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SOME OPERATORS AND CARLESON MEASURES

ON

WEIGHTED NORM SPACES

By

Dangsheng Gu

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ABSTRACT

SOME OPERATORS AND CARLESON MEASURES ON WEIGHTED NORM SPACES

By

Dangsheng Gu

Suppose $(\mathbf{X}, \nu, \mathbf{d})$ is a homogeneous space. Hörmander has constructed a maximal operator to study problems involving Carleson measures in this situation. In particular examples of homogeneous spaces, for example, in $\mathbf{R}^N$ and in the unit ball of $\mathbf{C}^N$, a maximal averaging operator has proved to be useful. The first goal of this paper is to study the weighted norm inequalities for the Hörmander maximal operator and the generalization of the maximal averaging operator. Using the concept of the "balayée" of a measure, we characterize those positive measures μ on $\mathbf{X}^+ = \mathbf{X} \times \mathbf{R}^+$ such that the inequality $\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$, where $q < p$, holds for the Hörmander maximal operator H_ν, and those positive measures μ on $\mathbf{X}$ such that the similar inequality $\|M_{\nu,r} f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$, where $q < p$, holds for the maximal averaging operator $M_{\nu,r}$ defined by

$$M_{\nu,r} f(x) = \sup_{t \geq r} \frac{1}{\nu(B(x,t))} \int_{B(x,t)} |f(u)| d\nu(u),$$

where $B(x,t)$ is the ball centered at x with radius t.

The second goal of this paper is to study the analytic functions on the unit ball

of $\mathbf{C}^N$. Let $\mathbf{U}$ be the unit ball in $\mathbf{C}^N$ and Ω be a positive measure on $\mathbf{U}$ satisfying Békollé's B_α^p condition for some $\alpha > -1$. The first result of this part is a Carleson measure theorem for weighted Bergman spaces. We characterize those positive measures μ on $\mathbf{U}$ such that $\|f\|_{L^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$ $(1 < p \leq q)$ for any function f in the weighted Bergman space $A^p(\Omega)$. The second result concerns the Bergman operator on weighted mixed norm spaces. Using an interpolation theorem between the L^p spaces on $\mathbf{U}$ and the L^p spaces on the boundary of $\mathbf{U}$ with different weights, we prove that for some weights satisfying Békollé's B_α^p condition, the Bergman operator induces a bounded projection on the weighted mixed norm space on $\mathbf{U}$. Thus we are able to identify the dual of those weighted mixed norm spaces of analytic functions.

To my parents and my wife.

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INTRODUCTION

The purpose of this work is to study several operators acting on the spaces of functions in a homogeneous space $(\mathbf{X}, \nu, d)$.

A homogeneous space $(\mathbf{X}, \nu, d)$ can be defined as a quasi-metric space $(\mathbf{X}, d)$ with a positive measure ν on $\mathbf{X}$ satisfying the following condition:

There is a constant C_ν, $C_\nu > 1$, such that

$$0 < \nu(B(x, 2r)) \leq C_\nu \nu(B(x, r)) \leq \infty$$

for all $r > 0$ and any $x \in \mathbf{X}$, where $B(x, r) = \{y | d(x, y) < r\}$.

We shall study the following operators.

1. **Hörmander maximal operator** H_ν. An operator defined on the space of locally integrable functions on $\mathbf{X}$ which maps a function f on $\mathbf{X}$ to a function $H_\nu f$ on $\mathbf{X} \times \mathbf{R}^+$:

   $$(H_\nu f)(x, t) = \sup \frac{1}{\nu(B(y, s))} \int_{B(y,s)} |f(u)| d\nu(u),$$

   where the supremum is taken over all balls $B(y, s) \supset B(x, t)$.

2. **maximal averaging operator** $M_{\nu,r}$. An operator defined on the space of locally integrable functions on $\mathbf{X}$ which maps a function f on $\mathbf{X}$ to a function $M_{\nu,r} f$ on $\mathbf{X}$.

   $$M_{\nu,r} f(x) = \sup_{t \geq r} \frac{1}{\nu(B(x, t))} \int_{B(x,t)} |f(u)| d\nu(u).$$

3. **The analytic embedding operator** I. The restriction of the identity operator to the Bergman spaces in the unit ball $\mathbf{U}$ of $\mathbf{C}^N$:

$$If = f.$$

4. **The Bergman operator** T_β. An operator defined on the space of integrable functions on the unit ball of $\mathbf{C}^N$:

$$T_\beta f(z) = \begin{pmatrix} N+\beta \\ N \end{pmatrix} \int_{\mathbf{U}} K_\beta(z,w) f(w) dm_\beta(w), \qquad z \in \mathbf{U},$$

where

$$K_\beta(z,w) = (1 - \langle z, w \rangle)^{-N-1-\beta} \quad \beta > -1.$$

The first half of this paper is devoted to the study of the maximal operators H_ν and $M_{\nu,r}$. The problem that we are concerned with is to characterize those measures μ defined on $\mathbf{X}^+$ in the H_ν case and on $\mathbf{X}$ in the $M_{\nu,r}$ case, respectively, such that the corresponding operator is bounded from $L^p(\Omega)$ to $L^q(\mu)$, where Ω is a "weighted measure" on $\mathbf{X}$ defined by $d\Omega = \omega d\nu$ with a positive weight function ω. We shall refer to these two problems as problem I and problem II, respectively.

We first consider the Hörmander operator.

For $\omega = 1$, the unweighted case, when $1 < p = q < \infty$, the solutions of problem I are known as the "Carleson measures". In [7], Carleson characterized those finite positive measures μ on the unit ball $\mathbf{U}$ in $\mathbf{C}^1$ such that

$$\left(\int_{\mathbf{U}} |U(z)|^p d\mu\right)^{1/p} \leq C \|f\|_{H^p}$$

for every function f in the Hardy space H^p $(0 < p < \infty)$, where $U(z)$ is the Poisson integral of f. He showed that the above inequality holds if and only if $\mu(S) \leq Ch$ for every set of the form

$$S = \{re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\}.$$

Such a measure μ is now often called a Carleson measure. In order to generalize Carleson's result, Hõrmander [11] introduced the operator H_ν. Using the Marcinkiewicz interpolation theorem and a simple covering argument, he proved that the Carleson measures are the solutions to the problem I when $1 < p = q < \infty$.

In [9], Duren extended Carleson's theorem to the indexes $0 < p \leq q < \infty$. He proved that, for $0 < p \leq q < \infty$

$$\left(\int_{\mathbf{U}} |U(z)|^q d\mu\right)^{1/q} \leq C\|f\|_{H^p}$$

for every f in H^p, if and only if $\mu(S) \leq Ch^\alpha$, where $1 \leq \alpha = q/p$. Such a measure is called an α -Carleson measure.

In general, an α -Carleson measure on $\mathbf{X}^+$ with respect to a positive Borel measure λ on $\mathbf{X}$ is a measure μ on $\mathbf{X}^+$ such that

$$|\mu|(T(B(x,t))) \leq C[\lambda(B(x,t))]^\alpha,$$

where

$$T(B(x,t)) = \{(y,s) \in \mathbf{X}^+ | B(y,s) \subset B(x,t)\}$$

is the "tent" over the ball $B(x,t)$. We shall see that, using Hõrmander's idea, it is not hard to show that if $1 < p \leq q < \infty$ and $\alpha = q/p$, then the α -Carleson measures are the solutions to the problem I.

For the weighted case, when $p = q$, $\mathbf{X} = \mathbf{R}^N$ and $\nu = m$, where m is the Lebesgue measure, the problem I has been solved by Francisco J. Ruiz and José L. Torrea [21].

In the case ω satisfies Muckenhoupt's A_p condition, it will be shown that, similar to the unweighted case, the solutions to the range $1 < p \leq q < \infty$ are the α -Carleson measures with respect to Ω.

The difficult part is the case when $0 < q < p < \infty$. It is natural to guess that the solution must be an extension of α -Carleson measure with respect to Ω. Using the concept of the "balayée" of a measure μ as employed by E. Amar and A. Bonami [1], we are able to prove the following theorem which is contained in Theorem 2.9:

Theorem 1 *Let $0 < \alpha < 1$, and let $q > 0$, $p > 1$, $q/p = \alpha$. Let μ be a positive measure on $\mathbf{X}^+$. Suppose $\omega \in A_p$ and set $d\Omega = \omega d\nu$. Then there is a constant C such that*

$$\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$ if and only if

$$\sup_{r>0} \frac{\mu(TB(x,r))}{\Omega(B(x,r))} \in L^{\frac{1}{1-\alpha}}(\Omega). \tag{1}$$

Note that if $q = p$, then $\alpha = 1$ and the condition (1) shows that μ is an α-Carleson measure with respect to Ω. Therefore we have an unified approach to the solutions of problem I.

The above result enables us to extend Carleson's theorem to the weighted Hardy spaces $H^p(\Omega)$ with p, q in the range $1 < p \leq q < \infty$ and in the range $p > 1$, $0 < q < p$. It turns out that the solutions to the Carleson measure problem on the weighted Hardy spaces $H^p(\Omega)$ are the same as the solutions of problem I. The results for unweighted Hardy spaces when $0 < q < p < \infty$ were obtained by Videnskii [26] in the one dimension case and by Luecking [15] in higher dimension case.

Now we consider the maximal averaging operator.

In order to study problem II in a general homogeneous space, we first introduce the following concept:

We shall call a measure μ on $\mathbf{X}$ an α-Carleson measure with respect to a positive measure λ on $\mathbf{X}$ if there exits a fixed $r > 0$ and a constant C_r such that for any ball $B(x, r)$ centered at x with radius r,

$$|\mu|(B(x,r)) \leq C_r[\lambda(B(x,r))]^\alpha.$$

The reason to call such a measure an α-Carleson measure is that V. L. Oleinik and B. S. Pavlov [18] have proved the following theorem which is an analogue of the Carleson's theorem mentioned in the discussion of the Hörmander maximal operator:

Suppose $\mathbf{U}$ *is the unit ball of* $\mathbf{C}^1$. *Then for* $1 < p \leq q < \infty$,

$$\left(\int_{\mathbf{U}} |f|^q d\mu\right)^{1/q} \leq C\left(\int_{\mathbf{U}} |f|^p dm\right)^{1/p}$$

if and only if

$$\mu(E(z)) \leq C[m(E(z))]^\alpha$$

for every $z \in \mathbf{U}$ *and any function* f *in the Bergman space* A^p , *where* $E(z)$ *is a "suitable" subset of* $\mathbf{U}$ *and* $\alpha = q/p \geq 1$.

Similar characterizations were studied by Hastings [10] for the polydics $\mathbf{D}^\mathbf{N}$ and by Cima and Wogen [8] for the unit ball $\mathbf{U}$ of $\mathbf{C}^\mathbf{N}$.

We shall refer to the problem of characterizing those measures μ on $\mathbf{X}$ such that the inequality

$$\|f\|_{L^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$$

holds for all functions in weighted Bergman spaces $A_p(\Omega)$ as the "Carleson measure problem on $\mathbf{X}$".

The reason to study the operator $M_{\nu,r}$ is that many functions, for example subharmonic functions, are controlled by the operator $M_{\nu,r}$ and that Carleson measures can be applied in the study of the operator $M_{\nu,r}$.

Applying similar ideas used in the study of problem I, one can show that, in a homogeneous space, if ω satisfies the condition A_p, then the solutions to problem II when $1 < p \leq q < \infty$ are those measures μ on $\mathbf{X}$ satisfying

$$|\mu|(B(x,r)) \leq C_r[\Omega(B(x,r))]^{\alpha}$$

for any x with fixed r.

When $p > 1$ and $q < p$, we prove the following characterization theorem which is contained in Theorem 3.11:

Theorem 2 *Suppose μ is a positive measure on $\mathbf{X}$ and suppose $\omega \in A_p$. Let $q > 0$, $p > 1$, $q/p = \alpha < 1$. Then*

$$\left(\int |M_{\nu,r}f|^q d\mu\right)^{1/q} \leq C\left(\int |f|^p d\Omega\right)^{1/p} \qquad f \in L^p(\Omega),$$

if and only if

$$\frac{\mu(B(x,r))}{\Omega(B(x,r))} \in L^{\frac{1}{1-\alpha}}(\Omega) < \infty. \tag{2}$$

When $p = q$, then $\alpha = 1$ and the condition (2) implies that μ is an α-Carleson measure on $\mathbf{X}$. Therefore we have reached an unified approach to the solution of problem II.

Using the method of the proof of Theorem 2, we are able to characterize those measures on a general homogeneous space such that the Hardy-Littlewood maximal operator is bounded from $L^p(\Omega)$ to $L^q(\mu)$ $(1 < p < \infty, 0 < q < \infty)$. When $1 < p \leq q < \infty$ and $\mathbf{X} = \mathbf{R}^N$, such a characterization have been obtained by E. Sawyer [24].

In the second half of this paper, we shall restrict ourselves to a special homogeneous space, the unit ball $\mathbf{U}$ of $\mathbf{C}^N$. We shall always consider the measure defined by $dm_\beta = (1-r^2)^\beta dm$, $\beta > -1$, as the "unweighted" measure in $\mathbf{U}$, and shall refer to the "weighted" measure as the form $d\Omega = \omega dm_\beta$.

As in the previous discussion, we have seen that Carleson measures play an important role in the study of maximal operators. Our third problem, which will be referred to as problem III, is to determine the sufficient and necessary conditions under which the embedding operator is bounded from $A^p(\Omega)$ to $L^q(\mu)$. This is, in fact, equivalent to solving the Carleson measure problem on $\mathbf{U}$, or, to set up a Carleson measure theorem in the weighted Bergman spaces.

As we have mentioned before, for the unweighted case, when $1 < p \leq q < \infty$, the problem III was solved by Oleinik and Pavlov [18] in the one dimension case. The higher dimension case was solved by Cima and Wogen [8] for $q = p = 2$, and was generalized by Luecking [14] to $0 < p \leq q < \infty$. When $1 < q < p < \infty$, it was solved by Luecking [15].

In the weighted case, a general technique to find a sufficient condition such that

$$\|f\|_{A^p(\mu)} \leq C\|f\|_{A^p(\Omega)}$$

was obtained by Luecking [14].

In this paper, we solve the problem for those weights ω satisfying Békollé's $B^p_\beta(\omega)$ conditions studied by Békollé in [3]. We prove the following theorem which will be restated as Theorem 4.7 in chapter 4:

Theorem 3 *Let $\alpha \geq 1$ and let $1 < p \leq q < \infty$ such that $q/p = \alpha$. Suppose ω satisfies the $B^p_\beta(\omega)$ condition. Then*

$$\|f\|_{A^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$$

for any f in the weighted Bergman spaces $A^p(\Omega)$ if and only if there is a r, $1 > r > 0$, such that

$$\mu(E(a,r)) \leq C_r[\Omega(E(a,r))]^\alpha$$

for any $a \in \mathbf{U}$, where $E(a,r)$ is the psudohyperbolic ball centered at x with radius r.

The last problem in this paper concerns the boundedness of the Bergman operator on the weighted mixed norm spaces in the unit ball of $\mathbf{C}^N$. We shall refer this problem to problem IV.

In [3], Békollé found a necessary and sufficient condition for weight functions such that the Bergman operator is bounded on the corresponding weighted L^p spaces in the unit ball of $\mathbf{C}^N$. In [13], M. Jevtić proved that there are bounded projections from general mixed norm spaces onto the weighted mixed norm spaces of analytic functions with the normal-function weights. The projections he studied are very similar to the Bergman operator. Here, we show that the Bergman operator is bounded on weighted L^p spaces on the boundary of the unit ball of $\mathbf{C}^N$ with normal-function weights. Then we determine the weighted mixed norm spaces on the unit ball of $\mathbf{C}^N$ as the interpolation spaces between weighted L^p spaces on the unit ball of $\mathbf{C}^N$ and the weighted L^p spaces on the boundary of the unit ball $\mathbf{C}^N$ with different weights. These facts enable us to prove that the Bergman operator is bounded on weighted mixed norm spaces with radial weights satisfying Békollé's conditions. The main result of this part is the following theorem which is contained in Theorem 5.16:

Theorem 4 *Suppose $p \leq q \leq \infty$, $1 < p < \infty$, and that φ is a normal function.If a radial function $\omega(r)$ on [0,1) satisfies condition $B^p_\alpha(\varphi^p(r)\omega(r))$:*

$$\int_{1-h}^1 \omega(r)\varphi^p(r)(1-r^2)^\alpha r^{2N-1}dr$$
$$\times \ \left(\int_{1-h}^1 \omega^{-\frac{p'}{p}}(r)\varphi^{-p'}(r)(1-r^2)^\alpha r^{2N-1}dr\right)^{\frac{p}{p'}} \leq Ch^{(\alpha+1)p}$$

for all $0 < h < 1$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Then, for $\frac{1}{q} + \frac{1}{q'} = 1$,

(1) T_α *is bounded on* $L^{p,q}(\varphi^q \omega (1-r^2)^\alpha)$;

(2) T_α *is bounded on* $L^{p',q'}(\varphi^{-q'} \omega^{-\frac{q'}{q}} (1-r^2)^\alpha)$.

As an application, we show that Jevtić's result is a special case of our result. Using the Bergman operator, we have obtained several duality theorems of weighted mixed norm spaces.

Our exposition is organized in the following way.

We start by introducing the homogeneous spaces and analytic function spaces on the unit ball of $\mathbf{C}^N$ and some of their basic properties, the A_p and B_α^p weights, definitions and notations of operators and the concept of a "balayée" of a measure. This is done in chapter 1, immediately after this introduction.

In chapter 2, we first collect some results concerning the α-Carleson measures on $\mathbf{X}^+$ with $\alpha \geq 1$. Then we present the main result concerning the boundedness of the Hörmander operator from $L^p(\Omega)$ to $L^q(\mu)$ when $q < p$. The extension of the Carleson measure theorem in weighted Hardy spaces is presented in the last section of chapter 2.

Chapter 3 is devoted to study the maximal averaging operator.

A Carleson measure theorem in the weighted Bergman space is presented in chapter 4. As its application, we discuss the multipliers between different weighted Bergman spaces.

The last chapter is devoted to the study of the Bergman operator in weighted mixed norm spaces.

Chapter 1

PRELIMINARY

We introduce the homogeneous space and some of its basic properties in the first section. Some notations and basic facts concerning the analytic functions in the unit ball of $\mathbf{C}^N$ are presented in the second section.

§1.1 Homogeneous Space

Let $\mathbf{X}$ be a topological space with a positive measure ν. Let d be a real-valued function in $\mathbf{X} \times \mathbf{X}$. We shall call the triple $(\mathbf{X}, \nu, d)$ a homogeneous space if it satisfies the following conditions:

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x) > 0 \qquad \text{if } x \neq y$;
3. there is a constant C_d such that $d(x, z) \leq C_d[d(x, y) + d(y, z)]$ for all x, y and z;
4. given a neighborhood N of a point x, there is a r, $r > 0$, such that the sphere $B(x, r) = \{y | d(x, y) < r\}$ with center at x is contained in N;

5. the spheres $B(x,r) = \{y | d(x,y) < r\}$ are measurable and there is a constant C_ν, $C_\nu > 1$, such that

$$0 < \nu(B(x,2r)) \leq C_\nu \nu(B(x,r)) \leq \infty$$

for all r and x.

A measure satisfying condition 5 is called a doubling measure. The doubling measure ν has the following property:

For any $K > 0$, there is a constant $C_K > 0$ such that

$$\nu(B(x,Kr)) \leq C_K \nu(B(x,r))$$

for all x and r.

The family of balls in a homogeneous space satisfies the following geometric properties:

Lemma 1.1 *Let $a > 0$. Then there is a constant $C > 0$ such that if $r \leq ar'$ and $B(x,r) \cap B(y,r') \neq \phi$, then $B(x,r) \subset B(y,Cr')$.*

Lemma 1.2 *Let F be a family of $\{B(x,r)\}$ of balls with bounded radii. Then there is a countable subfamily $\{B(x_i,r_i)\}$ consisting of pairwise disjoint balls such that each ball in F is contained in one of the balls $B(x_i,br_i)$, where $b = 3C_d^2$ and C_d is the constant in condition 3 .*

For the proof of Lemma 1.1 and 1.2 , see A. P. Calderón [6].

Lemma 1.3 *Let μ be a positive measure in $\mathbf{X}$. Let $\alpha \geq 1$. If there is a $r_0 > 0$ such that $\mu(B(x,r_0)) \leq C[\nu(B(x,r_0))]^\alpha$ for any $x \in \mathbf{X}$, then for any $r \geq r_0$,*

$$\mu(B(x,r)) \leq CC_b^\alpha[\nu(B(x,r))]^\alpha,$$

where C_b depends only on the constant b in Lemma 1.2 and the constant C_d in condition 3 of the definition of homogeneous space.

Proof: Let $r > r_0$ and let

$$E = \{B(y, \frac{r_0}{b}) : y \in B(x,r)\},$$

where b is as in Lemma 1.2. Then

$$B(x,r) \subset \cup_E B(y, \frac{r_0}{b}).$$

By Lemma 1.2 , there exists $\{y_i\} \subset B(x,r)$ such that $B(x,r) \subset \cup B(y_i, r_0)$ and $\{B(y_i, \frac{r_0}{b})\}_{i=1}^{\infty}$ is a disjoint family. Note that

$$\cup B(y_i, \frac{r_0}{b}) \subset B(x, C_d(r + \frac{r_0}{b})) \subset B(x, 2C_d r)$$

since we may assume $b \geq 1$. By the doubling property of ν, there is a C_1 such that

$$\nu(B(y_i, r_0)) \leq C_1 \nu(B(y_i, \frac{r_0}{b})).$$

Thus

$$\begin{aligned}
& \mu(B(x,r)) \\
\leq\ & \sum_{i=1}^{\infty} \mu(B(y_i, r_0)) \\
\leq\ & C \sum_{i=1}^{\infty} [\nu(B(y_i, r_0))]^{\alpha} \\
\leq\ & C C_1^{\alpha} [\sum_{i=1}^{\infty} \nu(B(y_i, \frac{r_0}{b}))]^{\alpha} \\
\leq\ & C C_1^{\alpha} [\nu(B(x, 2C_d r))]^{\alpha} \\
\leq\ & C C_b^{\alpha} [\nu(B(x,r))]^{\alpha}.
\end{aligned}$$

The proof is complete.

Suppose $\omega(x) \geq 0$ is a positive locally integrable function on $\mathbf{X}$. We say that a measure Ω, defined by $d\Omega = \omega d\nu$, satisfies Muckenhoupt's A_p condition relative to ν if for any ball B,

$$\int_B \omega d\nu [\int_B \omega^{-\frac{1}{p-1}} d\nu]^{p-1} \leq C_\omega [\nu(B)]^p \qquad 1 < p < \infty;$$

$$\int_B \omega d\nu \leq C_\omega \nu(B) essinf_{x\in B} \omega(x) \qquad p = 1.$$

Note that if ω satisfies the condition A_p for some $p > 1$, then Ω is a doubling measure. In fact, by Hõlder's inequality and the fact that ν is a doubling measure, we have

$$\begin{aligned}
& \Omega(B(x, 2r)) \\
= & \int_{B(x,2r)} \omega d\nu \\
\leq & C_\omega \frac{[\nu(B(x,2r))]^p}{[\int_{B(x,2r)} \omega^{-\frac{1}{p-1}} d\nu]^{p-1}} \\
\leq & C_\omega C_\nu \frac{[\nu(B(x,r))]^p}{[\int_{B(x,r)} \omega^{-\frac{1}{p-1}} d\nu]^{p-1}} \\
\leq & C_\omega C_\nu \frac{\int_{B(x,r)} \omega d\nu [\int_{B(x,r)} \omega^{-\frac{1}{p-1}} d\nu]^{p-1}}{[\int_{B(x,r)} \omega^{-\frac{1}{p-1}} d\nu]^{p-1}} \\
= & C_\omega C_\nu \int_{B(x,r)} \omega d\nu \\
= & C_\omega C_\nu \Omega(B(x,r))
\end{aligned}$$

for any $x \in \mathbf{X}$ and $r > 0$.

By Hõlder's inequality, the condition A_p implies the condition A_q if $q > p$. In [6], A. P. Calderón proved the following theorem:

Theorem 1.4 *Suppose that all continuous functions with bounded support is dense in $L^1(\nu)$, then the A_p condition implies the A_γ condition for some $\gamma < p$.*

In this paper, we shall always assume that the class of compactly supported continuous functions is dense in the space of integrable functions $L^1(\nu)$.

Definition 1.5 *Let Ω be any positive measure on $\mathbf{X}$. The Hardy-Littlewood maximal operator is defined by*

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{\Omega(B(x,t))} \int_{B(x,t)} |f| d\Omega.$$

Let $\mathbf{X}^+ = \mathbf{X} \times \mathbf{R}^+$ with the product topology. Denote

$$T(B(x,t)) = \{(y,s) \in \mathbf{X}^+ | B(y,s) \subset B(x,t)\}.$$

Let Ω be a positive measure on $\mathbf{X}$. Following the notation of E. Amar and A. Bonami [1], for $0 \leq \alpha < \infty$, we shall call a Borel measure μ on $\mathbf{X}^+$ an α-Carleson measure relative to Ω if

$$|\mu|(T(B(x,t))) \leq C[\Omega(B(x,t))]^\alpha.$$

Definition 1.6 *Let Ω be a positive measure on $\mathbf{X}$. For $f \geq 0$, define*

$$S_\Omega(x,y,t) = \frac{1}{\Omega(B(x,t))} \chi_{B(x,t)}(y).$$

$$(S_\Omega f)(x,t) = \int_{\mathbf{X}} S_\Omega(x,y,t) f(y) d\Omega(y).$$

Definition 1.7 *The Hörmander maximal operator is defined by*

$$(H_\Omega f)(x,t) = \sup \frac{1}{\Omega(B(y,s))} \int_{B(y,s)} |f(u)| d\Omega(u),$$

where the supremum is taken over all balls $B(y,s) \supset B(x,t)$.

Definition 1.8 *The nontangential maximal operator on $\mathbf{X}^+$ is defined by*

$$N(u)(x) = \sup\{|u(y,t)| : d(x,y) \leq t\} = \sup\{|u(y,t)| : (y,t) \in \Gamma(x)\},$$

where u is a function in $\mathbf{X}^+$ and

$$\Gamma(x) = \{(y,t) : d(y,x) \leq t\}.$$

Definition 1.9 *The weighted Hardy space is defined by*

$$H^p(\Omega) = \{u : u \textit{ is harmonic in } \mathbf{R}_+^{N+1},\ N(u)(x) \in L^p(\Omega)\}$$

with $\|u\|_{H^p(\Omega)} = \|N(u)\|_{L^p(\Omega)}$.

Definition 1.10 *Let* $0 \le \alpha < \infty$ *and let* μ *be a Borel measure on* $\mathbf{X}^+$. *Define*

$$S_\Omega^*\mu(y) = \int_{\mathbf{X}^+} S_\Omega(x,y,t)d\mu(x,t).$$

$$V_\Omega^\alpha = \{\mu : |\mu|T(B(x,t)) \le C[\Omega(B(x,t))]^\alpha\}.$$

$$W_\Omega^\alpha = \{\mu : S_\Omega^*|\mu| \in L^{\frac{1}{1-\alpha}}(\Omega)\}.$$

We shall call $S_\Omega^*|\mu|$ the balayée of μ with respect to Ω. For $0 < \alpha < 1$, W_Ω^α is the complex interpolation space $(V_\Omega^0, V_\Omega^1)_\alpha$ (see [1]).

§1.2 Analytic Function Spaces on the Unit Ball of $\mathbf{C}^N$

Let $\mathbf{U}$ denote the unit ball in $\mathbf{C}^N$, $N \ge 1$. Denote by m Lebesgue measure on $\mathbf{C}^N = \mathbf{R}^{2N}$ normalized so that $m(\mathbf{U}) = 1$. For $\alpha \ge -1$, let $dm_\alpha = c_\alpha(1-|z|^2)^\alpha dm$ with c_α chosen so that when $\alpha > -1, m_\alpha(\mathbf{U}) = 1$. Denote by ν_0 the surface measure on the boundary $\mathbf{S}$ of $\mathbf{U}$ normalized so that $\nu_0(\mathbf{S}) = 1$.

A positive continuous function $\varphi(r)$ on [0,1) is normal if there exist a, b, $0 < a < b$, such that

$$\begin{aligned} &(i)\ \tfrac{\varphi(r)}{(1-r)^a} \textit{ is non-increasing},\ \lim_{r\to 1^-} \tfrac{\varphi(r)}{(1-r)^a} = 0. \\ &(ii)\ \tfrac{\varphi(r)}{(1-r)^b} \textit{ is non-decreasing},\ \lim_{r\to 1^-} \tfrac{\varphi(r)}{(1-r)^b} = \infty. \end{aligned} \tag{1.1}$$

We shall denote $\hat{b} = inf\{b : b \ satisfies \ (ii) \ of \ (1.1)\}$.

The functions $\{\varphi \ , \psi \ \}$ will be called a normal pair if φ is normal and if for some b satisfying (1.1), there exists $\lambda > b$, such that

$$\varphi(r)\psi(r) = (1 - r^2)^\lambda \qquad\qquad 0 \leq r < 1. \tag{1.2}$$

If φ is normal, then there exists ψ such that $\{ \ \varphi \ , \psi \ \}$ is a normal pair and then ψ is normal [23].

For $z = (z_1, z_2, \ldots, z_N)$ and $w = (w_1, w_2, \ldots, w_N)$ in $\mathbf{C}^N$, let

$$< z, w >= \sum_{i=1}^{N} z_i \bar{w}_i$$

so that $|z|^2 =< z, z >$. Following [19], for $a \in \mathbf{U}$, $a \neq 0$, let Φ_a denote the automorphism of $\mathbf{U}$ taking 0 to a defined by

$$\Phi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{\frac{1}{2}} Q_a z}{1- < z, a >},$$

where P_a is the projection of $\mathbf{C}^N$ onto the one-dimensional subspace spanned by a and $Q_a = I - P_a$.

For $a \in \mathbf{U}$, let

$$K(a) \equiv \{\Phi_a(z) : Re < z, a > \leq 0\},$$

then [20]

$$m_\alpha(K(a)) \sim (1 - |a|^2)^{\alpha+N+1}.$$

Define the pseudohyperbolic metric ρ on $\mathbf{U}$ by

$$\rho(z, \xi) = |\Phi_\xi(z)|.$$

For $0 < r < 1$, let

$$E(a,r) = \{z \in \mathbf{U} : \rho(z,a) < r\}.$$

Then we have

$$m_\alpha(E(a,r)) \sim m_\alpha(K(a)) \sim (1-|a|)^{N+1+\alpha}.$$

For basic properties of $K(a)$ and $E(a,r)$, see [16] and [20].

Let $1 < p < \infty$. For a positive function $\omega \in L^1(\mathbf{U}, dm_\alpha)$, the $B^p_\alpha(\omega)$ condition is the following:

There is a constant C such that for every $K = K(a)$, $a \in \mathbf{U}$,

$$\int_K \omega dm_\alpha \Big(\int_K \omega^{-\frac{p'}{p}} dm_\alpha\Big)^{\frac{p}{p'}} \leq C m^p_\alpha(K),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In the case ω is a radial function, that is, $\omega(r)$ is a measurable function on $[0,1)$, using the fact that $K(a)$ is "nearly"

$$S(\zeta,h) = \{z \in \mathbf{U} : |1 - <z,\zeta>| < h\}$$

for $\zeta = \frac{a}{|a|}$, $h = 1-|a|$ (see [16, p.321]), the condition $B^p_\alpha(\omega)$ can be written in the form

$$\begin{aligned}&\int_{1-h}^1 \omega(r)(1-r^2)^\alpha r^{2N-1} dr \\ \times\ &\Big[\int_{1-h}^1 \omega(r)^{-\frac{p'}{p}}(1-r^2)^\alpha r^{2N-1} dr\Big]^{\frac{p}{p'}} \leq C h^{(\alpha+1)p}.\end{aligned}$$

Let $1 < p < \infty$. For a positive function $\omega \in L^1(\mathbf{U}, dm_\alpha)$, the C_p condition is the following:

There is a constant C such that for any $E = E(a,r)$, $a \in \mathbf{U}$,

$$\int_E \omega dm \left(\int_E \omega^{-\frac{p'}{p}} dm\right)^{\frac{p}{p'}} \leq Cm^p(E).$$

The condition C_p is a consequence of B^p_α for any $\alpha > -1$ [16].

Let $A(z)$ be a non-negative measurable function on $\mathbf{U}$ and $B(r)$, $C(r)$ be non-negative measurable functions on $[0,1)$ such that

$$|\{r \in [0,1) : C(r) = 0\}| = 0,$$

where $|E|$ denotes the Lebesgue measure of E in $\mathbf{R}^1$. For a measurable function f on $\mathbf{U}$ and $z \in \mathbf{S}$, let

$$\|f_r\|^p_{A,p} = \int_{\mathbf{S}} |f(rz)|^p A(rz) d\nu_0(z), \qquad 0 \leq r < 1,\ 1 \leq p \leq \infty.$$

Since $|f(rz)|^p A(rz)$ is a measurable function on $\mathbf{U}$, $\|f_r\|^p_{A,p}$ is a measurable function on $[0,1)$ (see [20, p.150]).

Definition 1.11 *Let*

$$\begin{aligned} \|f\|^q_{L^{p,q}(A,B)} &= \int_0^1 \|f_r\|^q_{A,p} B(r) r^{2N-1} dr \qquad 1 \leq q < \infty, \\ \|f\|_{L^{p,\infty}(A,C)} &= \sup_{r \in [0,1)} \|f_r\|_{A,p} C(r). \end{aligned}$$

The mixed norm spaces are defined by

$$\begin{aligned} L^{p,q}(A,B) &= \{f : \|f\|_{A,B,p,q} < \infty\}. \\ L^{p,\infty}(A,C) &= \{f : \|f\|_{A,C,p,\infty} < \infty\}. \end{aligned}$$

We shall denote $H(\mathbf{U})$ the space of analytic functions on $\mathbf{U}$ and

$$H^{p,q}(A,B) = L^{p,q}(A,B)\bigcap H(\mathbf{U});$$

$$L^{p,\infty}(A,C) = L^{p,\infty}(A,C)\bigcap H(\mathbf{U}).$$

In the case $A = A(r)$ is a radial function, and $B(r) = \omega(r)(1-r^2)^\alpha$, $C(r) \equiv 1$, denote

$$L^{p,q}(A^{q/p}\omega(1-r^2)^\alpha) = L^{p,q}(A,B),$$

$$L^{p,\infty}(A) = L^{p,\infty}(A,C),$$

and

$$H^{p,q}(A^{q/p}\omega(1-r^2)^\alpha) = H^{p,q}(A,B),$$

$$H^{p,\infty}(A) = H^{p,\infty}(A,C).$$

In the case $A(z) = \omega(z)$, $B = (1-r^2)^\alpha$, $C(r) \equiv 1$, and $p = q$, we have

Definition 1.12 *The weighted Bergman spaces are defined by*

$$A^p(\omega dm_\alpha) = H^{p,p}(\omega(z),(1-r^2)^\alpha).$$

Let

$$K_\alpha(z,w) = (1-<z,w>)^{-N-1-\alpha}$$

with $\alpha > -1$, z, $w \in \mathbf{U}$. The Bergman operator T_α is defined by [19]

$$T_\alpha f(z) = \begin{pmatrix} N+\alpha \\ N \end{pmatrix} \int_{\mathbf{U}} K_\alpha(z,w)f(w)dm_\alpha(w) \qquad z \in \mathbf{U}.$$

Define

$$T_\alpha^* f(z) = \begin{pmatrix} N+\alpha \\ N \end{pmatrix} \int_{\mathbf{U}} |K_\alpha(z,w)| f(w) dm_\alpha(w) \qquad z \in \mathbf{U}.$$

Note that T_α^* is a linear operator.

In [3], B. Békollé proved the following:

Theorem 1.13 *T_α^* is bounded on $L^p(\omega dm_\alpha)$ if and only if ω satisfies $B_\alpha^p(\omega)$ condition.*

Chapter 2

HÖRMANDER MAXIMAL OPERATOR AND CARLESON MEASURES ON $\mathbf{X}^+$

In this chapter, we restrict ourselves to the space $\mathbf{X}^+ = \mathbf{X} \times \mathbf{R}^+$ where $\mathbf{X}$ is a homogeneous space. We study the characterization of measures μ on $\mathbf{X}^+$ such that the inequality $\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ holds for the maximal operator H_ν studied by Hõrmander. The solution when $q < p$ utilizes the concept of the "balayée" of the measure μ. Using this characterization we extend Duren's Carleson measure theorem to the weighted Hardy spaces.

In the first section we collect the results for α-Carleson measures with $\alpha \geq 1$. We shall prove the main result of this chapter in section 2 and section 3. In the last section we shall prove a Carleson measure theorem on weighted Hardy space.

§2.1 α-Carleson Measures on $\mathbf{X}^+$ with $\alpha \geq 1$

In this section, we always assume μ is a positive measure.

The method of the proof of following theorem is essentially due to Hõrmander [11], which gives a relation between an α-Carleson measure and the L^q-norm of the operator H_Ω.

Theorem 2.1 *Let $\alpha \geq 1$, $p > 1$. Suppose that Ω is a positive doubling measure on* $\mathbf{X}$. *Then $\mu \in V_\Omega^\alpha$ if and only if*

$$\|H_\Omega f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}, \qquad f \in L^p(\Omega)$$

where $q/p = \alpha$.

Proof: That $\|H_\Omega f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ implies $\mu \in V_\Omega^\alpha$ follows from the standard argument by taking $f = \chi_{B(x,t)}$.

For each $n > 0$, we define

$$(H_\Omega^n f)(x,t) = \sup_{s \leq n, B(y,s) \supset B(x,t)} \frac{1}{\Omega(B(y,s))} \int_{B(y,s)} |f(u)| d\Omega(u)$$

and we shall show that the inequality above holds with H_Ω replaced by H_Ω^n with C independent of n. Once this is established, the theorem will follow by letting n tend to infinity.

It is clear that H_Ω^n is of type (∞, ∞). If we can show that H_Ω^n is also of weak type $(1, \alpha)$, then the conclusion will follow from Marcinkiewicz interpolation theorem.

Let $\lambda > 0$ and let $E = \{(x,t) \in \mathbf{X}^+ : H_\Omega^n f(x,t) > \lambda\}$. For each $(x,t) \in E$, there is a ball $B(y,r)$ containing x such that $n \geq r \geq t$ and

$$\frac{1}{\Omega(B(y,r))} \int_{B(y,r)} |f(u)| d\Omega(u) > \lambda.$$

Let $\mathbf{B}$ be the collection of all such balls and let $\{B(y_i, r_i)\}$ be the countable subfamily of pairwise disjoint balls of $\mathbf{B}$ as in Lemma 1.2 . Then $\bigcup_{\mathbf{B}} B(y,r) \subset \bigcup B(y_i, br_i)$ and that each $B \in \mathbf{B}$ is contained in one of $B(y_i, br_i)$.

It is clear that $E \subset \bigcup T(B(y_i, br_i))$. Therefore

$$\begin{aligned}
\mu(E) &\leq \mu(\bigcup T(B(y_i, br_i))) \\
&\leq \sum \mu(T(B(y_i, br_i))) \\
&\leq C \sum (\Omega(B(y_i, br_i)))^\alpha \\
&\leq C \sum (\Omega((B(y_i, r_i)))^\alpha \\
&\leq \frac{C}{\lambda^\alpha} \sum (\int_{B(y_i, r_i)} |f| d\Omega)^\alpha \\
&\leq \frac{C}{\lambda^\alpha} (\sum \int_{B(y_i, r_i)} |f| d\Omega)^\alpha \\
&\leq \frac{C}{\lambda^\alpha} (\int |f| d\Omega)^\alpha.
\end{aligned}$$

That is H_Ω^n is of weak type $(1, \alpha)$. The conclusion follows.

Next we give a similar estimate to the operator H_ν.

Let $\gamma > 1$ and $d\Omega = \omega d\nu$. If $\omega \in A_\gamma$, by Hõlder's inequality, it is easy to show that

$$(H_\nu f)(x,t) \leq C[H_\Omega(|f|^\gamma)]^{1/\gamma},$$

where C only depends on the A_γ condition. Thus we have:

Theorem 2.2 *Let $\alpha \geq 1$. If $\omega \in A_p$ and let $d\Omega = \omega d\nu$, then $\mu \in V_\Omega^\alpha$ if and only if*

$$\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}, \qquad f \in L^p(\Omega)$$

for any $p > 1$, $q > 0$, such that $q/p = \alpha$.

Proof: That $\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ implies $\mu \in V_\Omega^\alpha$ follows from the standard argument by taking $f = \chi_{B(x,t)}$.

Now suppose $\mu \in V_\Omega^\alpha$.

Since $p > 1$, by Theorem 1.4, there is a $1 < \gamma < p$, such that $\omega \in A_\gamma$. Note that $\omega \in A_p$ implies that Ω is a doubling measure. Therefore

$$
\begin{aligned}
&\int_{\mathbf{X}^+} |H_\nu f|^q d\mu \\
&\leq C \int_{\mathbf{X}^+} |H_\Omega |f|^\gamma|^{\frac{q}{\gamma}} d\mu \\
&\leq C[\int_{\mathbf{X}^+} |f|^p d\Omega]^{q/p}.
\end{aligned}
$$

The last inequality follows from Theorem 2.1, since $\frac{\frac{q}{\gamma}}{\frac{p}{\gamma}} = q/p = \alpha$ and $\frac{p}{\gamma} > 1$. The proof is complete.

The next lemma is due to E. Amar and A. Bonami [1].

Lemma 2.3 *Let μ be a positive measure on $\mathbf{X}^+$. Let*

$$
g(y) = \int_{\mathbf{X}^+} S_\Omega(x, y, t) d\mu(x, t).
$$

If we define

$$
\lambda(E) = \int_E S_\Omega(1/g)(x, t) d\mu(x, t),
$$

then

$$
\lambda \in V_\Omega^1.
$$

Proof: We need to show that for any ball B

$$
\int_{T(B)} S_\Omega(1/g)(x, t) d\mu(x, t) \leq C\Omega(B).
$$

By definition

$$
\begin{aligned}
I &= \int_{T(B)} S_\Omega(1/g)(x, t) d\mu(x, t) \\
&= \int_{\mathbf{X}^+} \chi_{T(B)}(x, t)[\int_{\mathbf{X}} S_\Omega(x, y, t) \frac{1}{g(y)} d\Omega(y)] d\mu(x, t) \\
&= \int_{\mathbf{X}} \frac{1}{g(y)} [\int_{\mathbf{X}^+} \frac{\chi_{T(B)}(x, t)\chi_{B(x,t)}(y)}{\Omega(B(x, t))} d\mu(x, t)] d\Omega(y).
\end{aligned}
$$

Since $(x,t) \in T(B)$ and $y \in B(x,t)$ imply that $B(x,t) \subset B$ and $y \in B$, then

$$\chi_{T(B)}(x,t)\chi_{B(x,t)}(y) \leq \chi_B(y)\chi_{B(x,t)}(y).$$

Thus

$$\begin{aligned} I &\leq \int_B \frac{1}{g(y)}[\int_{\mathbf{X}^+} \frac{\chi_{B(x,t)}(y)}{\Omega(B(x,t))} d\mu(x,t)]d\Omega(y) \\ &= \int_B d\Omega(y) \\ &= \Omega(B). \end{aligned}$$

The proof is complete.

The last theorem of this section is due to Calderón in [6].

Theorem 2.4 *If $1 < p < \infty$, $d\Omega = \omega d\nu$ with $\omega \in A_p$, then*

$$[\int |M_\nu f|^p d\Omega]^{1/p} \leq C[\int |f|^p d\Omega]^{1/p}$$

for $f \in L^p(\Omega)$.

§2.2 Hörmander Maximal Operator and Space W_Ω^α

The following theorem shows the relation between the Hörmander maximal operator and the space of "balayées".

Theorem 2.5 *Let $0 < \alpha < 1$, and let $q > 0$, $p > 1$, $q/p = \alpha$. Let μ be a positive measure on $\mathbf{X}^+$. Suppose $\omega \in A_p$ and set $d\Omega = \omega d\nu$. If $\mu \in W_\Omega^\alpha$ then there is a constant C such that*

$$\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$.

Conversely, let $0 < q < p < \infty$ *and let* $\alpha = q/p$. *Suppose that* Ω *is a doubling measure on* $\mathbf{X}$. *If*

$$\|S_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$, *then* $\mu \in W_\Omega^\alpha$.

Proof: Suppose $\mu \in W_\Omega^\alpha$ and $q/p = \alpha$, $p > 1$. Let

$$g(y) = \int_{\mathbf{X}+} S_\Omega(x, y, t)d\mu(x, t).$$

Then $\mu \in W_\Omega^\alpha$ implies $g \in L^{\frac{1}{1-\alpha}}(\Omega)$. Note that by Hölder's inequality

$$[S_\Omega(1/g)(x,t)]^{-1} \leq (S_\Omega g)(x,t).$$

If $f \in L^p(\Omega)$, then

$$\begin{aligned}
& \int_{\mathbf{X}+} |H_\nu f|^q d\mu \\
= & \int_{\mathbf{X}+} |H_\nu f|^q [S_\Omega(1/g)(x,t)]^{-1} S_\Omega(1/g)(x,t)d\mu(x,t) \\
\leq & \int_{\mathbf{X}+} |H_\nu f|^q (S_\Omega g)(x,t) S_\Omega(1/g)(x,t)d\mu(x,t) \\
\leq & [\int_{\mathbf{X}+} |H_\nu f|^p S_\Omega(1/g)(x,t)d\mu(x,t)]^{q/p} \\
& \times[\int_{\mathbf{X}+} |(S_\Omega g)(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)d\mu(x,t)]^{1-q/p} \\
\leq & [\int_{\mathbf{X}+} |H_\nu f|^p S_\Omega(1/g)(x,t)d\mu(x,t)]^{q/p} \\
& \times[\int_{\mathbf{X}+} |(H_\Omega g)(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)d\mu(x,t)]^{1-q/p} \\
= & A \times B.
\end{aligned}$$

By Lemma 2.3, $S_\Omega(1/g)(x,t)\mu \in V_\Omega^1$. It follows from Theorem 2.2 that

$$A \leq C[\int_{\mathbf{X}} |f|^p d\Omega]^{q/p}$$

and from Theorem 2.1 that

$$B \leq C[\int_{\mathbf{X}} |g|^{\frac{1}{1-\alpha}} d\Omega]^{1-q/p}.$$

Therefore

$$\begin{aligned}
&\int_{\mathbf{X}+} |H_\nu f|^q d\mu \\
\leq\ & C[\int_{\mathbf{X}} |f|^p d\Omega]^{q/p} [\int_{\mathbf{X}} |g|^{\frac{1}{1-\alpha}} d\Omega]^{1-q/p} \\
\leq\ & C\|f\|^q_{L^p(\Omega)}.
\end{aligned}$$

For the converse, suppose that Ω is a doubling measure on $\mathbf{X}$, and that

$$\|S_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$. From the definition of W^α_Ω, we need to show $g \in L^{\frac{1}{1-\alpha}}(\Omega)$.

Let f be in $L^{p/q}(\Omega)$ which is the dual of $L^{\frac{1}{1-\alpha}}(\Omega)$. For any $y \in B(x,t)$, by Lemma 1.1 and the fact that Ω is a doubling measure, we have

$$(S_\Omega f)(x,t) \leq C M_\Omega f(y).$$

Hence

$$\begin{aligned}
&|(S_\Omega f)(x,t)|^{1/q} \\
\leq\ & \frac{C}{\nu(B(x,t))} \int_{B(x,t)} |M_\Omega f(y)|^{1/q} d\nu(y) \\
=\ & C S_\nu(|M_\Omega f|^{1/q})(x,t).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\int_{\mathbf{X}} g(y) f(y) d\Omega| \\
\leq\ & \int_{\mathbf{X}+} |(S_\Omega f)(x,t)| d\mu(x,t) \\
=\ & \int_{\mathbf{X}+} |(S_\Omega f)(x,t)|^{(1/q)q} d\mu(x,t)
\end{aligned}$$

$$\leq C \int_{\mathbf{X}^+} [S_\nu(|M_\Omega f|^{1/q})]^q d\mu(x,t)$$

$$\leq C[\int_{\mathbf{X}} (M_\Omega f)^{p/q} d\Omega]^{q/p} \qquad (\textit{by the hypothesis})$$

$$\leq C[\int_{\mathbf{X}} |f|^{p/q} d\Omega]^{q/p} < \infty.$$

Since $p/q > 1$, the last inequality follows from a similar argument used in the proof of Theorem 2.1, we leave the details to the reader. Therefore $g \in L^{\frac{1}{1-\alpha}}(\Omega)$. The proof is complete.

Corollary 2.6 *Let $0 < q < p$, $1 < p < \infty$ such that $\alpha = q/p$. Let $f \in L^p(\mathbf{R}^N)$ and let U(x,t) denote the Poisson integral of f. Let μ be a positive measure and let m denote the Lebesgue measure on $\mathbf{R}^N$. Then $\mu \in W^\alpha_m$ if and only if there is a constant C such that*

$$(\int |U(x,t)|^q d\mu)^{1/q} \leq C(\int |f|^p dm)^{1/p}.$$

Proof: It suffices to prove the theorem for positive functions $f \geq 0$.

Let m denote the Lebesgue measure on $\mathbf{R}^N$ and let

$$P(x,t) = \frac{C_N t}{(|x|^2 + t^2)^{\frac{N+1}{2}}}$$

be the Poisson kernel in $\mathbf{R}^{N+1}_+$. Let $U(x,t)$ be the Poisson integral of f. Then there exist C_1, C_2 such that

$$C_1 S_m f(x,t) \leq U(x,t) \leq C_2 H_m f(x,t)$$

for all (x,t).

Therefore the conclusion follows immediately from Theorem 2.5 .

Remark:

1. Corollary 2.6 is still true when $\mathbf{R}^{N+1}_+$ is replaced by the unit ball of $\mathbf{C}^1$. We leave the details to the reader.

2. In Corollary 2.6, the space $(\mathbf{R}_+^{N+1}, m)$ can be replaced by the homogeneous space $(\mathbf{R}_+^{N+1}, \omega dm, d)$ under the assumptions of Theorem 2.5 .

§2.3 Another Characterization of W_Ω^α

Let Ω be a positive measure on $\mathbf{X}$ defined by $d\Omega = \omega d\nu$. Let

$$K_\mu(x) = \sup_{r>0} \frac{|\mu|(TB(x,r))}{\Omega(B(x,r))}.$$

Theorem 2.7 *Let* $0 < \alpha < 1$. *Suppose* Ω *is a doubling measure on* $\mathbf{X}$. *Then*

$$W_\Omega^\alpha \subset \{\mu : K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)\}.$$

On the other hand, suppose $0 < \alpha < 1$ *and* $\omega \in A_\gamma$ *for some* $\gamma \geq 1$. *then*

$$W_\Omega^\alpha \supset \{\mu : K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)\}.$$

Proof: Suppose $\mu \in W_\Omega^\alpha$. Then $S_\Omega^*|\mu| \in L^{\frac{1}{1-\alpha}}(\Omega)$. We may assume that μ is positive. Then for any $y \in \mathbf{X}$ and $r > 0$,

$$\begin{aligned}
&\frac{1}{\Omega(B(y,r))}\int_{B(y,r)} S_\Omega^*|\mu|(s)d\Omega(s) \\
= &\frac{1}{\Omega(B(y,r))}\int_{\mathbf{X}^+}\int_{\mathbf{X}} \frac{\chi_{B(y,r)}(s)\chi_{B(x,t)}(s)}{\Omega(B(x,t))}d\Omega(s)d\mu(x,t) \\
= &\frac{1}{\Omega(B(y,r))}\int_{\mathbf{X}^+}\int_{\mathbf{X}} \frac{\chi_{B(y,r)\bigcap B(x,t)}(s)}{\Omega(B(x,t))}d\Omega(s)d\mu(x,t) \\
= &\frac{1}{\Omega(B(y,r))}\int_{\mathbf{X}^+} \frac{\Omega(B(y,r)\bigcap B(x,t))}{\Omega(B(x,t))}d\mu(x,t) \\
\geq &\frac{1}{\Omega(B(y,r))}\int_{TB(y,r)} \frac{\Omega(B(y,r)\bigcap B(x,t))}{\Omega(B(x,t))}d\mu(x,t).
\end{aligned}$$

Since if $(x,t) \in TB(y,r)$, then $B(x,t) \subset B(y,r)$. Thus

$$\begin{aligned}
&\frac{1}{\Omega(B(y,r))}\int_{B(y,r)} S_\Omega^*|\mu|(s)d\Omega(s) \\
\geq\; &\frac{1}{\Omega(B(y,r))}\int_{TB(y,r)} d\mu(x,t) \\
=\; &\frac{\mu(TB(y,r))}{\Omega(B(y,r))}.
\end{aligned}$$

Therefore $M_\Omega(S_\Omega^*|\mu|)(y) \geq K_\mu(y)$. By Theorem 2.4, if $S_\Omega^*|\mu| \in L^{\frac{1}{1-\alpha}}(\Omega)$, then $M_\Omega(S_\Omega^*|\mu|) \in L^{\frac{1}{1-\alpha}}(\Omega)$. Hence $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$.

Conversely, suppose $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$ and $\omega \in A_\gamma$. We first prove the following:

Lemma 2.8 $\{S_\Omega(\frac{1}{K_\mu})(x,t)\}\mu \in V_\Omega^1$.

Proof: Given any $B(y,r)$, we need to prove that

$$\int_{TB(y,r)} S_\Omega(\frac{1}{K_\mu})(x,t)d\mu(x,t) \leq C\Omega(B(y,r))$$

with C independent of y and r.

Note that if $s \in B(x,t)$ and $(x,t) \in TB(y,r)$, then $s \in B(y,r)$. By Lemma 1.1, there are C_1, $C_2 > 0$ independent of s, y and r, such that

$$B(y,r) \subset B(s, C_1 r) \subset B(y, C_2 r).$$

Since Ω is a doubling measure, we have

$$\begin{aligned}
\frac{1}{K_\mu(s)} &\leq \frac{\Omega(B(s,C_1r))}{\mu(TB(s,C_1r))} \\
&\leq \frac{\Omega(B(y,C_2r))}{\mu(TB(y,r))} \\
&\leq C\frac{\Omega(B(y,r))}{\mu(TB(y,r))}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\int_{TB(y,r)} S_\Omega(\frac{1}{K_\mu})(x,t)d\mu(x,t) \\
= &\int_{TB(y,r)} \frac{1}{\Omega(B(x,t))}\int_{B(x,t)} \frac{d\Omega(s)}{K_\mu(s)} d\mu(x,t) \\
\leq &\int_{TB(y,r)} C\frac{\Omega(B(y,r))}{\mu(TB(y,r))} d\mu(x,t) \\
= &\ C\Omega(B(y,r)).
\end{aligned}
$$

The proof of the lemma is complete.

Now, similar to the proof of the first part of Theorem 2.5 (with g replaced by K_μ), for any $f \in L^\gamma(\Omega)$, take $q < \gamma$ such that $\frac{q}{\gamma} = \alpha$, we have $\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^\gamma(\Omega)}$. Then since we may assume $\gamma > 1$, the second part of Theorem 2.5 implies that $\mu \in W_\Omega^\alpha$. The proof of Theorem 2.7 is complete.

Combining Theorem 2.5 and Theorem 2.7, we have proven the following:

Theorem 2.9 *Let $0 < \alpha < 1$, and let $q > 0$, $p > 1$, $q/p = \alpha$. Let μ be a positive measure on $\mathbf{X}^+$. Suppose $\omega \in A_p$ and set $d\Omega = \omega d\nu$. If $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$, then there is a constant C such that*

$$\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$.

Conversely, let $0 < q < p < \infty$ and let $\alpha = q/p$. Suppose that Ω is a doubling measure on $\mathbf{X}$. If

$$\|S_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$, then $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$.

§2.4 Carleson Measure Theorem on Weighted Hardy Spaces

On $\mathbf{R}^N$, let Ω be a doubling measure such that $d\Omega = \omega dm$, where m denotes the Lebesgue measure. Recall that the weighted Hardy space is defined by

$$H^p(\Omega) = \{u : u \textit{ is harmonic in } \mathbf{R}_+^{\mathbf{N+1}},\ N(u)(x) \in L^p(\Omega)\}$$

with $\|u\|_{H^p(\Omega)} = \|N(u)\|_{L^p(\Omega)}$.

Lemma 2.10 *Let*

$$\Gamma(x) = \{(y,t) : d(x,y) \le t\}.$$

(1) If $(y,t) \in \Gamma(x)$, for any function f defined on $\mathbf{X}$, we have

$$(H_\Omega f)(y,t) \le C M_\Omega f(x).$$

(2) For any x, we have

$$N(H_\Omega f)(x) \le C M_\Omega f(x).$$

Proof: Without lost of generality, we may assume $f \ge 0$. we have

$$(H_\Omega f)(y,t) = \sup_{B(z,s)\supset B(y,t)} \frac{1}{\Omega(B(z,s))} \int_{B(z,s)} f(u) d\Omega(u).$$

Since for any $(y,t) \in \Gamma(x)$ and $B(z,s) \supset B(y,t)$, we have $x \in B(y,t) \subset B(z,s)$. Therefore, by Lemma 1.1 there are constants $C_1 > C_2 > 0$ independent of x, y, z, s and t such that

$$B(z,s) \subset B(x, C_2 s) \subset B(z, C_1 s).$$

Since Ω is a doubling measure, there is constant A such that

$$\Omega(B(z,s)) \ge A\Omega(B(z, C_1 s)) \ge A\Omega(B(x, C_2 s)).$$

Therefore

$$\begin{aligned}
& \frac{1}{\Omega(B(z,s))}\int_{B(z,s)} f(u)d\Omega(u) \\
\leq\; & C\frac{1}{\Omega(B(x,C_1 s))}\int_{B(x,C_1 s)} f(u)d\Omega(u) \\
\leq\; & CM_\Omega f(x).
\end{aligned}$$

The conclusion (1) follows from the above inequality.

The conclusion (2) follows from (1) and the definition of operator N.

Theorem 2.11 *Let $\alpha \geq 1$. Let Ω be a doubling measure on $\mathbf{X}$. Then $\mu \in V_\Omega^\alpha$ if and only if*

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}$$

for any measurable function u satisfying $N(u)(x) \in L^p(\Omega)$ with $q/p = \alpha$.

In particular, if $\mathbf{X} = \mathbf{R}^N$ and $d\Omega = \omega dm$, then

(1) Suppose $\omega \in A_p$. If $p > 1$ and $\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}$ for any harmonic function $u(x,t)$ satisfying $N(u) \in L^p(\Omega)$, then $\mu \in V_\Omega^\alpha$;

(2) Suppose $\omega \in A_r$ for some $r \geq 1$. If $p \leq 1$ and $\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}$ for any subharmonic function satisfying $N(u) \in L^p(\Omega)$, then $\mu \in V_\Omega^\alpha$.

Proof: Suppose $p > 1$ and $\mu \in V_\Omega^\alpha$. If $y \in B(x,t)$, then

$$|u(x,t)| \leq N(u)(y).$$

Thus

$$\begin{aligned}
& H_\Omega(N(u))(x,t) \\
\geq\; & \frac{1}{\Omega(B(x,t))}\int_{B(x,t)} N(u)(y)d\Omega(y) \\
\geq\; & |u(x,t)|.
\end{aligned}$$

Therefore

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|H_\Omega(N(u))\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}.$$

The last inequality follows from Theorem 2.1 .

For $p \leq 1$, take $r > 0$ such that $p/r > 1$. Let $G(x,t) = |u(x,t)|^r$, then

$$NG(x) = |N(u)(x)|^r \in L^{p/r}(\Omega).$$

The conclusion follows from the case $p > 1$.

The "only if" part follows by letting $u(y,s) = \chi_{T(B(x,t))}(y,s)$.

We now prove the particular case.

(1) Let $\chi_{B(y,s)}$ be the characteristic function of $B(y,s)$. Let $U(x,t)$ be the Poisson integral of $\chi_{B(y,s)}$. Then there are C_1, $C_2 > 0$ such that

$$C_1 H_m(x,t) \geq U(x,t) \geq C_2 S_m(\chi_{B(y,s)})(x,t)$$

for all (x,t). Thus if $(x,t) \in TB(y,s)$, then

$$U(x,t) \geq C_2 S_m(\chi_{B(y,s)})(x,t) \geq C_2.$$

Hence

$$(\mu(TB(y,s)))^{1/q} \leq C\|U\|_{L^q(\mu)}.$$

By Lemma 2.10,

$$N(H_m \chi_{B(y,s)})(x) \leq C M_m(\chi_{B(y,s)}).$$

Therefore

$$\begin{aligned} & (\mu(TB(y,s)))^{1/q} \\ \leq\ & C\|U\|_{L^q(\mu)} \\ \leq\ & C\|N(U)\|_{L^p(\Omega)} \\ \leq\ & C\|N(H_m \chi_{B(y,s)})\|_{L^p(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&\leq C\|M_m(\chi_{B(y,s)})\|_{L^p(\Omega)} \qquad (Lemma\ 2.10) \\
&\leq C\|\chi_{B(y,s)}\|_{L^p(\Omega)} \\
&= C(\Omega(B(y,s)))^{1/p}.
\end{aligned}
$$

The last inequality follows from Theorem 2.4 .

(2) Suppose $p \leq 1$, $\omega \in A_r$ for some $r \geq 1$ and suppose

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}$$

for all subharmonic functions with $N(u) \in L^p(\Omega)$. Let $l > r$. For any harmonic function $u \in L^l(\Omega)$, take $k \geq 1$ such that $l/k = p$. Then

$$G(x,t) = |u(x,t)|^k$$

is subharmonic and $N(G) = |N(u)|^k \in L^p(\Omega)$. Thus

$$
\begin{aligned}
&\|u\|_{L^{l\alpha}(\mu)} \\
&= \|G\|^{1/k}_{L^{\alpha l/k}(\mu)} \\
&= \|G\|^{1/k}_{L^{\alpha p}(\mu)} \\
&\leq C\|G\|^{1/k}_{L^p(\Omega)} \\
&= C\|u\|_{L^l(\Omega)}.
\end{aligned}
$$

The conclusion follows from the case $p > 1$.

We now turn to the main result of this section:

Theorem 2.12 *Let* $0 < \alpha < 1$ *and let* $q/p = \alpha$. *Then*

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}$$

for all $u(x,t)$ *satisfying* $N(u) \in L^p(\Omega)$ *if and only if* $\mu \in W^\alpha_\Omega$.

In particular, if $\mathbf{X} = R^N$ and $d\Omega = \omega dm$, where m denotes the Lebesgue measure, then

(1) $\mu \in W_\Omega^\alpha$ implies $\|u(x,t)\|_{L^q(\mu)} \le C\|N(u)\|_{L^p(\Omega)}$;

(2) Suppose $\omega \in A_p$. If $p > 1$ and $\|u(x,t)\|_{L^q(\mu)} \le C\|N(u)\|_{L^p(\Omega)}$ for all harmonic functions $u(x,t)$ satisfying $N(u) \in L^p(\Omega)$, then $\mu \in W_\Omega^\alpha$;

(3) Suppose $\omega \in A_r$ for some $r \ge 1$. If $p \le 1$ and $\|u(x,t)\|_{L^q(\mu)} \le C\|N(u)\|_{L^p(\Omega)}$ for all subharmonic functions satisfying $N(u) \in L^p(\Omega)$, then $\mu \in W_\Omega^\alpha$.

Proof: We only prove the special case. The proof for the general case is similar.

(1) Suppose $\mu \in W_\Omega^\alpha$. Let g be the balayée of μ w.r.t. Ω as in Lemma 2.3. Note that by Hölder's inequality

$$[S_\Omega(1/g)(x,t)]^{-1} \le (S_\Omega g)(x,t).$$

Then

$$\begin{aligned}
& \int_{\mathbf{X}^+} |u(x,t)|^q d\mu \\
= \ & \int_{\mathbf{X}^+} |u(x,t)|^q [S_\Omega(1/g)(x,t)]^{-1} S_\Omega(1/g)(x,t) d\mu(x,t) \\
\le \ & \int_{\mathbf{X}^+} |u(x,t)|^q (S_\Omega g)(x,t) S_\Omega(1/g)(x,t) d\mu(x,t) \\
\le \ & [\int_{\mathbf{X}^+} |u(x,t)|^p S_\Omega(1/g)(x,t) d\mu(x,t)]^{q/p} \\
& \times [\int_{\mathbf{X}^+} |(S_\Omega g)(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t) d\mu(x,t)]^{1-q/p} \\
\le \ & (\int_{\mathbf{X}^+} |u(x,t)|^p S_\Omega(1/g)(x,t) d\mu)^{q/p} \\
& \times (\int_{\mathbf{X}^+} |H_\Omega(g)(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t) d\mu)^{1-q/p} \\
\le \ & C(\int_{\mathbf{X}} |N(u)|^p d\Omega)^{q/p}.
\end{aligned}$$

The last inequality follows from Theorem 2.11 and Theorem 2.2 since by Lemma 2.3, $S_\Omega(1/g)(x,t)\mu \in V_\Omega^1$.

(2) Suppose for all harmonic functions $u(x,t)$ with $N(u) \in L^p(\Omega)$, we have

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|N(u)\|_{L^p(\Omega)}.$$

Suppose $p > 1$ and that g is as above. Note that similar to the proof of Theorem 2.5, for any $y \in B(x,t)$, by Lemma 1.1 and the fact that Ω is a doubling measure, we have

$$(S_\Omega f)(x,t) \leq C M_\Omega f(y).$$

Hence

$$\begin{aligned}
& |(S_\Omega f)(x,t)|^{1/q} \\
\leq\ & \frac{C}{\Omega(B(x,t))}\int_{B(x,t)} |M_\Omega f(y)|^{1/q} d\Omega(y) \\
=\ & C S_\Omega(|M_\Omega f|^{1/q})(x,t).
\end{aligned}$$

Let $f \in L^{p/q}(\Omega)$. Then

$$\begin{aligned}
& |\int_{\mathbf{X}} g(y)f(y)d\Omega(y)| \\
\leq\ & \int_{\mathbf{X}+} [S_\Omega|f|(x,t)]d\mu \\
\leq\ & \int_{\mathbf{X}+} [(S_\Omega|f|)^{1/q}]^q d\mu \\
\leq\ & C\int_{\mathbf{X}+} [S_\Omega((M_\Omega|f|)^{1/q})]^q d\mu \\
\leq\ & C\int_{\mathbf{X}+} |U((M_\Omega|f|)^{1/q})|^q d\mu,
\end{aligned}$$

where $U((M_\Omega|f|)^{1/q})$ denotes the Poisson integral of $(M_\Omega|f|)^{1/q}$. Then by the hypothesis,

$$\begin{aligned}
& |\int_{\mathbf{X}} g(y)f(y)d\Omega(y)| \\
\leq\ & C(\int_{\mathbf{X}} |N[U((M_\Omega|f|)^{1/q})]|^p d\Omega)^{q/p} \\
\leq\ & C(\int_{\mathbf{X}} |N[H_m((M_\Omega|f|)^{1/q})]|^p d\Omega)^{q/p}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\int_{\mathbf{X}} |M_m[(M_\Omega|f|)^{1/q}]|^p d\Omega\Big)^{q/p} \qquad \textit{(by Lemma 2.10)}\\
&\le C\Big(\int_{\mathbf{X}} (M_\Omega|f|)^{p/q} d\Omega\Big)^{q/p}\\
&\le C\Big(\int_{\mathbf{X}} |f|^{p/q} d\Omega\Big)^{q/p} \le \infty.
\end{aligned}
$$

The last two inequalities follow from Theorem 2.4 since $p > 1$, $p/q > 1$ and $\omega \in A_p$. Therefore $g \in L^{\frac{1}{1-\alpha}}(\Omega)$, that is, $\mu \in W_\Omega^\alpha$.

(3) Similar to the proof of particular case (2) of Theorem 2.11 .

Chapter 3

MAXIMAL AVERAGING OPERATOR AND CARLESON MEASURES ON X

In this chapter, we characterize those measures μ such that the maximal averaging operator defined on a homogeneous space $(\mathbf{X}, \nu, d)$ is bounded from $L^p(\Omega)$ to $L^q(\mu)$ with $0 < q < p$, where Ω is a measure on $\mathbf{X}$ satisfying Muckenhoupt's A_p condition. In the proof, we use the "balayée" of measure μ with respect to Ω which is an analogue of the balayée defined on $\mathbf{X}^+$.

We shall collect some results for α-Carleson measures on $\mathbf{X}$ with $\alpha \geq 1$ in the first section. In the second section we shall discuss some properties of the space of "balayées" on $\mathbf{X}$. The ideas there follow directly from the paper of E. Amar and A. Bonami [1]. The main result of this chapter will be presented in the last section.

§3.1 α-Carleson Measures on X with $\alpha \geq 1$

In this and the next chapter, we shall state our results in the following generality. The role of the family $\{E(x) : x \in \mathbf{X}\}$ below will vary in different situations that we will subsequently study.

Let $(\mathbf{X}, \nu)$ be a measurable space satisfying the following condition:

For any $x \in \mathbf{X}$, there is a ν-measurable subset $E(x)$ containing x and a ν- measurable subset $E^2(x) \supset E(x)$ with the following properties:

(1) $\nu(E(x)) > 0$;

(2) "Doubling property":

$$\nu(E^2(x)) \leq C_\nu \nu(E(x));$$

(3) "Covering property":

If $B \subset \mathbf{X}$ and if $A \subset \cup_{x \in B} E(x)$, then there exists $\{x_i\}_{i=1}^{\infty} \subset B$ such that $\{E_i\}$ (where $E_i = E(x_i)$) is a disjoint family and $A \subset \cup_{i=1}^{\infty} E^2(x_i)$.

We now give the definition of α-Carleson measure on $\mathbf{X}$:

Definition 3.1 *Let $(\mathbf{X}, \nu)$ be as above. Let μ be a measure on $\mathbf{X}$ and let Ω be a positive measure on $\mathbf{X}$. If*

$$|\mu|(E^2(x)) \leq C[\Omega(E^2(x))]^\alpha$$

for any $x \in \mathbf{X}$, where $\infty > \alpha \geq 0$, then we call μ an α-Carleson measure w. r. t. Ω.

Let

$$V_\Omega^\alpha = \{\mu : |\mu|(E^2(x)) \leq C[\Omega(E^2(x))]^\alpha\}$$

with

$$\|\mu\|_{V_\Omega^\alpha} = inf\{C : |\mu|(E^2(x)) \leq C[\Omega(E^2(x))]^\alpha\}.$$

It is not hard to see that V_Ω^α becomes a linear normed space.

For $f \in L^1_{loc}(\nu)$, define

$$mf(x) = \sup_{x \in E(y)} \frac{1}{\nu(E(y))} \int_{E(y)} |f| d\nu.$$

Lemma 3.2 *Let $\alpha \geq 1$ and μ be a positive measure on $\mathbf{X}$. Suppose that $(\mathbf{X}, \nu)$ satisfies the assumptions made in the beginning of this section. If μ is an α-Carleson measure w. r. t. ν, then*

$$(\int |mf|^q d\mu)^{1/q} \leq C\|\mu\|_{V_\nu^\alpha}^{1/p}(\int |f|^p d\nu)^{1/p} \qquad f \in L^p(\nu)$$

for any $1 < p \leq q < \infty$ such that $q/p = \alpha$.

Proof: Suppose μ is an α-Carleson measure w. r. t. ν.

For $f \in L^\infty(\nu)$, it is clear that

$$\|mf\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\nu)}.$$

If we can show that m is of weak type $(1, \alpha)$, the conclusion will follow from Marcinkiewicz interpolation theorem.

Let $\lambda > 0$, $A = \{x \in \mathbf{X} | mf(x) > \lambda\}$. Then for any $x \in A$, there exists y such that $x \in E(y)$ and

$$\lambda < \frac{1}{\nu(E(y))} \int_{E(y)} |f| d\nu.$$

The covering property implies that there is $\{y_i\}_{i=1}^\infty \subset \mathbf{X}$, such that $\{E(y_i)\}$ is a disjoint family, $A \subset \cup_{i=1}^\infty E^2(y_i)$ and $\lambda < \frac{1}{\nu(E(y_i))} \int_{E(y_i)} |f| d\nu$.

Let $E_i = E(y_i)$, $E_i^2 = E^2(y_i)$. Then

$$\begin{aligned}
& \mu(A) \\
\leq\ & \mu(\cup_{i=1}^\infty (E_i^2)) \\
\leq\ & \sum_{i=1}^\infty \mu(E_i^2) \\
\leq\ & \|\mu\|_{V_\nu^\alpha} \sum_{i=1}^\infty [\nu(E_i^2)]^\alpha \\
\leq\ & C_\nu^\alpha \|\mu\|_{V_\nu^\alpha} \sum_{i=1}^\infty [\nu(E_i)]^\alpha \qquad \textit{(doubling property)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C_\nu^\alpha \|\mu\|_{V_\nu^\alpha}[\sum_{i=1}^{\infty} \nu(E_i)]^\alpha \qquad (\alpha \geq 1) \\
&\leq C_\nu^\alpha \|\mu\|_{V_\nu^\alpha}[\sum_{i=1}^{\infty} \frac{1}{\lambda}\int_{E_i} |f| d\nu]^\alpha \\
&\leq C_\nu^\alpha \|\mu\|_{V_\nu^\alpha}[\frac{1}{\lambda}\int |f| d\nu]^\alpha.
\end{aligned}
$$

Hence m is of weak type $(1,\alpha)$. By the Marcinkiewicz interpolation theorem, if $1/p = \theta$, $1/q = \frac{\theta}{\alpha}$, $0 < \theta < 1$, then

$$\|mf\|_{L^q(\mu)} \leq C\|\mu\|_{V_\nu^\alpha}^{1/p}\|f\|_{L^p(\nu)},$$

where C only depends on doubling constant C_ν. The proof is complete.

Remark: The doubling property and covering property can be replaced by the assumption: if $A \subset \cup_{x\in B} E(x)$, then there is $\{x_i\} \subset B$ such that $A \subset \cup_{i=1}^{\infty} E(x_i)$ and the sequence $E(x_i)$ can be distributed in N families of disjoint subfamilies.

Definition 3.3 *Let $0 < \alpha < \infty$ and let Ω be any positive measure on $\mathbf{X}$. Fix $r > 0$, define*

$$
\begin{aligned}
&P_{\Omega,r}(x,y) = \frac{1}{\Omega(B(x,r))}\chi_{B(x,r)}(y); \\
&P_{\Omega,r}f(x) = \int P_{\Omega,r}(x,y)f(y)d\Omega; \\
&M_{\Omega,r}f(x) = \sup_{B(y,t)\supset B(x,r)} \frac{1}{\Omega(B(y,t))}\int_{B(y,t)} |f| d\Omega; \\
&P_{\Omega,r}^*\mu(y) = \int P_{\Omega,r}(x,y)d\mu(x); \\
&V_{\Omega,r}^\alpha = \{\mu : |\mu|(B(x,2C_d r)) \leq C[\Omega(B(x,2C_d r))]^\alpha\}; \\
&W_{\Omega,r}^\alpha = \{\mu : P_{\Omega,r}^*|\mu| \in L^{\frac{1}{1-\alpha}}(\Omega)\}; \\
&V^0 = \{\mu : |\mu|(\mathbf{X}) < \infty.\}
\end{aligned}
$$

Amar and Bonami have used the term “balayée” in a different but similar context to describe the function $P_{\Omega,r}^*\mu$ (see Definition 1.10 in Chapter 1). We shall adopt their usage and call $P^*_{\Omega,r}|\mu|$ the balayée of μ w. r. t. (Ω, r). Under the norm

$$\|\mu\|_{W^{\alpha}_{\Omega,r}} = \|P^*_{\Omega,r}\mu\|_{L^{\frac{1}{1-\alpha}}(\Omega)},$$

$W^{\alpha}_{\Omega,r}$ becomes a linear normed space.

Let $(\mathbf{X}, \nu, d)$ be a homogeneous space. Let $\omega(x) \geq 0$, $\omega \in L^1_{loc}(\nu)$ be such that the measure Ω, defined by $d\Omega = \omega(x)d\nu$, is a doubling measure. Note that $(\mathbf{X}, \Omega, d)$ is also a homogeneous space.

Fix $r > 0$. Let $E(x) = B(x, r)$ and $E^2(x) = B(x, 2C_d r)$, then

$$\cup_{x \in E(y)} E(y) \subset B(x, 2C_d r).$$

It follows from Lemma 1.1 and Lemma 1.2 that the assumptions of Lemma 3.2 are satisfied by the space $(\mathbf{X}, \Omega, d)$. We shall call a measure μ on $\mathbf{X}$ an α-Carleson measure with respect to (Ω, r) if there is a constant $C_r > 0$ such that

$$\mu(E^2(x)) \leq C_r[\Omega(E(x))]^{\alpha}$$

for every $x \in \mathbf{X}$.

Theorem 3.4 *Let $1 \leq \alpha < \infty$, and let $1 < p \leq q$ such that $q/p = \alpha$. Let μ be a positive measure and Ω be a positive doubling measure. Then*

$$(\int |M_{\Omega,r} f|^q d\mu)^{1/q} \leq C(\int |f|^p d\Omega)^{1/p}$$

if and only if $\mu(B(x, 2C_d r)) \leq C_r[\Omega(B(x, 2C_d r))]^{\alpha}$ for every $x \in \mathbf{X}$.

Proof: Note that $(\mathbf{X}, \Omega, d)$ is a homogeneous space.

We only prove the "if" part. Fix $R > r$ and define

$$M^R_{\Omega,r} f(x) = \sup_{B(y,t) \supset B(x,r), t \leq R} \frac{1}{\Omega(B(y,t))} \int_{B(y,t)} |f| d\Omega.$$

The conclusion will follow by taking $R \to \infty$, if we can prove that μ is an α-Carleson measure w. r. t. (Ω, r) implies

$$(\int |M^R_{\Omega,r} f|^q d\mu)^{1/q} \leq C(\int |f|^p d\Omega)^{1/p}$$

with C independent of R.

But this is a consequence of the proof of Lemma 3.2 with the applications of Lemma 1.2 and Lemma 1.3. We leave the details to the reader. The proof is complete.

From the above proof, it is clear that if $\mu B(x, 2C_d r) \leq C_1[\Omega B(x, 2C_d r)]^\alpha$ with C_1 independent of r, then $\|M_{\Omega,r} f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ with C independent of r. Letting $r \to 0$, we have the following:

Corollary 3.5 *Let $1 \leq \alpha < \infty$, and let $1 < p \leq q$ such that $q/p = \alpha$. Let μ be a positive measure and Ω be a positive doubling measure. Then*

$$(\int |M_\Omega f|^q d\mu)^{1/q} \leq C(\int |f|^p d\Omega)^{1/p}$$

if and only if for any $r > 0$, $\mu(B(x, 2C_d r)) \leq C_1[\Omega(B(x, 2C_d r))]^\alpha$ for any $x \in \mathbf{X}$ with C_1 independent of r.

Now we turn to two-weight norm problem.

Let μ be a positive measure in $(\mathbf{X}, \nu, d)$.

Theorem 3.6 *Let $\alpha \geq 1$ and $p > 1$. If $\omega \in A_p$ and $d\Omega = \omega d\nu$, then for any $q \geq p$ such that $q/p = \alpha$,*

$$(\int |M_{\nu,r} f|^q d\mu)^{1/q} \leq C(\int |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega)$$

if and only if $\mu(B(x, 2C_d r)) \leq C_r[\Omega(B(x, 2C_d r))]^\alpha$ for any $x \in \mathbf{X}$.

Proof: The "only if" part follows from taking $f = \chi_{B(x,2C_d r)}$.

Conversely, suppose $\mu \in V^{\alpha}_{\Omega,r}$. Since $\omega \in A_p$, by Theorem 1.4, there is a γ, $\gamma < p$ such that $\omega \in A_\gamma$. By Hölder's inequality, we have

$$M_{\nu,r}f(x) \le C[M_{\Omega,r}(|f|^\gamma)(x)]^{\frac{1}{\gamma}},$$

where C only depends on A_γ constant.

Note that by Hölder's inequality, $\omega \in A_p$ implies that Ω is a doubling measure. Thus

$$\begin{aligned} & \int |M_{\nu,r}f|^q d\mu \\ \le\ & C\int [M_{\Omega,r}|f|^\gamma]^{\frac{q}{\gamma}} d\mu \\ \le\ & C[\int |f|^p d\Omega]^{q/p} \end{aligned}$$

with C depends on A_γ constant and the constant in the conclusion of Theorem 3.4. The last inequality follows from Theorem 3.4 , since $\frac{q}{\gamma} > 1$, and $\frac{\frac{q}{\gamma}}{\frac{p}{\gamma}} = \alpha \ge 1$. The proof is complete.

Similar to Corollary 3.5, if $\mu B(x, 2C_d r) \le C_1[\Omega B(x, 2C_d r)]^\alpha$ with C_1 independent of r, then we have

Corollary 3.7 *Let $p \ge 1$ and $\alpha \ge 1$. If $\omega \in A_p$, then for any $q \ge p$ such that $q/p = \alpha$,*

$$(\int |M_\nu f|^q d\mu)^{1/q} \le C(\int |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega)$$

if and only if for any $r > 0$, $\mu(B(x, 2C_d r)) \le C_1[\Omega(B(x, 2C_d r))]^\alpha$ for any $x \in \mathbf{X}$ with C_1 independent of r.

§3.2 Characterization of $W^{\alpha}_{\Omega,r}$ for $0 < \alpha < 1$

In [1], E. Amar and A. Bonami worked on $\mathbf{X}^+$ and showed that the space of "balayées" is the interpolation space between the space of bounded measures and the space of Carleson measures on $\mathbf{X}^+$. We shall prove that in our situation, the parallel result still holds. We shall show that the space $W^{\alpha}_{\Omega,r}$ with $0 < \alpha < 1$, is the complex interpolation space between V^0 and $V^1_{\Omega,r}$. The idea of the proof follows from E. Amar and A. Bonami.

In this section we always assume that Ω is a doubling measure on $(\mathbf{X}, \nu, d)$. Note that $(\mathbf{X}, \Omega, d)$ is also a homogeneous space.

Lemma 3.8 *If μ is a positive measure, for any $r > 0$, let*

$$g_r(y) = P^*_{\Omega,r}\mu(y).$$

Then there is a constant $C > 0$ independent of r such that if we define

$$\lambda_r(E) = \int_E P_{\Omega,r}(\frac{1}{g_r})d\mu,$$

then

$$\lambda_r(B(x, 2C_d r)) \le C\Omega(B(x, 2C_d r)).$$

Proof: Fix r, let $E(x) = B(x, r)$ and $E^2(x) = B(x, 2C_d r)$. It suffices to show that for any $x \in \mathbf{X}$,

$$\int_{E^2(x)} P_{\Omega,r}(\frac{1}{g_r})(y)d\mu(y) \le C\Omega(E^2(x))$$

with C independent of r. Note that

$$\chi_{E^2(x)}(y)\chi_{E(y)}(t) \le \chi_{B(x,3C_d^2 r)}(t)\chi_{E(y)}(t).$$

We have

$$\begin{aligned}
&\int_{E^2(x)} P_{\Omega,r}(\frac{1}{g_r})(y)d\mu(y) \\
=\ & \int_{E^2(x)} [\int P_{\Omega,r}(y,t)\frac{1}{g_r(t)}d\Omega(t)]d\mu(y) \\
=\ & \int \chi_{E^2(x)}(y) \int \frac{\chi_{E(y)}(t)}{\Omega(E(y))g_r(t)}d\Omega(t)d\mu(y) \\
=\ & \int \frac{1}{g_r(t)} \int \frac{\chi_{E^2(x)}(y)\chi_{E(y)}(t)}{\Omega(E(y))}d\mu(y)d\Omega(t) \\
\leq\ & \int \frac{1}{g_r(t)} \int \frac{\chi_{B(x,3C_d^2r)}(t)\chi_{E(y)}(t)}{\Omega(E(y))}d\mu(y)d\Omega(t) \\
=\ & \Omega(B(x,3C_d^2r)) \\
\leq\ & C\Omega(E^2(x)).
\end{aligned}$$

The last inequality follows from the doubling property of Ω and hence C depends only on doubling constant . The proof is complete.

Lemma 3.9 *If $\mu \in W^{\alpha}_{\Omega,r}$, then there exists positive $\mu_0 \in V^1_{\Omega,r}$ and $h \in L^p(\mu_0)$ such that*

$$\mu = h\mu_0,$$

where $1/p = 1 - \alpha$.

Proof: We may assume that μ is positive.

Take $\mu_0 = P_{\Omega,r}(\frac{1}{g_r})\mu$, $h = [P_{\Omega,r}(\frac{1}{g_r})]^{-1}$ with g_r defined in the previous lemma. By the assumption, $g_r \in L^p(\Omega)$.

By Schwarz inequality, $h \leq P_{\Omega,r}g_r$. It suffices to show $P_{\Omega,r}g_r \in L^p(\mu_0)$.

From the previous lemma, $\mu_0 \in V^1_{\Omega,r}$. Since $g_r \in L^p(\Omega)$, $P_{\Omega,r}g_r \leq M_{\Omega,r}g_r$, and $p > 1$, Lemma 3.2 implies that

$$\int |P_{\Omega,r}g_r|^p d\mu_0 \leq C \int |g_r|^p d\Omega < \infty.$$

The proof is complete.

We now prove the main result of this section.

Theorem 3.10

$$W^{\alpha}_{\Omega,r} = (V^0, V^1_{\Omega,r})_{\alpha}.$$

Proof: $W^{\alpha}_{\Omega,r} \hookrightarrow (V^0, V^1_{\Omega,r})_{\alpha}$ follows from Lemma 3.9 .

In fact, suppose $\mu \in W^{\alpha}_{\Omega,r}$. By Lemma 3.9, there exist $\mu_0 \in V^1_{\Omega,r}$ and $h \in L^p(\mu_0)$, $1/p = 1 - \alpha$, such that

$$\mu = h\mu_0.$$

Since $h \in (L^1(\mu_0), L^{\infty}(\mu_0))_{\alpha}$ and $L^1(\mu_0)$, $L^{\infty}(\mu_0)$ can be identified as a subspace of V^0 and $V^1_{\Omega,r}$, respectively, the conclusion follows.

Next show $(V^0, V^1_{\Omega,r})_{\alpha} \hookrightarrow W^{\alpha}_{\Omega,r}$.

Define a multilinear map by

$$T(f,g,h,\mu) = \int_{\mathbf{X}} (P_{\Omega,r}f)\overline{(P_{\Omega,r}g)}h\,d\mu.$$

Then on $L^2(\Omega) \times L^2(\Omega) \times L^{\infty}(\mu) \times V^1_{\Omega,r}$, by Schwarz inequality and Lemma 3.2, we have

$$\begin{aligned} & |T(f,g,h,\mu)| \\ \leq\ & \|h\|_{L^{\infty}(\mu)} \|P_{\Omega,r}f\|_{L^2(\mu)} \|P_{\Omega,r}g\|_{L^2(\mu)} \\ \leq\ & C\|\mu\|_{V^1_{\Omega}} \|h\|_{L^{\infty}(\mu)} \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \end{aligned}$$

Similarly, on $L^{\infty}(\Omega) \times L^{\infty}(\Omega) \times L^{\infty}(\mu) \times V^0$, we have

$$|T(f,g,h,\mu)| \leq C\|\mu\|_{V^0} \|h\|_{L^{\infty}(\mu)} \|f\|_{L^{\infty}(\Omega)} \|g\|_{L^{\infty}(\Omega)}.$$

By multilinear interpolation theorem [5, p.96], on

$$L^{2q}(\Omega) \times L^{2q}(\Omega) \times L^{\infty}(\mu) \times (V^0, V^1_{\Omega,r})_{\alpha},$$

where $1/q = \alpha$, we have

$$|T(f, g, h, \mu)| \le C\|\mu\|_{(V^0, V^1_{\Omega,r})_\alpha} \|h\|_{L^\infty(\mu)} \|f\|_{L^{2q}(\Omega)} \|g\|_{L^{2q}(\Omega)}.$$

Fix h such that $|h| = 1$ and $hd\mu = d|\mu|$. Let $f = g \in L^{2q}(\Omega)$. Then

$$\int_{\mathbf{X}} |P_{\Omega,r} f|^2 d|\mu| \le C\|\mu\|_{(V^0, V^1_{\Omega,r})_\alpha} \|f\|^2_{L^{2q}(\Omega)}.$$

Since Ω is a doubling measure, for any $y \in B(x,r)$, $|P_{\Omega,r} f(x)| \le CM_{\Omega,r} f(y)$, it follows that

$$\begin{aligned} |P_{\Omega,r} f(x)|^{1/2} &\le \frac{C}{\Omega(B(x,r))} \int_{B(x,r)} [M_{\Omega,r} f(y)]^{1/2} d\Omega(y) \\ &= CP_{\Omega,r}[(M_{\Omega,r} f)^{1/2}](x). \end{aligned}$$

Thus, if $f \in L^q(\Omega) = [L^{\frac{1}{1-\alpha}}(\Omega)]^*$, then

$$\begin{aligned} & \int_{\mathbf{X}} [P^*_{\Omega,r} |\mu|(y)] |f(y)| d\Omega(y) \\ \le\ & \int_{\mathbf{X}} P_{\Omega,r} |f|(x) d|\mu|(x) \\ \le\ & C \int_{\mathbf{X}} (P_{\Omega,r}[(M_{\Omega,r} f)^{1/2}](x))^2 d|\mu|(x) \\ \le\ & C\|\mu\|_{(V^0, V^1_{\Omega,r})_\alpha} \|(M_{\Omega,r} f)^{1/2}\|^2_{L^{2q}(\Omega)} \\ \le\ & C\|\mu\|_{(V^0, V^1_{\Omega,r})_\alpha} \|f\|_{L^q(\Omega)}. \end{aligned}$$

Therefore $(V^0, V^1_{\Omega,r})_\alpha \hookrightarrow W^\alpha_{\Omega,r}$. The proof is complete.

§3.3 Two-Weight Norm Inequalities

Let $(\mathbf{X}, \nu, d)$ be a homogeneous space. Let μ be a positive measure on $\mathbf{X}$ and let Ω be a positive measure on $\mathbf{X}$ defined by $d\Omega = \omega d\nu$. Let $r > 0$ be fixed. Define

$$K_r(x) = \frac{\mu(B(x,r))}{\Omega(B(x,r))}.$$

Theorem 3.11 *Fix $r > 0$. Suppose $p > 1$ and $\omega \in A_p$. Let $q > 0$ and $q/p = \alpha < 1$. If*

$$K_r \in L^{\frac{1}{1-\alpha}}(\Omega) < \infty,$$

then there is a $C_r > 0$ such that

$$(\int_{\mathbf{X}} |M_{\nu,r}f|^q d\mu)^{1/q} \leq C_r(\int_{\mathbf{X}} |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega).$$

Conversely, let $0 < q < p$ and $\alpha = q/p$. If Ω is a doubling measure on $\mathbf{X}$, and if

$$(\int_{\mathbf{X}} |P_{\nu,r}f|^q d\mu)^{1/q} \leq C_r(\int_{\mathbf{X}} |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega),$$

then

$$K_r \in L^{\frac{1}{1-\alpha}}(\Omega) < \infty.$$

Proof: Note that $\omega \in A_p$ implies that Ω is a doubling measure.

Suppose $\|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} < \infty$, and p, q, α as in the assumption. Let g_r be the balayée of μ w. r. t. (Ω, r) as in Lemma 3.8 . By Lemma 1.1, it is clear that there are constants A and B independent of r such that

$$AK_r \leq g_r \leq BK_r.$$

Then $g_r \in L^{\frac{1}{1-\alpha}}(\Omega)$. By Schwarz inequality, $[P_{\Omega,r}(\frac{1}{g_r})]^{-1}(x) \leq P_{\Omega,r}g_r(x)$. We have

$$\begin{aligned}
&\int |M_{\nu,r}f|^q d\mu \\
= &\int |M_{\nu,r}f|^q [P_{\Omega,r}(\frac{1}{g_r})(x)]^{-1} P_{\Omega,r}(\frac{1}{g_r})(x) d\mu(x) \\
\leq &\int |M_{\nu,r}f|^q [P_{\Omega,r}g_r(x)] P_{\Omega,r}(\frac{1}{g_r})(x) d\mu(x) \\
\leq &[\int |M_{\nu,r}f|^p [P_{\Omega,r}(\frac{1}{g_r})(x)] d\mu(x)]^{q/p} \\
&\times [\int |P_{\Omega,r}g_r(x)|^{\frac{1}{1-\alpha}} P_{\Omega,r}(\frac{1}{g_r})(x) d\mu(x)]^{1-q/p} \\
\leq &[\int |M_{\nu,r}f|^p [P_{\Omega,r}(\frac{1}{g_r})(x)] d\mu(x)]^{q/p} \\
&\times [\int |M_{\Omega,r}g_r(x)|^{\frac{1}{1-\alpha}} P_{\Omega,r}(\frac{1}{g_r})(x) d\mu(x)]^{1-q/p}.
\end{aligned}$$

By Lemma 3.8 , there is a constant C independent of r such that

$$\int_{B(x,2C_d r)} P_{\Omega,r}(\frac{1}{g_r})(y)d\mu(y) \leq C\Omega(B(x,2C_d r)).$$

It follows from Theorem 3.4 and Theorem 3.6 that

$$\begin{aligned}
&\int |M_{\nu,r}f|^q d\mu \\
\leq\ & C[\int |f|^p d\Omega]^{q/p}[\int |g_r|^{\frac{1}{1-\alpha}} d\Omega]^{1-q/p} \\
\leq\ & C\|g_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)}\|f\|^q_{L^p(\Omega)} \\
\leq\ & C\|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)}\|f\|^q_{L^p(\Omega)}
\end{aligned}$$

with C independent of r.

Conversely, suppose $\|M_{\nu,r}f\|_{L^q(\mu)} \leq C_r\|f\|_{L^p(\Omega)}$ with p, q and α as in the assumption. By the discussion at the beginning of above proof, we need to show that $\|g_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} < \infty$.

For $r > 0$ fixed and $x \in \mathbf{X}$, since Ω is a doubling measure, there is a constant C only depending on the doubling constant of Ω such that for any $y \in B(x,r)$,

$$P_{\Omega,r}|f|(x) \leq CM_{\Omega,r}f(y) \leq CM_\Omega f(y).$$

Thus

$$\begin{aligned}
[P_{\Omega,r}|f|(x)]^{1/q} &\leq \frac{C}{\nu(B(x,r))}\int_{B(x,r)}(M_\Omega f(y))^{1/q}d\nu(y) \\
&= CP_{\nu,r}[(M_\Omega f)^{1/q}](x).
\end{aligned}$$

Now if f belongs to $L^{p/q}(\Omega)$, the dual space of $L^{\frac{1}{1-\alpha}}(\Omega)$, we have

$$\begin{aligned}
&|\int g_r f d\Omega| \\
\leq\ & \int P_{\Omega,r}|f|(x)d\mu(x) \\
=\ & \int [P_{\Omega,r}|f|(x)]^{(1/q)q}d\mu(x)
\end{aligned}$$

$$\leq C\int (P_{\nu,r}[(M_\Omega f)^{1/q}](x))^q d\mu(x)$$
$$\leq CC_r[\int (M_\Omega f)^{p/q} d\Omega]^{q/p} \quad (by\ assumption)$$
$$\leq CC_r[\int |f|^{p/q} d\Omega]^{q/p} < \infty.$$

The last inequality follows from Corollary 3.5 with $\mu = \Omega$. Thus the constant C in the last inequality is independent of r. Therefore $g_r \in L^{\frac{1}{1-\alpha}}(\Omega)$ and $\|g_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \leq CC_r$. The proof is complete.

Next we turn to discuss Hardy-Littlewood maximal operator.

Note that under the assumption that continuous compact supported functions are dense in $L^1(\nu)$, Calderón showed [6] that if $\omega \in A_p$, then

$$\lim_{r\to 0} P_{\Omega,r} f(x) = f(x)$$

almost everywhere on $\mathbf{X}$. In particular, $M_\nu f(x) \geq |f(x)|$ almost everywhere. Then $\|M_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ implies $\|f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$. Therefore $d\mu = g d\Omega$ for some g. Now it not hard to prove that $\|M_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$ if and only if $g \in L^{\frac{1}{1-\alpha}}(\Omega)$.

In a general homogeneous space, applying the method used in the proof of Theorem 3.11, we can obtain the following two-weight norm inequality for Hardy-Littlewood maximal operator M_ν.

Theorem 3.12 *Let $\mathbf{X}$ be a general homogeneous space. Suppose $p > \gamma \geq 1$ and $\omega \in A_\gamma$. Let $q > 0$ and $q/p = \alpha < 1$. If*

$$\sup_{r>0} \|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \leq C < \infty,$$

then

$$(\int_{\mathbf{X}} |M_\nu f|^q d\mu)^{1/q} \leq C(\int_{\mathbf{X}} |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega).$$

Conversely, let $0 < q < p$ and $\alpha = q/p$. If Ω is a doubling measure on $\mathbf{X}$, and if for any $r > 0$

$$(\int_{\mathbf{X}} |P_{\nu,r} f|^q d\mu)^{1/q} \leq C(\int_{\mathbf{X}} |f|^p d\Omega)^{1/p} \qquad f \in L^p(\Omega),$$

with C independent of r, then

$$\sup_{r>0} \|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \leq C < \infty.$$

Proof: Suppose $\sup_{r>0} \|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \leq C < \infty$. From the proof of Theorem 3.11, we have

$$\int |M_{\nu,r} f|^q d\mu \leq C \sup_r \|K_r\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \|f\|^q_{L^p(\Omega)}$$

with C independent of r.

Now let $r \to 0$, since $M_{\nu,r} f$ increases, it follows from Fatou's lemma that

$$\|M_\nu f\|_{L^q(\mu)} \leq C \|f\|_{L^p(\Omega)}.$$

The converse part is a direct consequence of the proof of Theorem 3.11. The proof is complete.

Chapter 4

CARLESON MEASURE THEOREM IN WEIGHTED BERGMAN SPACES

Let **U** be the unit ball in $\mathbf{C}^N$ and Ω be a positive measure on **U** satisfying Békollé's B^p_β condition. We characterize those positive measures μ on **U** such that the inequality $\|f\|_{L^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$ $(1 < p \leq q)$ holds for any function f in the weighted Bergman space $A^p(\Omega)$. As an application, we characterize the multipliers from $A^p(\Omega)$ to $A^q(\Omega)$ $(q \geq p)$.

§4.1 Carleson Measure Theorem in Weighted Bergman Spaces

In [8], Cima and Wogen proved the following Carleson measure theorem for $A^2(dm_\beta)$ in the unit ball **U** of $\mathbf{C}^N$:

Theorem 4.1 *Let $\beta > -1$. Then*

$$\int_U |f|^2 d\mu \leq C \int_U |f|^2 dm_\beta$$

for any $f \in L^2(dm_\beta)$ if and only if for some fixed r, $0 < r < 1$,

$$\mu(E(a,r)) \leq C m_\beta(E(a,r)) \qquad a \in U.$$

In [14], Luecking developed a general technique to find a sufficient condition for $\|f\|_{L^p(\mu)} \leq C\|f\|_{L^p(\Omega)}$.

The following lemma is a generalization of Luecking's work in a homogeneous space.

Lemma 4.2 *Let $\delta > 0$ and let $(\mathbf{X}, \nu)$, $E(x)$, $\alpha \geq 1$ as in Lemma 3.2 . Let μ be a positive measure such that*

$$\mu(E^2(x)) \leq C_0[\nu(E^2(x))]^\alpha.$$

Let $C_1 > 0$. Then for $p > \delta$, $q/p = \alpha$, and any f satisfying

$$|f(x)|^\delta \leq \frac{C_1}{\nu(E(x))} \int_{E(x)} |f|^\delta d\nu \qquad x \in \mathbf{X},$$

there is a $C > 0$ such that

$$\|f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\nu)} \qquad f \in L^p(\nu).$$

Proof: We may assume $f \in L^p(\nu)$. Since $q \geq p > \delta$ and $\frac{\frac{q}{\delta}}{\frac{p}{\delta}} = \alpha$, Lemma 3.2 implies that

$$\begin{aligned}
& [\int |f|^q d\mu]^{1/q} \\
= \ & [\int (|f|^\delta)^{\frac{q}{\delta}} d\mu]^{1/q} \\
\leq \ & C[\int (m(|f|^\delta))^{\frac{q}{\delta}} d\mu]^{1/q} \\
\leq \ & C[\int (|f|^\delta)^{\frac{p}{\delta}} d\nu]^{1/p} \\
= \ & C[\int |f|^p d\nu]^{1/p}.
\end{aligned}$$

The proof is complete.

In this section, we shall work with the homogeneous space $(\mathbf{U}, \omega dm_\beta, \rho)$.

We shall refer all definitions and notations in this chapter to §1.2.

Recall that $\rho(z,\xi) = |\Phi_\xi(z)|$ is a metric on $\mathbf{U}$ and $E(x,r) = \{w \in \mathbf{U} : \rho(z,w) < r\}$.

As the second application of the generality stated in the beginning of section 3.1, chapter 3, we shall take $E(z) = E(z, r/3)$ for some fixed r, $0 < r < 1$, $E^2(z) = E(z,r)$. Then the A_γ ($\gamma > 1$) condition in the space $(\mathbf{U}, \omega dm_\beta, \rho)$ is equivalent to the C_γ condition defined in §1.2. By Hölder's inequality and the fact that m_β is a doubling measure, $d\Omega = \omega dm_\beta$ is a doubling measure. Therefore $(\mathbf{U}, \Omega, \rho)$ becomes a homogeneous space. From Lemma 1.2 , the assumptions of Lemma 3.2 are satisfied by $(\mathbf{U}, \Omega, \rho)$.

In [16, Lemma 3.1], D. Luecking proved the following:

Lemma 4.3 *If ω satisfies the C_γ condition for some $\gamma > 1$ and $d\Omega = \omega dm_\beta$, then for any f analytic in $\mathbf{U}$, any $q > 0$, and any $z \in \mathbf{U}$,*

$$|f(z)|^q \leq C \frac{\int_{E(z)} |f|^q d\Omega}{\Omega(E(z))}$$

with C depends only on β, γ, r, and C_γ constant.

From Lemma 4.3 and Lemma 4.2 , we have the following generalization of Theorem 4.1:

Theorem 4.4 *Let $\alpha \geq 1$. Let $p, q > 0$ such that $q/p = \alpha$. If ω satisfies the C_γ condition for some $\gamma > 1$ and μ is a positive α-Carleson measure w. r. t. (Ω, r), then for any $f \in A^p(\Omega)$*

$$[\int |f|^q d\mu]^{1/q} \leq C[\int |f|^p d\Omega]^{1/p}.$$

We next prove that being an α-Carleson measure is also a necessary condition for $\|f\|_{A^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$ if ω satisfies $B^p_\beta(\omega)$ condition.

We shall use the following well known facts in the proof of next two lemmas.

(1) For every $a \in \mathbf{B}$, $\Phi_a(0) = a$ and $\Phi_a(a) = 0$.

(2) The identity

$$1-<\Phi_a(z),\Phi_a(w)>=\frac{(1-<a,a>)(1-<z,w>)}{(1-<z,a>)(1-<a,w>)}$$

holds for all $z\in\bar{\mathbf{B}}$, $w\in\bar{\mathbf{B}}$.

(3) The identity

$$1-|\Phi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-<z,a>|^2}$$

holds for every $z\in\bar{\mathbf{B}}$.

(4) The real Jacobian of Φ_a at $z\in\mathbf{B}$ is

$$(J_R\Phi_a)(z)=\left(\tfrac{1-|a|^2}{|1-<z,a>|^2}\right)^{N+1}.$$

For the proof of these facts, see [19, p.26].

Lemma 4.5 *Let $a\in\mathbf{U}$ and $0<r<1$. Then*

$$\sup\{|1-<a,z>|:z\in E(a,r)\}=(1-|a|^2)(1-r|a|)^{-1}$$

Proof:

$$\begin{aligned}
&\sup\{|1-<a,z>|:z\in E(a,r)\}\\
=\ &\sup\{|1-<\Phi_a(0),\Phi_a(\lambda)>|:\lambda\in r\mathbf{U}\}\\
=\ &\sup\{|\frac{1-|a|^2}{1-<a,\lambda>}|:\lambda\in r\mathbf{U}\}\\
=\ &(1-|a|^2)(1-r|a|)^{-1}.
\end{aligned}$$

The proof is complete.

Recall that T_β is the Bergman operator and

$$T_\beta^*f(z)=\begin{pmatrix}N+\beta\\N\end{pmatrix}\int_{\mathbf{U}}|K_\beta(z,w)|f(w)dm_\beta(w)\qquad z\in\mathbf{U}.$$

Lemma 4.6 *Let $p > 1$ and $q/p = \alpha$. Suppose ω satisfies the $B^p_\beta(\omega)$ condition and $d\Omega = \omega dm_\beta$. Then $\|f\|_{A^q(\mu)} \le C\|f\|_{A^p(\Omega)}$ implies that for fixed $r > 0$*

$$\mu(E(a,r)) \le C_r[\Omega(E(a,r))]^\alpha$$

for any $a \in \mathbf{U}$.

Proof: Suppose for any $f \in A^p(\Omega)$, $\|f\|_{A^q(\mu)} \le C\|f\|_{A^p(\Omega)}$. For any $a \in \mathbf{U}$, take

$$f(z) = \int_{\mathbf{U}} \frac{\chi_{E(a,r)}(w)(1-<a,w>)^\beta}{(1-<z,w>)^{N+1+\beta}} dm(w).$$

Then $f(z)$ is analytic and

$$\begin{aligned}
\|f\|^p_{A^p(\Omega)} &= \int_{\mathbf{U}} \left| \int_{\mathbf{U}} \frac{\chi_{E(a,r)}(w)(1-<a,w>)^\beta}{(1-<z,w>)^{N+1+\beta}} dm(w)\right|^p d\Omega(z) \\
&\le \int_{\mathbf{U}} \left| \int_{\mathbf{U}} \frac{\chi_{E(a,r)}(w)|1-<a,w>|^\beta}{|1-<z,w>|^{N+1+\beta}(1-|w|^2)^\beta} dm_\beta(w)\right|^p d\Omega(z) \\
&= \|T^*_\beta(\chi_{E(a,r)}(w)|\frac{1-<a,w>}{1-|w|^2}|^\beta)(z)\|^p_{L^p(\Omega)}.
\end{aligned}$$

Since ω satisfies $B^p_\beta(\omega)$ condition, Theorem 1.13 implies that T^*_β is bounded on $L^p(\Omega)$. Hence

$$\begin{aligned}
\|f\|^p_{A^p(\Omega)} &\le C\|\chi_{E(a,r)}(w)|\frac{1-<a,w>}{1-|w|^2}|^\beta\|^p_{L^p(\Omega)} \\
&= C\int_{\mathbf{U}} \chi_{E(a,r)}(w)|\frac{1-<a,w>}{1-|w|^2}|^{\beta p} d\Omega(w) \\
&\le C\int_{\mathbf{U}} \chi_{E(a,r)}(w)\frac{\sup_{E(a,r)}|1-<a,w>|^{\beta p}}{|1-|w|^2|^{\beta p}} d\Omega(w) \\
&\le C\int_{\mathbf{U}} \chi_{E(a,r)}(w)\frac{(1-|a|^2)^{\beta p}(1-r|a|)^{-\beta p}}{|1-|w|^2|^{\beta p}} d\Omega(w) \\
&\le C_r\int_{E(a,r)} \frac{(1-|a|^2)^{\beta p}}{|1-|w|^2|^{\beta p}} d\Omega(w).
\end{aligned}$$

Since on $E(a,r)$, $(1-|w|^2) \sim (1-|a|^2)$, we have

$$\|f\|^p_{A^p(\Omega)} \le C_r\int_{E(a,r)} d\Omega(w) = C_r\Omega(E(a,r)).$$

On the other hand

$$\begin{aligned}
\|f\|^q_{A^q(\mu)} &= \int_{\mathbf{U}} \Big| \int_{\mathbf{U}} \frac{\chi_{E(a,r)}(w)(1-<a,w>)^\beta}{(1-<z,w>)^{N+1+\beta}} dm(w)\Big|^q d\mu(z) \\
&\geq \int_{E(a,r)} \Big| \int_{E(a,r)} \frac{(1-<a,w>)^\beta}{(1-<z,w>)^{N+1+\beta}} dm(w)\Big|^q d\mu(z).
\end{aligned}$$

Let $w = \Phi_a(\lambda)$ in the second integral, then $z = \Phi_a(\eta)$ for some η and $\lambda, \eta \in r\mathbf{U}$. Thus

$$\begin{aligned}
&\|f\|^q_{A^q(\mu)} \\
\geq\ & \int_{E(a,r)} \Big| \int_{r\mathbf{U}} \frac{(1-<\Phi_a(0),\Phi_a(\lambda)>)^\beta}{(1-<\Phi_a(\eta),\Phi_a(\lambda)>)^{N+1+\beta}} \Big(\frac{1-|a|^2}{|1-<\lambda,a>|^2}\Big)^{N+1} dm(\lambda)\Big|^q d\mu(z) \\
=\ & \int_{E(a,r)} \Big| \int_{r\mathbf{U}} \Big(\frac{1-|a|^2}{1-<a,\lambda>}\Big)^\beta \Big(\frac{(1-<\eta,a>)(1-<a,\lambda>)}{(1-|a|^2)(1-<\eta,\lambda>)}\Big)^{N+1+\beta} \\
& \times \Big(\frac{1-|a|^2}{|1-<\lambda,a>|^2}\Big)^{N+1} dm(\lambda)\Big|^q d\mu(z) \\
=\ & \int_{E(a,r)} \Big|(1-<\eta,a>)^{N+1+\beta} \int_{r\mathbf{U}} \frac{dm(\lambda)}{(1-<\lambda,\eta>)^{N+1+\beta}(1-<a,\lambda>)^{N+1}}\Big|^q d\mu(z) \\
\overset{\lambda = rt}{=}\ & \int_{E(a,r)} \Big|(1-<\eta,a>)^{N+1+\beta} \int_{\mathbf{U}} \frac{r^{2N} dm(t)}{(1-<rt,\eta>)^{N+1+\beta}(1-<a,rt>)^{N+1}}\Big|^q d\mu(z) \\
=\ & C \int_{E(a,r)} \Big|(1-<\eta,a>)^{N+1+\beta} r^{2N} \Big[T_0\Big(\frac{1}{(1-<rt,\eta>)^{N+1+\beta}}\Big)(ra)\Big]\Big|^q d\mu(z) \\
=\ & C \int_{E(a,r)} \Big|(1-<\eta,a>)^{N+1+\beta} r^{2N} \frac{1}{(1-<r^2a,\eta>)^{N+1+\beta}}\Big|^q d\mu(z) \\
\geq\ & C \int_{E(a,r)} \Big(|1-r|^{N+1+\beta} r^{2N} \frac{1}{2^{N+1+\beta}}\Big)^q d\mu(z) \\
\geq\ & C_r \mu(E(a,r)).
\end{aligned}$$

Since $\|f\|_{A^q(\mu)} \leq C\|f\|_{A^p(\Omega)}$, it follows that

$$\mu(E(a,r)) \leq C_r[\Omega(E(a,r))]^\alpha$$

with C_r only depends on r. The proof is complete.

Since the $B^p_\beta(\omega)$ condition implies the C_p condition, combining Theorem 4.4 and Lemma 4.6 , we have proved the following:

Theorem 4.7 *Let $\alpha \geq 1$ and $q \geq p > 1$ such that $q/p = \alpha$. Suppose ω satisfies $B^p_\beta(\omega)$ condition. Then*

$$\|f\|_{A^q(\mu)} \leq C\|f\|_{A^p(\Omega)} \qquad f \in A^p(\Omega),$$

if and only if $\mu(E(a,r)) \leq C_r[\Omega(E(a,r))]^{\alpha}$ *for any* $a \in \mathbf{U}$.

From Theorem 1.13, $\omega \in B^p_\beta(\omega)$ implies that $\|T_\beta f\|_{A^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$. Note that $T_\beta f$ is analytic, we have

Corollary 4.8 *Under the assumption of Theorem 4.7 , for any* $f \in L^p(\Omega)$,

$$\|T_\beta f\|_{A^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

if and only if $\mu \in V^{\alpha}_{\Omega,r}$.

We close this section by considering the case $q < p$.

Theorem 4.9 *Let* $\alpha = q/p < 1$ *and* $1 > r > 0$. *Then*

1. *If* $0 < q$, $p > max\{1, q\}$ *and* $\omega \in B^p_\beta(\omega)$, *then* $\frac{\mu(B(z,r))}{\Omega(B(z,r))} \in L^{\frac{1}{1-\alpha}}(\Omega)$ *for some* r *implies*

$$\|T_\beta f\|_{A^q(\mu)} \leq C\|f\|_{L^p(\Omega)} \qquad f \in L^p(\Omega).$$

In particular, $\frac{\mu(B(z,r))}{\Omega(B(z,r))} \in L^{\frac{1}{1-\alpha}}(\Omega)$ *for some* r *implies*

$$\|f\|_{A^q(\mu)} \leq C\|f\|_{A^p(\Omega)} \qquad f \in A^p(\Omega).$$

2. *If* $0 < q < p$ *and* Ω *is a doubling measure, then*

$$\|T^*_\beta f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)} \qquad f \in L^p(\Omega),$$

implies $\frac{\mu(B(z,r))}{\Omega(B(z,r))} \in L^{\frac{1}{1-\alpha}}(\Omega)$ *for any* $r > 0$.

Proof: 1. Since $T_\beta f$ is analytic, $\omega \in B^p_\beta(\omega)$ implies (by Lemma 4.3)

$$|T_\beta f(z)| \leq CM_{\Omega,r}[T_\beta f(z)].$$

Since $B^p_\beta(\omega)$ implies C_p which is equivalent to the A_p condition in $(\mathrm{U}, \omega dm_\beta, \rho)$. By Theorem 3.11 and Theorem 1.13 , if $\mu \in W^\alpha_{\Omega,r}$ and $\omega \in B^p_\beta(\omega)$ then

$$\begin{aligned} \|T_\beta f\|_{A^q(\mu)} &\leq C\|M_{\Omega,r}[T_\beta f]\|_{L^q(\mu)} \\ &\leq C\|T_\beta f\|_{A^p(\Omega)} \\ &\leq C\|f\|_{L^p(\Omega)}. \end{aligned}$$

2. For any $1 > r > 0$, by the fact that

$$m_\beta(E(a,r)) \sim (1-|a|^2)^{N+1+\beta}$$

and Lemma 4.5, there is a constant $C_r > 0$ such that $P_{m_\beta,r} f \leq C_r T^*_\beta |f|$. Therefore

$$\|P_{m_\beta,r} f\|_{L^q(\mu)} \leq C_r \|T^*_\beta |f|\|_{L^q(\mu)} \leq C_r \|f\|_{L^p(\Omega)}.$$

By Theorem 3.11, $\mu \in W^\alpha_{\Omega,r}$.

§4.2 Multipliers on Weighted Bergman Spaces

Let $M(p,\beta,\gamma)$ denote the collection of all functions f which multiply $A^p(\omega dm_\beta)$ into $A^p(\omega dm_\gamma)$, that is, $fg \in A^p(\omega dm_\gamma)$ for any $g \in A^p(\omega dm_\beta)$.

Let $N(p,q,\beta)$ denote the collection of all functions f which multiply $A^p(\omega dm_\beta)$ into $A^q(\omega dm_\beta)$.

In [25], G. D. Taylor proved that

(1) if $\beta > \gamma$, $M(2,\beta,\gamma) = \{0\}$;

(2) if $\beta \leq \gamma$, $M(2,\beta,\gamma) = \{f : f \text{ is analytic, } |f(z)| = O(1-|z|)^{\frac{\beta-\gamma}{2}}\}$.

In [2], K. R. M. Attele proved that

(1) if $p < q$, $N(p,q,\beta) = \{0\}$;

(2) if $p = q$, $N(p,p,\beta) = H^\infty$;

(3) if $p > q$, $N(p,q,\beta) = \{f \in A^\gamma(dm_\beta) : \frac{1}{\gamma} = 1/q - 1/p\}$.

The (3) of Attele's result has been generalized by Luecking (see [17]).

Applying Theorem 4.7 , we have the following results for the weighted Bergman spaces.

Theorem 4.10 *Let $1 < p < \infty$. Suppose ω satisfies the $B^p_\beta(\omega)$ condition. Then*

(1) if $\beta > \gamma$, $M(p,\beta,\gamma) = \{0\}$;

(2) if $\beta \leq \gamma$, $M(p,\beta,\gamma) = \{f : f$ is analytic, $|f(z)| = O(1-|z|)^{\frac{\beta-\gamma}{p}}\}$.

Proof: Since $f \in M(p,\beta,\gamma)$ if and only if for any $g \in A^p(\omega dm_\beta)$

$$\int |gf|^p \omega dm_\gamma \leq C \int |g|^p \omega dm_\beta,$$

from Theorem 4.7, we have that $f \in M(p,\beta,\gamma)$ if and only if for any $0 < r < 1$, there is a $C > 0$ depending only on r, such that for any $z \in \mathbf{U}$,

$$\int_{E(z,r)} |f|^p \omega dm_\gamma \leq C \int_{E(z,r)} \omega dm_\beta.$$

Let $d\Omega = \omega dm_\gamma$. Since $m_\alpha(E(z,r)) \sim (1-|z|^2)^\alpha$ for any $\alpha > 1$, the above inequality is equivalent to

$$\frac{1}{\Omega(E(z,r))} \int_{E(z,r)} |f|^p d\Omega \leq C(1-|z|^2)^{\beta-\gamma}.$$

Then Lemma 4.3 implies that

$$|f(z)|^p \leq C(1-|z|)^{\beta-\gamma}.$$

Conversely, it is clear that $|f(z)|^p \leq C(1-|z|)^{\beta-\gamma}$ for any $z \in \mathbf{U}$ implies that $f \in M(p,\beta,\gamma)$. Therefore , if $\beta > \gamma$, letting $|z| \to 1$, it follows that $f \equiv 0$; if $\beta \leq \gamma$, then $|f(z)| = O(1-|z|)^{\frac{\beta-\gamma}{p}}$. The proof is complete.

Let $H^\infty = \{f : f$ is a bounded analytic function in $\mathbf{U}\}$.

Theorem 4.11 *Let $p > 1$ and $q > 0$. If ω satisfies the condition $B^p_\beta(\omega)$, then $N(p,p,\beta) = H^\infty$ and $N(p,q,\beta) = \{0\}$ if $p < q$.*

Proof: Let $p \leq q$. By Hölder's inequality $B^p_\beta(\omega)$ implies $B^q_\beta(\omega)$. Let $d\Omega = \omega dm_\beta$. Similar to the proof of Theorem 4.10 , $f \in N(p,q,\beta)$ if and only if

$$|f(z)|^q \leq \frac{1}{\Omega(E(z,r))}\int_{E(z,r)} |f|^q \omega dm_\beta \leq C[\Omega(E(z,r))]^{q/p-1}$$

with C depending only on r.

If $p = q$, it is clear that $f \in H^\infty$; if $p < q$ letting $|z| \to 1$, it follows that $f(z) \equiv 0$ on $\mathbf{U}$. The proof is complete.

We close this section by giving an example of Theorem 4.10 and Theorem 4.11.

Let $\{\phi(r), \psi(r)\}$ be the normal pair defined in (1.1), Chapter 1. Let $\lambda > 0$ be the real number in (1.2), Chapter 1.

For a normal function $\phi(r)$, if $p > 1$, there exists a nonnegative number $t \geq 0$ such that $(\phi(r)(1-r)^{-t})^{-\frac{p'}{p}}$ is integrable in $L^1(dm_{\beta+t})$. We may assume that t is big enough.

We now prove that $W = \phi(r)(1-r)^{-t}$ satisfies $B^p_{\beta+t}(W)$.

In fact, fix $z_0 \in \mathbf{U}$, denote $K = K(z_0)$. Since $\frac{\phi(r)}{(1-r)^a}$ is non-increasing and if $z \in K(z_0)$, $|z| > |z_0|$.

$$\begin{aligned}
&\int_K W dm_{\beta+t}(z) \\
\leq\ & C\int_K \frac{\phi(r)}{(1-r)^a} dm_{\beta+t+a}(z) \\
\leq\ & C\frac{\phi(|z_0|)}{(1-|z_0|)^a}\int_K dm_{\beta+t+a}(z) \\
\leq\ & C\frac{\phi(|z_0|)}{(1-|z_0|)^a}(1-|z_0|^2)^{\beta+t+a+N+1} \\
=\ & C\phi(|z_0|)(1-|z_0|^2)^{\beta+t+N+1}.
\end{aligned}$$

The third inequality follows from $m_{\beta+t+a}(K) \sim (1-|z_0|^2)^{\beta+t+a+N+1}$.

Similarly, note that

$$[\phi(r)(1-r)^{-t}]^{-\frac{p'}{p}} = C(\psi(r))^{\frac{p'}{p}}(1-r)^{(t-\lambda)\frac{p'}{p}}.$$

Hence

$$\begin{aligned}
&\int_K W^{-\frac{p'}{p}} dm_{\beta+t}(z) \\
= \;& C\int_K (\psi(r))^{\frac{p'}{p}} dm_{\beta+t+(t-\lambda)\frac{p'}{p}}(z) \\
\leq \;& C(\psi(|z_0|))^{\frac{p'}{p}}(1-|z_0|^2)^{\beta+t+(t-\lambda)\frac{p'}{p}+N+1}.
\end{aligned}$$

Now it is clear that $B^p_{\beta+t}(W)$ is satisfied.

Since $A^p(\phi(r)dm_\beta) = A^p(Wdm_{\beta+t})$ and $A^q(\phi(r)dm_\gamma) = A^q(Wdm_{\gamma+t})$, Theorem 4.10 and Theorem 4.11 imply the following:

Theorem 4.12 *Consider the spaces $A^p(\phi(r)dm_\beta)$ and $A^q(\phi(r)dm_\gamma)$, where $\phi(r)$ is a normal function. Then*

(1) If $\beta > \gamma$, $M(p,\beta,\gamma) = \{0\}$;

(2) If $\beta \leq \gamma$, $M(p,\beta,\gamma) = \{f : f$ is analytic, $|f(z)| = O(1-|z|)^{\frac{\beta-\gamma}{p}}\}$;

(3) $N(p,p,\beta) = H^\infty$ and $N(p,q,\beta) = \{0\}$ if $p < q$.

Chapter 5

BERGMAN OPERATOR IN WEIGHTED MIXED-NORMED SPACES

In this chapter, we use an interpolation theorem between weighted norm spaces to determine the weighted mixed norm spaces on $\mathbf{U}$, the unit ball of $\mathbf{C}^N$, as the interpolation spaces between the L^p spaces on $\mathbf{U}$ and the L^p spaces on the boundary S of $\mathbf{U}$ with different weights. Using these facts, we prove that for some appropriate weights, the Bergman operator induces a bounded projection on the weighted mixed norm space. Thus we are able to identify the dual of those weighted mixed norm spaces of analytic functions.

In section 1 we give some preliminaries. In section 2 we prove an interpolation theorem of mixed norm spaces. We shall present the main result of this chapter in section 3. Several duality theorems will be presented in the last section.

§5.1 Preliminaries

We shall refer all definitions and notations in this chapter to §1.2.

Let $\{\varphi, \psi\}$ be the normal pair as in (1.2), Chapter 1. Suppose that $A = \varphi^p(r)$, $B = \omega(r)(1-r^2)^\alpha$, $C \equiv 1$. We shall need the following lemmas.

Lemma 5.1

$$\int_0^1 (1-\rho r)^{-\lambda}(1-r)^{-1}\varphi(r)dr \leq C\psi^{-1}(\rho) \qquad 0 \leq \rho < 1.$$

For the proof, see [23, p.291].

Lemma 5.2 *If* $t > 0$, $w \in \mathbf{U}$, *then*

$$\int_{\mathbf{S}} \frac{d\nu_0(z)}{|1-<z,w>|^{N+t}} = O(\frac{1}{(1-|w|)^t}).$$

For the proof, see [19, p.17].

Lemma 5.3 *For* $\gamma > -1$, *and* $m > 1+\gamma$,

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\gamma}dr \leq C(1-\rho)^{1+\gamma-m} \qquad 0 \leq \rho < 1.$$

For the proof, see [23, p.291].

Lemma 5.4 *For any* $f \in H^\infty$, $T_\alpha(f) = f$.

For the proof, see [19, p.121].

Let $L^{p,q}$ and $H^{p,q}$ as in Definition 1.11, §1.2.

Lemma 5.5 *For* $1 \leq p < \infty$, $1 \leq q < \infty$, $\|f_r - f\|_{H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)} \to 0$ *as* $r \to 1^-$.

This follows immediately from the dominated convergence theorem. (For details, see [22, Proposition 3.3]).

We shall use the following pairing between functions in $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ and functions in $L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$:

$$<f,g>=\int_{\mathbf{U}} f(z)\bar{g}(z)dm_\alpha(z). \tag{5.3}$$

In [4, p.304], A. Benedek and R. Panzone showed that the dual space of the mixed norm space $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ can be identified with $L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)$ under the pairing

$$<f,g>=\int_{\mathbf{U}} f(z)\bar{g}(z)\varphi^q\omega dm_\alpha(z).$$

Lemma 5.6 *For $1 \leq p < \infty$, $1 \leq q < \infty$, under the pairing (5.3), the dual of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ can be identified with $L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$.*

Proof: For any linear functional L of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$, there is a unique function $h \in L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)$ such that for any $f \in L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$,

$$L(f) = \int_{\mathbf{U}} f(z)\bar{h}(z)\varphi^q(r)\omega(r)dm_\alpha(z)$$

and $\|L\| = \|h\|_{L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)}$ [4].

Let

$$g = h\varphi^q(r)\omega(r).$$

Then

$$\begin{aligned}
& \|g\|^{q'}_{L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)} \\
= & \int_0^1 \left(\int_{\mathbf{S}} |g|^{p'}\varphi^{-p'}d\nu_0\right)^{\frac{q'}{p'}}\omega^{-\frac{q'}{q}}r^{2N-1}(1-r^2)^\alpha dr \\
= & \int_0^1 \left(\int_{\mathbf{S}} |h|^{p'}\varphi^{qp'}\omega^{p'}\varphi^{-p'}d\nu_0\right)^{\frac{q'}{p'}}\omega^{-\frac{q'}{q}}r^{2N-1}(1-r^2)^\alpha dr \\
= & \int_0^1 \left(\int_{\mathbf{S}} |h|^{p'}d\nu_0\right)^{\frac{q'}{p'}}\varphi^{qq'-q'}\omega^{q'-\frac{q'}{q}}r^{2N-1}(1-r^2)^\alpha dr \\
= & \|h\|_{L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)}.
\end{aligned}$$

Thus $g \in L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$ and

$$L(f) = \int_{\mathbf{U}} f(z)\bar{g}(z)dm_\alpha(z).$$

Conversely for any $g \in L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$, by Hölder's inequality

$$\int_{\mathbf{U}} f(z)\bar{g}(z)dm_\alpha(z) = L_g(f)$$

is a bounded linear functional on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$. The proof is complete.

Let $\mathbf{U}^{p,q}$ denote the unit ball of $L^{p,q}(B(r))$.

Lemma 5.7

$$\|f\|_{L^{p,q}(B)} = \sup_{g \in \mathbf{U}^{p',q'}} \int_{\mathbf{U}} f\bar{g}B(r)dm.$$

For the proof see [4, p.303].

Lemma 5.8 *Let* $0 < p < \infty$, $0 < q < \infty$. *Suppose* $\omega_1(r), \omega_2(r) \in L^1(dr)$ *are two positive functions on [0,1). If there exists a* $r_0 > 0$ *such that for* $r_0 < r < 1$, $\omega_1 \sim \omega_2$, *then*

$$H^{p,q}(\omega_1(r)) \sim H^{p,q}(\omega_2(r)).$$

Proof: There are $C_1, C_2 > 0$ such that if $r_0 < r < 1$, $C_1\omega_2(r) < \omega_1(r) < C_2\omega_2(r)$. Let $z \in \mathbf{S}$ and $f \in H^{p,q}(\omega_1(r))$. Let

$$I = \int_0^{r_0} (\int_{\mathbf{S}} |f|^p d\nu_0)^{\frac{q}{p}} \omega_1 r^{2N-1} dr.$$

Since if f is analytic, then $\int_{\mathbf{S}} |f(r_0 z)|^p d\nu_0$ is an increasing function of r. Thus

$$I \leq \int_0^{r_0} \omega_1(r) r^{2N-1} dr (\int_{\mathbf{S}} |f(r_0 z)|^p d\nu_0(z))^{\frac{q}{p}}.$$

Let

$$C(r_0) = (\int_{r_0}^1 \omega_2(\rho)\rho^{2N-1} d\rho)^{-1}.$$

Then

$$\begin{aligned}
I &= C(r_0) \int_{r_0}^1 I\omega_2(\rho)\rho^{2N-1} d\rho \\
&\leq C \int_{r_0}^1 (\int_{\mathbf{S}} |f(r_0 z)|^p d\nu_0(z))^{\frac{q}{p}} \omega_2(\rho)\rho^{2N-1} d\rho \\
&\leq C \int_{r_0}^1 (\int_{\mathbf{S}} |f(\rho z)|^p d\nu_0(z))^{\frac{q}{p}} \omega_2(\rho)\rho^{2N-1} d\rho \\
&\leq C\|f\|^q_{H^{p,q}(\omega_2)},
\end{aligned}$$

where C depends only on r_0.

Therefore

$$\begin{aligned}
&\|f\|^q_{H^{p,q}(\omega_1)} \\
=\;& \int_0^1 \Big(\int_{\mathbf{S}} |f|^p d\nu_0(z)\Big)^{\frac{q}{p}} \omega_1(r) r^{2N-1} dr \\
=\;& \Big(\int_0^{r_0} + \int_{r_0}^1\Big)\Big(\int_{\mathbf{S}} |f|^p d\nu_0(z)\Big)^{\frac{q}{p}} \omega_1^{2N-1} dr \\
\le\;& C(r_0)\|f\|^q_{H^{p,q}(\omega_2)} + C_2 \int_{r_0}^1 \Big(\int_{\mathbf{S}} |f|^p d\nu_0(z)\Big)^{\frac{q}{p}} \omega_2^{2N-1} dr \\
\le\;& (C(r_0) + C_2)\|f\|^q_{H^{p,q}(\omega_2)}.
\end{aligned}$$

Similarly

$$\|f\|^q_{H^{p,q}(\omega_2)} \le C\|f\|^q_{H^{p,q}(\omega_1)}.$$

The proof is complete.

§5.2 Interpolation Spaces

Throughout this section, we will follow the notations of [5].

We first list some basic definitions of real interpolation method.

Definition 5.9 *Let X_0, X_1 be two topological vector spaces. X_0, X_1 are said to be compatible if there is a Hausdorff topological vector space U such that X_0, X_1 are sub-spaces of U.*

Let $\bar{X} = (X_0, X_1)$ denote a compatible couple of two quasi-normed spaces X_0 andX_1.

Definition 5.10 *Let $a \in \sum_{i=0}^1 X_i$. Define*

$$\begin{aligned}
K(t,a) &= K(t,a;X_0,X_1) \\
&= inf\{\|a_0\|_{X_0} + t\|a_1\|_{X_1} : a = a_0 + a_1, a_0 \in X_0, a_1 \in X_1\}.
\end{aligned}$$

$$\begin{aligned}
\|a\|^q_{\theta,q,\bar{X}} &= \int_0^\infty [t^\theta K(t,a)]^q \frac{dt}{t} \qquad\qquad 0<\theta<1, 0<q\le\infty. \\
(X_0,X_1)_{\theta,q,\bar{X}} &= \{a \in \sum_{i=0}^{1} X_i : \|a\|_{\theta,q,\bar{X}} < \infty\}.
\end{aligned}$$

Theorem 5.11 *Let* $\bar{X} = (X_0, X_1), \bar{Y} = (Y_0, Y_1)$. *Suppose T is a linear map from $\sum_{i=0}^1 X_i \longrightarrow \sum_{i=0}^1 Y_i$ such that for any $a_i \in X_i$, $i = 0,1$,*

$$\|Ta_i\|_{Y_i} \le K_i \|a_i\|_{X_i}.$$

Then

$$T : (X_0, X_1)_{\theta,q,\bar{X}} \longrightarrow (Y_0, Y_1)_{\theta,q,\bar{Y}}$$

with

$$\|Ta\|_{\theta,q,\bar{Y}} \le K_0^{1-\theta} K_1^\theta \|a\|_{\theta,q,\bar{X}}.$$

For the proof, see [5].

Throughout this section, we will assume that $C(r)$ is a non-negative function on [0,1) such that $|\{r \in [0,1) : C(r) = 0\}| = 0$, where $|E|$ is the Lebesgue measure of E on [0,1). Suppose $B(r)$ is a non-negative function on [0,1) and suppose that the measure μ, defined by $d\mu = B(r)C(r)^{-\gamma} r^{2N-1} dr$, where $0 < \gamma < \infty$, is a σ-finite measure on [0,1). Let

$$\begin{aligned}
m(\rho, f) &= \mu\{r \in [0,1) : \|f_r\|_{A,p} C(r) > \rho\}. \\
f^*(t) &= inf\{\rho : m(\rho, f) \le t\}. \\
\|f\|^q_{\tau,q} &= \int_0^\infty [t^{\frac{1}{\tau}} f^*(t)]^q \frac{dt}{t} \qquad\qquad 0<\tau\le\infty, 0<q\le\infty.
\end{aligned}$$

The vector valued Lorentz space $L(p,\tau,q)$ is defined by

$$L(p,\tau,q) = \{f : \|f\|_{\tau,q} \le \infty\}.$$

For the properties of $m(\rho, f)$, f^* and $L(p,\tau,q)$, see [12] and [5].

Lemma 5.12

$$L(p,q,q) = L^{p,q}(A, BC^{q-r}).$$

Proof:

$$\begin{aligned} \|f\|_{q,q}^q &= \int_0^\infty |f^*(t)|^q dt \\ &= \int_0^1 [\|f_r\|_{A,p} C(r)]^q d\mu \\ &= \int_0^1 \|f_r\|_{A,p}^q B(r) C(r)^{q-\gamma} r^{2N-1} dr \\ &= \|f\|_{L^{p,q}(A,BC^{q-\gamma})}^q. \end{aligned}$$

The proof is complete.

Assuming $\bar{L} = (L^{p,q}(A,B), L^{p,\infty}(A,C))$ is a compatible couple, we have the following vector valued version of Theorem 5.2.1 of [5].

Theorem 5.13 *Suppose* $f \in L^{p,\gamma}(A,B) + L^{p,\infty}(A,C)$, $1 \le p < \infty, 0 < \gamma < \infty$. *Then*

(1) $K(t, f; L^{p,\gamma}(A,B), L^{p,\infty}(A,C)) \sim (\int_0^{t^\gamma} |f^*(s)|^\gamma ds)^{\frac{1}{\gamma}}$;

(2) *For* $\gamma < q \le \infty, \frac{1}{\tau} = \frac{1-\theta}{\gamma}$

$$(L^{p,\gamma}(A,B), L^{p,\infty}(A,C))_{\theta,q,L} = L(p,\tau,q).$$

Proof: (1) The proof will follow from the argument in the proof of Theorem 5.2.1 in [5] once we make a decomposition of f.

(i) " $\le$ " part. For $z \in \mathbf{S}$, let

$$f_0(rz) = \begin{cases} f(rz) - \frac{f^*(t^\gamma) f(rz)}{\|f_r\|_{A,p} C(r)} & \text{if } \|f_r\|_{A,p} C(r) > f^*(t^\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and let $f_1 = f - f_0$.

Let

$$E = \{r \in [0,1) : \|(f_0)_r\|_{A,p} C(r) \neq 0\}.$$

Then since f^* is non-increasing, we have

$$\mu(E) = m(f^*(t^\gamma), f) = |\{s : f^*(s) > f^*(t^\gamma)\}| \le t^\gamma$$

and $f^*(s)$ is constant on $[\mu(E), t^\gamma]$. Thus

$$\begin{aligned}
& K(t, f; L^{p,\gamma}(A,B), L^{p,\infty}(A,C)) \\
\le\ & \|f_0\|_{L^{p,\gamma}(A,B)} + t\|f_1\|_{L^{p,\infty}(A,C)} \\
=\ & [\int_E \|f(rz) - \frac{f^*(t^\gamma) f(rz)}{\|f_r\|_{A,p} C(r)}\|_{A,p}^\gamma B(r) r^{2N-1} dr]^{\frac{1}{\gamma}} + t f^*(t^\gamma) \\
=\ & [\int_E \|\|f_r\|_{A,p} C(r) - f^*(t)\|_{A,p}^\gamma B(r) C(r)^{-\gamma} r^{2N-1} dr]^{\frac{1}{\gamma}} + t f^*(t^\gamma) \\
=\ & [\int_0^{\mu(E)} (f^*(s) - f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} + [\int_0^{t^\gamma} (f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} \\
=\ & [\int_0^{t^\gamma} (f^*(s) - f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} + [\int_0^{t^\gamma} (f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} \\
\le\ & 3[\int_0^{t^\gamma} (f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}}.
\end{aligned}$$

(ii) " $\ge$ " part. Assume $f = f_0 + f_1$, $f_0 \in L^{p,\gamma}(A,B)$, and $f_1 \in L^{p,\infty}(A,C)$. Since

$$\|(f_0 + f_1)_r\|_{A,p} C(r) \le \|(f_0)_r\|_{A,p} C(r) + \|(f_1)_r\|_{A,p} C(r),$$

we have

$$m(\rho_1 + \rho_2, f) \le m(\rho_1, f_0) + m(\rho_2, f_1).$$

Hence

$$\{\rho_1 + \rho_2 : m(\rho_1 + \rho_2, f) \le s\} \supset \{\rho_1 + \rho_2 : m(\rho_1, f_0) \le (1-\epsilon)s; m(\rho_2, f_1) \le \epsilon s\}$$

for $0 < \epsilon < 1$. Since

$$\begin{aligned}
& \inf\{\rho_1 + \rho_2 : m(\rho_1, f_0) \le (1-\epsilon)s; m(\rho_2, f_1) \le \epsilon s\} \\
&= \inf\{\rho_1 : m(\rho_1, f_0) \le (1-\epsilon)s\} + \{\rho_2 : m(\rho_2, f_1) \le \epsilon s\}.
\end{aligned}$$

It follows that

$$f^*(s) \le f_0^*(1-\epsilon) + f_1^*(\epsilon s).$$

Thus

$$
\begin{aligned}
& [\int_0^{t^\gamma} (f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} \\
\leq\ & [\int_0^{t^\gamma} (f_0^*((1-\epsilon)s))^\gamma ds]^{\frac{1}{\gamma}} + [\int_0^{t^\gamma} (f^*(t^\gamma))^\gamma ds]^{\frac{1}{\gamma}} \\
\leq\ & [\int_0^{\infty} (f_0^*((1-\epsilon)s))^\gamma ds]^{\frac{1}{\gamma}} + t f_1^*(0) \\
\leq\ & [\int_0^{1} (\|(f_0)_r\|_{A,p} C(r))^\gamma B(r) C(r)^{-\gamma} r^{2N-1} dr]^{\frac{1}{\gamma}} (1-\epsilon)^{-\frac{1}{\gamma}} + t\|f_1\|_{L^{p,\infty}(A,C)} \\
=\ & (1-\epsilon)^{-\frac{1}{\gamma}} \|f_0\|_{L^{p,\gamma}(A,B)} + t\|f_1\|_{L^{p,\infty}(A,C)}.
\end{aligned}
$$

Let $\epsilon \longrightarrow 0$, we have proved $< 1 >$.

(2) See the proof of Theorem 5.2.1 of [5].

Let $\gamma = p, \theta = 1 - \frac{p}{q}, C(r) \equiv 1$, we have

Corollary 5.14

$$L^{p,q}(A,B) = (L^{p,p}(A,B), L^{p,\infty}(A,1))_{1-\frac{p}{q},q}$$

for $q > p$.

In particular

$$L^{p,q}(\varphi^q \omega (1-r^2)^\alpha) = (L^{p,p}(\varphi^p \omega (1-r^2)^\alpha), L^{p,\infty}(\varphi^p))_{1-\frac{p}{q},q}.$$

Proof: It follows from Lemma 5.12 and (2) of Theorem 5.13.

§5.3 Bergman Operator

In this section, we first prove that the Bergman operator is bounded on the weighted Hardy type spaces $L^{p,\infty}(\varphi^p)$, $1 \leq p \leq \infty$. Then our main result of this chapter will follow from this fact and Corollary 5.14.

Theorem 5.15 *Let φ be a normal function. Let $\hat{b} > 0$ be as defined after (1.1), Chapter 1. If $\alpha - \hat{b} > -1$, then T_α^* is bounded on $L^{p,\infty}(\varphi^p)$.*

Proof: Let $a > 0$ as in (1.1). $\zeta, \xi \in \mathbf{S}$ and $z = \rho\zeta, w = r\xi, 0 \leq \rho < 1, 0 \leq r < 1$. Let $k + l = N + \alpha + 1$, where k and l will be determined later.

For $1 < p < \infty$

$$\begin{aligned}
&\int_{\mathbf{S}} |T_\alpha^* f(z)|^p \varphi^p(\rho) d\nu_0(\zeta) \\
\leq\ & \int_{\mathbf{S}} [\int_{\mathbf{U}} \frac{|f(w)|}{|1- < z, w > |^{N+\alpha+1}} dm_\alpha(w)]^p \varphi^p(\rho) d\nu_0(\zeta) \\
=\ & \int_{\mathbf{S}} [\int_{\mathbf{U}} \frac{|f(w)|\varphi(r)}{|1- < z, w > |^{N+\alpha+1}\varphi(r)} dm_\alpha(w)]^p \varphi^p(\rho) d\nu_0(\zeta) \\
\leq\ & \int_{\mathbf{S}} [\int_{\mathbf{U}} \frac{|f(w)|^p \varphi(r)^p}{|1- < z, w > |^{kp}\varphi(r)} dm_\alpha(w)] \\
& \times [\int_{\mathbf{U}} \frac{dm_\alpha(w)}{|1- < z, w > |^{lp'}\varphi(r)}]^{\frac{p}{p'}} \varphi^p(\rho) d\nu_0(\zeta).
\end{aligned}$$

The second factor of the integrand is

$$\begin{aligned}
I &= [\int_{\mathbf{U}} \frac{dm_\alpha(w)}{|1- < z, w > |^{lp'}\varphi(r)}]^{\frac{p}{p'}} \\
&= [\int_0^1 (\int_{\mathbf{S}} \frac{c_\alpha d\nu_0(\xi)}{|1- < z, w > |^{lp'}}) \frac{(1-r^2)^\alpha}{\varphi(r)} r^{2N-1} dr]^{\frac{p}{p'}}.
\end{aligned}$$

Since $\alpha - \hat{b} > -1$, there is a $b > 0$ such that $\alpha - b > -1$. If $lp' - N > 0$, it follows from Lemma 5.2 that

$$\begin{aligned}
I &\leq C[\int_0^1 \frac{(1-r^2)^\alpha dr}{\varphi(r)(1-r\rho)^{lp'-N}}]^{\frac{p}{p'}} \\
&\leq C[\int_0^\rho \frac{(1-r)^a(1-r)^{\alpha-a} dr}{\varphi(r)(1-r\rho)^{lp'-N}} + \int_\rho^1 \frac{(1-r)^b(1-r)^{\alpha-b} dr}{\varphi(r)(1-r\rho)^{lp'-N}}]^{\frac{p}{p'}}.
\end{aligned}$$

If $lp' - N > \alpha - a + 1$, by lemma 5.3, since $\frac{(1-r)^a}{\varphi(r)}$ is non-decreasing, $\frac{(1-r)^b}{\varphi(r)}$ is non-increasing, we have

$$\begin{aligned}
& [\int_{\mathbf{U}} \frac{dm_\alpha(w)}{|1- < z, w > |^{lp'}\varphi(r)}]^{\frac{p}{p'}} \\
\leq\ & C[\frac{(1-\rho)^a}{\varphi(\rho)}(1-\rho)^{-lp'+N+\alpha-a+1} + \frac{(1-\rho)^b}{\varphi(\rho)}(1-\rho)^{-lp'+N+\alpha-b+1}]^{\frac{p}{p'}} \\
=\ & C[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\mathbf{S}} |T_\alpha^* f(z)|^p \varphi^p(\rho) d\nu_0(\zeta) \\
\leq\ & C[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}} \int_{\mathbf{S}} [\int_{\mathbf{U}} \frac{|f(w)|^p \varphi(r)^p}{|1-<z,w>|^{kp}\varphi(r)} dm_\alpha(w)] \varphi^p(\rho) d\nu_0(\zeta) \\
\leq\ & C[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}} \int_{\mathbf{U}} (\int_{\mathbf{S}} \frac{d\nu_0(\zeta)}{|1-<z,w>|^{kp}}) \frac{|f(w)|^p \varphi^p(\rho)\varphi^p(r)}{\varphi(r)} dm_\alpha(w) \\
\leq\ & C[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}} \int_{\mathbf{U}} \frac{1}{|1-\rho r|^{kp-N}} \frac{|f(w)|^p \varphi^p(\rho)\varphi^p(r)}{\varphi(r)} dm_\alpha(w) \\
\leq\ & C[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}} \|f\|_{L^{p,\infty}(\varphi^p)} \int_0^1 \frac{(1-r)^\alpha \varphi^p(\rho)}{(1-r\rho)^{kp-N}\varphi(r)} dr \\
\leq\ & C\|f\|_{L^{p,\infty}(\varphi^p)} [\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}]^{\frac{p}{p'}} \frac{(1-\rho)^{N+1+\alpha-kp}}{\varphi(\rho)} \varphi^p(\rho) \\
=\ & C\|f\|_{L^{p,\infty}(\varphi^p)}.
\end{aligned}$$

The third inequality holds if $kp - N > 0$, and the fifth inequality holds if

$$kp - N > \alpha - a + 1.$$

Therefore if we choose k, l such that

$$\begin{cases} kp - N > 0 \\ lp' - N > 0 \\ kp - N > \alpha - a + 1 \\ lp' - N > \alpha - a + 1 \end{cases}$$

or

$$\begin{cases} l + \frac{a}{p'} > \frac{N+1+\alpha}{p'} \\ k + \frac{a}{p} > \frac{N+1+\alpha}{p} \end{cases}$$

then $\|T_\alpha^* f\|_{L^{p,\infty}(\varphi^p)} \leq C\|f\|_{L^{p,\infty}(\varphi^p)}$. Since $k + l = N + 1 + \alpha$, we can let $l = \frac{N+1+\alpha}{p'}$, $k = \frac{N+1+\alpha}{p}$. The proof for $1 < p < \infty$ is complete.

For $p = 1$ and $p = \infty$, the arguments are similar. The proof is complete.

Remark:

1. T_α^* is not bounded on "unweighted Hardy type" spaces $L^{p,\infty}(1)$. In fact, $1 \in L^{p,\infty}(1)$ but $T_\alpha^*(1)$ is not in $L^{p,\infty}(1)$.

2. The condition $\alpha - \hat{b} > -1$ can not be omitted. In fact, take $\varphi(r) = (1-r)^c$ for some $c > 0$, then $\hat{b} = c$. Suppose $\alpha - c = -1$. Let $f = (1-r)^{-c}$. Then $f \in L^{p,\infty}(\varphi^p)$, but $T_\alpha^*(f)$ is undefined.

We now prove the main result of this chapter.

Theorem 5.16 *Suppose $p \leq q \leq \infty$, $1 < p < \infty$, φ is a normal function, and $\alpha - \hat{b} > -1$, where $\hat{b}$ is as defined after (1.1), Chapter 1. If a radial function $\omega(r)$ on [0,1) satisfies condition $B_\alpha^p(\varphi^p(r)\omega(r))$:*

$$\int_{1-h}^{1} \omega(r)\varphi^p(r)(1-r^2)^\alpha r^{2N-1}dr$$
$$\times \quad \left(\int_{1-h}^{1} \omega^{-\frac{p'}{p}}(r)\varphi^{-p'}(r)(1-r^2)^\alpha r^{2N-1}dr\right)^{\frac{p}{p'}} \leq Ch^{(\alpha+1)p} \tag{5.4}$$

for all $0 < h < 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Then, for $\frac{1}{q} + \frac{1}{q'} = 1$,

(1) *T_α^* is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$;*

(2) *T_α^* is bounded on $L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$.*

Proof: (1) From Corollary 5.14,

$$L^{p,q}(\varphi^q\omega(1-r^2)^\alpha) = (L^{p,p}(\varphi^p\omega(1-r^2)^\alpha), L^{p,\infty}(\varphi^p))_{1-\frac{p}{q},q}.$$

By Theorem 1.13 , (5.4) implies that T_α^* is bounded on $L^{p,p}(\varphi^p\omega(1-r^2)^\alpha))$. Then it follows from Theorem 5.15 and Corollary 5.14 that T_α^* is bounded on the space $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$.

(2) For $f \in L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$, $g \in L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$,

$$\begin{aligned}
& |\int_{\mathbf{U}} g(z)\overline{T_\alpha^* f(z)} dm_\alpha(z)| \\
\leq & \int_{\mathbf{U}} |g(z)||T_\alpha^* f(z)| dm_\alpha(z) \\
\leq & \int_{\mathbf{U}} T_\alpha^* |g(z)||f(w)| dm_\alpha(w) \\
\leq & C \| T_\alpha^* |g(z)| \|_{L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)} \|f\|_{L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)} \\
\leq & C \|g\|_{L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)} \|f\|_{L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)}.
\end{aligned}$$

The last inequality holds because of part $< 1 >$. Lemma 5.6 and Lemma 5.7 then implies that T_α^* is bounded on $L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$. The proof is complete.

By Lemma 5.5 we have the following corollary:

Corollary 5.17 *Under the same assumption of Theorem 5.16 , T_α is a bounded projection of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ onto $H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ and a bounded projection of $L^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$ onto $H^{p',q'}(\varphi^{-q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$ for $p \leq q \leq \infty, 1 < p < \infty$.*

Remark:

1. The example in remark 2 after the proof of Theorem 5.15 shows that, in general (We assume $\int_0^1 \omega(1-r)^\alpha dr < \infty$), in order to make T_α^* well defined in $L^{p,q}(\varphi^q\omega(1-r)^\alpha)$, we must have $\alpha - \hat{b} > -1$.

2. In order to make T_α^* a bounded operator, it is not necessary that ω and φ satisfy the condition (5.4). In fact, for $N = 1$, fix q and p with $q > p$, choose $c > 0$ and $\alpha > -1$ such that $\alpha - c > -1$ and $\alpha - c(q-p) < -1$. Let $\varphi(r) = (1-r)^c$ and $\omega(r) = (1-r)^{-cq}$. Then

$$\int_{1-h}^1 \omega\varphi^p(1-r^2)^\alpha dr = \infty,$$

so that φ and ω do not satisfy the condition $B_\alpha^p(\varphi^p\omega)$.

However, since $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha) = L^{p,q}((1-r^2)^\alpha)$, choose $\gamma > 0$ very small and let $\hat{\varphi}(r) = (1-r)^\gamma$, $\hat{\omega}(r) = (1-r)^{-c\gamma}$, then it is not hard to see that the condition $B_\alpha^p(\hat{\varphi}^p\hat{\omega})$ is satisfied. Theorem 5.16 then implies that T_α^* is bounded on $L^{p,q}(\hat{\varphi}^q\hat{\omega}(1-r^2)^\alpha) = L^{p,q}((1-r^2)^\alpha)$.

3. Suppose T_α is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$. Following the method Békollé used in [3, p.311], if we put $f(z) = \omega^{-\frac{q'}{q}}(r)\varphi^{-q'}(r)\chi_{K(a)}(z)$, it can be shown that φ and ω satisfy the condition $B_\alpha^q(\varphi^q\omega)$.

We next give an application of Theorem 5.16.

In [13] M. Jevtić showed the following:

Theorem 5.18 *For $1 \le p \le \infty$, $1 \le q \le \infty$, the transformation P defined by*

$$Pf(w) = \int_{\mathbf{U}} f(z)K_{\lambda-1}(z,w)\psi(r)(1-r^2)^{-\frac{1}{q}}dm(z),$$

where $w \in \mathbf{U}$, is bounded from $L^{p,q}(r^{1-2N})$ onto $H^{p,q}(\varphi^q r^{1-2N}(1-r)^{-1})$.

We now show that for $1 < p < \infty$, $1 < q < \infty$, this Theorem is a special case of Theorem 5.16 .

Let $\omega(r) = (1-r^2)^{-\lambda}r^{1-2N}$. For any $f \in L^{p,q}(r^{1-2N})$, define

$$F(z) = f(z)\psi(r)(1-r^2)^{\frac{1}{q}-\lambda}.$$

Then $F \in L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$ and

$$\|F\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})} = \|f\|_{L^{p,q}(r^{1-2N})}$$

Thus $Pf(w) = T_{\lambda-1}F(w)$. Since

$$H^{p,q}(\varphi^q r^{1-2N}(1-r^2)^{-1}) = H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1}),$$

Theorem 5.18 is now equivalent to the statement :

$T_{\lambda-1}$ is a bounded operator mapping from the space $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$ onto the space $H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$.

It suffices to prove the boundedness of the operator $T_{\lambda-1}$, since then it will follow from Lemma 5.4 and Lemma 5.5 that $T_{\lambda-1}$ is an onto map.

By Theorem 5.16 , for $q \geq p$, it suffices to verify (5.4). We have

$$\begin{aligned}
\varphi^{-p'}\omega^{-\frac{p'}{p}} &= \varphi^{-p'}(1-r^2)^{\lambda\frac{p'}{p}}r^{(2N-1)\frac{p'}{p}} \\
&= \varphi^{-p'}(1-r^2)^{\lambda(p'-1)}r^{(2N-1)(p'-1)} \\
&= \psi^{p'}r^{(2N-1)(p'-1)}(1-r^2)^{-\lambda}.
\end{aligned}$$

Thus (5.4) is equivalent to

$$\begin{aligned}
& \int_{1-h}^{1}\varphi^p(r)(1-r^2)^{-1}dr \\
\times & \left(\int_{1-h}^{1}\psi(r)^{p'}r^{(2N-1)p'}(1-r^2)^{-1}dr\right)^{\frac{p}{p'}} \leq Ch^{\lambda p}. \qquad (5.5)
\end{aligned}$$

Condition (5.5) will be verified by the following

Lemma 5.19 *For any normal pair $\{\varphi,\psi\}$ and t a non-negative real number,*

$$\int_{1-h}^{1}\varphi^p(r)(1-r)^{t-1}dr\left(\int_{1-h}^{1}\psi(r)^{p'}(1-r)^{t-1}dr\right)^{\frac{p}{p'}} \leq Ch^{(\lambda+t)p} \qquad (5.6)$$

for all $0 < h < 1$, as long as each factor makes sense.

Proof: Let $0 < h < 1$. Since $\frac{\varphi(r)}{(1-r)^a}$ is non-increasing, where $a > 0$ is as in (1.1),

$$\begin{aligned}
& \int_{1-h}^{1}\varphi^p(r)(1-r)^{t-1}dr \\
\leq\ & C\int_{1-h}^{1}\frac{\varphi^p(r)}{(1-r)^{ap}}(1-r)^{t-1+ap}dr \\
\leq\ & C\frac{\varphi^p(1-h)}{h^{ap}}\int_{1-h}^{1}(1-r)^{t-1+ap}dr \\
\leq\ & C\frac{\varphi^p(1-h)}{h^{ap}}h^{ap+t} \\
=\ & C\varphi^p(1-h)h^t.
\end{aligned}$$

Similarly

$$\left(\int_{1-h}^{1} \psi^{p'}(1-r)^{t-1}dr\right)^{\frac{p}{p'}} \leq C\psi^{p}(1-h)h^{(t)\frac{p}{p'}}.$$

Since $\varphi^p(1-h)\psi^p(1-h) = h^{\lambda p}$, (5.6) follows. The proof of the lemma is complete.

Since (5.6) implies (5.5), for $p \leq q$, Theorem 5.18 follows from Lemma 5.19 and Theorem 5.16 . For $q < p$, we have $q' > p'$. Since in (5.6), the position of p, p', φ, ψ are symmetric, (2) of Theorem 5.16 implies that $T^*_{\lambda-1}$ is bounded on the space $L^{p,q}(\psi^{-q}\omega^{-\frac{q}{q'}}(1-r^2)^{\lambda-1})$. Since

$$\varphi(r)\psi(r) = (1-r^2)^{\lambda},$$

and

$$\omega(r) = (1-r^2)^{-\lambda},$$

we have

$$\psi^{-q}\omega^{-\frac{q}{q'}}(1-r^2)^{\lambda-1} = \varphi^q\omega r^{(2N-1)\frac{q}{q'}+1}(1-r^2)^{\lambda-1}.$$

Therefore

$$L^{p,q}(\psi^{-q}\omega^{-\frac{q}{q'}}(1-r^2)^{\lambda-1}) = L^{p,q}(\varphi^q\omega r^{(2N-1)\frac{q}{q'}+1}(1-r^2)^{\lambda-1}).$$

On the other hand, since

$$L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1}) \subset L^{p,q}(\varphi^q\omega r^{(2N-1)\frac{q}{q'}+1}(1-r^2)^{\lambda-1})$$

and $T_{\lambda-1}f$ is analytic for $f \in L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$, we have

$$\begin{aligned}
&\|T_{\lambda-1}f\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})} \\
= \ &\|T_{\lambda-1}f\|_{H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})} \\
\leq \ &C\|T_{\lambda-1}f\|_{H^{p,q}(\varphi^q\omega r^{(2N-1)\frac{q}{q'}+1}(1-r^2)^{\lambda-1})} \qquad (\textit{ by Lemma } 5.8 \text{ }) \\
\leq \ &C\|f\|_{L^{p,q}(\varphi^q\omega r^{(2N-1)\frac{q}{q'}+1}(1-r^2)^{\lambda-1})} \\
\leq \ &C\|f\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})}.
\end{aligned}$$

Thus $T_{\lambda-1}$ is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$. Therefore Theorem 5.18 is also true for $q<p$. The proof is complete.

§5.4 Duality Theorems

The following theorem is the main result of this section.

Theorem 5.20 *Suppose $p \le q \le \infty$, $1<p<\infty$, and $\{\ \varphi,\ \psi\ \}$ is a normal pair, $\alpha+\lambda-\hat{b} > -1$, where $\hat{b}$ is as defined after (1.1) and λ is as in (1.2). Suppose $\omega(r)\ge 0$ satisfies*

$$\int_{1-h}^{1}\omega(r)\varphi^p(r)(1-r^2)^\alpha r^{2N-1}dr$$
$$\times\ \left(\int_{1-h}^{1}\omega^{-\frac{p'}{p}}(r)\varphi^{-p'}(r)(1-r^2)^\alpha r^{2N-1}dr\right)^{\frac{p}{p'}} \le Ch^{(\alpha+\lambda+1)p} \tag{5.7}$$

for all $0<h<1$, where $\frac{1}{p}+\frac{1}{p'}=1$. Then the dual of $H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ can be identified with $H^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$ under the pairing

$$<f,g>=\int_{\mathbf{U}}f(z)\bar{g}(z)dm_{\alpha+\lambda}(z). \tag{5.8}$$

More precisely, if $g\in H^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$ and if we define

$$L_g(f)=<f,g>$$

for all functions $f\in H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$, then $L_g\in[H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)]^*$ and

$$\|L_g\|\le C\|g\|_{H^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)}.$$

Conversely, given a linear functional $L\in[H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)]^*$, there is a unique

$$g\in H^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)$$

such that $L_g=L$ and

$$\|g\|_{H^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha)}\le C\|L\|.$$

Proof: Take $\tilde{\omega}(r) = \omega(r)(1-r^2)^{-\lambda}$. Then

$$\begin{aligned}
\varphi^{-p'}\tilde{\omega}^{-\frac{p'}{p}} &= \varphi^{-p'}\omega^{-\frac{p'}{p}}(1-r^2)^{\lambda\frac{p'}{p}} = \psi^{p'}\omega^{-\frac{p'}{p}}; \\
L^{p,q}(\varphi^q\omega(1-r^2)^\alpha) &= L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda}) \qquad (\textit{equal norm}); \\
L^{p',q'}(\psi^{q'}\omega^{-\frac{q'}{q}}(1-r^2)^\alpha) &= L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda}) \quad (\textit{equal norm}).
\end{aligned}$$

Thus ω satisfies (5.7) implies that $\tilde{\omega}$ satisfies (5.4). Theorem 5.16 then implies that $T^*_{\alpha+\lambda}$ is a bounded map on both $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$ and $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$.

Now, it suffices to prove that the dual of $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$ can be identified with $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$ under the pairing (5.8) for $p \leq q$.

Let $g \in L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$. It follows from Hölder's inequality that

$$|<f,g>| \leq C\|f\|_{L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})}\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})}.$$

So g defines a bounded linear functional L on $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$ and

$$\|L\| \leq C\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})}.$$

Conversely, let L be a bounded linear functional on $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. Then L can be extended to be a linear functional on $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. By Lemma 5.6, there exists a $h \in L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$ such that

$$L(f) = <f,h>$$

and

$$\|L\| \sim \|h\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})}.$$

Let

$$g = T_{\alpha+\lambda}h.$$

Theorem 5.16 implies that $g \in H^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$. Now for $f \in H^\infty(\mathbf{U})$, by Lemma 5.4 and Fubini's theorem, we have

$$L(f) = <f,h> = <T_{\alpha+\lambda}f,h> = <f,T_{\alpha+\lambda}(h)> = <f,g>.$$

By Lemma 5.5 , H^∞ is dense in $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. Then the continuity of L implies $L(f) =< f,g >$ for all $f \in H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. We also have

$$\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})} \leq C\|h\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})} \leq C\|L\|.$$

If $g \in L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$ defines a zero functional, then since the function $\overline{K_{\alpha+\lambda}(z,\cdot)} \in H^\infty(\mathbf{U})$, for any fixed $z \in \mathbf{U}$, we have, for some $C > 0$,

$$0 =< \overline{K_{\alpha+\lambda}(z,\cdot)}, g >= C\overline{g(z)}.$$

Hence $g \equiv 0$. So there is a one-to-one, continuous, linear transformation from $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-\frac{q'}{q}}(1-r^2)^{\alpha+\lambda})$ onto the dual space of $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. The proof is complete.

We next give some applications of Theorem 5.20.

In [16] D. Luecking used Theorem 1.13 to identify the dual of weighted Bergman spaces. He proved the following:

Theorem 5.21 *Suppose $\omega(z)$ satisfies*

$$\int_K \omega(z)dm_\eta(z)(\int_K \omega^{-\frac{p'}{p}}(z)dm_\gamma(z))^{\frac{p}{p'}} \leq Cm_\alpha^p(K) \tag{5.9}$$

for $1 < p < \infty$, where $\eta > -1$, $\gamma > -1$, $\alpha = \frac{\eta}{p}+\frac{\gamma}{p'}$ and K is the region in the condition $A_\alpha^p(\omega)$. Then the dual of $H^{p,p}(\omega(z)dm_\eta)$ can be identified with $H^{p',q'}(\omega(z)^{-\frac{p'}{p}}dm_\gamma)$ under the pairing (5.3).

In [13] M. Jevtić showed the following:

Theorem 5.22 *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Then the dual space of the space $H^{p,q}(\varphi^q r^{1-2N}(1-r^2)^{-1})$ can be identified with $H^{p',q'}(\psi^{q'} r^{1-2N}(1-r^2)^{-1})$ under the pairing*

$$< f,g >= \int_{\mathbf{U}} f(z)\bar{g}(z)dm_{\lambda-1}.$$

In Theorem 5.20, taking $\varphi = (1-r^2)^i$, $\psi = (1-r^2)^j$, $\alpha = -1$ in Theorem 5.20, where $i, j > 0$, we have $i + j = \lambda$. Then (5.7) becomes

$$\int_{1-h}^{1} \omega(r)(1-r^2)^{ip-1} r^{2N-1} dr \\ \times \left(\int_{1-h}^{1} \omega^{-\frac{p'}{p}}(r)(1-r^2)^{jp'-1} r^{2N-1} dr\right)^{\frac{p}{p'}} \leq C h^{\lambda p}. \quad (5.10)$$

It is not hard to see that

$$H^{p,q}(\varphi^q \omega (1-r^2)^{-1}) = H^{p,q}(\omega(1-r^2)^{iq-1})$$

and

$$H^{p',q'}(\psi^{q'} \omega^{-\frac{q'}{q}}(1-r^2)^{-1}) = H^{p',q'}(\omega^{-\frac{q'}{q}}(1-r^2)^{jq'-1}).$$

By Theorem 5.20, for $q \geq p$, if ω satisfies the condition (5.10), then the dual space of $H^{p,q}(\omega(1-r^2)^{iq-1})$ can be identified with $H^{p',q'}(\omega^{-\frac{q'}{q}}(1-r^2)^{jq'-1})$ under the pairing (5.8). Let $i = \frac{\eta+1}{q}$, $j = \frac{\gamma+1}{q'}$. Then $\eta = iq - 1$, $\gamma = jq - 1$ and $\lambda = i + j = t + 1$. These observations give the following:

Theorem 5.23 *For $1 < p \leq q < \infty$, $\eta > -1$, $\gamma > -1$, and $t > -1$ satisfy*

$$t = \frac{\eta}{q} + \frac{\gamma}{q'},$$

if $\omega(r)$ satisfies

$$\int_{1-h}^{1} \omega(r)(1-r^2)^{(\eta+1)\frac{p}{q}-1} r^{2N-1} dr \\ \times \left(\int_{1-h}^{1} \omega^{-\frac{p'}{p}}(r)(1-r^2)^{(\gamma+1)\frac{p'}{q'}-1} r^{2N-1} dr\right)^{\frac{p}{p'}} \leq C h^{(t+1)p}, \quad (5.11)$$

then under the pairing

$$< f, g > = \int_{\mathbf{U}} f \bar{g} \, dm_t \quad (5.12)$$

the dual of $H^{p,q}(\omega(1-r^2)^{\eta})$ can be identified with $H^{p',q'}(\omega^{-\frac{q'}{q}}(1-r^2)^{\gamma})$.

Proof: (5.11) is now equivalent to (5.10). From the above discussion, the proof is complete.

It is not hart to see that, for a radial weight function $\omega(r)$, Theorem 5.20 gives a generalization of Theorem 5.21 in mixed-norm spaces. In fact, in Theorem 5.23, taking $q = p$, we immediately get Theorem 5.21.

Next we show that Theorem 5.22 is a special case of Theorem 5.20 if $1 < p < \infty$, $1 < q < \infty$.

In fact, by Lemma 5.8 , it suffices to show that the dual of $H^{p,q}(\varphi^q(1-r^2)^{-1})$ can be identified with $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$ under the pairing

$$< f, g >= \int_{\mathbf{U}} f(z)\bar{g}(z)dm_{\lambda-1}.$$

Taking $\omega = 1$, $\alpha = -1$ in Theorem 5.20, (5.7) becomes

$$\int_{1-h}^{1} \varphi^p(r)(1-r^2)^{-1}r^{2N-1}dr(\int_{1-h}^{1} \psi^{p'}(1-r^2)^{-1}r^{2N-1}dr)^{\frac{p}{p'}} \leq Ch^{\lambda p}.$$

By Lemma 5.19 and the discussion after Lemma 5.19, any normal pair satisfies this inequality. Therefore for $q \geq p$, Theorem 5.22 follows from Theorem 5.20 immediately.

For $q < p$, then $q' > p'$. Using the duality argument, it cam be shown that the dual space of $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$ is $H^{p,q}(\varphi^q(1-r^2)^{-1})$. This implies that $H^{p,q}(\varphi^q(1-r^2)^{-1})$ is a closed subspace of $L^{p,q}(\varphi^q(1-r^2)^{-1})$. Since $L^{p,q}(\varphi^q(1-r^2)^{-1})$ is reflexive [4, p.306], it follows that the dual of $H^{p,q}(\varphi^q(1-r^2)^{-1})$ is $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$. We also get Theorem 5.22.

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