i^* - \hat{E}_i^*)y_{di}^* -$$

$$\frac{1}{2}\bar{E}_i^*(\bar{E}_i - \hat{E}_i)y_{di}, \quad (y_{di} = -jb_{di}),$$

$$i = 1, 2, \cdots, n,$$

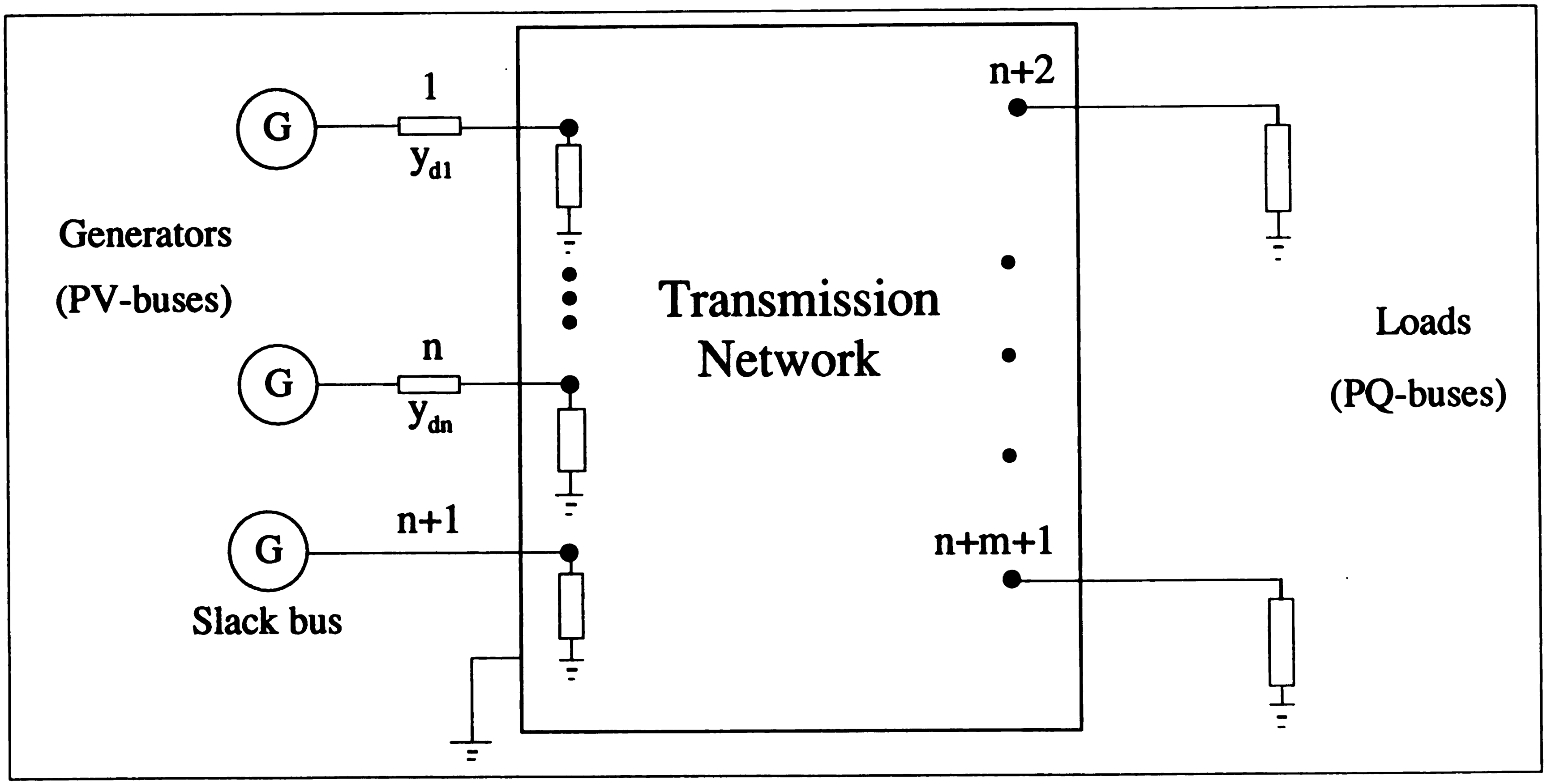


Figure 2.2. The main components of a power system including the direct axis synchronous reactance.

and

$$0 = \bar{E}_i \bar{E}_i^* - E_{Fi}^2, \quad (E_{qi} = E_{Fi}), \tag{2.2.3b}$$

$$i = 1, 2, \cdots, n.$$

At the i-th ($1 \le i \le n$) generator terminal, one obtains

$$0 = g_i(E, E^*) \tag{2.2.3c}$$

$$= \frac{1}{2}\hat{E}_i(\hat{E}_i^* - \bar{E}_i^*)y_{di}^* +$$

$$\frac{1}{2}\hat{E}_i^*(\hat{E}_i - \bar{E}_i)y_{di} +$$

$$\frac{1}{2}(\hat{E}_i \sum_{l=1}^{n+1} y_{il}^* \hat{E}_l^* + \hat{E}_i^* \sum_{l=1}^{n+1} y_{il} \hat{E}_l) +$$

$$\frac{1}{2}(\hat{E}_i \sum_{k=n+2}^{n+m+1} y_{ik}^* \bar{E}_k^* + \hat{E}_i^* \sum_{k=n+2}^{n+m+1} y_{ik} \bar{E}_k),$$

and

$$0 = h_i(E, E^*) \tag{2.2.3d}$$

$$= \frac{1}{2j}\hat{E}_i(\hat{E}_i^* - \bar{E}_i^*)y_{di}^* -$$

$$\frac{1}{2j}\hat{E}_i^*(\hat{E}_i - \bar{E}_i)y_{di} +$$

$$\frac{1}{2j}(\hat{E}_i \sum_{l=1}^{n+1} y_{il}^* \hat{E}_l^* - \hat{E}_i^* \sum_{l=1}^{n+1} y_{il} \hat{E}_l) +$$

$$\frac{1}{2j}(\hat{E}_i \sum_{k=n+2}^{n+m+1} y_{ik}^* \tilde{E}_k^* - \hat{E}_i^* \sum_{k=n+2}^{n+m+1} y_{ik} \tilde{E}_k).$$

At the k-th ($n+2 \le k \le n+m+1$) load bus, one has

$$0 = P_k^d - g_k(E, E^*) \quad (2.2.3e)$$

$$= P_k^d - \frac{1}{2}(\tilde{E}_k \sum_{i=1}^{n+1} y_{ki}^* \hat{E}_i^* + \tilde{E}_k^* \sum_{i=1}^{n+1} y_{ki} \hat{E}_i) -$$

$$\frac{1}{2}(\tilde{E}_k \sum_{l=n+2}^{n+m+1} y_{kl}^* \tilde{E}_l^* + \tilde{E}_k^* \sum_{l=n+2}^{n+m+1} y_{kl} \tilde{E}_l),$$

and

$$0 = Q_k^d - h_k(E, E^*) \quad (2.2.3f)$$

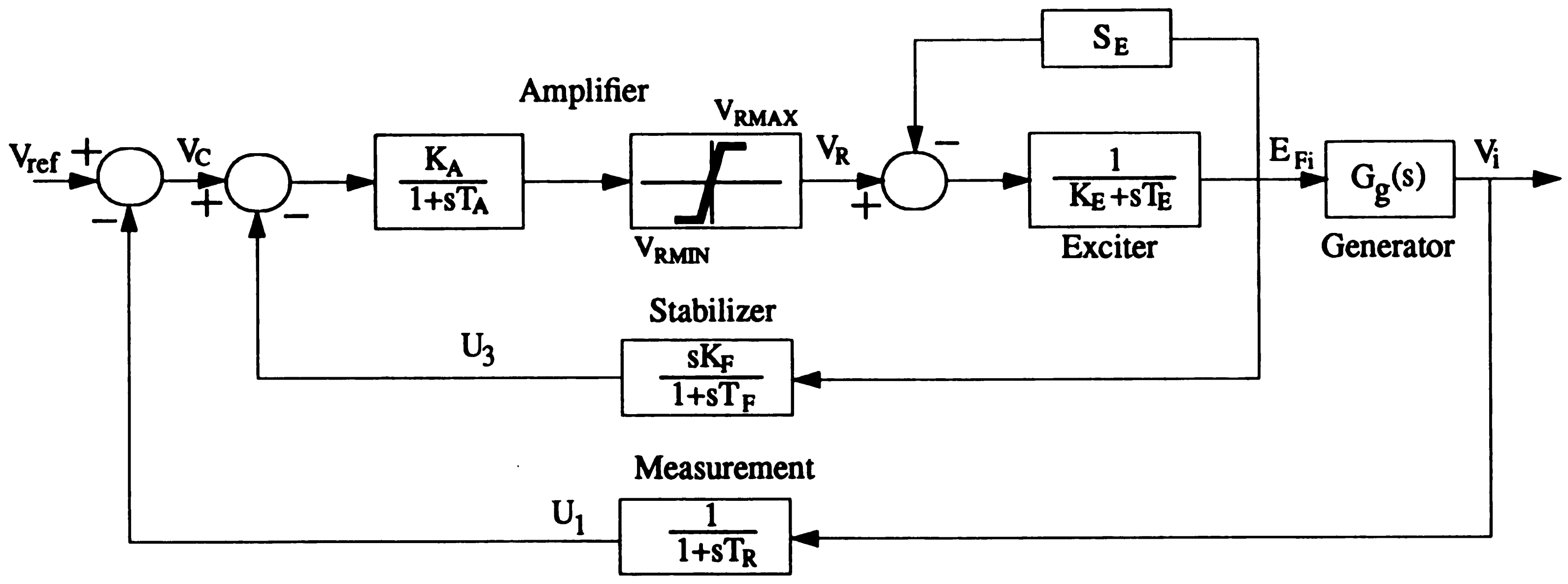
$$= Q_k^d - \frac{1}{2j}(\tilde{E}_k \sum_{i=1}^{n+1} y_{ki}^* \hat{E}_i^* - \tilde{E}_k^* \sum_{i=1}^{n+1} y_{ki} \hat{E}_i) -$$

$$\frac{1}{2j}(\tilde{E}_k \sum_{l=n+2}^{n+m+1} y_{kl}^* \tilde{E}_l^* - \tilde{E}_k^* \sum_{l=n+2}^{n+m+1} y_{kl} \tilde{E}_l).$$

2.3. The Model Augmented by the Excitation System

In the classical model of the swing equations, we simplify the model by assuming the flux linkage to be constant. This simplified model can be used to obtain approximate solutions for the full model. In some cases, experience indicates the simplification of the model to be reasonable and convenient. It is generally suggested, in transient stability studies, that generators close to the fault should be modeled in greater detail. The complexity of a generator model is completed by adding the flux decay dynamics in the rotor windings and the excitation system dynamics [75] (see Figure 2.3).

The action of the excitation system of the i-th ($1 \le i \le n$)generator can be described by the following differential equations



Typical data of the excitation system										
T_R	T_A	K_A	V_{RMAX}	V_{RMIN}	T_E	K_E	S_E	$S_{E0.75}$	T_F	K_F
0-0.06	0.06-0.2	25-50	+1.0	-1.0	0.5	-0.05	0.267	0.074	0.35-1.0	0.01-0.08

Figure2.3. Block diagram of an excitation system.

$$T_R \dot{U}_1 = -U_1 + V_i, \tag{2.3.1a}$$

$$T_F \dot{U}_3 = -U_3 + \frac{K_F V_R}{T_E} - \frac{K_F E_{Fi}(S_E + K_E)}{T_E}, \tag{2.3.1b}$$

$$T_A \dot{V}_R = K_A (V_{ref} - U_1 - U_3) - V_R, \tag{2.3.1c}$$

$$T_E \dot{E}_{Fi} = V_R - (S_E + K_E) E_{Fi}. \tag{2.3.1d}$$

The steady state equations of (2.3.1) are obtained by setting all derivative terms to zero. Thus

$$0 = U_1 - V_i, \tag{2.3.2a}$$

$$0 = U_3 - \frac{K_F V_R}{T_E} + \frac{K_F E_{Fi}(S_E + K_E)}{T_E}, \tag{2.3.2b}$$

$$0 = K_A (V_{ref} - U_1 - U_3) - V_R, \tag{2.3.2c}$$

$$0 = V_R - (S_E + K_E) E_{Fi}, \tag{2.3.2d}$$

$$i = 1, 2, \cdots, n.$$

It can be shown, after elimination of the variables U_1, U_3 and V_R from (2.3.2), that the terminal voltage magnitude V_i and the field voltage E_{Fi} of the i-th generator are constrained by the following algebraic equation

$$0 = (S_E + K_E) E_{Fi} + K_A V_i - K_A V_{ref}, \tag{2.3.3}$$

$$i = 1, 2, \cdots, n.$$

Solving for E_{Fi} from (2.3.3), we have

$$E_{Fi} = K_B V_{ref} - K_B V_i, \quad (where\ K_B = \frac{K_A}{S_E + K_E}), \tag{2.3.4}$$

$$i = 1, 2, \cdots, n.$$

Plugging (2.3.4) into (2.2.2a), (2.2.2b), and (2.2.2c), we obtain

$$0 = P_i^m - b_{di} K_B [V_{ref} V_i \sin(\delta_i - \Theta_i) - V_i^2 \sin(\delta_i - \Theta_i)], \tag{2.3.5a}$$

$$i = 1, 2, \cdots, n.$$

At the i-th ($1 \le i \le n$) terminal generator bus, (2.2.2b) yields

$$0 = g_i(\delta, E_q, \Theta, V, \Phi) \tag{2.3.5b}$$

$$= b_{di} K_B [V_{ref} V_i \sin(\Theta_i - \delta_i) - V_i^2 \sin(\Theta_i - \delta_i)] +$$

$$\textstyle\sum_{j=1}^{n+1} [g_{ij} V_i V_j \cos(\Theta_i - \Theta_j) + b_{ij} V_i V_j \sin(\Theta_i - \Theta_j)] +$$

$$\textstyle\sum_{k=n+2}^{n+m+1} [g_{ik} V_i V_k \cos(\Theta_i - \Phi_k) + b_{ik} V_i V_k \sin(\Theta_i - \Phi_k)],$$

and (2.2.2c) yields

$$0 = h_i(\delta, E_q, \Theta, V, \Phi) \tag{2.3.5c}$$

$$= b_{di} [V_i^2 - K_B V_{ref} V_i \cos(\Theta_i - \delta_i) + K_B V_i^2 \cos(\Theta_i - \delta_i)] +$$

$$\textstyle\sum_{j=1}^{n+1} [g_{ij} V_i V_j \sin(\Theta_i - \Theta_j) - b_{ij} V_i V_j \cos(\Theta_i - \Theta_j)] +$$

$$\textstyle\sum_{k=n+2}^{n+m+1} [g_{ik} V_i V_k \sin(\Theta_i - \Phi_k) - b_{ik} V_i V_k \cos(\Theta_i - \Phi_k)].$$

At the k-th ($n+2 \le k \le n+m+1$) load bus, (2.2.2d) yields

$$P_k{}^d = g_k(\delta, E_q, \Theta, V, \Phi)$$

$$= \sum_{i=1}^{n+1}[g_{ki}V_kV_i\cos(\Phi_k-\Theta_i) + b_{ki}V_kV_i\sin(\Phi_k-\Theta_i)] +$$

$$\sum_{l=n+2}^{n+m+1}[g_{kl}V_kV_l\cos(\Phi_k-\Phi_l) + b_{kl}V_kV_l\sin(\Phi_k-\Phi_l)], \qquad (2.3.5d)$$

and (2.2.2e) yields

$$Q_k{}^d = h_k(\delta, E_q, \Theta, V, \Phi)$$

$$= \sum_{i=1}^{n+1}[g_{ki}V_kV_i\sin(\Phi_k-\Theta_i) - b_{ki}V_kV_i\cos(\Phi_k-\Theta_i)] +$$

$$\sum_{l=n+2}^{n+m+1}[g_{kl}V_kV_l\sin(\Phi_k-\Phi_l) - b_{kl}V_kV_l\cos(\Phi_k-\Phi_l)]. \qquad (2.3.5e)$$

Let the complex variables $\bar{E}_i$, $\hat{E}_i$ and $\tilde{E}_k$ be defined by

$$\bar{E}_i := V_i\cos(\delta_i-\Theta_i) + j\ V_i\sin(\delta_i-\Theta_i),$$

$$\hat{E}_i := V_i\cos\Theta_i + j\ V_i\sin\Theta_i,$$

$$i = 1, 2, \cdots, n,$$

and

$$\tilde{E}_k := V_k\cos\Phi_k + j\ V_k\sin\Phi_k,$$

$$k = n+2, \cdots, n+m+1.$$

Then the static equations become

$$0 = P_i^m - \frac{1}{2}y_{di}K_B(V_{ref} - V_i)(\bar{E}_i - \bar{E}_i^*), \qquad (2.3.6a)$$

$i = 1, 2, \cdots, n,$

associated with

$$0 = \bar{E}_i \bar{E}_i^* - V_i^2, \qquad i = 1, 2, \cdots, n, \tag{2.3.6b}$$

and

$$0 = \hat{E}_i \hat{E}_i^* - V_i^2, \qquad i = 1, 2, \cdots, n. \tag{2.3.6c}$$

At the terminal generator buses, one has

$$0 = \frac{1}{2} y_{di} K_B (V_i - V_{ref})(\bar{E}_i - \bar{E}_i^*) + \tag{2.3.6d}$$

$$\frac{1}{2}(\hat{E}_i \sum_{l=1}^{n+1} y_{il}^* \hat{E}_l^* + \hat{E}_i^* \sum_{l=1}^{n+1} y_{il} \hat{E}_l) +$$

$$\frac{1}{2}(\hat{E}_i \sum_{k=n+2}^{n+m+1} y_{ik}^* \tilde{E}_k^* + \hat{E}_i^* \sum_{k=n+2}^{n+m+1} y_{ik} \tilde{E}_k),$$

$i = 1, 2, \cdots, n,$

and

$$0 = \frac{1}{2j} y_{di} [-2V_i^2 + K_B (V_{ref} - V_i)(\bar{E}_i + \bar{E}_i^*)] + \tag{2.3.6e}$$

$$\frac{1}{2j}(\hat{E}_i \sum_{l=1}^{n+1} y_{il}^* \hat{E}_l^* - \hat{E}_i^* \sum_{l=1}^{n+1} y_{il} \hat{E}_l) +$$

$$\frac{1}{2j}(\hat{E}_i \sum_{k=n+2}^{n+m+1} y_{ik}^* \tilde{E}_k^* - \hat{E}_i^* \sum_{k=n+2}^{n+m+1} y_{ik} \tilde{E}_k),$$

$i = 1, 2, \cdots, n.$

At the load buses, one obtains

$$0 = P_k^d - \frac{1}{2}(\tilde{E}_k \sum_{i=1}^{n+1} y_{ki}^* \hat{E}_i^* + \tilde{E}_k^* \sum_{i=1}^{n+1} y_{ki} \hat{E}_i) - \tag{2.3.6f}$$

$$\frac{1}{2}(\tilde{E}_k \sum_{l=n+2}^{n+m+1} y_{kl}^* \tilde{E}_l^* + \tilde{E}_k^* \sum_{l=n+2}^{n+m+1} y_{kl} \tilde{E}_l),$$

$k = n+2, \cdots, n+m+1,$

and

$$0 = Q_k^d - \frac{1}{2j}(\tilde{E}_k \sum_{i=1}^{n+1} y_{ki}^* \hat{E}_i^* - \tilde{E}_k^* \sum_{i=1}^{n+1} y_{ki} \hat{E}_i) - \tag{2.3.6g}$$

$$\frac{1}{2j}(\tilde{E}_k \sum_{l=n+2}^{n+m+1} y_{kl}^* \tilde{E}_l^* - \tilde{E}_k^* \sum_{l=n+2}^{n+m+1} y_{kl} \tilde{E}_l),$$

$k = n+2, \cdots, n+m+1.$

Chapter 3

ON THE NUMBER OF (EQUILIBRIUM) STEADY STATE SOLUTIONS OF POWER SYSTEMS

The fundamental theorem of algebraic geometry states that the number of the isolated solutions of n polynomial equations in n complex unknowns is bounded above by the total degree

$$d = \prod_{i=1}^{n} d_i ,$$

where d_i is the degree (i.e. the power of the highest ordered term) of the i-th polynomial. This result is referred to as the *Bézout* theorem, and the total degree d is often referred to as the *Bézout* number.

The load flow of a power system, expressed in the polynomial form (e.g. the rectangular-coordinate form (2.1.4) or the complex form (2.1.9)), comprise 2N polynomial equations each with degree $d_i = 2$. Consequently, the number of solutions is bounded by their *Bézout* number

$$d = \prod_{i=1}^{n} d_i = 2^{2N} .$$

To get a feel for these numbers, we choose N = 10 to get $2^{20} = 1{,}048{,}576$.

However, it will be shown that the load flow models of power systems belong to the class of deficient polynomial systems ([18], [19]) and thus the number of their finite solutions are bounded by $\binom{2N}{N}$. For N = 10, the bound on the number of solutions is $\binom{20}{10}$ = 184,756. The difference is 2^{20} - $\binom{20}{10}$ = 863,820.

The basic homotopy methods ([3]-[4], [35]-[38]) use 2^{2N} initial points to trace 2^{2N} homotopy curves. When applied to a deficient system such as a power system, however, at most $\binom{2N}{N}$ homotopy curves reach their finite solutions while the rest of the homotopy curves grow unbounded as t approaches 1. From a computational point of view, tracing the non-finite solution curves takes much more time than tracing the finite solution curves. This would consequently represent wasted computational time, and in some cases would cause serious difficulties in numerical computations.

In this chapter, we describe *deficient* systems and show that the (polynomial) models of power systems are *deficient*. For the various models of power systems, we also determine an upper bound on the number of solutions and conditions under which this upper bound can be reached.

3.1. A Deficient System and Its Associated Homogeneous System

Suppose we are solving a system of n polynomial equations in n unknowns, namely

$$F(x_1, \cdots, x_n) = \left[f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)\right]^T = 0. \quad (3.1.1)$$

Let the degree of the i-th polynomial equation f_i be d_i, where $i = 1, 2, \cdots, n$. Then we associate to F a homogeneous polynomial system, namely,

$$\tilde{F}(x_0, x_1, \cdots, x_n) := \left[\tilde{f}_1(x_0, x_1, \cdots, x_n), \cdots, \tilde{f}_n(x_0, x_1, \cdots, x_n)\right]^T \quad (3.1.2)$$

$$= \left[x_0^{d_1} f_1(x_1/x_0, \cdots, x_n/x_0), \cdots, x_0^{d_n} f_n(x_1/x_0, \cdots, x_n/x_0)\right]^T.$$

According to the *Bézout* theory, the number of the isolated solutions, counting multiplicity, of a homogeneous polynomial system $\tilde{F}$ (in the projective space CP^n) is expected to be $d_1 \times \ldots \times d_n$. However, the associated homogeneous system $\tilde{F}$ and the original polynomial system F are related by setting $x_0 = 1$, that is,

$$\tilde{F}(1, x_1, \cdots, x_n) := \left[\tilde{f}_1(1, x_1, \cdots, x_n), \cdots, \tilde{f}_n(1, x_1, \cdots, x_n)\right]^T \tag{3.1.3}$$

$$= \left[f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)\right]^T =: F(x_1, \cdots, x_n).$$

It has been shown that the solutions of the associated homogeneous system $\tilde{F} = 0$ in CP^n consist of both the finite solutions (for $x_0 \neq 0$) and the non-finite solutions (for $x_0 = 0$). The latter solutions are called "the solutions at infinity" [19]. For an example, we consider the following 2 polynomial equations $F(x_1, x_2)$ in 2 variables [19],

$$f_1(x_1, x_2) = x_1^2 - x_1 + x_2 - 2 = 0, \tag{3.1.4}$$

$$f_2(x_1, x_2) = x_1 x_2 - 4x_1 + 3x_2 - 4 = 0.$$

Its associated homogeneous system $\tilde{F}(x_0, x_1, x_2)$ is given by

$$\tilde{f}_1(x_0, x_1, x_2) = x_1^2 - x_0 x_1 + x_0 x_2 - 2x_0^2 = 0, \tag{3.1.5}$$

$$\tilde{f}_2(x_0, x_1, x_2) = x_1 x_2 - 4x_0 x_1 + 3x_0 x_2 - 4x_0^2 = 0.$$

The *Bézout* theorem actually states that the associated homogeneous system $\tilde{F}(x_0, x_1, x_2)$ of (3.1.5) has $d_1 \times d_2 = 2 \times 2 = 4$ isolated solutions in CP^2. These solutions include both the solutions for $x_0 \neq 0$ (the finite solutions) and the solutions for $x_0 = 0$ (i.e. the solutions at infinity). Therefore, the number of the original system solutions in C^2 (the two dimensional complex space) equals the number of the solutions of the associated homogeneous system minus the number of its solutions at infinity.

It can be checked by hand calculations that the associated homogeneous system (3.1.5) has exactly 4 solutions, one of which is non-finite in terms of $x_0 = 0$ (i.e. the solution at infinity), which is basically the linear space Z := $\{(x_1, x_2)| x_1 = 0\}$, and the remaining three are finite in terms of $x_0 \neq 0$ (these are the solutions of (3.1.4)). Therefore, the number of the original system solutions in C^2 (the two dimensional complex space) is 3 (: = 4-1).

In order to predict the number of solutions of (3.1.4) a priori, one must determine the "solutions at infinity" of the associated homogeneous system (3.1.5). The relationship between the original system (3.1.4) and the associated homogeneous system (3.1.5) is clearly that if $x_0 = 1$, then $\tilde{F}(x_0, x_1, x_2) = F(x_1, x_2)$.

3.2 Nonsingular Zeros on Projective Spaces

In geometric terms for general polynomial systems [18], if Z is the common zero set in the n-dimensional complex projective space CP^n of the associated homogeneous system $\tilde{F}$, then Z is the disjoint union

$$\{(1, x_1, \cdots, x_n)|\tilde{F}(x_0, x_1, \cdots, x_n) = 0\} \cup \{(0, x_1, \cdots, x_n)|\tilde{F}(x_0, x_1, \cdots, x_n) = 0\} .$$

Since $\tilde{F}(1, x_1, \cdots, x_n) = F(x_1, \cdots, x_n)$, Z is then the disjoint union of the zeros of F in C^n and the zeros in the "hyperplane at infinity" for $x_0 = 0$. Here the n-dimensional projective space is denoted by

$$CP^n := \{x = (x_0, x_1, \cdots, x_n)|x_i \in C, x \neq (0, 0, \cdots, 0)\}/\sim .$$

Here x $\sim$ y signifies x = cy for some nonzero $c \in C$.

Theorem ([18], [32]). Let $f_1, \cdots, f_n$ be polynomials in x=$(x_1, \cdots, x_n)$ of degrees $d_1, \cdots, d_n$, respectively, and suppose that the common zero set of the associated homogeneous system, say $Z(\tilde{f}_1, \cdots, \tilde{f}_n)$ in CP^n is a disjoint union of nonsingular points $y_1, \cdots, y_r$ and nonsingular linear spaces $Z_1, \cdots, Z_s$. Then

$$d_1 \times \cdots \times d_n = r + \sum_{i=1}^{s} [Z_i], \tag{3.2.1}$$

where for a linear space Z = CP^e, the equivalence [Z] is given by the coefficient of t^e in the power series [18]

$$(1+t)^{e-n} \prod_{i=1}^{n} (1+d_i t). \tag{3.2.2}$$

As an illustration of this theorem, consider the case when $d_1 = \ldots = d_n = 2$, then

$$\begin{aligned} (1+2t)^n (1+t)^{e-n} &= (1+t+t)^n (1+t)^{e-n} && (3.2.3) \\ &= \sum_{i=0}^{n} (1+t)^{n-i} t^i \binom{n}{i} (1+t)^{e-n} \\ &= \sum_{i=0}^{n} (1+t)^{e-i} t^i \binom{n}{i}. \end{aligned}$$

Thence $$[Z] = \sum_{i=0}^{e} \binom{n}{i}.$$

Definition. We say a point $y \in Z(\tilde{f}_1, \cdots, \tilde{f}_r)$ is nonsingular, if

$$rank_c \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_r)}{\partial(x_0, \cdots, x_n)}(y) = codim_y(Z(\tilde{f}_1, \cdots, \tilde{f}_r), CP^n), \tag{3.2.4}$$

where codim denotes complex codimension with $codim_y(Z, CP^n) = n - e$, and $e = dim_y Z$ signifies complex dimension of Z in CP^n. A variety Z is nonsingular if each point y of the variety Z is nonsingular [18].

3.3. Application to Quadratic Polynomial Systems

Consider the general quadratic polynomial system in 2N equations with 2N variables, namely

$$f_1 = \hat{L}_{11}\hat{L}_{12} = 0, \tag{3.3.1}$$

$$\vdots$$

$$f_{2N} = \hat{L}_{2N,1}\hat{L}_{2N,2} = 0,$$

where

$$\hat{L}_{i1} = \hat{L}_{i1}(x_1, \cdots, x_N) = a_{i1}x_1 + \ldots + a_{iN}x_N + \alpha_i$$

is a linear form in the variables $x_1, \cdots, x_N$ for $i = 1, 2, \cdots, 2N$, and

$$\hat{L}_{j2} = \hat{L}_{j2}(y_1, \cdots, y_N) = b_{j1}y_1 + \ldots + b_{jN}y_N + \beta_j$$

is a linear form in the variables $y_1, \cdots, y_N$ for $j = 1, \cdots, 2N$. Its associated homogeneous system is given by

$$\tilde{f}_1 = \tilde{L}_{11}\tilde{L}_{12} = 0, \tag{3.3.2}$$

$$\vdots$$

$$\tilde{f}_{2N} = \tilde{L}_{2N,1}\tilde{L}_{2N,2} = 0,$$

where

$$\tilde{L}_{i1} = \tilde{L}_{i1}(x_1, \cdots, x_N) = a_{i1}x_1 + \ldots + a_{iN}x_N + \alpha_i x_0,$$

$$i = 1, 2, \cdots, 2N,$$

and

$$\tilde{L}_{j2} = \tilde{L}_{j2}(y_1, \cdots, y_N) = b_{j1}y_1 + \ldots + b_{jN}y_N + \beta_j x_0,$$

$$j = 1, 2, \cdots, 2N.$$

Calculate the Jacobian matrix J(z) defined as

$$J(z) = \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_{2N})}{\partial(x_0, x_1, \cdots, x_N, y_1, \cdots, y_N)}.$$

With $x_0 = 0$, one obtains

$$J(z, x_0 = 0) = \tag{3.3.3}$$

$$\begin{bmatrix} \alpha_1 L_{12} + \beta_1 L_{11} & a_{11}L_{12} & \cdots & a_{1N}L_{12} & b_{11}L_{11} & \cdots & b_{1N}L_{11} \\ \alpha_2 L_{22} + \beta_2 L_{21} & a_{21}L_{22} & \cdots & a_{2N}L_{22} & b_{21}L_{21} & \cdots & b_{2N}L_{21} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2N}L_{2N,2} + \beta_{2N}L_{2N,1} & a_{2N,1}L_{2N,2} & \cdots & a_{2N,N}L_{2N,2} & b_{2N,1}L_{2N,1} & \cdots & b_{2N,N}L_{2N,1} \end{bmatrix},$$

where

$$L_{i1} = L_{i1}(x_1, \cdots, x_N) = a_{i1}x_1 + \ldots + a_{iN}x_N,$$

$$i = 1, 2, \cdots, 2N,$$

and

$$L_{j2} = L_{j2}(y_1, \cdots, y_N) = b_{j1}y_1 + \ldots + b_{jN}y_N,$$

$$j = 1, 2, \cdots, 2N.$$

The solutions at infinity of (3.3.2) are the solutions of the following equations obtained by ignoring all lower order degree terms of (3.3.1). Therefore the solutions at infinity of (3.3.2) are the solutions of

$$\bar{f}_1 = L_{11}L_{12} = 0, \cdots, \bar{f}_{2N} = L_{2N,1}L_{2N,2} = 0. \tag{3.3.4}$$

Let the coefficient matrices A and B be defined by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{2N,1} & a_{2N,2} & \cdots & a_{2N,N} \end{bmatrix}_{2N \times N}, \tag{3.3.5}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{2N,1} & b_{2N,2} & \cdots & b_{2N,N} \end{bmatrix}_{2N \times N}. \tag{3.3.6}$$

If all the $N \times N$ minors of A and B are nonsingular, *the solution sets at infinity* of (3.3.2), say Z_1 *and* Z_2, *consist of the two linear subspaces*

$$Z_1 = \{(0, x_1, \cdots, x_N, y_1, \cdots, y_N) \in CP^{2N} \mid x_1 = \ldots = x_N = 0\} = CP^{N-1},$$

and

$$Z_2 = \{(0, x_1, \cdots, x_N, y_1, \cdots, y_N) \in CP^{2N} \mid y_1 = \ldots = y_N = 0\} = CP^{N-1}.$$

The $codim(Z, CP^{2N})$ would be N+1 (=2N-(N-1)). Plugging *the solutions at infinity* Z_1 and Z_2 into the Jacobian matrix (3.3.3) and removing all zero columns from J(z), one obtains

$$\hat{J}(Z_1) = \begin{bmatrix} \alpha_1 L_{12} & a_{11}L_{12} & a_{12}L_{12} & \cdots & a_{1N}L_{12} \\ \alpha_2 L_{22} & a_{21}L_{22} & a_{22}L_{22} & \cdots & a_{2N}L_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2N} L_{2N,2} & a_{2N,1}L_{2N,2} & a_{2N,2}L_{2N,2} & \cdots & a_{2N,N}L_{2N,2} \end{bmatrix}, \tag{3.3.7}$$

and

$$\hat{J}(Z_2) = \begin{bmatrix} \beta_1 L_{11} & b_{11}L_{11} & b_{12}L_{11} & \cdots & b_{1N}L_{11} \\ \beta_2 L_{21} & b_{21}L_{21} & b_{22}L_{21} & \cdots & b_{2N}L_{21} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{2N} L_{2N,1} & b_{2N,1}L_{2N,1} & b_{2N,2}L_{2N,1} & \cdots & b_{2N,N}L_{2N,1} \end{bmatrix}. \tag{3.3.8}$$

Now define the extended matrices $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = \begin{bmatrix} \alpha_1 & a_{11} & a_{12} & \cdots & a_{1N} \\ \alpha_2 & a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2N} & a_{2N,1} & a_{2N,2} & \cdots & a_{2N,N} \end{bmatrix}_{2N \times (N+1)}, \tag{3.3.9}$$

and

$$\tilde{B} = \begin{bmatrix} \beta_1 & b_{11} & b_{12} & \cdots & b_{1N} \\ \beta_2 & b_{21} & b_{22} & \cdots & b_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{2N} & b_{2N,1} & b_{2N,2} & \cdots & b_{2N,N} \end{bmatrix}_{2N \times (N+1)}. \tag{3.3.10}$$

Then the following theorem holds.

Theorem 3.3. The system (3.3.1) has exactly r isolated (complex) solutions if all $N \times N$ minors of A of (3.3.5) and B of (3.3.6), and all $(N+1) \times (N+1)$ minors of the extended matrix $\tilde{A}$ of (3.3.9) and $\tilde{B}$ of (3.3.10) are nonsingular, where r is given by

$$r = 2^{2N} - [Z_1] - [Z_2] = 2^{2N} - \sum_{i=0}^{N-1}\binom{2N}{i} - \sum_{i=N+1}^{2N}\binom{2N}{i} = \binom{2N}{N},$$

and [Z] denotes the equivalence of Z.

Proof. Since $J(Z_1)$ and $J(Z_2)$ are of the same form, we only need to prove $rank_c J(Z_1) = codim(Z_1, CP^{2N}) = (N+1)$. Then the same conclusion on $J(Z_1)$ would apply to $J(Z_2)$. From the first assumption of the nonsingularity of all $N \times N$ minors of the matrix B, there are at most (N - 1) linear forms L_{j2} to be zero for $y \neq (0, 0, \cdots, 0)$. Suppose that the first $i \leq (N-1)$ linear forms L_{k2}, $k = 1, 2, \cdots, i$ are zero (If not, we can always reorder the polynomial equations (3.3.1) so that such a condition is satisfied.). Therefore, the rank of J (Z_1) can be determined by checking the rank of the following matrix

$$\bar{A} = \begin{bmatrix} \alpha_{i+1} & a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,N} \\ \alpha_{i+2} & a_{i+2,1} & a_{i+2,2} & \cdots & a_{i+2,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2N} & a_{2N,1} & a_{2N,2} & \cdots & a_{2N,N} \end{bmatrix}_{(2N-i)\times(N+1)} . \qquad (3.3.11)$$

From the assumption of nonsingularity of all $(N+1) \times (N+1)$ of $\tilde{A}$, the rank of the matrix $\bar{A}$ equals N+1. Consequently, Z_1 is nonsingular. Similarly, we can draw the same conclusion that Z_2 is nonsingular. By the Intersection Theory, system (3.3.1) has total r isolated solutions, where r is

$$r = 2^{2N} - [Z_1] - [Z_2] = 2^{2N} - \sum_{i=0}^{N-1}\binom{2N}{i} - \sum_{i=N+1}^{2N}\binom{2N}{i} = \binom{2N}{N}.$$

(Q.E.D)

3.4. The Classical Model of Power Systems

Intersection theory implies that if there are no "solutions at infinity", the number of solutions of the original system is exactly equal to the total degree d. And if "the solutions at infinity", which consist of the intersection of all linear subspaces Z_i ($1 \le i \le s$) are nonsingular, the original system has exactly r isolated solutions where r is given in (3.2.1).

In the case of the load flow equations of power systems, it will be shown that "the solutions at infinity" of the load flow equations exist because the load flow equations are deficient. Not all of *the solutions at infinity* of the load flow equations are nonsingular, however, because a power system is not always fully connected (Note: we say the system is fully connected if each bus is connected directly to all other buses.).

3.4.1. The all-PQ-bus load flow equations

In the PQ-bus networks, the injected complex power at each bus is specified. The maximum number of solutions of such systems can be attained if all the conditions of the following theorem are satisfied. We point out on the outset that the sufficient conditions are generically satisfied, i.e., they are satisfied for all but a measure-zero set of admittance values.

Theorem 3.4.1. There are exactly $\binom{2N}{N}$ isolated complex solutions of the load flow equations (2.1.7) for an all-PQ-bus N-node power system (excluding the slack bus), if all $k \times k$ ($k = 1, 2, \cdots, N$) minors of the extended complex admittance matrix $\tilde{Y}$ are nonsingular. Where the extended complex admittance matrix $\tilde{Y}$ is defined by

$$\tilde{Y} = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1N} & y_{1,N+1} \\ y_{21} & y_{22} & \cdots & y_{2N} & y_{2,N+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N1} & y_{N2} & \cdots & y_{NN} & y_{N,N+1} \end{bmatrix}. \tag{3.4.1.1}$$

Proof. Since all $k \times k$ ($k = 1, 2, \cdots, N$) minors of $\tilde{Y}$ are nonsingular, the solution set at infinity is $Z_1 \cup Z_2$, where

$$Z_1 := \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) \mid 0 = E_1 = \ldots = E_N\} = CP^{N-1},$$

and

$$Z_2 := \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) \mid 0 = E_1^* = \ldots = E_N^*\} = CP^{N-1},$$

with $codim(Z, CP^{2N})$ = 2N-(N-1) = N+1.

Calculating the matrices $J(Z_1)$ and $J(Z_2)$ with respect to $(E_0, E_1, \cdots, E_N^*)$ according to (3.2.4) from the associated homogeneous system of (2.1.7) and removing all zero columns from $J(Z_1)$ and $J(Z_2)$, one respectively obtains

$$\hat{J}(Z_1) = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_N \\ y_{1,N+1}E_1^* & y_{11}E_1^* & y_{12}E_1^* & \cdots & y_{1N}E_1^* \\ y_{2,N+1}E_2^* & y_{21}E_2^* & y_{22}E_2^* & \cdots & y_{2N}E_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N,N+1}E_N^* & y_{N1}E_N^* & y_{N2}E_N^* & \cdots & y_{NN}E_N^* \end{bmatrix}_{2N\times(N+1)}, \tag{3.4.1.2}$$

and

$$\hat{J}(Z_2) = \begin{bmatrix} y_{1,N+1}^*E_1 & y_{11}^*E_1 & y_{12}^*E_1 & & y_{1N}^*E_1 \\ y_{2,N+1}^*E_2 & y_{21}^*E_2 & y_{22}^*E_2 & \cdots & y_{2N}^*E_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N,N+1}^*E_N & y_{N1}^*E_N & y_{N2}^*E_N & \cdots & y_{NN}^*E_N \\ 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_N \end{bmatrix}_{2N\times(N+1)}, \tag{3.4.1.3}$$

where $a_k := \sum_{j=1}^{N} y_{kj}^* E_j^*$, and $b_k := \sum_{j=1}^{N} y_{kj} E_j$ for $k = 1, 2, \cdots, N$.

Consider the non-singularity of $\hat{J}(Z_2)$ first. Suppose that the first i rows (e.g. E_i=0, $0 \le i \le (N-1)$) become zero. It can be shown that there are at most $p=N-i-1$ rows of $\hat{J}(Z_2)$ with $b_j=0$ $(0 \le j \le P)$. In the worst case, suppose that b_j $(0 \le j \le N-i-1)$ become zero. Also assume that the load flow equations (2.1.7) are reordered in such a manner that the first i rows and the last p rows of $\hat{J}(Z_2)$ are zero. We obtain the reduced matrix by removing the first i rows and the last p rows from $\hat{J}(Z_2)$. From the assumptions of the theorem, the following reduced matrix

$$\begin{bmatrix} y^*_{i+1,N+1}E_{i+1} & y^*_{i+1,1}E_{i+1} & \cdots & y^*_{i+1,i+1}E_{i+1} & \cdots & y^*_{i+1,N}E_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y^*_{N,N+1}E_N & y^*_{N,1}E_N & \cdots & y^*_{N,i+1}E_N & \cdots & y^*_{NN}E_N \\ 0 & b_1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{i+1} & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)} \tag{3.4.1.4}$$

is nonsingular. Therefore, $rank_c J(Z_2)$ equals N+1. Consequently, Z_2 is nonsingular. Since $J(Z_1)$ and $J(Z_2)$ are of the same form, we can follow analogous procedures to prove that Z_1 is nonsingular. By the Intersection Theorem (see page 32 of this thesis), the number of isolated solutions is

$$r = 2^{2N} - [Z_1] - [Z_2] = 2^{2N} - \sum_{i=0}^{N-1}\binom{2N}{i} - \sum_{i=N+1}^{2N}\binom{2N}{i} = \binom{2N}{N}.$$

Thus, the proof of the theorem is completed. (Q.E.D)

3.4.1.1. Example: a fully-connected 3-bus power system structure

We now apply theorem 3.4.1 to investigate the number of solutions of the load flow equations for an all-PQ-bus fully-connected 3-node power system as shown in Figure 3.1. The system's extended complex admittance matrix $\tilde{Y}$ is given by

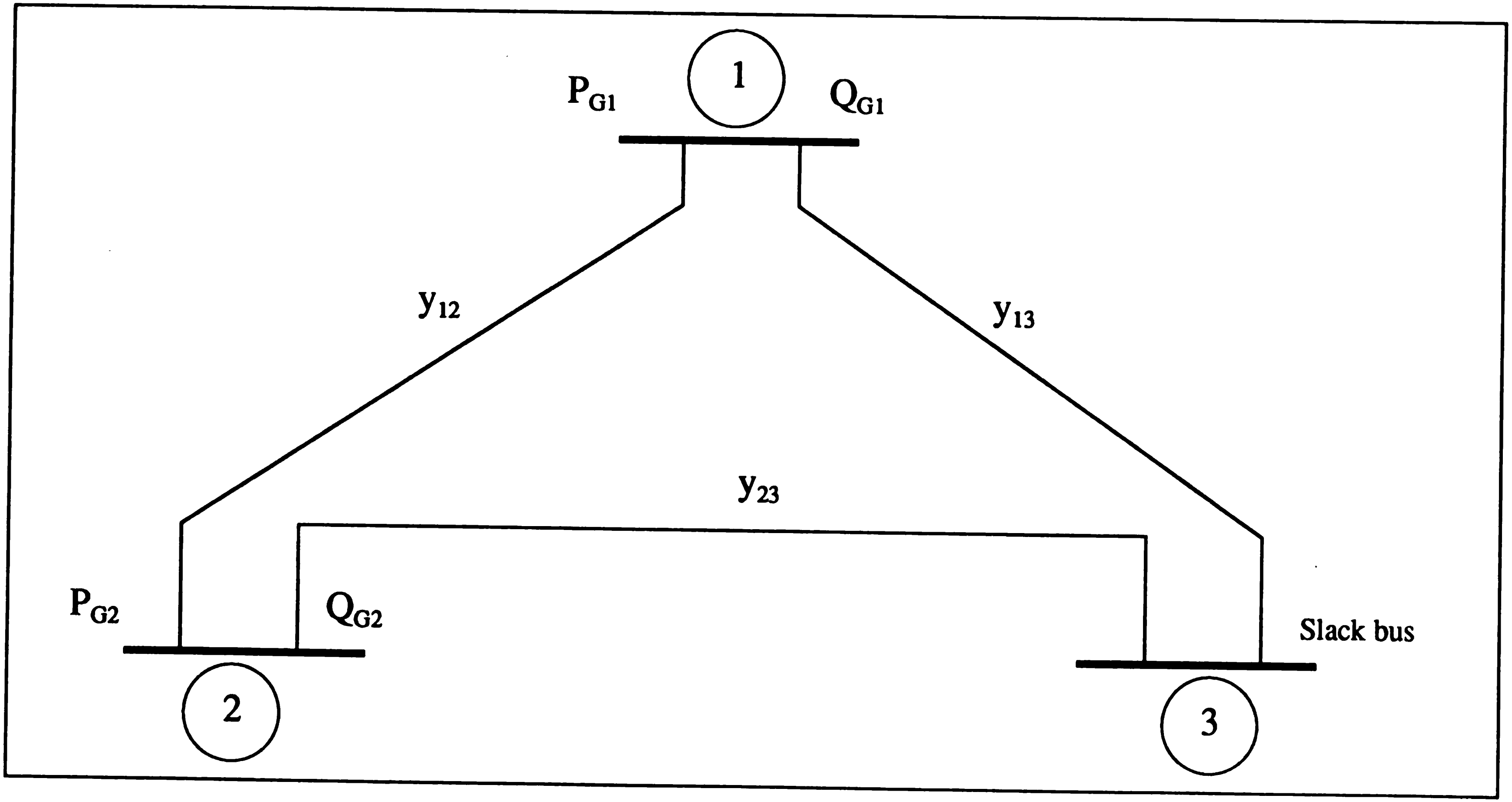


Figure 3.1. The 3-bus power system structure.

$$\tilde{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}.$$

The load flow equations for a general all-PQ-bus 3-node power system can be expressed in the complex polynomial form (2.1.7) as

$$E_1(y_{11}^*E_1^* + y_{12}^*E_2^* + y_{13}^*) - S_1 = 0, \tag{3.4.1.5a}$$

$$E_2(y_{21}^*E_1^* + y_{22}^*E_2^* + y_{23}^*) - S_2 = 0, \tag{3.4.1.5b}$$

$$E_1^*(y_{11}E_1 + y_{12}E_2 + y_{13}) - S_1^* = 0, \tag{3.4.1.5c}$$

$$E_2^*(y_{21}E_1 + y_{22}E_2 + y_{23}) - S_2^* = 0. \tag{3.4.1.5d}$$

Its associated homogeneous system is given by

$$E_1(y_{11}^*E_1^* + y_{12}^*E_2^* + y_{13}^*E_0) - S_1E_0^2 = 0, \tag{3.4.1.6a}$$

$$E_2(y_{21}^*E_1^* + y_{22}^*E_2^* + y_{23}^*E_0) - S_2E_0^2 = 0, \tag{3.4.1.6b}$$

$$E_1^*(y_{11}E_1 + y_{12}E_2 + y_{13}E_0) - S_1^*E_0^2 = 0, \tag{3.4.1.6c}$$

$$E_2^*(y_{21}E_1 + y_{22}E_2 + y_{23}E_0) - S_2^*E_0^2 = 0. \tag{3.4.1.6d}$$

The solutions at infinity, by definition, are the solutions of the following equations

$$E_1(y_{11}^*E_1^* + y_{12}^*E_2^*) = 0, \tag{3.4.1.7a}$$

$$E_2(y_{21}^*E_1^* + y_{22}^*E_2^*) = 0, \tag{3.4.1.7b}$$

$$E_1^*(y_{11}E_1 + y_{12}E_2) = 0, \tag{3.4.1.7c}$$

$$E_2^*(y_{21}E_1 + y_{22}E_2) = 0. \tag{3.4.1.7d}$$

Define the matrices A and B by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \quad B = \begin{bmatrix} y_{11}^* & y_{12}^* \\ y_{21}^* & y_{22}^* \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4.1.8}$$

It can be seen that if all $k \times k$ (k = 1, 2) minors of $\tilde{Y}$ are nonsingular, then all 2×2 minors of matrices A and B are nonsingular. Therefore, all *the solutions at infinity* consist of the two linear subspaces

$$Z_1 = \{(0, E_1, E_2, E_1^*, E_2^*): 0 = E_1 = E_2\},$$

and

$$Z_2 = \{(0, E_1, E_2, E_1^*, E_2^*): 0 = E_1^* = E_2^*\}.$$

Since both $codim(Z_1, CP^4)$ and $codim(Z_2, CP^4)$ are equal to 3 (= 4-1), Z_1 and Z_2 are nonsingular if $rank_c J(Z)$ equals 3. The singularity of Z_1 and Z_2 can be checked by the following steps. First, taking the partial derivative with respect to $(E_0, E_1, \cdots, E_2^*)$ of (3.4.1.6), one obtains

$$\left.\frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_4)}{\partial(E_0, E_1, \cdots, E_2^*)}\right|_{E_0=0} = \tag{3.4.1.9}$$

$$\begin{bmatrix} y_{13}^*E_1 & y_{11}^*E_1^* + y_{12}^*E_2^* & 0 & y_{11}^*E_1 & y_{12}^*E_1 \\ y_{23}^*E_2 & 0 & y_{21}^*E_1^* + y_{22}^*E_2^* & y_{21}^*E_2 & y_{22}^*E_2 \\ y_{13}E_1^* & y_{11}E_1^* & y_{12}E_1^* & y_{11}E_1 + y_{12}E_2 & 0 \\ y_{23}E_2^* & y_{21}E_2^* & y_{22}E_2^* & 0 & y_{21}E_1 + y_{22}E_2 \end{bmatrix}.$$

Plugging the solutions at infinity Z_1 and Z_2 into (3.4.1.9), one respectively obtains

$$J(Z_1) = \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_4)}{\partial(E_0, E_1, \cdots, E_2^*)}\Big|_{(E_0=E_1=E_2=0)} \qquad (3.4.1.10a)$$

$$= \begin{bmatrix} 0 & y_{11}^* E_1^* + y_{12}^* E_2^* & 0 & 0 & 0 \\ 0 & 0 & y_{21}^* E_1^* + y_{22}^* E_2^* & 0 & 0 \\ y_{13} E_1^* & y_{11} E_1^* & y_{12} E_1^* & 0 & 0 \\ y_{23} E_2^* & y_{21} E_2^* & y_{22} E_2^* & 0 & 0 \end{bmatrix}_{4\times 5},$$

and

$$J(Z_2) = \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_4)}{\partial(E_0, E_1, \cdots, E_2^*)}\Big|_{(E_0=E_1^*=E_2^*=0)} \qquad (3.4.1.10b)$$

$$= \begin{bmatrix} y_{13}^* E_1 & 0 & 0 & y_{11}^* E_1 & y_{12}^* E_1 \\ y_{23}^* E_2 & 0 & 0 & y_{21}^* E_2 & y_{22}^* E_2 \\ 0 & 0 & 0 & y_{11} E_1 + y_{12} E_2 & 0 \\ 0 & 0 & 0 & 0 & y_{21} E_1 + y_{22} E_2 \end{bmatrix}_{4\times 5}.$$

Since all the $k \times k$ ($1 \le k \le 2$) minors of $\tilde{Y}$ are nonsingular, both $rank_c J(Z_1)$ and $rank_c J(Z_2)$ equal 3 (:= N+1). Therefore, we conclude that system (3.4.1.5) has 6 [: = 2^4 - $[Z_1]$ - $[Z_2]$ = $\binom{4}{2}$] isolated complex solutions.

3.4.2. The all-PV-bus load flow equations

PV buses are mostly generation buses at which the injected active power is specified and held fixed by turbine settings. A voltage regulator holds the magnitude of the voltage fixed at a PV bus by automatically varying the generator field excitation. This variation anticipated to cause the generated reactive power to vary in such a way as to bring the terminal voltage magnitude to the specified value. The load flow equations in this case are described by (2.1.8). The number of solutions are specified by the following theorem.

Theorem 3.4.2. There are exactly $\begin{bmatrix} 2N \\ N \end{bmatrix}$ isolated complex solutions of the load flow equations (2.1.8) for an all-PV-bus N-node power system (excluding the slack bus), if the following conditions hold:

(a) all $k_1 \times k_1$ $(k_1 = 1, \cdots, N)$ minors of the complex admittance matrix Y are nonsingular,

(b) all $k_2 \times k_2$ minors of the extended matrix $\tilde{Y}$ are nonsingular,

where $k_2 = 1, \cdots, N/2$ for even N, and $k_2 = 1, \cdots, (N+1)/2$ for odd N, and

where the complex admittance matrix Y and the extended admittance matrix $\tilde{Y}$ are defined, respectively, by

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & y_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_{N1} & y_{N2} & \cdots & y_{NN} \end{bmatrix}$$

and

$$\tilde{Y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} & y_{1,N+1} \\ y_{21} & y_{22} & \cdots & y_{2N} & y_{2,N+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N1} & y_{N2} & \cdots & y_{NN} & y_{N,N+1} \end{bmatrix}.$$

Proof. The proof is essentially the same as the proof of Theorem 3.4.1. We first give the associated homogeneous system of (2.1.8) by

$$0 = E_1(\textstyle\sum_{j=1}^{N} y_{1j}^* E_j^* + y_{1,N+1}^* E_0) + \qquad (3.4.2.1a)$$

$$E_1^*(\textstyle\sum_{j=1}^{N} y_{1j} E_j + y_{1,N+1} E_0) - 2P_1 E_0^2,$$

$$0 = E_2(\sum_{j=1}^{N} y_{2j}^* E_j^* + y_{2,N+1}^* E_0) +$$

$$E_2^*(\sum_{j=1}^{N} y_{2j} E_j + y_{2,N+1} E_0) - 2P_2 E_0^2,$$

$$\vdots$$

$$0 = E_N(\sum_{j=1}^{N} y_{Nj}^* E_j^* + y_{N,N+1}^* E_0) +$$

$$E_N^*(\sum_{j=1}^{N} y_{Nj} E_j + y_{N,N+1} E_0) - 2P_N E_0^2,$$

$$0 = E_1 E_1^* - V_1^2 E_0^2, \qquad (3.4.2.1b)$$

$$0 = E_2 E_2^* - V_2^2 E_0^2,$$

$$\vdots$$

$$0 = E_N E_N^* - V_N^2 E_0^2.$$

It can be shown that if assumption (a) of Theorem 3.4.2 holds, then the solution sets at infinity of (3.4.2.1) are the union of the two disjoint linear subspaces Z_1 and Z_2, where these two sets are of the following forms:

$$Z_1 = \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) | \, 0 = E_1 = \ldots = E_N\} = CP^{N-1},$$

and

$$Z_2 = \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) | \, 0 = E_1^* = \ldots = E_N^*\} = CP^{N-1}.$$

It can be seen also that the $codim(Z, CP^{2N})$ equals N+1. In the following, we will show that the $rank_c J(Z_1)$ and $rank_c J(Z_1)$ are both equal to N+1. It then leads to [18] the nonsingularity of the solution sets at infinity, namely, Z_1 and Z_2. The polynomial equations of the load flow of the all-PV-bus power systems may appear more complicated if one compares them with the load flow equations (2.1.7) of the all-PQ-bus power systems. However, the proof of the nonsingularity of the zero sets at infinity, namely, Z_1 and Z_2 can be carried out by following the same procedures as in the proof of Theorem 3.4.1.

Similarly, calculating the J(z) at $E_0 = 0$ according to (3.2.4) and removing all zero columns from $J(Z_1)$, one obtains

$$\hat{J}(Z_1)\Big|_{E_0=0} = \tag{3.4.2.2}$$

$$\begin{bmatrix} y_{1N+1}E_1^* & a_1 + y_{11}E_1^* & y_{12}E_1^* & \cdots & y_{1N}E_1^* \\ y_{2N+1}E_2^* & y_{21}E_2^* & a_2 + y_{22}E_2^* & \cdots & y_{2N}E_2^* \\ \vdots & \vdots & \vdots & & \vdots \\ y_{NN+1}E_N^* & y_{N1}E_N^* & y_{N2}E_N^* & \cdots & a_N + y_{NN}E_N^* \\ 0 & E_1^* & 0 & \cdots & 0 \\ 0 & 0 & E_2^* & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E_N^* \end{bmatrix}_{2N\times(N+1)},$$

where $a_k := \sum_{j=1}^{N} y_{kj}^* E_j^*$, $k = 1, 2, \cdots, N$ as before.

Because of symmetry, it can be seen that $\hat{J}(Z_1)$ and $\hat{J}(Z_2)$ are of the same form. Therefore, we only need to check $\hat{J}(Z_1)$, then by similarity the same conclusion can be applied to $\hat{J}(Z_2)$.

For convenience, we separate $\hat{J}(Z_1)$ into two parts. The first N rows of $\hat{J}(Z_1)$ are called part 1, and the last N rows of $\hat{J}(Z_1)$ are called part 2. Now consider the solutions at infinity $\{Z_1 \mid E_{k_1}^* = 0, E_{k_2}^* \neq 0$ for $k_1 = 1, \cdots, i,\ k_2 = i+1, \cdots, N,\ 0 \le i < N\}$. From assumption (a) of Theorem 3.4.2, it follows that there are at most (N-i-1) rows of part 1 in which a_k become zero. Because of $E_{k_1}^* = 0$ and $E_{k_2}^* \neq 0$ ($k_1 = 1, \cdots, i,\ k_2 = i+1, \cdots, N$), all the rows containing only $E_{k_1}^*$ are removed from $\hat{J}(Z_1)$, and all the rows and the columns that share the element $E^*_{k_2}$ in part 2 are removed from the matrix $\hat{J}(Z_1)$. Call the remaining matrix $\bar{J}(Z_1)$.

$$\bar{J}(Z_1) = \begin{bmatrix} 0 & a_1 & 0 & \cdot\cdot & 0 \\ 0 & 0 & a_2 & \cdot\cdot & 0 \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ 0 & 0 & 0 & \cdot\cdot & a_i \\ y_{i+1,N+1}E^*_{i+1} & y_{i+1,1}E^*_{i+1} & y_{i+1,2}E^*_{i+1} & \cdot\cdot & y_{i+1,i}E^*_{i+1} \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ y_{N,N+1}E^*_N & y_{N1}E^*_N & y_{N2}E^*_N & \cdot\cdot & y_{N,i}E^*_N \end{bmatrix}_{N\times(i+1)} . \qquad (3.4.2.3)$$

Therefore, we can simply check the rank of $\bar{J}$. It can be shown that the nonsingularity of Z_1 is verified by showing the rank of $\bar{J}$ to be i+1 (:= N+1-(N-i)).

The matrix $\bar{J}(Z_1)$ can further be simplified by removing all zero rows from it. There are two cases to be considered.

(1) i < N - i.

For convenient, we assume that the first i rows of $\bar{J}$ become zeros because of $a_j = 0$ ($1 \le j \le i$) (this is the worst case), these zero rows are removed from the matrix $\bar{J}$. The new matrix is denoted by $\hat{J}_1 := \hat{J}_1(Z_1)$

$$\hat{J}_1(Z_1) = \begin{bmatrix} y_{i+1,N+1}E^*_{i+1} & y_{i+1,1}E^*_{i+1} & y_{i+1,2}E^*_{i+1} & \cdot\cdot & y_{i+1,i}E^*_{i+1} \\ y_{i+2,N+1}E^*_{i+2} & y_{i+2,1}E^*_{i+2} & y_{i+2,2}E^*_{i+2} & \cdot\cdot & y_{i+2,i}E^*_{i+2} \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ y_{N,N+1}E^*_N & y_{N,1}E^*_N & y_{N,2}E^*_N & \cdot\cdot & y_{N,i}E^*_N \end{bmatrix}_{(N-i)\times(i+1)} . \qquad (3.4.2.4)$$

Clearly, the number of rows are greater than or at least equal to the number of columns of $\hat{J}_1$. By assumption (b) of the theorem, we conclude that the $rank_c \hat{J}_1$ equals i+1.

(2) i ≥ N-i.

From assumption (a) of the theorem, there are in this case at most (N-i-1) rows with $a_j = 0$. For simplicity, we suppose that the first (N-i-1) rows (this is the worst case) become zero, and are removed from $\bar{J}$. The remaining matrix is denoted by $\tilde{J} := \tilde{J}(Z_1)$

$$\tilde{J}(Z_1) = \begin{bmatrix} 0 & 0 & \cdots & a_{N-i} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & a_i \\ y_{i+1,N+1}E^*_{i+1} & y_{i+1,1}E^*_{i+1} & \cdots & y_{i+1,N-i}E^*_{i+1} & \cdots & y_{i+1,i}E^*_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N,N+1}E^*_N & y_{N,1}E^*_N & \cdots & y_{N,N-i}E^*_N & \cdots & y_{N,i}E^*_N \end{bmatrix}_{(i+1)\times(i+1)} . \quad (3.4.2.5)$$

Since a_k $(N-i \le k \le i)$ are nonzero, we remove all the rows and the columns that share a_j from $\tilde{J}$. The remaining $(N-i) \times (N-i)$ matrix is denoted by $\hat{J}_2$. Therefore

$$\hat{J}_2(Z_1) = \begin{bmatrix} y_{i+1,N+1}E^*_{i+1} & y_{i+1,1}E^*_{i+1} & y_{i+1,2}E^*_{i+1} & \cdots & y_{i+1,N-i-1}E^*_{i+1} \\ y_{i+2,N+1}E^*_{i+2} & y_{i+2,1}E^*_{i+2} & y_{i+2,2}E^*_{i+2} & \cdots & y_{i+2,N-i-1}E^*_{i+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N,N+1}E^*_N & y_{N,1}E^*_N & y_{N,2}E^*_N & \cdots & y_{N,N-i-1}E^*_N \end{bmatrix}_{(N-i)\times(N-i)} . \quad (3.4.2.6)$$

Obviously, from assumption (b) of the theorem, $rank_c \hat{J}_2$ equals N-i. Therefore, $rank_c \hat{J}(Z_1)$ equals N+1 (*i.e.* $(N-i)+(N-i)+(i-(N-i-1))=N+1$). This infers that Z_1 is nonsingular. The same conclusion can be obtained for Z_2 by taking exactly the same procedures as above. From the Intersection Theory, system (2.1.8) has total r isolated solutions in total. Where r is given by

$$r = 2^{2N} - [Z_1] - [Z_2] = 2^{2N} - \sum_{i=0}^{N-1}\binom{2N}{i} - \sum_{i=N+1}^{2N}\binom{2N}{i} = \binom{2N}{N}.$$

This completes the proof of the theorem. (Q.E.D)

3.4.3. General load flow equations of power systems

In general, a power system consists of both PQ and PV buses. The load flow (2.1.9) is governed by both the PQ-bus and the PV-bus load flow equations combined. In the following, we give explicit sufficient conditions on the (complex) admittance matrix of the load flow network which would ensure that the load flow has a precise (complex) number of solutions. We point out on the outset that the sufficient conditions are generically satisfied, i.e., they are satisfied for all but a measure-zero set of admittance values. From a practical viewpoint, however, we may consider that the "physical" admittance matrix for a fully-connected network satisfies the sufficient conditions naturally. We can follow almost the same procedures to analyze the number of solutions for such a system. First we rewrite the general load flow equations (2.1.9) as follows:

At the PV buses:

$$0 = E_1(\textstyle\sum_{j=1}^{N+1} y_{1j}^* E_j^*) + E_1^*(\sum_{j=1}^{N+1} y_{1j} E_j) - 2P_1, \qquad (3.4.3.1a)$$

$$\vdots$$

$$0 = E_{N_g}(\textstyle\sum_{j=1}^{N+1} y_{N_g,j}^* E_j^*) + E_{N_g}^*(\sum_{j=1}^{N+1} y_{N_g,j} E_j) - 2P_{N_g},$$

$$0 = E_1 E_1^* - V_1^2, \qquad (3.4.3.1b)$$

$$\vdots$$

$$0 = E_{N_g} E_{N_g}^* - V_{N_g}^2.$$

At the PQ buses:

$$0 = E_{N_g+1}(\sum_{j=1}^{N+1} y^*_{N_g+1,j} E^*_j) - S_{N_g+1}, \quad (3.4.3.1c)$$

$$\vdots$$

$$0 = E_N(\sum_{j=1}^{N+1} y^*_{Nj} E^*_j) - S_N,$$

$$0 = E^*_{N_g+1}(\sum_{j=1}^{N+1} y_{N_g+1,j} E_j) - S^*_{N_g+1}, \quad (3.4.3.1d)$$

$$\vdots$$

$$0 = E^*_N(\sum_{j=1}^{N+1} y_{Nj} E_j) - S^*_N.$$

There are 2N complex variables in 2N polynomial equations. The first $2N_g$ equations are the PV-bus load flow equations, and the last $2(N-N_g)$ equations are the PQ-bus load flow equations. The following theorem holds.

Theorem 3.4.3. There are exactly $\binom{2N}{N}$ isolated complex solutions of the load flow equations (3.4.3.1) for an N-node PV- and PQ-bus power system (excluding the slack bus), if the following conditions hold:

(a) all $k_1 \times k_1$ $(k_1 = 1, \cdots, N)$ minors of the complex admittance matrix Y are nonsingular,

(b) all $k_2 \times k_2$ minors of the extended matrix $\tilde{Y}$ are nonsingular,

where $k_2 = 1, \cdots, \{N - N_g/2\}$ for even N_g, and $k_2 = 1, \cdots, \{N - (N_g-1)/2\}$ for odd N_g.

The proof. of this theorem is given in the appendix. Note that the proof of Theorem 3.4.1 and 3.4.2 immediately follow from Theorem 3.4.3.

While Theorems 3.4.1, 3.4.2, and 3.4.3 conclude the exact number of solutions when the stated assumptions hold, they say nothing when those assumptions are not satisfied. The following theorem gives an upper bound on the number of solutions of a general power system.

Theorem 3.4.4. The number of isolated solutions of the load flow equations for an N-node power system (excluding the slack bus) is bounded above by $\binom{2N}{N}$.

Proof. Let $\hat{T}(\tilde{E})$ denote the load flow equations of an N-node power system excluding the slack bus, where $\tilde{E}$ is defined as $\tilde{E} = [E_1, \cdots, E_N, E_1^*, \cdots, E_N^*]$. And let $\tilde{T}$ denote its associated homogeneous system. Define a system $\hat{S}(\tilde{E})$ as follows:

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (\sum_{i=1}^{N} a_{1i} E_i + a_{1,N+1}) \times (\sum_{i=1}^{N} b_{1i} E_i^* + b_{1,N+1}) \\ (\sum_{i=1}^{N} a_{2i} E_i + a_{2,N+1}) \times (\sum_{i=1}^{N} b_{2i} E_i^* + b_{2,N+1}) \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ (\sum_{i=1}^{N} a_{2N,i} E_i + a_{2N,N+1}) \times (\sum_{i=1}^{N} b_{2N,i} E_i^* + b_{2N,N+1}) \end{bmatrix}, \qquad (3.4.3.2)$$

where the complex constants a_{ij} and b_{ij} $(1 \le i \le 2N,\ 1 \le j \le N+1)$ are chosen randomly. The random choice ensures that $\hat{S}(\tilde{E})$ has $\binom{2N}{N}$ distinct solutions.

Let $\tilde{S}$ denote the associated homogeneous system to $\hat{S}$. Let $Z(\tilde{S})$ denote the solution set of the associated homogeneous system $\tilde{S}$. Similarly, let $Z(\tilde{T})$ denote the solution set of the associated homogeneous system $\tilde{T}$. Clearly,

(1) the points of $Z(\tilde{S})$ lying at infinity are nonsingular as a consequence to the random choice of the coefficients (a_{ij}) and (b_{ij}), and

(2) every point of $Z(\tilde{S})$ at infinity is also a point of $Z(\tilde{T})$.

Now define a homotopy function, $H(\tilde{E}, t)$,

$$H(\tilde{E}, t) = \begin{bmatrix} H_1(\tilde{E}, t) \\ \cdots \\ H_n(\tilde{E}, t) \end{bmatrix} = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}), \qquad (3.4.3.3)$$

By Theorem 2.2 in [19], for almost all choices of the complex coefficients (a_{ij}), (b_{ij})

($1 \le i \le 2N$, $1 \le j \le N+1$) and c, every one of the smooth and disjoint $\begin{bmatrix} 2N \\ N \end{bmatrix}$ solution paths of $H(\tilde{E}, t) = 0$, parameterized by $t \in [0,1)$, beginning at the zeros of $\hat{S}$, will either converge to an isolated solution of $\hat{T}(\tilde{E}) = 0$, or grows unbounded to a "zero at infinity", as t approaches 1. For each isolated solution of $\hat{T}(\tilde{E}) = 0$, there must be a solution path of $H(\tilde{E}, t) = 0$ converging to it. This implies that the system $\hat{T}(\tilde{E})$ has at most $\begin{bmatrix} 2N \\ N \end{bmatrix}$ isolated solutions. Thus, the theorem is proved.

3.5. The Model with Internal and Terminal Buses of a Generator

For convenient, we renumber the generator buses and load buses in the following way: the i-th generator's internal stator-based voltage is E_i ($1 \le i \le n$), and its terminal voltage is denoted by E_l ($n+1 \le l \le 2n$), and the k-th load-bus voltage is denoted by E_k ($2n+1 \le k \le 2n+m$). The slack bus is taken as {2n+m+1}. In this arrangement, the static polynomial equations (2.2.3) become:

$$0 = P_i^m - P_i^e \tag{3.5.1a}$$

$$= P_i^m - \frac{1}{2}[E_i(E_i^* - E_{n+i}^*)y_{i,n+i}^* +$$

$$E_i^*(E_i - E_{n+i})y_{i,n+i}], \quad (y_{i,n+i} = y_{di}),$$

$$i = 1, 2, \cdots, n,$$

$$0 = E_i E_i^* - E_{Fi}^2,$$

$$i = 1, 2, \cdots, n.$$

At the generator terminals, (2.2.3c) yields

$$0 = g_l(E, E^*) \qquad (3.5.1b)$$

$$= \frac{1}{2}[E_l(E_l^* - E_{l-n}^*)y_{l,l-n}^* + E_l^*(E_l - E_{l-n})y_{l,l-n}] +$$

$$\frac{1}{2}(E_l \textstyle\sum_{i=n+1}^{2n+m+1} y_{li}^* E_i^* + E_l^* \sum_{i=n+1}^{2n+m+1} y_{li} E_i)$$

$$l = n+1, n+2, \cdots, 2n,$$

and (2.2.3d) yields

$$0 = h_l(E, E^*) \qquad (3.5.1c)$$

$$= \frac{1}{2j}[E_l(E_l^* - E_{l-n}^*)y_{l,l-n}^* - E_l^*(E_l - E_{l-n})y_{l,l-n}] +$$

$$\frac{1}{2j}(E_l \textstyle\sum_{i=n+1}^{2n+m+1} y_{li}^* E_i^* - E_l^* \sum_{i=n+1}^{2n+m+1} y_{li} E_i)$$

$$l = n+1, n+2, \cdots, 2n.$$

At the load bus, (2.2.3e) yields

$$0 = P_k^d - g_k(E, E^*) \qquad (3.5.1d)$$

$$= P_k^d - \frac{1}{2}(E_k \textstyle\sum_{i=n+1}^{2n+m+1} y_{ki}^* E_i^* + E_k^* \sum_{i=n+1}^{2n+m+1} y_{ki} E_i),$$

$$k = 2n+1, \cdots, 2n+m,$$

and (2.2.3f) yields

$$0 = Q_k^d - h_k(E, E^*) \qquad (3.5.1e)$$

$$= Q_k^d - \frac{1}{2j}(E_k \sum_{i=n+1}^{2n+m+1} y_{ki}^* E_i^* - E_k^* \sum_{i=n+1}^{2n+m+1} y_{ki} E_i),$$

$$k = 2n+1, \cdots, 2n+m.$$

It can be seen that system (3.5.1) is of the form of (2.1.9) in 4n+2m polynomial equations with (4n+2m) unknowns. From Theorem 3.4.4, the number of complex solutions of the polynomial equations (3.5.1) is bounded above by $\begin{bmatrix} 4n+2m \\ 2n+m \end{bmatrix}$.

3.6. The Model Augmented by the Excitation System

For system (2.3.6) which has 5n+2m polynomial equations in (5n+2m) unknowns, the upper bound on the number of complex solutions can be determined by the following theorem.

Theorem 3.6. Let n denote the number of generators excluding a slack bus. Let m denote the number of load buses. The load flow equations (2.3.6) including both flux decay in the field winding of the rotor and the excitation system have at most $\sum_{i=2n+m}^{3n+m} \begin{bmatrix} 5n+2m \\ i \end{bmatrix}$ isolated (complex) solutions.

__**Proof**__. The theorem can be proved by the homotopy method. As the proof of Theorem 3.4.4 in the previous section. Let $\hat{T}(E)$ denote the static equations (2.3.6). We define

$$E = \left[V_1, \cdots, V_n, \bar{E}_1, \cdots, \bar{E}_m, \bar{E}_1^*, \cdots, \bar{E}_m^* \right].$$

Let $\tilde{T}$ denote its associated homogeneous system. Let $Z(\tilde{T})$ denote the set of zeros of the associated homogeneous system $\tilde{T}$.

Construct the "initial" starting system as

$$\hat{S}(E) = \begin{bmatrix} [\sum_{i=1}^{n}(a_{1i}V_i + a_{1,n+i}\bar{E}_i + a_{1,2n+i}\hat{E}_i) \\ + \sum_{j=1}^{m}(a_{1,3n+j}\tilde{E}_j) + \alpha_1] \\ \times[\sum_{i=1}^{n}(b_{1i}V_i + b_{1,n+i}\bar{E}_i^* + b_{1,2n+i}\hat{E}_i^*) \\ + \sum_{j=1}^{m}(b_{1,3n+j}\tilde{E}_j^*) + \beta_1] \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ [\sum_{i=1}^{n}(a_{5n+2m,i}V_i + a_{5n+2m,n+i}\bar{E}_i + a_{5n+2m,2n+i}\hat{E}_i) \\ + \sum_{j=1}^{m}(a_{5n+2m,3n+j}\tilde{E}_j) + \alpha_{5n+2m}] \\ \times[\sum_{i=1}^{n}(b_{5n+2m,i}V_i + b_{5n+2m,n+i}\bar{E}_i^* + b_{5n+2m,2n+i}\hat{E}_i^*) \\ + \sum_{j=1}^{m}(b_{5n+2m,3n+j}\tilde{E}_j^*) + \beta_{5n+2m}] \end{bmatrix}, \qquad (3.6.1)$$

where all the complex constants (a_{ij}), (b_{ij}), (α_i), and (β_i) $(1 \le i \le 5n+2m,\ 1 \le j \le 3n+m)$ are chosen randomly. It can be shown that $\hat{S}(E)$ has $\sum_{i=2n+m}^{3n+m} \binom{5n+2m}{i}$ isolated (complex) solutions.

Let $Z(\tilde{S})$ denote the set of solutions of the associated homogeneous system $\tilde{S} = 0$. Analogous to the steps in the proof of theorem 3.4.4, it can be shown that

(1) the sets of solutions at infinity of $\tilde{S}$, consisting of the two subsets

$$Z_1 = \{(0, E) \in CP^{5n+2m} \mid 0 = V, 0 = \bar{E}_1, \cdots, \tilde{E}_m\} = CP^{2n+m-1}$$

and

$$Z_2 = \{(0, E) \in CP^{5n+2m} \mid 0 = V, 0 = \bar{E}_1^*, \cdots, \tilde{E}_m^*\} = CP^{2n+m-1}$$

are nonsingular due to randomly choosing all the complex constants (a_{ij}), (b_{ij}), (α_i), and (β_i) $(1 \le i \le 5n+2m,\ 1 \le j \le 3n+m)$, and

(2) every point of the sets Z_1 and Z_2 is also a point of $Z(\tilde{T})$.

Therefore, by Theorem 2.2 in [19], we conclude that by the random choice of the complex constants (a_{ij}), (b_{ij}), (α_i), (β_i), and c, every one of the smooth and disjoint $\sum_{i=2n+m}^{3n+m} \binom{5n+2m}{i}$ solution paths of

$$H(E, t) = (1-t)c\hat{S}(E) + t\hat{T}(E) = 0$$

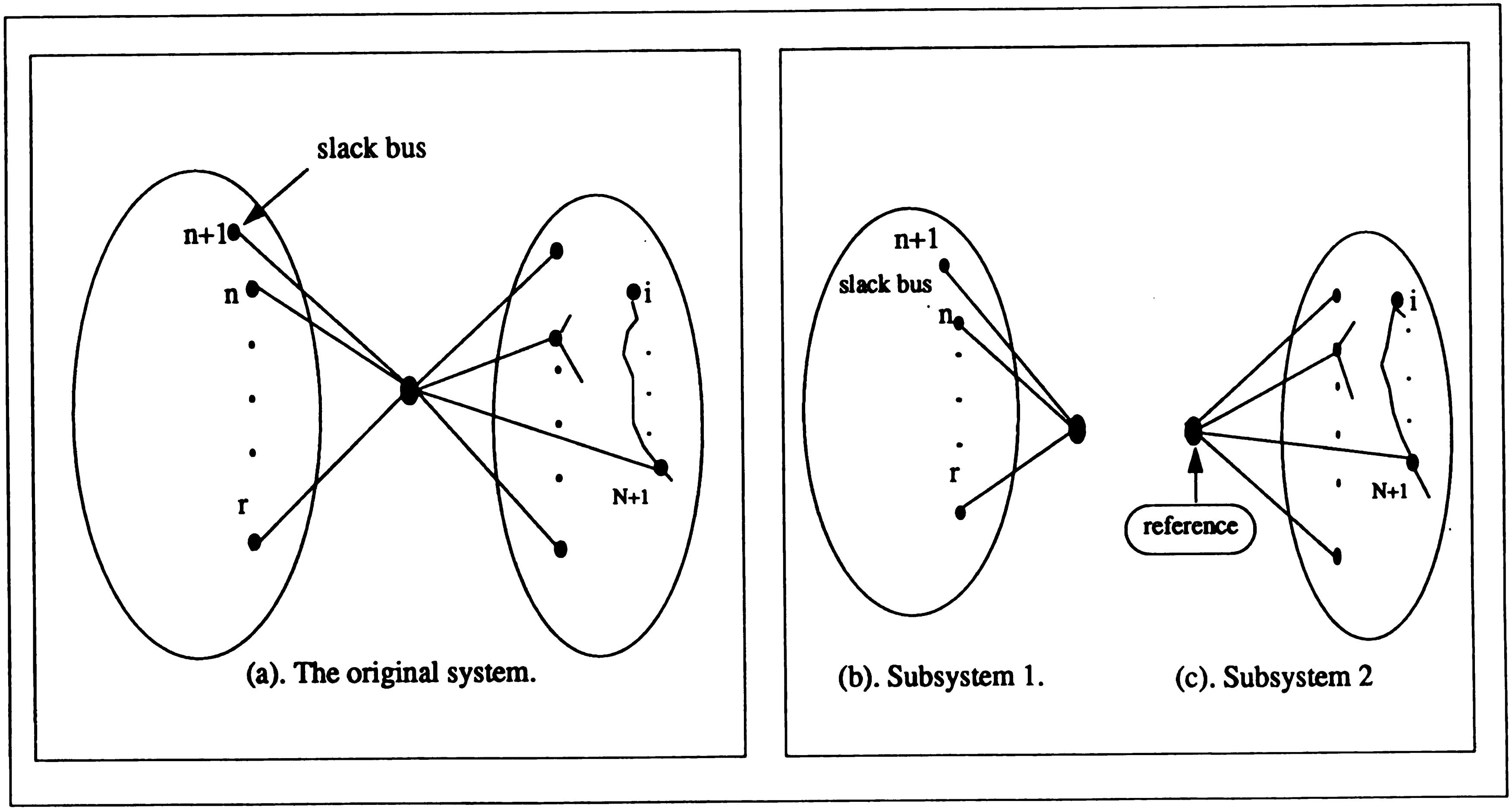
emanating from the zeros of $\hat{S}$, will either converge to an isolated solutions of $\hat{T}(E) = 0$, or grows unbounded to a "zero at infinity", as t approaches 1. For each isolated solution of $\hat{T}(E) = 0$, there must be a solution path of $H(E, t) = 0$ converging to it. Thus, the proof of the theorem is completed.

3.7. A Structure of Special Power Systems

The conditions given in Theorems 3.4.1, 3.4.2, and 3.4.3 can not be satisfied for a system which is not fully connected. Therefore, the number of solutions of the load flow equations for such a system may be much smaller than the number $\binom{2N}{N}$. In the following, we will discuss special power system structures in which there is only one node connecting two subsystems (see Figure 3.2 for example).

3.7.1. The cluster method

Theoretically speaking, an N+1-node original system can be separated into subsystems, and the solutions of the load flow equations of the original system are bounded by the product of the upper bound on the number of solutions of the load flow equations of the subsystems. As an illustration, the power system network given in Figure 3.2(a) is separated into two parts called subsystem 1 (Figure 3.2(b)) and subsystem 2 (Figure 3.2(c)) by re-choosing the common bus as a new reference bus. One may determine the number of the solutions of the load flow equations of subsystem 1 and subsystem 2, if possible. Then the product of the number of solutions of the load flow equations of subsystem 1 and subsystem 2 will determine a reduced upper bound on the number of the solutions of the load flow equations of the original system (Figure 3.2(a)). Thus the total number of solutions of the load flow equations of the original system is bounded above by the upper bound on the number of solutions of the load flow equations of subsystem 1 *times* the upper bound on the number



(a). The original system.

(b). Subsystem 1.

(c). Subsystem 2

Figure 3.2 . The partitioning of a network.

of solutions of the load flow equations of subsystem 2. In fact, the following theorem holds.

Theorem 3.7. The number of complex solutions of the load flow equations of the original system shown schematically in Figure 3.2(a) is bounded above by the upper bound on the number of solutions of the load flow equations of subsystem 1 (Figure 3.2(b)) times the upper bound on the number of solutions of the load flow equations of subsystem 2 (Figure 3.2(c)) with the common node as a new reference bus.

proof. The validity of the theorem is obvious. Assume that subsystem 1 is solvable. For each known solution, say E_c, we solve for the solutions of subsystem 2 by re-choosing the common node as a reference bus. Then, all the solutions of the original system are included in the combination of all the solutions of subsystem 1 with all the solutions of subsystem 2. This implies that the number of solutions of the load flow equations of the original system is bounded above by the upper bound on the number of solutions of the load flow equations of subsystem 1 "times" the upper bound on the number of solutions of the load flow equations of subsystem 2. Thus, the proof of the theorem is completed.

The importance of Theorem 3.7 is that we can partition the original system into as many subsystems as possible if the original system consists of subsystems which can be separated by one (shared) node. Then the total number of solutions of the load flow equations of the original system is bounded above by the product of the upper bounds on the number of solutions of the load flow equations of all the subsystems. The results are appealing from practical view points since power networks are sparsely connected, i.e. each bus is connected to relatively few (neighboring) nodes. Note that one may extend the applicability to multiple sub-systems whereby each two subsystems are connected via only one node.

3.7.2. Examples of power system networks

Here are three examples to be used to illustrate the procedures and to demonstrate their capabilities.

3.7.2.1. A not-fully-connected 3-bus network

A simple 3-bus network is shown in Figure 3.3(a). It can be seen that there are only two lines interconnected in serious between the buses. Since two lines are sharing a common bus, we can separated it into two subsystems as shown in Figure 3.3(b) and 3.3(c). By Theorem 3.4.4, the upper bound on the number of solutions for each subsystem is 2. Therefore, the number of solutions of the load flow equations for the system shown in Figure 3.3(a) is bounded above by 4.

3.7.2.2. A 7-bus network

The second example of a 7-bus network is given in Figure 3.4. Since it is not completely interconnected, by Theorem 3.4.4 we only can say that the number of complex solutions of the load flow equations is bounded above by $\binom{12}{6} = 924$.

We separate the system into three parts motivated by Theorem 3.7 by taking the shared bus as a reference bus for both subsystem 2 and subsystem 3. Now we can easily calculate the upper bound on the number of solutions of the load flow equations for each of them. Subsystem 1 is a 4-node system, by Theorem 3.4.4 it has at most 20 solutions. Subsystem 2 is a 3-node system in which the upper bound on the number of solutions is 6. The last subsystem contributes at most 2 solutions. Therefore, the total number of solutions of the load flow equations of the 7-bus network shown in Figure 3.4(a) is bounded above by $20 \times 6 \times 2 = 240$.

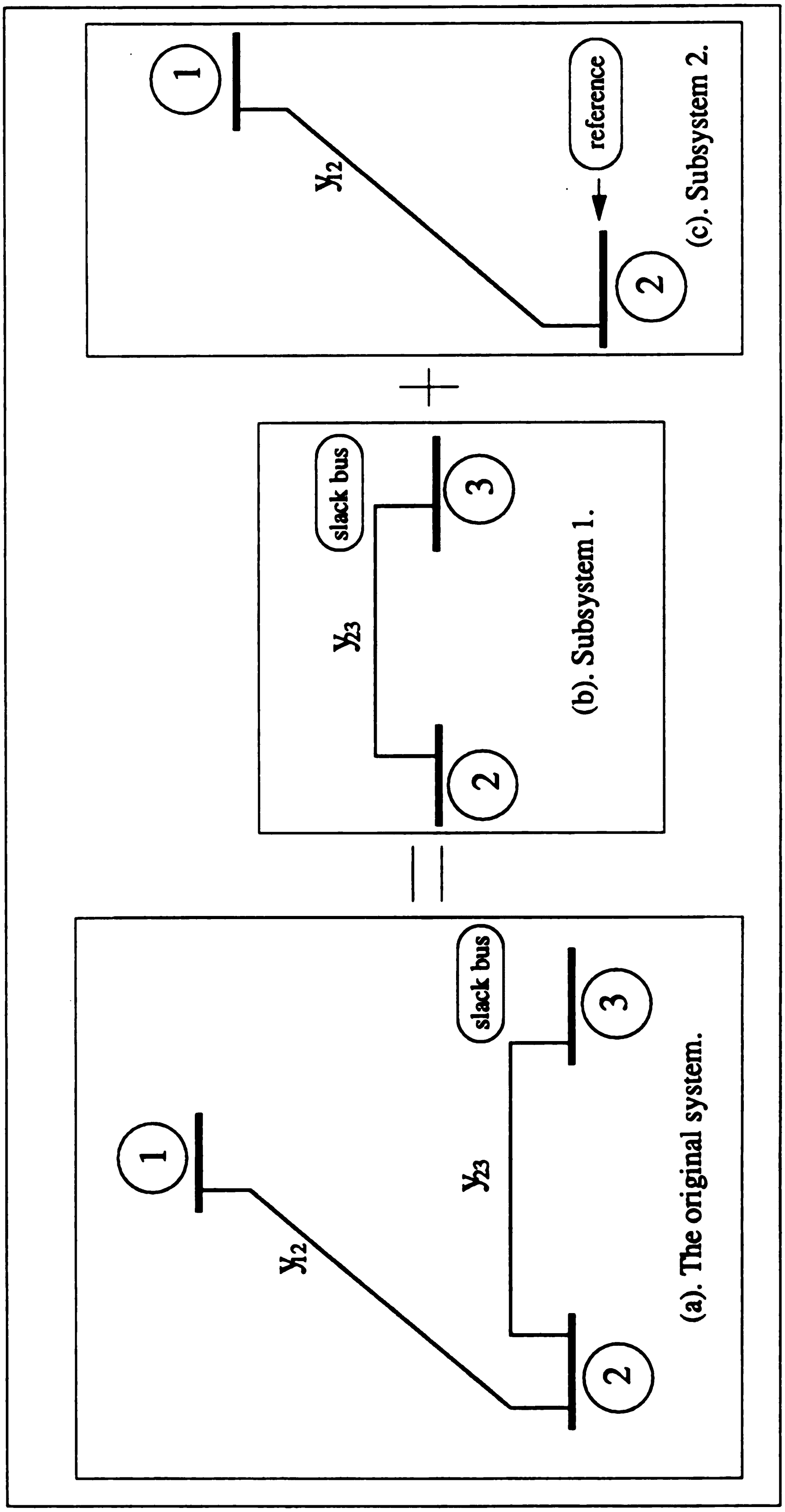


Figure 3.3. The not-fully-connected 3-bus structure.

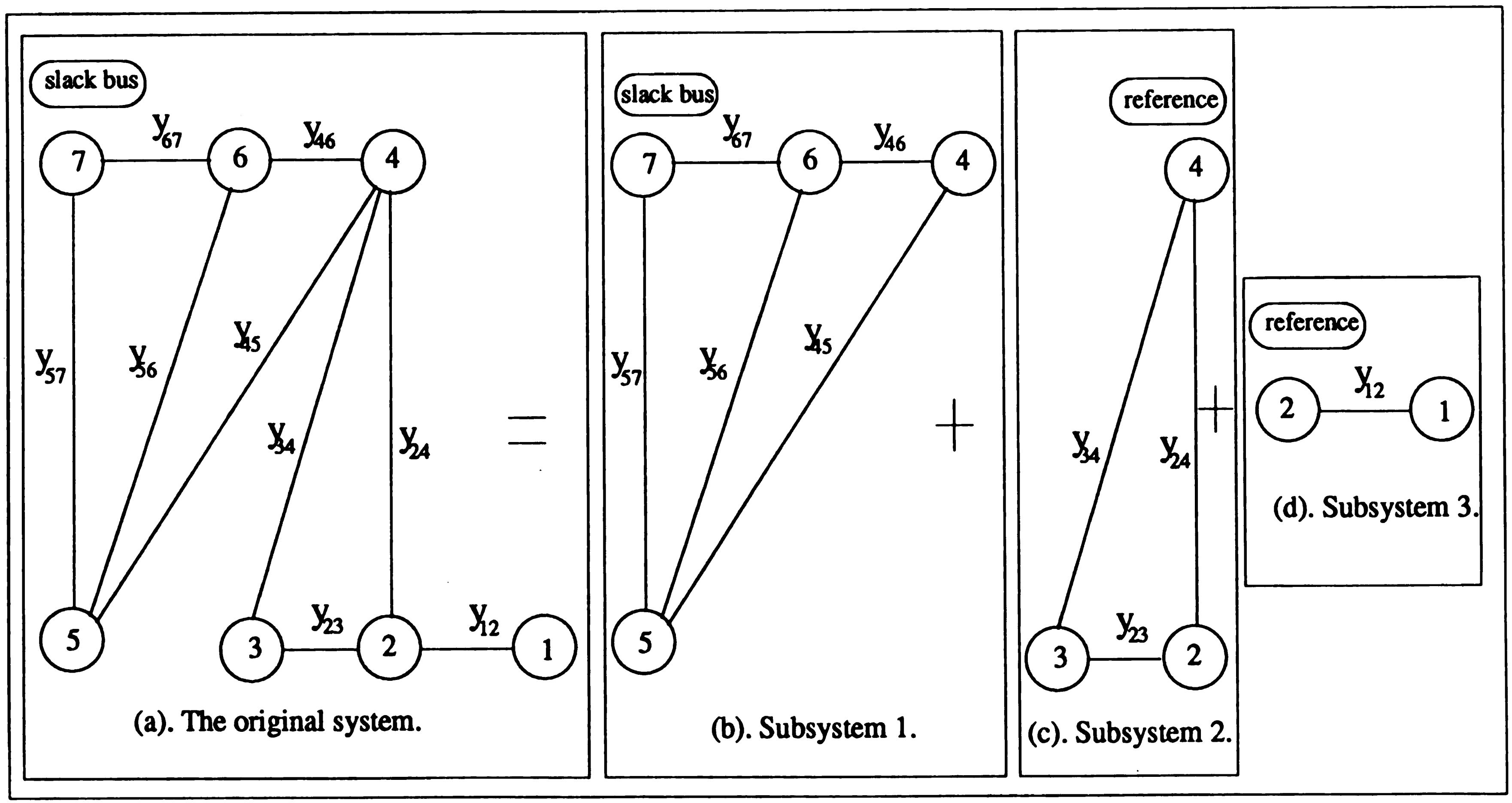


Figure 3.4. The partitioning of the 7-bus network.

3.7.2.3. The model with internal and terminal buses of a generator

Since the internal bus and the terminal bus at each generator bus of the model given in Figure 2.2 are only connected by single line with the direct axis synchronous reactance, thus we can separate the original system shown in Figure 2.2 into the n+1 subsystems in which each of the first n subsystems is a 2-bus network which has single line connecting by the direct axis synchronous reactance y_{di} $(1 \le i \le n)$, and the (n+1)th subsystem is an N+1-node (N=n+m) system with the slack bus as n+1. Clearly, each of the first n subsystems contributes at most 2 complex solutions, and subsystem n+1 has at most $\begin{bmatrix} 2N \\ N \end{bmatrix}$. Therefore, from Theorem 3.7, the number of complex solutions of the polynomial equations (3.5.1) is bounded above by $2^n \begin{bmatrix} 2(n+m) \\ n+m \end{bmatrix} \le \begin{bmatrix} 4n+2m \\ 2n+m \end{bmatrix}$.

3.8. Summary

We use powerful results from Algebraic Geometry, and the Intersection Theory to show that a power system of 2N (polynomial) load-flow equations in 2N complex unknowns has fewer than, and in many cases only a small fraction of, the total degree (i.e., the B*ézout* number) of solutions. In fact, the number of solutions of the load-flow equations for a given general N-node power system excluding the slack bus is bounded above by or equal to $N_n = \begin{bmatrix} 2N \\ N \end{bmatrix}$. We also give sufficient conditions in which this bound can be reached. Loosely speaking, these sufficient conditions mean that the power network is fully connected (i.e., every node is connected to all other nodes).

Many practical power networks, however, are not fully connected, and in fact connected only via a common node or a branch. We have also stated a theorem that is the foundation of a cluster method to determine a "tighter" upper bound on the number of solutions for power systems with special structures or topologies.

The theorems and the method developed in this paper are applicable to load flow models that are augmented by the excitation system as well; and thus can be used in investigating the voltage collapse problem due to the load flow/or the excitation system.

Chapter 4

THE SPECIAL HOMOTOPY METHOD

We have presented models for the steady state equations of power systems in polynomial form, and developed the theorems and the cluster method to predict the number of solutions of the static equations of power systems. In light of the multiplicity of solutions, one is interested in obtaining all possible solutions to the steady state equations. Practically speaking, only stable solutions are of interest. However, all or at least the so-called type-one (or index-one) solutions are desired in the study of (transient) stability ([6]-[10], [33]).

Traditional methods are not capable of solving for all the steady state solutions. The most common and acceptable numerical method is the Newton method and its variants. The Newton-like method is also known to fail in the finding of the all possible solutions to the load flow problems, see [25], for instance. A good initial guess is very helpful to the success of this (traditional) method.

Solving all solutions of a power system of polynomial equations has been almost impossible until the advent of the *globally convergent probability-one basic homotopy method* ([19], [43]). The basic homotopy method has been proven to be superior to Newton-like (traditional) method in systematical implementation of the numerical computations both in the sequential and parallel machines to take advantage of massively parallel computers. The basic homotopy method is also globally convergent in the sense that one can arbitrarily choose any set of initial guesses, this method can converge to all possible solutions from the starting initial guesses with probability one. Here we briefly describe the homotopy method as applied to solving systems of polynomial equations and underline some of its properties.

4.1. The Basic Homotopy Method

Solving a target system $\hat{T}(Z)$ of n polynomial equations in n complex variables may be represented as

$$\hat{T}(Z) = \begin{bmatrix} \hat{T}_1(z_1, z_2, \cdots, z_n) \\ \hat{T}_2(z_1, z_2, \cdots, z_n) \\ \cdots \\ \hat{T}_n(z_1, z_2, \cdots, z_n) \end{bmatrix} = 0, \tag{4.1.1}$$

where Z is an n-dimensional complex vector $[z_1, \cdots, z_n]$. Let the degree of $\hat{T}_i$ be d_i for $1 \le i \le n$. It can be seen that if n = 1, then there are d_1 complex roots including multiplicity roots. In general, (4.1.1) has at most $d = d_1 \times d_2 \times \ldots \times d_n$ isolated roots.

Consider an "initial" starting system $\hat{S}(Z)$ of polynomial equations,

$$\hat{S}(Z) = \begin{bmatrix} a_1 z_1^{d_1} - b_1 \\ \cdots \\ a_n z_n^{d_n} - b_n \end{bmatrix} = 0, \tag{4.1.2}$$

where $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are $2n$ complex constants. Each polynomial, $a_i z_i^{d_i} - b_i = 0$, has exactly d_i distinct complex roots and can be easily obtained. It can be shown that $\hat{S}(Z)$ has $d = \prod_{i=1}^{n} d_i$ distinct complex roots.

Define a homotopy function, $H(Z, t)$,

$$H(Z, t) = \begin{bmatrix} H_1(Z, t) \\ \cdots \\ H_n(Z, t) \end{bmatrix} = (1-t)\, \hat{S}(Z) + t\, \hat{T}(Z), \tag{4.1.3}$$

where t is a real parameter varying from 0 to 1. Clearly, we have

$$H(Z, 0) = \hat{S}(Z),$$

and

$$H(Z, 1) = \hat{T}(Z).$$

In [24], it is shown that for almost every choice of the constants a_i and b_i, for $1 \leq i \leq n$, $\hat{T}(Z) = 0$ can be solved by tracing the solution curves of the following equation $H(Z, t) = 0$.

Pictorially, we may think of having d known distinct roots at t = 0 (based on (4.1.2)). As t gradually increases from 0 to 1, the value of each root will be changed according to (4.1.3) and a curve based on the changing values is formed. When t = 1, the final value is one of the desired roots. These curves are called *homotopy curves.*

To solve a target system of polynomials using the homotopy method, a numerical approach is used to trace all the homotopy curves. The following properties of the homotopy curve delineated in [43] are essential in tracing the curve.

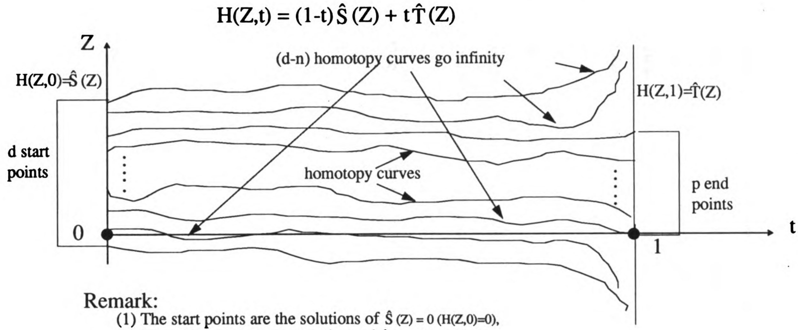
(P1) The solution set of $H(Z, t) = 0$ has d disjoint smooth homotopy curves for $t \in [0,1)$. In other words, from each root of the "initial" starting system

$$H(Z, 0) = \hat{S}(Z) = 0,$$

there emanates a smooth curve.

(P2) The homotopy curve is always forward, i.e., as t increases, the length of the curve increases correspondingly.

(P3) The homotopy curve does not stop or cease in midway, i.e. it is well-defined over $t \in [0,1)$.



Remark:
(1) The start points are the solutions of $\hat{S}(Z) = 0$ ($H(Z,0)=0$),
(2) The end points are the solutions of $\hat{T}(Z)=0$ ($H(Z,1)=0$),
(3) Homotopy curve is smooth, forward,
(4) Homotopy curve doesn't stop in midway,
(5) If $\hat{T}(Z)$ has p roots, then there are exact p homotopy curves end at $t = 1$.

Figure 4.1. The homotopy curves.

(P4) If $\hat{T}(Z)$ has exactly p roots including multiplicity, then there are exactly p end points of these curves at $t = 1$.

In the degenerate case, however, the number of roots is less than d. In this case, the rest of the curves will go to infinity as t approaches 1 (see Figure 4.1).

It may seem that the homotopy method is inefficient for solving large scale polynomial systems. However, the robustness, stability, and accuracy properties of the homotopy method make it the only and best choice for solving polynomial and certain nonlinear systems.

4.1.1. Applications to power systems

Two power systems examples are presented here to demonstrate that all possible solutions of the load flow equations can be computed simultaneously using the basic homotopy method.

We now describe the application of the basic homotopy method. Let the "initial" starting system $\hat{S}(\tilde{E})$ be constructed as

$$\hat{S}(\tilde{E}) = \begin{bmatrix} a_1 E_1^2 - b_1 \\ \cdots \\ a_N E_N^2 - b_N \\ a_{N+1} E_1^{*2} - b_{N+1} \\ \cdots \\ a_{2N} E_N^{*2} - b_{2N} \end{bmatrix}. \tag{4.1.4a}$$

Obviously, (4.1.4a) has $d = 2^{2N}$ distinct complex roots.

Define the homotopy function

$$H(\tilde{E}, t) = (1-t)\,\hat{S}(\tilde{E}) + t\,\hat{T}(\tilde{E}), \tag{4.1.4b}$$

where $\tilde{E} = [E_1, \cdots, E_N, E_1^*, \cdots, E_N^*]$ is the (complex) voltage of the given load flow equations in the complex form of the classical model. Consequently, to solve for all the solutions of the load flow equations $\hat{T}(\tilde{E}) = 0$, a numerical approach is used to trace all 2^{2N} homotopy curves. The following properties of the homotopy curves are essential in successful curve-tracing. For "almost all" $(a, b) \in C^{2N} \times C^{2N}$ with a = $(a_1, \cdots, a_{2N})$ and b = $(b_1, \cdots, b_{2N})$,

(a) the solution set of $H(\tilde{E}, t) = 0$ consists of smooth 1-d manifolds parameterized by $t \in [0, 1)$, and

(b) each isolated solution of $\hat{T}(\tilde{E}) = 0$ is reached by a finite path emanating from a solution of $\hat{S}(\tilde{E}) = 0$.

4.1.1.1. A 3-bus numerical example

For a 3-node example showed in Figure 4.2, the load flow equations $\hat{T}(\tilde{E})$ for a 3-node system is of the 4 polynomial equations (bus 3 as slack bus) in 4 complex variables. In the complex form, $\hat{T}(\tilde{E})$ is expressed by

$$\hat{T}(\tilde{E}) = \begin{bmatrix} E_k \sum_{i=1}^{3} (y_{ki}^* E_i^*) + E_k^* \sum_{i=1}^{3} (y_{ki} E_i) - 2P_{Gk} \\ k = 1,2 \\ E_1 E_1^* - V_1^2 \\ E_2 E_2^* - V_2^2 \end{bmatrix}. \tag{4.1.5}$$

Consider the "initial" starting system $\hat{S}(\tilde{E})$ as

$$\hat{S}(\tilde{E}) = \begin{bmatrix} a_1 E_1^2 - b_1 \\ a_2 E_2^2 - b_2 \\ a_3 E_1^{*2} - b_3 \\ a_4 E_2^{*2} - b_4 \end{bmatrix} = 0, \tag{4.1.6}$$

where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are complex constants which are chosen at random.

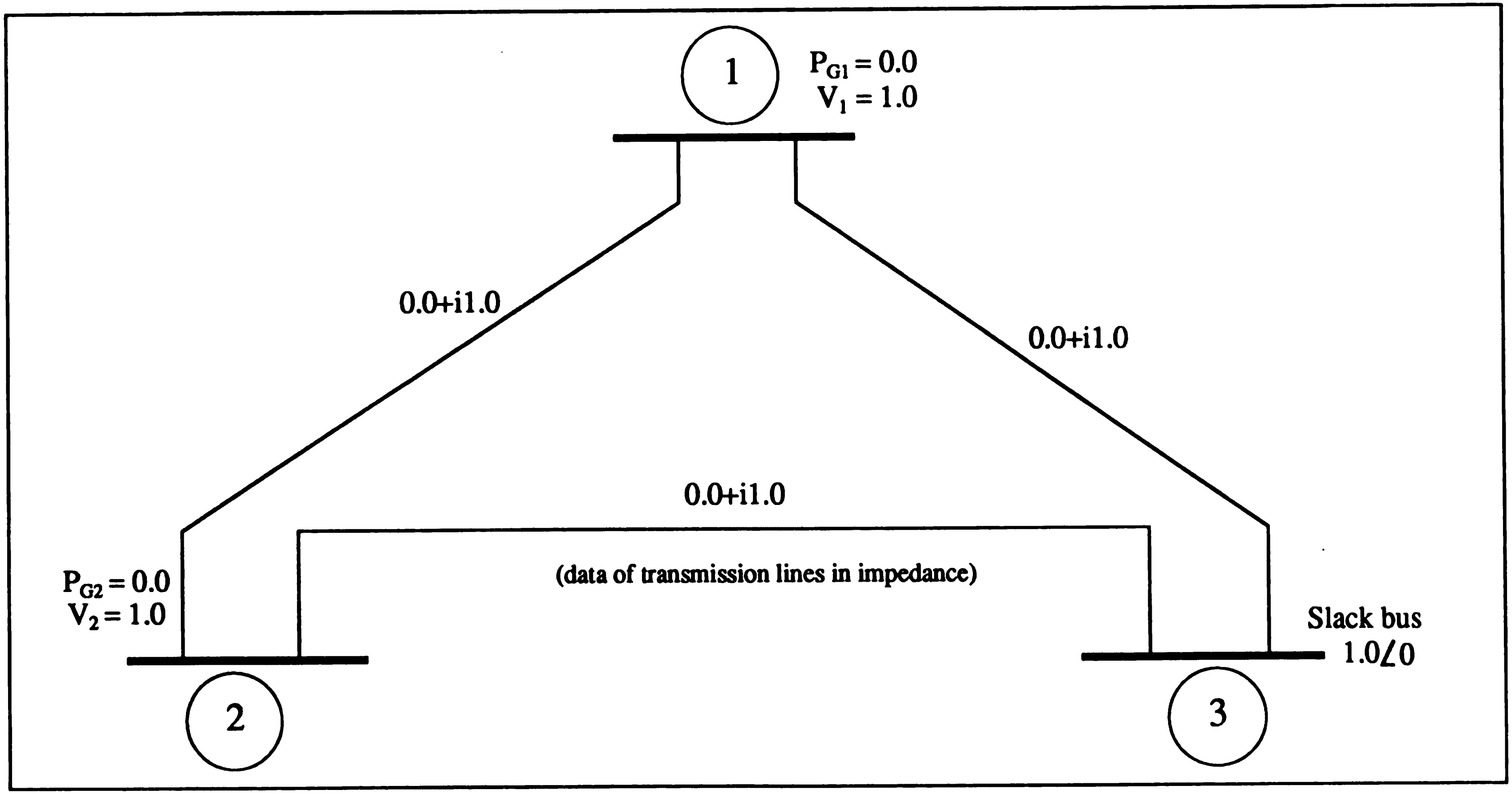


Figure 4.2. The fully-connected 3-bus example.

Obviously, $\hat{S}(\tilde{E})$ has $d = 2^4 = 16$ distinct complex roots. Each quadratic polynomial of the form, $a_i E_i^2 - b_i = 0$, contributes 2 distinct easily obtained complex roots.

Defining the homotopy function

$$H(\tilde{E}, t) = (1-t)\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}),$$

we have executed our computation by tracing 16 homotopy curves. For each of the 2^4 solutions of $\hat{S}(\tilde{E}) = 0$ acting as an initial point, we use the Newton-Raphson iterative method to compute the solution in each time interval. And thus tracing a homotopy curve from $t = 0$ to $t = 1$, we obtain 6 finite solutions. Table 4.1 lists the 6 (finite and isolated) solutions of $\hat{T}(\tilde{E}) = 0$ for a 3-bus example shown in Figure 4.2.

4.1.1.2. A 5-bus power system network

For a 5-bus network depicted in Figure 4.3, the "initial" starting system $\hat{S}(\tilde{E})$ is given by

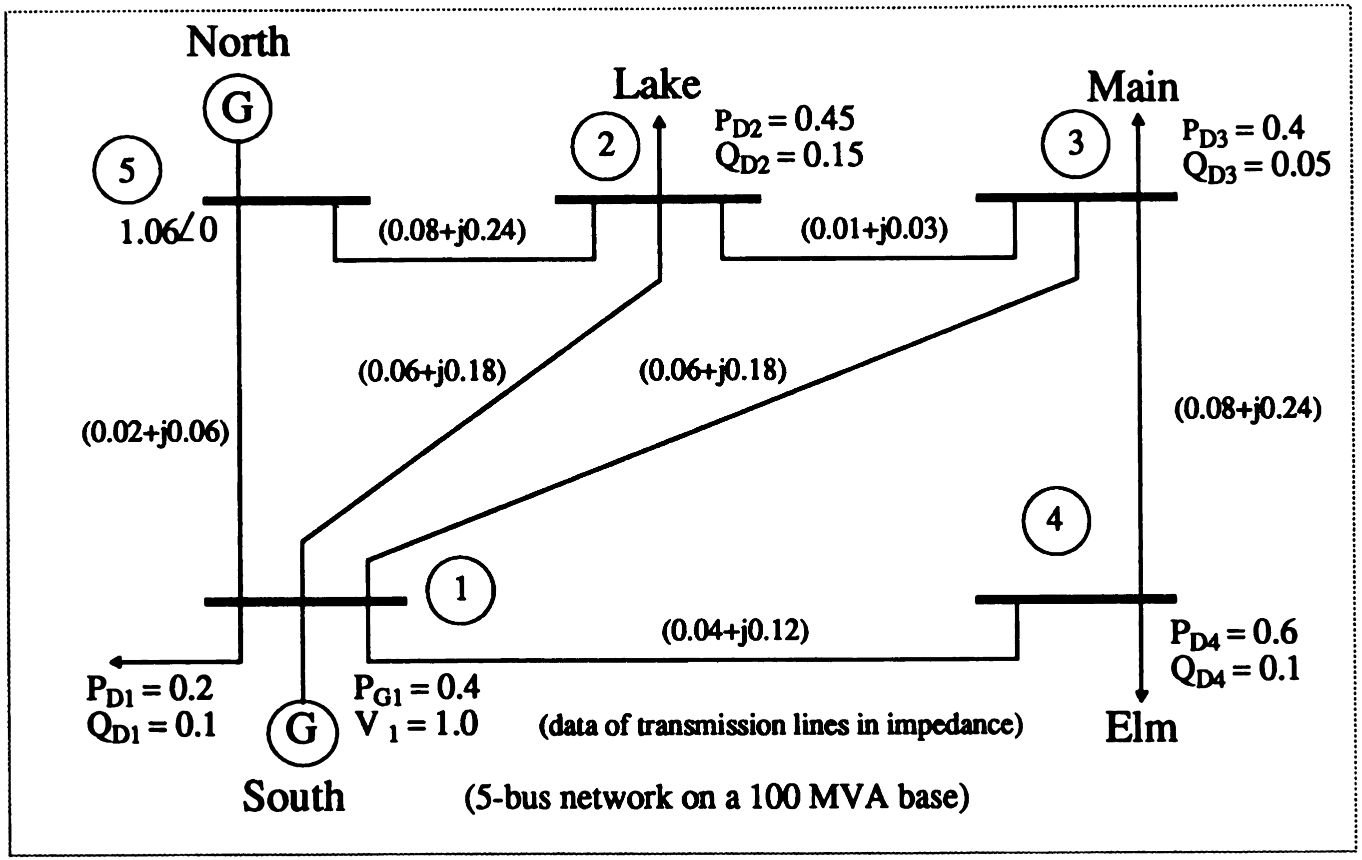
$$\hat{S}(\tilde{E}) = \begin{bmatrix} a_1 E_1^2 - b_1 \\ \cdots \\ a_4 E_4^2 - b_4 \\ a_5 E_1^{*2} - b_5 \\ \cdots\cdot \\ a_8 E_4^{*2} - b_8 \end{bmatrix}, \tag{4.1.7}$$

where a = (a_i) and b = (b_i), $i = 1, \ldots, 8$, are randomly chosen complex constants. Clearly, $\hat{S}(\tilde{E})$ has $d = 2^8 = 256$ distinct complex roots (each quadratic polynomial of the form, $a_i E_i^2 - b_i = 0$, contributing 2 distinct easily obtained complex roots).

The load flow equations $\hat{T}(\tilde{E})$ of a 5-bus network, expressed in the complex form, is the following 8 polynomial equations in 8 complex variables (bus 5 as a slack bus)

Table 4.1. The solutions of the 3-bus example (16 initial points)

	The complex voltages ($i = \sqrt{-1}$)		
The number of solutions	Bus 1	Bus 2	Bus 3
1	-1.000004+i0.000005	-0.999996-i0.000013	1.000000+i0.000000
2	-0.500043-i0.866014	-0.499973+i0.866002	1.000000+i0.000000
3	-0.499997+i0.866022	-0.500015-i0.866011	1.000000+i0.000000
4	-1.000158+i0.000008	1.000662-i0.000235	1.000000+i0.000000
5	0.999962+i0.00002	-0.999988-i0.000002	1.000000+i0.000000
6	0.999996-i0.000012	1.000012-i0.000022	1.000000+i0.000000



Bus code	Admittance to ground (y_p)
1	0.0+i0.0850
2	0.0+i0.0550
3	0.0+i0.0550
4	0.0+i0.0400
5	0.0+i0.0550

Figure 4.3. The 5-bus network.

$$\hat{T}(\tilde{E}) = \begin{bmatrix} E_1\sum_{i=1}^{5} y_{1i}^* E_i^* + E_1^* \sum_{i=1}^{5} y_{1i} E_i - 2P_{G1} \\ \cdots \\ E_4\sum_{i=1}^{5} y_{4i}^* E_i^* + E_4^* \sum_{i=1}^{5} y_{4i} E_i - 2P_{D4} \\ E_1 E_1^* - V_1^2 \\ E_2\sum_{i=0}^{4} y_{2i}^* E_i^* - E_2^* \sum_{i=1}^{5} y_{2i} E_i - 2jQ_{D2} \\ \cdots \\ E_4\sum_{i=1}^{5} y_{4i}^* E_i^* - E_4^* \sum_{i=1}^{5} y_{4i} E_i - 2jQ_{D4} \end{bmatrix}. \tag{4.1.8}$$

To solve the polynomial equations (4.1.8) using the homotopy method,

$$H(\tilde{E}, t) = (1-t)\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}),$$

a numerical approach is used to trace all 256 homotopy curves. The 54 homotopy curves converge to the finite solutions in which the 10 solutions are the system solutions of $\hat{T}(\tilde{E}) = 0$ listed in the Table 4.2.

4.2. The Special Homotopy Method

It has been shown that the load flow equations of a power system are *deficient*. That is, the number of solutions can turn out to be a lot smaller than the *Bézout* number. When using the "basic homotopy method" to solve for the solutions to the polynomial load flow equations of power system, we need to trace all the total degree d (the *Bézout* number) homotopy curves in order to obtain all possible isolated solutions. This would represent wasted computational time. We will present the special homotopy method to reduce the number of homotopy curves to be traced.

4.2.1. The classical model

Since the number of solutions to the load flow equations described by (2.1.7), (2.1.8), and (2.1.9) is at most $\begin{bmatrix} 2N \\ N \end{bmatrix}$, by the "basic homotopy method" we need to trace all 2^{2N} homotopy curves. There are at most $\begin{bmatrix} 2N \\ N \end{bmatrix}$ homotopy curves which converge to solutions, and the rest of the homotopy curves (at least $2^{2N} - \begin{bmatrix} 2N \\ N \end{bmatrix}$) will go to

Table 4.2. The solutions of the 5-bus network (256 initial points)

	The complex voltages($i=\sqrt{-1}$)				
	Bus 1	Bus 2	Bus 3	Bus 4	Bus 5
1	0.999357-i0.035972	0.984019-i0.079810	0.980454-i0.085040	0.966785-i0.097607	1.060000+i0.000000
2	-0.624255-i0.781215	-0.171782-i0.338276	-0.232411-i0.341746	-0.066198+i0.006724	1.060000+i0.000000
3	0.958810-i0.284049	0.177415-i0.086687	0.004245-i0.029796	0.578685-i0.251457	1.060000+i0.000000
4	0.923751-i0.383017	0.169494-i0.100567	0.002590-i0.036740	0.014910-i0.079989	1.060000+i0.000000
5	-0.754947-i0.655802	-0.324692-i0.386832	-0.418581-i0.418821	-0.656479-i0.517419	1.060000+i0.000000
6	0.956775-i0.290830	0.012190-i0.031881	0.146346-i0.113375	0.629834-i0.279274	1.060000+i0.000000
7	-0.596508-i0.802607	-0.058515-i0.021717	-0.176828-i0.124688	-0.482070-i0.508367	1.060000+i0.000000
8	0.950539-i0.310645	0.027180-i0.049306	0.007989-i0.048910	0.573043-i0.273089	1.060000+i0.000000
9	-0.498083-i0.867119	-0.069495-i0.054408	-0.135533-i0.095611	-0.075543-i0.001951	1.060000+i0.000000
10	0.977655-i0.210234	0.778095-i0.176122	0.722734-i0.179552	0.018326-i0.054892	1.060000+i0.000000

infinity [3].

It has been shown, by the proof of Theorem 3.4.4, that the special homotopy method only needs to trace $\begin{bmatrix} 2N \\ N \end{bmatrix}$ homotopy curves, and all possible solutions of the load flow equations can be obtained by following $\begin{bmatrix} 2N \\ N \end{bmatrix}$ roots of the "initial" starting system $\hat{S}(\tilde{E})$ which is given by

$$\hat{S}(\tilde{E}) = \begin{bmatrix} \hat{S}_1(\tilde{E}) \\ \hat{S}_2(\tilde{E}) \\ \cdots \\ \cdots \\ \cdots \\ \hat{S}_{2N}(\tilde{E}) \end{bmatrix} = \begin{bmatrix} (\sum_{i=1}^{N} a_{1i} E_i + a_{1,N+1}) \\ \times(\sum_{i=1}^{N} b_{1i} E_i^* + b_{1,N+1}) \\ (\sum_{i=1}^{N} a_{2i} E_i + a_{2,N+1}) \\ \times(\sum_{i=1}^{N} b_{2i} E_i^* + b_{2,N+1}) \\ \cdots \\ \cdots \\ \cdots \\ (\sum_{i=1}^{N} a_{2N,i} E_i + a_{2N,N+1}) \\ \times(\sum_{i=1}^{N} b_{2N,i} E_i^* + b_{2N,N+1}) \end{bmatrix}, \tag{4.2.1.1}$$

where the complex matrices a = (a_{ij}) and b = (b_{ij}) $(1 \le i \le 2N,\ 1 \le j \le N+1)$ are chosen randomly. Since the "initial" starting system $\hat{S}(\tilde{E})$ is trivial, we can easily calculate the solutions by setting the first half of the N $\hat{S}_i$'s to be zero to solve for $E_1, E_2, \ldots, E_N$ and setting the second half of the N $\hat{S}_j$'s $(j \ne i)$ to be zero to solve for $E_1^*, E_2^*, \cdots, E_N^*$. Then all the solutions of $\hat{S}(\tilde{E}) = 0$ are obtained by the combinations of all the values of $(E_1, E_2, \cdots, E_N)$ with all the values of $(E_1^*, E_2^*, \cdots, E_N^*)$. It can be shown that $\hat{S}(\tilde{E}) = 0$ has exactly $\begin{bmatrix} 2N \\ N \end{bmatrix}$ roots. Thus, the number of homotopy curves that need to be traced is $\begin{bmatrix} 2N \\ N \end{bmatrix}$.

Let $\hat{T}(\tilde{E})$ be the steady state polynomial equations of power systems excluding the model with excitation system. We define the homotopy function

$$H(\tilde{E}, t) = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}). \tag{4.2.1.2}$$

To solve for the polynomial equations (4.2.1.2), a numerical approach is used to trace the homotopy curves.

Let $s_i = t_i - t_{i-1}$ be stepping distance from t_{i-1} to the next sampling point t_i. Initially, we have t_0. The value of $\tilde{E}(t_0)$, which defines the initial point of the curve, is a root of $\hat{S}(\tilde{E}) = 0$. Now we want to solve $\tilde{E}(t_1)$ for $t_1 = t_0 + s_1$. In general, given $\tilde{E}(t_{i-1})$, we have to solve for $\tilde{E}(t_i)$ for $t_i = t_{i-1} + s_i$. For s_i sufficiently small, the root of the homotopy function at t_{i-1} is near the root of the homotopy function at t_i. Thus, to solve for the root at t_i, we choose the root at time t_{i-1} as initial guess. The collection of the roots traces a homotopy curve emanating from a root of $\hat{S}(\tilde{E}) = 0$. The Newton-Raphson iterative method fits well in executing the "local" computation at every t_i.

In a serial computer, the stepping distance is usually chosen to be a constant (i.e., $s_i = s$ for all i). If the value of s is large, two undersirable facts could happen. One is that the Newton-Raphson method may not converge; the other is the possibility of curve-merging. Consequently, some roots may be missed. Therefore, it is common to begin with a small or a conservative choice of a fixed stepping distance. Either increasing the number of sampling points due to a small s or increasing the possibility of repetitive computation due to a large s, more computation time is wasted.

In order to speedup the computation, one may wish to consider a greater stepping distance. To expedite the finding of a homotopy curve, we use dynamic stepping interval. The largest allowable value of s is 0.2. When divergence or curve merging is detected, the stepping size decreases automatically until convergence is achieved or curve-merging is prevented.

We have executed our computation using the approaches discussed. For each of the 2^{2N} solutions of $\hat{S}(\tilde{E}) = 0$ acting as an initial point, we use the Newton-Raphson iterative method to compute the solution in the each time interval. And thus tracing a homotopy curve from $t = 0$ to $t = 1$.

Specializing the deficient polynomials for a 3-bus example shown in Figure 4.2, we choose $\hat{S}(\tilde{E})$ as follows:

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (a_{11}E_1 + a_{12}E_2 + a_{13})(b_{11}E_1^* + b_{12}E_2^* + b_{13}) \\ (a_{21}E_1 + a_{22}E_2 + a_{23})(b_{21}E_1^* + b_{22}E_2^* + b_{23}) \\ (a_{31}E_1 + a_{32}E_2 + a_{33})(b_{31}E_1^* + b_{32}E_2^* + b_{33}) \\ (a_{41}E_1 + a_{42}E_2 + a_{43})(b_{41}E_1^* + b_{42}E_2^* + b_{43}) \end{bmatrix} . \qquad (4.2.1.3)$$

The matrices a = (a_{ij}) and b = (b_{ij}) $(1 \le i \le 4,\ 1 \le j \le 3)$ are the randomly chosen complex constants.

Since $\hat{S}(\tilde{E})$ of (4.2.1.3) has totally 6 isolated solutions, we need to trace 6 homotopy curves according to (4.2.1.2) with t increasing from t=0 to t=1. All the 6 solutions are obtained by tracing the 6 homotopy curves beginning at the solutions of $\hat{S}(\tilde{E})$ = 0. These 6 solutions are listed in Table 4.3.

We redo the 5-bus network shown in Figure 4.3. Let $\hat{S}(\tilde{E})$ be constructed as

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (\sum_{i=1}^{4} a_{1i}E_i + a_{15}) \\ \times(\sum_{i=1}^{4} b_{1i}E_i^* + b_{15}) \\ (\sum_{i=1}^{4} a_{2i}E_i + a_{25}) \\ \times(\sum_{i=1}^{4} b_{2i}E_i^* + b_{25}) \\ \cdots \\ \cdots \\ (\sum_{i=1}^{4} a_{8,i}E_i + a_{85}) \\ \times(\sum_{i=1}^{4} b_{8,i}E_i^* + b_{85}) \end{bmatrix} . \qquad (4.2.1.4)$$

It has been shown that for almost all choices of the complex constants (a_{ij}), (b_{ij}) $(1 \le i \le 8,\ 1 \le j \le 5)$, and c each isolated solution of $\hat{T}(\tilde{E}) = 0$ in (4.1.8) is obtained by tracing a finite path of

$$H(\tilde{E}, t) = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E})$$

Table 4.3. The solutions of the 3-bus example (6 initial points)

	The complex voltages ($i = \sqrt{-1}$)		
The number of solutions	Bus 1	Bus 2	Bus 3
1	-1.000890+i0.000024	-0.999581-i0.000076	1.000000+i0.000000
2	-0.500047-i0.866081	-0.500029+i0.866121	1.000000+i0.000000
3	-0.499994+i0.866294	-0.500116-i0.866399	1.000000+i0.000000
4	-0.999750+i0.000064	0.99328-i0.000103	1.000000+i0.000000
5	0.999828-i0.000451	-0.999796+i0.000382	1.000000+i0.000000
6	0.999989-i0.000005	0.999997-i0.000011	1.000000+i0.000000

Table 4.4. The solutions of the 5-bus network (70 initial points)

	The complex voltages($i=\sqrt{-1}$)				
	Bus 1	Bus 2	Bus 3	Bus 4	Bus 5
1	0.999402-i0.036007	0.984070-i0.079836	0.980510-i0.085069	0.966644-i0.097647	1.060000+i0.000000
2	-0.624277-i0.781209	-0.171789-i0.338292	-0.232418-i0.341761	-0.066201+i0.006722	1.060000+i0.000000
3	0.958811-i0.284049	0.177417-i0.086695	0.004246-i0.029796	0.578686-i0.251457	1.060000+i0.000000
4	0.923763-i0.383013	0.169492-i0.100657	0.002602-i0.036749	0.014914-i0.080006	1.060000+i0.000000
5	-0.754962-i0.655766	-0.324688-i0.386791	-0.418579-i0.418775	-0.656506-i0.517354	1.060000+i0.000000
6	0.956775-i0.290830	0.012190-i0.031881	0.146346-i0.113375	0.629834-i0.279274	1.060000+i0.000000
7	-0.596509-i0.802619	-0.058505-i0.021701	-0.176827-i0.124706	-0.482080-i0.508378	1.060000+i0.000000
8	0.950514-i0.310641	0.027148-i0.049232	0.007967-i0.048894	0.573023-i0.273086	1.060000+i0.000000
9	-0.498098-i0.867128	-0.069475-i0.054406	-0.135547-i0.095631	-0.075544-i0.001957	1.060000+i0.000000
10	0.977654-i0.210227	0.778094-i0.176117	0.722732-i0.179548	0.018322-i0.054893	1.060000+i0.000000

emanating from a solution of $\hat{S}(\tilde{E}) = 0$. The 54 finite paths converge to the complex solutions, and the 10 computed system solutions of the 5-bus network are listed in Table 4.4. Observe the slight numerical difference in the values of the solutions in Table 4.4 as compared to the values of the solutions in Table 4.2.

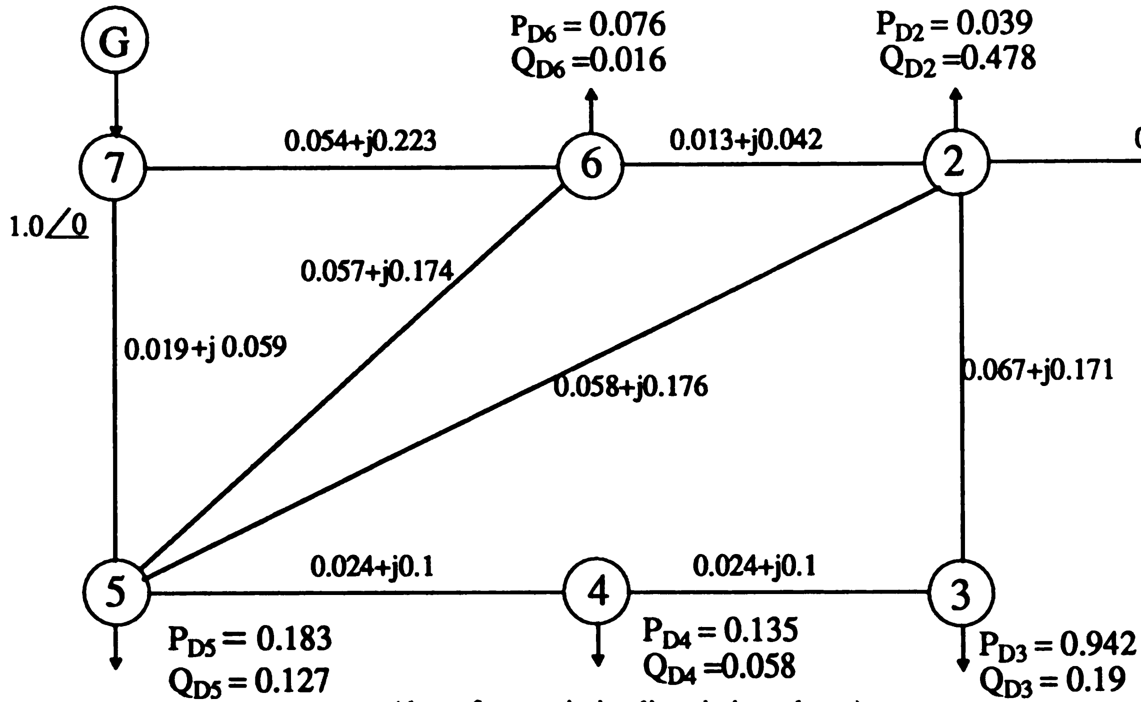
From the computational point of view, the basic homotopy method needs to trace 2^{2N} homotopy curves to obtain all possible solutions of the load flow equations for an N-node power systems excluding the slack bus. The amount of computational effort is exponentially increasing with the size of a system. Consequently, the basic homotopy method is only limited to a smaller-sized system.

Since the special homotopy method reduces the number of homotopy curves from 2^{2N} to $\begin{bmatrix} 2N \\ N \end{bmatrix}$, it makes possible to solve for the load flow solutions of a 7-bus network shown in Figure 4.4.

The load flow equations for a 7-bus network with bus 7 as a slack bus expressed in the complex form are given by

$$\hat{T}(\tilde{E}) = \begin{bmatrix} E_1 \sum_{i=1}^{7} y_{1i}^* E_i^* - S_1 \\ \cdots \\ \cdots \\ \cdots \\ E_6 \sum_{i=1}^{7} y_{6i}^* E_i^* - S_6 \\ E_1^* \sum_{i=1}^{7} y_{1i} E_i - S_1^* \\ \cdots \\ \cdots \\ \cdots \\ E_6^* \sum_{i=1}^{7} y_{6i} E_i - S_6^* \end{bmatrix} . \qquad (4.2.1.5)$$

where E_i $(1 \leq i \leq 6)$ denotes the complex voltage at i-th bus, and $S_k = P_i + jQ_i$ denotes the complex power injected into the i-th bus. The superscript * denotes the complex conjugate.



Bus code	Admittance to ground (y_p)
1	0.0+i0.0000
2	0.0+i0.0424
3	0.0+i0.0273
4	0.0+i0.0200
5	0.0+i0.0717
6	0.0+i0.0484
7	0.0+i0.0510

(data of transmission lines in impedance)

(data on a 100MVA base)

Figure 4.4. The 7-bus network.

Table 4.5. The solutions of the 7-bus network (924 initial points)

The complex voltages ($i=\sqrt{-1}$)				
	The number of solutions			
	1	2	3	4
Bus 1	1.084697+i0.092841	-0.060593+i0.278604	0.710975+i0.185251	0.009488+i0.342643
Bus 2	0.976052-i0.053103	0.545264-i0.061268	0.588767-i0.055038	0.431067-i0.052518
Bus 3	0.910192-i0.135705	0.522620-i0.185173	0.103935-i0.138090	0.174681-i0.173834
Bus 4	0.936708-i0.095927	0.643021-i0.127462	0.400388-i0.101473	0.417709-i0.121529
Bus 5	0.972258-i0.043687	0.779316-i0.053751	0.724505-i0.041423	0.687687-i0.047862
Bus 6	0.979085-i0.047188	0.644572-i0.057840	0.665836-i0.051382	0.548858-i0.051997
Bus 7	1.000000+i0.000000	1.000000+i0.000000	1.000000+i0.000000	1.00000+i0.000000

Consider the "initial" system

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (\sum_{i=1}^{6} a_{1i}E_i + a_{17}) \\ \times(\sum_{i=1}^{6} b_{1i}E_i^* + b_{17}) \\ (\sum_{i=1}^{6} a_{2i}E_i + a_{27}) \\ \times(\sum_{i=1}^{6} b_{2i}E_i^* + b_{27}) \\ \cdots \\ \cdots \\ \cdots \\ (\sum_{i=1}^{6} a_{12,i}E_i + a_{12,7}) \\ \times(\sum_{i=1}^{6} b_{12,i}E_i^* + b_{12,7}) \end{bmatrix}. \tag{4.2.1.6}$$

Using the special homotopy method to solve for $\hat{T}(\tilde{E}) = 0$, we follow the homotopy curves emanating from the roots of the trivial system $\hat{S}(\tilde{E})$ to the desired solutions of H($\tilde{E}$, 1) = 0 according to (4.2.1.2). All the 288 complex solutions including 4 system solutions are obtained by tracing total 924 homotopy curves. Those 4 system solutions are listed in the Table 4.5.

4.2.2. The model with internal and terminal buses of a generator

In this model, the steady state equations (3.5.1) include the effect of the direct axis synchronous reactances which do not appear in the classical model. Let N be defined, at this time, as N = 2n+m. Then to solve for the system (3.5.1), we can use the same $\hat{S}(\tilde{E})$ of (4.2.1.1) as the "initial" starting system. It has been shown that for a random choice of complex constants a = (a_{ij}) and b = (b_{ij}) $(1 \le i \le 2N,\ 1 \le j \le N+1)$, all possible solutions of (3.5.1) will be obtained by tracing $\begin{bmatrix} 2N \\ N \end{bmatrix}$ homotopy curves according to (4.2.1.2) which emanate from the solutions of $\hat{S}(\tilde{E}) = 0$. Here a 4-bus example shown in Figure 4.5 is used to demonstrate the procedures. Where bus 1 is considered as the internal bus of the generator, and bus 2 is considered as the terminal bus of the generator. Bus 4 is taken as a slack bus, and bus 3 is treated as load bus. The admittance between bus 1 and bus 2 signifies the direct axis synchronous admittance.

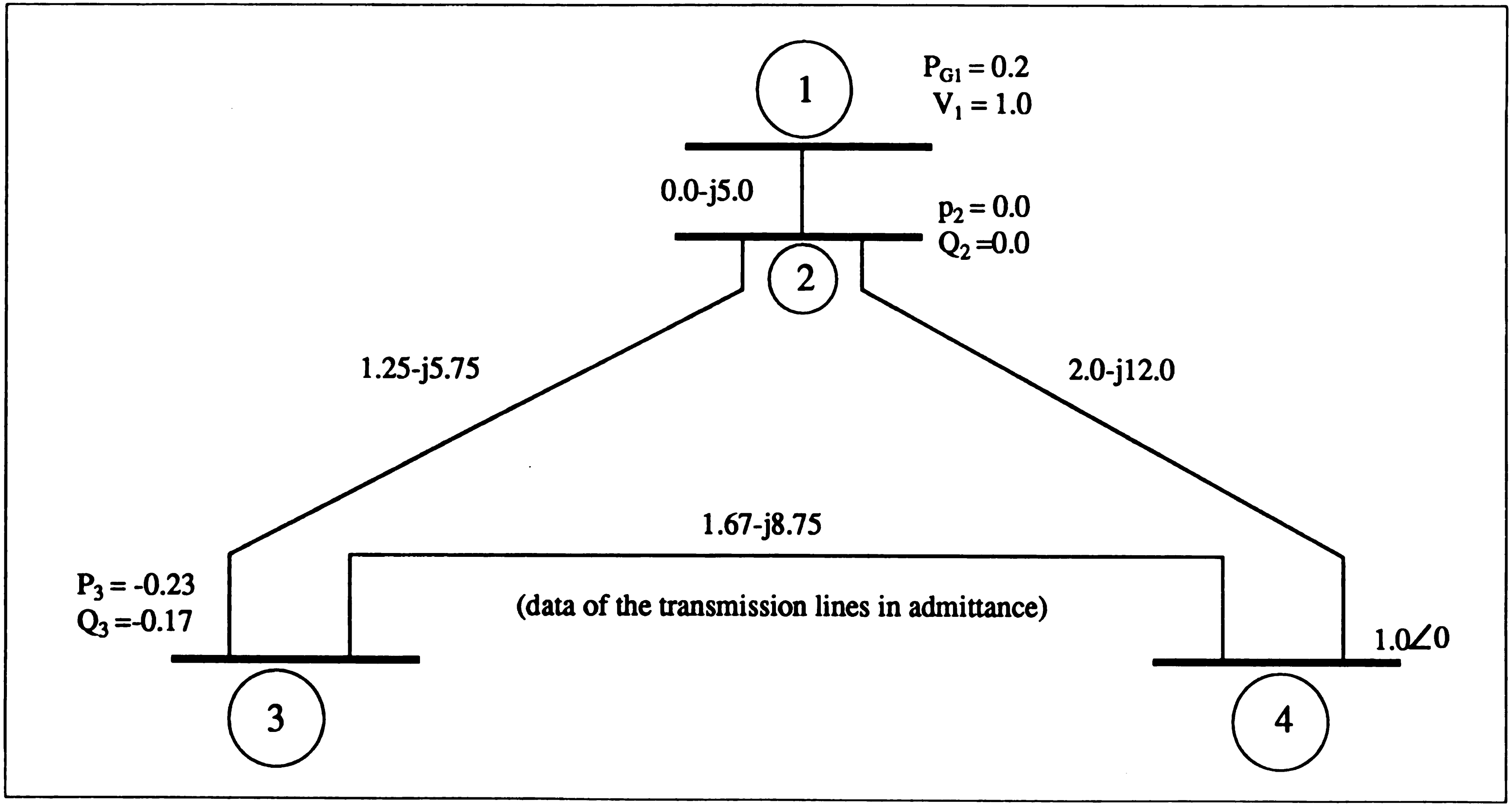


Figure 4.5. The 4-bus example.

Table 4.6. The solutions of the 4-bus example (20 initial points)

	The complex voltages($i = \sqrt{-1}$)			
The number of solutions	Bus 1	Bus 2	Bus 3	Bus 4
1	0.998871+i0.047512	0.997372+i0.007396	0.984209-i0.010106	1.000000+i0.000000
2	0.999424+i0.033957	0.749976-i0.014542	0.016332-i0.014539	1.000000+i0.000000
3	-0.999698-i0.024648	0.514467+i0.052699	0.788389+i0.001508	1.000000+i0.000000
4	-1.000049+i0.001889	0.319561+i0.039393	0.020935-i0.017451	1.000000+i0.000000

The steady state equations of a 4-node network depicted in Figure 4.5 in the polynomial form can be expressed as

$$\hat{T}(\tilde{E}) = \begin{bmatrix} E_1\sum_{i=1}^{2}(y_{1i}^* E_i^*) + E_1^*\sum_{i=1}^{2}(y_{1i}E_i) - 2P_{G1} \\ E_1E_1^* - V_1 \\ E_2\sum_{i=1}^{4}(y_{2i}^*E_i^*) - S_2 \\ E_2^*\sum_{i=1}^{4}(y_{2i}E_i) - S_2^* \\ E_3\sum_{i=2}^{4}(y_{3i}^*E_i^*) - S_3 \\ E_3^*\sum_{i=2}^{4}(y_{3i}E_i) - S_3^* \end{bmatrix}, \tag{4.2.2.1}$$

where $S_k = P_k + jQ_k$ denote the complex injected power at node k (k=2,3).

Choose an "initial" starting system $\hat{S}(\tilde{E})$ as

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (\sum_{i=1}^{3}a_{1i}E_i + a_{14}) \\ \times(\sum_{i=1}^{3}b_{1i}E_i^* + b_{14}) \\ (\sum_{i=1}^{3}a_{2i}E_i + a_{24}) \\ \times(\sum_{i=1}^{3}b_{2i}E_i^* + b_{24}) \\ \cdots \\ \cdots \\ \cdots \\ (\sum_{i=1}^{3}a_{6i}E_i + a_{64}) \\ \times(\sum_{i=1}^{3}b_{6i}E_i^* + b_{64}) \end{bmatrix}, \tag{4.2.2.2}$$

where the complex constants a = (a_{ij}) and b = (b_{ij}) ($1 \le i \le 6$, $1 \le j \le 4$) are chosen at random.

Define the homotopy function as

$$H(\tilde{E}, t) = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}). \tag{4.2.2.3}$$

Finally, the 12 complex finite solutions are obtained at t=1 by tracing 20 solution paths of $H(\tilde{E}, t)$ from t = 0. We obtain the 4 computed system solutions listed in Table 4.6.

4.2.3. The model augmented by the excitation system

The model, as described above, is completed by adding the excitation system in it. Thus, the steady state polynomial representation $\hat{T}(\tilde{E})$ includes additional two kinds

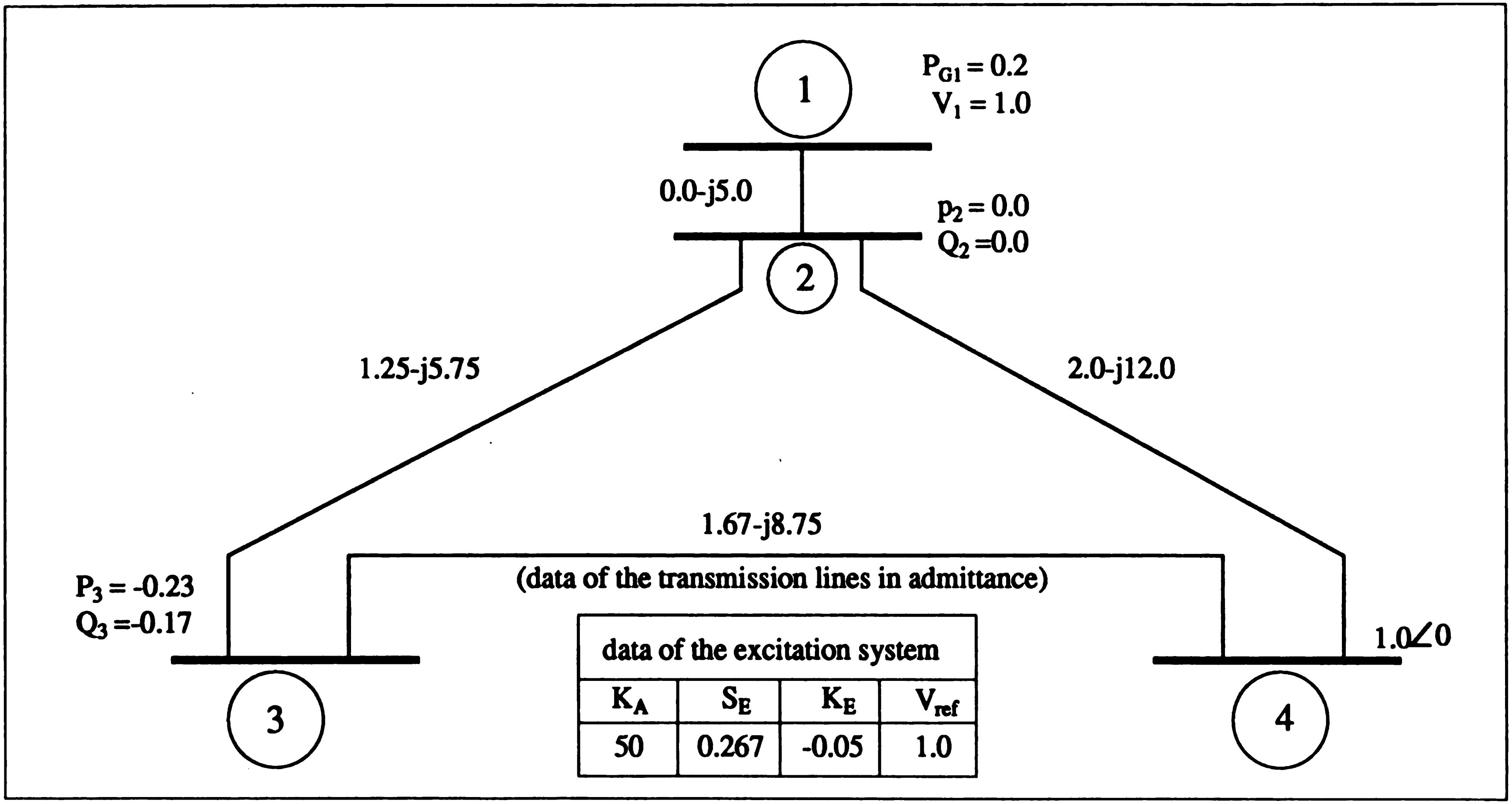


Figure 4.6. The 4-bus network with an excitation system.

of terms in which one term is contributed by the direct axis synchronous susceptances in generators and the other term is created by the excitation systems.

It has been shown, in the proof of Theorem 3.6, that the solution paths of $H(\tilde{E}, t) = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E})$ beginning at the solutions of $\hat{S}(\tilde{E}) = 0$ will converge to all possible solutions of $\hat{T}(\tilde{E}) = 0$. For a 4-bus example shown in Figure 4.6, let E_1 denote $\bar{E}_1$. Let E_2 denote $\hat{E}_1$, and let E_3 denote $\tilde{E}_1$. Then the steady state equations $\hat{T}(\tilde{E})$ including the excitation system are given by

$$\hat{T}(\tilde{E}) = \begin{bmatrix} y_{12}K_B(V_1 - V_{ref})(E_1 - E_1^*) - 2P_{G1} \\ E_1E_1^* - V_1^2 \\ E_2E_2^* - V_1^2 \\ y_{21}K_B(V_{ref} - V_1)(E_1 - E_1^*) + \\ E_2\sum_{i=2}^{4}(y_{2i}^*E_i^*) + E_2^*\sum_{i=2}^{4}(y_{2i}E_i) \\ y_{21}[2V_1^2 - K_B(V_{ref} - V_1)(E_1 + E_1^*)] + \\ E_2\sum_{i=2}^{4}(y_{2i}^*E_i^*) - E_2^*\sum_{i=2}^{4}(y_{2i}E_i) \\ E_3\sum_{i=2}^{4}(y_{3i}^*E_i^*) - S_3 \\ E_3^*\sum_{i=2}^{4}(y_{3i}E_i) - S_3^* \end{bmatrix} = 0, \qquad (4.2.3.1)$$

where $S_3 = P_3 + jQ_3$ denote the complex injected power at node 3.

In fact, the polynomial equations $\hat{T}(\tilde{E})$ of (4.2.3.1) represent the load flow equations of a 3-bus power system network by choosing bus 1 as a generator bus including both excitation system and the the direct axis synchronous susceptance between the terminals bus and internal bus of the generator. According to the proof of the theorem, we can choose an starting system in the following:

$$\hat{S}(\tilde{E}) = \begin{bmatrix} (\sum_{i=1}^{3}a_{1i}E_i + a_{14}V_1 + a_{15}) \\ \times(\sum_{i=1}^{3}b_{1i}E_i^* + b_{14}V_1 + b_{15}) \\ (\sum_{i=1}^{3}a_{2i}E_i + a_{24}V_1 + a_{25}) \\ \times(\sum_{i=1}^{3}b_{2i}E_i^* + b_{24}V_1 + b_{25}) \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ (\sum_{i=1}^{3}a_{7i}E_i + a_{74}V_1 + a_{75}) \\ \times(\sum_{i=1}^{3}b_{7i}E_i^* + b_{74}V_1 + b_{75}) \end{bmatrix}, \qquad (4.2.3.2)$$

Table 4.7. The solutions of the 4-bus network with an excitation system

(70 initial points)

	The complex voltages ($i = \sqrt{-1}$)			
The number of solutions	E_1	E_2	E_3	E_4
1	0.994876+i0.040288	0.995661+i0.007700	0.983517-i0.010006	1.000000+i0.000000
2	-1.031188-i0.005564	-0.967705-i0.356270	0.111095-i0.136940	1.000000+i0.000000
3	-1.009485-i0.017998	1.005059-i0.096128	0.014168-i0.013481	1.000000+i0.000000
4	-1.003716-i0.038852	1.004449+i0.006124	0.987069-i0.010525	1.000000+i0.000000
5	0.990565+i0.018753	0.986745-i0.088917	0.014313-i0.013546	1.000000+i0.000000
6	0.969792+i0.005750	-0.915954-i0.318682	0.140960-i0.135487	1.000000+i0.000000
7	0.970319+i0.005852	-0.905574-i0.348553	0.043414-i0.079872	1.000000+i0.000000
8	-1.030793-i0.005635	-0.957077-i0.382849	0.042014-i0.088780	1.000000+i0.000000

where all complex constants (a_{ij}), and (b_{ij}) $(1 \le i \le 7,\ 1 \le j \le 5)$ are chosen randomly.

Similarly, we define the homotopy function

$$H(\tilde{E}, t) = (1-t)c\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}). \qquad (4.2.3.3)$$

Tracing the homotopy curves emanating from the roots of $\hat{S}(\tilde{E})$, we eventually obtained 8 system solutions which are listed in Table 4.7.

4.3. Summary

Solving for all possible solutions of the load flow equations of power systems has been impossible until the advent of the homotopy methods. In this chapter, we present the special homotopy method that reduces the computational complexity (comparing with the basic homotopy methods), and still guarantees in finding all the steady state (equilibrium) solutions of various levels of detailed (nonlinear) models of power systems.

Chapter 5

THE IMBEDDING-BASED METHOD SOLVING FOR ROOTS OF DEFICIENT SYSTEMS

The homotopy continuation methods are globally convergent, i.e., one may choose any set of initial guesses, and by successively incrementing a parameter t, such methods converge to all solutions with probability one. However, the serious problem of the current homotopy methods arise in their implementation. Even for a small-sized polynomial system, say $\hat{T}(x)$, the homotopy methods may require the tracing of many useless homotopy curves that would not lead to solutions of the system $\hat{T}(x)$. The number of the (useless) homotopy curves grows exponentially to the point where efficient computation is rendered impossible even on very fast and/or parallel computers. As an example, we consider the deficient systems of the load flow equations of power systems. The number of paths that need to be traced even by "the special homotopy method" for a 5-bus system [3] is 70 [= $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$]. This number increases rapidly to 924 [= $\begin{bmatrix} 12 \\ 6 \end{bmatrix}$] when the number of buses is 7 (i.e., a 7-bus system). This means that we must follow all 924 paths in order to obtain the 4 system solutions.

From computational point of view, most of the computer computational effort in tracing the homotopy curves is spent tracing the curves which do not converge! That is, more computational effort is allocated for "useless" homotopy curves which would not lead to a solution of the target system of polynomials $\hat{T}(x)$. In contrast, the computational time spent to trace convergent homotopy curves is relatively small. Consequently, it is detrimental from a computational point of view to get rid of nonconvergent homotopy curves at any cost.

5.1. The Basic Imbedding Method

Solving a system of n polynomial equations in n variables may be expressed as

$$\hat{T}(x) = \begin{bmatrix} \hat{T}_1(x) \\ \hat{T}_2(x) \\ \cdots \\ \hat{T}_n(x) \end{bmatrix} = 0, \tag{5.1.1}$$

where each $\hat{T}_i$, $i = 1, 2, \cdots, n$ is a polynomial in $x = (x_1, \cdots, x_n)$ of degree d_i.

The basic idea of the imbedding methods for computing a solution x^* of (5.1.1) consists of "embedding" system (5.1.1) into a family of maps

$$H(x, t) = 0, \quad t \in [0,1]; \quad H: C^n \times [0,1] \rightarrow C^n \tag{5.1.2a}$$

defined by a homotopy or one-parameter embedding function H satisfying the two properties

$$H(x(0), 0) = 0 \quad \textit{for a given } x(0), \tag{5.1.2b}$$

and

$$H(x(1), 1) = \hat{T}(x^*) = 0 \quad \textit{for all } x^*. \tag{5.1.2c}$$

Therefore, the homotopy curves are the solution paths that emanate from the given vector x(0) to the unknown solution vector $x(1) = x^*$.

Some well-known homotopy methods derived from the embedding function (5.1.2) which guarantee finding all possible solutions with probability one are listed as follows.

(1) The basic homotopy method

The number of homotopy curves that need to be traced equals the total degree d (the B*ézout* number)

$$d = \prod_{i=1}^{n} d_i,$$

where d_i for $1 \le i \le n$ denotes the degree of the i-th component $\hat{T}_i$ of the target system $\hat{T}(x)$. When applied to the load flow equations of power systems, we must trace 2^{2N} homotopy curves in order to obtain all possible solutions of the load flow equations for an N-node power system excluding the slack bus.

(2) The special homotopy method

The special homotopy method reduces the number of homotopy curves to be traced from 2^{2N} to $\begin{bmatrix} 2N \\ N \end{bmatrix}$ in finding all possible solutions of the load flow equations of N-node power systems excluding the slack bus.

(3) The basic imbedding method [34]

Start with the system $\hat{T}(x) = 0$ of n (nonlinear) polynomial equations $\hat{T}_1, \hat{T}_2, \cdots, \hat{T}_n$ in n unknown variables $x_1, \cdots, x_n$ and choose an embedding parameter $t \in C$. The system $\hat{T}(x)$ can always be written in the following form.

$$\hat{T}(x) = \hat{S}(x) + F(x) \quad \hat{T}: C^n \rightarrow C^n \tag{5.1.3}$$

with

$$\hat{S}(x) = \begin{bmatrix} \hat{S}_1(x_1) \\ \hat{S}_2(x_2) \\ \cdots \\ \hat{S}_n(x_n) \end{bmatrix}, \tag{5.1.4}$$

and

$$F(x) = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ \cdots \\ f_n(x_1, \cdots, x_n) \end{bmatrix}, \tag{5.1.5}$$

where $\deg\{\hat{S}_i(x_i)\}$ is required to be greater than, or at least equal to, $\deg\{f_i(x)\}$, $i = 1, 2, \cdots, n$. Moreover, the function $\hat{S}_i$, $i = 1, 2, \cdots, n$ has only simple zeros.

The problem of finding all the zeros of such a system has been solved by Drexler [34] using a homotopy continuation method.

Giving a system $\hat{T}(x) = 0$, one can choose an embedding parameter t, and construct the homotopy continuation method in the following way:

$$H(x, t) = \hat{S}(x) + tF(x), \tag{5.1.6}$$

$$H: C^n \times C \to C^n; H(x, 1) = \hat{T}(x).$$

It has been shown [34] that, for almost every curve W in the complex plane C, simple solutions of $\hat{S}(x) = 0$ generate (bounded, continuous, and differentiable) homotopy curves that converge to all the solutions of the system H(x, 1) = $\hat{S}(x)$ + F(x) = 0 as t approaches 1+i0. These curves are parameterized by the complex embedding parameter $t \in C$, connecting 0 + i0 and 1 +i0. The justification of this method relies on relatively sophisticated ideas from algebraic geometry and transversality theory [34]. However, a drawback is that $\hat{S}_i(x_i)$ is at least of the same degree as $f_i(x)$; so for a heavily deficient system, of which the number of solutions is only a small fraction of the total degree d, only a few of the solution curves reach the system solutions and the rest of the solutions, which do not converge to the system solutions, will go to infinity. This would represent wasted computational time, and cause serious problems for numerical computations.

5.2. The Imbedding-based Method: Practical Heuristic Approach

It is of great interest from a computational view-point to reduce the number of homotopy curves traced. It is desirable to trace a number of curves that is comparable to the number of the actual (complex) solutions of a given polynomial system $\hat{T}(x)$. For deficient systems where their structures may reveal a "tight" bound on the number of solutions, we suggest to use an initial polynomial system $\hat{S}(x)$ having simple solutions comparable in number to the actual solutions of $\hat{T}(x) = 0$. Such an drastically reducing the number of homotopy curves traced.

In the following we describe the "imbedding-based" method that is based on our experience with many simulations. Although this formulation may not have a theoretical base, we have found it to work satisfactorily in many simulations of power system examples. Indeed, our approach renders the homotopy methods practically feasible in the sense that it reduces the number of traced homotopy curves to a number comparable to the number of actual solutions of a given (target) system $\hat{T}(x)$. We begin by describing the proposed imbedding based approach.

Let $\hat{T}(x)$: $C^n \rightarrow C^n$ be a polynomial system. Let C[0,1) (excluding the point 1 + i0) denote a set of complex curves from 0 + i0 to 1 + i0. Let $\hat{S}(x)$ be of the form

$$\hat{S}(x) = \begin{bmatrix} \hat{S}_1(x) \\ \hat{S}_2(x) \\ \cdots \\ \cdots \\ \cdots \\ \hat{S}_n(x) \end{bmatrix} = \begin{bmatrix} \bar{S}_1(x) - b_1 \\ \bar{S}_2(x) - b_2 \\ \cdots \\ \cdots \\ \cdots \\ \bar{S}_n(x) - b_n \end{bmatrix} . \qquad (5.2.1)$$

Assume that $\hat{S}(x)$ has only simple zeros. Consider the homotopy

$$H(x,t) = \hat{S}(x) + tF(x), \; H: C^n \times C \rightarrow C^n; \qquad (5.2.2)$$

$$\hat{T}(x) = \hat{S}(x) + F(x).$$

Then for random choices of complex coefficients $b_i \in C$ $(1 \le i \le n)$ and $t \in C[0,1]$. The zero set

$$\{(x, t) \in C^n \times C[0,1) \mid H(x, t) = 0\}$$

consists of solution paths $x_1(t), \cdots, x_n(t)$ emanating from the roots of H(x, 0) = $\hat{S}(x)$ at t = 0. The zeros of $\hat{T}(x)$ may be obtained by tracing solution paths to t = 1.

The boundness of solution paths $x_1(t), \cdots, x_n(t)$ have been discussed by Drexler in [34]. In the following we explain the elements of the results and their implications.

As in [34], we begin by using the elimination theory to compute a "Resultante" function which is only a function of a single variable, e.g., x_k. The coefficients of such function are finite degree polynomial of the (complex) imbedding parameter t. The poles of the "Resultante" function occur when the leading (polynomial) coefficient of the "Resultante" equals zero. Since the coefficient is a polynomial in the complex parameter t, the coefficient becomes zero only at finite values of t. Consequently, for almost all curves in the complex plane parameterized by t and connecting the points 0 + i0 and 1 + i0, the "Resultante" function has "no poles". This follows since the zeros of the "Resultante" function of each x_k is the same as the k-th "component" or projection of the zeros of the homotopy function H(x, t). The homotopy solution curves, therefore, do not go to infinity (in the sense of some norm) for almost all curves traced by the parameter t in the complex plane.

The approach actually says that if a strict bound on the number of complex solutions of the target system $\hat{T}(x)$ is known, then all the solutions of the target system $\hat{T}(x)$ may be obtained by tracing N_f (the number of complex zeros of F(x)) solution paths, even with the assumption $\deg\{\hat{S}_i(x)\} \le \deg\{\hat{T}_i(x)\}$.

The number of solutions of deficient systems, such as power systems, is bounded by $\begin{pmatrix} 2N \\ N \end{pmatrix}$, where N is the number of buses excluding the slack bus. This result is obtained for the load flow equations of power systems including transmittances. Yet, based on numerous simulations and applications of several homotopies to many power system example models, this upper bound is usually too large. Indeed, the topology or the figuration of a power system limits the actual number of solutions of this deficient system. In the following, we apply the proposed approach to the same 3-bus example without complete interconnection and the same 7-bus example of power systems.

5.3. Numerical Examples of the Load Flow Equations

The following two examples will be used to illustrate the computational procedures and to present a considerable saving in the computational effort.

5.3.1. A not-fully-connected 3-bus power system network

For a 3-bus example shown in Figure 5.1, the load flow equations in complex form can be expressed as

$$\hat{T}(\tilde{E}) = \begin{bmatrix} E_k \sum_{i=1}^{3}(y_{ki}^* E_i^*) + E_k^* \sum_{i=1}^{3}(y_{ki} E_i) - 2P_{Gk} \\ k = 1,2 \\ \\ E_1 E_1^* - V_1^2 \\ E_2 E_2^* - V_2^2 \end{bmatrix}. \tag{5.3.1}$$

Where $\tilde{E} = [E_1, \cdots, E_N, E_1^*, \cdots, E_N^*]$. E^* is the complex conjugate of E.

We have proved, in chapter 3, that the system shown in Figure 5.1 has exactly 4 solutions. Therefore, we only need to follow 4 solution curves by the imbedding-based method.

Consider the polynomial equations $\hat{S}(\tilde{E})$ in the following:

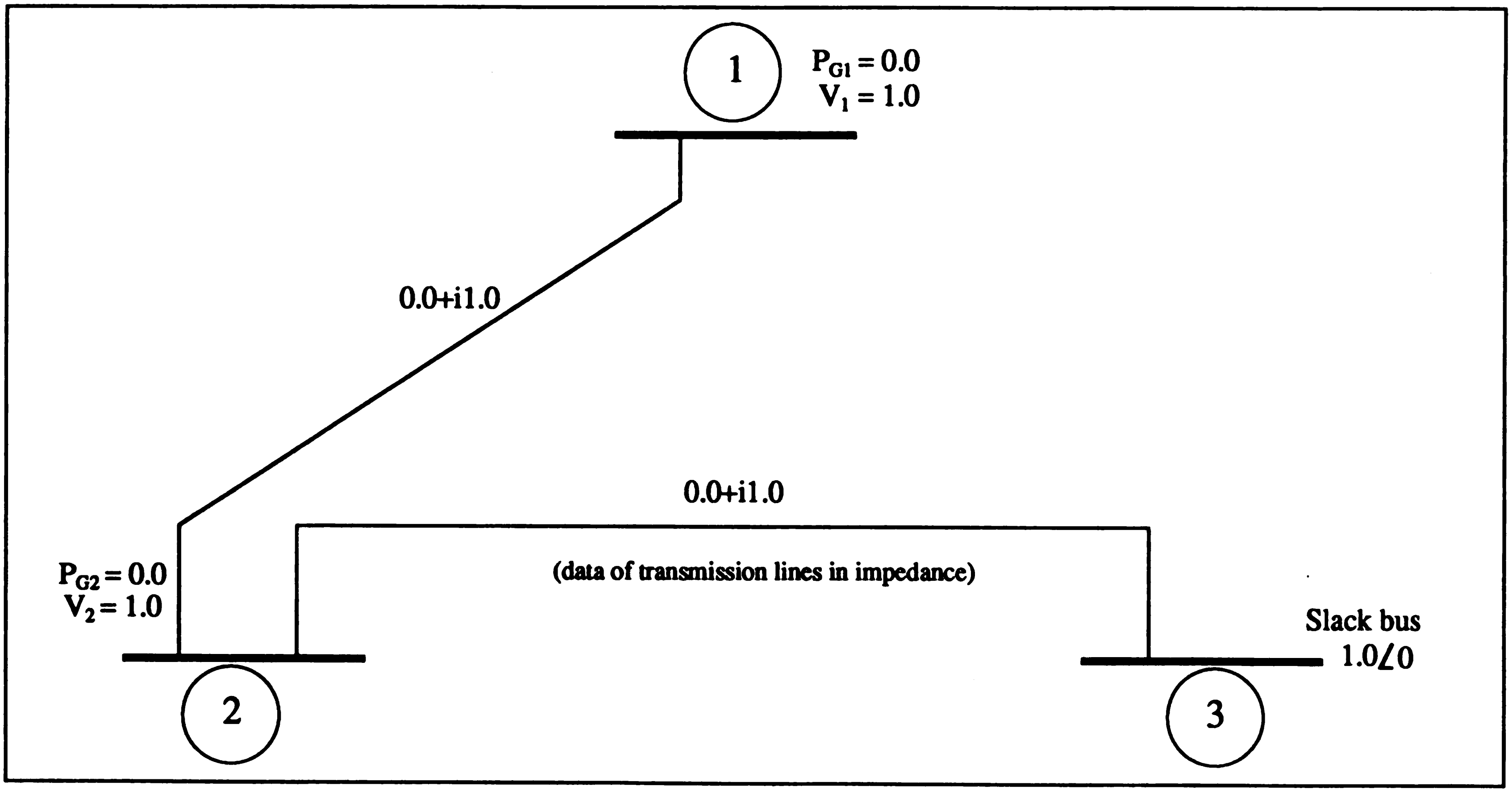


Figure 5.1. The not-fully-connected 3-bus example.

$$\hat{S}(\tilde{E}) = \begin{bmatrix} a_1E_1^2 - b_1 \\ a_2E_2^2 - b_2 \\ a_3E_1^* - b_3 \\ a_4E_2^* - b_4 \end{bmatrix}, \tag{5.3.2}$$

where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are complex constants which are chosen randomly. Obviously, $\hat{S}(\tilde{E})$ has d = 2^2 = 4 distinct complex roots. Each of the first two quadratic polynomial of the form $a_i E_i^2 - b_i = 0$ contributes 2 distinct easily computed complex solutions.

Define a homotopy continuation function as

$$H(\tilde{E}, t) = (1-t)\hat{S}(\tilde{E}) + t\hat{T}(\tilde{E}): \quad t \in C[0,1]. \tag{5.3.3}$$

To solve the polynomial equations (5.3.3), numerically, 4 solution curves need to be traced. Let a complex parameter t be chosen as $t = re^{j\Theta}$, where r is a real parameter with $r \in [0,1]$, and Θ is a real parameter which varies from Θ_0 to 0. Initially, we have $r=0$ and $\Theta = \Theta_0$ where Θ_0 is randomly chosen. Let s_1 be stepping distance from $r=0$ to the next sampling point r_1. Let t_1 be defined as

$$t_1 = (0 + s_1)e^{j\Theta_0(1-s_1)} = r_1e^{j\Theta_1}.$$

In general, given t_{i-1}, we have

$$t_i = (r_{i-1} + s_i)e^{j(\Theta_{i-1} - \Theta_0 s_i)}.$$

Clearly, Θ will be zero when r reaches 1. So the complex function $re^{j\Theta}$ is a good candidate for such a complex parameter t. In general, from the value $\tilde{E}(t_{i-1})$, we need to solve for $\tilde{E}(t_i)$ for $r_i = r_{i-1} + s_i$ and $\Theta_i = \Theta_{i-1} - \Theta_0 s_i$. For s_i sufficiently small, the zero of the homotopy function at t_{i-1} is closed to the zero of the homotopy function at t_i. The Newton-Raphson iterative method can be used to execute the "local" computation at every t_i.

Because of the quadratic convergence of the Newton-Raphson algorithm, there is less motivation to use "acceleration factors" [78] in Newton-Raphson iteration at every t_i. However, for the case of nonconvergent iteration, acceleration factor can be used to render a divergent case as convergent. The technique is

$$[\tilde{E}]^{(l+1)} = [\tilde{E}]^{(l)} - \alpha J^{-1}[H(\tilde{E}, t_i)]^{(l)}$$

at the l-th iteration. The quantity α is the acceleration factor and it generally lies in the range $0.2 \le \alpha \le 1.3$. Acceleration factors below 1.00 actually slow convergence, and acceleration factors above 1.00 will speed convergence.

In numerical computation, if the value of s is large, there is the possibility of curve-merging. And thus some roots may be missed. Consequently, it is common to begin with a small or a conservative choice of a fixed stepping distance. Either increasing the number of sampling points due to a small s or increasing the possibility of repetitive computation due to a large s, more computation time is wasted.

Table 5.1 lists the 4 solutions of the load flow equations of a 3-bus example obtained as the 4 solution curves of (5.3.3) evolved toward their values at t = 1.

5.3.2. A 7-bus power system network

For the 7-bus network depicted in Figure 4.4 of chapter 4, $\hat{T}(\tilde{E})$, expressed in complex form, is the following 12 polynomial equations in 12 complex variables (bus 7 is taken as a slack bus) as given in (4.2.1.5).

Since the number of complex solutions of such a 7-bus network is 288 which has been shown by numerical simulation in Chapter 4, by the imbedding-based method, we can follow 288 solution paths in the finding of the system solutions. Let $\hat{S}(\tilde{E})$ be defined as follows:

Table 5.1. The solutions of the not-fully-connected 3-bus example by the imbedding-based method (4 initial points)

The complex voltages ($i=\sqrt{-1}$)			
The number of solutions	Bus 1	Bus 2	Bus 3
1	-0.999997+i0.000001	1.000000+i0.000000	1.000000+i0.000000
2	-1.000000+i0.000002	-1.000000-i0.000000	1.000000+i0.000000
3	1.000005+i0.000002	-1.000000+i0.000000	1.000000+i0.000000
4	1.000007+i0.000002	1.000001+i0.000002	1.000000+i0.000000

Table 5.2. The solutions of the 7-bus network by the imbedding-based method (288 initial points)

The complex voltages ($i = \sqrt{-1}$)				
	The number of solutions			
	1	2	3	4
Bus 1	1.083577+i0.087879	-0.059637+i0.279155	0.710857+i0.183129	0.010602+i0.342747
Bus 2	0.974166-i0.057748	0.544340-i0.062712	0.587903-i0.057031	0.430691-i0.053562
Bus 3	0.908440-i0.138914	0.521564-i0.186110	0.103805-i0.138434	0.174715-i0.174183
Bus 4	0.935369-i0.098453	0.642245-i0.128212	0.400156-i0.101957	0.417658-i0.121879
Bus 5	0.971378-i0.045560	0.778857-i0.054320	0.724215-i0.042066	0.687571-i0.048225
Bus 6	0.977600-i0.050760	0.643849-i0.058951	0.665181-i0.052886	0.548572-i0.052788
Bus 7	1.000000+i0.000000	1.000000+i0.000000	1.000000+i0.000000	1.00000+i0.000000

$$\hat{S}(\tilde{E}) = \begin{bmatrix} E_1^3 - b_1 \\ E_2^3 - b_2 \\ E_3^2 - b_3 \\ E_4^2 - b_4 \\ E_5^2 - b_5 \\ E_6^2 - b_6 \\ E_1^{*2} - b_7 \\ E_2^* - b_8 \\ E_3^* - b_9 \\ \cdots\cdots \\ E_6^* - b_{12} \end{bmatrix}, \tag{5.3.4}$$

where all complex coefficients and parameters are randomly chosen. Since each of the first two equations of (5.3.4) contributes 3 complex solutions, and each equation with degree 2 contributes 2 complex solutions, thus the total number of the complex solutions of $\hat{S}(\tilde{E})$=0 is $3^2 \times 2^5 \times 1^4 = 288$.

Solving for $T(\tilde{E}) = 0$ by the imbedding-based method described above, we trace all 288 solution paths of $H(\tilde{E}, t) = 0$ as t increases from 0 to 1. We obtain the 4 system solutions which are listed in the Table 5.2.

5.4. Summary

In this chapter, we present an imbedding-based method which is shown to be computationally efficient to calculate the system solutions of a power system. This method can be used when an upper bound on the number of complex solutions of deficient system is known. Consequently, one can choose an "initial" starting system $\hat{S}(x)$, of which the number of solutions is near but greater than, or at least equal to, the number of solutions of the target system $\hat{T}(x)$. We use a 3-bus and a 7-bus power systems to demonstrate the computational capabilities.

Chapter 6

THE HOMOTOPY-BASED METHOD TO DEFICIENT SYSTEMS

In the previous chapter, we suggested the imbedding-based method in finding the solutions of deficient systems. The imbedding-based method allows us to choose t only in such a way that t is a curve in the plane of complex numbers connecting the points 0 + i0 and 1 + i0 continuously. Consequently, some serious problems may arise in implementation of the algorithm and numerical computational efficiency. The following homotopy-based method, as it will be shown, is to use real parameter t with $t \in [0,1]$. The proposed homotopy-based method still keeps the properties of the imbedding-based method, but with a real parameter t.

6.1. The Practical Heuristic Approach

Solve a system of n polynomial equations in n unknowns

$$\hat{T}(x) = \left[\hat{T}_1(x), \hat{T}_2(x), \cdots, \hat{T}_n(x)\right]^T = 0, \tag{6.1.1}$$

where each $\hat{T}_k$, $k = 1, 2, \cdots, n$ is a polynomial in $x = (x_1, \cdots, x_n) \in C^n$ of degree d_k.

Let $\hat{S}(x,b)$ be constructed as

$$\hat{S}(x,b) = \begin{bmatrix} \hat{S}_1(x_1, b_1) \\ \hat{S}_2(x_2, b_2) \\ \cdots \\ \cdots \\ \cdots \\ \hat{S}_n(x_n, b_n) \end{bmatrix} : \quad x_k, b_k \in C, \tag{6.1.2}$$

with $\quad \hat{S}_k(x_k, b_k) = x_k^{d_{s_k}} - b_k, \quad 1 \le k \le n.$

Let R(x, a) be defined as

$$R(x,a) = \begin{bmatrix} r_1(x_1, a_1) \\ r_2(x_2, a_2) \\ \cdots \\ \cdots \\ \cdots \\ r_n(x_n, a_n) \end{bmatrix} : \quad x_k, a_k \in C, \tag{6.1.3}$$

with $\quad r_k(x_k, a_k) = a_k x_k^{d_{r_k}}, \quad d_{r_k} = \max\{d_k, d_{s_k}\}, \quad 1 \le k \le n.$

Define H : $C^n \times [0,1] \times C^n \times C^n \to C^n$ by

$$H(x, t, b, a) = \begin{bmatrix} h_1(x, t, b, a) \\ \cdots \\ \cdots \\ \cdots \\ h_n(x, t, b, a) \end{bmatrix} \tag{6.1.4}$$

$$= (1-t)\hat{S}(x, b) + t\hat{T}(x) + t(1-t)R(x, a),$$

with a = $(a_1, \cdots, a_n) \in C^n$ and b = $(b_1, \cdots, b_n) \in C^n$.

Some Results:

Let N_T denote the number of zeros of $\hat{T}(x)$, or an upper bound on the number of zeros of $\hat{T}(x)$. Assume $d_s = \prod_{k=1}^{n} d_{s_k} \geq N_T$. For a random choices of (b, a), except for a set of measure zero, in C^n, the zero set

$$H^{-1}(0) = \{(x, t) \in C^n \times [0,1) \mid H(x, t, a, b) = 0\}$$

consists of disjoint analytic solution paths emanating from the roots of H(x, 0) = $\hat{S}(x)$ at t = 0. The zeros of $\hat{T}(x)$ may be obtained by tracing solution paths to t = 1.

The theoretical results that follow can be shown by following the same procedures delineated by Chow, Mallet-Paret, and Yorke in [43]. In the following, we will exploit the steps in A.P. Morgan [36].

(A) 0 is a regular value of H(., ., b) restricted to $C^n \times [0,1)$ for almost all choices of $b \in C^n$. Consequently, from the Implicit Function Theorem, it results in that $H^{-1}(0)$ consists of smooth, disjoint paths.

(B) Let $\tilde{H}$ denote the highest order homogeneous terms of H. 0 is a regular value of $\tilde{H}$(., ., a) on $\{(x,t) | x \neq 0\}$ for almost all $a \in C^n$. It turns out that the solution paths of H(x,t) are bounded on $t \in (0,1)$.

(C) dt/ds ≠ 0 on solution paths $H^{-1}(0)$, where s denotes the arc length of a homotopy curve, implies that no path can be homeomorphic to a circle. Therefore, each path (x(s), t(s)) would eventually go to infinity or would lead to a root of H(., 1) when t approaches 1.

The proof of the nonsingularity of the Jacobian matrix $\partial H(x, t)/\partial x$ of the homotopy function $H(x, t)$ with respect to x at x_s and t_s needs the following Transversality Theorem, where x_s is a root of $H(x, t)$ at t_s.

Transversality Theorem ([24], [35], [43]). Let $V \subseteq R^q$, $U \subseteq R^m$ be open, and let F: $V \times U \rightarrow R^p$ be C^r where r > max{0, m-p}. If $0 \in R^p$ is a regular value of F, then for almost all $a \in V$, except in a set of measure 0 in R^q, 0 is a regular value of F_a: $U \rightarrow R^p$, where $F_a(x) = F(x, a)$.

In applying the Transversality Theorem above, we regard the complex space as a real space of two dimensions. We now apply this theorem to the homotopy function H defined in (6.1.4). Since $h_{b_k}(.,.,b)$ with respect to b_k ($b^T=[b_1, \cdots, b_n]$)

$$h_{b_k}(.,.,b) = \frac{\partial h_k(.,.,b)}{\partial b_k} \tag{6.1.5}$$

$$= -(1-t) \neq 0, \quad (0 < t < 1,\ 1 \le k \le n).$$

Therefore, the Jacobian matrix

$$H_b(.,.,b) = \begin{bmatrix} h_{b_1}(.,.,b) \\ \cdots \\ \cdots \\ \cdots \\ h_{b_n}(.,.,b) \end{bmatrix} \tag{6.1.6}$$

is of full rank equal to 2n. Hence, by the Transversality Theorem, 0 is a regular value of H(., ., b) for almost all $b \in C^n$.

The proof of statement (B) can be carried out as follows [24]. Given the domain

$$\begin{aligned} W = \{(x_1, \cdots, x_n) \in C^n \mid & x_{i_1} \neq 0, \cdots, x_{i_j} \neq 0, \\ & i_1, i_2, \cdots, i_j \in (1, 2, \cdots, n) \text{ and } x_k = 0 \\ & \text{for } k \neq (i_1, \cdots, i_j),\ 1 \le j \le n\}. \end{aligned}$$

We illustrate the proof of statement (B) for the simple case

$$W = \{(x_1, \cdots, x_i) \in C^i \mid x_j \neq 0 \text{ for } j = 1, \cdots, i\}.$$

Let

$$\tilde{H}(x,t,a) = \left[\tilde{h}_1(x,t,a), \cdots, \tilde{h}_n(x,t,a)\right] \tag{6.1.7}$$

with

$$\tilde{H}: C^n \times (0,1) \times C^n \to C^n \tag{6.1.8}$$

be comprised of the highest order homogeneous terms of H, that is

$$\tilde{h}_k(x,t,a_k) = (1-t)\tilde{S}_k(x) + t\tilde{T}_k(x) + t(1-t)r_k(x,a_k), \qquad 1 \le k \le n,$$

where for $1 \le k \le n$, $\tilde{S}_k$, and $\tilde{T}_k$, are the homogeneous part of $\hat{S}_k$, and $\hat{T}_k$, respectively, consisting of all terms with degree d_{r_k}. Now, since

$$\frac{\partial \tilde{h}_j}{\partial a_j} = t(1-t)\frac{\partial r_j}{\partial a_j} = t(1-t)x_j^{d_{r_j}} \ne 0, \qquad 1 \le j \le i, \tag{6.1.9}$$

we conclude that the Jacobian of $\tilde{H}$ has real-rank 2i at solutions for almost all $a \in C^n$. Therefore, 0 is a regular value of $\tilde{H}$(., ., a) on $W \times (0,1)$ for almost all $a \in C^n$. From the lemma 2.3 of [43], $\tilde{H}(x_0, t_0) = 0$ implies that $x_0 = 0$. Hence, all the solution paths emanating from the roots of $\hat{S}(x)$ are bounded by a constant $k(t_0) > 0$ with any $t_0 \in [0,1)$. If not the case, then there exists $t_0 \in [0,1)$ and a sequence $(x(s), t(s)) \in C^n \times [0, t_0]$ where $H(x(s), t(s)) = 0$ and $|x(s)| \to \infty$. We may suppose $\frac{x(s)}{|x(s)|} \to w \in C^n$ as $t(s) \to \tau \in [0, t_0]$. Since $h_k - \tilde{h}_k$ is a polynomial in x of degree less than d_{r_k}, and $\tilde{h}_k$ is homogeneous of degree d_{r_k} it follows that

$$|x(s)|^{-d_{r_k}}[\tilde{h}_k(x(s), t(s)) - h_k(x(s), t(s))] \to 0, \tag{6.1.10a}$$

and so

$$\tilde{h}_k\left(\frac{x(s)}{|x(s)|}, t(s)\right) = |x(s)|^{-d_{r_k}} \tilde{h}_k(x(s), t(s)) \qquad (6.1.10b)$$

$$= |x(s)|^{-d_{r_k}} [\tilde{h}_k(x(s), t(s)) - h_k(x(s), t(s))] \to 0.$$

Hence, $\tilde{h}(w, \tau) = 0$. But $w \neq 0$, this contradicts lemma 2.3 [43].

Now let (x (s), t(s)) be a local parametrization of a solution path with respect to the arc length s. We can assume [36] that dx/ds and dt/ds are not both zero at s for which (x(s), t(s)) is defined. Then from H(x(s), t(s)) = 0, we have

$$\frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial t}\frac{dt}{ds} = 0. \qquad (6.1.11)$$

Since $\partial H/\partial x$ is nonsingular, dt/ds = 0 implies that dx/ds = 0. Thus, dt/ds is always nonzero. This means that x can be parametrized by t such that x (t) consists of a path when t increases from t=0 to t=1. The monotonicity of t implies that no path can be homeomorphic to a circle.

6.2. Numerical Examples of Power Systems

We redo a 4-bus and a 7-bus examples to illustrate its procedures and to demonstrate their capabilities.

For a 4-bus example shown in Figure 4.5, we have shown, from chapter 3, that the system (3.5.1) has at most

$$2^n \binom{2(n+m)}{n+m}$$

complex solutions. Therefore, we only need to trace

$$2^n \binom{2(n+m)}{n+m} = 2 \times \binom{4}{2} = 12$$

solution paths.

For given the steady-state equations $\hat{T}(\tilde{E})$ in (4.2.2.1), we construct an "initial" starting system as

$$\hat{S}(\tilde{E}) = \begin{bmatrix} E_1^3 - b_1 \\ E_2^2 - b_2 \\ E_3^2 - b_3 \\ E_1^* - b_4 \\ E_2^* - b_5 \\ E_3^* - b_6 \end{bmatrix}, \tag{6.2.1}$$

where (b_i) $(1 \le i \le 6)$ are the randomly chosen complex constants. Obviously, $\hat{S}(\tilde{E})$ has 12 complex solutions because the first equation $E_1^3 - b_1 = 0$ contributes 3 solutions and the each equation with degree 2 contributes 2 solutions.

Define a homotopy function $H(\tilde{E}, t)$ as follows:

$$H(\tilde{E}, t) = (1-t)\,\hat{S}(\tilde{E}) + t\,\hat{T}(\tilde{E}) + t(1-t)\,R(\tilde{E}). \tag{6.2.2}$$

Where $R(\tilde{E})$ is given by

$$R(\tilde{E}) = \begin{bmatrix} a_1 E_1^3 \\ a_2 E_2^2 \\ a_3 E_3^2 \\ a_4 E_4^2 \\ a_5 E_5^2 \\ a_6 E_6^2 \end{bmatrix}. \tag{6.2.3}$$

We trace the 12 solution paths of (6.2.2) emanating from the roots of $\hat{S}(\tilde{E})$ for a random choice of complex coefficients (a_i) and (b_i) $(1 \le i \le 6)$. We obtain the 4 system solutions which are listed in Table 6.1. (We use acceleration factor in the range $0.2 \le \alpha \le 1.3$. More details can be found in chapter 5.)

For a 7-bus example shown in Figure 4.4, the load flow equations are given in (4.2.1.5). Construct $\hat{S}(\tilde{E})$ as

Table 6.1. The solutions of the 4-bus example by the homotopy-based method (12 initial points)

	The complex voltages ($i = \sqrt{-1}$)			
The number of solutions	Bus 1	Bus 2	Bus 3	Bus 4
1	0.998868+i0.047518	0.997370+i0.007397	0.984201-i0.010106	1.000000+i0.000000
2	0.999423+i0.033876	0.749975-i0.014560	0.016335-i0.014541	1.000000+i0.000000
3	-0.999217-i0.024801	0.514578+i0.052647	0.788435+i0.001488	1.000000+i0.000000
4	-0.999969+i0.001821	0.319584+i0.039395	0.020961-i0.017464	1.000000+i0.000000

$$\hat{S}(\tilde{E}) = \begin{bmatrix} E_1^3 - b_1 \\ E_2^3 - b_2 \\ E_3^2 - b_3 \\ E_4^2 - b_4 \\ E_5^2 - b_5 \\ E_6^2 - b_6 \\ E_1^{*2} - b_7 \\ E_2^* - b_8 \\ E_3^* - b_9 \\ \cdots\cdots \\ E_6^* - b_{12} \end{bmatrix}, \qquad (6.2.4)$$

where the complex constant numbers (b_i) ($1 \le i \le 12$) are chosen at random. It can be seen that the total number of the complex solutions of $\hat{S}(\tilde{E})=0$ is $3^2 \times 2^5 \times 1^4 = 288$.

Let $R(\tilde{E})$ be chosen as

$$R(\tilde{E}) = \begin{bmatrix} a_1 E_1^3 \\ a_2 E_2^3 \\ a_3 E_3^2 \\ a_4 E_4^2 \\ a_5 E_5^2 \\ a_6 E_6^2 \\ a_7 E_1^{*2} \\ \cdots\cdots \\ a_{12} E_6^{*2} \end{bmatrix}. \qquad (6.2.5)$$

Since the system has 288 finite solutions, thus we only need to trace 288 solution paths according to (6.2.2). By the homotopy-based method. We obtain the 4 system solutions listed in Table 6.2. (Similarly, we also use acceleration factor in the range $0.2 \le \alpha \le 1.3$ to avoid divergent problem.)

Table 6.2. The solutions of the 7-bus network by the homotopy-based method (288 initial points)

The complex voltages ($i=\sqrt{-1}$)				
	The number of solutions			
	1	2	3	4
Bus 1	1.084615+i0.092783	-0.060330+i0.278557	0.710984+i0.185250	0.009490+i0.342659
Bus 2	0.975997-i0.053053	0.545183-i0.061400	0.588776-i0.055036	0.431049-i0.052528
Bus 3	0.910134-i0.135589	0.522311-i0.185301	0.103953-i0.138085	0.174627-i0.173866
Bus 4	0.936664-i0.095837	0.642794-i0.127592	0.400400-i0.101470	0.417675-i0.121550
Bus 5	0.972229-i0.043647	0.779230-i0.053837	0.724511-i0.041422	0.687673-i0.047869
Bus 6	0.979045-i0.047145	0.644483-i0.057962	0.665842-i0.051380	0.548843-i0.052006
Bus 7	1.000000+i0.000000	1.000000+i0.000000	1.000000+i0.000000	1.00000+i0.000000

6.3. Summary

In this chapter, the homotopy-based method is developed and shown to be computationally efficient compared to current homotopy methods. This method can also be used when the number of solutions of a polynomial deficient system is known. Consequently, one can choose an "initial" starting system $\hat{S}(x)$, of which the number of solutions can be exactly equal to the actual number of solutions of the target system $\hat{T}(x)$. We use a 4-bus and a 7-bus power system networks to demonstrate the computational capabilities and efficiency of the algorithm.

Chapter 7

CONCLUSIONS AND SUGGESTIONS

The fundamental theorem of algebraic geometry states that the number of isolated complex solutions of 2N polynomials in 2N complex variables is bounded by the total degree d = $\prod_{i=1}^{2N} d_i$ of polynomials. This is the statement of the B*ézout* theorem. When using the "basic homotopy method" to solve for the solutions to 2N polynomials, such as 2N load flow equations of power system, we need to trace all the total degree d (the B*ézout* number) homotopy curves in finding all possible solutions.

7.1. Conclusions

In this thesis, we use powerful results from algebraic geometry and homotopy methods to determine the number of (complex) solutions for various models of power systems. We develop the theorems to predict the upper bound on the number of solutions of the full-fledged steady-state equations for various levels of the detailed models of power systems. We give sufficient conditions under which this bound can be reached. We observe that the sufficient conditions are naturally satisfied for fully connected power system with generic coefficients. We also develop a cluster method to predict the number of solutions for not-fully-connected special power systems.

Basic homotopy method is computationally expensive for finding of solutions to the (equilibrium) steady state equations of power systems. The special homotopy method is applied to reduce the computational complexity and to *guarantee* finding all the solutions to the load flow equations of power systems with probability one. We develop the imbedding-based and the homotopy-based methods that pertain to procedures to simplify the computations in finding the solutions of power systems. Moreover, these procedures are directly implementable on digital sequential and parallel processors.

7.2. Suggestions

An application of these theorems and the methods developed in this thesis yields explicit quantitative information regarding the number of complex solutions for given (polynomial) equations of power systems. However, for very deficient systems this upper bound may still be too large. From the numerical results of the power system examples, the number of the real system solutions of the load flow equations of power systems depends on the values of the system parameters for a given structure, while the number of the complex solutions is changed only with the system structure. An important open question is: what is the upper bound on the number of the real system solutions of the load flow equation for a given N-node power grid?

Solving all solutions of the load flow equations of power systems has been almost impossible until the advent of the globally convergent homotopy method. However, the current homotopy continuation methods are computationally expensive for finding the solutions to very deficient systems. The amount of computational effort grows exponentially to the point where efficient computation is rendered impossible even on very fast and/or parallel computers. Consequently, the current homotopy continuation methods become incapable for a larger-sized *deficient* system. An important question facing mathematicians and scientific researchers is to develop a method with a theoretical proof to reduce the (computer) computational complexity in the finding of all (or some of) the solutions of the so-called *deficient* systems. Even for a large-sized power system, the methods can still compute all (or some of) the solutions.

APPENDIX

Appendix

The Proof of Theorem 3.4.3

For the system (3.4.3.1), its associated homogeneous system is given by

$$0 = E_1(\textstyle\sum_{j=1}^{N} y_{1j}^* E_j^* + y_{1,N+1}^* E_0) + \qquad \text{(A-1.1a)}$$

$$E_1^*(\textstyle\sum_{j=1}^{N} y_{1j} E_j + y_{1,N+1} E_0) - 2P_1 E_0^2,$$

.

.

.

$$0 = E_{N_g}(\textstyle\sum_{j=1}^{N} y_{N_g,j}^* E_j^* + y_{N_g,N+1}^* E_0) +$$

$$E_{N_g}^*(\textstyle\sum_{j=1}^{N} y_{N_g,j} E_j + y_{N_g,N+1} E_0) - 2P_{N_g} E_0^2,$$

$$0 = E_1 E_1^* - V_1^2 E_0^2, \qquad \text{(A-1.1b)}$$

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$$0 = E_{N_g} E_{N_g}^* - V_{N_g}^2 E_0^2,$$

$$0 = E_{N_g+1}(\textstyle\sum_{j=1}^{N} y_{N_g+1,j}^* E_j^* + y_{N_g+1,N+1}^* E_0) - S_{N_g+1} E_0^2, \qquad \text{(A-1.1c)}$$

.

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$$0 = E_N(\textstyle\sum_{j=1}^{N} y_{Nj}^* E_j^* + y_{N,N+1}^* E_0) - S_N E_0^2,$$

$$0 = E^*_{N_s+1}(\sum_{j=1}^{N} y_{N_s+1,j}E_j + y_{N_s+1,N+1}E_0) - S^*_{N_s+1}E_0^2, \tag{A-1.1d}$$

$$\vdots$$

$$0 = E^*_N(\sum_{j=1}^{N} y_{Nj}E_j + y_{N,N+1}E_0) - S^*_N E_0^2.$$

The *Bézout* theorem says that the solutions of the associated homogeneous system (A-1.1) include both the solutions for $E_0 \neq 0$ (i.e. the finite solutions) of the original system (3.4.3.1) and the solutions for $E_0 = 0$ (i.e. the solutions at infinity). Therefore, the number of the original system solutions equals the number of the solutions of the associated homogeneous system minus the number of the solutions at infinity.

However, the solutions at infinity of (A-1.1) are the solutions with $E_0 = 0$, namely, the solutions of

$$0 = E_1(\sum_{j=1}^{N} y^*_{1j}E^*_j) + E^*_1(\sum_{j=1}^{N} y_{1j}E_j), \tag{A-1.2a}$$

$$\vdots$$

$$0 = E_{N_s}(\sum_{j=1}^{N} y^*_{N_s,j}E^*_j) + E^*_{N_s}(\sum_{j=1}^{N} y_{N_s,j}E_j),$$

$$0 = E_1E^*_1, \tag{A-1.2b}$$

$$\vdots$$

$$0 = E_{N_s}E^*_{N_s},$$

$$0 = E_{N_s+1}(\sum_{j=1}^{N} y^*_{N_s+1,j}E^*_j), \tag{A-1.2c}$$

$$\vdots$$

$$0 = E_N(\sum_{j=1}^{N} y^*_{Nj}E^*_j),$$

$$0 = E^*_{N_s+1}(\textstyle\sum_{j=1}^{N} y_{N_s+1,j}E_j), \tag{A-1.2d}$$

.

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$$0 = E^*_N(\textstyle\sum_{j=1}^{N} y_{Nj}E_j).$$

From assumption (a) of the theorem, we can see that the solution set of (A-1.2) consist of two linear subspaces, say, Z_1 and Z_2 which are the disjoint union of

$$Z_1 = \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) \mid 0 = E_1 = \ldots = E_N\} = CP^{N-1},$$

and

$$Z_2 = \{(0, E_1, \cdots, E_N, E_1^*, \cdots, E_N^*) \mid 0 = E_1^* = \ldots = E_N^*\} = CP^{N-1}.$$

The codimension of Z_1 and Z_2 in the complex projective space CP^{2N} satisfy the equality $codim(Z_1, CP^{2N}) = codim(Z_2, CP^{2N}) = N+1$.

In the following, we will show through tedious calculations that if all the conditions of assumption (b) of the theorem hold, then the subspaces Z_1 and Z_2 are non-singular. Consequently, by the fundamental theorem in (3.2.1), we conclude that the number of isolated complex solutions of the polynomial system (3.4.3.1) (or (2.1.9)) is given by

$$r = 2^{2N} - [Z_1] - [Z_2] = 2^{2N} - \textstyle\sum_{i=0}^{N-1}\binom{2N}{i} - \sum_{i=N+1}^{2N}\binom{2N}{i} = \binom{2N}{N}.$$

We now begin the lengthy calculations. We first reorder system (A-1.1) in the following way for convenience.

$$0 = E_1(\textstyle\sum_{j=1}^{N} y^*_{1j}E^*_j + y^*_{1,N+1}E_0) + \tag{A-1.3a}$$

$$E^*_1(\textstyle\sum_{j=1}^{N} y_{1j}E_j + y_{1,N+1}E_0) - 2P_1E_0^2 = \tilde{f}_1,$$

.

.

.

$$0 = E_{N_s}(\textstyle\sum_{j=1}^{N} y^*_{N_s,j}E^*_j + y^*_{N_s,N+1}E_0) +$$

$$E_{N_s}^* (\textstyle\sum_{j=1}^{N} y_{N_s,j} E_j + y_{N_s,N+1} E_0) - 2P_{N_s} E_0^2 = \tilde{f}_{N_s},$$

$$0 = E_{N_s+1}^* (\textstyle\sum_{j=1}^{N} y_{N_s+1,j} E_j + y_{N_s+1,N+1} E_0) - S_{N_s+1}^* E_0^2 = \tilde{f}_{N_s+1}, \qquad \text{(A-1.3b)}$$

$$\vdots$$

$$0 = E_N^* (\textstyle\sum_{j=1}^{N} y_{Nj} E_j + y_{N,N+1} E_0) - S_N^* E_0^2 = \tilde{f}_N .$$

$$0 = E_1 E_1^* - V_1^2 E_0^2 = \tilde{f}_{N+1}, \qquad \text{(A-1.3c)}$$

$$\vdots$$

$$0 = E_{N_s} E_{N_s}^* - V_{N_s}^2 E_0^2 = \tilde{f}_{N+N_s},$$

$$0 = E_{N_s+1} (\textstyle\sum_{j=1}^{N} y_{N_s+1,j}^* E_j^* + y_{N_s+1,N+1}^* E_0) - S_{N_s+1} E_0^2 = \tilde{f}_{N+N_s+1}, \qquad \text{(A-1.3d)}$$

$$\vdots$$

$$0 = E_N (\textstyle\sum_{j=1}^{N} y_{Nj}^* E_j^* + y_{N,N+1}^* E_0) - S_N E_0^2 = \tilde{f}_{2N} .$$

The solution sets Z_1 and Z_2 at infinity are nonsingular, if

$$J(Z_1) = rank_c \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_{2N})}{\partial(E_0, \cdots, E_N, E_1^*, \cdots, E_N^*)}(x) = codim_x(Z_1, CP^{2N}) = N+1$$

for every point $x \in Z_1$, and

$$J(Z_2) = rank_c \frac{\partial(\tilde{f}_1, \cdots, \tilde{f}_{2N})}{\partial(E_0, \cdots, E_N, E_1^*, \cdots, E_N^*)}(y) = codim_y(Z_2, CP^{2N}) = N+1$$

for every point $y \in Z_2$.

Calculating the Jacobian matrices $J(Z_1)$ and $J(Z_2)$ of (A-1.3) and removing all the zero columns, we respectively obtain

$$
\hat{J}(Z_1) = \begin{bmatrix}
e_{1,N+1} & a_1+e_{11} & \cdots & e_{1,N_g} & e_{1,N_g+1} & \cdots & e_{1,N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
e_{N_g,N+1} & e_{N_g,1} & & a_{N_g}+e_{N_g,N_g} & e_{N_g,N_g+1} & & e_{N_g,N} \\
e_{N_g+1,N+1} & e_{N_g+1,1} & \cdots & e_{N_g+1,N_g} & e_{N_g+1,N_g+1} & \cdots & e_{N_g+1,N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
e_{N,N+1} & e_{N,1} & \cdots & e_{N,N_g} & e_{N,N_g+1} & \cdots & e_{N,N} \\
0 & E_1^* & & 0 & 0 & & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & & E_{N_g}^* & 0 & & 0 \\
0 & 0 & \cdots & 0 & a_{N_g+1} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & & 0 & 0 & & a_N
\end{bmatrix}_{2N\times(N+1)}, \quad \text{(A-1.4)}
$$

and

$$
\hat{J}(Z_2) = \begin{bmatrix}
f_{1,N+1} & b_1+f_{11} & & f_{1,N_g} & f_{1,N_g+1} & & f_{1,N} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
f_{N_g,N+1} & f_{N_g,1} & & b_{N_g}+f_{N_g,N_g} & f_{N_g,N_g+1} & & f_{N_g,N} \\
0 & 0 & \cdots & 0 & b_{N_g+1} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & b_N \\
0 & E_1 & & 0 & 0 & & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & & E_{N_g} & 0 & & 0 \\
f_{N_g+1,N+1} & f_{N_g+1,1} & \cdots & f_{N_g+1,N_g} & f_{N_g+1,N_g+1} & \cdots & f_{N_g+1,N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
f_{N,N+1} & f_{N,1} & & f_{N,N_g} & f_{N,N_g+1} & & f_{N,N}
\end{bmatrix}_{2N\times(N+1)}, \quad \text{(A-1.5)}
$$

where we have set $a_i := \sum_{j=1}^{N} y_{ij}^* E_j^*$, $b_i := \sum_{j=1}^{N} y_{ij} E_j$, $e_{ik} := y_{ik} E_i^*$, and $f_{ik} := y_{ik}^* E_i$ $(1 \le i, k \le N)$.

It is known, by the definition in 3.2, that Z_1 and Z_2 are nonsingular if $rank_c \hat{J}(Z_1)$ and $rank_c \hat{J}(Z_2)$ are both equal to N+1. It should be observed that $\hat{J}(Z_1)$ and $\hat{J}(Z_2)$ have the same form simply by reordering the rows of $\hat{J}(Z_2)$. Therefore, $\hat{J}(Z_1)$ and $\hat{J}(Z_2)$ have the same rank. Hence, we only need to check the rank of $\hat{J}(Z_1)$. In $\hat{J}(Z_1)$, now we assume that the first $E^*_{k_1} = 0$ ($0 \le k_1 \le i$, $i \le N-1$) and the remaining $E^*_{k_2} \ne 0$ ($k_2 \ne k_1$). From assumption (a) of the theorem, at most $(N-i-1)$ of the a_j terms are equal to zero. There are now two cases to be discussed.

(A) $i \le N_g$.

For convenience, we partition $\hat{J}(Z_1)$ into two parts. The first N rows of $\hat{J}(Z_1)$ are called Part 1, and the remaining N rows are called Part 2.

(a) $i < N-i$.

We consider the worst case in which (1) the first i rows of Part 2 become zero because of $E^*_{k_1} = 0$ ($0 \le k_1 \le i$), and (2) the first i rows of Part 1 and the last $\{N-i-1-i\}$ rows of Part 2 become zero because $E^*_{k_1} = 0$ ($0 \le k_1 \le i$) and at most $(N-i-1)$ of the a_j terms equal to zero. Note that the condition $N-N_g-(N-i-1-i) \ge 0$ is required, otherwise, we take $N-N_g-(N-i-1-i) = 0$. Remove all zero rows from $\hat{J}(Z_1)$ and remove all rows and columns that share the element E_{k_p} ($k_p \ne k_1$) from $\hat{J}(Z_1)$. Then remove all rows and columns that have a common element a_p ($p \ne j$) in Part 2 from $\hat{J}(Z_1)$. The final reduced matrix is denoted by $\tilde{J}_1(Z_1)$

$$\tilde{J}_1(Z_1) = \begin{bmatrix} e_{i+1,N+1} & e_{i+1,1} & \cdots & e_{i+1,i} & e_{i+1,2i+2} & \cdots & e_{i+1,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{N_g,N+1} & e_{N_g,1} & \cdots & e_{N_g,i} & e_{N_g,2i+2} & \cdots & e_{N_g,N} \\ e_{N_g+1,N+1} & e_{N_g+1,1} & \cdots & e_{N_g+1,i} & e_{N_g+1,2i+2} & \cdots & e_{N_g+1,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{N,N+1} & e_{N,1} & \cdots & e_{N,i} & e_{N,2i+2} & \cdots & e_{N,N} \end{bmatrix}_{(N-i)\times(N-i)} . \quad \text{(A-1.6)}$$

Since the condition $N-N_g-(N-i-1-i) \ge 0$ implies $i \ge (N_g-1)/2$, the matrix $\tilde{J}_1(Z_1)$ is nonsingular from assumption (b) of the theorem. Therefore, we have

$rank_c \hat{J}(Z_1) = N+1$ [$:= (N-i)+(N_g-i)+(N-N_g-(N-i-1-i))$].

(b) $i \geq N-i$.

In this case, we still assume that the first i rows of Part 2 become zero since $E^*_{k_1} = 0$ $(0 \leq k_1 \leq i)$. We also assume $a_j = 0$ $(0 \leq j \leq N-i-1)$ (i.e. we are considering the worst case). Similarly, we first remove all zero rows from $\hat{J}(Z_1)$. We then remove all rows and columns that share the element E_{k_p} $(k_p \neq k_1)$ in Part 2 from $\hat{J}(Z_1)$. Finally, we remove all rows and columns that have a common element a_p $(p \neq j)$ from $\hat{J}(Z_1)$. The remaining matrix is given by

$$\tilde{J}_2(Z_1) = \begin{bmatrix} e_{i+1,N+1} & e_{i+1,1} & \cdots & e_{i+1,N-i-1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{N_g,N+1} & e_{N_g,1} & \cdots & e_{N_g,N-i-1} \\ e_{N_g+1,N+1} & e_{N_g+1,1} & \cdots & e_{N_g+1,N-i-1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{N,N+1} & e_{N,1} & \cdots & e_{N,N-i-1} \end{bmatrix}_{(N-i)\times(N-i)} \tag{A-1.7}$$

The condition $i \geq N-i$ implies $N-i \leq N/2$. Therefore, the matrix $\tilde{J}_2(Z_1)$ is non-singular if condition (b) of the theorem holds. We conclude that $rank_c \hat{J}(Z_1) = N+1$ [$:= (N-i)+(N-(N-i-1))$].

(B) $i > N_g$.

In the following, we will follow the same procedures as above to show the non-singularity of Z_1 and Z_2.

(a) $i < N-i$.

In the worst case, we assume that $E^*_{k_1} = 0$ $(0 \leq k_1 \leq i)$. We also assume that $a_j = 0$ $(0 \leq j \leq N-i-1)$. Remove all zero rows from $\hat{J}(Z_1)$, and remove all rows and columns that have a common element a_p $(p \neq j)$ from $\hat{J}(Z_1)$. The remaining matrix is denoted by $\tilde{J}_3(Z_1)$

$$\tilde{J}_3(Z_1) = \begin{bmatrix} e_{i+1,N+1} & e_{i+1,1} & \cdots & e_{i+1,N-i-1} \\ e_{i+2,N+1} & e_{i+2,1} & \cdots & e_{i+2,N-i-1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N,N+1} & e_{N,1} & \cdots & e_{N,N-i-1} \end{bmatrix}_{(N-i)\times(N-i)} . \tag{A-1.8}$$

From $i > N_g$, we have $N-i < N-N_g < N-N_g/2$. Therefore, the matrix $\tilde{J}_3(Z_1)$ is nonsingular if assumption (b) of the theorem holds. Consequently, we prove that $rank_c \hat{J}(Z_1)$ equals $N+1$ [$:= (N-i+N-(N-i-1))$].

(b) $i \geq N-i$.

We still consider the worst case in which $E^*_{k_1} = 0$ $(0 \leq k_1 \leq i)$, and $a_j = 0$ $(0 \leq j \leq N-i-1)$. Similarly, we remove all zero rows from $\hat{J}(Z_1)$, and remove all rows and columns that have a common element a_p $(p \neq j)$ from $\hat{J}(Z_1)$. The remaining matrix is denoted by $\tilde{J}_4(Z_1)$.

$$\tilde{J}_4(Z_1) = \begin{bmatrix} e_{i+1,N+1} & e_{i+1,1} & \cdots & e_{i+1,N-i-1} \\ e_{i+2,N+1} & e_{i+2,1} & \cdots & e_{i+2,N-i-1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N,N+1} & e_{N,1} & \cdots & e_{N,N-i-1} \end{bmatrix}_{(N-i)\times(N-i)} . \tag{A-1.9}$$

Obviously, the condition $i \geq N-i$ implies $N-i \leq N/2$. We conclude that the matrix $\tilde{J}_4(Z_1)$ is nonsingular if assumption (b) of the theorem is satisfied. Therefore, the rank of $\hat{J}(Z_1)$ equals $N+1$ [$:= (N-i+N-(N-i-1))$].

Since the matrices $\hat{J}(Z_1)$ and $\hat{J}(Z_2)$ are of the same form, we can follow the same procedure to prove that $rank_c \hat{J}(Z_2)$ equals N+1. Consequently, we prove that Z_1 and Z_2 are nonsingular if assumption (b) of the theorem holds. This completes the proof.

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