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REAL TORIC MANIFOLDS

By

Radhouane Sellami

A DISSERTATION

Submitted to
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ABSTRACT

REAL TORIC MANIFOLDS

By

Radhouane Sellami

In this thesis, we study toric manifolds as a particular case of the monomial manifolds, and give an identification of the two structures under weak conditions. Toric manifolds of dimension r have a $(\mathbf{Z}_2)^r$ action, while their complexifications have a T^r action and the two actions on the real and the corresponding complex toric manifolds have the same orbit space. For $r = 2$ or 3, the manifolds with T^r action are well studied, and we use the known results about them to classify the dimensional 2 toric manifolds and give a characterization up to surgery and connected sum with $\mathbf{R}P^3$ of the 3-dimensional case. Also we give a Heegard characterization of the orientable toric 3 manifolds and get a restriction on the manifolds which can support a toric structure..

To my grandparents, my parents, my wife, and my son youssef.

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CHAPTER 1

Introduction

Algebraic spaces are topological spaces modelled on algebraic sets with the glueing functions being birational isomorphisms. If in addition, the charts are nonsingular then the space is a smooth manifold.

Toric manifolds are a particular case of nonsingular algebraic spaces,where the glueing maps are monomials [9, 8, 5].

In [1] S.Akbulut discussed the real algebraic structures on smooth manifolds where a real algebraic space $(\mathbf{X}, \{\phi_\alpha\})$ is a topological space $\mathbf{X}$ with a collection of imbeddings $\phi_\alpha : S_\alpha - S'_\alpha \longrightarrow \mathbf{X}$ such that $S'_\alpha \subset S_\alpha$ are real algebraic sets, the images of ϕ_α cover $\mathbf{X}$ and

i Each $T_{\alpha\beta} = S'_\alpha \cup \phi_\alpha^{-1}(\mathbf{X} - \text{image } \phi_\beta)$ is a real algebraic set

ii Each $\phi_\beta^{-1}\phi_\alpha : (S_\alpha - T_{\alpha\beta}) \longrightarrow (S_\beta - T_{\beta\alpha})$ is a birational isomorphism.

And in particular he considered the rational algebraic structures where a rational algebraic space is a real algebraic space with the extra condition that $S_\alpha = \mathbf{R}^m$, and he asked the question:Is every smooth compact manifold diffeomorphic to a nonsingular rational space?

In this paper, we look at a particular case of rational structures, namely the case where the transition maps are monomials. It turns out that the toric manifolds [9] admit this monomial structure i.e the real compact toric manifolds of dimension n are obtained by glueing copies of $\mathbf{R}^n$ using monomials as transition maps (so they are a particular case of rational manifolds). Conversely, here we give a necessary and sufficient condition for a monomial manifold to be toric. Also we show that the toric structure is not developable. Then we specialize in the toric structures on compact 2 and 3 dimensional manifolds. The coefficients of the monomials are determined by a collection of integral nonsingular cones of dimension n, called a fan. Starting with the same fan, we construct a complex toric manifold by glueing copies of $\mathbf{C}^n$ together with the transition maps being the same monomials as in the real case, this gives a natural complexification of the real toric manifolds. These complex manifolds admit a $\mathbf{T}^n$ smooth action on them, this action induces an action of $(\mathbf{Z}_2)^n$ on the corresponding real toric manifolds and the orbit spaces of the two actions are equal, while the isotropy groups in the real case are $(\mathbf{Z}_2)^n \cap$ (isotropy groups in the complex case).

For the cases we are interested in, the orbit spaces are dual to the fans, hence by starting from a fan we obtain directly the orbit space, without the need to identify the manifold.

If $n = 2$, the orbit spaces are D^2 with weights on the boundary S^1 [10]. The 2-dimensional toric manifolds are obtained by glueing four copies of the orbit spaces along the boundary, and we are able to identify all compact real toric manifolds of dimension 2.

If $n = 3$ the orbit spaces are D^3 with weighted graphs G on the boundary S^2 [7], the corresponding toric manifolds are obtained by glueing eight copies of the orbit spaces following the informations given by the weights on S^2 [7]. The graphs corresponding to orientable manifolds are colored by only 4 colors, and to identify the orientable toric manifolds (and hence to partially answer the question of S.Akbulut in the case of toric structures), we glue the eight copies of the orbit space along the cells surrounding a vertex in the graph, so that we obtain a 3 ball with a graph P_G on its boundary. The cells on P_G are identified two by two. We bore out small cylinders around the edges of the graph G as done in [12] to obtain a Heegaard representation of the manifold.

We show that all orientable compact toric 3 manifolds are obtained from $\mathbf{R}P^3$ by a sequence of blowing up points (the fixed points of the $(\mathbf{Z}_2)^3$ action), which corresponds to connected summing with $\mathbf{R}P^3$, and 1/2 surgeries along some special circles corresponding to edges in the graph. Choices of these circles are importants since all 3-manifolds can be obtained from $\sharp_k S^1 \times S^2$ by 1/2 surgeries [2]. We use this representation to draw some conclusions about the homology groups of these manifolds and show that some 3-manifolds such as lens spaces $L(2s+1, q)$ can not admit toric structures.

CHAPTER 2

Real Toric Manifolds

2.1 Construction

In this section we will recall how toric manifolds are constructed from rational cones [9]. Let $N = \mathbf{Z}^r$, $M = \mathrm{Hom}(N, \mathbf{Z})$, $N_{\mathbf{R}} = N \otimes \mathbf{R} \simeq \mathbf{R}^r$, $M_{\mathbf{R}} = M \otimes \mathbf{R}$ and let $<,>$ denote the duality pairing of M and N as well as its extension to $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$. N and M are groups in the obvious way. We assume througout the paper that all splittings $N_{\mathbf{R}} = V \oplus W$ (where V and W are rational subspaces) have the further property $N = (N \cap V) \oplus (N \cap W)$ and likewise splittings in $M_{\mathbf{R}}$.

Definition 2.1 *1. A subset σ of $N_{\mathbf{R}}$ is a convex polyhedral cone, and denoted in short by* **crpc** *if there exists a finite set of vectors $\{n_1, \ldots, n_s\}$ of $N_{\mathbf{R}}$ such that $\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_s$.*

2. A convex polyhedral cone in $N_{\mathbf{R}}$ is called rational if its generating vectors $\{n_1, \ldots, n_s\}$ are in N.

3. Such a σ is called strongly convex rational polyhedral cone and denoted in short by **scrpc** *if $\sigma \cap (-\sigma) = \{0\}$.*

4. For a **crpc** *σ we define $\dim \sigma$ to be the dimension of the vector subspace of $N_{\mathbf{R}}$ generated by σ*

Definition 2.2 *Let σ be an* **scrpc** *in $N_{\mathbf{R}}$. Then*

$$\begin{aligned}
\sigma^{\vee} &= \{y \in M_{\mathbf{R}} \mid <y,x> \geq 0 \ \forall x \in \sigma\} \\
&= \{y \in M_{\mathbf{R}} \mid <y,n_i> \geq 0 \ \forall i = 1,\ldots,s\} \ \textit{is called the dual of } \sigma \\
\sigma^{\perp} &= \{y \in M_{\mathbf{R}} \mid <y,x> = 0 \ \forall x \in \sigma\}
\end{aligned}$$

$int\sigma$ is the usual interior of σ regarded as a subset of the real vector space $\mathbf{R}\sigma$.

Remark :

By theorem 19.1 [11] there exist $m_1, \cdots, m_t$ in $M_{\mathbf{R}}$ such that $\sigma^{\vee} = \mathbf{R}_{\geq 0}m_1 + \cdots + \mathbf{R}_{\geq 0}m_t$ and since σ is rational i.e the generators $\{n_1, \ldots, n_t\}$ are in N, it is easily seen that the m_i can be chosen to be in M, hence we get that $\sigma^{\vee}$ is a **crpc** but not necessarily strongly convex.

Proposition 2.1 *Let σ be an* **scrpc** *in $N_{\mathbf{R}}$ then*

1. $(\sigma^{\vee})^{\vee} = \sigma$.

2. $(\sigma \cap \sigma')^{\vee} = \sigma^{\vee} + \sigma'^{\vee}$.

3. $x \in int\sigma \Leftrightarrow <x,y>> 0 \ \ \forall y \in \sigma^{\vee} \setminus \sigma^{\perp} \ \ \Leftrightarrow \sigma^{\vee} \cap \{x\}^{\perp} = \sigma^{\perp}$.

Proof: Theorem A.1, lemma A.4 in [9]. □

Definition 2.3 *Let σ be an* **scrpc** *in $N_{\mathbf{R}}$, define $S_{\sigma} = M \cap \sigma^{\vee} = \{y \in M \mid <y,x> \geq 0 \ \ \forall x \in \sigma\}$*

Proposition 2.2 *Let σ be an* **scrpc**, *then*

1. *S_{σ} is a subsemigroup of M.*

2. *S_{σ} is finitely generated as a semigroup.*

3. *S_{σ} generates M as a group.*

Proof: (Prop 1.1 in [9]). □

Definition 2.4 *Let σ be an scrpc in $N_{\mathbf{R}}$. Define*

$$U_\sigma = \{u : S_\sigma \longrightarrow \mathbf{R} \mid u(m+m') = u(m)u(m') \text{ and } u(0) = 1\}$$

Remarks :

Let $S_\sigma = \mathbf{Z}^{\geq 0}m_1 + \cdots + \mathbf{Z}^{\geq 0}m_p$ for some $(m_1, \ldots, m_p) \subset M$, then every u in U_σ is completely determined by $(u(m_1), \ldots, u(m_p))$ i.e

$$\begin{array}{rcl} U_\sigma & \longrightarrow & \mathbf{R}^p \\ u & \longmapsto & (u(m_1), \ldots, u(m_p)) \end{array}$$

defines a coordinate system on U_σ.

Proposition 2.3 *If we identify U_σ with its image in $\mathbf{R}^p$, then*

$$U_\sigma = \{(x_1, \ldots, x_p) \in \mathbf{R}^p \mid x_1^{\alpha_1} \ldots x_p^{\alpha_p} = x_1^{\beta_1} \ldots x_p^{\beta_p} \text{ for all} \alpha_i, \beta_i \in \mathbf{Z}_{\geq 0} \text{ with} \sum \alpha_i m_i = \sum \beta_i m_i\}$$

so U_σ is an algebraic subset of $\mathbf{R}^p$.

Proof : (Prop 1.2 of [9]). □

2.2 Nonsingularity

Proposition 2.4 *U_σ is nonsingular iff σ is generated by a $\mathbf{Z}$ subbasis of N.*

Definition 2.5 *We call a cone generated by a $\mathbf{Z}$ subbasis of N a nonsingular cone.*

Proof :

($\Leftarrow$) Let $\sigma = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_p$ where $(n_i)_{i=1}^p$ is a $\mathbf{Z}$ subbasis of N, we complete this subbasis to $\{n_1, \ldots, n_r\}$ a $\mathbf{Z}$ basis of N and let $\{m_1, \ldots, m_r\}$ be the dual $\mathbf{Z}$ basis in M, then $\sigma^\vee = \mathbf{R}_{\geq 0} m_1 + \cdots + \mathbf{R}_{\geq 0} m_p + \mathbf{R} m_{p+1} + \cdots + \mathbf{R} m_r$, and for u in U_σ, $u(m_i) \neq 0$ for $p+1 \leq i \leq r$, since m_i and $-m_i$ are in $\sigma^\vee$ for such i. Therefore $U_\sigma = \underbrace{\mathbf{R} \times \cdots \times \mathbf{R}}_{p} \times \underbrace{\mathbf{R}^* \times \cdots \times \mathbf{R}^*}_{r-p}$.

($\Rightarrow$) Let $N'_{\mathbf{R}}$ be the smallest vector subspace of $N_{\mathbf{R}}$ generated by σ, then $N_{\mathbf{R}} = N'_{\mathbf{R}} \oplus V$ where $V \cong \sigma^\perp$. We will first show that we can assume $N'_{\mathbf{R}} = N_{\mathbf{R}}$. Otherwise, let $M'_R = \mathrm{Hom}(N'_{\mathbf{R}}, \mathbf{R})$, we can view $M'_{\mathbf{R}}$ as a subspace of $M_{\mathbf{R}}$ by letting $m(V) = 0 \ \forall m \in M'_{\mathbf{R}}$, hence $M'_{\mathbf{R}} \cong M_{\mathbf{R}}/\sigma^\perp$, so $M_{\mathbf{R}} \cong M'_{\mathbf{R}} \oplus \sigma^\perp$, and $M \cong M' \oplus (M \cap \sigma^\perp)$. Since $\sigma \subset N'_{\mathbf{R}}$ and if $\sigma^\sqcup$ denote its dual in $M'_{\mathbf{R}}$, then $\sigma^\sqcup$ is the image of $\sigma^\vee$ under the above identification. Let $S'_\sigma = M' \cap \sigma^\sqcup$ hence $S_\sigma = M \cap \sigma^\vee \cong (M \cap \sigma^\perp) \times S'_\sigma$. Let $(m_1, \ldots, m_q, m'_1, \ldots, m'_p)$ be a family of generators for S_σ where $(m_1, \ldots, m_q)$ are chosen to form a basis for the vector space $\sigma^\perp$, and $(m'_1, \ldots, m'_p)$ are generators for S'_σ. Since there is no relation between $(m_i)_i$ and $(m'_j)_j$ then $U_\sigma = U \times U'_\sigma$ where

$$\begin{aligned} U &= \{u : M \cap \sigma^\perp \longrightarrow \mathbf{R}^* \mid u(m+m') = u(m)u(m') \text{ and } u(0) = 1\} \\ &\cong \underbrace{\mathbf{R}^* \times \cdots \times \mathbf{R}^*}_{q} \end{aligned}$$

Therefore U_σ is nonsingular iff U'_σ is nonsingular. So we assume that σ generates $N_{\mathbf{R}}$ i.e $\dim \sigma = r$, hence $\sigma^\vee$ is strongly convex because $(\sigma^\vee) \cap (-\sigma^\vee) = \sigma^\perp = \{0\}$, hence $0 \in U_\sigma$. Let

$$U_\sigma = \{(x_1, \ldots, x_p) \in \mathbf{R}^p \mid x_1^{\alpha_1} \ldots x_p^{\alpha_p} = x_1^{\beta_1} \ldots x_p^{\beta_p} \text{ for all} \alpha_i, \beta_i \in \mathbf{Z}_{\geq 0} \text{ with} \sum \alpha_i m_i = \sum \beta_i m_i\}$$

Let $\{m_i\}_1^p$ be a minimal set of generators of S_σ so that there is no i such that $m_i = \sum_{\substack{j=1 \\ j \neq i}}^{p} a_j m_j$ and $a_j \in \mathbf{Z}_{\geq 0}$. We prove that $p = r$.

Let $\alpha_{i1} m_1 + \cdots + \alpha_{ip} m_p = \beta_{i1} m_1 + \cdots + \beta_{ip} m_p$ with the condition that if $\alpha_{ij} > 0$ then $\beta_{ij} = 0$ and if $\beta_{ij} > 0$ then $\alpha_{ij} = 0$. We remark that U_σ contains $U_{\{0\}} \cong \mathbf{R}^{*r}$ as an

open subset, hence $\dim U_\sigma$ is r. If U_σ is nonsingular then it is nonsingular at 0, thus there exists a finite number of polynomials $\{f_i = x_1^{\alpha_{i1}} \dots x_p^{\alpha_{ip}} - x_1^{\beta_{i1}} \dots x_p^{\beta_{ip}}\}$ in $I(U_\sigma)$ such that $rank\left(\frac{\partial f_i}{\partial x_j}|_0\right) = p - r$, and if $p \neq r$ then $rank\left(\frac{\partial f_1}{\partial x_j}|_0\right) > 0$.
Assume without loss of generality that $\frac{\partial f_1}{\partial x_1} \neq 0$ with $f_1(x_1, \dots, x_p) = x_1^{\alpha_{11}} \dots x_p^{\alpha_{1p}} - x_1^{\beta_{11}} \dots x_p^{\beta_{1p}}$, hence

$$\left.\frac{\partial f_1}{\partial x_1}\right|_0 = \alpha_{11} x_1^{\alpha_{11}-1} \dots x_p^{\alpha_{1p}} - \beta_{11} x_1^{\beta_{11}-1} \dots x_p^{\beta_{1p}}\Big|_0 \neq 0$$

If $\alpha_{11} = 0$ then $\frac{\partial f_1}{\partial x_1}|_0 = -\beta_{11} 0^{\beta_{11}-1} 0^{\beta_{12}} \dots 0^{\beta_{1p}}$ which is different of zero only if $\beta_{11} = 1$ and $\beta_{ij} = 0 \ \forall j = 2 \dots p$ i.e $m_1 = \alpha_2 m_2 + \dots + \alpha_p m_p$ which contradict the hypothesis. Therefore $\alpha_{11} \geq 1$ then by assumption $\beta_{11} = 0$, but this is just the same replacing α_{11} by β_{11}, hence there is no relation between (m_i), i.e (m_i) form a basis for M. Hence (n_i) form a basis for N. □

From now on we assume that all our cones are nonsingular.

2.3 Toric Manifolds

Definition 2.6 *Let σ be a* **scrpc** *in $N_{\mathbf{R}}$. A subset τ of σ is called a face of σ (denoted $\tau < \sigma$) iff there exists m_0 in $\sigma^\vee$ such that $\tau = \sigma \cap \{m_0\}^\perp$*

Proposition 2.5 *1. Since σ is rational then m_0 can be chosen to be in S_σ.*

2. By definition τ is also an scrpc.

Proof : (Prop 1.3 of [9]).

Proposition 2.6 *If $\tau < \sigma$ are nonsingular cones then there exists $\{n_1, \dots, n_r\}$ a $\mathbf{Z}$ basis of N such that $\sigma = \mathbf{R}_{\geq 0} n_1 + \dots + \mathbf{R}_{\geq 0} n_p$ and $\tau = \mathbf{R}_{\geq 0} n_1 + \dots + \mathbf{R}_{\geq 0} n_s$ with $1 \leq s \leq p$.*

Proposition 2.7 *If $\tau < \sigma$ so that $\tau = \sigma \cap \{y\}^{\perp}$ for some y in S_σ then $\tau^\vee = \sigma^\vee + \mathbf{R}_{\geq 0}(-y)$, and $S_\tau = S_\sigma + \mathbf{Z}_{\geq 0}(-y)$.*

Proof:

$$\begin{aligned} \tau < \sigma \Leftrightarrow \tau &= \sigma \cap \{y\}^{\perp} \text{ for some } y \in S_\sigma \\ &= \sigma \cap (\mathbf{R}_{\geq 0}(y))^\vee \cap (\mathbf{R}_{\geq 0}(-y))^\vee \end{aligned}$$

hence $\tau^\vee = \sigma^\vee + \mathbf{R}_{\geq 0}(y) + \mathbf{R}_{\geq 0}(-y)$ by proposition (2.1) but $y \in \sigma^\vee$ hence $\tau^\vee = \sigma^\vee + \mathbf{R}_{\geq 0}(-y)$. □

Proposition 2.8 *If $\tau = \sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2, then $S_\tau = S_{\sigma_1} + S_{\sigma_2}$.*

Proof:

($\supseteq$) $\tau^\vee = (\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$, hence $S_\tau = M \cap (\sigma_1 \cap \sigma_2)^\vee = M \cap (\sigma_1^\vee + \sigma_2^\vee) \supset (M \cap \sigma_1^\vee) + (M \cap \sigma_2^\vee) = S_{\sigma_1} + S_{\sigma_2}$.

($\subseteq$) The proof is by induction on $\dim \sigma_1 + \dim \sigma_2$. We assume that $\sigma_i \not< \sigma_j$. Then $\mathrm{int}\sigma_1 \cap \mathrm{int}\sigma_2 = \emptyset$. By the separation theorem in [11] there exists a hyperplane H of $N_{\mathbf{R}}$ such that σ_1 is contained in one of the closed half spaces limited by H, and σ_2 is contained in the other closed half space, and since $\mathrm{int}\sigma_1 \cap \mathrm{int}\sigma_2 = \emptyset$ we can assume that they do not both lie in H. Now let $H = \{m_0\}^{\perp}$ for some m_0 in $M_{\mathbf{R}}$, then since $\tau \subset H$ and τ is a rational cone, m_0 can be chosen to be in M, so σ_1 and σ_2 lie on mutually opposite sides with respect to the hyperplane $\{m_0\}^{\perp}$, so that $\sigma_1 \subset \{x \in N_{\mathbf{R}} \mid < x, m_0 > \geq 0\}$, hence $m_0 \in S_{\sigma_1}$, and
$\sigma_2 \subset \{x \in N_{\mathbf{R}} \mid < x, m_0 > \leq 0\} = \{x \in N_{\mathbf{R}} \mid < x, -m_0 > \geq 0\}$ hence $(-m_0) \in S_{\sigma_2}$. $\tau = \sigma_1 \cap \sigma_2 \subset \sigma_i \cap \{m_0\}^{\perp}$ for $i = 1, 2$.
Let $\sigma_i' = \sigma_i \cap \{m_0\}^{\perp}$, note that since σ_1 and σ_2 don't both lie in H then $\dim \sigma_1' + \dim \sigma_2' < \dim \sigma_1 + \dim \sigma_2$. Then $S_{\sigma_1'} = S_{\sigma_1} + \mathbf{Z}_{\geq 0}(-m_0) \subset S_{\sigma_1} +$

$S_{\sigma_2}, S_{\sigma_2'} = \mathbf{Z}_{\geq 0}(m_0) + S_{\sigma_2} \subset S_{\sigma_1} + S_{\sigma_2}$.Hence $S_{\sigma_1'} + S_{\sigma_2'} \subset S_{\sigma_1} + S_{\sigma_2}$.

Since $\tau = \sigma_1' \cap \sigma_2'$, the induction hypothesis implies that $S_\tau \subset S_{\sigma_1'} + S_{\sigma_2'}$. □

Proposition 2.9 *Let σ be an* **scrpc** *in $N_\mathbf{R}$, and $\tau < \sigma$ then U_τ is an open subset of U_σ.*

Proof : Since we are assuming that our cones are nonsingular, we give a proof for that case only. Let $\sigma = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_p$ where $n_1, \ldots, n_r$ is a $\mathbf{Z}$ -basis for $\mathbf{N_R}$ hence $\sigma^\vee = \mathbf{R}_{\geq 0}m_1 + \cdots + \mathbf{R}_{\geq 0}m_p + \mathbf{R}m_{p+1} + \cdots + \mathbf{R}m_r$, and let $\tau < \sigma$. Without loss of generality we can assume $\tau = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_s for s \leq p$, then we have $\tau^\vee = \mathbf{R}_{\geq 0}m_1 + \cdots + \mathbf{R}_{\geq 0}m_s + \mathbf{R}m_{s+1} + \cdots + \mathbf{R}m_r$, so $S_\sigma = \mathbf{Z}_{\geq 0}m_1 + \cdots + \mathbf{Z}_{\geq 0}m_p + \mathbf{Z}m_{p+1} + \cdots + \mathbf{Z}m_r$, $S_\tau = \mathbf{Z}_{\geq 0}m_1 + \cdots + \mathbf{Z}_{\geq 0}m_s + \mathbf{Z}m_{s+1} + \cdots + \mathbf{Z}m_r$, and $U_\sigma \simeq \underbrace{\mathbf{R} \times \ldots \times \mathbf{R}}_{p} \times \underbrace{\mathbf{R}^* \times \ldots \times \mathbf{R}^*}_{r-p}$, $U_\tau \simeq \underbrace{\mathbf{R} \times \ldots \times \mathbf{R}}_{s} \times \underbrace{\mathbf{R}^* \times \ldots \times \mathbf{R}^*}_{r-s}$. □

Definition 2.7 *A fan in $N_\mathbf{R}$ is a nonempty set Δ of* **scrpc** *in $N_\mathbf{R}$ such that:*

i *If $\sigma \in \Delta$ and $\tau < \sigma$, then $\tau \in \Delta$.*

ii *If $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$.*

Theorem 2.1 *Let Δ be a fan in $N_\mathbf{R}$. Then we can naturally glue $\{U_\sigma, \sigma \in \Delta\}$ together to obtain a manifold $X_\Delta = \cup_{\sigma \in \Delta} U_\sigma$.*

Proof : We have to prove that X_Δ is Hausdorff, which is equivalent to proving that the map

$$\begin{array}{ccc} X_\Delta & \longrightarrow & X_\Delta \times X_\Delta \\ u & \longmapsto & (u, u) \end{array}$$

is closed.

The only problem is with the identification of U_τ in U_{σ_1} with U_τ in U_{σ_2} where

$\tau = \sigma_1 \cap \sigma_2$. Now let $S_{\sigma_1} = \mathbf{Z}_{\geq 0}m_1 + \cdots + \mathbf{Z}_{\geq 0}m_p$ and $S_{\sigma_2} = \mathbf{Z}_{\geq 0}m'_1 + \cdots + \mathbf{Z}_{\geq 0}m'_q$, then by proposition 2.8 $S_\tau = \mathbf{Z}_{\geq 0} + \cdots + \mathbf{Z}_{\geq 0}m_p + \mathbf{Z}_{\geq 0}m'_1 + \cdots + \mathbf{Z}_{\geq 0}m'_q$ (we do not exclude the cases $m_i = -m_j$ or $m'_i = -m'_j$). We identify U_τ (resp U_{σ_1}, respU_{σ_2}) with its image in $\mathbf{R}^{p+q}$ (resp $\mathbf{R}^p$, resp $\mathbf{R}^q$). Thus U_τ and $U_{\sigma_1} \times U_{\sigma_2}$ can be regarded as closed subsets of $\mathbf{R}^{p+q}$ and since U_τ is contained in $U_{\sigma_1} \times U_{\sigma_2}$ therefore U_τ is closed in $U_{\sigma_1} \times U_{\sigma_2}$. $\square$

Remark :

1. $\{0\} \in \Delta$, then $U_{\{0\}} = \{u : M \longrightarrow \mathbf{R} \mid u(m+m') = u(m)u(m')\ u(0) = 1\}$, and since $M = \mathbf{Z}m_1 + \cdots + \mathbf{Z}m_r$, we have $U_{\{0\}} \cong \mathbf{R}^* \times \cdots \times \mathbf{R}^*$.

   We will denote $U_{\{0\}}$ by $\mathcal{T}$.

2. For every σ in Δ we have $\{0\} < \sigma$, then $\mathcal{T}$ is open in U_σ for every σ in Δ. Therefore $\mathcal{T}$ is open in X_Δ.

3. Let t be in $\mathcal{T}$ and u be in U_σ. Define tu by $(tu)(m) = t(m)u(m)$ for m in S_σ; This defines an action of $\mathcal{T}$ on U_σ and by natural glueing on X_Δ. So the real toric manifolds are a particular case of manifolds with $\mathbf{R}^* \times \cdots \times \mathbf{R}^*$ action, and in particular a $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ action. We will see that there are manifolds which have $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ action but they are not toric manifolds.

Example :

Let $\sigma_1 = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2$, where $\{n_1, n_2\}$ is the canonical basis for $\mathbf{R}^2$, and $\sigma_2 = \mathbf{R}_{\geq 0}(an_1 + bn_2) + \mathbf{R}_{\geq 0}(cn_1 + dn_2)$ where $ad - bc = 1$. Then $\sigma_1^\vee = \mathbf{R}_{\geq 0}m_1 + \mathbf{R}_{\geq 0}m_2$, $S_{\sigma_1} = \mathbf{Z}_{\geq 0}m_1 + \mathbf{Z}_{\geq 0}m_2$, $\sigma_2^\vee = \mathbf{R}_{\geq 0}(dm_1 - cm_2) + \mathbf{R}_{\geq 0}(-bm_1 + am_2)$. So $u \in U_{\sigma_1}$ is determined by $(u(m_1), u(m_2))$ and $u \in U_{\sigma_2}$ is determined by $(u(dm_1 - cm_2), u(-bm_1 + am_2))$.

And we have the following commutative diagram :

$$\begin{array}{ccc}
\mathcal{T} \times (U_{\sigma_1} \cap U_{\sigma_2}) \subset U_{\sigma_1} & \longrightarrow & (U_{\sigma_1} \cap U_{\sigma_2}) \subset U_{\sigma_1} \\
(t_1, t_2), (x_1, x_2) & \longmapsto & (t_1 x_1, t_2 x_2) \\
\downarrow & & \downarrow \\
\mathcal{T} \times (U_{\sigma_1} \cap U_{\sigma_2}) \subset U_{\sigma_2} & \longrightarrow & (U_{\sigma_1} \cap U_{\sigma_2}) \subset U_{\sigma_2} \\
(t_1^d t_2^{-c}, t_1^{-b} t_2^a), (x_1^d x_2^{-c}, x_1^{-b} x_2^a) & \longmapsto & (t_1^d x_1^d t_2^{-c} x_2^{-c}, t_1^{-b} x_1^{-b} t_2^a x_2^a)
\end{array}$$

Definition 2.8 *Let σ be an* **scrpc** *in $N_{\mathbf{R}}$. Define:*

$$\mathbf{orb}\sigma = \{u : M \cap \sigma^{\perp} \longrightarrow \mathbf{R}^* \textit{ group homomorphism}\}$$

Remark :

$\mathbf{orb}\sigma$ is canonically embedded in U_σ by letting $u(m) = 0$ for any m in S_σ not in $M \cap \sigma^{\perp}$.

Proposition 2.10 *Let Δ be a fan in $N_{\mathbf{R}}$.*

For every $\sigma \in \Delta$ let $\mathbf{orb}\sigma$ *be as above. Then we have:*

1. *Every $\mathcal{T}$ orbit in X_Δ is of this form and in this way.*

2. *If $\tau < \sigma$ then $\mathbf{orb}\tau \subset U_\sigma$.*

3. $\forall \sigma \in \Delta, \quad U_\sigma = \sqcup_{\tau < \sigma} \mathbf{orb}\tau$.

4. *Δ is in one to one correspondence with the set of $\mathcal{T}$ orbits in X_Δ.*

5. *For $\sigma, \tau \in \Delta \quad \tau < \sigma \Leftrightarrow \mathbf{orb}\sigma \subset \overline{\mathbf{orb}\tau}$.*

6. $\overline{\mathbf{orb}\tau} = \sqcup_{\tau < \sigma} \mathbf{orb}\sigma$.

7. $\mathbf{orb}(0) = \mathcal{T}$.

Proof : (Prop 1.6 [9]). □

Example : Let $\sigma = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2$, then $\Delta = \{\{0\}, \tau_1 = \mathbf{R}_{\geq 0}n_1, \tau_2 = \mathbf{R}_{\geq 0}n_2, \sigma\}$ is a fan. An element u in U_σ is determined by $(u(m_1), u(m_2))$ and $\mathbf{orb}\{0\} = \mathbf{R}^* \times \mathbf{R}^*$, $\mathbf{orb}\tau_1 = \{u : M \cap \tau_1^\perp \longrightarrow \mathbf{R}^*,$ group homomorphism$\} = \{u : \mathbf{Z}m_2 \longrightarrow \mathbf{R}^*$ group homomorphism$\} \cong 0 \times \mathbf{R}^*$, $\mathbf{orb}\tau_2 = \mathbf{R}^* \times 0$ and $\mathbf{orb}\sigma = 0 \times 0$.

2.4 Compactness

Proposition 2.11

$$X_\Delta \textit{ is compact} \Leftrightarrow \begin{cases} \Delta \textit{ is finite} \\ \textit{and } \mid \Delta \mid = N_{\mathbf{R}} \textit{ where } \mid \Delta \mid = \cup_{\sigma \in \Delta} \sigma \end{cases}$$

Proof : $(\Rightarrow)$ X_Δ is compact.

Let $\Delta' = \{$maximal dimensional cones in $\Delta\}$ then $X_\Delta = \cup_{\sigma \in \Delta'} U_\sigma$ and X_Δ is not covered by any proper subset of Δ', hence

$$X_\Delta \text{ compact} \Rightarrow \Delta' \text{ is finite} \Rightarrow \Delta \text{ is finite.}$$

Let $n \in N$ and $\lambda \in \mathbf{R}^*$, define

$$\begin{aligned} \gamma_n(\lambda) : \ M &\longrightarrow \ \mathbf{R}^* \\ m &\longmapsto \ \lambda^{<n,m>} \end{aligned}$$

Then $\gamma_n(\lambda) \in \mathcal{T}$ for every λ.

X_Δ is compact $\Rightarrow \lim_{\lambda \to 0} \gamma_n(\lambda) \in X_\Delta \Rightarrow \exists \sigma \in \Delta$ such that $\lim_{\lambda \to 0} \gamma_n(\lambda) \in U_\sigma$. i.e $\lim_{\lambda \to 0} \lambda^{<n,m>} \in U_\sigma \Rightarrow < n, m > \ \geq 0 \ \forall m \in S_\sigma \Rightarrow n \in \sigma$ for some $\sigma \in \Delta \Rightarrow \mid \Delta \mid = N_{\mathbf{R}}$.

$(\Leftarrow)$ Let $T = \{u \in \mathcal{T} \mid u(m) = \pm 1, \forall m \in M\} \simeq (\mathbf{Z}_2)^r$ and

$\tau = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_t$, where $\{n_1 \ldots n_r\}$ is a basis of N,

$\tau^\vee = \mathbf{R}_{\geq 0}m_1 + \cdots + \mathbf{R}_{\geq 0}m_t + \mathbf{R}m_{t+1} + \cdots + \mathbf{R}m_r$. Let $u \in \mathbf{orb}\tau$ so that $u = (\underbrace{0, \ldots, 0}_{t}, u(m_{t+1}), \ldots, u(m_r))$ with $u(m_i) \neq 0$ for $i = t+1, \ldots, r$, we then have:

$$\begin{aligned}
\mathbf{orb}\tau/T &\cong \{(0, \ldots, 0, x_{t+1}, \ldots, x_r) \text{ where } x_i > 0\} \\
&\cong \mathbf{R}^{>0}n_{t+1} + \cdots + \mathbf{R}^{>0}n_r \\
&\cong \mathbf{R}n_{t+1} + \cdots + \mathbf{R}n_r \qquad (\text{ using the function } -\log) \\
&\cong N_{\mathbf{R}}/\mathbf{R}\tau
\end{aligned}$$

Since T is a compact group and acts on X_Δ, it is enough to prove that X_Δ/T is compact.

$$\begin{aligned}
X_\Delta/T &= \bigcup_{\rho\in\Delta}(\mathbf{orb}\rho/T) \\
&= \bigcup_{\rho\in\Delta}(N_{\mathbf{R}}/\mathbf{R}\rho) \\
&= \bigcup_{\rho\in\Delta}\Big(\bigcup_{\substack{\sigma\in\Delta' \\ \rho<\sigma}}(\sigma+\mathbf{R}\rho)/\mathbf{R}\rho\Big) \qquad (\text{since } \mid \Delta' \mid = N_{\mathbf{R}}) \\
&= \bigcup_{\sigma\in\Delta'}\Big(\bigcup_{\substack{\rho\in\Delta \\ \rho<\sigma}}(\sigma+\mathbf{R}\rho)/\mathbf{R}\rho\Big) \\
&= \bigcup_{\sigma\in\Delta'} w_\sigma
\end{aligned}$$

It is enough to show that $w_\sigma = \bigcup_{\substack{\rho\in\Delta \\ \rho<\sigma}}(\sigma + \mathbf{R}\rho)/\mathbf{R}\rho$ is compact $\forall\sigma \in \Delta'$ (since Δ is finite).

Let $\sigma = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_r$ be a nonsingular cone, and $\rho = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_t$ be a face of σ. The image of $(\sigma + \mathbf{R}\rho)/\mathbf{R}\rho$ under the identification $N_{\mathbf{R}}/\mathbf{R}\rho \cong \mathbf{orb}\rho/T$ is given by:

$$\begin{aligned}
(\sigma+\mathbf{R}\rho)/\mathbf{R}\rho &\cong (\mathbf{R}n_1 + \cdots + \mathbf{R}n_t + \mathbf{R}_{\geq 0}n_{t+1} + \cdots + \mathbf{R}_{\geq 0}n_r)/\mathbf{R}n_1 + \cdots + \mathbf{R}n_t \\
&\cong \mathbf{R}_{\geq 0}n_{t+1} + \cdots + \mathbf{R}_{\geq 0}n_r
\end{aligned}$$

$$\cong (0,1]n_{t+1} + \cdots + (0,1]n_r$$

$$\cong \{(0,\ldots,0,x_{t+1},\ldots,x_r) \mid 0 < x_i \leq 1\}$$

hence

$$\bigcup_{\substack{\rho\in\Delta \\ \rho<\sigma}} (\sigma + \mathbf{R}\rho)/\mathbf{R}\rho \cong \bigcup_{S\subset\{1,\ldots,r\}} \{(x_1,\ldots,x_r) \mid x_i = 0 \text{ if } i \in S, \text{ and } 0 < x_i \leq 1 \text{ otherwise }\}$$
$$\cong [0,1]^r$$

which is compact. □

So the compact real toric manifolds of $\dim r$ correspond to complete nonsingular fans in $\mathbf{R}^r$.

2.5 Equivariant isomorphisms

Let $(N,\Delta),(N',\Delta')$ be fans. $N \simeq \mathbf{Z}^r, N' \simeq \mathbf{Z}^{r'}$

Definition 2.9 *A map of fans $(N,\Delta) \longrightarrow (N',\Delta')$ constitutes of a $\mathbf{Z}$ linear homomorphism $\varphi : N \longrightarrow N'$ whose scalar extension $\varphi : N_{\mathbf{R}} \longrightarrow N'_{\mathbf{R}}$ satisfies the property*

$$\forall\sigma \in \Delta \ \ \exists\, \sigma' \in \Delta' \ \textit{ such that } \ \varphi(\sigma) \subset \sigma'$$

Theorem 2.2 *Let $\varphi : (N,\Delta) \longrightarrow (N',\Delta')$ be a map of fans. Then φ gives rise to a smooth map $\varphi_* : X_\Delta \longrightarrow X'_{\Delta'}$ which is equivariant with respect to the actions of $\mathcal{T}$ and $\mathcal{T}'$ on X_Δ and $X'_{\Delta'}$ respectively. Conversely : If $\overline{f} : \mathcal{T} \longrightarrow \mathcal{T}'$ is a group homomorphism and $f : X_\Delta \longrightarrow X'_{\Delta'}$ is a map equivariant with respect to $\overline{f}$, then there exists a unique $\mathbf{Z}$ linear homomorphism $\varphi : N \longrightarrow N'$ which gives rise to a map of fans $\varphi : (N,\Delta) \longrightarrow (N',\Delta')$ such that $f = \varphi_*$.*

Proof : Let $\varphi : N \longrightarrow N'$ be a $\mathbf{Z}$ homomorphism, we define :

$$\begin{aligned} \varphi^* : \ M' &\longrightarrow \ M \\ m' &\longmapsto \ \varphi^*(m') \text{ where } <\varphi^*(m'), n> = <m', \varphi(n)> \end{aligned}$$

Suppose $\varphi(\sigma) \subset \sigma'$, then $\forall n \in \sigma, \forall m' \in \sigma'^{\vee} \ <\varphi^*(m'), n> = <m', \varphi(n)> \ \geq 0$ therefore $\varphi^*(\sigma'^{\vee}) \subset \sigma^{\vee}$ and $\varphi^*(S'_{\sigma'}) \subset S_\sigma$ Define $\varphi_* : X_\Delta \longrightarrow X'_{\Delta'}$ as follows:

Let $u \in X_\Delta$ then there exists $\sigma \in \Delta$ such that $u \in U_\sigma$, let $\sigma' \in \Delta'$ such that $\varphi(\sigma) \subset \sigma'$, then define $\varphi_*(u)$ in $U'_{\sigma'}$ such that $\varphi_*(u)(m') = u(\varphi^*(m'))$ for every m in $S'_{\sigma'}$. We have to prove that such φ_* is well defined and equivariant: So let $u \in U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2} = U_\tau$ let $\tau' \in \Delta'$ such that $\varphi(\tau) \subset \tau'$ then φ_* is well defined in $U'_{\tau'}$ hence φ_* is well defined. Now we prove the equivariance of φ_*: $\varphi_*(tu)(m') = tu(\varphi^*(m')) = t(\varphi^*(m'))u(\varphi^*(m')) = \varphi_*(t)(m')\varphi_*(u)(m') = (\varphi_*(t)\varphi_*(u))(m')$, hence φ_* is equivariant.

Conversely: Let $\overline{f} : \mathcal{T} \longrightarrow \mathcal{T}'$ be a group homomorphism and define

$$\begin{aligned} \varphi^* : \ M' \cong \operatorname{Hom}(\mathcal{T}', \mathbf{R}^*) \ &\longrightarrow M \cong \operatorname{Hom}(\mathcal{T}, \mathbf{R}^*) \\ m' \qquad\qquad &\longmapsto \varphi^*(m')(t) = m'(\overline{f}(t)) \end{aligned}$$

then we get

$$\begin{aligned} \varphi : \ N &\longrightarrow \ N' \\ n &\longmapsto \ \varphi(n) \text{ where } <\varphi(n), m'> = <n, \varphi^*(m')> \end{aligned}$$

Let $f : \mathcal{T} \longrightarrow \mathcal{T}'$ be an equivariant map with respect to $\overline{f}$, we want to prove that there exists a unique $\mathbf{Z}$ linear morphism φ such that $\forall \sigma \in \Delta, \exists \sigma' \in \Delta'$ such that $\varphi(\sigma) \subset \sigma'$. Let $\sigma \in \Delta$, then $\exists \sigma' \in \Delta'$ such that $f(\mathbf{orb}\sigma) \subset \mathbf{orb}\sigma'$ by equivariance of f . Let $\tau < \sigma$ and let $\tau' \in \Delta'$ such that $f(\mathbf{orb}\tau) \subset \mathbf{orb}\tau'$. Since $\tau < \sigma$ then $\mathbf{orb}\sigma \subset \overline{\mathbf{orb}\tau} \Rightarrow f(\mathbf{orb}\sigma) \subset f(\overline{\mathbf{orb}\tau}) \subset \overline{f(\mathbf{orb}\tau)} \subset \overline{\mathbf{orb}\tau'}$. Since $f(\mathbf{orb}\sigma) \subset (\mathbf{orb}\sigma')$ we get $\mathbf{orb}\sigma' \subset \overline{\mathbf{orb}\tau'}$ which is equivalent to $\tau' < \sigma'$. But $U_\sigma = \cup_{\tau<\sigma}\mathbf{orb}\tau$, hence $f(U_\sigma) \subset \cup_{\tau<\sigma} f(\mathbf{orb}\tau) \subset \cup_{\tau'<\sigma'}\mathbf{orb}\tau' = U'_{\sigma'}$. Now let $n \in \sigma$ then $\lim_{\lambda\to 0}\gamma_n(\lambda) \in U_\sigma$, therefore

$$
\begin{aligned}
[f\circ\gamma_n(\lambda)](m') &= [\overline{f}(\gamma_n(\lambda))](m') \quad (\text{because } \ \gamma_n(\lambda) \in \mathcal{T} \text{and} f = \overline{f} \text{on} \mathcal{T})\\
&= m'[\overline{f}(\gamma_n(\lambda))]\\
&= \varphi^*(m')[\gamma_n(\lambda)]\\
&= \lambda^{<\varphi(n),m'>}\\
&= [\gamma_{\varphi(n)}(\lambda)](m').
\end{aligned}
$$

i.e $f\circ\gamma_n = \gamma_{\varphi(n)}$, but f is continuous hence $\lim_{\lambda\to 0}\gamma_{\varphi(n)}(\lambda) = f(\lim_{\lambda\to 0}\gamma_n(\lambda)) \in U'_{\sigma'}$ (since $\lim_{\lambda\to 0}\gamma_n(\lambda) \in U_\sigma$ and $f(U_\sigma) \subset U'_{\sigma'}$), therefore $\varphi(n) \in \sigma'$, so $\varphi(\sigma) \subset \sigma'$. □

Remarks :

1. Suppose $\sigma^\vee = \mathbf{Z}_{\geq 0}m_1 + \cdots + \mathbf{Z}_{\geq 0}m_p$ and $\sigma'^\vee = \mathbf{Z}_{\geq 0}m'_1 + \cdots + \mathbf{Z}_{\geq 0}m'_q$ and $\varphi^*(m'_i) = \sum a_{ij}m_j$ where $a_{ij} \in \mathbf{Z}^{\geq 0}$, let $u \in U_\sigma$ and $u = (u(m_1),\ldots,u(m_p)) = (x_1,\ldots,x_p)$ then $\varphi_*(u) = (x_1^{a_{11}}\ldots x_p^{a_{1q}},\ldots,x_1^{a_{q1}}\ldots x_p^{a_{qp}})$ i.e the equivariant maps are represented by monomials.

2. By the construction of φ_* we see that X_Δ and $X'_{\Delta'}$ are equivariantly diffeomorphic if and only if φ is an isomorphism between N and N' such that for every $\sigma \in \Delta$ there exists σ' such that $\varphi(\sigma) = \sigma'$. So the classification of closed real toric manifolds of $\dim r$ up to equivariant homeomorphism is equivalent to the classification of complete nonsingular fans in $\mathbf{Z}^r$ up to fan isomorphism.

This ends our review of toric manifolds. In the next chapters we will show how they arise naturally from special types of rational structures, and discuss their classification in dimensions 2 and 3.

CHAPTER 3

Monomial Structures and Uniformization

3.1 Monomial Structures

Let Δ be a complete nonsingular fan in $\mathbf{R}^r$. Then X_Δ is a compact manifold of $\dim r$. Let σ (resp σ') be maximal cones in Δ, and let $\{n_1, \ldots, n_r\}$ (resp $\{n'_1, \ldots, n'_r\}$) be the generating basis for σ (resp σ'). Without loss of generality we assume that $\{n_1, \ldots, n_r\}$ is the canonical basis for $\mathbf{R}^r$, then $n'_i = \sum_{j=1}^r a_{ij} n_j$ with $a_{ij} \in \mathbf{Z}$; letting $A = (a_{ij})$ we get $n'_i = A^t n_i$ and since (n_i) and (n'_i) are $\mathbf{Z}$ bases, then $\det A = \pm 1$ (without loss of generality we assume it is $+1$).

Let $\sigma^\vee = \mathbf{R}_{\geq 0} m_1 + \cdots + \mathbf{R}_{\geq 0} m_r$ and $\sigma'^\vee = \mathbf{R}_{\geq 0} m'_1 + \cdots + \mathbf{R}_{\geq 0} m'_r$ be the dual cones.
Therefore $\{< m'_i, n'_i >= 1$ and $< m'_i, n'_j >= 0$ if $i \neq j\}$ $\Leftrightarrow$
$\{< m'_i, A^t n_i >= 1$ and $< m'_i, A^t n_j >= 0$ if $i \neq j\}$ $\Leftrightarrow$
$\{< A m'_i, n_i >= 1$ and $< A m'_i, n_j >= 0$ if $i \neq j\}$ $\Leftrightarrow$ $\{m_i = A m'_i\}$, hence letting $\tau = \sigma \cap \sigma'$ we have:

$$\begin{aligned} \varphi : \quad U_\tau = U_\sigma \cap U_{\sigma'} \subset U_{\sigma'} &\longrightarrow U_\tau \subset U_\sigma \\ (u(m'_1), \ldots, u(m'_r)) &\longmapsto (u(m_1), \ldots, u(m_r)) \end{aligned}$$

So if we let $(x_1, \ldots, x_r) = (u(m'_1), \ldots, u(m'_r))$ then $u(m_i) = u(a_{1i}m'_1 + a_{2i}m'_2 + \cdots + a_{ri}m'_r) = u(m'_1)^{a_{1i}} \ldots u(m'_r)^{a_{ri}} = x_1^{a_{1i}} \ldots x_r^{a_{ri}}$ i.e φ is defined by:

$$(x_1, \ldots, x_r) \longmapsto (x_1^{a_{11}} x_2^{a_{21}} \ldots x_r^{a_{r1}}, \ldots, x_1^{a_{1r}} \ldots x_r^{a_{rr}})$$

So we see that toric manifolds have a monomial structure, that is they are covered by charts with the transition maps being monomials, and by the construction above, we see that the transition maps are completely defined by the coordinates of the generating vectors of the different maximal cones of the fan.

Now considering the transition map $\varphi : U_\tau(\subset U_{\sigma'}) \longrightarrow U_\tau(\subset U_\sigma)$ we remark that $\varphi_{|U_{\{0\}}} : U_{\{0\}} \cong (\mathbf{R}^*)^r \longrightarrow U_{\{0\}} \cong (\mathbf{R}^*)^r$ is a homeomorphism (since $\det A = 1$). The following theorem shows that the toric manifolds are a special case of the rational manifolds defined in [ ], with the rational functions being replaced by monomials.

Theorem 3.1 *A compact manifold M is toric iff M is covered by a finite number of charts (U_i, φ_i) such that*

1. $U_i \overset{\varphi_i}{\simeq} \mathbf{R}^r$.

2. $\varphi_i \circ \varphi_j^{-1}$ *are monomials.*

3. U_i *is dense in M for every i.*

Lemma 3.1 *A monomial φ is a transformation map for a compact toric manifold if and only if φ and φ^{-1} are homeomorphisms on their respective domain of definition.*

Proof :

($\Longrightarrow$) We consider $\varphi : U_\tau \longrightarrow U_\tau$ as above. The only possibility to extend φ is along some of the coordinate axes (since φ is defined on$(\mathbf{R}^*)^r$) but

$$\begin{array}{rcl} \varphi : & U_\tau & \longrightarrow U_\tau \\ & (x_1 \ldots x_r) & \longmapsto (x_1^{a_{11}} \ldots x_r^{a_{r1}}, \ldots, x_1^{a_{1r}} \ldots x_r^{a_{rr}}) \end{array}$$

so φ can be extended only to the i^{th} axes where $a_{ij} \geq 0 \; \forall j$, without loss of generality assume $i = 1$. Suppose $\mathbf{R} \times \mathbf{R}^* \times \cdots \times \mathbf{R}^* \not\subset U_\tau$ i.e $0 \times \mathbf{R}^* \times \cdots \times \mathbf{R}^* \not\subset U_\tau$ and that φ is defined on $0 \times \mathbf{R}^* \times \cdots \times \mathbf{R}^*$, which means that $a_{11}, \ldots, a_{1r} \geq 0$ then we have either one of the three cases:

(i) All $a_{1i} = 0$ then $\det A = 0$ contradiction.

(ii) A unique $a_{1i_0} \neq 0 \Rightarrow a_{1i_0} = 1$ and $n'_1 = n_{i_0}$, therefore $\mathbf{R}_{\geq 0} n'_1 < \tau$ hence $x_1 \in \mathbf{R}$ i.e $0 \times \mathbf{R}^* \times \cdots \times \mathbf{R}^* \subset U_\tau$.

(iii) There exists at least two j_1, j_2 such that $a_{1j_1}, a_{1j_2} > 0$. Therefore n'_1 is in the interior of a face of σ namely the face generated by the n'_j where $a_{1j} > 0$, but this contradicts the definition of a fan. Therefore φ is defined only on U_τ and by the same method φ^{-1} is defined on U_τ.

($\Longleftarrow$) Let

$$\begin{array}{rccl} \varphi: & A & \longrightarrow & B \\ & (x_1, \ldots, x_r) & \longmapsto & (x_1^{a_{11}} \ldots x_r^{a_{r1}}, \ldots, x_1^{a_{1r}} \ldots x_r^{a_{rr}}) \end{array}$$

be a homeomorphism from A to B and such that φ can not be defined on a set bigger than A, and φ^{-1} can not be defined on a set larger than B. φ is a homeomorphism from $\mathbf{R}^{*r}$ onto itself, hence $\det(a_{ij}) = \pm 1$, we assume it is 1. We will prove that the cones $\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_r$ and $\sigma' = \mathbf{R}_{\geq 0} n'_1 + \cdots + \mathbf{R}_{\geq 0} n'_r$ where $n'_i = \sum_j a_{ij} n_j$ intersect on a face.

Suppose that φ is defined on $\mathbf{R} \times \mathbf{R}^* \times \cdots \times \mathbf{R}^*$, so $a_{1j} \geq 0 \;\; \forall j$. We have the following cases:

1. All $a_{1j} = 0$, then $\det(a_{ij}) = 0$, which is not acceptable.

2. There exists a unique $a_{1j_1} \neq 0$, hence $a_{1j_1} = 1$ i.e $n'_1 = n_{j_1}$.

3. There exist at least two j_1, j_2 such that $a_{1j_1}, a_{1j_2} > 0$. Without loss of generality we can assume that φ is of the following form:

$$\varphi(x_1, \ldots, x_r) = (x_1^{a_{11}} \ldots x_r^{a_{r1}}, \ldots, x_1^{a_{1s}} \ldots x_r^{a_{rs}}, x_2^{a_{2s+1}} \ldots x_r^{a_{rs+1}}, \ldots, x_2^{a_{2r}} \ldots x_r^{a_{rr}})$$

Hence $\varphi(0, x_2, \ldots, x_r) = (0, \ldots, 0, x_2^{a_{2s+1}} \ldots x_r^{a_{rs+1}}, \ldots, x_2^{a_{2r}} \ldots x_r^{a_{rr}})$ i.e $\varphi(0 \times \mathbf{R}^* \times \cdots \times \mathbf{R}^*) \subset (0 \times \cdots \times 0 \times \mathbf{R}^* \times \cdots \times \mathbf{R}^*)$ injectively which is impossible.

Now if φ is defined on $\mathbf{R}^s \times \mathbf{R}^{*r-s}$ then $n'_k = n_{i_k}$ for $1 \leq k \leq s$ with $i_k \neq i_l$ if $k \neq l$, so $\sigma \cap \sigma' = \mathbf{R}_{\geq 0} n'_1 + \cdots + \mathbf{R}_{\geq 0} n'_s = \mathbf{R}_{\geq 0} n_{i_1} + \cdots + \mathbf{R}_{\geq 0} n_{i_s}$ which is a face of both σ and σ' because of the nonsingularity of the cones.

So we proved that no face of σ' is in the interior of a face of σ unless it is equal to it. And we prove the same result for the faces of σ using φ^{-1} □

Example : Let

$$\begin{aligned} \varphi : \quad \mathbf{R}^* \times \mathbf{R}^* &\longrightarrow \mathbf{R}^* \times \mathbf{R}^* \\ (x, y) &\longmapsto (x^2 y^{-1}, x^{-1} y) \end{aligned}$$

φ is a monomial which is a homeomorphism on its domain of definition, but φ is not a toric transition map because:

$$\begin{aligned} \varphi^{-1} : \quad \mathbf{R}^* \times \mathbf{R}^* &\longrightarrow \mathbf{R}^* \times \mathbf{R}^* \\ (x, y) &\longmapsto (xy, xy^2) \end{aligned}$$

can be extended to $\mathbf{R}^2$ but not as a homeomorphism. Infact φ arises from the cones: $\sigma = \mathbf{R}^{\geq 0} n_1 + \mathbf{R}^{\geq 0} n_2$ and $\sigma' = \mathbf{R}^{\geq 0}(2n_1 - n_2) + \mathbf{R}^{\geq 0}(-n_1 + n_2)$ and these two cones do not intersect along a face.

Proof of the theorem:

($\Rightarrow$) this is verified by construction.

($\Leftarrow$) Having a covering of M verifying the three conditions we need to exhibit a fan that corresponds to M:

Let $\Phi = \varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \subset \mathbf{R}^r \longrightarrow \varphi_1(U_1 \cap U_2) \subset \mathbf{R}^r$ if we prove that $\varphi_2(U_1 \cap U_2)$ is the domain of definition of Φ (as a monomial in $\mathbf{R}^r$) we would have two r-dimensional cones, and they would be intersecting along a face by the previous lemma and by repeating the process for all i, j we get a fan that corresponds to M. So the only thing we have to prove is that $\varphi_2(U_1 \cap U_2)$ is the domain of definition of Φ (the same procedure will apply for Φ^{-1}).

($\subseteq$) this is verified by definition.

($\supseteq$) let $x \in$ domain $\Phi \subset \mathbf{R}^r = \varphi_2(U_2)$. We want to prove that $x \in \varphi_2(U_1 \cap U_2)$ i.e $\varphi_2^{-1}(x) \in U_1 \cap U_2$. U_1 and U_2 are open dense in M, therefore $U_1 \cap U_2$ is dense in M, therefore there exists a sequence $(t_n)_n \subset U_1 \cap U_2$ such that (t_n) converges to $\varphi_2^{-1}(x)$. Let $y_n = \varphi_2(t_n)$ and $z_n = \varphi_1(t_n) = \varphi_1(\varphi_2^{-1}(y_n)) = \Phi(y_n)$. Since $\lim_{n\to\infty} t_n = \varphi_2^{-1}(x)$ then $\lim_{n\to\infty} y_n = x$ therefore $\lim_{n\to\infty} \Phi(y_n) = \Phi(x)$ i.e $\lim_{n\to\infty} z_n = \Phi(x)$, but z_n and $\Phi(x)$ are in $\varphi_1(U_1) \simeq \mathbf{R}^r$ therefore $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \varphi_1^{-1}(z_n) = \varphi_1^{-1}(\Phi(x))$ hence $\varphi_1^{-1}(\Phi(x)) = \varphi_2^{-1}(x)$ therefore $\Phi(x) = \varphi_1 \circ \varphi_2^{-1}(x)$ i.e $x \in \varphi_2(U_1 \cap U_2)$. $\square$

3.2 Uniformization

Let S be a topological space, G a group of homeomorphisms of S such that every g in G is determined by its action on any open subset of S [6].

Definition 3.1 *A topological space M is said to be uniformized by (S,G) if there exist an atlas $(U_\alpha, \varphi_\alpha)_\alpha$ covering M such that $\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subset S$ is a homeomorphism* *and* $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ *are restrictions of elements of* G.

Consider the following bundle ξ with base M and fiber S

$$E_\xi = \cup_\alpha (U_\alpha \times S)/_\sim \text{ where } (x, s_\alpha) \sim (x', s_\beta) \text{ iff } \begin{cases} x = x' \ \text{in} U_\alpha \cap U_\beta \\ s_\alpha = g_{\alpha\beta} s_\beta \end{cases}$$

The bundle ξ is determined by its characteristic class $\rho : \Pi_1(M) \longrightarrow G$ which is called the holonomy representation of the uniformization. Let $K = \ker\rho, M_K =$ the corresponding regular covering space of M; let $p : M_K \longrightarrow M$ then $\rho_K = \rho \circ p_* :$ $\Phi_1(M_K) \longrightarrow G$ is trivial, hence $p_*\xi$ is trivial on M_K i.e $E_{p_*\xi} \simeq M_K \times S$. Let

$$\begin{aligned} \sigma : \ M &\longrightarrow E_\xi \\ x &\longmapsto [x, \varphi_\alpha(x)] \end{aligned}$$

and define $\sigma_K : M_k \longrightarrow E_{p_*\xi}$, $\delta_K = P_2 \circ \sigma_K : M_K \rightarrow E_{p_*\xi} \rightarrow S$

Definition 3.2 *δ_K is a local homeomorphism called the developing map of the uniformization.*

Theorem 3.2 *The toric structure is not uniformizable.*

Proof : Assume that the toric structure correspond to a uniformization by (S, G), where G is isomorphic to a subgroup of the group of monomials so that G is discrete, and we compute the corresponding holonomy. Let M be a toric manifold, consider the bundle ξ as above and let σ be a curve in M from x_0 to x_1 , denote by S_t the fiber over $\sigma(t)$. Let $h_0 : S \longrightarrow S_0$ be the identification map, then there exists a bundlemap $h : I \times S \longrightarrow E$ such that $h(0, s) = h_0(s)$ and $ph(t, s) = \sigma(t)$.
Denote $\sigma^\sharp = h_0 \circ h_1^{-1} : S_1 \longrightarrow S_0$ where $h_t(s) = h(t, s)$. Since G is discrete, $\sigma^\sharp$ depends only on the homotopy class of σ.

If σ is a closed curve then $\sigma^\sharp : S_0 \longrightarrow S_0$ and we can regard $\sigma^\sharp$ as an element of G.
Let $\sigma \subset U_\alpha$ be a curve from x_0 to x_1 and $h_0(s) = \Phi_\alpha(x_0, s)$ where $\Phi_\alpha : U_\alpha \times S \longrightarrow p^{-1}(U_\alpha)$ is a trivialization chart for the bundle over U_α, then $h(s,t) = \Phi_\alpha(\sigma(t), s) = \Phi_{\alpha,\sigma(t)}(s)$ hence $\sigma^\sharp = \Phi_{\alpha,\sigma(0)}\Phi^{-1}_{\alpha,\sigma(1)}$ where s is in S_1. If σ is closed then $\sigma^\sharp = 1$.
If $\sigma = \sigma_1.\sigma_2$ with $\sigma_i \subset U_i$ then $\sigma^\sharp = \sigma_1^\sharp \cdot \sigma_2^\sharp = \Phi_{\alpha,\sigma_1(0)}\Phi^{-1}_{\alpha,\sigma_1(1)}\Phi_{\beta,\sigma_2(0)}\Phi^{-1}_{\beta,\sigma_2(1)}$. Working with the associated principal bundle we have:

$$\begin{aligned} \Phi_1 : \ U_1 \times G &\longrightarrow p^{-1}(U_1) \\ (x,g) &\longmapsto [x,g] \end{aligned}$$

and

$$\begin{aligned} \Phi_{1,x} : \ G &\longrightarrow p^{-1}(x) \\ g &\longmapsto [x,g] \end{aligned}$$

where $[x,g] = [x, g_{12}g]$ if $x = U_1 \cap U_2$ with $g_{12} = \varphi_1 \circ \varphi_2^{-1}$.
Let $\sigma = \sigma_1 \cdot \sigma_2$ with $\sigma_i \subset U_i$ and $\sigma_2(1) = \sigma_1(0)$, then $\sigma^\sharp \in G$ acts on the fiber $G_{\sigma_1(0)} = G_{\sigma_2(1)}$

$$\begin{aligned} \sigma^\sharp[\sigma_2(1), g] &= \Phi_{1,\sigma_1(0)}\Phi^{-1}_{1,\sigma_1(1)}\Phi_{2,\sigma_2(0)}\Phi^{-1}_{2,\sigma_2(1)}[\sigma_2(1), g] \\ &= \Phi_{1,\sigma_1(0)}\Phi^{-1}_{1,\sigma_1(1)}\Phi_{2,\sigma_2(0)}(g) \\ &= \Phi_{1,\sigma_1(0)}\Phi^{-1}_{1,\sigma_1(1)}[\sigma_2(0), g] \\ &= \Phi_{1,\sigma_1(0)}\Phi^{-1}_{1,\sigma_1(1)}[\sigma_1(1), g_{12}g] \\ &= \Phi_{1,\sigma_1(0)}(g_{12}g) \\ &= [\sigma_1(0), g_{12}g] \\ &= [\sigma_2(1), g] \end{aligned}$$

i.e $\sigma^\sharp = 1$.

Now since M is toric then the charts are dense in M, hence $g_{\alpha\beta} \circ g_{\beta\gamma}$ is defined on an open set of M and we get $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ (which is not the case in general since if $U_\alpha \cap U_\beta \cap U_\gamma = \emptyset$ we cannot define $g_{\alpha\beta} \circ g_{\beta\gamma}$).

By the same method as above we prove that $\sigma^\sharp = 1$ for a general σ therefore:

$$\begin{aligned} \rho: \ \Pi_1(X, x_0) &\longrightarrow G \\ \sigma \quad &\longmapsto \sigma^\sharp \end{aligned}$$

is trivial which implies that $E \simeq M \times S$ hence the map

$$\begin{array}{cccccc} \delta: & M & \longrightarrow & E & \longrightarrow & S \\ & x & \longmapsto & (x, \varphi_\alpha(x)) & \longmapsto & \varphi_\alpha(x) \end{array}$$

is a local homeomorphism since the holonomy is trivial.

If M were a compact manifold then δ would be a covering map so we see that all compact toric manifolds are covering spaces of S which can not happen since as we see in the next chapter in the example of dimension 2 the torus and $\mathbf{R}P^2$ are toric varieties and obviously they can not cover the same space. □

Remark :

We could have concluded the non uniformizability of the toric structures by observing that if M has an (S, G) structure then any covering space of M has it also. But we wanted to show that this is the case because of the holonomy triviality .

CHAPTER 4

Study of The 2 and 3 Dimensional Cases

4.1 Complex Toric manifolds

We consider the compact complex toric manifolds, they arise in the same way as the real ones. In this section we will give a short overview of these manifolds, for more details see [9]. Let Δ be a nonsingular complete fan in $N_{\mathbf{R}} \simeq \mathbf{R}^r$ and let σ be in Δ. Define $U_\sigma^{\mathbf{C}} = \{u : S_\sigma \longrightarrow \mathbf{C} \mid u(m+m') = u(m)u(m')\ u(0) = 1\}$ and $X_\Delta^{\mathbf{C}} = \cup_{\sigma\in\Delta} U_\sigma^{\mathbf{C}}$. $X_\Delta^{\mathbf{C}}$ is a simply connected compact manifold of dimension 2r and the transition maps are monomials with the same coefficients as in the real case. $(\mathbf{C}^*)^r$ acts smoothly on $X_\Delta^{\mathbf{C}}$, therefore we have a smooth action of the r-dimensional torus T^r on $X_\Delta^{\mathbf{C}}$, since $T^r \subset (\mathbf{C}^*)^r$. The action of $(\mathbf{C}^*)^r$ is determined by the same coefficients as in the real case, and we have similar results for the complex case as in proposition 2.10 (see Prop 1.6 in [9]).

Proposition 4.1

$$X_\Delta^{\mathbf{C}}/T^r = X_\Delta/(\mathbf{Z}_2)^r$$

Proof : Since $U_\sigma^{\mathbf{C}}$ (resp U_σ) is invariant under the action of $(\mathbf{C}^*)^r$ (resp $(\mathbf{R}^*)^r$) then it is invariant under the action of T^r (resp $(\mathbf{Z}_2)^r$), and the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Z}_2^r \times U_\sigma & \longrightarrow & U_\sigma \\ \downarrow & & \downarrow \\ T^r \times U_\sigma^{\mathbf{C}} & \longrightarrow & U_\sigma^{\mathbf{C}} \end{array}$$

Therefore it is enough to prove that $U_\sigma^{\mathbf{C}}/T^r = U_\sigma/(\mathbf{Z}_2)^r$, and also since $X_\Delta^{\mathbf{C}} = \cup_{\sigma\in\Delta'} U_\sigma^{\mathbf{C}}$ where $\Delta' = \{\text{maximal cones of } \Delta\}$, we need only to consider σ to be a maximal cone. So it suffices to show that $\mathbf{C}^r/T^r = \mathbf{R}^r/(\mathbf{Z}_2)^r$ under the action defined by the following commutative diagram

$$\begin{array}{ccc} (\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2) \times (\mathbf{R} \times \cdots \times \mathbf{R}) & \longrightarrow & \mathbf{R} \times \cdots \times \mathbf{R} \\ \downarrow & & \downarrow \\ (S^1 \times \cdots \times S^1) \times (\mathbf{C} \times \cdots \times \mathbf{C}) & \longrightarrow & \mathbf{C} \times \cdots \times \mathbf{C} \\ (t_1 \ldots t_r)(x_1 \ldots x_1) & \longmapsto & (t_1^{a_{11}} \ldots t_r^{a_{1r}} x_1, \ldots, t_1^{a_{r1}} \ldots t_r^{a_{rr}} x_r) \text{ with } \det(a_{ij}) = 1. \end{array}$$

Let us denote the equivalence classes in the complex case by $[x]$ and in the real case by $[x]_{\mathbf{R}}$ so we need to show:

$$\begin{cases} \forall x \in \mathbf{C}^r; \exists\, y \in \mathbf{R}^r \text{ such that } [x] = [y] \\ \text{and} \\ \forall y \in \mathbf{R}^r \subset \mathbf{C}^r \quad [y]_{\mathbf{R}} \subset [y] \end{cases}$$

So let $(x_1, \ldots, x_r) \in \mathbf{C}^r$, then there exists $(s_1, \ldots, s_r) \in T^r$ such that $(s_1x_1, \ldots, s_rx_r) \in \mathbf{R}^r$ and since $\det(a_{ij}) = 1$, there exists $(t_1, \ldots, t_r) \in T^r$ such that $(t_1^{a_{11}} \ldots t_r^{a_{1r}}, \ldots, t_1^{a_{r1}} \ldots t_r^{a_{rr}}) = (s_1, \ldots, s_r)$. Therefore for every x in $\mathbf{C}^r$ there exists y in $\mathbf{R}^r$ such that $T^r(x) = T^r(y)$ i.e $[x] = [y]$. And since $(\mathbf{Z}_2)^r \subset T^r$ then $\mathbf{Z}_2^r(x) \subset T^r(x)$ for every $x \in \mathbf{R}^r$ i.e $[x]_{\mathbf{R}} \subset [x]$, therefore $\mathbf{C}^r/T^r = \mathbf{R}^r/(\mathbf{Z}_2)^r$ and hence $X_\Delta^{\mathbf{C}}/T^r = X_\Delta/(\mathbf{Z}_2)^r$. $\square$

Proposition 4.2 *The isotropy groups of the action of T^r on $X_\Delta^{\mathbf{C}}$ are tori subgroups of T^r.*

Proof : Since every $U_\sigma^{\mathbf{C}}$ is invariant under T^r it is enough to prove the result in $U_\sigma^{\mathbf{C}}$ where σ is a maximal cone in Δ. The action is given by:

$$\begin{array}{ccc} (S^1 \times \cdots \times S^1) \times (\mathbf{C} \times \cdots \times \mathbf{C}) & \longrightarrow & \mathbf{C} \times \cdots \times \mathbf{C} \\ (e^{i\theta_1}, \ldots, e^{i\theta_r})(x_1, \ldots, x_r) & \longmapsto & (e^{i\sum a_{1j}\theta_j} x_1, \ldots, e^{i\sum a_{rj}\theta_j} x_r) \end{array}$$

with $\det(a_{ij}) = 1$. Obviously the points that are fixed by some subgroups of T^r are the ones that have some x_i's equal to zero, thus if $x_{i_k} = 0$ for $k = 1, \ldots, p$ and $x_{i_k} \neq 0$ otherwise, then the isotropy group corresponding to such point is $I_{\{i_1,\ldots,i_p\}} = \{(e^{i\theta_1}, \ldots, e^{i\theta_r}) \mid \sum_j a_{ij}\theta_j \equiv 0 \bmod 2\pi \ \ \forall\, i \neq i_1, \ldots, i_p\}$.
Let $A = (a_{ij})$, then since $A \in SL(r, \mathbf{Z})$ we have

$$\begin{array}{ccc} S^r & \longrightarrow & S^r \\ (e^{i\theta_1}, \ldots, e^{i\theta_r}) & \longmapsto & (e^{i\sum a_{1j}\theta_j}, \ldots, e^{i\sum a_{rj}\theta_j}) = (e^{i\psi_1}, \ldots, e^{i\psi_r}) \end{array}$$

is a change of coordinates of S^r, so $I_{\{i_1,\ldots,i_p\}} = \{(e^{i\psi_1}, \ldots, e^{i\psi_r}) \mid e^{i\psi_k} = 1 \ \text{ for } \ k \neq i_1, \ldots, i_p\}$. Therefore $I_{\{i_1,\ldots,i_p\}}$ is a torus of dimension p. $\square$

Remark :

Since the action of $(\mathbf{Z}_2)^r$ on X_Δ is just a restriction of the action of T^r on $X_\Delta^{\mathbf{C}}$, the isotropy groups of the first action are just $(\mathbf{Z}_2)^r$ intersecting the isotropy groups of the second action.

4.2 Classification of Compact Toric Manifolds of Dimension two

In [10] P.Orlik and F.Raymond studied the action of the 2-torus on simply connected closed 4 manifolds, we start this by a brief description of the effective smooth action of T^2 on a closed simply connected 4-manifold and then we give an overview of their results:

1. If the isotropy group of x is T^2 i.e x is a fixed point then the slice at x (which is the fiber over x of the normal disc bundle of the orbit) is a 4 disc and T^2 acts on it by a rotation in two planes by (m_1, n_1) and (m_2, n_2) with $m_1 n_2 - m_2 n_1 = \pm 1$ and the image of x in M^* is an isolated boundary point.

2. If the isotropy group of a point x in M is a circle subgroup of T^2 denoted $(m, n) = \{(e^{i\theta_1}, e^{i\theta_2} \in T^2 \mid m\theta_1 + n\theta_2 \equiv 0 \bmod 2\pi \text{ and } gcd(m, n) = 1\}$, then the slice at x is a 3 disc, the isotropy group (m, n) acts on it by rotation, the image of the orbit in M^* is a boundary point.

3. If the isotropy group of x is e, so the orbit of x is a torus, then the slice is a 2 disc and the image of the orbit is an interior point.

Theorem 4.1 *(Theorem 1.12 in [10]). If T^2 acts effectively and smoothly on a 4 manifold M without boundary, such that there are no nontrivial finite isotropy groups, and such that the set of fixed points and points of circle isotropy groups is not empty, then the orbit space is a 2 manifold with boundary, with weights identifying the isotropy groups.*

In section 4.4 of [10] they prove that under the hypothesis of theorem 1.12 the interior points correspond to principal orbits and the boundary points correspond to orbit with circle isotropy groups or isolated fixed points. In section 5 of [10] they studied the action of T^2 on closed simply connected 4-manifolds and they proved:

Theorem 4.2 *(Lemma 5.1 in [10]). The action has fixed points and M^* is a 2-disk with interior points corresponding to principal orbits, and the boundary points correspond to orbits with circle isotropy groups or isolated fixed points.*

So if $f_1, \ldots, f_t$ denote the fixed points and f_i^* their images in M^* then the arc S_i^* between f_i^* and f_{i+1}^* on ∂M^* represents a 2 sphere S_i and if we denote its stability group by $(a_i, b_i) = \{(\alpha, \beta) \in T^2 \mid \alpha^{a_i}\beta^{b_i} = 1\}$, we get a representation for M^* as shown in Fig. 4.1. where $\begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix} = \pm 1$. The determinant condition arises because

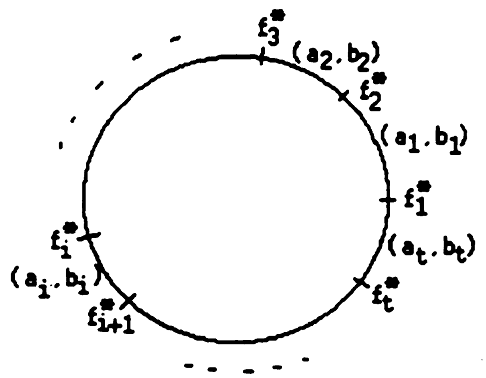

Figure 4.1. Representation of the orbit space

the action of T^2 on $X_\Delta^{\mathbf{C}}$ is differentiable then by corollary VI.2.4 and the definition of local smooth actions in [3], the restriction of the toric action to a neighborhood of a fixed point is equivalent to an orthogonal action of T^2 on D^4 i.e to:

$$\begin{array}{ccc} T^2 \times D^4 & \longrightarrow & D^4 \\ (t_1, t_2)(x, y) & \longmapsto & (t_1^{m_1} t_2^{n_1} x, t_1^{m_2} t_2^{n_2} y) \end{array}$$

This action where all the m_i and n_i are integers has the orbit space represented in Fig. 4.2. Now we translate these results to the case of two dimensional toric manifolds:

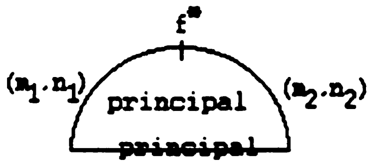

Figure 4.2. Orbit space of the orthogonal action of T^2 on D^4

We start the study of the two dimensional toric manifolds by fixing the coordinates in $(\mathbf{Z}_2)^2$ as follows: We consider an element (t_1, t_2) in $(\mathbf{Z}_2)^2$ to be $(t(m_1), t(m_2))$ where m_1 and m_2 are the duals of the canonical basis of $\mathbf{R}^2$. Let Δ be a complete nonsingular 2 fan, let $\sigma = \mathbf{R}_{\geq 0}(an_1 + bn_2) + \mathbf{R}_{\geq 0}(cn_1 + dn_2)$ be a maximal cone in Δ with $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$, then U_σ is isomorphic to $\mathbf{R}^2$, and the action of $(\mathbf{Z}_2)^2$ on U_σ is given by $(t_1, t_2)(u_1, u_2) = (t_1^d t_2^{-c} u_1, t_1^{-b} t_2^{a} u_2)$ (see example in page 11), the origin is the unique fixed point in U_σ, and since $U_\sigma \cap U_{\sigma'}$ for σ and σ' in Δ' does not contain the origin of either one of them, then there is a one to one correspondence between the set of fixed points and Δ'. The different proper isotropy groups are $(d, -c) \cap (\mathbf{Z}_2)^2$ and $(-b, a) \cap (\mathbf{Z}_2)^2$, where

$$(d, -c) \cap (\mathbf{Z}_2)^2 = \{(t_1, t_2) \in (\mathbf{Z}_2)^2 \mid t_1^d t_2^{-c} = 1\} = \begin{cases} \mathbf{Z}_2 \times 1 \text{ if } d \text{ even and } c \text{ odd} & \text{denoted } 10 \\ 1 \times \mathbf{Z}_2 \text{ if } d \text{ odd and } c \text{ even} & \text{denoted } 01 \\ \{(-1,-1),(1,1)\} \text{ if } d, c \text{ odd} & \text{denoted } 11 \end{cases}$$

By the results in [10] presented above and the fact that the orbit space is the same in the real and complex toric manifolds we get a presentation for $X^*_\Delta = X_\Delta/(\mathbf{Z}_2)^2$ as shown in Fig. 4.3. We call such a picture a colored graph (or graph for short) and will be denoted by G, the labels $01, 10, 11$ are the colors.

Remarks :

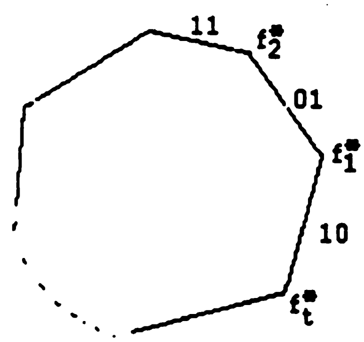

Figure 4.3. X^*_Δ

1. No two adjacent edges on the graph have the same color, since two adjacent coloring correspond to the isotropy groups of the action of $(\mathbf{Z}_2)^2$ on U_σ for a maximal cone and the determinant condition does not allow this to happen.

2. The number of edges is equal to the number of one dimensional cones and the number of fixed points is equal to the number of maximal cones so we have a duality picture: if we represent the fan and the orbit space corresponding to it on the same picture we get Fig. 4.4, where the notation $\overline{a}$ in the figure represents the class of a modulo 2.

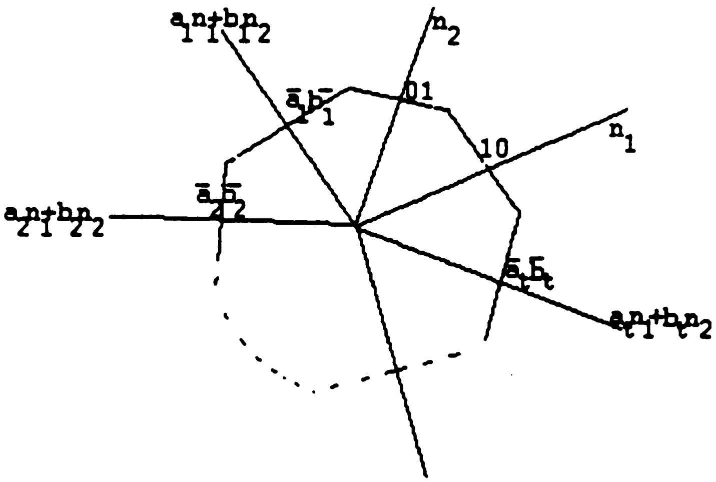

Figure 4.4. Duality picture

4.3 Cross Sections

Definition 4.1 *A cross section for $\pi : M \longrightarrow M^*$ is a continuous map $S : M* \longrightarrow M$ such that $\pi \circ s$ is the identity on M^*.*

Lemma 4.1 *If $(\mathbf{Z}_2)^2$ acts on a 2 manifold such that $M^* \simeq D^2$ with D^2 colored as shown in Fig. 4.5, then there is a cross section to this action. Furthermore if a cross section is given on an arc $A \subset S^-$ (where S^- is the horizontal segment in the figure), then it can be extended to all of D^2.*

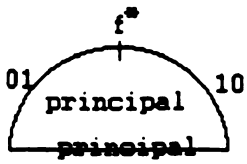

Figure 4.5. Orbit space of the action of $(\mathbf{Z}_2)^2$ on D^2

Proof : M is obtained from M^* by glueing 4 copies of D^2 along parts of S^+ (S^+ is the upper half circle of the boundary)two by two in the way shown in Fig. 4.6. Obviously

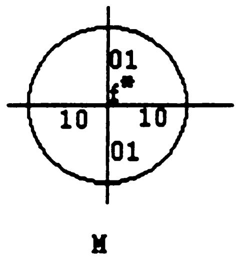

Figure 4.6. The disk

a cross section is just the choice of one quarter of M. And if a cross section is given in $A \subset S^-$, then this amounts to just indicating which quarter of M is chosen, and therefore the cross section is extended to that quarter. □

Theorem 4.3 *If* $(\mathbf{Z}_2)^2$ *acts on a closed 2-manifold such that* $M^* \simeq D^2$ *with all interior points of* D^2 *correspond to principal orbits, and points on the boundary correspond to either fixed points or orbits with 10 , 01 or 11 stability groups, then there is a cross section.*

Proof : Let M^* be as shown in Fig. 4.7(a) with t edges, then we cut D^2 into t cones C_i such that every cone contains a unique fixed point f_i^* as shown in Fig. 4.7(b). Then by the above lemma there exists a cross section along C_1, and this cross section is defined along an arc of the southern boundary of C_2, therefore by the same lemma, it can be extended to C_2, and continuing the same procedure to the following cones, we see that we can extend the cross section to all of D^2. □

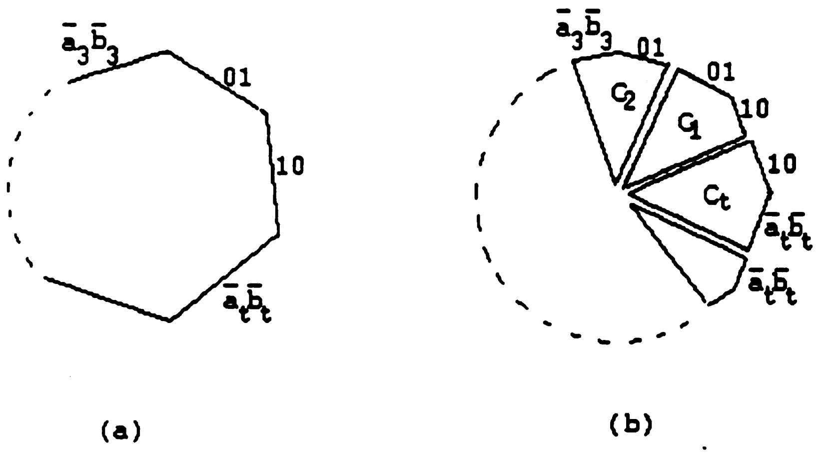

Figure 4.7. M^* and cutting it into cones

Theorem 4.4 *If $(\mathbf{Z}_2)^2$ acts on a compact 2-manifold with boundary such that $M^* \simeq D^2$. If $D^2 \setminus S^+$ consists of principal orbits, and points on S^+ correspond to either fixed points or orbits with 10 , 01, 11 stability groups, then there is a cross section to this action..*

Proof : same as the boundaryless case. □

Definition 4.2 *Let $(\mathbf{Z}_2)^2$ act on M and M' with M^* and M'^* being as in theorem 4.3 then a homeomorphism between M^* and M'^* which carries the weights of M^* onto the weights of M'^* isomorphically is called a weight preserving homeomorphism.*

Theorem 4.5 *Suppose $(\mathbf{Z}_2)^2$ acts on two closed 2 manifolds M, N such that M^*, N^* satisfy the condition of the theorem above and that there is a weight preserving homeomorphism $h^* : M^* \longrightarrow N^*$ then there is an equivariant homeomorphism $h : M \longrightarrow N$.*

Proof : This follows from theorem 4.1 and theorem 3.3 chapter I in [3]. □

Remark :

Theorem 4.5 means that if we start with a closed graph and change the colors using a bijection of $\{10, 01, 11\}$ onto itself, then we have the same manifold. So we can assume that we have a fixed point f in M where a neighborhood of f^* in M^* is as shown in Fig. 4.5. We remark that this is just the same assumption done at the end of chapter 2 for the fans.

4.4 Classification

Definition 4.3 *Let G be a graph, then cutting an edge out of G means to replace G by a new graph G' as shown in Fig. 4.8.*

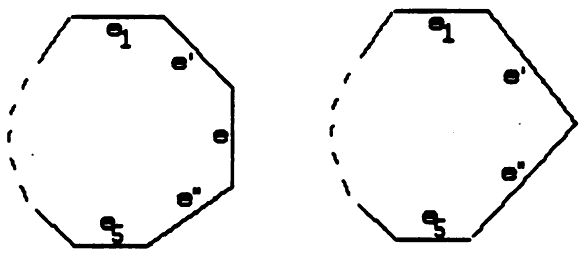

Figure 4.8. Cutting an edge

Now we see which colored graphs correspond to fans:

Proposition 4.3 *Let t be the number of fixed points in the manifold.*

If $t = 3$ and $t = 4$, the possible graphs and their dual fans are shown in Fig. 4.9.

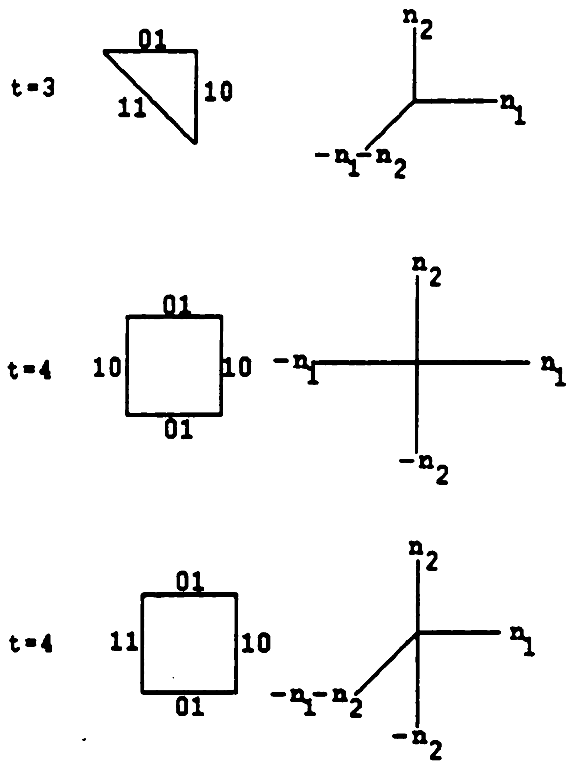

Figure 4.9. The graphs and their dual fans with $t = 3, 4$

Proposition 4.4 *If $t \geq 5$ then a graph is dual to a nonsingular complete fan iff it is colored by the three colors.*

Proof : ($\Rightarrow$) Let G be dual to Δ with $t \geq 5$ then by (Proof of theorem 8.2 in [8]) there exists n_i such that $n_i = n_{i-1} + n_{i+1}$, and since $\det(n_{i-1}, n_i) = 1, \det(n_i, n_{i+1}) = 1$ then the parity of n_{i-1} and n_{i+1} are different otherwise the parity of n_i would be 00, also the parity of n_i is different from the parity of the two others by the determinant condition, therefore we have the three colors.

($\Leftarrow$) If G is colored by 3 colors then wlog we have the two cases shown in Fig. 4.10. So we cut out the edge e in the first case, and we are still left with three colors, and in

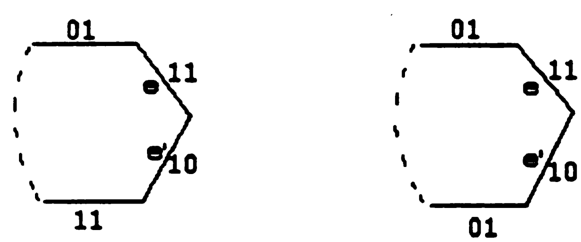

Figure 4.10. Two different cases for the graph with $t \geq 5$

case 2 we can cut out either edges e or e', but we make sure that the cut out edge will still leave us with three colors. We keep doing this operation until we get the triangle with three colors which was seen for $t = 3$, then, we start with the fan corresponding to $t = 3$ and for each step of the above operation (beginning from its last step) we introduce the sum of the two vectors dual to the two edges surrounding the removed edge until we get our graph back and we get a dual fan for it. $\square$

Example : See Fig. 4.11.

Now we see which manifolds correspond to these graphs:

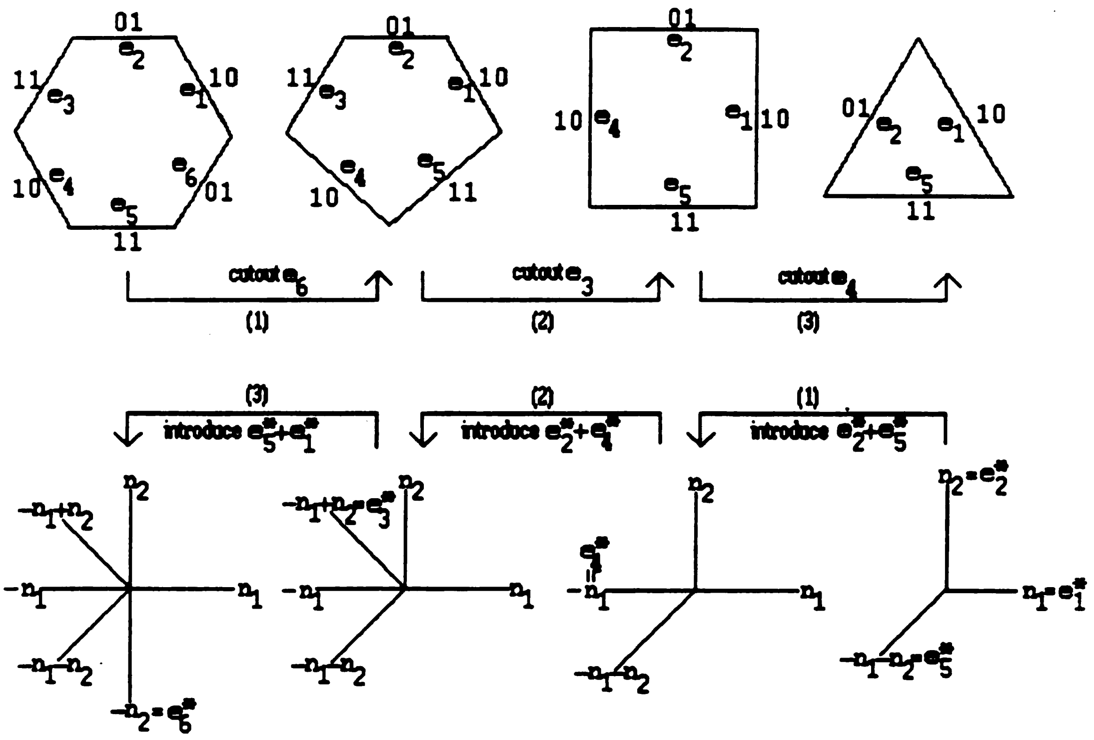

Figure 4.11. Example of reducing G to the triangle and the dual action on fans

Let G be a colored graph dual to a fan Δ, then X_Δ is obtained by glueing 4 copies of G along the edges. The 4 copies correspond to images of G under the action of the different elements of $(\mathbf{Z}_2)^2$. Let us denote $1 = (1,1)G$, $2 = (1,-1)G$, $3 = (-1,1)G$ and $4 = (-1,-1)G$. Therefore, for example, a side of 1 whose color is 10, is identified with the same side of 2, and a side of 2 whose color is 11, is identified with the same side of 4. To mark these informations on the graphs, we let

$1\overset{+}{0}$ (resp $1\bar{0}$) denote the color 10 for 1 and 2 (resp 3 and 4)

$\overset{+}{0}$ 1 (resp $\bar{0}$ 1) denote the color 01 for 1 and 3 (resp 2 and 4)

11 (resp -11) denote the color 11 for 1 and 4 (resp 2 and 3)

And now we can determine the 2 dimensional toric manifolds.

Proposition 4.5 *If $t = 3$ then $M_G \simeq \mathbf{R}P^2$.*

If $t = 4$ then there are two cases, and M_G is either T^2 or the Klein bottle as shown in Fig. 4.12.

Proposition 4.6 *If $t \geq 5$ then $M_G \simeq \sharp_{t-2}\mathbf{R}P^2$.*

Proof : By proposition 4.4, G is colored by three colors, hence it looks as shown in Fig. 4.13(a), but G_2 corresponds to $\mathbf{R}P^2 \setminus D^2$, and G_1 corresponds to $M_1 \setminus D^2$ for some manifold M_1 as shown in Fig. 4.13(b). Therefore G corresponds to $M_1 \sharp \mathbf{R}P^2$, and by the proof of proposition 4.4, we have $M_G \simeq \sharp_{t-2}\mathbf{R}P^2$. □

4.5 Dimension 3 Compact Toric Manifolds

In [7] D. Mac Gavrin studied the action of the 3-torus on simply connected closed 6 manifolds, we start this section with a brief description of the effective smooth action of T^3 on closed simply connected 6-manifolds, and then we give an overview of his results:

1. If the isotropy group of x is T^3 i.e x is a fixed point then the slice at x is a 6 disc, and T^3 acts on it by a rotation in three planes by $T(a_{11}, a_{12}, a_{13}), T(a_{21}, a_{22}, a_{23}), T(a_{31}, a_{32}, a_{33})$ with $\det(a_{ij}) = \pm 1$ and where $T(a_{k1}, a_{k2}, a_{k3}) = \{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) \mid \sum a_{lj}\varphi_j \equiv 0(2\pi) \text{ for } l \neq k\}$.The image of x in M^* is an isolated boundary point.

2. If the isotropy group of a point x in M is a 2-torus then the orbit of x is a circle, the slice is a 5-disc. The action of the isotropy group on the slice is a

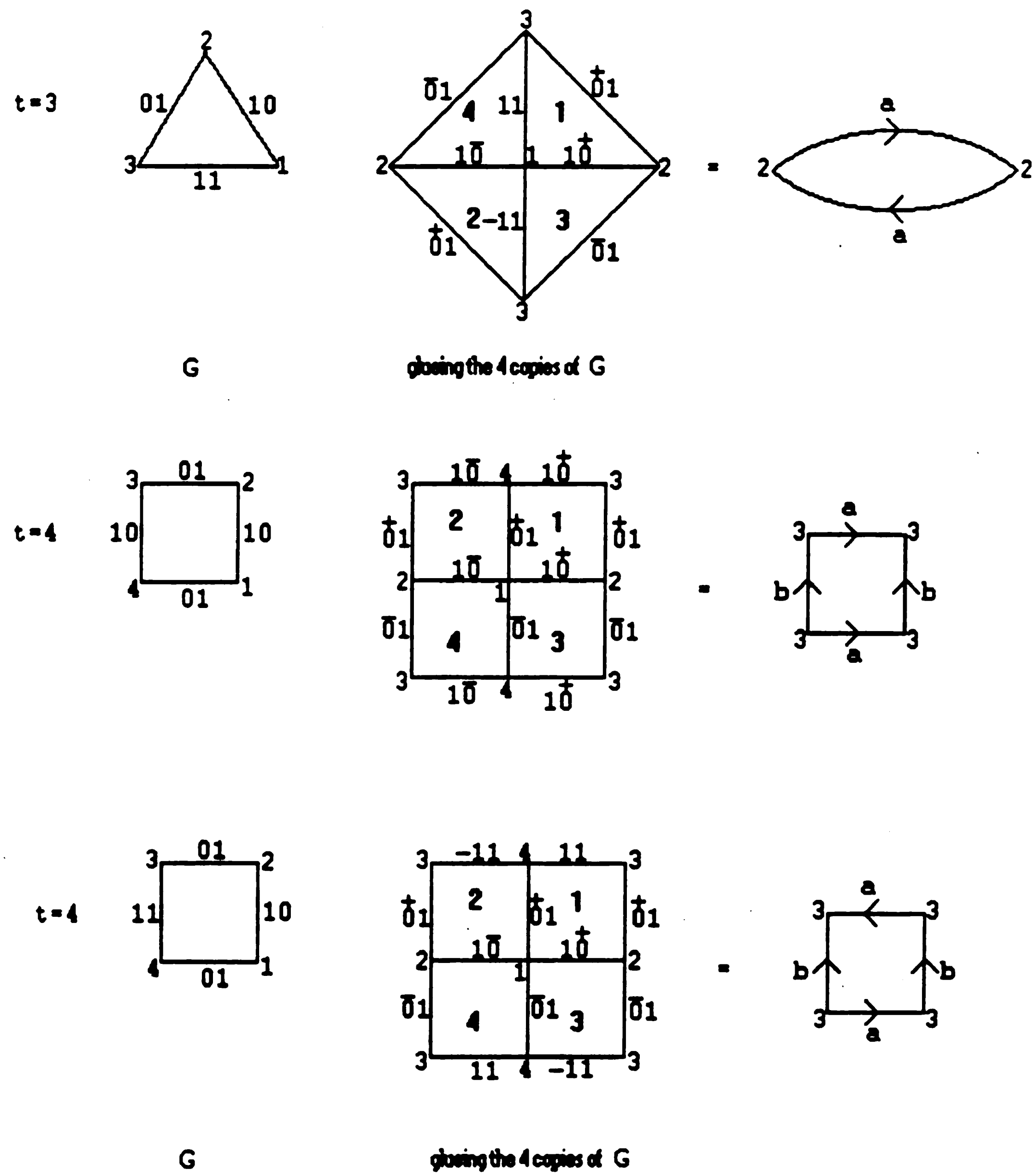

Figure 4.12. Obtaining M_G from G when $t = 3, 4$

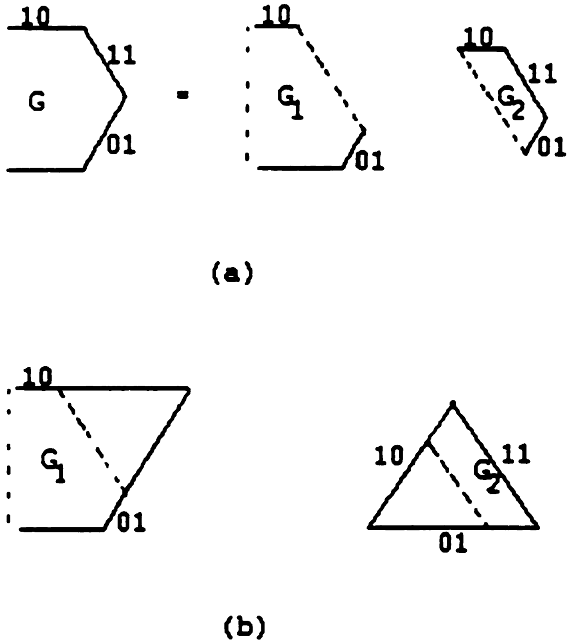

Figure 4.13. Reducing M to $M_1 \sharp \mathbf{R}P^2$

rotation in two planes by $T(a,b,c), T(a',b',c')$. The image of the orbit in M^* is a boundary point.

3. If the isotropy group of x is circle then the orbit is a 2-torus, the slice is a 4-disc. The action of the isotropy group on the slice is a rotation and the image of the orbit in M^* is a boundary point.

4. If the isotropy group of x is e then the orbit is T^3, the slice is a 3-disc. The image of the orbit in M^* is an interior point.

Theorem 4.6 *(lemma 4.5 [7])*

If T^3 acts smoothly and effectively on a compact connected simply connected 6 manifold M and the only stability groups are torus subgroups of T^3, then the orbit space is simply connected 3 manifold, with the points on the boundary of M^* are orbits of isotropy type T^1, T^2 or T^3 and interior points are principal orbits. The weighted orbit space M^* can be described by a graph G on the boundary of M^*, the vertices will correspond to the fixed points, the points on the edges will be orbits with T^2 stability groups and the points in the cells correspond to orbits with T^1 stability groups.

Theorem 4.7 *If the manifold is closed, then the principal orbits are only in the interior of M^*. If in addition, ∂M^* is connected, then $M^* \simeq D^3$.*

Notation : If M is closed and $M^* \simeq D^3$, we let $G_{\mathbf{C}}$ denote the orbit space as well as the graph and $M_{G_{\mathbf{C}}}$ denote the manifold.

Proposition 4.7 *The orthogonal action of T^3 on D^6 given by:*

$$\begin{aligned} T^3 \times D^6 &\longrightarrow D^6 \\ (e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, r_3 e^{i\theta_3}) &\longmapsto (r_1 e^{i(\theta_1 + a_1\varphi_1 + b_1\varphi_2 + c_1\varphi_3)}, \\ &\qquad r_2 e^{i(\theta_2 + a_2\varphi_1 + b_2\varphi_2 + c_2\varphi_3)}, \\ &\qquad r_3 e^{i(\theta_3 + a_3\varphi_1 + b_3\varphi_2 + c_3\varphi_3)}) \end{aligned}$$

is a smooth action, it is effective iff $\det|(a_i),(b_i),(c_i)| = \pm 1$. *The orbit space is given in Fig. 4.14 where* $G_j = \{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) | a_j\varphi_1 + b_j\varphi_2 + c_j\varphi_3 \equiv 0(2\pi)\}$, *and* $T_i = G_j \cap G_k$ *where* i, j, k *are all distinct.*

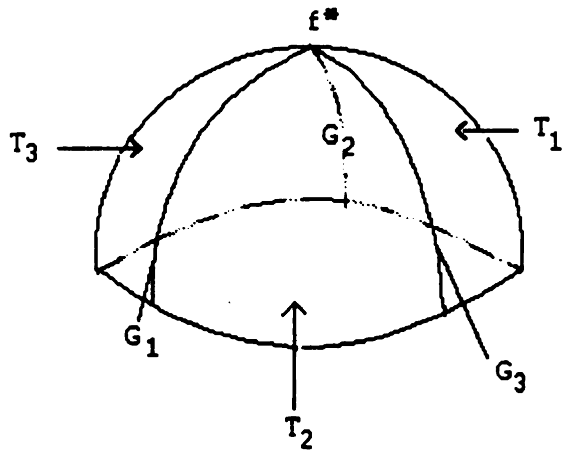

Figure 4.14. $(D^6)^*$

Proposition 4.8 *Let* T^3 *act on* M^6 *as in theorem 4.6, then by corollary VI.2.4 and definition of local smooth action in [9], the restriction of the action to a neighborhood of a fixed point is equivalent to the orthogonal action defined above.*

Corollary 4.1 *In any graph corresponding to such action , there are exactly three edges emanating from each vertex and exactly three cells that meet at each vertex.*

Now we translate these informations to the case of toric manifolds. Let Δ be a complete nonsingular 3 fan, and let $\sigma = \mathbf{R}_{\geq 0}(\sum_1^3 a_i n_i) + \mathbf{R}_{\geq 0}(\sum_1^3 b_i n_i) + \mathbf{R}_{\geq 0}(\sum_1^3 c_i n_i)$

be a maximal cone in Δ with $\det|a_i, b_i, c_i| = 1$ then

$$\begin{aligned}\sigma^\vee &= \mathbf{R}^{\geq 0}\left(\underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{a'_1} m_1 + \underbrace{\begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix}}_{a'_2} m_2 + \underbrace{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}_{a'_3} m_3\right) + \\ &\quad \mathbf{R}^{\geq 0}\left(\underbrace{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}}_{b'_1} m_1 + \underbrace{\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}}_{b'_2} m_2 + \underbrace{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}_{b'_3} m_3\right) + \\ &\quad \mathbf{R}^{\geq 0}\left(\underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{c'_1} m_1 + \underbrace{\begin{vmatrix} b_1 & a_1 \\ b_3 & a_3 \end{vmatrix}}_{c'_2} m_2 + \underbrace{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}_{c'_3} m_3\right)\end{aligned}$$

$U_\sigma^{\mathbf{C}} \simeq \mathbf{C}^3$, and the action of T^3 on $U_\sigma^{\mathbf{C}}$ is given by:

$$(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})(x, y, z) = (e^{i\sum_1^3 a'_j\varphi_j}x, e^{i\sum_1^3 b'_j\varphi_j}y, e^{i\sum_1^3 c'_j\varphi_j}z)$$

Hence, for each maximum cone corresponds a unique fixed point i.e a vertex in $G_{\mathbf{C}}$; a T^2 isotropy group would be of the form

$$\{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) | 0 \leq \varphi_i \leq 2\pi \text{ and } \sum_1^3 a'_i\varphi_i \equiv 0(2\pi)\}$$

and a T^1 isotropy group would be of the form

$$K = \{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) | 0 \leq \varphi_i \leq 2\pi \text{ and } \sum_1^3 a'_i\varphi_i \equiv 0(2\pi), \sum_1^3 b'_i\varphi_i \equiv 0(2\pi)\}$$

so K is determined by the vectors $u = (a'_1, a'_2, a'_3)$ and $v = (b'_1, b'_2, b'_3)$ which are orthogonal to (c_1, c_2, c_3).

Proposition 4.9 *K is completely determined by (c_1, c_2, c_3) and will be denoted $T(c_1, c_2, c_3)$.*

Proof : Let $X = (x_1, x_2, x_3)$ be an integral vector orthogonal to (c_1, c_2, c_3), then $X = \alpha u + \beta v$ and since $\det |(a'_i), (b'_i), (c'_i)| = 1$ then α and β are integers, let $Y = (y_1, y_2, y_3)$ be another integral vector orthogonal to (c_1, c_2, c_3) and such that the 2-minors of the matrix $[(X), (Y)]$ are relatively prime (hence there exists an integral vector Z such that $\det |(X), (Y), (Z)| = 1$), then because of the determinant conditions, and the fact that u, v, X, Y are in the same plane, it is easy to see that

$$K = \{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) | 0 \leq \varphi_i \leq 2\pi \text{ and } \sum_1^3 x_i\varphi_i \equiv 0(2\pi), \sum_1^3 y_i\varphi_i \equiv 0(2\pi)\}$$

□

Therefore to each 1-dimensional cone of Δ is associated a T^1 isotropy group, to each 2-dimensional cone is associated a T^2 isotropy group as follows: if $\tau = \mathbf{R}_{\geq 0}(a_{11}n_1 + a_{12}n_2 + a_{13}n_3) + \mathbf{R}_{\geq 0}(a_{21}n_1 + a_{22}n_2 + a_{23}n_3)$ then there exists an integral vector (a_{31}, a_{32}, a_{33}) such that $\det(a_{ij}) = 1$, so let the isotropy group $G = \{(t_1, t_2, t_3) \in T^3 \mid t_1^{a_{31}} t_2^{a_{32}} t_3^{a_{33}} = 1\}$. G is easily proved to be uniquely determined by (a_{11}, a_{12}, a_{13}) and (a_{21}, a_{22}, a_{23}), and to each 3- dimensional cone is associated a fixed point, hence we have a duality between Δ and $G_{\mathbf{C}}$ as follows:

The 1-dimensional cones of Δ are half lines emanating from 0, each half line is supported by its generating vector, the 2 dimensional cones are membranes that are bounded by two 1 dimensional cones so that when we intersect Δ with S^2 we get a triangulation of S^2 whose edges are equal to $S^2 \cap$ (2dim cones) and the vertices are equal to $S^2 \cap$ (1dim cones) . So we can represent Δ as a weighted triangulation of S^2 where the weights are adjoined to the vertices, the weights are the coordinates of the respective generators of the 1 dimensional cones. To represent Δ on the plane we project S^2 stereographically from a vertex (usually the considered vertex is adjacent to a maximum number of vertices).

Example : let

$$\begin{aligned}\Delta \;=\{\;& \mathbf{R}_{\geq 0}n_1+\mathbf{R}_{\geq 0}n_2+\mathbf{R}_{\geq 0}n_3;\; \mathbf{R}_{\geq 0}n_1+\mathbf{R}_{\geq 0}n_2+\mathbf{R}_{\geq 0}(-n_1-n_2-n_3);\\ & \mathbf{R}_{\geq 0}n_1+\mathbf{R}_{\geq 0}n_3+\mathbf{R}_{\geq 0}(-n_1-n_2-n_3);\; \mathbf{R}_{\geq 0}n_2+\mathbf{R}_{\geq 0}n_3+\mathbf{R}_{\geq 0}(-n_1-n_2-n_3);\\ & \text{the faces of these cones}\}\end{aligned}$$

Now $G_{\mathbf{C}}$ is obtained from Δ as the dual graph on S^2 and the weights of $G_{\mathbf{C}}$ are determined from the weights of Δ as shown in Fig. 4.15.

Remark : By construction of $G_{\mathbf{C}}$ from Δ we see that $G_{\mathbf{C}}$ is connected and hence

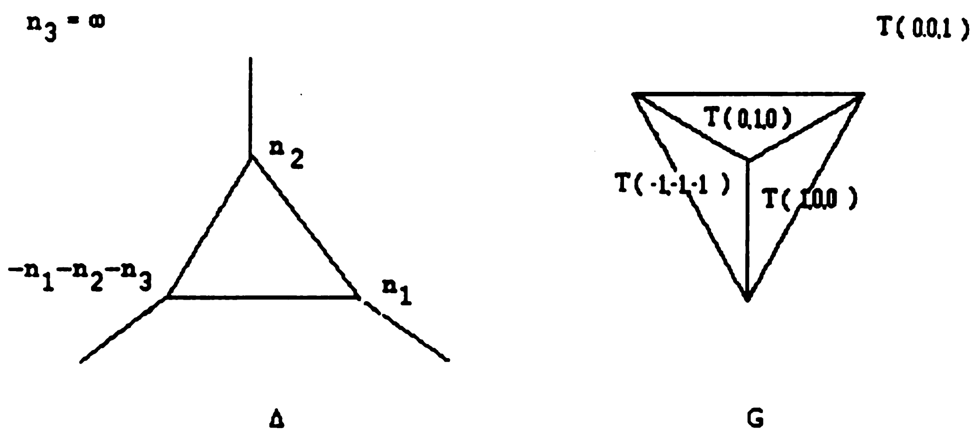

Figure 4.15. Example

$(X_\Delta^{\mathbf{C}})^* \simeq D^3.$

In the real case the $\mathbf{Z}_2$ isotropy groups are of the form

$$\begin{aligned} K \cap (\mathbf{Z}_2)^3 &= \{((-1)^{k_1}, (-1)^{k_2}, (-1)^{k_3}) | \sum a_i' k_i \equiv 0(2), \sum b_i' k_i \equiv 0(2)\} \\ &= \{((-1)^{k_1}, (-1)^{k_2}, (-1)^{k_3}) | \sum \overline{a_i' k_i} = 0, \sum \overline{b_i' k_i} = 0\} \text{ where } \overline{a} = \text{class of } a \bmod 2 \end{aligned}$$

but since we have $\sum a_i' c_i = 0$ and $\sum b_i' c_i = 0$ and $\det |(a_i'), (b_i'), (c_i')| = 1$ then $\overline{k_i} = \overline{c_i}$. Hence $T(c_1, c_2, c_3) \cap (\mathbf{Z}_2)^3 = \{(1,1,1); ((-1)^{\overline{c_1}}, (-1)^{\overline{c_2}}, (-1)^{\overline{c_3}})\}$ and it will be denoted by $\overline{c}_1\overline{c}_2\overline{c}_3$.

So in the example above we get G (we denote by G the graph in the real case) as in Fig. 4.16.

Remark : The isotropy groups around a fixed point verify the determinant condition

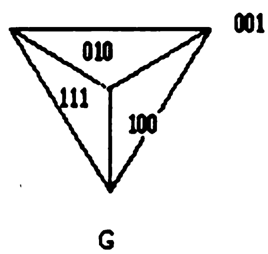

Figure 4.16. G

hence for example we can not have $\{100, 010, 110\}$ as colors around a fixed point .

4.6 Cross Sections

Lemma 4.2 *If* $(\mathbf{Z}_2)^3$ *acts on a 3-manifold* M *such that* $M^* \simeq D^3$ *with* D^3 *colored as shown in Fig. 4.17 where the interior points and points on* S^- *correspond to principal orbits and* $\det|(\overline{a_i}), (\overline{b_i}), (\overline{c_i})| = \pm 1$, *then the action has a cross section and if a cross section is given on a disc* $D \subset S^-$ *then it can be extended to all of* D^3.

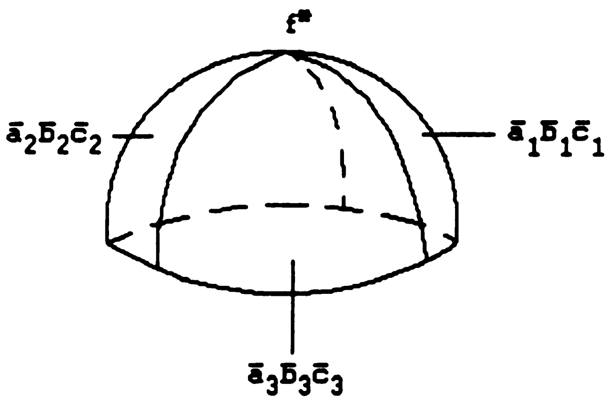

Figure 4.17. $(D^3)^*$

Proof : M is obtained by glueing eight copies of D^3 two by two along the weighted cells and a cross section is just a choice of one copy among the eight. □

Theorem 4.8 *If* $(\mathbf{Z}_2)^3$ *acts on a closed 3-manifold* M *such that* $M^* \simeq D^3$ *with all interior points of* D^3 *correspond to principal orbits, and we have a graph on the boundary as described in the previous section, then the action has a cross section.*

Proof : We just cut D^3 into cones, with each cone containing one fixed point in its base as shown in Fig. 4.18. Then by the same argument as in dimension 2 we have a cross section. □

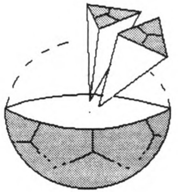

Figure 4.18. Cutting G into cones

Theorem 4.9 *If* $(\mathbf{Z}_2)^3$ *acts on a compact 3 manifold with boundary such that* $M^* \simeq D^3$. *If* $D^3 \setminus S^+$ *consists of principal orbits and* S^+ *has a graph on it, then the action has a cross section.*

Proof : The proof is similar to the boundaryless case. □

Theorem 4.10 *If* $(\mathbf{Z}_2)^3$ *acts on two 3 manifolds* M, N *such that* M^* *and* N^* *satisfy the conditions of one of the two theorems above and if there is a weight preserving homeomorphism between* M^* *and* N^* *then* M *and* N *are equivariantly homeomorphic.*

Proof : The proof is similar to the 2 dimensional case. □

Note : By the above theorem we assume that there is a fixed point where G looks as in Fig. 4.19 which means for Δ that there exists a maximal cone σ in Δ whose generating vectors form the canonical basis for $\mathbf{R}^3$.

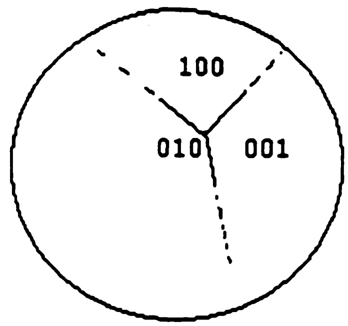

Figure 4.19. G

4.7 Orientation

Theorem 4.11 *Let G be a graph on S^2 as before. Then M_G is orientable iff 100,010,001 and 111 are the only colors in G.*

Corollary 4.2 *A 3 dimensional compact toric manifold X_Δ is orientable iff every generator vector of Δ has the parity 100,010,001 or 111.*

Proof of the theorem : Let G be given, then we have a part H of G that has the representation shown in Fig. 4.20 where T is a subgroup of order 2 of $(\mathbf{Z}_2)^3$ and by the determinant condition we know that T corresponds to one of the following four colors: 100, 110, 101, 111. But we can represent H as in Fig. 4.21. By theorem 4.10, both X_1 and X_2 yield D^3 with the obvious action in the case of X_2, and in the case of X_1 the action is given by:

$$\begin{aligned} (\mathbf{Z}_2)^3 \times D^1 \times D^2 &\longrightarrow D^1 \times D^2 \\ (t_1, t_2, t_3)(x, y, z) &\longmapsto (t_1 x, t_1^a t_2 y, t_1^b t_3 z) \end{aligned}$$

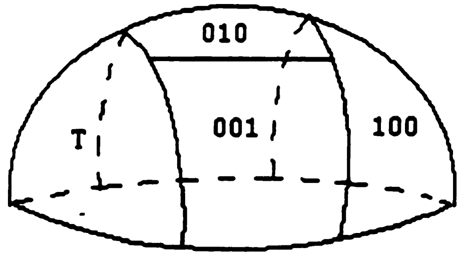

Figure 4.20. *H*

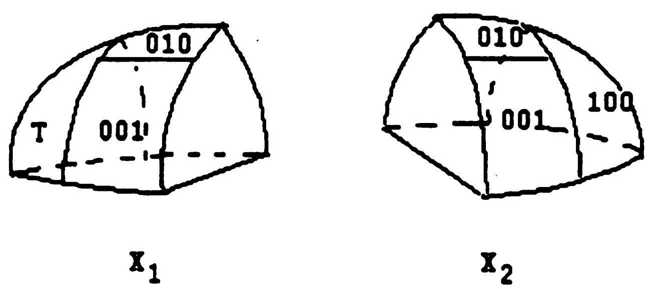

Figure 4.21. Cutting *H*

where a and b are 1 or 0 depending on T. Also by the cross section theorem we have that $X_1 \cap X_2$ corresponds to $S^0 \times D^2$ with the induced action from either one (the two actions are similar on $S^0 \times D^2$), hence M_H is the union of two copies of $D^1 \times D^2$ glued together along $S^0 \times D^2$ i.e M_H is a D^2 bundle over S^1. Such bundles are classified by $\pi_0(O_2) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$. Let

$$\begin{array}{rcl} f: \ (S^0 \times D^1 \times D^1) \subset M_{X_2} & \longrightarrow & (S^0 \times D^1 \times D^1) \subset M_{X_1} \\ (x, y, z) & \longmapsto & (x, x^a y, x^b z) \end{array}$$

f is obviously an equivariant homeomorphism, and the action of $(\mathbf{Z}_2)^3$ on $M_{X_1} \cup_f M_{X_2}$ has H as orbit space therefore $M_H = M_{X_1} \cup_f M_{X_2}$ by theorem 4.10.

Therefore, M_H is orientable $\Leftrightarrow$ the bundle is trivial $\Leftrightarrow$

$$\begin{array}{rcl} \overline{f_x} = f(x, -, -): \quad \mathbf{R}^2 & \longrightarrow & \mathbf{R}^2 \\ (y, z) & \longmapsto & (x^a y, x^b z) \end{array}$$

correspond to the identity element in $\pi_0(O_2) \Leftrightarrow a = b = 0$ or $a = b = 1 \Leftrightarrow T = 100$ or $T = 111$.

Now assume that M is orientable and since any edge of G touches exactly four cells (see Fig. 4.22). Then by the above discussion T_4 is either T_1 or $T_1T_2T_3$ (this notation means that the nontrivial element of T_4 is the product of the nontrivial elements of $T_1T_2T_3$) and since the product of any three of the colors 100, 010 , 001 , 111 is the fourth we have that G is colored by the 4 colors only.

Conversely let G be colored by the 4 colors only. We have eight copies of G, each one is the image of G under the reflection by an element of $(\mathbf{Z}_2)^3$. We glue these eight copies along the 3 faces surrounding a vertex (call it the **central vertex**) so that we get a weighted 3 ball P_G, the glueing is made such that a 100 face (010, 001, 111

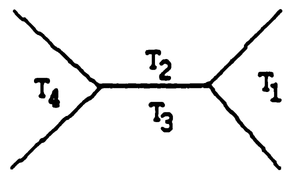

Figure 4.22. An edge

resp) of αG is identified with the same face of βG iff $(-1,1,1)\alpha = \beta\,((1,-1,1)\alpha = \beta, (1,1,-1)\alpha = \beta, (-1,-1,-1)\alpha = \beta$ resp) then we choose orientation for G (i.e for the interior of G and the faces) and reflect these orientations to the other copies so that we obtain a coherent orientation on P_G. To prove that M_G is orientable, it is enough to prove that any two identified faces have opposite orientations but as we remarked above, a face of αG is identified with the same face of βG iff β is the product of α by (-1,1,1) ; (1,-1,1) . (1,1-1) or (-1,-1,-1) , and all of them are odd reflections i.e they reverse the orientations. □

Notation : We denote the eight copies of G by:

1 = (1,1,1)G, 2 =(-1,1,1)G , 3=(1,-1,1)G , 4=(1,1-1)G , 5=(-1,- 1,1)G, 6=(-1,1,-1)G, 7=(1,-1,-1)G and 8=(-1,-1,-1)G.

Remark : We notice that in the glueing process of the eight copies of G, we obtain 4 distinct copies of each cell which are attached along their edges, in the same way as in the 2 dimensional case. Let C be a cell of G with k edges and let n be the generator of its dual cone in Δ and $n_1, \ldots, n_k$ are the generators of the respective dual cones of the adjacent cells to C with $\det(n_i, n_{i+1}, n) = 1$. Since a $\mathbf{Z}$ change of basis will change the weights of G bijectively we assume that n is on the z axis. If $n_i = (a_i, b_i, c_i)$ then $a_i b_{i+1} - b_i a_{i+1} = 1$. And as we have seen in dimension 2, if $k \geq 5$ we have at least 3 different colors on the adjacent cells of C. Therefore in the case of an orientable toric manifold, a cell with more than four edges has exactly three colors surrounding it.

4.8 Heegaard Diagrams for Orientable Toric 3-Manifolds

Let Δ be a fan corresponding to an orientable compact toric manifold, and G its dual graph. G is colored by only 4 colors 100, 010, 001 and 111; let n denote the number of cells of G, it is equal to the number of generating vectors of Δ, hence the number of cells of P_G is equal to $8(n-3)$ and M_G ($= X_\Delta$) is obtained from P_G by pairwise identification of its cells. If we consider the decomposition of M_G into the eight copies of G and bore out the vertices of the decomposition by the procedure of boring out a small ball surrounding each vertex except the central vertex, likewise we bore out the edges of the decomposition (except the edges which have the central vertex as an end point) by boring out small full cylinders about them where the cylinders connect the balls surrounding the end points of the edges, we obtain a handlebody $\mathcal{H}_1$. The subspace which is remaining after one has bored out $\mathcal{H}_1$ is a handlebody $\mathcal{H}_2$, obviously it is the one obtained from P_G by identifying small discs inside the cells (one disc for each cell). $\mathcal{H}_1$ and $\mathcal{H}_2$ have genus $h = 4(n-3)$. M_G is obtained by glueing $\mathcal{H}_1$ and $\mathcal{H}_2$ along their common boundaries, the glueing is defined by a homeomorphism between $\partial\mathcal{H}_1$ and $\partial\mathcal{H}_2$, and since $\mathcal{H}_1$ and $\mathcal{H}_2$ are handlebodies, such a homeomorphism is determined by the images of the boundary of the pairwise identified discs in $\partial\mathcal{H}_1$.

But we have seen, in proposition 4.8, that locally (in the neighbohood of a vertex), the cells look like coordinate planes, the edges like the coordinate axes and the vertex like the origin; also the eight copies of a cell C are identified two by two, and the four unidentified copies form a plane whose intersection with $\partial\mathcal{H}_1$ is just the image of the boundary of the disc in C under the identification homeomorphism, hence we don't need to represent $\partial\mathcal{H}_1$ as a full 3 dimensional handlebody, it is enough to draw the generating circles of $\mathcal{H}_1$ with the vertices, and, to mark the planes corresponding to the cells, we just draw their intersection with the balls around the vertices and mark the cooresponding weights on them.

Remarks :

1. The skeleton or the generating circles of $\mathcal{H}_1$ can be obtained from the graph G by just deleting the central vertex and all the edges that have it as an end point. We call the circles represented by edges: **edge circles**, and the circles that arise from the cell boundaries as **cellular circles**.

2. A vertex in G which is the endpoint of an edge that has the central vertex as the other endpoint is the center of only one plane in the Heegaard representation (H.D for short), this plane corresponds to the cell which does not have the central vertex on its boundary.

3. A vertex which is in the boundary of the same cell as the central vertex but not the same edge is the center of two perpendicular planes in the H.D.

4. A vertex that does not share a cell with the central vertex is the center of three planes.

5. An edge which is in the boundary of a cell C carries two of the four unidentified copies of C on one half of the circle it represents, and the two other copies on the other half.

6. If an edge is in the boundary of a cell that contains the central vertex then the edge carries only one cell in the H.D.

7. If an edge is in the boundary of two cells that do not contain the central vertex then the edge carries two cells in the H.D.

Example 1 : Let Δ and G be as in Fig. 4.15, then P_G, its skeleton and $\mathcal{H}_1$ are as shown in Fig. 4.23.

Example : **2** Let Δ and G be as shown in Fig. 4.24(a), then P_G is as shown in Fig. 4.24(b) and $\mathcal{H}_1$ are as shown in Fig. 4.25.

4.9 Surgery

Suppose that G is as given in Fig. 4.26(a), where the T_i's are the 4 different colors, then as we have seen in the proof of theorem 4.11, the indicated region is $D^2 \times S^1$. We may do equivariant surgery on the circle by replacing the solid torus X_1 around it with another solid torus X_2, to obtain a new manifold with the orbit space shown in Fig. 4.26(b), (we can do that, because the boundary of the two exchanged parts is the same). The corresponding picture of this surgery in the H.D is shown in Fig. 4.27. A meridian curve of X_2 is for example the one that is running over the boundary of the D-square ($1D \rightarrow 3D \rightarrow 4D \rightarrow 2D$). Its image in X_1 is running twice along the longitude and once around the meridian, hence the surgery coefficient is $\frac{1}{2}$, i.e the surgery operations we are doing are $\frac{1}{2}$ surgeries. Also by construction, the surgery is performed only on the edge circles of $\mathcal{H}_1$ which have a half twist in their normal bundles; the twist is indicated in $\mathcal{H}_1$ by the planes carried by the edge circle.

Theorem 4.12 *If M_G is an orientable closed toric manifold, then it is obtained from* $\mathbf{R}P^3$ *by a series of connected sums of* $\mathbf{R}P^3$ *and the above surgery.*

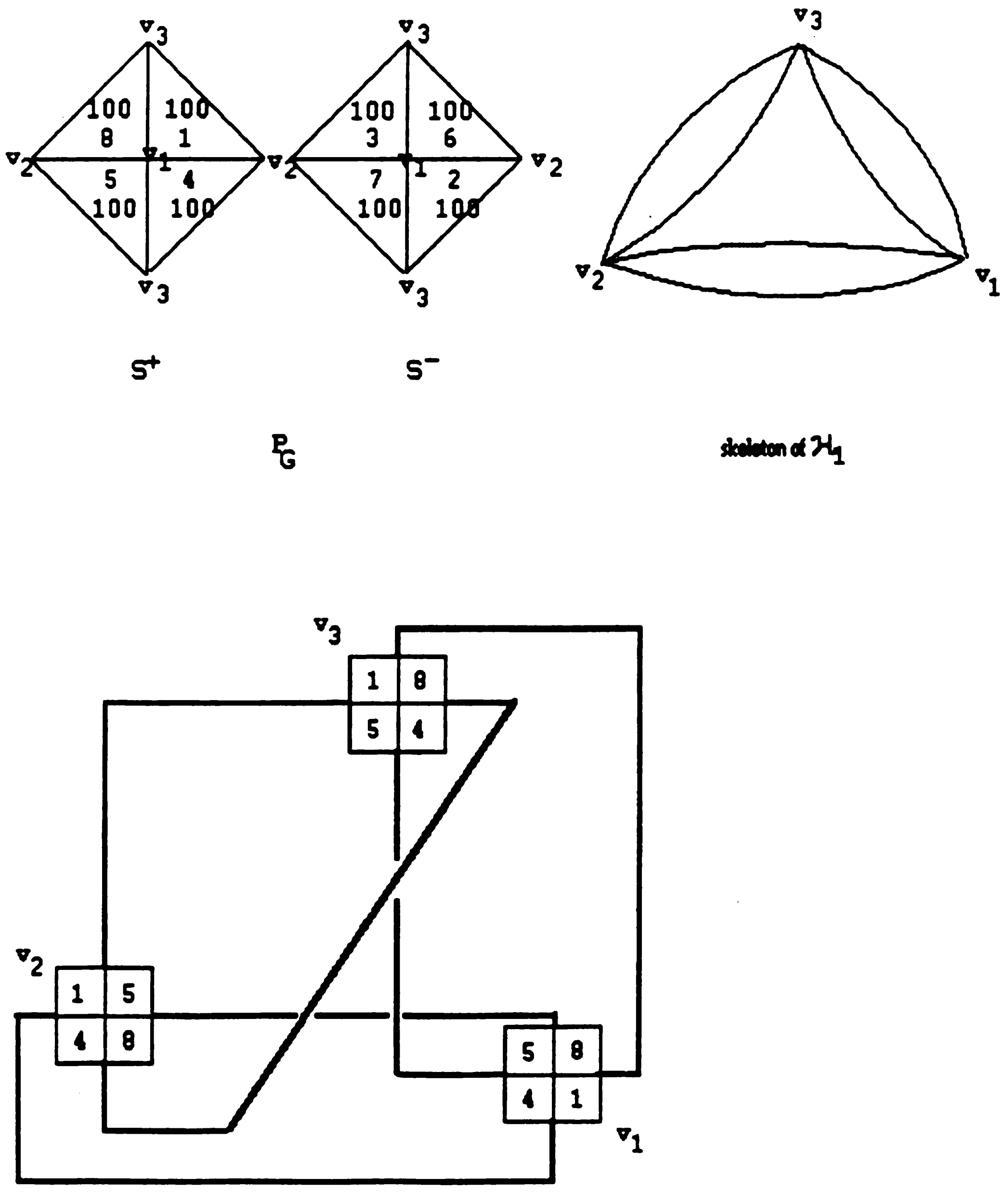

Figure 4.23. $\mathbf{R}P^3$

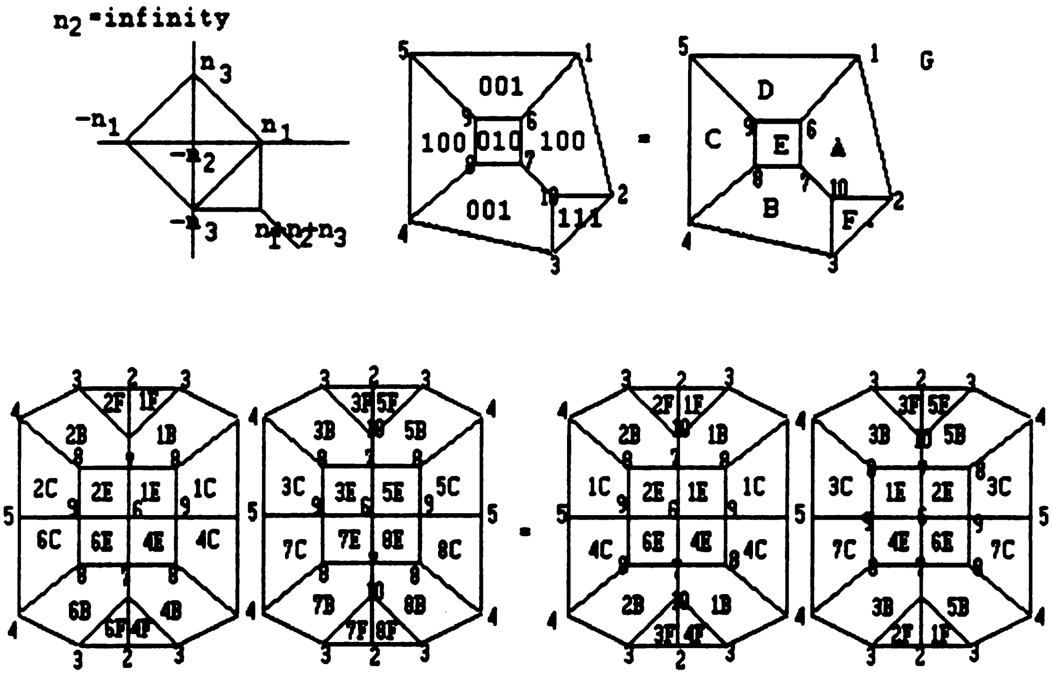

Figure 4.24. $\Delta, GandP_G$

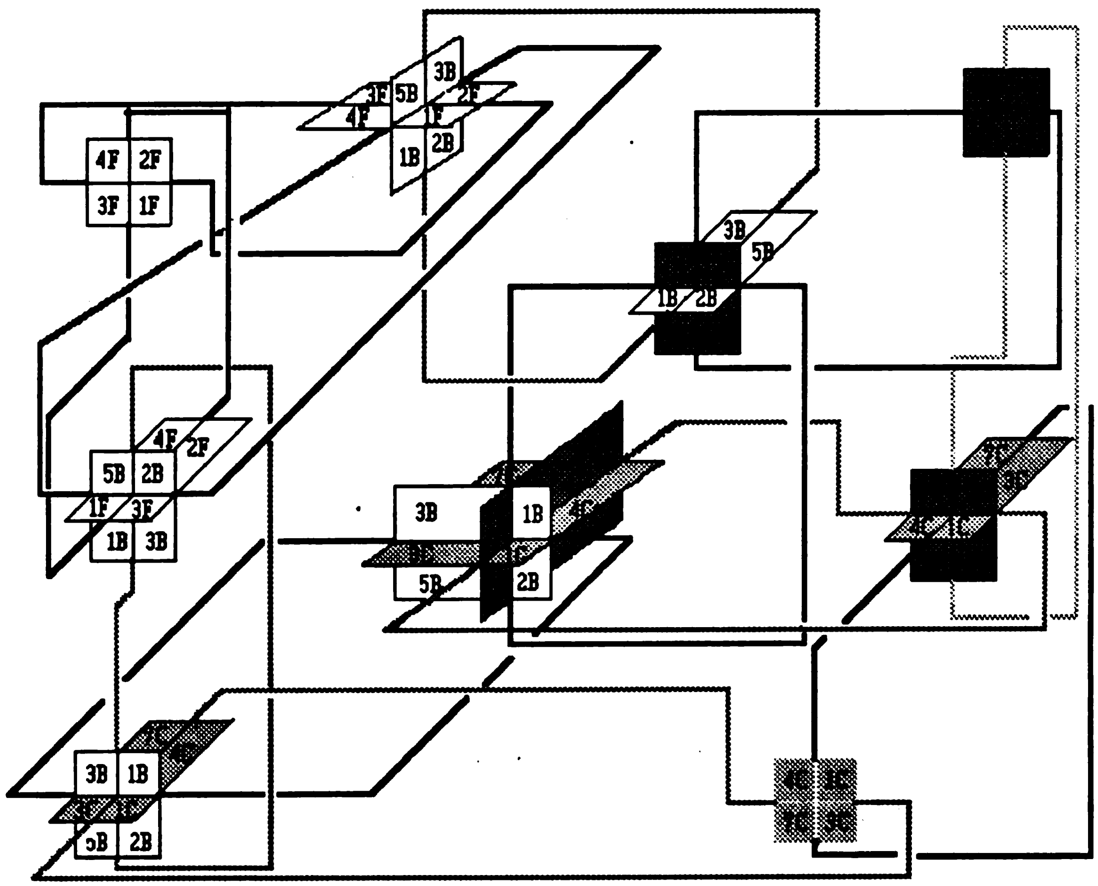

Figure 4.25. $\mathcal{H}_1$

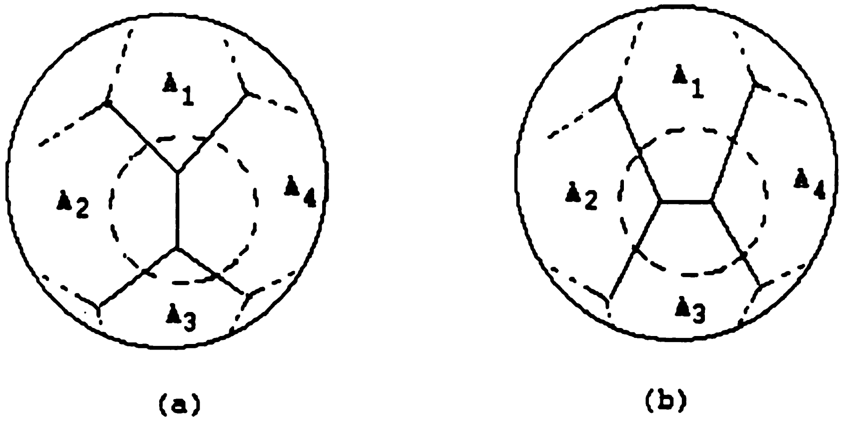

Figure 4.26. Surgery

Proof : we assume that G is different from the tetrahedron, which corresponds to $\mathbf{R}P^3$. Since there are four colors, then G is as in Fig. 4.28(a). Then we do the surgery on an edge of the cell C_1 in the same manner as for the cutout of edges in dimension 2 so that the new cell C_1' still has 3 different adjacent colors. We continue doing the surgeries until we end up with the modified cell C_1 having exactly 3 edges as shown in Fig. 4.28(b). But this corresponds to a connected sum with $\mathbf{R}P^3$. We remove the $\mathbf{R}P^3$ only if we still have four colors on the graph, otherwise we have to change to another cell and do the same work on it. After we remove the $\mathbf{R}P^3$, the graph will be as shown in Fig. 4.28(c). After doing this, the graph G has one less cell and is still colored by four colors. We keep repeating the same process until we obtain a graph with only four cells which is the tetrahedron and the associated manifold is $\mathbf{R}P^3$. □

Remarks :

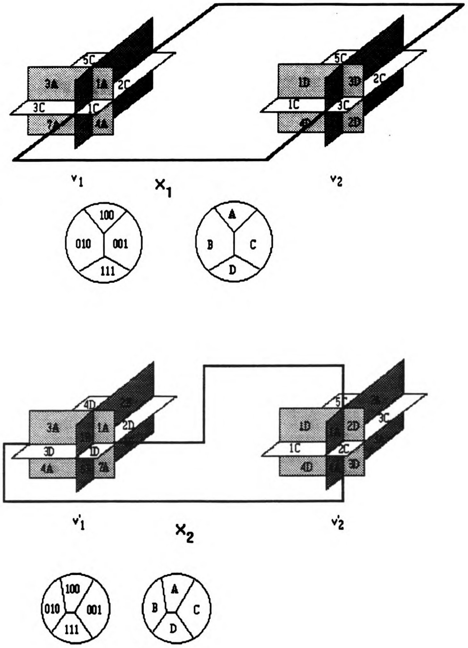

Figure 4.27. $\mathcal{H}_1$ for the surgery

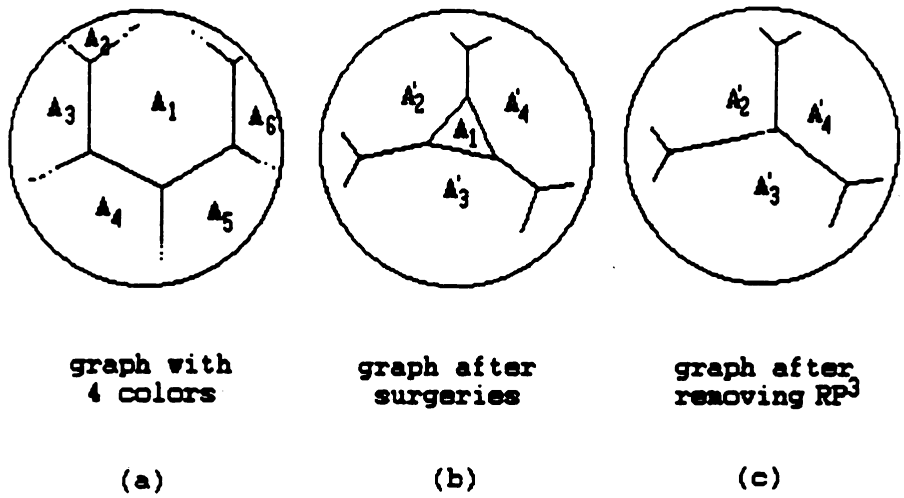

Figure 4.28. Reducing G

1. In case G has only three colors, the only possibility for G to correspond to a fan is that G is the cube and hence M_G is T^3. In that case we connect sum T^3 with $\mathbf{R}P^3$ obtaining a new graph with a cell having five sides and we proceed with the moves above.

2. The graphs obtained by the different moves in the process to reduce a given graph to the tetrahedron may not be dual to fans. It is not known if every triangulation of the sphere is supported by a nonsingular fan.

Proposition 4.10 *Let Δ be a nonsingular fan, let $\tau = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_s$ be in Δ.*

We remark that if $\tau < \sigma$ then there exists $\sigma' < \sigma$ such that $\sigma = \tau + \sigma'$ with $\sigma' \cap \tau = \{0\}$. Define $n_0 = n_1 + \cdots + n_s$ and $\tau_i = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_{i-1} + \mathbf{R}_{\geq 0}n_0 + \mathbf{R}_{\geq 0}n_{i+1} + \cdots + \mathbf{R}_{\geq 0}n_s$ for $1 \leq i \leq s$. We then let $\sigma_i = \tau_i + \sigma'$ and $\Delta_\tau = (\Delta \setminus \{\sigma \in \Delta | \tau < \sigma\}) \cup \{faces\ of \sigma_i | \sigma \in \Delta, \tau < \sigma, 1 \leq i \leq s\}$.

Then X_{Δ_τ} is obtained by blowing up the closed submanifold $\overline{orb\tau}$.

Proof : See Prop. 1.26 in [9]. □

Remarks :

1. The corresponding move in dimension 3 for the fan and its dual graph are as shown in Fig. 4.29.

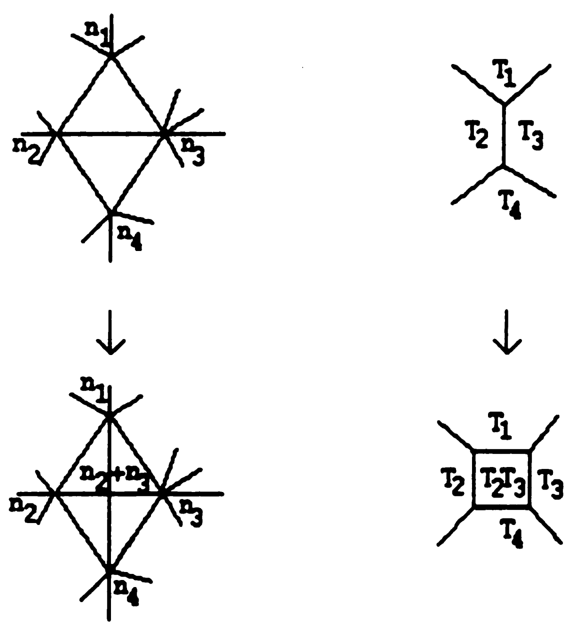

Figure 4.29. Blowing up

2. The 1/2 surgery move is a blowing up followed by a blowing down the appropriate circle. We can do such consecutive moves only if $T_1T_4 = T_2T_3$ i.e $T_4 = T_1T_2T_3$ and to obtaine a toric manifold we need to have $n_1 + n_4 = n_2 + n_3$ so we get a nonsingular fan.

3. In [4] Danilov showed that any toric manifold is obtained from $\mathbf{R}P^3$ by a sequence of blowing up and down along points and circles as with every step of the sequence corresponding to a toric manifold.

4.10 First Homology Groups for Orientable Toric Manifolds

A representation of the first homology group of an orientable compact toric 3-manifold is obtained from its HD as follows: $\mathcal{H}_1$ is a $4(n-3)$ handlebody, so we have $4(n-3)$ generators corresponding to the generating circle of $\mathcal{H}_1$ and $4(n-3)$ relations corresponding to the boundaries of the cellular disks (n = number of generating vectors of the fan = number of cells of G).
Every cell in G except the ones carrying the central vertex yield 4 relations, one for each of the unidentified four copies. The relations are obtained by running over the boundaries of the cellular disks. The generators that appear in these relations correspond to the edges and the cellular circle of the cell.
We orient the generating circle of $\mathcal{H}_1$ as follows: in the skeleton of $\mathcal{H}_1$, we orient the edge circle so that the cellular circles are given a coherent orientation i.e the arrows in the cellular circles have the same direction, so that an edge circle has an opposite orientation to any cellular circle it contributes to.
Every generator corresponding to an edge appears exactly in two or four relations, depending on the number of planes it carries, these generators appears with coefficient 1, but each generator corresponding to a cellular circle appear in the four relations obtained from its cell, and its coefficient is -1. Therefore if we write the coefficients of the different generators in a matrix where the columns correspond to the relations, and the rows correspond to the generators, we will have a square $4(n-3) \times 4(n-3)$ matrix

A with entries being 0 or ± 1, with every row containing two or four 1's and the other coefficients in that row are 0, or it contains four -1's and the other coefficients are 0. This matrix is equivalent to a diagonal integral matrix D ($A \simeq D \Leftrightarrow \exists P, Q$ invertible matrices over $\mathbf{Z}$ such that $D = PAQ$) , where the diagonal is $\{d_1, \ldots, d_r, 0, \ldots, 0\}$ with $d_i \neq 0 \; \forall i$ and $d_i | d_j$ if $i \leq j$. So the rank of H_1 is $4(n-3) - r$ and its torsion coefficients are the d_i's.

We have A is equivalent to a matrix D whose entries in the first column are either 2's or 4's (by adding all the columns of A and replacing it with the first column), hence $2|\det D$ i.e if rank$H_1 = 0$ then H_1 has an element of torsion 2.

Corollary 4.3 S^3 *and L(p,q) where p is odd are not toric manifolds.*

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BIBLIOGRAPHY

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