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Floer homology for Connected Sums of homology 3-Spheres

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FLOER HOMOLOGY FOR CONNECTED SUMS OF HOMOLOGY 3-SPHERES

By

Weiping Li

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ABSTRACT

FLOER HOMOLOGY
FOR CONNECTED SUMS
OF HOMOLOGY 3-SPHERES

By

Weiping Li

Supervising Professor: Ronald Fintushel

In this thesis, we try to understand a Mayer-Vietoris principle for Floer homology. Floer homology is defined from a chain complex whose chain groups are roughly generated from the $SU(2)$-irreducible representations. And boundary maps depend on the 1-dimensional moduli space of self-dual connections on the (homology 3-sphere)$\times\mathbf{R}$. For Floer homology on connected sums, it relies on understanding the gluing procedure on noncompact 4-manifolds with almost-harmonic 2-forms in the gluing region. A particular gluing data and analysis are introduced. The splitting and perturbation effected on 1-dimensional moduli spaces are also considered.

Using this gluing result and the much simpler calculation of the spectral flow of the Chern-Simons Hessian for the connected sums we are able to calculate Floer homology in several examples.

To my grandparents.

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Contents

Chapter 1

Introduction

Floer homology is a mod 8-graded homology theory for homology three spheres which relates Donaldson's polynomial invariants in the relative and absolute cases via a Mayer-Vietoris principle. It is defined from a chain complex whose chain groups are (roughly) built from the $SU(2)$-representations of the fundamental group of the homology sphere. These can often be straightforward to compute. (They rely "only" on linear analysis.) The boundary operators, however, depend on nonlinear analysis, namely, the structure of the 1-dimensional moduli space of self-dual connections on the (homology sphere)$\times\mathbf{R}$.

The first calculations of Floer homology were carried out by R.Fintushel and R.Stern who computed Floer homology for Brieskorn homology spheres and outlined a program for their calculation for all Seifert fibered homology spheres. A natural question is to ask about the Floer homology of a connected sum of homology 3-spheres Y_0 and Y_1. The difficult point is understanding the structure of the 1-dimensional moduli space of anti-self-dual connections on the tube $(Y_0\#Y_1)\times\mathbf{R}$. This relies on understanding the (Taubes) gluing procedure on noncompact 4-manifolds with almost-harmonic 2-forms in the gluing region. Each connection in a connected component of a moduli space $\mathcal{M}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#\beta,\alpha'\#\beta')$ must limit asymptotically to flat $SU(2)$-connections on $Y_0\#Y_1$ which in turn correspond to representations such as $\alpha\#\beta$ where α is an $SU(2)$-representation of $\pi_1(Y_0)$ and β of $\pi_1(Y_1)$. We have proved:

Theorem 1.0.1 *: For appropriate metrics on* $(Y_0\#Y_1)\times\mathbf{R}$, *any 1-dimensional anti-self-*

dual moduli space takes the form $\mathcal{M}^1_{(Y_0\# Y_1)\times\mathbf{R}}(\alpha\#\beta,\alpha'\#\beta)$ *(or* $(\alpha\#\beta,\alpha\#\beta')$*).*

In other words, given a anti-self-dual connection A in $\mathcal{M}^1_{Y_0\times\mathbf{R}}(\alpha,\alpha')$ and a flat connection $B\in\mathcal{M}^0_{Y_1\times\mathbf{R}}(\beta,\beta)$ (constant in t), they can be grafted together to produce a self-dual connection on $(Y_0\# Y_1)\times\mathbf{R}$. Furthermore, each 1-dimensional moduli space on $(Y_0\# Y_1)\times\mathbf{R}$ arises via this construction (perhaps with the role of Y_0 and Y_1 reversed).

Using this theorem and the (much simpler) calculation of the spectral flow of the Chern-Simons Hessian for the connected sum we are able to calculate Floer homology in several examples.

Theorem 1.0.2 *1. For the connected sum of Poincaré 3-sphere with itself, the Floer homology is*

$$HF_0 = Z_2\oplus Z_2,\quad HF_1 = Z,\quad HF_2 = Z,\quad HF_3 = Z\oplus Z$$

$$HF_4 = Z_2\oplus Z_2,\quad HF_5 = 0,\quad HF_6 = 0,\quad HF_7 = Z\oplus Z$$

2. For $\Sigma(2,3,7)\#\Sigma(2,3,7)$

$$HF_0 = Z_2\oplus Z_2,\quad HF_1 = 0,\quad HF_2 = 0,\quad HF_3 = Z\oplus Z\oplus Z$$

$$HF_4 = Z_2\oplus Z_2,\quad HF_5 = 0,\quad HF_6 = 0,\quad HF_7 = Z\oplus Z\oplus Z$$

3. For $\Sigma(2,3,5)\#\Sigma(2,3,7)$

$$HF_0 = Z,\quad HF_1 = Z\oplus Z\oplus Z,\quad HF_2 = Z_2\oplus Z_2,\quad HF_3 = 0$$

$$HF_4 = Z,\quad HF_5 = Z\oplus Z\oplus Z,\quad HF_6 = Z_2\oplus Z_2,\quad HF_7 = 0$$

At one time it was conjectured that the Floer homology was actually mod 4 rather than mod 8 graded. Example 1 above shows that this conjecture is false.

Chapter 2

Floer homology of homology 3-spheres

2.1 Floer homology

In this subsection, we will give a brief description of gauge theory on 3-manifolds and review the definition of Floer homology. For more see [4], [10], [12], and [15].

Let Y be a homology 3-sphere, i.e. an oriented closed 3-dimensional smooth manifold with $H_1(Y,Z)=0$, and let $P \to Y$ be a smooth principal $SU(2)$-bundle. (Since $c_2(P)=0$, this bundle is trivial.) Fix a trivialization $Y \times SU(2)$ of P and let θ be the associated trivial connection. Denote the Sobolev L^p_k space of connections on P by $\mathcal{A}(P)$. It has a natural affine structure with underlying vector space $\Omega^1(Y, adP)$ where adP is the adjoint bundle. $\mathcal{A}(P)$ is acted upon by the gauge group of bundle automorphisms of P which can be identified with $\mathcal{G} = Aut(P) = L^p_{k+1}(\Omega^0(Y, adP))$. Here we need $k+1 > \frac{3}{p}$ so that we can form the quotient space of gauge equivalence classes $\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}$. The irreducible connections (those for which the stabilizer of the action of $\mathcal{G}$ is Z_2) form an open and dense subspace $\mathcal{B}^*(P)$ of $\mathcal{B}(P)$. The space $\mathcal{B}^*(P)$ has the structure of a Banach manifold with

$$T_a\mathcal{B}^*(P) \equiv \{\alpha \in L^p_k(\Omega^1(Y, adP)) |\ d_a^*\alpha = 0\}$$

where d_a^* is the L^2-adjoint of d_a (covariant derivative on sections of adP) with respect to some metric on Y.

The Chern-Simons functional $cs : \mathcal{A}(P) \to \mathbf{R}$ is defined as

$$cs(a) = \frac{1}{2}\int_Y tr(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

It satisfies $cs(g \cdot a) = cs(a) + 2\pi deg(g)$ for gauge transformations $g : Y \to SU(2)$. Thus cs is well-defined on $\tilde{\mathcal{B}}(P) = \mathcal{A}(P)/\{g \in \mathcal{G} : deg(g) = 0\}$ and it descends to a function

$$cs : \mathcal{B}(P) \to R/2\pi Z$$

which plays the role of a Morse function in defining Floer homology. Its differential is

$$dcs(a)(\alpha) = \int_Y tr(F_a \wedge \alpha),$$

and so its critical set consists of the flat connections $\mathcal{R}(\mathcal{B}(P)) = \{a \in \mathcal{B}(P) | \quad F_a = 0\}$. (Here F_a is the curvature 2-form on Y.) It is well-known that $\mathcal{R}(\mathcal{B}(P))$ is in 1-1 correspondence with $\mathcal{R}(Y) = Hom(\pi_1(Y), SU(2))/adSU(2)$, the $SU(2)$-representations of $\pi_1(Y)$ mod conjugacy. Given any metric on Y, the Hodge star operator applied to the curvature F_a gives a vector field $f(a) = \star F_a \in L^p_k(\Omega^1(Y, adP))$. In fact because $f(g\cdot a) = g\cdot f(a)\cdot g^{-1}$, $f(a)$ is a section of the bundle with fiber $T_a\mathcal{B}^*(P)$. A representation $\alpha \in \mathcal{R}(Y)$ is called *nondegenerate* if the twisted cohomology $H^1(Y; ad\alpha) = 0$. This is the same as requiring that $\ker df(a) = \ker \star d_a = 0$ where $\star d_a$ is the Hessian of the Chern-Simons functional.

Note that a 1-parameter family $\{a(t) | \quad t \in \mathbf{R}\}$ of connections on P gives rise to a connection A with vanishing t-component on the trivial $SU(2)$ bundle over $Y \times \mathbf{R}$. Floer's crucial observation is that trajectories of the vector field f, i.e. the flow lines of $\frac{\partial a}{\partial t} + f(a(t)) = 0$ or $\frac{\partial a}{\partial t} = \star F(a(t))$, can be identified with instantons A on $Y \times \mathbf{R}$ and $A|_{Y\times\{t\}} = a(t)$. A trajectory flow "connects" two flat connections on Y if and only if the Yang-Mills energy of the trajectory (as a connection on $Y \times \mathbf{R}$ with trivial component in the $\mathbf{R}$ direction) is finite. One needs that all zeros of f are nondegenerate and that their stable and unstable manifolds intersect transversally in smooth finite dimensional manifolds. Floer has shown that one can perturb the Chern-Simons functional to make the trajectory flow "Morse-Smale" type (see [15] Lemma 2c.1, Proposition 2c.1 and 2c.2

). These perturbations are based on Wilson loop functions. For the rest of this paper, we assume that the Chern-Simon functional has been so perturbed. Then all irreducible representations are isolated and nondegenerate. Since $\mathcal{R}(Y)$ is compact, it is then also finite.

Fix a Riemannian metric on Y. For any connection a in the trivial real 3-plane bundle over Y, define the elliptic operator

$$D_a : (\Omega^1 \oplus \Omega^0)(Y, adSU(2)) \to (\Omega^1 \oplus \Omega^0)(Y, adSU(2))$$

by $D_a(\alpha, \beta) = (\star d_a\alpha - d_a\beta, -d_a^*\alpha)$. For a nondegenerate representation $\alpha \in \mathcal{R}(Y)$ the Floer grading $\mu(\alpha) \in Z_8$ is defined to be the spectral flow $SF(\alpha, \theta)$ of the family of operators D_{a_t} with the asymptotic values $\lim_{t \to -\infty} a_t = \alpha$, and $\lim_{t \to +\infty} a_t = a_\theta$, the element of $\mathcal{R}(\mathcal{B}(P))$ corresponding to α. (We also denote a_α by α.) The grading $\mu(\alpha)$ is well-defined mod 8 on $\mathcal{B}(P)$ independent of the choice of path a_t. Define the weighted Sobolev space $L^P_{k,\delta}$ on sections ξ of a bundle over $Y \times \mathbf{R}$ to be the L^p_k Sobolev space of $e_\delta \cdot \xi$ where $e_\delta(y,t) = e^{\delta|t|}$ for $|t| \geq 1$. For δ sufficiently small (we will be more precise in §3) and any $SU(2)$ connection A on trivial bundle over $Y \times \mathbf{R}$, the anti-self-duality operator

$$d_A^* \oplus d_A^+ : L^p_{k+1,\delta}(\Omega^1(Y \times \mathbf{R}, adP)) \to L^p_{k,\delta}((\Omega^0 \oplus \Omega^2_+)(Y \times \mathbf{R}, adP))$$

is Fredholm. We say that A is *regular* if $d_A^* \oplus d_A^+$ is surjective. In terms of the complex:

$$L^p_{k+1,\delta}(\Omega^0(Y \times \mathbf{R}, adP)) \xrightarrow{d_A} L^p_{k,\delta}(\Omega^1(Y \times \mathbf{R}, adP)) \xrightarrow{d_A^+} L^p_{k-1,\delta}(\Omega^2_+(Y \times \mathbf{R}, adP))$$

A is regular means that $H^0_A = 0$ (irreducible) and $H^2_A = 0$ (generic). For a nondegenerate critical point α of cs, the spectral flow is $SF(\alpha, \theta) = Index(d_A^* \oplus d_A^+)(\alpha, \theta)$, the Atiyah-Patodi-Singer index of the anti-self-duality operator over $Y \times \mathbf{R}$. So

$$\mu(\alpha) \equiv Index(d_A^* \oplus d_A^+)(\alpha, \theta) \mod 8$$

where A is any family of connections $\{a(t)\} \in \mathcal{B}(P)$ over Y with $a(+\infty) = \theta, a(-\infty) = a_\alpha$ (see [15] or [12]). Floer's chain group $C_j(Y)$ is defined to be the free module generated by irreducible flat connections α with $\mu(\alpha) = j$ mod8.

Note: Changing the orientation of Y switches the sign of cs and hence the spectrum of the Hessian reverses, so $-\mu_{-Y}(\alpha) = 3 - (-\mu_Y(\alpha))$ mod 8. I.e. $\mu_{-Y}(\alpha) = 5 - \mu_Y(\alpha)$ mod 8.

Define $\mathcal{M}_{Y\times\mathbf{R}}$ to be the moduli space of anti-self-dual connections on $Y \times \mathbf{R}$ and let $\mathcal{M}(\alpha,\beta)$ be the subspace of those A such that $\lim_{t\to-\infty} A = \alpha$, $\lim_{t\to+\infty} A = \beta$ for fixed flat connections α and β. It is a smooth canonically oriented manifold which has dimension congruent to $\mu(\alpha) - \mu(\beta)$ (mod 8). The moduli space $\mathcal{M}(\alpha,\beta)$ has finitely many connected components each of which admits a proper, free $\mathbf{R}$-action arising from translations in $Y \times \mathbf{R}$. If $\mu(\alpha) - \mu(\beta) = 1$ (mod 8), let $\mathcal{M}^1(\alpha,\beta)$ be the union of 1-dimensional components of $\mathcal{M}(\alpha,\beta)$. Further perturbations make all the $\mathcal{M}^1(\alpha,\beta)$ regular. Then $\mathcal{M}^1(\alpha,\beta)/\mathbf{R}$ will be a compact oriented 0-manifold, i.e. it is a finite set of signed points. The differential $\partial : C_j \to C_{j-1}$ of Floer's chain complex is defined by

$$\partial\alpha = \sum_{\beta\in C_{j-1}} \#\hat{\mathcal{M}}(\alpha,\beta)\beta$$

where $\hat{\mathcal{M}}(\alpha,\beta) = \mathcal{M}^1(\alpha,\beta)/\mathbf{R}$ and $\#\hat{\mathcal{M}}(\alpha,\beta)$ is the algebraic number of points. The sign in this formula can be counted by transporting the orientation on the normal bundle of the unstable manifold of α along the trajectory flow into the stable manifold of β. If this agrees with the natural orientation on the stable manifold of β, the trajectory gets the sign $+1$, otherwise -1. Floer has shown that $\partial^2 = 0$. Hence $\{C_j, \partial\}_{j\in Z_8}$ is a chain complex graded by $\mathbf{Z}_8$. The homology of this complex is Floer homology, denoted by HF_j. Floer has shown that it is independent of the choice of metric on Y and of perturbations (see [4], [10], [15]).

The connected sum $Y = Y_0\#Y_1$ of two homology 3-spheres is again a homology 3-sphere. Its fundamental group $\pi_1(Y_0\#Y_1)$ is the free product of $\pi_1(Y_0)$ and $\pi_1(Y_1)$. There are four types of $SU(2)$ representations of $\pi_1(Y_0\#Y_1)$:

$$(1)\ \theta = \theta_0\#\theta_1, \quad (2)\ \theta_0\#\alpha_1, \quad (3)\ \alpha_0\#\theta_1, \quad (4)\ \alpha_0\#\alpha_1$$

where the α_i are irreducible representations of $\pi_1(Y_i)$ and θ_i is the trivial representation of $\pi_1(Y_i), i = 0, 1$. These four types of representations correspond to equivalence classes of flat connections glued together by the clutching map which forms the trivial $SU(2)$ bundle over the connected sum from the bundles on the punctured summands. In each case we have a family $a_0\#a_1$ of flat connections parametrized by a copy of $SU(2)$, which can be identified with the automorphisms of a fiber over a point in the gluing region.

Two elements of this family corresponding to automorphisms ρ_0, ρ_1 are gauge equivalent if and only if $\rho_0\rho_1^{-1}$ extends to an element of the isotropy group Γ_{a_0} or Γ_{a_1}. Thus the corresponding family of gauge equivalence classes is $SU(2)/\Gamma_{a_0} \times \Gamma_{a_1}$. Since $\Gamma_\theta \equiv SU(2)$ and $\Gamma_a = Z_2$ for a irreducible, the first three types of representations gives rise to a unique gauge equivalence class, whereas the last type of representation gives a copy of $SO(3)$ for each pair of irreducible representations. In §3 we show that all trajectories between these representations are obtained by grafting together existing trajectories from each side. Thus one needs to compute the spectral flow along such trajectories. This is done in the next subsection.

2.2 Spectral flow

Consider irreducible representations $\alpha, \beta \in \mathcal{R}(Y)$ and let $\{a_t\}$ be a 1-parameter family of $SU(2)$-connections on Y joining α to β. Let A be the corresponding connection over $Y \times \mathbf{R}$. Recall that the spectral flow $SF(\alpha, \beta)$ is (modulo 8) the index of the Fredholm operator $D_A = d_A^{*\delta} \oplus d_A^+$ on the weighted Sobolev space with sufficiently small weight δ. Then the Floer grading,

$$\mu(\alpha) \equiv Index D_A(\alpha, \theta) \pmod 8 \tag{2.1}$$

One can consider the calculation of the index of the anti-self-duality operator as a boundary value problem with Atiyah-Patodi-Singer global boundary conditions ([3]). We have

$$Index(d_A^{*\delta} \oplus d_A^+)(\alpha, \beta) = -2\int_{Y \times \mathbf{R}} p_1(A) - \frac{h_\beta + \rho_\beta(0)}{2} + \frac{-h_\alpha + \rho_\alpha(0)}{2} \tag{2.2}$$

where $p_1(A)$ is the Pontryagin form, the term h_β is the sum of the dimensions of $H^i(Y, V_\beta)$, $i = 0, 1$, and ρ_β is the ρ- invariant of the signature operator $\star d_{a_\beta} - d_{a_\beta}\star$ over Y restricted to even forms (cf.[12]). An application of the signature formula to $Y \times I$ shows that $\rho_\alpha = \rho_\alpha(0)$ is independent of the Riemannian metric on Y and is an orientation-preserving diffeomorphism invariant of Y and α.

Lemma 2.2.1 *For $\alpha_i \in \mathcal{R}(Y_i)$ irreducible, we have*

1. $\rho_{\alpha_0 \# \alpha_1}(0) = \rho_{\alpha_0}(0) + \rho_{\alpha_1}(0)$.

2. $h_{\alpha_0\#\alpha_1} = h_{\alpha_0} + h_{\alpha_1} + 3$,

$h_{\alpha_0\#\theta_1} = h_{\alpha_0}$,

$h_{\theta_0\#\alpha_1} = h_{\alpha_1}$,

$h_{\theta_0\#\theta_1} = 3$.

Proof: (1) Consider the cobordism X built by attaching a 1-handle to $(Y_0 \amalg Y_1) \times \{1\}$ in $(Y_0 \amalg Y_1) \times I$. The boundary of X is $Y_0\#Y_1 \amalg -Y_0 \amalg -Y_1$. Note that $\pi_1(X) = \pi_1(Y_0\#Y_1)$. So there are natural inclusions $\mathcal{R}(Y_i) \to \mathcal{R}(Y_0\#Y_1)$ such that the pair (α_0, α_1) can be extended to a unitary representation of $\pi_1(Y_0\#Y_1)$. (In fact, if the α_i are both irreducible, there is an $SO(3)$-family of such extensions.) By Theorem 2.4 in [3], we have

$$\rho_{\alpha_0\#\alpha_1}(Y_0\#Y_1) - \rho_{\alpha_0\#\alpha_1}(Y_0 \amalg Y_1) = 2\text{sign}(X) - \text{sign}_{\alpha_0\#\alpha_1}(X),$$

where $H^2(X) = 0$ and $H^2(X; ad\alpha) = 0$. So we get the signatures satisfying $\text{sign}(X) = 0$, $\text{sign}_{\alpha_0\#\alpha_1}(X) = 0$. Thus $\rho_{\alpha_0\#\alpha_1}(Y_0\#Y_1) = \rho_{\alpha_0}(Y_0) + \rho_{\alpha_1}(Y_1)$.

(2) Since α_0, α_1 are both irreducible, we have the betti numbers $h^0_{\alpha_i} = 0, i = 0, 1$, and similarly $h^0_{\alpha_0\#\alpha_1} = 0$. The Mayer-Vietoris sequence gives:

$$0 \to H^0(S^2, adSU(2)) \to H^1_{\alpha_0\#\alpha_1}(Y_0\#Y_1, adSU(2))$$

$$\to H^1_{\alpha_0}(Y_0, adSU(2)) \oplus H^1_{\alpha_1}(Y_1, adSU(2)) \to 0$$

and so $h_{\alpha_0\#\alpha_1} = h_{\alpha_0} + h_{\alpha_1} + 3$.

Clearly $h_{\theta_0\#\theta_1} = 3$. So we consider the case of θ_0 and α_1 ,where α_1 is irreducible. We have $h^0_{\theta_0\#\alpha_1} = 0, h^1_{\theta_0} = 0$. Again applying the Mayer-Vietoris sequence

$$0 \to H^0_{\theta_0}(Y_0, adSU(2)) \oplus H^0_{\alpha_1}(Y_1, adSU(2)) \to H^0(S^2, adSU(2))$$

$$\to H^1_{\theta_0\#\alpha_1}(Y_0\#Y_1, adSU(2)) \to H^1_{\theta_0}(Y_0, adSU(2)) \oplus H^1_{\alpha_1}(Y_1, adSU(2)) \to 0$$

and using $h^1_{\theta_0} = 0$, we have

$$(3 + 0) - 3 + h^1_{\theta_0\#\alpha_1} - (0 + h^1_{\alpha_1}) = 0$$

i.e. $h_{\theta_0\#\alpha_1} = h_{\alpha_1}$. ∎

Lemma 2.2.2 *For irreducible representations $\alpha_i \in \mathcal{R}(Y_i)$, we have the following addition property for the Floer grading μ:*

$$\mu(\alpha_0\#\alpha_1) = \mu(\alpha_0) + \mu(\alpha_1)$$

$$\mu(\theta_0\#\alpha_1) = \mu(\alpha_1);\ \mu(\alpha_0\#\theta_1) = \mu(\alpha_0).$$

Proof: For computing $\mu(\alpha_i)$ we can use any connections A_i over $Y_i \times \mathbf{R}$ which interpolate between θ_i and α_i. We choose A_i to be flat on the regions $B^3 \times \mathbf{R}$ used to make the connected sum $(Y_0\#Y_1) \times \mathbf{R}$. So the A_i's match to give a connection $A_1\#A_2$ over $(Y_0\#Y_1) \times \mathbf{R}$ which interpolates from $\theta_0\#\theta_1$ to $\alpha_0\#\alpha_1$. By definition, $\mu(\alpha_0\#\alpha_1) = Index D_A(\alpha_0\#\alpha_1, \theta_0\#\theta_1)\text{mod}8$. Then by equation(2.2)

$$\mu(\alpha_0\#\alpha_1) = -2\int_{Y\times\mathbf{R}} p_1(A_1\#A_2) - \frac{h_{\alpha_0\#\alpha_1} - \rho_{\alpha_0\#\alpha_1}}{2} - \frac{h_{\theta_0\#\theta_1} + \rho_{\theta_0\#\theta_1}}{2},$$

where $Y = Y_0\#Y_1$. From our choice of A_i, $p_1(A_1\#A_2) = p_1(A_1)+p_1(A_2)$. Since $\rho_{\theta_0\#\theta_1} = 0$ and $\rho_{\theta_i} = 0$, our result follows from Lemma 2.2.1. Similarly one checks that $\mu(\theta_0\#\alpha_1) = \mu(\alpha_1)$ and $\mu(\alpha_0\#\theta_1) = \mu(\alpha_0)$. ■

Similarly one shows:

Proposition 2.2.3 *For all $\beta_i \in \mathcal{R}(Y_i)$ and $\alpha_i \in \mathcal{R}^*(Y_i)$*

$$Index D_A(\alpha_0\#\alpha_1, \beta_0\#\beta_1) = Index D_A(\alpha_0, \beta_0) + Index D_A(\alpha_1, \beta_1) + 3. \tag{2.3}$$

Theorem 2.2.4 *(Fintushel, Stern [12]) Let $\mathcal{R}_\alpha$ be a connected component of $\mathcal{R}(Y)$. Suppose that $\mathcal{R}_\alpha$ is a manifold, that $\star d_a$ is normally nondegenerate on $\mathcal{R}_\alpha$, and let $g : \mathcal{R}_\alpha \to \mathbf{R}$ be a Morse function. Then the critical points of g are basis elements of the instanton chain complex. Such a critical point b has grading*

$$\mu(b) = \mu(\mathcal{R}_\alpha) - \mu_g(b) \tag{2.4}$$

where $\mu_g(\alpha)$ is the Morse index of b relative to g.

We end of this section by giving the following remark which we will use to do calculations in §3 and §4.

Remark: If $1 = \mu(\alpha_0 \# \alpha_1) - \mu(\beta_0 \# \beta_1) = (\mu(\alpha_0) - \mu(\beta_0)) + (\mu(\alpha_1) - \mu(\beta_1))$, one gets that either $\mu(\alpha_0) - \mu(\beta_0) = 0$ or $= 1$. This means that if $A_i (i = 0, 1)$ is an anti-self-dual connection interpolating from α_i to β_i then one of the A_i is a constant flat anti-self-dual connection on Y_i ($A_i(t) = \alpha_i \in \mathcal{R}(Y_i)$ for all $t \in \mathbf{R}$) and the other A_j lives in a 1-dimensional moduli space $\mathcal{M}^1_{Y_j}(\alpha_j, \beta_j)$.

Chapter 3

Grafting

The essential step in the calculation of the Floer homology of a connected sum of homology 3-spheres Y_0, Y_1 is in understanding the structure of the 1-dimensional moduli space of anti-self-dual connections on $(Y_0 \# Y_1) \times \mathbf{R}$. This relies on grafting together anti-self-dual connections on noncompact 4-manifolds. The major problem is the existence of harmonic 2-forms in the gluing region. The difficult point is obtaining estimates on the overlap relating the "merged" metric with the original metrics g_i on Y_i. For the merged metric g we will take a weighted average. The usual Rayleigh quotient for first eigenvalue involves the d^{*g} operator, and in order to get a uniform bound on the first eigenvalue on the connected sum from one on each side, we have to compare d^{*g} and d^{*g_i}. These operators involve the derivative term of the weighted average with no control for gluing parameter ε(the neck-length). Thus we adopt Donaldson and Sullivan's technique for building a right inverse directly (cf. [11]).

We begin by looking at a special feature of the **R**-action on the equivalence classes of connections which will give us a particular way of solving the anti-self-duality equation

$$F_A^+ + (d_A^+ + d_A^{*g})a + a \wedge a = 0$$

uniquely on the subspace of $\Omega^1_{ad}(Y \times \mathbf{R})$, which is perpendicular to H^1_A. Then we show that for all balanced 1-dimensional self-dual connections on a single homology 3-sphere $\times \mathbf{R}$ there is a uniform lower eigenvalue. Using the parametric method to construct the right inverse on the connected sum and applying the inverse function theorem, we are able to prove a gluing and splitting theorem for 1-dimensional anti-self-dual connections over $(Y_0 \# Y_1) \times \mathbf{R}$.

Throughout this section we assume that the anti-self-duality operator is regular. (As we have mentioned above, this can always be achieved by a compact perturbation of the anti-self-duality operator. For the sake of simplicity we shall ignore the perturbation.)

3.1 Properties of balanced connections

Let Y be a closed, connected, oriented, smooth homology 3-sphere. For $\delta \geq 0$ (to be determined), let $e_\delta : Y \times \mathbf{R} \to \mathbf{R}$ be a smooth positive function with $e_\delta(y,t) = e^{\delta|t|}$ for $|t| \geq 1$. Let E be an $SU(2)$-vector bundle over $Y \times \mathbf{R}$ with a translationally invariant metric and metric-preserving connection. Then following [15], [18], [19], and [29], we define the weighted Sobolev space $L^p_{k,\delta}$ on sections ξ of E to be the L^p_k Sobolev space of $e_\delta \cdot \xi$. To define Banach manifolds $\mathcal{B}(a,b)$ of paths connecting a and b in $\mathcal{B}_Y$ (the $L^2_1 - SU(2)$ connections over Y modulo L^2_2-gauge equivalence), choose any smooth representatives of $a, b \in \mathcal{A}_Y$ and a connection C (as below) on $Y \times \mathbf{R}$ which coincides with a for $t \leq -1$ and with b for $t \geq 1$. Then

$$\mathcal{A}_\delta(a,b) = C + L^p_{1,\delta}(\Omega^1_{ad}(Y \times \mathbf{R}))$$

is an affine space and is independent of the choice of C. The corresponding gauge group is:

$$\begin{aligned} \mathcal{G}_\delta \;=\; & \{g \in L^p_{2,loc}(Y \times \mathbf{R}, SU(2)) \mid \text{there exists } T > 0, \\ & \xi \in L^p_{2,\delta}(\Omega^0_{ad}(Y \times \mathbf{R})) \text{ such that } g = \exp \xi \text{ for } |t| \geq T\}. \end{aligned}$$

We need $p > 2$ to construct the orbit space $\mathcal{B}_{Y \times \mathbf{R}} = \mathcal{A}^{1,p}_\delta / \mathcal{G}^{2,p}_\delta$.

Proposition 3.1.1 *1. Let*

$$D_a : L^p_{1,\delta}(\Omega^1 \oplus \Omega^0)(Y, adSU(2)) \to L^p_{0,\delta}(\Omega^1 \oplus \Omega^0)(Y, adSU(2))$$

be the operator $D_a(\alpha,\beta) = (\star d_a \alpha - d_a \beta, -d_a^* \alpha)$. There exists a positive λ_0 such that for all $a \in \mathcal{R}^*(Y)$ the eigenvalues of D_a satisfy $|\lambda(D_a)| \geq \lambda_0$.

2. If $F(A)$ is in L^p for $p \geq 2$, then there is a constant C_A such that

$$\sup |F_A|_{y,t} \leq C_A e^{-\gamma |t|}.$$

where $\gamma = \gamma(\lambda_0) > 0$, and C_A is continuous in A.

Proof: The first is from [15], and the second is in [10] (see 4.1). ∎

Choose a positive $\delta < \min\{\lambda_0, \frac{\gamma}{2}\}$ and a finite action connection C over $Y \times \mathbf{R}$ with limiting values a, b at $Y \times \{\pm\infty\}$, and use it to define the $L^p_{k,\delta}$ norm as above. Let us denote $\|u\|_{L^p_{1,\delta}(A)} = \|\nabla_A u\|_{L^p_{0,\delta}} + \|u\|_{L^p_{0,\delta}}$ (and $\|u\|_{L^p_{1,\delta}} = \|u\|_{L^p_{1,\delta}(C)}$).

Definition 3.1.2 : *The balancing function $b : \mathcal{B}_{Y\times\mathbf{R}} \to \mathbf{R}$ is given by the equation:*

$$\int_{-\infty}^{b(A)} \|F(A)\|^2_{L^2(Y)} = \int_{b(A)}^{\infty} \|F(A)\|^2_{L^2(Y)}.$$

(So the value $b(A)$ is the time which splits the action of A in half.)

Lemma 3.1.3 *1. Shifting the connection A in the t-direction, $A(t) \to A(t \pm s)$, one has*

$$b(A(t+s)) = b(A(t)) - s, \qquad b(A(t-s)) = b(A(t)) + s$$

2. Let $\mathcal{B}_0 = b^{-1}(0)$ be the space of equivalence classes of connections whose action is balanced at 0. Then there is a one-to-one map from $\mathcal{B}_0$ to $\mathcal{B}_s = b^{-1}(s)$ for any $s \in \mathbf{R}$.

3. If A is not a constant flat connection, the derivative of b is

$$D_A b(a) = \int_{-\infty}^{+\infty} < \frac{sign(t - b(a))}{\|F(A)\|^2_{L^2(Y\times\mathbf{R})}} d_A^* F_A, a > .$$

Proof: (1) is proved by a change of variable. (2) follows from (1). For (3):

$$\int_{-\infty}^{b(A+sa)} \|F(A+sa)\|^2_{L^2(Y)} = \int_{b(A+sa)}^{+\infty} \|F(A+sa)\|^2_{L^2(Y)}$$

Taking the derivative with respect to s at $s = 0$ and combining the terms, one has

$$\|F(A)\|^2_{L^2(Y\times\mathbf{R})} D_A b(a) = \int_{b(A)}^{+\infty} < d_A^* F_A, a > - \int_{-\infty}^{b(A)} < d_A^* F_A, a > .$$

Now $\|F(A)\|^2_{L^2(Y\times\mathbf{R})} = 0$ if and only if $-\frac{\partial a}{\partial t}dt + F_a = 0$, i.e. if and only if A is a constant flat connection, contrary to our hypothesis. Thus (3) follows. ■

Definition 3.1.4 *Set the balanced moduli space* $\mathcal{M}^{bal}_{Y\times\mathbf{R}} = \{A \in \mathcal{M}_{Y\times\mathbf{R}} \subset \mathcal{B}_{Y\times\mathbf{R}} \mid \ b(A) = 0\}$.

Lemma 3.1.5 *For* $A \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}, y \in Y$, *and each* $p \geq 2$, *there exist constants* M_0, C_1, C_2 *independent of* A *such that*

(i) If $\dim\mathcal{M}_{Y\times\mathbf{R}} \leq 1$, *then* $\mathcal{M}^{bal}_{Y\times\mathbf{R}}$ *is compact, and*

$$\int_{Y\times\mathbf{R}} e^{p\delta|t|}|F_A|^p < C_1 \qquad \int_{B^3_y(\epsilon)\times\mathbf{R}} e^{p\delta|t|}|F_A|^p < C_2\epsilon^3.$$

(ii) If $\dim\mathcal{M}_{Y\times\mathbf{R}} < 8$, *then* $\|F_A\|_{L^\infty(Y\times\mathbf{R})} \leq M_0$.

Proof: (i) No sequence of connections in $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ can converge weakly to a limit plus an instanton bubble, since bubbling needs $\dim\mathcal{M}_{Y\times\mathbf{R}} \geq 8$. The only other way a sequence in $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ can fail to have a convergent subsequence is for there to exist a subsequence $\{A_n\}$ limiting weakly to a disjoint union of connections $A_{-\infty} \in \mathcal{M}_{Y\times\mathbf{R}}(a,b), A_0 \in \mathcal{M}_{Y\times\mathbf{R}}(b,c), A_{+\infty} \in \mathcal{M}_{Y\times\mathbf{R}}(c,d)$ where a,b,c,d denote limiting values and at least one of of $A_{-\infty}$, $A_{+\infty}$ is not constant flat (otherwise $\{A_n\}$ actually converges to A_0). If, say, $A_{+\infty}$ is not constant flat then $\dim\mathcal{M}_{Y\times\mathbf{R}}(c,d) \geq 1$. Since each A_n is balanced, the limit, $A_{-\infty} \amalg A_0 \amalg A_{+\infty}$ is also balanced, and it follows that $\dim\mathcal{M}_{Y\times\mathbf{R}}(a,b) + \dim\mathcal{M}_{Y\times\mathbf{R}}(b,c) \geq 1$. This is impossible since the dimension of the moduli space $\mathcal{M}_{Y\times\mathbf{R}}(a,d)$ which contains the A_n is equal to 1. Thus $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ is compact.

There exists a constant C independent of A such that $C_A \leq C$ for all $A \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ from compactness where C_A is the constant in Proposition 3.1.1(2). The inequalities follow from a straightforward calculation by using $\sup|F_A| \leq Ce^{-\gamma|t|}$.

(ii) Suppose not. Then there exists a sequence $\{A_n\} \in \mathcal{M}_{Y\times\mathbf{R}}$ with $\|F_{A_n}\|_{L^\infty(Y\times\mathbf{R})} > n$. Thus we have (y_n, t_n) such that $|F_{A_n}|_{(y_n,t_n)} = n$. Let $A'_n = A_n(t - t_n)$ (rescaling). So $|F_{A'_n}|_{(y_n,0)} = n$. Applying Uhlenbeck's compactness theorem on the compact space $Y \times [-1,1]$ shows that there exists a subsequence $\{A_i\}$ with a bubble point, and this requires $\dim\mathcal{M}_{Y\times\mathbf{R}} \geq 8$, contradicting our assumption. ■

Remark: For any $A \in \mathcal{A}_\delta(a,b)$, there is a positive constant $M_{(C,A)}$ such that

$$M^{-1}_{(C,A)}\|u\|_{L^p_{1,\delta}(A)} \leq \|u\|_{L^p_{1,\delta}} \leq M_{(C,A)}\|u\|_{L^p_{1,\delta}(A)}.$$

If $A \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ and $\dim\mathcal{M}_{Y\times\mathbf{R}} \leq 1$, then $M_{(C,A)} \leq C_4$ where C_4 is a constant independent of A from Lemma 3.1.5.

Proposition 3.1.6 *The space $\mathcal{B}^{bal}_{Y\times\mathbf{R}} = \{A \in \mathcal{B}_{Y\times\mathbf{R}} |\ b(A) = 0\}$ of balanced connections is a smooth manifold with codimension 1 and the moduli space $\mathcal{M}^1_{Y\times\mathbf{R}}$ is transversal to $\mathcal{B}^{bal}_{Y\times\mathbf{R}}$.*

Proof: Since an arbitrary $A' \in \mathcal{M}^1_{Y\times\mathbf{R}}$ is not a constant flat connection, it has a translate A under the $\mathbf{R}$-action which lies in $\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$. Note that $\|F_A\|^2 \neq 0$. Let $A = a(t)$, then if $0 = d^*_A F_A = -(d_a\star\frac{\partial a}{\partial t} + \frac{\partial\star F_a}{\partial t}) \wedge dt + d_a\star F_a$ we get $\star F_a = 0$. Since A is anti-self-dual $\frac{\partial a}{\partial t} = \star F_a = 0$ and this A is constant flat connection. But this is not true, so the normal vector

$$\frac{sign(t)\cdot d^*_A F_A}{\|F_A\|^2}$$

to $T\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ at A is nontrivial. By the implicit function theorem for Banach spaces, we have that $\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} = b^{-1}(0)$ is a smooth codimension 1 Banach submanifold, and moreover $D_A b : T_A N \to T_0\mathbf{R}$ is an isomorphism where $T_A N$ is the subspace of $T_A\mathcal{B}_{Y\times\mathbf{R}}$ spanned by this nontrivial normal vector. Notice that derivative of b along $\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ is zero. We may consider

$$T_A\mathcal{B}_{Y\times\mathbf{R}} \cong T_A\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} \times T_A N.$$

Since $\mathcal{B}_{Y\times\mathbf{R}} \cong \mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} \times \mathbf{R}$ and $D_t b(A) = \pm Id$ in the time direction, we may identify $T_A N \cong T_0\mathbf{R} \cong (T_A\mathcal{B}_{Y\times\mathbf{R}})_t$ the tangent space to $\mathcal{B}_{Y\times\mathbf{R}}$ at A in the time direction.

For $A \in \mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, the cohomology H^1_A is a 1-dimensional space. We claim that it contains $\{A(t+s) : s \in \mathbf{R}\}$. We have

$$H^1_A = \{A(t) + sa(t) : s \in \mathbf{R}, d^{*\delta}_A a = 0, d^+_A a = 0\}.$$

Define $f(s,u) = A(t)+sa(t)-A(t-u)$. Then $f(0,0) = 0$, and $\frac{\partial f}{\partial u}(0,0) = A'(t) \neq 0$, since A is not a constant connection. Hence the implicit function theorem gives a local coordinate $u = u(s)$ in a neighborhood of $(0,0)$ such that $f(s,u(s)) = 0$. I.e. $A(t)+sa(t) = A(t-u(s))$ in time-translation form. Let S be the subset of $\mathbf{R}$ defined by

$$S = \{s \in \mathbf{R} : \text{there exists } u(s) \text{ such that } f(s,u(s)) = 0\}.$$

Then S is nonempty (since it contains 0), open (by the implicit function theorem) and closed (since $f(s,u(s))$ is continuous in s). Therefore $S = \mathbf{R}$, and so $H^1_A = \{A(t+s) : s \in \mathbf{R}\}$. Hence H^1_A intersects $T_A\mathcal{B}^{\text{bal}}_{Y\times\mathbf{R}}$ transversely in the point $\{[A]\}$. The Kuranishi technique then implies that locally, solutions of the anti-self-duality equation live in a 1-dimensional moduli space parameterized by H^1_A, i.e. by time-translation. ■

3.2 Smallest eigenvalue on $Y \times \mathbf{R}$

(i) Some analytical facts

Let d_A denote the covariant derivative corresponding to the connection A and $d_A^{*\delta} = e_\delta^{-1} d_A^* e_\delta$ be the adjoint of d_A with respect to the $L^2_{0,\delta}$-norm. Floer has proved the following in [15].

Proposition 3.2.1 (Floer) (i) *For positive δ, $\mathcal{G}_\delta$ is a Banach Lie group with Lie algebra (which can be identified with) $L^p_{2,\delta}(\Omega^0_{ad}(Y \times \mathbf{R}))$.*

(ii) *The quotient space $\mathcal{B}_\delta(a,b) = \mathcal{A}^*_\delta(a,b)/\mathcal{G}_\delta$ is a smooth Banach manifold with tangent spaces*

$$T_{[A]}\mathcal{B}_\delta(a,b) = \{\alpha \in L^p_{1,\delta}(\Omega^1_{ad}(Y \times \mathbf{R})) \mid d_A^{*\delta}\alpha = 0\}.$$

(iii) *The 2-form F_A^- representing the anti-self-dual part of the curvature of A is smooth and $\mathcal{G}_\delta$-equivariant.*

(iv) *If $\delta > 0$ is smaller than the smallest nonzero absolute value of an eigenvalue of D_a or D_b, then for any anti-self-dual connection $A \in \mathcal{B}_\delta(a,b)$ the anti-self-duality operator*

$$D_A^\delta = d_A^{*\delta} \oplus d_A^+ : L^p_{1,\delta}\Omega^1_{ad}(Y \times \mathbf{R}) \to L^p_{0,\delta}(\Omega^0_{ad} \oplus \Omega^2_{ad,+})(Y \times \mathbf{R})$$

is Fredholm. Furthermore, $D^\delta_A = \frac{\partial}{\partial t} + D^\delta_{a_t}$ *where*

$$D^\delta_{a_t} = \begin{pmatrix} \star d_a & -d_a \\ -d_a^* & \delta \end{pmatrix}$$

self-adjoint on $\Omega^1_{ad}(Y) \oplus \Omega^0_{ad}(Y)$ *where* $\star$ *is the Hodge operator on the 3-manifold* Y*. If* a *and* b *are irreducible nondegenerate flat connections, then one can take* $\delta = 0$*.*

(v) *Let* $\mathcal{M}$ *be the moduli space of all equivalence classes of nonflat anti-self-dual connections* A *on* $Y \times \mathbf{R}$ *whose action* $\|\frac{\partial A}{\partial t}\|_2^2$ *is finite. There is a first category set of metrics on* Y *such that the anti-self-duality operator* D^δ_A *is surjective for all* $A \in \mathcal{M} \cap \mathcal{B}_\delta$*.*

Remark: Proposition 3.2.1(v) implies that $(D^\delta_A)^*$ has trivial kernel. From the ellipticity of the anti-self-duality operator we have

$$C_A \|v \oplus u\|_{L^p_{1,\delta}} \leq \|(D^\delta_A)^*(v \oplus u)\|_{L^p_{0,\delta}}$$

for $v \oplus u \in (\Omega^0_{ad} \oplus \Omega^2_{ad,+})(Y \times \mathbf{R})$. Thus, for any such $A \in \mathcal{M}^k_{Y\times\mathbf{R}} \cap \mathcal{B}_\delta$, by taking $p = 2, v = 0$, there is a positive real number $C(A)$ such that

$$C(A) \cdot \int_{Y\times\mathbf{R}} e^{2\delta\cdot|t|}\,|u|^2 \leq \int_{Y\times\mathbf{R}} e^{2\delta\cdot|t|}\,|d_A^{*\delta} u|^2 \tag{3.1}$$

for all $u \in \Omega^2_{ad,+}(Y \times \mathbf{R})$.

The following definitions are combined from [9], [10], and [15].

Definition 3.2.2 : *An ideal anti-self-dual connection (trajectory) over* $Y \times \mathbf{R}$*, of Chern number* k*, is a pair*

$$(A; (x_1, ..., x_l)) \in \mathcal{M}^{k-l}_{Y\times\mathbf{R}}(a,b) \times S^l(Y \times \mathbf{R})$$

where A *is a point in* $\mathcal{M}^{k-l}_{Y\times\mathbf{R}}(a,b) \cap \mathcal{B}_\delta$ *and* $(x_1, ..., x_l)$ *is a multiset of degree* l *(unordered* l*-tuple) of points of* $Y \times \mathbf{R}$*.*

Let $\{A_n\}, n \in \mathbf{N}$, be a sequence of connections of charge k on the $SU(2)$ bundle P over $Y \times \mathbf{R}$. We say that the gauge equivalence classes $\{A_n\}$ *converge weakly* to a limiting ideal anti-self-dual connection $(A; (x_1, ..., x_l))$ if

(i) The action densities converges as measures, i.e. for any continuous function on $Y \times \mathbf{R}$,

$$\int_{Y\times\mathbf{R}} f|F(A_n)|^2 d\mu \to \int_{Y\times\mathbf{R}} f|F(A)|^2 d\mu + 8\pi^2 \sum_{i=1}^{l} f(x_i).$$

(ii) there are bundle maps

$$\rho_n : P|_{Y\times\mathbf{R}\setminus\{x_1,\ldots,x_l\}} \to P|_{Y\times\mathbf{R}\setminus\{x_1,\ldots,x_l\}}$$

such that $\rho_n^*(A_n)$ converges to A in C^∞ on compact subsets of the punctured manifold.

Definition 3.2.3 : *Let a and b be flat $SU(2)$ connections over Y. A chain of connections $(B_1, \ldots , B_n)$ from a to b is a finite set of connections over $Y \times \mathbf{R}$ which limit to flat connections c_{i-1}, c_i as $t \to \mp\infty$ such that $a = c_0$, $c_n = b$, and B_i connects c_{i-1}, c_i for $0 \leq i \leq n$.*

We say that the sequence $\{A_\alpha\} \in \mathcal{M}^k_{Y\times\mathbf{R}}(a, b)$ is *(weakly) convergent* to the chain of connections $(B_1, \ldots, B_n)$ if there is a sequence of n-tuples of real numbers $\{t_{\alpha,1} \leq \ldots \leq t_{\alpha,n}\}_\alpha$, such that $t_{\alpha,i} - t_{\alpha,i-1} \to \infty$ as $\alpha \to \infty$, and if, for each i, the translates $t^*_{\alpha,i} A_\alpha = A_\alpha(\circ - t_{\alpha,i})$ converge weakly to B_i.

We need to combine the notion of chain connection with the notion of an ideal connection.

Definition 3.2.4 : *An ideal chain connection joining flat connections a and b over Y is a set*

$$(A_j; x_{j1}, \ldots, x_{jl_j})_{1\leq j\leq J}$$

where $(A_j)_{1\leq j\leq J}$ is a chain connection and for each j, $(A_j; x_{j1}, \ldots, x_{jl_j})$ is an ideal connection.

In this set-up, there is a version of the Uhlenbeck compactness theorem. We state it in a form proved by Floer in [15].

Theorem 3.2.5 (Uhlenbeck compactness on $Y \times \mathbf{R}$) *Let $A_\alpha \in \mathcal{M}^k_{Y\times\mathbf{R}} \cap \mathcal{B}_\delta(a_\alpha, b_\alpha)$ be a sequence of anti-self-dual connections with uniformly bounded action. Then there exists a subsequence converging to an ideal chain connection*

$$(A_j; x_{j1}, \ldots, x_{jl_j})_{1\le j\le J}.$$

Moreover, one has

$$\sum_{j=1}^{J}(k_j + l_j) = k, \qquad c_2(A_j) = k_j \text{ (not necessarily an integer).}$$

(For more discussion and details, we refer the reader to [10] and [15].)

(ii) Smallest eigenvalue estimates

(a) We want to prove the existence of a uniform lower bound for the eigenvalues of $\Delta^{2,+}_A$ for all balanced 1-dimensional anti-self-dual connections A over $Y \times \mathbf{R}$ which are asymptotically flat at the ends.

Theorem 3.2.6 *Suppose $\dim \mathcal{M}_{Y\times\mathbf{R}} = 1$. Then there exists a positive constant C such that for all $A \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}$, and for all $p \ge 2, u \in L^p_{0,\delta}(\Omega^2_{ad,+}(Y\times\mathbf{R}))$ we have*

$$C \cdot \int_{Y\times\mathbf{R}} e^{p\delta\cdot|t|}\,|u|^p \le \int_{Y\times\mathbf{R}} e^{p\delta\cdot|t|}\,|d^{*\delta}_A u|^p.$$

Proof: For $p = 2$, the result follows from inequality 3.1 and Lemma 3.1.5. For $p > 2$, we use the inequality in the remark after Proposition 3.2.1 for $v = 0$. The constant C_A is continuous in A. Hence the result follows by using Lemma 3.1.5. ■

Remark: The above estimate also holds when A is the trivial connection. On a fixed homology 3-sphere (i.e. with a fixed Riemannian metric) the standard Laplacian on the self-dual 2-forms has a strictly positive first eigenvalue by the Hodge Theorem (see Chapter 5). We can use this to get the bounded right inverse for d^+. This will allow us to glue one side 1-dimensional trajectory flow together with trivial connection on the other side.

(b) The flattening construction: We first describe a special gauge suited to our constructions. Fix $A \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, and choose a trivialization of the fiber at a base point $y \in Y$. Parallel transport first along the $\mathbf{R}$-direction, and then outward in normal coordinates in Y at each fixed time slice. This defines a gauge for $A \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ which we call the *cylindrical gauge*. In this gauge $A_t = 0$ on $\{y\} \times \mathbf{R}$, and $A_r = 0$ where r is the radius on Y centered at y.

Lemma 3.2.7 *In the cylindrical gauge in $B^3_y(\varepsilon) \times \mathbf{R}$, we have $|A(x,t)| \leq r\|F_A\|_\infty$.*

Proof: Let (x_1, x_2, x_3, t) be coordinates in $B^3_y(\varepsilon) \times \mathbf{R}$. For $1 \leq i \leq 3$, we have $|A_i(x,t)| \leq \frac{r}{2}\max_{|(x,t)|<r}|F(x,t)|$ (c.f. [31]). Since $A_r = \sum_{k=1}^3 x_k A_k = 0$ we get $\sum_{k=1}^3 x_k \frac{\partial A_k}{\partial t} = 0$, thus $\sum_{k=1}^3 x_k F_{kt} = r\frac{\partial}{\partial r}A_t - \sum_{k=1}^3 x_k \frac{\partial A_k}{\partial t} = r\frac{\partial}{\partial r}A_t$. Also $\int_0^r \frac{\partial}{\partial r}A_t = A_t(x,t) - A_t(y,t) = A_t(x,t)$. Thus

$$|A_t(x,t)| \leq |\int_0^r \sum_{k=1}^3 \frac{x_k}{r} F_k t| \leq r\max_{|(x,t)|<r}|F(x,t)|.$$

■

We next need to describe how to flatten a connection $A \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ along $B_y(r_0) \times \mathbf{R}$. Let $\chi = \chi(r_0, \varepsilon)$ be a smooth cutoff function satisfying

$$\chi \equiv 0 \quad \text{on } B_y(r_0), \quad \chi \equiv 1 \quad \text{on } Y \setminus B_y(r_0 + \varepsilon) \ \text{ and } \ |d\chi| \leq \frac{C_0}{\varepsilon}$$

for some constant C_0.

Definition 3.2.8 *For $A \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}$ define $\tilde{A} \in \mathcal{B}_{Y\times\mathbf{R}}$ to be the connection on E which is equal to A outside $B_y(r_0+\varepsilon)$ and on $B_y(r_0+\varepsilon)$ is $\tilde{A} = \chi \cdot A$ as connection matrix in the local trivialization of E given by the cylindrical gauge.*

Lemma 3.2.9 *There exist ε_0 and C (independent of A) such that for $0 < \varepsilon < \varepsilon_0$ and any $A \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}$ with $\dim\mathcal{M}_{Y\times\mathbf{R}} \leq 1$ and any $p, q \geq 2$*

$$\|\tilde{A} - A\|_{L^q_{0,\delta}(Y\times\mathbf{R})} \leq C\varepsilon^{\frac{3+q}{q}}, \qquad \|F^+_{\tilde{A}}\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \leq C\varepsilon^{\frac{3}{p}}$$

Proof: Take $\chi = \chi(\varepsilon, \varepsilon)$ and $\tilde{A} = \chi \cdot A$, we have $F^+_{\tilde{A}} = (d\chi \wedge A)_+ + (\chi^2 - \chi)(A \wedge A)_+$ since A is anti-self-dual. This has support on $B^3(2\varepsilon) \times \mathbf{R}$, and using Lemma 3.2.7 and Proposition 3.1.1, we have the pointwise bound

$$|F^+_{\tilde{A}}| \le C_0\varepsilon^{-1}|A| + |A|^2 \le C_0\varepsilon^{-1}2\varepsilon|F_A| + 4\varepsilon^2|F_A|^2 \le C_0'|F_A| \le C_0'Ce^{-\gamma|t|}$$

where from Lemma 3.1.5 $C_0'C$ is independent of A. Hence

$$\|F^+_{\tilde{A}}\|_{L^p_{0,\delta}(Y\times\mathbf{R})} = \left(\int_{B^3(2\varepsilon)\times\mathbf{R}} |e^{\delta|t|}F^+_{\tilde{A}}|^p\right)^{\frac{1}{p}} \le C_2(\delta)\varepsilon^{\frac{3}{p}}.$$

The bound on $\tilde{A} - A$ is similar, $|\tilde{A} - A| \le |A| \le 2\varepsilon|F_A| \le C\varepsilon e^{-\gamma|t|}$. Thus the result follows. ∎

(c) **The Neighborhood of $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$** : Assume throughout this subsection that dimension of moduli space $\dim\mathcal{M}_{Y\times\mathbf{R}} \le 1$. Fix $p, q > 2$. We are going to show that the uniform lower eigenvalue estimate also holds for nearby anti-self-dual connections.

Definition 3.2.10 : *Set*

$$U_{\delta_1} = \{B \in \mathcal{B}_{Y\times\mathbf{R}} | \text{there exists a } A \in \mathcal{M}^{bal}_{Y\times\mathbf{R}} \text{ such that} \|A - B\|_{L^q_{0,\delta}} < \delta_1, \|F^+_B\|_{L^p_{0,\delta}} < \delta_1\}$$

Note that Lemma 3.2.9 implies that if $A \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$, then for sufficiently small ε the flattened connection $\tilde{A}$ lies in U_{δ_1}.

Lemma 3.2.11 *There exists δ_0 such that for $0 < \delta_1 < \delta_0$ there is a C_5 independent of δ_1 such that*

$$\|u\|_{L^p_{1,\delta}(Y\times\mathbf{R})} \le C_5\|(d^+_B)^{*_\delta}u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \quad \text{for all } B \in U_{\delta_1}$$

Proof: $\|(d^+_B)^{*_\delta}u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \ge \|(d^+_A)^{*_\delta}u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} - \|(A - B) *_\delta u\|_{L^p_{0,\delta}(Y\times\mathbf{R})}$ where A is an element in $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$ which is δ_1-close to B.

$$\|(A - B) *_\delta u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \le \|A - B\|_{L^q_{0,\delta/2}}\|u\|_{L^4_{0,\delta/2}} \le C\delta_1\|u\|_{L^p_{1,\delta}}$$

by Hölder's inequality and the weighted Sobolev embedding theorem [19].

Since A is anti-self-dual, the Weitzenböck formula gives $d_A^+(d_A^+)^{*_\delta} = \nabla_A^{*_\delta}\nabla_A + R$, which implies that

$$\|u\|_{L^p_{1,\delta}} \leq C_4\|u\|_{L^p_{1,\delta}(A)} \leq C_4C(p)\|(d_A^-)^{*_\delta}u\|_{L^p_{0,\delta}} + C\|u\|_{L^p_{0,\delta}} \leq \tilde{C}\|(d_A^+)^{*_\delta}u\|_{L^p_{0,\delta}} \tag{3.2}$$

The first inequality is from the remark after Lemma 3.1.5, and the last from Theorem 3.2.6. Choosing δ_0 such that $C\tilde{C}\delta_0 < \frac{1}{2}$, we have

$$\|(d_B^+)^{*_\delta}u\|_{L^p_{0,\delta}} \geq \frac{1}{2}\|(d_A^+)^{*_\delta}u\|_{L^p_{0,\delta}} \tag{3.3}$$

Thus from (3.2) and (3.3), we have

$$\|u\|_{L^p_{1,\delta}} \leq \tilde{C}\|(d_A^+)^{*_\delta}u\|_{L^p_{0,\delta}} \leq 2\tilde{C}\|(d_B^+)^{*_\delta}u\|_{L^p_{0,\delta}}$$

∎

From Lemma 3.2.11 and the weighted Sobolev embedding theorem $L^p_{1,\delta} \hookrightarrow L^q_{0,\delta}$, for $\frac{1}{4} + \frac{1}{q} \geq \frac{1}{p}$, the bounded right inverse operator $Q_B(= (d_B^+)^{*_\delta}(d_B^+(d_B^+)^{*_\delta})^{-1})$ satisfies

$$\|Q_Bu\|_{L^q_{0,\delta}} \leq C\|Q_Bu\|_{L^p_{1,\delta}} \leq C\|u\|_{L^p_{0,\delta}} \quad \text{for all } B \in U_{\delta_1}.$$

(d) **Changing metrics** : We want to show that there is also bounded right inverse for flattened connections with metric C^0 close to the original metric. Pick a point $y_0 \in Y_0$. For simplicity we assume that the metric on Y_0 is flat in the 3-ball $B_3(r_0+\varepsilon)$ centered at y_0 with radius $r_0+\varepsilon$. For $r_1 < r_0$, let $N_{\varepsilon',r_0,r_1}(g_0)$ be the set of Riemannian metrics g on $Y_0 \setminus B_3(r_1)$ which satisfy

(i) $g = g_0$ on $Y_0 \setminus B_3(r_0)$;

(ii) $\|g - g_0\|_{C^0} < \varepsilon'$ on $B_3(r_0) \setminus B_3(r_1)$.

The annulus $B_3(r_0) \setminus B_3(r_1)$ will be used as the gluing region in forming connected sums.

(1) Let π_+^g be the projection onto self-dual 2-forms with respect to the metric g. Note that π_+^g is a continuous map with respect to the metrics, i.e. $\|\pi_+^g - \pi_+^{g_0}\| \leq C\|g - g_0\|_{C^0}$.

(2) For the metric g_0 on Y_0, there is right inverse Q_0 for the operator $d_{A_0}^{+g_0}$. Let $S = d_{A_0}^{-g_0} Q_0$. Then

$$d_{A_0}^{+g_0} Q_0 u_0 = u_0, \quad \|Su_0\|_{L^p_{0,\delta}(g_0)} \leq C_p \|u_0\|_{L^p_{0,\delta}(g_0)}.$$

where $\|\cdot\|_{L^p_{0,\delta}(g_0)}$ indicates the Sobolev space with metric g_0 for forms with support in $(Y_0 \setminus B_3(r_1)) \times \mathbf{R}$.

(3) For $g \in N_{\varepsilon', r_0, r_1}(g_0)$, the $L^p_{0,\delta}$-norms are equivalent, i.e.

$$C_{\varepsilon'}^{-1} \|u_0\|_{L^p_{0,\delta}(g_0)} \leq \|u_0\|_{L^p_{0,\delta}(g)} \leq C_{\varepsilon'} \|u_0\|_{L^p_{0,\delta}(g_0)}$$

where $C_{\varepsilon'} \to 1$ as $\varepsilon' \to 0$.

Lemma 3.2.12 *For self-dual 2-forms u_0 with support in the $(Y_0 \setminus B_3(r_1)) \times \mathbf{R}$ and $g \in N_{\varepsilon', r_0, r_1}(g_0)$ with sufficiently small ε', $d_{A_0}^{+g}$ has right inverse Q_0^g with*

$$\|Q_0^g u\|_{L^p_{1,\delta}(g)} \leq C \|u\|_{L^p_{0,\delta}(g)}.$$

Also, $\|Q_0^g u\|_{L^q_{0,\delta}(g)} \leq C \|u\|_{L^p_{0,\delta}(g)}$ for $\frac{1}{4} + \frac{1}{q} \geq \frac{1}{p}$.

Proof: We will construct the right inverse by arranging that $d_{A_0}^{+g} Q_0 - Id$ is a contraction mapping on $L^p_{0,\delta}(g)(\Omega^2_+(Y_0 \times \mathbf{R}))$. We have $d_{A_0}^{+g} Q_0 u_0 = d_{A_0}^{+g_0} Q_0 u_0 + (d_{A_0}^{+g} - d_{A_0}^{+g_0}) Q_0 u_0$ and from (2)

$$(d_{A_0}^{+g} Q_0 - Id) u_0 = (d_{A_0}^{+g} - d_{A_0}^{+g_0}) Q_0 u_0.$$

By the definitions of g and the flattening construction for A_0 with $\chi|_{[0,r_0]} \equiv 0$, one has

$$d_{A_0}^{+g} - d_{A_0}^{+g_0} = d^{+g} - d^{+g_0} = (\pi_+^g - \pi_+^{g_0})(d^{+g_0} + d^{-g_0}).$$

From (1), (2), and (3) above we have

$$\|(d_{A_0}^{+g} Q_0 - Id) u_0\|_{L^p_{0,\delta}(g)} \leq C C_{\varepsilon'}^2 \varepsilon' (1 + C_p) \|u_0\|_{L^p_{0,\delta}(g)}.$$

For ε' small enough that $C C_{\varepsilon'}^2 \varepsilon' (1 + C_p) < \frac{1}{2}$, the operator $d_{A_0}^{+g} Q_0$ is invertible, and the right inverse for $d_{A_0}^{+g}$ is $Q_0^g = Q_0 (d_{A_0}^{+g} Q_0)^{-1}$. ∎

For $B \in U_{\delta_1}$ and $g \in N_{\varepsilon', r_0, r_1}(g_0)$, we also get a bounded right inverse for the operator d_B^{+g} by combining the proof of Lemma 3.2.11 and Lemma 3.2.12.

3.3 Structure of the trajectory flow on the connected sum

(i) Forming the connected sum

(a) Let Y_i be an oriented homology 3-sphere with Riemannian metrics $g_i, i = 0, 1$. Choose basepoints $y_i \in Y_i$ and suppose for simplicity that the metrics g_i on Y_i are flat in neighborhoods of the y_i. Using these flat metrics we identify neighborhoods of the points y_i in Y_i with neighborhoods of zero in the tangent spaces $T_{y_i}Y_i$. Precisely, for any real numbers $\varepsilon, T > 0$, we set $N_{y_i}(\varepsilon, T) = \{(r, \theta) : T^{-1}\varepsilon \le r \le T\varepsilon\} \subset T_{y_i}Y_i \setminus \{0\}$, where ε eventually will be made small and $T(> 1)$ is another parameter (to be fixed later in the proof) with $T\varepsilon$ less than half the radius of injectivity of y_i. Then define

$$f_{\varepsilon,T} : N_{y_0}(\varepsilon, T) \longrightarrow N_{y_1}(\varepsilon, T)$$

by $f_{\varepsilon,T}(r, \theta) = (-r + T\varepsilon + T^{-1}\varepsilon, \theta)$. Let $U_i \subset Y_i$ be the annulus centered at y_i with inner radius $r_1 = T^{-1}\varepsilon$ and outer radius $r_0 = T\varepsilon$. The "linear inversion" map $f_{\varepsilon,T}$ taking the inner radius of U_0 to the outer radius of U_1 induces an orientation-reversing diffeomorphism from U_0 to U_1. Let $Y_i' \subset Y_i$ be the open set obtained by removing the $T^{-1}\varepsilon$ ball about y_i. Then, in the usual sense, we define the connected sum $Y = Y(\varepsilon, T)$ to be

$$Y = Y_0 \# Y_1 = Y_0' \cup_{f_{\varepsilon,T}} Y_1'$$

where the annuli U_i are identified by $f_{\varepsilon,T}$.

(b) Let $(Y_i, g_i), i = 0, 1$, be oriented Riemannian 3-manifolds as in (a). To construct a Riemannian metric on $Y_0 \# Y_1$, we fix a cutoff function, $\phi \in C^\infty([0, +\infty))$, which satisfies

$$\phi|_{[0,T^{-1}\varepsilon]} \equiv 0, \quad \phi(\frac{T + T^{-1}}{2}\varepsilon) = \frac{1}{2}, \quad \text{and} \quad \phi|_{[T\varepsilon,+\infty)} \equiv 1.$$

Definition 3.3.1 : *The Riemannian metric g on the connected sum $Y_0 \# Y_1$ is defined as follows:*

On $Y_i \setminus B_{y_i}(T\varepsilon)$, set $g = g_i$ for $i = 0, 1$.

On the overlap annulus $N_{y_0}(\varepsilon, T) \cong N_{y_1}(\varepsilon, T)$, $g = \phi g_0 + (1-\phi) f^*_{\varepsilon,T} g_1 = \phi g_0 + f^*_{\varepsilon,T}(\phi g_1)$ (because of the linearity of $f_{\varepsilon,T}$).

Lemma 3.3.2 *Let ε' be the constant of Lemma 3.2.12. There exists $T_0 > 1$ such that for all $1 < T \le T_0$ $\exists \varepsilon$ with $T\varepsilon < \frac{1}{2}$injectivity radius, we have $N_{\varepsilon',T\varepsilon,T^{-1}\varepsilon}(g_i) \neq \emptyset$. Furthermore we have $N_{\varepsilon',T\varepsilon,T^{-1}\varepsilon}(g_0) \cap N_{\varepsilon',T\varepsilon,T^{-1}\varepsilon}(g_1) \neq \emptyset$.*

Proof: We just use the metric from Definition 3.3.1 and calculate the C^0 norm of $g - g_0$ on the annular region.

$$g_{rr} = (g_0)_{rr} \quad g_{\theta\theta} = \{\phi + (1-\phi)(-1 + \frac{(T+T^{-1})\varepsilon}{r})^2\}(g_0)_{\theta\theta}$$

Then $\forall T \leq T_0$

$$\|g - g_0\|_{C^0} \leq \max\{T^4 - 1, |T^{-4} - 1|\}$$

Choose T_0 close to 1 enough to make $\|g - g_0\|_{C^0} \leq \varepsilon'$. Then the result follows. ∎

Remark: Lemma 3.3.2 tells us that we may glue the two manifolds by an orientation-reversing isometry on the tiny overlap region. Therefore for forms u supported on Y_i', we have

$$\frac{1}{2}\|u\|_{L^p_{0,\delta}(g_i)(Y_i')} \leq \|u\|_{L^p_{0,\delta}(g)(Y_0 \# Y_1)} \leq 2\|u\|_{L^p_{0,\delta}(g_i)(Y_i')}.$$

(c) We next use the $SU(2)$-bundles P_i over Y_i to define a bundle P over Y. Using the projection map $\pi_1 : Y_i \times \mathbf{R} \to Y_i$, we pull back the bundles P_i to get bundles $\pi_1{}^*(P_i)$ over $Y_i \times \mathbf{R}$. Let A_0 be a flat connection on $Y_0 \times \mathbf{R}$, constant in the sense that $A_0(t) = \alpha \in \mathcal{R}(Y_0)$ for all $t \in \mathbf{R}$, and let A_1 be an anti-self-dual trajectory from β to γ (i.e. a anti-self-dual connection lying in a one dimensional moduli space) on $Y_1 \times \mathbf{R}$. Set $\tilde{A}_i = \chi A_i$, $i = 0, 1$ (using the flattening procedure on each side as in section 3.2 (ii) (b) with $\chi = \chi(T\varepsilon, \varepsilon)$).

Choose an $SU(2)$-isomorphism of the fibers:

$$\rho : (P_0)_{y_0} \longrightarrow (P_1)_{y_1}$$

Using the flat structures $\tilde{A}_i$ (both are flat on the overlap), we can spread out this isomorphism by parallel transport to give a bundle isomorphism g_ρ between the P_i over the identified part (an annulus or conformally spherical tube) covering $f_{\varepsilon,T}$. We call such a bundle isomorphism g_ρ a gluing map. Use this gluing map to construct a bundle $P_0 \cup_\rho P_1$ over $Y = Y_0 \#_{\varepsilon,T} Y_1$ and also the pull-back bundle $\pi_1^*(P_0 \cup_\rho P_1) = E(\rho)$ over $Y \times \mathbf{R}$. The gluing map g_ρ respects the connections $\tilde{A}_i$ so we get an induced connection, $A_\rho = A_0 \#_\rho A_1$ on $E(\rho)$. Thus $A_\rho = A_0 \#_\rho A_1$ is $\tilde{A}_i$ on $(Y_i \setminus B_3(T^{-1}\varepsilon)) \times \mathbf{R}$. Note that A_ρ is trivial over the region identified by the gluing map.

The connections A_ρ, for different ρ, are not in general gauge equivalent (even though the bundles $E(\rho)$ are obviously isomorphic). Let Γ_{A_i} be the isotropy group of A_i over $Y_i \times \mathbf{R}$ and let $\Gamma = \Gamma_{A_0} \times \Gamma_{A_1}$. The equivalence classes of connections constructed in this way are in one-to-one correspondence with

$$\mathrm{Hom}_{SU(2)}((P_0)_{y_0}, (P_1)_{y_1}) = SU(2)/\Gamma,$$

the space of "gluing parameters". When the A_i are irreducible, $\Gamma = \{\pm 1\}$ so the space of gluing parameters is $SO(3)$.

The following proposition can be found in the text of Donaldson and Kronheimer ([9] page 286, for a proof see [6] Lemma 4.31, page 314).

Proposition 3.3.3 *The connections A_{ρ_1}, A_{ρ_2} are gauge equivalent if and only if the parameters ρ_1, ρ_2 are in the same orbits of the action of Γ on $SU(2)$.*

The following proposition follows from the above Lemma 3.3.2 and Lemma 3.2.12. Recall the constants ε_0 of Lemma 3.2.9 and T_0 of Lemma 3.3.2.

Proposition 3.3.4 *For $0 < \varepsilon < \varepsilon_0$ and $1 < T < T_0$, there is a constant C independent of ε such that the operator $d^{+g}_{A_\rho}$ has a bounded right inverse G with*

$$\|Gu\|_{L^p_{1,\delta}(g)(Y_0 \#_{\varepsilon,T} Y_1)} \le C \|u\|_{L^p_{0,\delta}(g)(Y_0 \#_{\varepsilon,T} Y_1)}$$

and

$$\|Gu\|_{L^q_{0,\delta}(g)} \le C\|u\|_{L^p_{0,\delta}(g)} \quad \frac{1}{4} + \frac{1}{q} \ge \frac{1}{p}.$$

Proof: The right inverse for $d^{+g}_{\tilde{A}_i}$ is Q^g_i. Then using the definition of A_ρ, we define $Gu = Q^g_0 u_0 + Q^g_1 u_1$ which is the right inverse for the operator $d^{+g}_{A_\rho}$. Here $u_0 = \eta u$, $u_1 = (1-\eta)u$ and η is a smooth cutoff function on the annulus $U_0 \cap U_1$ which obeys $\eta|_{[0,T^{-1}\varepsilon]} = 0$ and $\eta|_{\{T\varepsilon \le r\}} = 1$. ∎

(ii) Gluing and splitting

Our goal is to deform the "almost anti-self-dual" connection A_ρ to a nearby anti-self-dual connection $A_\rho + a_\rho$. This entails solving the non-linear anti-self-duality equation

$$F^+(A_\rho) + d^+_{A_\rho} a + (a \wedge a)_+ = 0.$$

The upshot of Proposition 3.3.4 is that we are able to solve the linearized anti-self-duality equation $d^+_A u = b$ over $Y = Y_0 \#_{\varepsilon,T} Y_1$, as long as A is irreducible ($H^0_A = 0$) and regular ($H^2_A = 0$), and furthermore there are estimates on the solution of the corresponding linearized equation which are independent of ε. We shall use the inverse function theorem to deform the almost anti-self-dual connection A_ρ.

Lemma 3.3.5 *(c.f. [15]) Let $f : E \to F$ be a C^1 map between Banach spaces. Assume that in the first order Taylor expansion $f(\xi) = f(0) + Df(0)\xi + N(\xi)$, $Df(0)$ has a finite dimensional kernel and a right inverse G such that for $\xi, \zeta \in E$*

$$\|GN(\xi) - GN(\zeta)\|_E \leq C(\|\xi\|_E + \|\zeta\|_E)\|\xi - \zeta\|_E$$

for some constant C. Let $\delta_1 = (8C)^{-1}$. Then if $\|Gf(0)\|_E \leq \frac{\delta_1}{3}$, there exists a C^1-function

$$\phi : K_{\delta_1} \to ImG$$

with $f(\xi + \phi(\xi)) = 0$ for all $\xi \in K_{\delta_1}$ and furthermore we have estimate

$$\|\phi(\xi)\|_E \leq \frac{4}{3}\|Gf(0)\|_E + \frac{1}{3}\|\xi\|_E$$

where $K_{\delta_1} = KerDf(0) \cap \{\xi \in E : \|\xi\|_E < \delta_1\}$.

Applying Lemma 3.3.5 to $f(a) = F^+(A_\rho) + d^+_{A_\rho} a + (a \wedge a)_+$ with $f(0) = F^+(A_\rho), N(a) = (a \wedge a)_+$, $Df(0) = d^+_{A_\rho}$ (with the bounded right inverse G from the Proposition 3.3.4), $E = L^p_{1,\delta} \cap L^q_{0,\delta}(T_{A_\rho}\mathcal{B})$ and $F = L^p_{0,\delta}(\Omega^2_+(Y \times \mathbf{R}, ad))$, we have the following

Theorem 3.3.6 *Let $Y_i (i = 0, 1)$ be homology 3-sphere and $A_i \in \mathcal{M}^{bal}_{Y_i \times \mathbf{R}}$. Assume $dim\mathcal{M}_{Y_0 \times \mathbf{R}} = 0$, $dim\mathcal{M}_{Y_1 \times \mathbf{R}} = 1$. Let ε_0 be the constant of Lemma 3.2.9. Then if $0 < \varepsilon < \varepsilon_0$ and $1 < T < T_0$,(T_0 is choosed from Lemma 3.3.2) we can deform A_ρ to a smooth anti-self-dual connection over $(Y_0 \#_{\varepsilon,T} Y_1) \times \mathbf{R}$.*

Proof: Using Proposition 3.3.4 and Lemma 3.2.9, we have

$$\|GF^+(A_\rho)\|_{L^q_{0,\delta}} \le C\|F^+(A_\rho)\|_{L^p_{0,\delta}} \le C(\|F^+(\tilde{A}_0)\|_{L^p_{0,\delta}} + \|F^+(\tilde{A}_1)\|_{L^p_{0,\delta}} \le C_3\varepsilon^{\frac{3}{p}}$$

and $N(a) - N(b) = ((a-b)\wedge a)_+ + (b\wedge(a-b))_+$. We use weighted Hölder inequality and Lemma 7.2 in [19]

$$\|((a-b)\wedge a)_+\|_{L^p_{0,\delta}} \le \|a-b\|_{L^q_{0,\frac{\delta}{2}}}\|a\|_{L^4_{0,\frac{\delta}{2}}} \le C_\delta\|a-b\|_{L^q_{0,\delta}}\|a\|_{L^q_{0,\delta}}$$

where $C_\delta = c(\mathrm{Vol}(Y_0) + \mathrm{Vol}(Y_1) + 1)/\delta$. So

$$\|GN(a) - GN(b)\|_{L^q_{0,\delta}} \le CC_\delta\|a-b\|_{L^q_{0,\delta}}(\|a\|_{L^q_{0,\delta}} + \|b\|_{L^q_{0,\delta}}).$$

Thus by Lemma 3.3.5 with $\delta_1 = (8CC_\delta)^{-1}$, there exist $\phi : H^1_{A_\rho} \to \mathrm{Im}G$ with $f(\xi + \phi(\xi)) = 0$, here $\phi(A_\rho) = a_\rho$. So $A_\rho + a_\rho$ is anti-self-dual connection over $(Y_0\#_{\epsilon,T}Y_1)\times\mathbf{R}$ with $\|a_\rho\|_{L^q_{0,\delta}}$ small, and is smooth by standard elliptic regularity (cf.[21]). ∎

Remarks: (i) The restriction on dimensions of moduli spaces is from Lemma 3.2.9 and Proposition 3.3.4 to be able to get the bounded right inverse. Also from the proof above we can glue two 1-dimensional anti-self-dual connections into a 2-dimensional anti-self-dual connection.

(ii) Using the remark after Theorem 3.2.6 and the construction in Proposition 3.3.4, we also can deform the $A_0\#A_1$ into anti-self-dual connection when one of A_i is trivial.

To incorporate the gluing parameter $SO(3)$, we apply the parameterized version of Lemma 3.3.5 which states that the solution depends smoothly on the parameters and is well-behaved under gauge transformations (see [5] Chapter X). That gives the description of a model for an open subset in the moduli space $\mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}$.

Theorem 3.3.7 *Given a constant flat anti-self-dual connection A_0 and a 1-dimensional anti-self-dual connection A_1 with each D_{A_i} surjective in the weighted Sobolev space, then for small enough ε and all gluing parameters ρ, there is a smooth anti-self-dual connection $(A_0\#_\rho A_1) + a_\rho(t)$. If ρ_1, ρ_2 are in the same orbit under the Γ action on the space of gluing parameters $SU(2)$, the corresponding anti-self-dual connections are gauge equivalent.*

The restrictions on ε and T imposed in Theorem 3.3.6 mean that the "neck" region of the connected sum must be narrow with very small radius. Conversely, when our metric satisfies these conditions, we can characterize the anti-self-dual solutions found by our gluing construction. Define

$$Gl_\varepsilon : \mathcal{M}^{i_0}_{Y_0\times\mathbf{R}} \times SO(3) \times \mathcal{M}^{i_1}_{Y_1\times\mathbf{R}} \to \mathcal{B}_{(Y_0\#Y_1)\times\mathbf{R}}$$

by $Gl_\varepsilon(A_0, \rho, A_1) = A_0\#_\rho A_1$ as in §3.3 (i) (c), where $i_0 \geq 0, i_1 \geq 0, i_0 + i_1 = 1$. Now for $\delta_2 > 0$ in the proof of Theorem 3.3.6, let $U_{\delta_2}(\varepsilon) \subset \mathcal{B}_{Y\times\mathbf{R}}$ be the open set

$$U_{\delta_2}(\varepsilon) = \{A | \inf_{B\in \mathrm{Im} Gl_\varepsilon} \|A - B\|_{L^q_{0,\delta}(g)((Y_0\#_{\varepsilon,T}Y_1)\times\mathbf{R})} < \delta_2, \|F^+_A\|_{L^p_{0,\delta}(g)((Y_0\#_{\varepsilon,T}Y_1)\times\mathbf{R})} < \delta_2\}.$$

The solutions to the anti-self-duality equation obtained from Theorem 3.3.6 lie in $U_{\delta_2}(\varepsilon)$, and any element in U_{δ_2} can be deformed to a unique anti-self-dual connection by Lemma 3.3.5 (The uniqueness follows from the contraction mapping principle on $T\mathcal{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$).

Theorem 3.3.8 *For ε, T as in Theorem 3.3.6, any point in $U_{\delta_2}(\varepsilon) \cap \mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}(g_\varepsilon)$ can be represented by a connection A of the form $A_0\#_\rho A_1 + \phi(A_0\#_\rho A_1)$ where A_i is (0 or 1)-dimensional anti-self-dual connection on $\mathcal{M}_{Y_i\times\mathbf{R}}$ and ϕ is the C^1-diffeomorphism in the proof of Theorem 3.3.6 with $\|\phi(A_0\#_\rho A_1)\|_{L^q_{0,\delta}} < \delta_2$.*

Proof: Suppose the contrary. Then there exists a sequence $\varepsilon_n \to 0$ with $\varepsilon_n < \varepsilon_0$, $\{[A_n]\} \in U^c_{\delta_2}(\varepsilon_n) \cap \mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}(g_{\varepsilon_n})$ where $U^c_{\delta_2}(\varepsilon_n)$ is complement of U_{δ_2}, i.e. A_n are not in such a form.

By Uhlenbeck's compactness theorem applied to the balanced anti-self-dual connections, we have a subsequence converging to $A_0 \vee A_1$, where A_i is a anti-self-dual connection on $(Y_i \setminus \{y_i\}) \times \mathbf{R}$ (since 1-dimensional moduli space is compact up to time-translation by Lemma 3.1.5). The connection A_i has a singularity along a line $\{y_i\} \times \mathbf{R}$. Since this is codimension 3 , it can be removed by Sibner's theorem [26]. Let the extended anti-self-dual connections still be denoted by A_i. By the flattening construction, for small enough ε_n we have

$$\tilde{A}_i(\varepsilon_n) = \chi(T\varepsilon_n, \varepsilon_n)A_i \quad \text{with} \quad \|\tilde{A}_i(\varepsilon_n) - A_i\|_{L^q_{0,\delta}(g_n)(Y_i\times\mathbf{R})} < \frac{\delta_2}{8},$$

and let $A_\rho(\varepsilon_n) = A_0 \#_{\rho,\varepsilon_n} A_1$ as in §3.3 (i) (c). Then

$$\|A_n - A_\rho(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R})}$$

$$\begin{aligned}
&\leq \|A_n - \tilde{A}_0(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} + \|A_n - \tilde{A}_1(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_1\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} \\
&\leq \sum_{i=0}^{1} \|A_n - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} + \|\tilde{A}_i(\varepsilon_n) - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})}
\end{aligned}$$

For n large enough we have $\|A_n - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} < \frac{\delta_2}{8}$ from convergence. Thus $\|A_n - A_\rho(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R})} < \frac{\delta_2}{2}$. Since $A_n \in \mathcal{M}^1_{(Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R}}(g_n)$, of course $F^+(A_n) = 0$, so we have $A_n \in U_{\frac{\delta_2}{2}}(\varepsilon_n)$ which contradicts $A_n \in U^c_{\delta_2}(\varepsilon_n)$.

Thus for sufficiently small ε, a 1-dimensional moduli space can be represented by the one deformed from the gluing process. ■

Corollary 3.3.9 *Under the assumption of Theorem 3.3.8, there is a unique small solution to the anti-self-duality equation. So $U_{\delta_2}(\varepsilon) \cap \mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}$ is equal to the image of the gluing map.*

This is the main analytic result of this thesis. We summarize this section in the following

Theorem 3.3.10 *Suppose A_i is an anti-self-dual connection on $Y_i \times \mathbf{R}$, and we consider the connected sums $Z = (Y_0\#_{\varepsilon,T}Y_1) \times \mathbf{R}$ for fixed $0 < \varepsilon < \varepsilon_0, 1 < T < T_0$. Then for sufficiently small ε one has maps*

$$\bigsqcup_{i+i'=1} \mathcal{M}^1_{Y_i\times\mathbf{R}}(\alpha_i, \beta_i) \times_{SO(3)} \mathcal{M}^0_{Y_{i'}\times\mathbf{R}}(\gamma_{i'}, \gamma_{i'}) \xrightarrow{\text{Gluing}} \mathcal{M}^1_Z(\alpha_i\#\gamma_{i'}, \beta_i\#\gamma_{i'})$$

and

$$\mathcal{M}^1_Z(\alpha_i\#\gamma_{i'}, \beta_i\#\gamma_{i'}) \xrightarrow{\text{Splitting}} \bigsqcup_{i+i'=1} \mathcal{M}^1_{Y_i\times\mathbf{R}}(\alpha_i, \beta_i) \times_{SO(3)} \mathcal{M}^0_{Y_{i'}\times\mathbf{R}}(\gamma_{i'}, \gamma_{i'}).$$

which are inverse maps to each other.

Note that the splitting operation takes an anti-self-dual connection $A \in \mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}$ with asymptotic values $\alpha_i\#\gamma_{i'}$ to $\beta_i\#\gamma_{i'}$ one obtained by deforming $A_0\#_\rho A_1$ where

$$(A_i, A_{i'}) \in \mathcal{M}^1_{Y_i\times\mathbf{R}}(\alpha_i, \beta_i) \times \mathcal{M}^0_{Y_{i'}\times\mathbf{R}}(\gamma_{i'}, \gamma_{i'})$$

and they are glued together by constant (in $\mathbf{R}$) gluing parameter $\rho \in SO(3)$.

Chapter 4

The Floer homology of $Y_0\#Y_1$

4.1 General properties of the Floer homology of $Y_0\#Y_1$

From Chapter 2 we know that the nontrivial representations of $\pi(Y_0\#Y_1)$ in $SU(2)$ fall into two classes: the nondegenerate representations $\alpha_0\#\theta_1, \theta_0\#\alpha_1$ and the degenerate representations $\alpha_0\#_\rho\alpha_1, \rho \in SO(3)$. From Lemma 2.2.2 we have the gradings $\mu(\theta_0\#\alpha_1) \equiv \mu(\alpha_1), \mu(\alpha_0\#\theta_1) \equiv \mu(\alpha_0)$ and $\mu(\alpha_0\#\alpha_1) \equiv \mu(\alpha_0) + \mu(\alpha_1)$ (mod 8). To compute the Floer homology we first need to perturb the $SO(3)$ components $\mathcal{C}(\alpha_0\#\alpha_1) = \{\alpha_0\#_\rho\alpha_1 \mid \rho \in SO(3)\}$ to make the critical points nondegenerate. This is accomplished as follows. (See [12]).

Each $\mathcal{C}(\alpha_0\#\alpha_1)$ is a smooth closed submanifold of $\mathcal{B}_{Y_0\#Y_1}$; so we have the following exact sequence which splits in L^2:

$$0 \to T\mathcal{C}(\alpha_0\#\alpha_1) \to T\mathcal{B}_{Y_0\#Y_1}|_{\mathcal{C}(\alpha_0\#\alpha_1)} \to N\mathcal{C}(\alpha_0\#\alpha_1) \to 0.$$

The Chern-Simons Hessian is degenerate along $T\mathcal{C}(\alpha_0\#\alpha_1)$ but is nondegenerate in the normal direction. Identify the normal bundle $N(\alpha_0\#\alpha_1)$ of $\mathcal{C}(\alpha_0\#\alpha_1)$ with the total space of $N\mathcal{C}(\alpha_0\#\alpha_1)$, and let

$$N_\varepsilon(\alpha_0\#\alpha_1) = \{u \in N\mathcal{C}(\alpha_0\#\alpha_1) \mid \|u\|_{L^2} < \varepsilon\}.$$

Let $\varepsilon_1 < \varepsilon_2$ and define a cutoff function χ on $\mathcal{B}_{Y_0\#Y_1}$ such that $\chi = 1$ on $N_{\varepsilon_1}(\alpha_0\#\alpha_1)$ and $\chi = 0$ on the complement of $N_{\varepsilon_2}(\alpha_0\#\alpha_1)$. Let $\pi : N(\alpha_0\#\alpha_1) \to \mathcal{C}(\alpha_0\#\alpha_1)$ be the projection. Let $g : SO(3) \to \mathbf{R}$ be the standard Morse function with one critical point in each dimension. Set $\varepsilon_3 = (\varepsilon_2 - \varepsilon_1)^2$ and define

$$E(a) = \varepsilon_3 \chi(a) \cdot g(\pi(a)) : \mathcal{B}_{Y_0 \# Y_1} \to \mathbf{R}$$

and let e be the L^2 gradient of E. On $N_{\varepsilon_1}(\alpha_0 \# \alpha_1)$ we have $e = \nabla g$, on the complement of $N_{\varepsilon_2}(\alpha_0 \# \alpha_1), e = 0$, and on $N_{\varepsilon_2}(\alpha_0 \# \alpha_1) \setminus N_{\varepsilon_1}(\alpha_0 \# \alpha_1)$:

$$\|e(a)\|_{C^0} \geq \varepsilon_3(|d\chi| \|g(a)\|_{C^0} - |\chi| \|\nabla g(a)\|_{C^0}) \geq \varepsilon_3(\frac{C}{\varepsilon_2 - \varepsilon_1} \|g(a)\|_{C^0} - \|\nabla g(a)\|_{C^0}).$$

This points out that provided we choose $\varepsilon_2 - \varepsilon_1$ sufficiently small, there will be no critical point of e inside $N_{\varepsilon_2}(\alpha_0 \# \alpha_1) \setminus N_{\varepsilon_1}(\alpha_0 \# \alpha_1)$. Set $\tilde{f}(a) = *F_a + e(a)$ over $\mathcal{B}_{Y_0 \# Y_1}$. In [12] Lemma 4.1 it is shown that the zeros of $\tilde{f}$ are the zeros of $*F_a$ outside of $N_{\varepsilon_2}(\alpha_0 \# \alpha_1)$ and inside $N_{\varepsilon_2}(\alpha_0 \# \alpha_1)$ they are the critical points of g, and all zeros of $\tilde{f}$ are nondegenerate.

After performing this perturbation for each $\mathcal{C}(\alpha_0 \# \alpha_1)$ in $\mathcal{R}(Y_0 \# Y_1)$, the Floer chain groups $C_j(Y_0 \# Y_1)$ are the free abelian groups generated by elements

- $\alpha_0 \# \theta_1$ in grading $\mu(\alpha_0)$.
- $\theta_0 \# \alpha_1$ in grading $\mu(\alpha_1)$.
- $(\alpha_0 \# \alpha_1)_j$ in grading $\mu(\alpha_0) + \mu(\alpha_1) - j, \quad j = 0, 1, 2, 3$.

(See §2 and [12].)

The boundary operator of $C_*(Y_0 \# Y_1)$ is computed by counting the trajectory flow line spaces $\hat{\mathcal{M}}^1_{Y_0 \# Y_1}(a, b)$ for a, b generators in the list above.

We start with a graded differential group $C_* = \oplus_q C_q$, $q \in Z_8$, with $\partial C_q \subset C_{q-1}$, where $C_q = \{a \in \mathcal{R}^*(Y_0 \# Y_1) |\ \mu(a) = q\}$. There is an associated filtration compatible with the grading, $F_p C_q \subset F_{p+1} C_p$ (increasing filtration). Define the $F_p C_*$ as follows:

$$F_p C_* = \oplus_{k \leq p} \{Z < a > \mid a = (a_0 \# a_1)_j,\ \mu(a_0 \# a_1) = k,\ a_i \in \mathcal{R}(Y_i), i = 0, 1\}.$$

Note that the perturbed Chern-Simons functional is non-decreasing along the gradient flows (anti-self-dual connections). It follows that Floer's boundary map $\partial : F_p C_q \hookrightarrow F_p C_{q-1}$ preserves the filtration. Because $\mathcal{R}^*(Y_0 \# Y_1)$ is compact each $E^1_{p,q}$ is finitely generated. Note that $\alpha_0 \# \theta_1$ generates a free Z-summand of $E^1_{p,q}$ for $p = \mu(\alpha_0), q = 0$, similarly $\theta_0\#\alpha_1$ gives a free Z-summand of $E^1_{p,q}$ for $p=\mu(\alpha_1), q=0$. An $SO(3)$ component coming from $\alpha_0\#\alpha_1$ gives free Z-summands of $E^1_{p,q}$ for $p=\mu(\alpha_0)+\mu(\alpha_1)$ and $q=0$ and -3, and a Z_2-summand $E^1_{\mu(\alpha_0)+\mu(\alpha_1),-2}$.

In particular the filtration is bounded. Since $\bigcup F_pC_* = C_*$ we get a spectral sequence.

Theorem 4.1.1 *There is a (fourth quadrant) spectral sequence with*

$$E^1_{p,q} \cong H_{p+q}(F_pC_*/F_{p-1}C_*)$$

which converges to $HF_*(C_*,\partial)$.

∎

Clearly $d_r=0$ for $r\geq 5$, i.e. the spectral sequence $(E^r_{*,*}, d_r)$ collapses at 5th term. Therefore $E^5_{*,*}$ gives the graded Floer homology $HF_*(C_*(Y_0\#Y_1),\partial)$. The next section will be devoted to a complete description of d_1 which will be enough to calculate some examples. The description of d_2, d_4 will be the subject of a future paper. We end up this section by showing $d_3=0$.

Proposition 4.1.2 *In the spectral sequence $(E^r_{*,*}, d_r)$ built from the filtration of Floer's chain complex for connected sums of homology 3-spheres, the third differential map $d_3: E^3_{p,q} \to E^3_{p-3,q+2}$ is zero.*

Proof: Note that $E^1_{p,q}=0$ for $q>0$ and $q\leq -4$. Also, we have seen above that $E^1_{p,-1}=0$. So the only possible nonzero case is $d_3: E^3_{p,-2}\to E^3_{p-3,0}$. Suppose $d_3(\alpha\#\beta)_2 = n(\alpha'\#\beta')_0 \neq 0$. Then for the boundary operator ∂ of $C_*(Y_0\#Y_1)$, the coefficient of $(\alpha'\#\beta')_0$ in $\partial(\alpha\#\beta)_2$ is $n\neq 0$. Then $\partial^2(\alpha\#\beta)_1 = 2\partial(\alpha\#\beta)_2 \neq 0$, which is impossible. ∎

4.2 Description of d_1

We are going to define two special maps which are not used in the definition of Floer homology.

Definition 4.2.1 *$d : C_1(Y) \to Z < \theta >$ is defined by*

$$d\alpha = \#\hat{\mathcal{M}}^1(\alpha, \theta)\theta$$

and $\delta : Z < \theta > \to C_4(Y)$ is defined by

$$\delta\theta = \sum_{\beta \in C_4(Y)} \#\hat{\mathcal{M}}^1(\theta, \beta)\beta.$$

Lemma 4.2.2 $d\partial = 0, \quad \partial\delta = 0.$

Proof: This is proved in [10]. The proof is similar to the proof that $\partial^2 = 0$ in the Floer complex. ■

Let $\partial_i : C_*(Y_i) \to C_{*-1}(Y_i), i = 0, 1$ be the boundary map of the Floer chain complexes of $Y_i, i = 0, 1$. The main result of this section is a description of d_1 for the spectral sequence of filtration built from the Floer chain complex $C_*(Y_0 \# Y_1)$. Our result is:

Theorem 4.2.3 *The differential d_1 of the spectral sequence $(E^1_{*,*}, d_1)$ for the Floer chain complex $C_*(Y_0 \# Y_1)$ is given in terms of the listed basis in §4.1 by*

$$d_1 = \partial_0 \otimes Id_1 \pm Id_0 \otimes \partial_1 \pm \partial_3 + d \otimes Id_1 \pm Id_0 \otimes d + \delta \otimes Id_1 \pm Id_0 \otimes \delta,$$

where ∂_3 is boundary map of the standard cellular chain complex of $SO(3)$.

The notation in Theorem 4.2.3 and the determination of signs can best be explained by two simple examples:

$$d_1(\alpha\#\beta)_i = (\partial_0\alpha\#\beta)_i + (-1)^{\mu(\alpha)}(\alpha\#\partial_1\beta)_i + (-1)^{\mu((\alpha\#\beta)_i)}\partial_3(\alpha\#\beta)_i$$

$$d_1(\alpha_0\#\theta_1) = \partial_0\alpha\#\theta_1 + (-1)^{\mu(\alpha)}(\alpha_0\#\delta\theta_1)_0.$$

We have extended our notation here in the obvious way: $(\sum m_j\gamma_j)\#\beta = \sum m_j(\gamma_j\#\beta)$.

Theorem 4.2.3 gives a full description of d_1. It is built from the contributions from each homology 3-sphere and the gluing parameter space $SO(3)$. The trivial connections of both homology 3-spheres also contribute via the special maps $d_i, \delta_i, i = 0, 1$. Now we proceed to the proof of Theorem 4.2.3.

Let $\alpha\#_{\rho_i}\beta$ correspond to $(\alpha\#\beta)_i$, where ρ_i is in the gluing parameter space $SO(3)$. For the boundaries in the Floer chain complex we only consider 1-dimensional trajectory flows with finite action. Fix metrics g_0, g_1 and parameters ε, T which satisfy the hypotheses of Theorem 3.3.6 and Theorem 3.3.8. This fixes a metric on $Y_0\#Y_1$ by §3.3 (i) (b). If $A_1 \in \mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\gamma)$ and $A_0 \in \mathcal{M}^0_{Y_0\times\mathbf{R}}(\alpha,\alpha)$ is constant, $A_0(t) = \alpha$, then for fixed $\rho \in SO(3)$, Theorem 3.3.6 gives a unique element in $\mathcal{M}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma)$ and the Splitting Theorem 3.3.8 gives the converse. Therefore we have

$$\#\hat{\mathcal{M}}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma) = \#\hat{\mathcal{M}}^1_{Y_1\times\mathbf{R}}(\beta,\gamma).$$

Let $\hat{\mathcal{M}}^{1,\varepsilon_3}_{(Y_0\#Y_1)\times\mathbf{R}}$ denote the quotient of the moduli space of perturbed anti-self-dual connections by time-translation. We want to show that

$$\#\hat{\mathcal{M}}^{1,\varepsilon_3}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma) = \#\hat{\mathcal{M}}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma).$$

The following proposition says that the compact perturbation we used in the construction neither changes the algebraic number of trajectory flows between critical points $\alpha\#_\rho\beta$ and $\alpha\#_\rho\gamma$ nor does it create new 1-dimensional trajectory flows.

Proposition 4.2.4 *For ε_3 sufficiently small, we have*

(1) If $\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\gamma) \neq \emptyset$, then for $\rho \in SO(3)$ a critical point for the Morse function $g : SO(3) \to \mathbf{R}$

$$\#\hat{\mathcal{M}}^{1,\varepsilon_3}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma) = \#\hat{\mathcal{M}}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta, \alpha\#_\rho\gamma).$$

(2) If $\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\gamma) = \emptyset$, then $\mathcal{M}^{1,\varepsilon_3}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_{\rho_0}\beta, \alpha\#_{\rho_1}\gamma) = \emptyset$.

Proof: (1) Using the identification between trajectory flows and anti-self-dual connections, we consider the bundle $\mathcal{A}^{\mathrm{bal}}_{(Y_0\#Y_1)\times\mathbf{R}} \times_{\mathcal{G}} \Omega^2_+((Y_0\#Y_1)\times\mathbf{R}, ad) \to \mathcal{B}_0(\alpha\#\beta, \alpha\#\gamma)$. The self-dual curvature F_A^+ induces a section of this bundle whose zeros are

$$\hat{\mathcal{M}}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#\beta, \alpha\#\gamma) = \mathcal{M}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#\beta, \alpha\#\gamma)/\mathbf{R}.$$

For 1-dimensional anti-self-dual connections, this is a finite set of points. The orientation on these points is precisely the orientation on the trajectory flow used in defining the boundary map of Floer homology. Thus the algebraic number of zeros of F_A^+ is

$\#\hat{\mathcal{M}}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#\beta,\alpha\#\gamma)$. Consider a trajectory with asymptotic values $\alpha\#_\rho\beta,\alpha\#_\rho\gamma$. Consider $s\in I=[0,1]$

$$\Psi_{s,A}: I\times\mathcal{B}_0(\alpha\#_\rho\beta,\alpha\#_\rho\gamma)\to I\times\mathcal{A}^{\mathrm{bal}}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha\#_\rho\beta,\alpha\#_\rho\gamma)\times_{\mathcal{G}}\Omega^2_+((Y_0\#Y_1)\times\mathbf{R},ad)$$

$$(s,A)\longmapsto F_A^+ + \Psi_A + s\Phi_A$$

where $\Phi_A=\frac{1}{2}(*_g e(a(t))+e(a(t))\wedge dt)$, and where $\Psi_A=\psi(a(t))$ is a perturbation constructed so that $\Psi_{s,A}$ has transverse zeros. For $s=1$, the zeros of $F_A^+ + \Psi_A + \Phi_A$ are the solutions of the perturbed anti-self-duality equation

$$\frac{\partial a(t)}{\partial t}-*_g F_{a(t)}-\psi(a(t))-e(a(t))=0.$$

Let $\mathcal{M}^{1,s}$ denote the zeros of $\Psi_{s,A}$. Then $0,1\in I$ are regular values of the projection $\pi_1:\mathcal{M}^{1,s}\to I$ since the 1-dimensional moduli space is transverse to $\mathcal{B}_0$ by Proposition 3.1.6. So the parametrized moduli space $\mathcal{M}^{1,s}$ is a one dimensional submanifold of $I\times\mathcal{B}_0(\alpha\#_\rho\beta,\alpha\#_\rho\gamma)$ with oriented boundary components $-\mathcal{M}^{1,0}$ and $\mathcal{M}^{1,1}$. Each $\mathcal{M}^{1,s}$ is compact by [15, Theorem 3]. This means that $\mathcal{M}^{1,0}$ and $\mathcal{M}^{1,1}$ are oriented cobordant; so (1) follows.

(2) Suppose $\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\gamma)=\emptyset$ but there exists a 1-dimensional trajectory flow between $\alpha\#_{\rho_0}\beta,\alpha\#_{\rho_1}\gamma$ after performing our perturbations. We have a solution A_{ε_3} of the perturbed anti-self-duality equation, and A_{ε_3} lives in a 1-dimensional moduli space with asymptotic values $\alpha\#_{\rho_0}\beta,\alpha\#_{\rho_1}\gamma$. So $F^+_{A_{\varepsilon_3}}=-\frac{1}{2}(e(a_{\varepsilon_3}(t))+e(a_{\varepsilon_3}(t))\wedge dt)$. We have

$$\|GF^+_{A_{\varepsilon_3}}\|_{L^q_{0,\delta}}\le\|Ge(a_{\varepsilon_3}(t))\|_{L^q_{0,\delta}}\le C\cdot\varepsilon_3.$$

The last inequality holds because we have a uniform bound for G by Proposition 3.3.4 and because in constructing our perturbation, we have used a smooth Morse function. By choosing ε_3 small enough so that $C\cdot\varepsilon_3\le\frac{\delta_1}{3}$, then we can (by Lemma 3.3.5 and Theorem 3.3.6) deform A_{ε_3} to an anti-self-dual connection A with asymptotic values $\alpha\#_{\rho_0}\beta,\alpha\#_{\rho_1}\gamma$. For any metric on $Y_0\#Y_1$ for which the splitting Theorem 3.3.10 holds, A is obtained from $A_0\#_\rho A_1$ where $A_0\in\mathcal{M}^0_{Y_0\times\mathbf{R}}(\alpha,\alpha)$ and $A_1\in\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\gamma)$. This contradicts the hypothesis. ∎

Proposition 4.2.5 *For generators a, b of $C_*(Y_0\#Y_1)$, elements of $\hat{\mathcal{M}}^{1,\varepsilon_3}_{(Y_0\#Y_1)}(a,b)$ (1-dimensional trajectory flow lines) occur only as follows:*

1. *If both α_k, β_k are irreducible, there is a 1-dimensional trajectory from $(\alpha_0\#\alpha_1)_i$ to $(\beta_0\#\beta_1)_j$ if and only if $i = j$ and either*

   (i) $\alpha_0 = \beta_0$ and there is a 1-dimensional trajectory from α_1 to β_1, or

   (ii) $\alpha_1 = \beta_1$ and there is a 1-dimensional trajectory from α_0 to β_0.

2. *There is a 1-dimensional trajectory from $(\alpha_0\#\alpha_1)_i$ to $\beta_0\#\theta_1$ if and only if $\alpha_0 = \beta_0$ and there is a 1-dimensional trajectory from α_1 to θ_1 and $i = 0$. A similar statement holds for $\theta_0\#\beta_1$.*

3. *There is a 1-dimensional trajectory from $\alpha_0\#\theta_1$ to $(\beta_0\#\beta_1)_j$ if and only if $\alpha_0 = \beta_0$ and there is a 1-dimensional trajectory from θ_1 to β_1 and $j = 0$. A similar statement holds for $\theta_0\#\alpha_1$.*

4. *There is a 1-dimensional trajectory from $\alpha_0\#\theta_1$ to $\beta_0\#\theta_1$ if and only if there is a 1-dimensional trajectory from α_0 to β_0. A similar statement holds for $\theta_0\#\alpha_1$ and $\theta_0\#\beta_1$.*

5. *There is no 1-dimensional trajectory from $\alpha_0\#\theta_1$ to $\theta_0\#\beta_1$ or from $\theta_0\#\alpha_1$ to $\beta_0\#\alpha_1$.*

Proof: (1) It follows from the proof of Proposition 4.2.4 and the remark after it, that for $\rho \in SO(3)$

$$\#\hat{\mathcal{M}}^{1,\varepsilon_3}_{(Y_0\#Y_1)\times\mathbf{R}}((\alpha_0\#\alpha_1)_i, (\beta_0\#\beta_1)_j) = \#\hat{\mathcal{M}}^{1}_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha_0\#_\rho\alpha_1, \beta_0\#_\rho\beta_1).$$

For any metric on $Y_0\#Y_1$ for which Theorem 3.3.10 can be applied, we have that any $A \in \mathcal{M}^1_{(Y_0\#Y_1)\times\mathbf{R}}(\alpha_0\#_\rho\alpha_1, \beta_0\#_\rho\beta_1)$ is obtained from $A_0\#_\rho A_1$ where $A_k, k = 0, 1$ which are (0 or 1)-dimensional anti-self-dual connections in $\mathcal{M}_{Y_k\times\mathbf{R}}$. Hence $\alpha_k = \beta_k$ or there is a 1-dimensional trajectory flow from α_k to β_k which is realized by A_k. So

$$0 \le \mu(\alpha_k) - \mu(\beta_k) \le 1$$

and

$$\mu((\alpha_0\#\alpha_1)_i) - \mu((\beta_0\#\beta_1)_j) = 1, \quad \mu(\alpha_0\#\alpha_1) - \mu(\beta_0\#\beta_1) = 1.$$

Therefore we have that $i = j$ and one side is constant and the other comes from a 1-dimensional trajectory flow. (2), (3) and (4) follow by a similar argument.

(5) By Theorem 3.3.8, there is no 1-dimensional trajectory flow from $\alpha_0\#\theta_1$ to $\theta_0\#\beta_1$. ■

A straightforward calculation shows:

Lemma 4.2.6 $\partial_i\partial_3 = \partial_3\partial_i, i = 0, 1.$

■

Proof of Theorem 4.2.3: In Proposition 4.2.5, we have listed all possibilities for 1-dimensional trajectory flows. So the boundary map for $C_*(Y_0\#Y_1)$ includes all possibilities. We need to check that $d_1^2 = 0$.

(1) For basis element $(\alpha\#\beta)_i$ with α, β both irreducible, we have

$$d_1(\alpha\#\beta)_i = (\partial_0\alpha\#\beta)_i + (-1)^{\mu(\alpha)}(\alpha\#\partial_1\beta)_i + (-1)^{\mu(\alpha\#\beta)_i}\partial_3(\alpha\#\beta)_i$$

(a) If $\mathcal{M}^1_{Y_0\times\mathbf{R}}(\alpha,\theta_0) = \emptyset$ and $\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\theta_1) = \emptyset$, by the definition of d_1 we have

$$d_1^2(\alpha\#\beta)_i = \partial[(\partial_0\alpha\#\beta)_i] + \partial[(-1)^{\mu(\alpha)}(\alpha\#\partial_1\beta)_i] + \partial[(-1)^{\mu(\alpha\#\beta)_i}\partial_3(\alpha\#\beta)_i] =$$

$$\begin{aligned}&(\partial_0^2\alpha\#\beta)_i + (-1)^{\mu(\partial_0\alpha)}((\partial_0\alpha\#\partial_1\beta)_i + (-1)^{\mu(\partial_0\alpha\#\beta)_i}\partial_3((\partial_0\alpha\#\beta)_i)\\ &+(-1)^{\mu(\alpha)}\{(\partial_0\alpha\#\partial_1\beta)_i + (-1)^{\mu(\alpha)}(\alpha\#\partial_1^2\beta)_i + (-1)^{\mu((\alpha\#\partial_1\beta)_i)}\partial_3((\alpha\#\partial_1\beta)_i)\}\\ &+(-1)^{\mu(\alpha\#\beta)_i}\{\partial_3(\partial_0\alpha\#\beta)_i + (-1)^{\mu(\alpha)}\partial_3(\alpha\#\partial_1\beta)_i + (-1)^{\mu((\partial_3(\alpha\#\beta)_i))}\partial_3^2((\alpha\#\beta)_i)\}\end{aligned}$$

We use Lemma 4.2.6 in the last equality. Clearly it shows $d_1^2(\alpha\#\beta)_i = 0$.

(b) If $\mathcal{M}^1_{Y_0\times\mathbf{R}}(\alpha,\theta_0) \neq \emptyset$ and $\mathcal{M}^1_{Y_1\times\mathbf{R}}(\beta,\theta_1) = \emptyset$. Note that $\partial_3(\partial_0\alpha\#\beta) = 0$. By Proposition 4.2.5 (2), $i = 0$ in this case. Then we have

$$\begin{aligned}&d_1^2(\alpha\#\beta)_0 = (d\partial_0\alpha\#\beta) + (\partial_0\alpha\#\partial_1\beta)\\ &+(-1)^{\mu(\alpha)}(\partial_0\alpha\#\partial\beta) + (-1)^{2\mu(\alpha)}(\alpha\#\partial_1^2\beta)_0 + (-1)^{\mu(\alpha)+\mu((\alpha\#\partial_1\beta)_0)}\partial_3(\alpha\#\partial_1\beta)_0\\ &+(-1)^{\mu(\alpha\#\beta)_0}\partial_3((\partial_0\alpha\#\beta) + (-1)^{\mu(\alpha)}(\alpha\#\partial_1\beta)_0) + (-1)^{\mu(\alpha\#\beta)_0+\mu(\partial_3(\alpha\#\beta)_0)}\partial_3^2(\alpha\#\beta)_0\end{aligned}$$

Then it shows that $d_1^2(\alpha\#\beta)_0 = 0$. For $\partial_0\alpha \neq \theta_0, \partial_1\beta = \theta_1$ and $\partial_0\alpha = \theta_0, \partial_1\beta = \theta_1$, the $d_1^2 = 0$ follows from a similar argument.

(2) For another kind of basis like $\alpha_0\#\theta_1$ or $\theta_0\#\alpha_1$, We verify one of them, $\alpha_0\#\theta_1$. The other one is similar. By Proposition 4.2.5 we have

$$d_1(\alpha_0\#\theta_1) = \partial_0\alpha_0\#\theta_1 + (-1)^{\mu(\alpha_0)}(\alpha\#\delta\theta_1)_0.$$

Based on the definition of Floer homology, the trivial connection $\theta_0\#\theta_1$ of $Y_0\#Y_1$ is not taken into account. So $\partial_0\alpha_0 \neq \theta_0$.

$$d_1^2(\alpha_0\#\theta_1) = \partial_0^2\alpha\#\theta_1 + (-1)^{\mu(\partial_0\alpha_0)}(\partial_0\alpha_0\#\delta\theta_1)_0$$

$$+(-1)^{\mu(\alpha)}(\partial_0\alpha\#\delta\theta_1)_0 + (\alpha\#\partial_1\delta\theta_1)_0 + (-1)^{\mu(\alpha)+\mu((\alpha\#\delta\theta_1)_0)}\partial_3(\alpha_0\#\delta\theta_1)_0.$$

Using the definition of ∂_3 , Lemma 4.2.2 and Lemma 4.2.6, we have the differential d_1 satisfying $d_1^2 = 0$. ∎

4.3 Examples

Even with our formulas of §4.1, examples remain difficult to compute since it is an extremely nontrivial problem for a fixed homology sphere Y to compute all possible 1-dimensional moduli spaces $\mathcal{M}^1_{Y\times\mathbf{R}}(\alpha,\beta), \mathcal{M}^1_{Y\times\mathbf{R}}(\alpha,\theta), \mathcal{M}^1_{Y\times\mathbf{R}}(\theta,\alpha)$.

Example 1: Calculation of $HF_*(\Sigma(2,3,5)\#\Sigma(2,3,5))$

Let us consider the connected sum of the Poincaré 3-sphere with itself. The Floer homology of the Poincaré 3-sphere is generated by $\alpha \in C_1, \beta \in C_5$ ($\dim\mathcal{M}_Y(\alpha,\theta) = 1, \dim\mathcal{M}_Y(\beta,\theta) = 5$) with all the boundary differentials trivial, so for $\Sigma(2,3,5)$:

$$C_1 = HF_1 \cong Z, \quad C_5 = HF_5 \cong Z, \quad C_j = HF_j = 0, \quad j \neq 1,5 \pmod 8.$$

This follows from work of Fintushel and Stern [12].

Proposition 4.3.1 *Let Y be the Poincaré 3-sphere $\Sigma(2,3,5)$ and let α be the irreducible representation with $\mu(\alpha) = 1$. Then the 1-dimensional moduli space of anti-self-dual connections with asymptotic values α, θ (denoted by $\mathcal{M}^1_Y(\alpha,\theta)$) is nonempty and $\#\mathcal{M}^1_Y(\alpha,\theta)$ $= \pm 1$, where "$\#$" denotes a count with sign.*

Proof: Let X be the 4-manifold obtained by plumbing the (negative definite) E_8-diagram. Then $\partial X = Y$. The intersection form of X is the form E_8. Let P be the principal $SO(3)$-bundle over X with $p_1 = 2$ and w_2 represented by the Poincaré dual of one of the 2-spheres corresponding to the nodes of the E_8-diagram. The moduli space $\mathcal{M}_X(\theta)$ of anti-self-dual connections with asymptotic value trivial (θ) on the boundary Y has dimension one. There are two kinds of ends of $\mathcal{M}_X(\theta)$. The first corresponds to reducible anti-self-dual connections on P. These are in one to one correspondence with

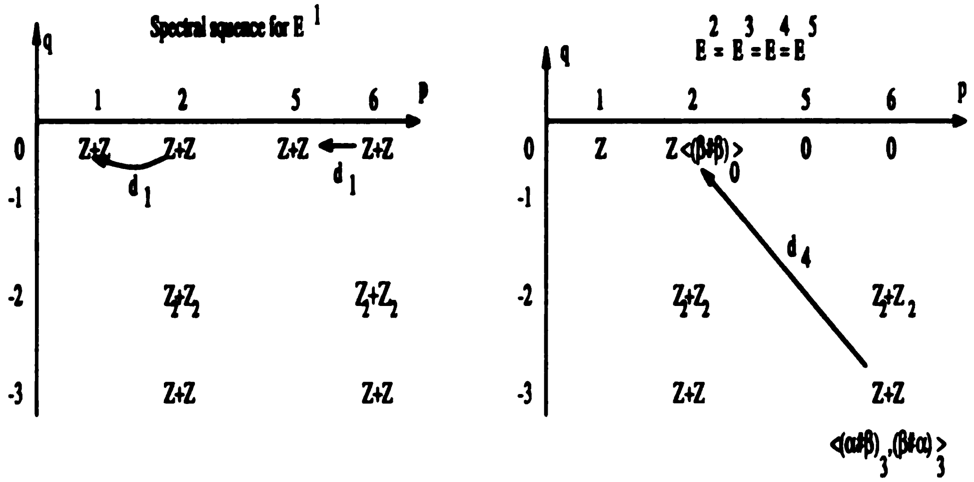
$$\{\pm e \in H^2(X, Z)| \;\; e^2 = -2, \;\; e \equiv w_2(P) \;(\mathrm{mod}\; 2)\}.$$

There is a unique such reducible anti-self-dual connection (cf.[13]). The other ends come from splittings $\mathcal{M}^0_X(\alpha)\#\mathcal{M}^1_Y(\alpha, \theta)$, where $\mathcal{M}^0_X(\alpha)$ is a compact zero dimensional moduli space with asymptotic value α. Let $n_\alpha = \#\mathcal{M}^0_X(\alpha)$. Thus $\mathcal{M}_X(\theta)$ is an oriented (see [7]) 1-dimensional manifold whose noncompact components fall into two classes. First, those components such that neither end comes from a reducible connection. Each end of such a moduli space corresponds to an $\mathcal{M}^1_Y(\alpha, \theta)$ which contributes $\pm n_\alpha$ to the counting $\#\mathcal{M}^1_Y(\alpha, \theta)$. The other end of this same component of $\mathcal{M}_X(\theta)$ contributes $\mp n_\alpha$ to the count; so they cancel out. There is one other noncompact component of $\mathcal{M}_X(\theta)$ with ends corresponding to the unique reducible anti-self-dual connection on P and the remaining component of $\mathcal{M}^1_Y(\alpha, \theta)$. Thus $\#\mathcal{M}^1_Y(\alpha, \theta) = \pm 1$. ■

We next want to calculate $HF_*(\Sigma(2,3,5)\#\Sigma(2,3,5))$ by using the spectral sequence. From §4.1 we have $E^1_{*,*}$ as in the figure. As we described in §4.2, $\partial_i = 0, \delta_i = 0, i = 0, 1$ and the only nontrivial contribution is from $d_{Y_i}\alpha = \pm\theta$ and ∂_3 from the gluing parameter $SO(3)$. I.e. $d_1 = 1 \otimes d_{Y_1} \pm d_{Y_0} \otimes 1 + \partial_3$. Thus we have

$$E^2_{1,0} = Z < \alpha\#\theta \pm \theta\#\alpha >, \qquad E^2_{2,0} = Z < (\beta\#\beta)_2 >,$$
$$E^2_{p,-2} = Z \oplus Z, \;\; \text{for } p = 2, 6, \qquad E^2_{p,-1} = Z_2 \oplus Z_2 \;\; \text{for } p = 2, 6,$$
$$E^2_{p,q} = 0 \;\; \text{otherwise}.$$

Obviously, $d_2 = 0$ and $d_3 = 0$ as well from Proposition 4.1.2. Thus $E^2_{*,*} = E^3_{*,*} = E^4_{*,*}$. The $d_4 : E^4_{6,-3} \to E^4_{2,0}$ is the only possible nontrivial differential. But $E^4_{2,0}$ is generated



by $(\beta\#\beta)_2$ and $\dim\mathcal{M}(\alpha\#\beta, \beta\#\beta) = -4$, so $d_4 = 0$. Hence the spectral sequence is convergent at $E^2_{*,*}$.

Theorem 4.3.2 *For the connected sum of Poincaré 3-sphere with itself, the Floer homology is*

$$HF_0 = Z_2 \oplus Z_2, \quad HF_1 = Z, \quad HF_2 = Z, \quad HF_3 = Z \oplus Z$$

$$HF_4 = Z_2 \oplus Z_2, \quad HF_5 = 0, \quad HF_6 = 0, \quad HF_7 = Z \oplus Z$$

Note that this theorem shows that Floer homology is not in general 4-periodic.

Example 2: Calculation of $HF_*(\Sigma(2,3,7)\#\Sigma(2,3,7))$

From [12], we have

$$HF_1(\Sigma(2,3,7)) = C_1(\Sigma(2,3,7)) = Z < a >, \quad HF_5(\Sigma(2,3,7)) = C_5(\Sigma(2,3,7)) = Z < b >$$

$$\dim\mathcal{M}_{\Sigma(2,3,7)}(a,\theta) \equiv 3, \quad \dim\mathcal{M}_{\Sigma(2,3,7)}(b,\theta) \equiv 7, \quad \dim\mathcal{M}_{\Sigma(2,3,7)}(b,a) \equiv 4, \bmod 8,$$

and we get

$$HF_0 = Z_2 \oplus Z_2 \quad HF_1 = 0 \quad HF_2 = Z \quad HF_3 = Z \oplus Z \oplus Z$$

$$HF_4 = Z_2 \oplus Z_2 \quad HF_5 = 0 \quad HF_6 = Z \quad HF_7 = Z \oplus Z \oplus Z$$

Example 3: Calculation for $HF_*(\Sigma(2,3,5)\#\Sigma(2,3,7))$.

An easy calculation shows

$$HF_0 = Z \quad HF_1 = Z \oplus Z \oplus Z \quad HF_2 = Z_2 \oplus Z_2 \quad HF_3 = 0$$

$$HF_4 = Z \quad HF_5 = Z \oplus Z \oplus Z \quad HF_6 = Z_2 \oplus Z_2 \quad HF_7 = 0.$$

Chapter 5

On spectral properties

5.1 The Laplacian on a connected sum

As we have explained in section 2, in order to calculate the Floer homology for a connected sum, one has to cut and paste anti-self-dual connections by cut-off functions which make the connection trivial on the tube $S^2 \times I \times \mathbf{R}$. Then the uniform lower boundedness of the Laplacian on anti-self-dual 2-forms on the tube plays an important role for solving the resulting anti-self-duality equation for the glued connection. We will show that the uniform lower bound on the tube goes to 0 as we stretch the length of the interval I to ∞.

We consider a homology 3-sphere Y with an open 3-ball, $B_y(r)$, removed, where r is the radius of the ball centered at y, and boundary of this manifold is a 2-sphere S^2. Let $\Omega^p(Y \setminus B_y(r), S^2)$ denote the space of smooth p-forms on $Y \setminus B_y(r)$ which vanish on S^2, and let Δ^p be the Laplacian acting on p-forms.

Lemma 5.1.1 *1. The operator $\Delta^p : L^2_2(\Omega^p(Y \setminus B_y(r), S^2)) \to L^2_0(\Omega^p(Y \setminus B_y(r), S^2))$ is injective.*

2. The volume form ω_0 of S^2 is the unique generator for $H^2_{dR}(S^2 \times I)$, up to a constant.

We omit the proof, which follows from Hodge theory.

Suppose Y_0 and Y_1 are oriented homology 3-spheres with Riemannian metrics which are flat in fixed small balls. We use an identification of these balls to define a connected sum $Y_0 \# Y_1$. Thus if we have locally oriented Euclidean coordinates ξ centered at $y_0 \in Y_0$ and η at $y_1 \in Y_1$, the identification map on an annular region is given by the conformal equivalence $\eta = f_\varepsilon(\xi) = \varepsilon^2 \overline{\xi}/|\xi|^2$. Here $\xi \to \overline{\xi}$ is any reflection and ε is a parameter which will eventually be made sufficiently small. We introduce another parameter N such that $N \cdot \varepsilon$ is less than the radius of injectivity of both Y_0 and Y_1.

Let $U_i \subset Y_i$ be the annulus centered at y_i with inner radius $N^{-1}\varepsilon$ and outer radius $N\varepsilon$. The conformal equivalence map induces a diffeomorphism from U_0 to U_1. We let $Y_i' \subset Y_i$ be the open set obtained by removing the $N^{-1}\varepsilon$ ball about y_i. Then, in the usual sense, we define the connected sum $Y = Y(\varepsilon)$ to be

$$Y = Y_0 \# Y_1 = Y_0' \cup_{f_\varepsilon} Y_1'$$

where the annuli U_i are identified by f_ε.

We shall also need to use another model for the connected sum. This depends on the conformal equivalence:

$$d : S^2 \times \mathbf{R} \longrightarrow R^3 \setminus \{0\}$$

given in 'polar coordinates' by $d(s,t) = s \cdot e^t$. Under this map the annulus U with radii $N\varepsilon, N^{-1}\varepsilon$ goes over to the tube $d^{-1}(U) = S^2 \times (\log \varepsilon - \log N, \log \varepsilon + \log N)$. Thus we can think of the connected sum as being formed by deleting the points y_i from Y_i, regarding punctured neighborhoods as half cylinders and identifying the cylinders by a reflection. (See [9])

From this point of view, we are working with the manifold $Y_i \setminus B_{y_i}(N^{-1}\varepsilon)$ and in the annular region we relabel the length by shifting $\log \varepsilon$ units and setting $T = \log N$. Thus we have identified the annular region with the tube $S^2 \times [-T, T]$. Define a smooth cut-off function β with support in $[-T, T]$ and $\beta = 1$ on $[-T + \varepsilon, T - \varepsilon]$.

Proposition 5.1.2 *Let $(Y \setminus B_y(r), \partial)$ be a homology 3-sphere with a 3-ball removed and fix a cut-off function β on the neck as above. Then there exists a unique $\eta \in \Omega^2(Y \setminus B_y(r), \partial)$ such that*

$$\Delta \eta = -\beta'' \omega_0$$

$$\eta = 0 \quad on \quad \partial$$

Remarks:

1. Δ is the Laplacian on $\Omega^2(Y \setminus B_y(r), \partial)$. To prove this proposition we shall need minimize the Lagrangian $L(\eta) = \int_{Y_0} |d\eta|^2 + |d^*\eta|^2 + 2\beta'' \omega_0 \wedge *\eta$, whose formal variational equation is the equation we expect to solve.

2. Injectivity of the Laplacian gives the nonzero first eigenvalue

$$\lambda_1 = \inf \frac{\|d^*\eta\|_{L^2}^2 + \|d\eta\|_{L^2}^2}{\|\eta\|_{L^2}^2} > 0.$$

   We also have the Poincaré inequality

$$\int_{Y_0} \eta \wedge *\eta \leq C \cdot \int_{Y_0} d\eta \wedge *d\eta + d^*\eta \wedge *d^*\eta.$$

Proof: The proof will follow the standard variational minimizing method which will give us a weak solution and then elliptic regularity will imply that the solution is smooth. This will give $\eta_0 \in C^\infty(\Omega^2(Y \setminus B_y(r), \partial))$ and $\Delta\eta_0 = -\beta''\omega_0$. So $\Delta(\beta\omega_0 - \eta_0) = 0$, $(\beta\omega_0 - \eta_0)|_\partial = \omega_0$. So $\beta\omega_0 - \eta_0$ is a harmonic 2-form with the correct boundary condition for patching the harmonic 2-forms together. We will follow the proof in [21] page 294 and [25].

We first show that for $\eta \in L_1^2(\Omega^2(Y \setminus B_y(r), \partial))$, the Lagrangian

$$L(\eta) = \int_{Y \setminus B_y(r)} |d\eta|^2 + |d^*\eta|^2 + 2\beta''\omega_0 \wedge *\eta$$

has a lower bound. Because the inequality $2ab \leq t^{-1}a^2 + tb^2$ holds for all positive t we have

$$|2(-\beta''\omega_0, \eta)| \leq t^{-1}\| - \beta''\omega_0\|_{L^2}^2 + t\|\eta\|_{L^2}^2.$$

Thus

$$L(\eta) \geq \|d\eta\|_{L^2}^2 + \|d^*\eta\|_{L^2}^2 - t^{-1}\| - \beta''\omega_0\|_{L^2}^2 - t\|\eta\|_{L^2}^2$$

for all $\eta \in L_1^2(\Omega^0(Y \setminus B_y(r), \partial))$.

Next use the Poincaré inequality to continue the inequality

$$\begin{aligned} &\geq (1 - tC)(\|d\eta\|_{L^2}^2 + \|d^*\eta\|_{L^2}^2) - t^{-1}\|\beta''\omega_0\|_{L^2}^2 \\ &= \frac{1}{2}(\|d\eta\|_{L^2}^2 + \|d^*\eta\|_{L^2}^2) - 2C\|\beta''\omega_0\|_{L^2}^2 \qquad (\text{letting } t = \tfrac{1}{2C}) \\ &\geq -2C\|\beta''\omega_0\|_{L^2}^2 \end{aligned}$$

Since β, ω_0 are fixed, and we have **Rang** $L \subset [-2C\|\beta''\omega_0\|_{L^2}^2, \infty)$, we have the desired lower bound. Hence $L_0 = \inf\{L(\eta_k) : \eta_k \in L_2^2(\Omega^2(Y \setminus B_y(r), \partial))\}$ exists.

Now choose a sequence $\{\eta_k \in L_2^2(\Omega^2(Y \setminus B_y(r), \partial))\}$ such that $L(\eta_k) \to L_0$ as $k \to \infty$. The above inequalities show that

$$\begin{aligned} \|d\eta_k\|_{L^2}^2 + \|d^*\eta_k\|_{L^2}^2 &\leq 2L_0 + 4C\|\beta''\omega_0\|_{L^2}^2 + 1, \text{ and} \\ \|\eta_k\|_{L^2}^2 &\leq C(\|d\eta_k\|_{L^2}^2 + \|d^*\eta_k\|_{L^2}^2) \\ &\leq C(2L_0 + 4C\|\beta''\omega_0\|_{L^2}^2 + 1) \end{aligned}$$

for $k > N$. An a priori estimate (see Morrey chapter 6 in [21], Appendix in [9] and Nirenberg [23] page 153) tells us that

$$\|\eta_k\|_{L_2^2} \leq C'(\|\beta''\omega_0\|_{L^2}^2 + \|\eta_k\|_{L^2}).$$

So using the above inequalities we see that the sequence $\{\eta_k\}$ is uniformly bounded in L_2^2 norm, since the unit ball in L_2^2 is weakly compact. Thus there exists a subsequence

$$\eta_{k'} \xrightarrow{L_2^2} \eta_0 \in L_2^2(\Omega^2(Y \setminus B_y(r)))$$

Since Y_0 is a 3-dimensional compact manifold, $L_2^2 \hookrightarrow C^0$ is a compact embedding; so we may assume that $\{\eta_{k'}\} \to \eta_0$ in C^0. Thus $\eta_0 = 0$ on ∂Y_0, and hence $\eta_0 \in L_2^2(\Omega^2(Y \setminus B_y(r), \partial))$. Weak convergence gives that $L(\eta_0) \leq \lim L(\eta_{k'}) = L_0$ which means that η_0 minimizes L over $L_2^2(\Omega^2(Y \setminus B_y(r), \partial))$.

Hence for any $\xi \in C^\infty(Y \setminus B_y(r), \partial)$ and $t \geq 0, L(\eta_0) \leq L(\eta_0 + t\xi)$

$$L(\eta_0 + t\xi) = L(\eta_0) + 2t \int_{Y \setminus B_y(r)} (d\eta_0 \wedge *d\xi + d^*\eta_0 \wedge *d^*\xi + \beta''\omega_0 \wedge *\xi) + O(t^2).$$

where $O(t^2) = t^2(\|d\xi\|_{L^2}^2 + \|d^*\xi\|_{L^2}^2)$ Since this holds for all $t \geq 0$,

$$\begin{aligned} 0 &\leq (d\eta_0, d\xi) + (d^*\eta_0, d^*\xi) + (\beta''\omega_0, \xi) \\ &= (d^*d\eta_0, \xi) + (\eta_0, d\xi)_\partial + (dd^*\eta_0, \xi) - (d^*\eta_0, \xi)_\partial + (\beta''\omega_0, \xi) \\ &= (\Delta\eta_0 + \beta''\omega_0, \xi) \end{aligned}$$

Replacing ξ by $-\xi$ shows that $\Delta\eta_0 + \beta''\omega_0 = 0$ in the weak sense, and elliptic regularity then implies that $\eta_0 \in C^\infty(\Omega^2(Y \setminus B_y(r), \partial))$ and $\Delta\eta_0 = -\beta''\omega_0$. This completes the proof.

Note: Note that we now have $\Delta(\beta\omega_0 - \eta_0) = 0$, $(\beta\omega_0 - \eta_0)|_{\partial Y_0 \setminus B_{y_0}(N^{-1}\varepsilon)} = \omega_0$, $\partial = S^2 \times \{T - \varepsilon\}$.

Using this proposition, we have solutions for $\Delta\eta_i = -\beta''\omega_0$, $\eta_i|_{\partial_i} = 0$ for $i = 0, 1$ where $\partial_i = S^2 \times \{T - \varepsilon\}$. Define

$$u = \begin{cases} \beta\omega_0 - \eta_0 & \text{on } (Y_0 \setminus B_{y_0}(N^{-1}\varepsilon), \partial_0) \\ \omega_0 & \text{on } S^2 \times [-T + \varepsilon, T - \varepsilon] \\ \beta\omega_0 - \eta_1 & \text{on } (Y_1 \setminus B_{y_1}(N^{-1}\varepsilon), \partial_1). \end{cases}$$

We have $u \in C^0(\Omega^2(Y_0 \# Y_1))$, and $\Delta u = 0$ in the weak sense. Choose the eigenfunction, say μ, corresponding to the first eigenvalue for the one dimensional Laplacian with Dirichlet boundary condition on $[-\frac{N}{\varepsilon}, \frac{N}{\varepsilon}]$. We use the 2-form $\mu \cdot u$ which has compact support on $Y \times \mathbf{R}$. In this case, the $L^2_{0,\delta}$ norm is equivalent to L^2 norm, so the operator $\Delta^2_{Y \times \mathbf{R}}$ is $-\frac{\partial^2}{\partial t^2} + \Delta_Y$. This means that the first eigenvalue of $\Delta^2_{Y \times \mathbf{R}}$ is $\leq (\frac{\pi}{2\frac{N}{\varepsilon}})^2$. Let $w = \mu u + \star\mu u \wedge dt \in \Omega^2_+(Y_0 \# Y_1 \times \mathbf{R})$, where $\star$ is the 3-dimensional Hodge star operator. Since μu is perpendicular to $\star\mu u \wedge dt$ and $\Delta^+ = \frac{1}{4}\Delta$ on Ω^2_+, we have

$$(\mu u, \Delta^+(\mu u))_{L^2} = \frac{1}{4}(\mu u, -\mu'' u)_{L^2} = C \cdot (\frac{\varepsilon}{N})^2 (\mu u, \mu u)_{L^2}$$

where $'$ denotes the derivative with respect to t — the $\mathbf{R}$ factor. So the same holds for the self-dual 2-form w. This implies that the first nonzero eigenvalue for Δ^+ is less than or equal to $C \cdot (\frac{\varepsilon}{N})^2$; so in particular it approaches 0 as the length of the tube goes to ∞. Another way to see this fact is to work directly with the weighted Sobolev space $L^2_{0,\delta}$. Then one has the one dimensional Laplacian

$$-\frac{d^2}{dt^2} + \delta \operatorname{sign}(t) \frac{d}{dt}$$

with Dirichlet boundary conditions on $[-\frac{N}{\varepsilon}, \frac{N}{\varepsilon}]$, and one sees that for this boundary value problem there is no zero eigenvalue. Then the eigenfunction of the first nonzero eigenvalue λ is

$$f = a e^{\frac{\delta + k}{2}|t|} + b e^{\frac{\delta - k}{2}|t|}$$

where $k = \sqrt{\delta^2 - 4\lambda}$. We have $f \in C^1$, and $f(t) = f(-t)$; so $\frac{df}{dt}(0) = 0$. Together with the boundary condition, this gives us

$$\frac{\delta + k}{2} \cdot a + \frac{\delta - k}{2} \cdot b = 0, \quad a e^{\frac{\delta + k}{2} \cdot \frac{N}{\varepsilon}} + b e^{\frac{\delta - k}{2} \cdot \frac{N}{\varepsilon}} = 0.$$

Combine these equations to get

$$\frac{k}{\delta} = \tanh(\frac{k}{2} \cdot \frac{N}{\varepsilon}).$$

Stretching the tube length to ∞ corresponds to letting $\varepsilon \to 0$ or $N \to \infty$, which means that $k \to \delta$, and thus $\lambda \to 0$. (In T. Mrowka's thesis [22] he proves the existence of a lower bound for the first eigenvalue of this operator when the tube length is fixed.) The fact that the first eigenvalues of Δ^+ go to 0 as the tube gets stretched to ∞ gives an "obstruction" to solving the anti-self-duality equation when one connects two anti-self-dual connections on $Y_0 \times \mathbf{R}$ and $Y_1 \times \mathbf{R}$. One may try to build an obstruction bundle as Taubes did in the case of a compact 4-manifold with an indefinite intersection form [28], but the following propositions tell us that we cannot do the same thing for a noncompact 4-manifold.

Proposition 5.1.3 *If $H^i(Y) = 0$, then $L^2_{0,\delta}H^{i+1}(Y \times \mathbf{R}) = 0$.*

Proof: This follows from Lefschetz duality and a theorem ([3],Proposition (4.9)) of Atiyah-Patodi-Singer.

For example, if Y is a homology 3-sphere, then $L^2_{0,\delta}H^2(Y \times \mathbf{R}) = 0$, and $L^2_{0,\delta}H^3(Y \times \mathbf{R}) = 0$; so we do not have an $L^2_{0,\delta}$ harmonic 2-form. However we do get the small eigenvalue for the Laplacian on self-dual 2-forms from Proposition 5.1.2.

In the statement of the next proposition we let σ denote "spectrum".

Proposition 5.1.4 *Let $\lambda_0 = \min\{\sigma(\Delta^i_Y), \sigma(\Delta^{i-1}_Y)\}$, then we have $\sigma(\Delta^i_{Y\times\mathbf{R}}) = [\lambda_0, +\infty)$. For a compact 3-manifold Y the spectrum consists of eigenvalues, and for the noncompact manifold $Y \times \mathbf{R}$ the spectrum is essential.*

Remark: This proposition points out that it is impossible to form an obstruction bundle in the manner of Taubes with a finite-dimensional fiber.

Proof: In this proof we will drop the weighted Sobolev norm notation and denote it simply by $\|\cdot\|$. Recall that $\lambda \in \sigma(A)$ for a self-adjoint operator A on a Hilbert space if and only if for arbitrary positive ε, there exists a nonzero u_ε such that

$$\|(A - \lambda I)u_\varepsilon\| \leq \varepsilon \cdot \|u_\varepsilon\|$$

(c.f. [33] Theorem 5.24).

Given $\lambda \in [\lambda_0, +\infty)$, write $\lambda = \lambda_0 + \mu$, $\mu \geq 0$. It is well-known that in $L^2_{0,\delta}$-norm the operator $\Delta_t = -\frac{\partial}{\partial t}(e^{-\delta|t|}\frac{\partial}{\partial t}e^{\delta|t|})$ has essential spectrum $[0,\infty)$ (see [33] section 10.2). So there exists f_ϵ such that

$$\|(\Delta_t - \mu I)f_\epsilon\| \leq \varepsilon \cdot \|f_\epsilon\|$$

If $\omega_1 + \omega_2 dt \in \Omega^i(Y \times \mathbf{R})$, then one has

$$\Delta^i_{Y\times\mathbf{R}}(f_\epsilon(\omega_1 + \omega_2 dt)) = \Delta_t f_\epsilon(\omega_1 + \omega_2 dt) + f_\epsilon(\Delta^i_Y \omega_1 + \Delta^{i-1}_Y \omega_2 dt).$$

Suppose that $\lambda_0 = \min \sigma(\Delta^{i-1}_Y)$. Let $\omega_1 = 0$ and ω_2 an eigenform corresponding to λ_0. Then $\Delta^{i-1}_Y \omega_2 = \lambda_0 \omega_2$. Therefore we have

$$\begin{aligned}
&\|\Delta^i_{Y\times\mathbf{R}}(f_\epsilon \omega_2 dt) - (\lambda_0 + \mu) f_\epsilon \omega_2 dt\| \\
&\quad = \|(\Delta_t f_\epsilon - \mu f_\epsilon)\omega_2 dt + f_\epsilon(\Delta^{i-1}_Y \omega_2 - \lambda_0 \omega_2) dt\| \\
&\quad \leq \varepsilon \|f_\epsilon \omega_2 dt\|
\end{aligned}$$

because of our choice of f_ϵ. So $\lambda_0 + \sigma(\Delta_t) \subset \sigma(\Delta^i_{Y\times\mathbf{R}})$, i.e. $[\lambda_0, +\infty) \subset \sigma(\Delta^i_{Y\times\mathbf{R}})$. When $\lambda_0 = \min \sigma(\Delta^i_Y)$, we get the same result by choosing ω_1 an eigenform corresponding to λ_0, i.e. $\Delta^i_Y \omega_1 = \lambda_0 \omega_1$, $\omega_2 = 0$.

Next we need to show that $\sigma(\Delta^i_{Y\times\mathbf{R}}) \subset [\lambda_0, +\infty)$. Given any $\lambda \in \sigma(\Delta^i_{Y\times\mathbf{R}})$ there is a u_ϵ with $\|(\Delta^i_{Y\times\mathbf{R}} - \lambda I)u_\epsilon\| \leq \varepsilon \cdot \|u_\epsilon\|$. Write $u_\epsilon = \omega_1(\varepsilon) + \omega_2(\varepsilon)dt$, where $\omega_1(\varepsilon)$ and $\omega_2(\varepsilon)dt$ are in $L^2_{0,\delta}\Omega^i(Y \times \mathbf{R})$ and are mutually orthogonal. Then also $\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon)$ and $\Delta^i_{Y\times\mathbf{R}}\omega_2(\varepsilon)dt$ are orthogonal. Thus we have the

$$\begin{aligned}
&\|\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) - \lambda\omega_1(\varepsilon) + \Delta^i_{Y\times\mathbf{R}}\omega_2(\varepsilon)dt - \lambda\omega_2(\varepsilon)dt\|^2 \\
&\quad = \|\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) - \lambda\omega_1(\varepsilon)\|^2 + \|\Delta^i_{Y\times\mathbf{R}}\omega_2(\varepsilon)dt - \lambda\omega_2(\varepsilon)dt\|^2 \\
&\quad \leq \varepsilon^2 \|u_\epsilon\|^2 \\
&\quad \leq \varepsilon^2 (\|\omega_1(\varepsilon)\|^2 + \|\omega_2(\varepsilon)dt\|^2)
\end{aligned}$$

We have

$$\text{either} \qquad \|\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) - \lambda\omega_1(\varepsilon)\|^2 \leq \varepsilon^2 \|\omega_1(\varepsilon)\|^2$$

$$\text{or} \qquad \|\Delta^i_{Y\times\mathbf{R}}\omega_2(\varepsilon)dt - \lambda\omega_2(\varepsilon)dt\|^2 \leq \varepsilon^2 \|\omega_2(\varepsilon)dt\|^2$$

In the first case where $\|\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) - \lambda\omega_1(\varepsilon)\|^2 \le \varepsilon^2\|\omega_1(\varepsilon)\|^2$.

Let $\{\phi_i\}$ be an orthonormal basis of $L^2_{0,\delta}\Omega^i(Y)$ consisting of eigenforms of Δ^i_Y with eigenvalues $\{\lambda_i\}$, and write $\omega_1(\varepsilon) = \sum_i f_i(t,\varepsilon)\phi_i$.

$$\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) = \sum_j \Delta^i_{Y\times\mathbf{R}} f_j(t,\varepsilon)\phi_j = \sum_j (\Delta_t f_j(t,\varepsilon)\cdot\phi_j + \lambda_j f_j(t,\varepsilon)\phi_j)$$

$$\Delta^i_{Y\times\mathbf{R}}\omega_1(\varepsilon) - \lambda\omega_1(\varepsilon) = \sum_j (\Delta_t f_j(t,\varepsilon)\cdot\phi_j + \lambda_j f_j(t,\varepsilon)\phi_j - \lambda f_j(t,\varepsilon)\phi_j).$$

Since the basis $\{\phi_i\}$ is orthonormal, we have

$$\sum_j \|\Delta_t f_j(t,\varepsilon) + (\lambda_j - \lambda) f_j(t,\varepsilon)\|^2 \le \varepsilon^2 \sum_j \|f_j(t,\varepsilon)\|^2.$$

Thus there exists a k such that

$$\|\Delta_t f_k(t,\varepsilon) + (\lambda_k - \lambda) f_k(t,\varepsilon)\|^2 \le \varepsilon^2\|f_k(t,\varepsilon)\|^2.$$

where $\|f_k(t,\varepsilon)\| \ne 0$.

If $\lambda < \lambda_0$, recall $\lambda \in \sigma(\Delta^i_{Y\times\mathbf{R}})$, so we have $\lambda_k - \lambda \ge \lambda_0 - \lambda > 0$, therefore $\lambda - \lambda_k < 0$; so $\lambda - \lambda_k$ is not in the spectrum of Δ_t, i.e. $\Delta_t - (\lambda - \lambda_k)I$ is invertible. Thus there is a positive constant c (independent of ε) satisfying

$$c\cdot\|f_k(t,\varepsilon)\| \le \|(\Delta_t - (\lambda - \lambda_k)I) f_k(t,\varepsilon)\| \le \varepsilon\cdot\|f_k(t,\varepsilon)\|.$$

Choosing a small enough ε we get a contradiction. So $\lambda \ge \lambda_0$, i.e. $\sigma(\Delta^i_{Y\times\mathbf{R}}) \subset [\lambda_0, +\infty)$.

Similarly one can reach the same conclusion in the second case. So we have shown that $\sigma(\Delta^i_{Y\times\mathbf{R}}) \subset [\lambda_0, +\infty)$ in both cases. This completes the proof that $\sigma(\Delta^i_{Y\times\mathbf{R}}) = [\lambda_0, +\infty)$. ■

In particular, for a fixed homology 3-sphere Y (i.e with a fixed Riemannian metric), the spectrum is strictly positive, i.e.

$$\min\{\sigma(\Delta^2_{Y\times\mathbf{R}}), \sigma(\Delta^1_{Y\times\mathbf{R}})\} = \lambda_0 > 0.$$

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