This is to certify that the

dissertation entitled

Classification of the smooth closed manifolds up to blowup

presented by

Grigory MIKHALKIN

has been accepted towards fulfillment
of the requirements for

PhD degree in Mathematics

Major professor

Date 23/Feb/93

CLASSIFICATION OF THE SMOOTH CLOSED MANIFOLDS UP TO BLOWUP

By

Grigory MIKHALKIN

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

Advisor: Professor S.Akbulut

1993

ABSTRACT

CLASSIFICATION OF THE SMOOTH CLOSED MANIFOLDS UP TO BLOWUP

By

Grigory MIKHALKIN

The dissertation is devoted to the problem of classification of the smooth manifolds. The dissertation contains the classification of the smooth manifolds up to the blowups.

The boundary of the tubular neighbourhood of a submanifold in a manifold is a sphere bundle over the submanifold. The tubular neighbourhood of a submanifold in a manifold is a mapping cylinder of the projection of the sphere bundle given by the boundary of the tubular neighbourhood of the submanifold. The blowup of a manifold along its submanifold is the replacing of the tubular neighbourhood of the submanifold by the mapping cylinder of the projectivisation of this sphere bundle. In the other words we replace the submanifold by the space of linear directions in the tangent space of the manifold normal to the submanifold. This operation is well-known in algebraic geometry.

The main theorem of the dissertaion is that any two smooth closed connected manifolds of the same dimension are equivalent up to blowups.

ACKNOWLEDGEMENTS

The author would like to thank S.Akbulut, V.Kharlamov, A.Marin and O.Viro for their attention to the work and for the pointing out of the errors in previous versions.

The author would like to thank S.Akbulut and A.Marin for the useful comments and discussions that led to the generalisation of the proof in 4-dimensional case to the case of an arbitrary dimension.

Contents

List of Figures

INTRODUCTION

J.Nash in [8] proved that every smooth closed manifold is diffeomorphic to a component of a real algebraic variety and completed his paper with a conjecture that every smooth closed connected manifold is diffeomorphic to a rational real algebraic nonsingular variety. If a rational real algebraic nonsingular surface is orientable then its genus has to be less then 2 (see e.g. [6]). So this "algebraic" Nash conjecture is not true in dimension 2. It was observed in [2] and [4] that there is a topological version of the Nash conjecture: *Every smooth closed connected manifold can be obtained from* S^n *by a sequence of blowings up and blowings down.* The topological Nash conjecture obviously holds for 2-manifolds since blowing up a surface has an affect of connecting summing with $\mathbb{R}P^2$. In [2] and [4] the topological Nash conjecture was proven for 3-manifolds.

The dissertation contains a proof of the topological Nash conjecture for any dimension.

1 The statement of the theorem

Let M be a smooth manifold and L be a proper smooth submanifold of M. By $\nu_M(L)$ we denote the normal bundle of L in M. Let $\tilde{L}$ be the projectivisation of $\nu_M(L)$. In other words L is the $\mathbb{R}P^{dimM-dimL-1}$-bundle over L associated to $\nu_M(L)$. Let $\nu_M(\tilde{L})$ be the tautological I-bundle over $\tilde{L}$ defined by the projectivisation $p : \tilde{L} \to L$. Then we have a natural diffeomorfism $\mu : \nu_M(\tilde{L}) - \tilde{L} \to \nu_M(L) - L$ which can be extended to a map $\pi : \nu_M(\tilde{L}) \to \nu_M(L)$ by the identity $\pi|_{\tilde{L}} = p$. Note that we may view $\nu_M(L)$ as a tubular neighbourhood of L in M.

Definition. Map $f : \tilde{M} \to M$ is called the *blowup* of a smooth manifold M along a smooth submanifold $L \subset M$ if $\tilde{M}$ is the result of gluing of $\nu_M(\tilde{L})$ to $M - L$ with μ, the projection f is defined by $f|_{M-L} = id$ and $f|_{\nu_M(\tilde{L})} = \pi$. Smooth manifold $\tilde{M}$ is called the *result of blowup* of M along L, we will also denote $\tilde{M}$ by $B(M, L)$. Submanifold $L \subset M$ is called the *center of blowup* f. Submanifold $\tilde{L} \subset \tilde{M}$ is called the *exceptional divisor* of f. We say that $\tilde{V} \subset \tilde{M}$ is the *proper transform* of a closed subset $V \subset M$, if $f(\tilde{V}) = V$, $\tilde{V}$ is closed in $\tilde{M}$, and $\tilde{V} - \tilde{L}$ is dense in $\tilde{V}$ [1].

Definition. The *multiblowup* is a sequence of blowups

$$\tilde{M} = M_k \to \ldots \to M_0 = M$$

Manifold $\tilde{M}$ is called the *result of multiblowup* of M.

Definition. Two manifolds M and N of the same dimension are called *blowup equivalent* (or *m-equivalent* [4], or *topologically birationally equivalent* [2]), if there exists a sequence $M = M_0, M_1, \ldots, M_n = N$ such that for any $j \in \{1, \ldots, n\}$ either M_j is the result of blowup of M_{j-1} along some submanifold of M_{j-1} or M_{j-1} is the result of blowup of M_j along some submanifold of M_j. The sequence $M =$

[1] If V is a submanifold transverse to L then we can define the proper transform much simpler, as the union of $V - L$ and the projectivisation of $\nu_V(L \cap V)$

$M_0, M_1, \ldots, M_n = N$ is called the *blowup sequence.*

Definition. We say that a *pair* of manifolds $(\tilde{M}, \tilde{V})$ is the *result of blowup* of a pair of manifolds (M, V) along $L \subset M$ if $\tilde{M}$ is the result of blowup of M along L and $\tilde{V}$ is the proper transform of V. Similarly, two *pairs* of manifolds (M, V) and (N, W) are called *blowup equivalent*, if there exists a sequence $(M, V) = (M_0, V_0), (M_1, V_1), \ldots, (M_n, V_n) = (N, W)$ such that for any $j \in \{1, \ldots, n\}$ either (M_j, V_j) is the result of blowup of (M_{j-1}, V_{j-1}) along some submanifold of M_{j-1} or (M_{j-1}, V_{j-1}) is the result of blowup of (M_j, V_j) along some submanifold of M_j.

Definition. A smooth manifold M of dimension n is called *topologically rational*, if M is blowup equivalent to the standard sphere S^n.

Theorem 1 *If M and N are closed smooth connected manifolds of the same dimension then M is blowup equivalent to N*

In other words *every closed smooth connected manifold is topologically rational.* Theorem 1 follows from Lemma 1 and Lemma 3.

2 Each cobordism class contains at least one topologically rational manifold

Lemma 1 (A.Marin) *Every manifold M is cobordant to a topologically rational manifold*

Proof. The cobordism group is generated by products of non-singular hypersurfaces $\mathbb{R}H_{p,q}$ of bidegree (1,1) in $\mathbb{R}P^p \times \mathbb{R}P^q, p \leq q$ (see Exercise 16.F of [7]). Lemma 2 implies that we need only to show that $\mathbb{R}H_{p,q}$ is topologically rational. It suffices to prove that $\mathbb{R}H_{p,q}$ is blowup equivalent to $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$.

The hypersurface $\mathbb{R}H_{p,q}$ is given by the equation

$$x_0y_0 + x_1y_1 + \ldots + x_py_p = 0$$

where $[x_0 : \ldots : x_p] \times [y_0 : \ldots : y_q]$ are standard bihomogeneous coordinates of $\mathbb{R}P^p \times \mathbb{R}P^q$. It is easy to see that $\mathbb{R}H_{p,q}$ is non-singular. Let $p : \mathbb{R}H_{p,q} - \mathbb{R}S \to \mathbb{R}P^{p-1} \times \mathbb{R}P^q$ be the map given by the equation

$$p([x_0 : \ldots : x_p] \times [y_0 : \ldots : y_q]) = [x_0 : \ldots : x_{p-1}] \times [y_0 : \ldots : y_q]$$

where $[x_0 : \ldots : x_{p-1}] \times [y_0 : \ldots : y_q]$ are standard bihomogeneous coordinates of $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$ and $\mathbb{R}S$ is the subvariety of $\mathbb{R}H_{p,q}$ given by the system of equations $x_0 = \ldots = x_{p-1} = 0$ ($\mathbb{R}S = \mathbb{R}H_{p,q} \cap [0 : \ldots : 0 : 1] \times \mathbb{R}P^q$). In $[0 : \ldots : 0 : 1] \times \mathbb{R}P^q$ manifold $\mathbb{R}S$ is given by equation $y_p = 0$, thus $\mathbb{R}S$ is a non-singular subvariety of $\mathbb{R}H_{p,q}$ diffeomorphic to $\mathbb{R}P^{q-1}$. We can view p geometrically. Consider $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$ as the submanifold of $\mathbb{R}P^p \times \mathbb{R}P^q$ given by the equation $x_p = 0$. Let $L_{(x,y)}$ be the line in $\mathbb{R}P^p \times \{y\}$ through (x, y) and $([0 : \ldots : 0 : 1], y)$. We define $p(x, y)$ as the intersection point of $L_{(x,y)}$ and $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$.

Let $f : \mathbb{R}F \to \mathbb{R}H_{p,q}$ be the blowup of $\mathbb{R}H_{p,q}$ along $\mathbb{R}S$ and $\mathbb{R}\tilde{S} \in \mathbb{R}F$ be the exceptional divisor of f. We want to find a smooth map $\tilde{p} : \mathbb{R}F \to \mathbb{R}P^{p-1} \times \mathbb{R}P^q$ such that $\tilde{p}|_{\mathbb{R}F - \mathbb{R}\tilde{S}} = p \circ f$. By definition, the points of $\mathbb{R}\tilde{S}$ are lines tangent to $\mathbb{R}H_{p,q}$ passing through $\mathbb{R}S$ and normal to $\mathbb{R}S$. Since $\mathbb{R}S = \mathbb{R}H_{p,q} \cap [0 : \ldots : 0 : 1] \times \mathbb{R}P^q$ is non-singular, these lines are transverse to $[0 : \ldots : 0 : 1] \times \mathbb{R}P^q$. For $z \in \mathbb{R}\tilde{S}$ passing through $([1 : 0 : \ldots : 0], y)$ we define $\tilde{p}(z)$ to be the intersection of $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$ with the projection of z along $[0 : \ldots : 0 : 1] \times \mathbb{R}P^q$ onto $\mathbb{R}P^p \times \{y\}$. Evidently, $\tilde{p}$ is a smooth well-defined map.

We can define a partial inverse of $\tilde{p}$ as a map $q : \mathbb{R}P^{p-1} \times \mathbb{R}P^q - \mathbb{R}T \to \mathbb{R}F$, where

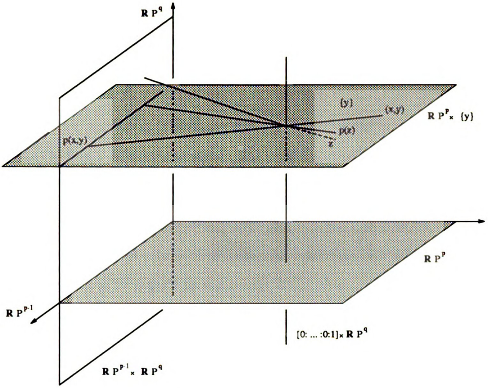

Figure 1: $\mathbb{R}P^p \times \mathbb{R}P^q$

$\mathbb{R}T$ is given by the equations

$$\begin{cases} y_p = 0 \\ x_0 y_0 + \ldots + x_{p-1} y_{p-1} = 0 \end{cases}$$

in the following way. Let $q(x,y)$ be the intersection point $A_{(x,y)}$ of $\mathbb{R}H_{p,q}$ and the line $L_{(x,y)}$ in $\mathbb{R}P^p \times \{y\}$ connecting the points (x,y) and $([0:\ldots:0:1],y)$ if $A_{(x,y)} \notin \mathbb{R}S$ (i.e. if $y_p \neq 0$). Point $A_{(x,y)}$ is unique if $L_{(x,y)} \not\subset \mathbb{R}H_{p,q}$ since $\mathbb{R}H_{p,q}$ is a hyperplane in $\mathbb{R}P^p \times \{y\}$. To define $q(x,y)$ in the case of $A_{(x,y)} \in \mathbb{R}S$ (i.e. when $y_p = 0$) we note that if $L_{(x,y)} \not\subset \mathbb{R}H_{p,q}$ then $L_{(x,y)}$ determines a line $l_{(x,y)}$ in the quotient of tangent plane to $\mathbb{R}H_{p,q}$ at $([0:\ldots:0:1],y)$ by tangent plane to $[0:\ldots:0:1] \times \mathbb{R}P^q$ at $([0:\ldots:0:1],y)$. We put $q(x,y) = l_{(x,y)} \in \mathbb{R}\tilde{S} \subset \mathbb{R}F$. Note that $L_{(x,y)} \subset \mathbb{R}H_{p,q}$ iff $([0:\ldots:0:1],y),(x,y) \in \mathbb{R}H_{p,q}$, i.e. iff $y_p = 0$ and $x_0 y_0 + \ldots + x_{p-1} y_{p-1} = 0$, thus q is a smooth well-defined map on $\mathbb{R}P^{p-1} \times \mathbb{R}P^q - \mathbb{R}T$. By construction q is the inverse of $\tilde{p}$.

Let $g : \mathbb{R}G \to \mathbb{R}P^{p-1} \times \mathbb{R}P^q$ be the blowup of $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$ along $\mathbb{R}T$ and $\mathbb{R}\tilde{T}$ be the exceptional divisor of g.

$$\begin{tikzcd}
B(\mathbb{R}H_{p,q}, \mathbb{R}S) = \mathbb{R}F \arrow[rr, "\tilde{q}", "\approx"'] \arrow[d, "f"'] \arrow[drr] & & \mathbb{R}G = B(\mathbb{R}P^{p-1} \times \mathbb{R}P^q, \mathbb{R}T) \arrow[d, "g"] \\
\mathbb{R}H_{p,q} \supset \mathbb{R}H_{p,q} - \mathbb{R}S \arrow[rr, "p"'] & & \mathbb{R}P^{p-1} \times \mathbb{R}P^q
\end{tikzcd}$$

Now we want to find a diffeomorphism $\tilde{q} : \mathbb{R}G \to \mathbb{R}F$ extending q. Let $\tilde{q}|_{\mathbb{R}G - \mathbb{R}\tilde{T}} = q \circ g$. By definition, the points of $\mathbb{R}\tilde{T}$ are lines tangent to $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$ passing through $\mathbb{R}T$ and normal to $\mathbb{R}T$. If such a line does not lie in $\{x\} \times \mathbb{R}P^q$ and in $\mathbb{R}P^{p-1} \times \{y\}$ then we can consider this line as a curve C of bidegree (1,1) in $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$. Let D be the surface generated in $\mathbb{R}P^p \times \mathbb{R}P^q$ by lines in $\mathbb{R}P^p \times \{y\}$ connecting $([0 : \ldots : 0 : 1], y)$ and $C \cap \mathbb{R}P^p \times \{y\}$ for all y with $C \cap \mathbb{R}P^p \times \{y\} \neq \emptyset$. We claim that D is a surface of bidegree (1,1) in $\mathbb{R}P^p \times \mathbb{R}P^q$, indeed, D is given in $\mathbb{R}P^p \times \mathbb{R}P^q$ by the same system of equations as C in $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$. The intersection $\mathbb{R}H_{p,q} \cap D$ is a curve of bidegree (1,1) in D and, by construction, if $(x, y) \in \mathbb{R}T$ then $L_{(x,y)} \subset D$. Therefore, $\mathbb{R}H_{p,q} \cap D$ is reducible and consists of two intersecting lines. Define $\tilde{q}(x, y)$ as the intersection point $A_{(x,y)}$ of these lines if $A_{(x,y)} \notin \mathbb{R}S$ and as the line $l_{(x,y)}$ (see the construction of q) if $A_{(x,y)} \in \mathbb{R}S$. Map $\tilde{q}$ is a smooth injective map onto $\mathbb{R}F$, therefore, $\mathbb{R}F$ and $\mathbb{R}G$ are diffeomorphic ($\mathbb{R}G$ is compact) and $\mathbb{R}H_{p,q}$ is blowup equivalent to $\mathbb{R}P^{p-1} \times \mathbb{R}P^q$

□

Proposition 2.1 *If M is blowup equivalent to M' and N is blowup equivalent to N' then $M \times N$ is blowup equivalent to $M' \times N'$*

Proof. It is easy to see that if $M = M_0, M_1, \ldots, M_n = M'$ and $N = N_0, N_1, \ldots, N_m = N'$ are blowup sequences then $M \times N = M_0 \times N_0, M_1 \times N_0, \ldots, M_n \times N_0, M_n \times N_1, \ldots, M_n \times N_m = M' \times N'$ is also a blowup sequence □

Lemma 2 *The product of two topologically rational manifolds is topologically rational*

Proof. Note that it is enough to prove that $S^1 \times S^n$ is m-equivalent to S^{n+1} for any $n \geq 2$ (the classification of surfaces implies that manifolds of dimension less then 3 are topologically rational). Once we prove this we get that $S^p \times S^q$ is m-equivalent to $S^1 \times S^{p-1} \times S^q$ with the help of Proposition 2.1, by induction we get that $S^p \times S^q$ and S^{p+q} are equivalent to $S^1 \times S^1 \times \ldots \times S^1$ and, therefore, are equivalent to each other, i.e. the product of S^p and S^q is topologically rational and Proposition 2.1 implies the lemma.

Let us now show that $S^1 \times S^n$ is m-equivalent to S^{n+1}. Consider the result X of blowup of S^{n+1} along standard S^{n-1}. Manifold X is diffeomorphic to $S^1 \tilde{\times} S^n$. To see this we represent S^{n+1} as a join $S^1 * S^{n-1}$, then X consists of spheres $\{x\} * S^{n-1} \cup \{-x\} * S^{n-1}, x \in S^1$. Thus we see that X is an S^n-bundle over $\mathbb{R}P^1 \approx S^1$ diffeomorphic to the fiberwise join of the trivial S^{n-2} bundle ϵ_{n-2} over S^1 and the non-trivial S^0-bundle μ over S^1. By the same arguments, the result Y of blowup of X along $\epsilon_{n-2} \approx S^1 \times S^{n-2}$ is diffeomorphic to the fiberwise join of the trivial S^{n-3}-bundle over $S^1 \times S^1$ and a non-trivial S^0-bundle over $S^1 \times S^1$.

Consider $S^1 \times S^n$ now. We can represent $S^1 \times S^n$ as the fiberwise join of ϵ_{n-2} and the trivial S^0 bundle over S^1. By the same arguments, the result of blowup of $S^1 \times S^n$ along $\epsilon_{n-2} \approx S^1 \times S^{n-2}$ is diffeomorphic to the fiberwise join of the trivial S^{n-3}-bundle over $S^1 \times S^1$ and a non-trivial S^0-bundle over $S^1 \times S^1$.

Note that all the non-trivial S^0-bundle over $S^1 \times S^1$ are isomorphic, every S^0-bundle is determined by its first Stiefel-Whitney class and for any two non-zero elements $\alpha, \beta \in H^1(S^1 \times S^1; \mathbb{Z}_2)$ there exists a self-diffeomorphism $f : S^1 \times S^1 \to S^1 \times S^1$ such that $f^*(\beta) = \alpha$. Thus, $S^1 \times S^n$ is blowup equivalent to $S^1 \tilde{\times} S^n$ and therefore to S^{n+1} $\square$

3 Cobordant manifolds are blowup equivalent

Lemma 3 *If smooth connected closed manifolds M and N are cobordant then they are blowup equivalent*

Let W be a cobordism between M and N, then W admits a handlebody decomposition. A handlebody decomposition of W determines a sequence of manifolds $M = M_0, M_1, \dots, M_n = N$ such that M_j is the result of surgery of M_{j-1}, $j \in \{1, \dots, n\}$. Let $P_{j-1} \in M_{j-1}$ denote the surgery sphere (i.e. the boundary of the core of the j-th handle) and let $Q_j \subset M_j$ denote the dual surgery sphere (i.e. the boundary of the cocore of the j-th handle). Evidently we may assume that every M_j is connected.

Proposition 3.1

a) *Sphere P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1} iff*

$$dim H_*(M_j; \mathbb{Z}_2) \geq dim H_*(M_{j-1}; \mathbb{Z}_2)$$

b) *Sphere P_{j-1} is not $\mathbb{Z}_2$-homologous to zero in M_{j-1} iff*

$$dim H_*(M_j; \mathbb{Z}_2) = dim H_*(M_{j-1}; \mathbb{Z}_2) - 2$$

Proof. Consider the exact sequences of pairs (M_{j-1}, P_{j-1}) and (M_j, Q_j)

$$\dots \to H_k(P_{j-1}; \mathbb{Z}_2) \xrightarrow{i_k} H_k(M_{j-1}; \mathbb{Z}_2) \to H_k(M_{j-1}, P_{j-1}; \mathbb{Z}_2) \to \dots$$
$$\dots \to H_k(Q_j; \mathbb{Z}_2) \xrightarrow{j_k} H_k(M_j; \mathbb{Z}_2) \to H_k(M_j, P_j; \mathbb{Z}_2) \to \dots$$

Hence

$$0 \to H_k(M_{j-1}; \mathbb{Z}_2)/im i_k \to H_k(M_{j-1}, P_{j-1}; \mathbb{Z}_2) \to ker i_{k-1} \to 0$$
$$0 \to H_k(M_j; \mathbb{Z}_2)/im j_k \to H_k(M_j, Q_j; \mathbb{Z}_2) \to ker j_{k-1} \to 0$$

Note that $H_k(M_{j-1}, P_{j-1}; \mathbb{Z}_2) = H_k(M_j, Q_j; \mathbb{Z}_2)$ and therefore

$$\begin{aligned}&dim H_*(M_j; \mathbb{Z}_2) - dim H_*(M_{j-1}; \mathbb{Z}_2) = \\ &(dim(im(j_*)) - dim(ker(j_*))) - (dim(im(i_*)) - dim(ker(i_*)))\end{aligned}$$

If P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1} then $dim(im(i_*)) = 1, dim(ker(i_*)) = 1$, if not then $dim(im(i_*)) = 2, dim(ker(i_*)) = 0$. If Q_j is $\mathbb{Z}_2$-homologous to zero in M_j then $dim(im(j_*)) = 1, dim(ker(j_*)) = 1$, if not then $dim(im(j_*)) = 2, dim(ker(j_*)) = 0$. Thus, it is sufficient to prove that either P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1} or Q_j is $\mathbb{Z}_2$-homologous to zero in M_j (or both). Consider the boundary $\partial U \approx S^{dimP_{j-1}-1} \times S^{dimQ_j-1}$ of the tubular neighbourhood U of P_{j-1} in M_{j-1}. Note that $\partial U = \partial(M_{j-1} - U)$. Therefore, if $k : \partial U \to M_{j-1} - U$ is the inclusion map then

$$dim(ker(k_*)) = \frac{1}{2} dim H_*(\partial U; \mathbb{Z}_2) = 2$$

It follows that either $S^{dimP_{j-1}-1} \times \{pt\}$ is $\mathbb{Z}_2$-homologous to zero in $M_{j-1} - U$ and then P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1}, or $\{pt\} \times S^{dimQ_j-1}$ is $\mathbb{Z}_2$-homologous to zero in $M_{j-1} - U$ and then Q_j is $\mathbb{Z}_2$-homologous to zero in M_j $\square$

Remark 3.2 (A.Marin) It is not always true that either P_{j-1} is *not* $\mathbb{Z}_2$-homologous to zero in M_{j-1} or Q_j is *not* $\mathbb{Z}_2$-homologous to zero in M_j, if $dimP_{j-1} = dimQ_j$ then $S^{dimP_{j-1}-1} \times \{pt\}$ may be $\mathbb{Z}_2$-homologous to $\{pt\} \times S^{dimQ_j-1}$ in $M_{j-1} - U$ (consider, for instance the standard $(+1)$-surgery of S^3).

Definition. We say that the j-th handle of W is *odd*, if P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1} and the $\mathbb{Z}_2$-self-linking number of P_{j-1} equipped with the surgery trivialization of the normal bundle in M_{j-1} is not equal to zero (evidently, $dimW = dimM + 1 = 2dimP_{j-1} + 2$ in this case).

The following proposition implies that the case when both P_{j-1} is $\mathbb{Z}_2$-homologous to zero in M_{j-1} and Q_j is $\mathbb{Z}_2$-homologous to zero in M_j is exactly the case of odd handle.

Proposition 3.3 *The j-th handle of W is odd iff*

$$dimH_*(M_{j-1};\mathbb{Z}_2) = dimH_*(M_j;\mathbb{Z}_2)$$

Proof. By definition, the j-th handle is odd iff $S^{dimP_{j-1}-1} \times \{pt\}$ is $\mathbb{Z}_2$-homologous to $\{pt\} \times S^{dimQ_j-1}$, therefore, the proposition follows from the proof of Proposition 3.1 $\square$

Proposition 3.4 *The number of odd handles of any handlebody decomposition of W is congruent modulo 2 to $\chi(W) + \frac{\chi(M)+\chi(N)}{2}$*

Proof. Proposition 3.1 and Proposition 3.3 follow that this number is congruent modulo 2 to $dimH_*(W;\mathbb{Z}_2) - \frac{1}{2}(dimH_*(M;\mathbb{Z}_2) + dimH_*(N;\mathbb{Z}_2)$ that is equivalent to the proposition $\square$

Proposition 3.5 *If two manifolds M and N are cobordant then there exists a cobordism W between M and N and a handlebody decomposition of W containing no odd handles*

Proof. If $dimW$ is odd then W may not contain odd handles. Suppose that $dimW = 2k$ is even. If the jth handle of W is odd then we attach $\mathbb{R}P^{2k}$ to the tubular neighbourhood of M_j in W to get the new cobordism $W' \approx W\#\mathbb{R}P^{2k}$. Manifold $\mathbb{R}P^{2k}$ admits a standard handlebody decomposition containing exactly one handle of index k. By Propositon 3.4 this is an odd handle (since $\chi(\mathbb{R}P^{2k} = 1)$. The handlebody decomposition of W and the standard handlebody decomposition of $\mathbb{R}P^{2k}$ induce a handlebody decomposition of W'. By switching the order of the jth handle and the handles of $\mathbb{R}P^{2k}$ of index less than k we can make the two odd handles adjanced, these operations do not change the oddness of the handles. Then we slide ode of two adjanced via another. After this both of them become even, the first one is even because the self-intersection number is even, the second one is even because its surgery sphere is not $\mathbb{Z}_2$-homologous to zero after the attaching of the new handle □

Without loss of generality we may assume that the handlebody decomposition of W consists of a single handle (otherwise we proceed with induction). By Proposition 3.5 we may assume that the handle is not odd. Suppose that the handle is of index p, $S^{p-1} \approx P \subset M$ is the surgery sphere (the boundary of the core of the handle of W) and $S^{q-1} \approx Q \subset N$ is the dual surgery sphere (the boundary of the cocore of the handle of W). In particular, $dim(M) = p + q - 1$. By Proposition 3.1 and Proposition 3.3 we may assume that P is $\mathbb{Z}_2$-homologous to zero in M and Q is not $\mathbb{Z}_2$-homologous to zero in N (if by chance P is not $\mathbb{Z}_2$-homologous to zero in M and Q is $\mathbb{Z}_2$-homologous to zero we turn W upside down).

Lemma 4 *There exists a manifold $\tilde{N}$ such that $\tilde{N} - Q$ is the result of multiblowup of $N - Q$ and a smooth p-dimensional submanifold $\tilde{W} \subset \tilde{N}$ transversal to Q and such that $\tilde{W} \cap Q = \{q\}$, where q is a point.*

Proof. By the Nash theorem [8] we may assume that N is a non-singular algebraic variety. Since Q is not $\mathbb{Z}_2$-homologous to zero in N, there exists $\beta \in H_p(N; \mathbb{Z}_2)$ such that $\beta.[Q] = 1 \in \mathbb{Z}_2$. By a theorem of Thom (Théorème III.2 of [9]) β is representable by smooth map $f : B \to N$, where B is a smooth p-manifold and $f_*([B]) = \beta$. By the Akbulut-King normalization theorem (Theorem 2.8.3 of [3]) β is representable by a subalgebraic set of dimension p in N (i.e. there exists a non-singular component Z of an algebraic variety and a degree 1 rational map $F : Z \to N$ such that $F_*([Z]) = \beta$). Denote by W the Zariski closure of $F(Z)$. Since $F(Z)$ is subalgebraic (and, therefore, semialgebraic by the Tarski-Seidenberg theorem), $dimW = dimZ = p$. By changing Q with a small isotopy we can assume that Q intersects W transversally in non-singular points of W.

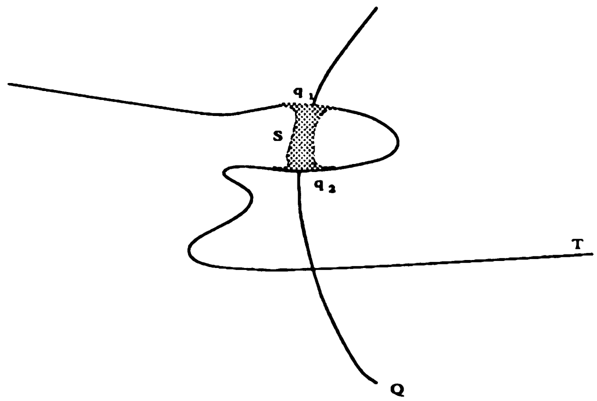

Figure 2: Cancelling of pairs of the intersection points

By the Hironaka theorem [5] there exists a multiblowup of N with centers not intersecting the transforms of Q such that the proper transform of W is a smooth submanifold of the result $\tilde{N}$ of the multiblowup. Therefore, the proper transform T of $F(Z)$ is a smooth submanifold of $\tilde{N}$ intersecting $Q \subset \tilde{N}$ transversally at odd number of points (the fact that T is a submanifold without the boundary follows from the local version of Corollary 2.3.3 of [3]). Let $q_1, \ldots, q_{2k+1}$ be the points of $T \cap Q$. Let $\gamma_1, \ldots, \gamma_k$ be the disjoint paths in Q connecting q_1 to q_2, q_3 to q_4, ..., q_{2k-1} to q_{2k}. Then, the tubular neighbourhood S of $\gamma_1 \cup \ldots \cup \gamma_k$ such that $\tilde{W} = T \cup \partial S - intS$ may be smoothened to a p-dimensional submanifold of $\tilde{N}$ intersecting Q transversally at $q = q_{2k+1}$ $\square$

Proposition 3.6 *There exists a manifold M' such that $M' - P$ is the result of multiblowup of $M - P$ and $P = \partial V$ for a submanifold $V \subset M'$*

Proof. Since $M - P = N - Q$, the multiblowup from Lemma 4 produces the required multiblowup of M, the complement of the small disk neighbourhood of q in $\tilde{W}$ gives V such that $P = \partial V$ $\square$

Now we may assume that $P = \partial V \subset M$.

Definition. Let $P \approx S^{p-1}$ be the submanifold of M equipped with a trivialization τ

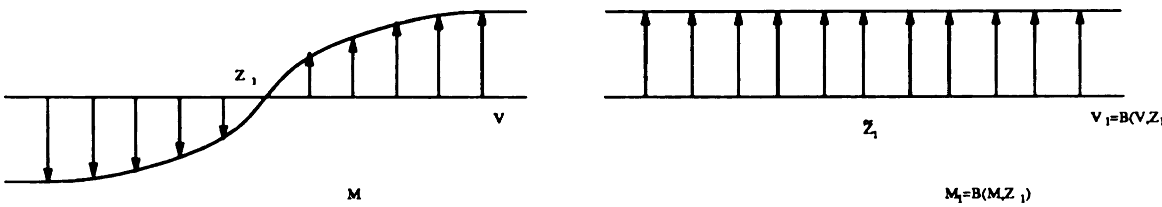

Figure 3: Blowup of the zero section

of $\nu_M(P)$. Let us denote by $\chi_\tau(M,P)$ the result of surgery of M along P. If $P = \partial V$ for some submanifold V we say that the surgery trivialization τ is *compatible with* V if there exists a trivialization τ_V of $\nu_M(V)$ such that $\tau_V|_P$ and the natural trivialization of $\nu_V(P)$ form τ.

Proposition 3.7 *There exists a manifold M' such that $M' - P$ is the result of multiblowup of $M - P$, $P = \partial V'$ and the surgery trivialization of $\nu_{M'}(P)$ is compatible with V.*

Proof. Let $\epsilon_1, \ldots, \epsilon_q$ be the trivialization of $\nu_M(P)$. Let ξ_1' be a generic section of $\nu_M(V)$ extending ϵ_1. The zero set of ξ_1' is a smooth submanifold Z_1 of V. Let $M_1 = B(M, Z_1)$ and let $V_1 \subset M_1$ be the proper transform of V, $V_1 \approx B(V, Z_1)$. Then ϵ_1 extends to a non-vanishing section ξ_1 of $\nu_{M_1}(V_1)$ (see [1] for explicit proof of this).

We now proceed by induction. Suppose now that there exists M_k such that $M_k - P$ is the result of multiblowup of $M - P$ and $\epsilon_1, \ldots, \epsilon_k$ extend to non-vanishing linearly independent sections $\xi_1, \ldots, \xi_k$ of $\nu_{M_k}(V_k)$.

Let ξ_{k+1}' be a generic section of $\nu_{M_k}(V_k)$ such that $\xi_1, \ldots, \xi_k, \xi_{k+1}'$ are linearly independent. Let $Z_{k+1} \subset V_k$ be the zero set of ξ_{k+1}'. Let $\bar{M}_k = B(M_k, Z_{k+1})$, let $\bar{V}_k$ be the proper transform of V_k, and let E be the exceptional divisor of $B(M_k, Z_{k+1})$. It is easy to see that ϵ_{k+1} extends to a non-vanishing section $\bar{\xi}_{k+1}$ of $\nu_{\bar{M}_k}(\bar{V}_k)$ and sections $\epsilon_1, \ldots, \epsilon_k$ extend to sections $\bar{\xi}_1, \ldots, \bar{\xi}_k$ of $\nu_{\bar{M}_k}(\bar{V}_k)$ transverse to $\bar{V}_k$ and vanishing exactly on $E \cap \bar{V}_k$. The projectivisation F of the orthogonal complement of

$\xi_1, \ldots, \xi_k$ in $\nu_{M_k}(Z_{k+1})$ is contained in E and contains $E \cap \bar{V}_k$. Let $M_{k+1} = B(\bar{M}_k, F)$. Sections $\epsilon_1, \ldots, \epsilon_{n+1}$ now extend to some non-vanishing linearly independent sections (produced by $\bar{\xi}_1, \ldots, \bar{\xi}_{k+1}$). since F is tangent to $\bar{\xi}_{k+1}$. Induction completes the proof of the lemma □

The following lemma is the main tool that enables us to blow down submanifolds inside the ambient manifold.

Lemma 5 *Let $\tilde{V}$ be a submanifold of M. If $\tilde{V} = B(V, K)$ and $\nu_M(\tilde{V})$ is trivial then there exists a manifold M' containing submanifold V and such that (M', V) is blowup equivalent to $(M, \tilde{V})$*

Addendum 6 *If a trivialization τ of $\nu_M(\tilde{V})|_{\partial\tilde{V}}$ extends to a trivialization of $\nu_M(\tilde{V})$ then the induced by τ trivialization of $\nu_{M'}(V)|_{\partial V}$ extends to a trivialization of $\nu_{M'}(V)$*

Proof of Lemma 5. Let $\tilde{K} \subset \tilde{V} \subset \tilde{M}$ be the exceptional divisor of $B(V, \tilde{K})$. Let $\tilde{M} = B(M, \tilde{K})$. The proper transform of $\tilde{V}$ in $\tilde{M}$ is diffeomorphic to $\tilde{V}$ (since the center of the blowup is of codimension 1 in $\tilde{V}$), we shall denote the proper transform of $\tilde{V}$ in $\tilde{M}$ still by $\tilde{V}$. The bundle $\nu_{\tilde{M}}(\tilde{V})$ is isomorphic to the tensor product of the trivial vector bundle and the 1-dimensional vector bundle $\eta_{\tilde{K}}$ over $\tilde{V}$ dual to $\tilde{K}$ (cf. proof of Proposition 3.7). By the adjunction formula, the normal bundle $\nu_{\tilde{V}}(\tilde{K})$ is isomorphic to $\eta_{\tilde{K}}|_{\tilde{K}}$. Hence $\nu_{\tilde{M}}(\tilde{K}) = (\epsilon^{q-1} \otimes \nu_{\tilde{V}}(\tilde{K})) \oplus \nu_{\tilde{V}}(\tilde{K}) = \epsilon^q \otimes \nu_{\tilde{V}}(\tilde{K})$. Therefore, since $dim(\nu_{\tilde{V}}(\tilde{K})) = 1$, the projectivisation E of the normal bundle of $\tilde{K}$ in $\tilde{M}$ is diffeomorphic to $\tilde{K} \times \mathbb{R}P^{q-1}$, where the diffeomorphism is induced by the trivialization of $\nu_M(\tilde{V})$, and the section F given by the normal bundle of $\tilde{K}$ in $\tilde{V}$ extends to some trivialization of $E \approx \tilde{K} \times \mathbb{R}P^{q-1}$.

Let $\bar{M} = B(\tilde{M}, \tilde{K})$. The proper transform $\bar{V}$ of $\tilde{V}$ in $\bar{M}$ is diffeomorphic to $\tilde{V}$ ($\tilde{K}$ is of codimension 1 in $\tilde{V}$). Therefore, the normal bundle of F in $\tilde{V}$ is diffeomorphic to

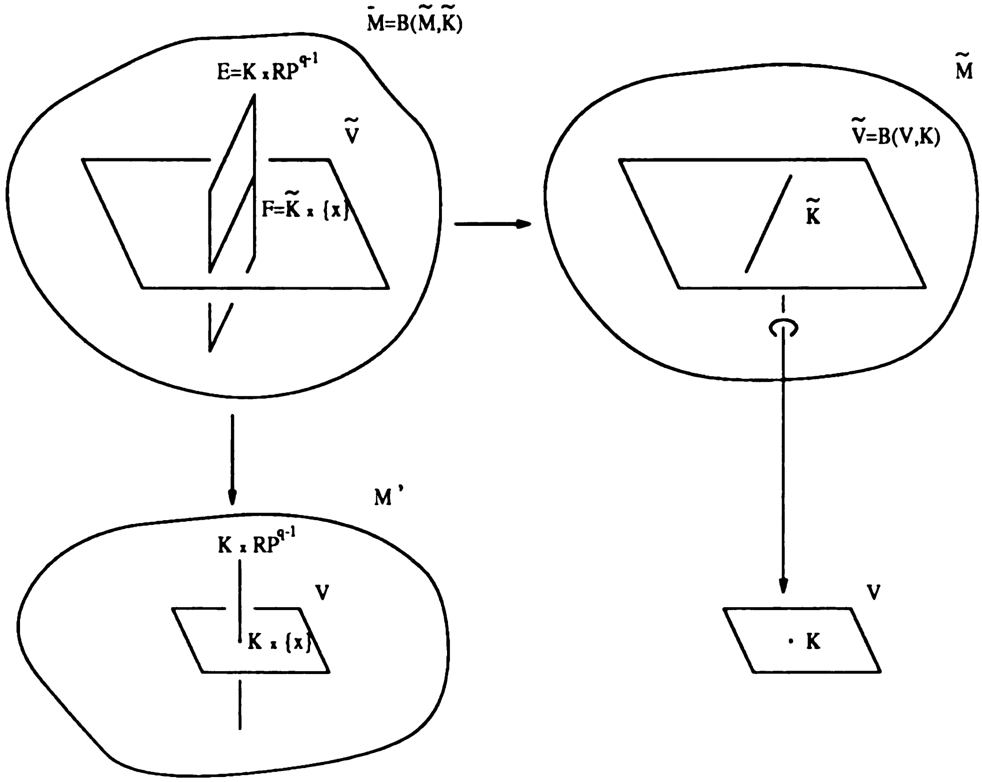

Figure 4: Proof of Lemma 5

the normal bundle of the exceptional divisor of blowup. But $F \approx \tilde{K} \times \{x\}$ for some $x \in \mathbb{R}P^{q-1}$ and, therefore, for any $y \in \mathbb{R}P^{q-1}$ the restriction of the normal bundle of E in $\bar{M}$ to $\tilde{K} \times \{y\}$ is diffeomorphic to the normal bundle of the exceptional divisor of a blowup. In other words the tubular neighbourhood of $\tilde{K} \times \mathbb{R}P^{q-1}$ in $\bar{M}$ is diffeomorphic to the I-bundle $(\tilde{K} \times \mathbb{R}P^{q-1})\tilde{\times} I \approx (\tilde{K} \tilde{\times} I) \times \mathbb{R}P^{q-1}$ which is canonical over each fiber of the map $\tilde{K} \times \mathbb{R}P^{q-1} \to K \times \mathbb{R}P^{q-1}$. It follows that $\bar{M} = B(M', K \times \mathbb{R}P^{q-1})$ for some manifold M' so that the exceptional divisor of $B(M', K \times \mathbb{R}P^{q-1})$ is E and $\bar{V}$ is the proper transform of submanifold $V \subset M'$ □

Proof of Addendum 6. We need to prove that $\epsilon_1, \ldots, \epsilon_{q-1}$ extend to a trivialization of $\nu_{M'}(V')$. We see that $\epsilon_1, \ldots, \epsilon_{q-1}$ extend to a trivialization of $\nu_{\bar{M}}(\bar{V})$, since to get $\bar{M}$ we blow M up twice along $\tilde{K} \subset \tilde{V}$ and by assumption $\epsilon_1, \ldots, \epsilon_{q-1}$ extend to a trivialization of $\nu_M(\tilde{V})$. Let $\xi_1, \ldots, \xi_{q-1}$ be the trivialization of $\nu_{\bar{M}}(\bar{V})$ induced by the trivialization of $\nu_M(\tilde{V})$. Then $\xi_1, \ldots, \xi_{q-1}$ extend $\epsilon_1, \ldots, \epsilon_{q-1}$. Since we defined the diffeomorphism $E \approx \tilde{K} \times \mathbb{R}P^{q-1}$ using $\xi_1, \ldots, \xi_{q-1}$, the fibers of the blowup $\bar{M} \to M'$ are tangent to $\xi_1, \ldots, \xi_{q-1}$ (i.e. the trivialization of $\nu_E(F)$ given by the restriction $(E, F) \to (K \times \mathbb{R}P^{q-1}, K \times \{x\})$ of the blowup is equivalent to $\xi_1, \ldots, \xi_{q-1}$). It follows that $\xi_1, \ldots, \xi_{q-1}$ is induced by some trivialization $\psi_1, \ldots, \psi_{q-1}$ of the normal bundle of V' in M'. Therefore, $\psi_1, \ldots, \psi_{q-1}$ extend $\epsilon_1, \ldots, \epsilon_{q-1}$ $\square$

Proof of Lemma 3 and Theorem 1. By Proposition 3.6 we need only to show that if N is the result of surgery $\chi_\tau(M, \partial V)$, where V is a submanifold of M, then N is blowup equivalent to M. Note that V is of positive codimension in M since N is connected. By Proposition 3.7 we may assume that τ is compatible with V.

We prove this by induction. Since $dimV < dimM$ we may assume that there exists a blowup sequence between V and D^p ($\partial V \approx S^{p-1}$). If $V \approx D^p$ then $N \approx M \# S^P \times S^{q-1}$ (τ is compatible with V) and, by Lemma 2, N is blowup equivalent to M. We want to construct, using the blowup sequence $V = V_0, \ldots, V_l = D^p$, a sequence $M = M_0, \ldots, M_l$ such that V_j is a submanifold of M_j, (M_j, V_j) is blowup equivalent to (M_{j-1}, V_{j-1}) and τ is compatible with V_j, $j \in \{1, \ldots, l\}$ (note that $P = \partial V = \partial V_1, = \ldots = \partial V_l$). This suffices for the proof since this implies that $N = \chi_\tau(M, \partial V)$ is blowup equivalent to $\chi_\tau(M_l, \partial V_l) \approx M_l \# S^p \times S^{q-1}$.

We construct such a sequence by induction. If $V_{j-1} = B(V_j, K_j)$ then we apply Lemma 5 to (M_{j-1}, V_{j-1}) and get M_j. By Addendum 6 τ is compatible with V_j. If $V_j = B(V_{j-1}, L_{j-1})$ then let $M_j' = B(M_{j-1}, L_{j-1})$, then V_j is the proper transform of V_{j-1}. By Proposition 3.7 there exists the result $(\bar{M}_j, \bar{V}_j)$ of multiblowup of (M_j', V_j) such that τ is compatible with $\bar{V}_j$. Then there is the multiblowup $\bar{V}_j = W_k \to \ldots \to W_0 = V_j$. Now we apply Lemma 5 consequently k times to construct M_j so that (M_j, V_j) is blowup equivalent to $(\bar{M}_j, \bar{V}_j)$ and, therefore, to (M_{j-1}, V_{j-1}). By

Addendum 6 the surgery trivialization τ is compatible with V_j □

4 Summary

Theorem 1 is proven with the help of the classification of the smooth manifolds up to cobordism due to R.Thom. We prove first that every smooth closed connected manifold is cobordant to a topologically rational manifold and then we prove that two cobordant smooth closed connected manifolds are blowup equivalent.

References

[1] S.Akbulut, H.King. Submanifolds and homology of nonsingular algebraic varieties. American Journal of Math., (1985), 45-83

[2] S.Akbulut, H.King. Rational structures on 3-manifolds. Pacif.J.Math., **150** (1991), 201-214.

[3] S.Akbulut, H.King. Topology of Real Algebraic Sets. Mathematical Sciences Research Institute Publications **25**. Springer 1992.

[4] R.Benedetti, A.Marin. Déchirures de variéts de dimension trois et la conjecture de Nash rationalité en dimension trois. Comment. Math. Helv., **67** (1992), 514-545

[5] H.Hironaka. Resolutions of singularities of an algebraic variety over a field of characteristic zero. Ann. of Math., **79** (1964), 109-326.

[6] V.Kharlamov. The topological type of non singular surfaces in $\mathbb{R}P^3$ of degree four, Funk. Anal. and its Appl., **10** (1976), 295-305.

[7] J.Milnor, J.Stasheff. Characteristic classes. Annals of Mathematics Studies, **76**, Princeton 1974.

[8] J.Nash. Real algebraic manifolds. Ann. of Math., **56** (1952), 405-421.

[9] R.Thom. Quelques proprietes globales de varieties differentiables. Comment. Math. Helv., **28** (1954), 17-86.