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ABSTRACT

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In this paper we construct first order and second order piecewise polynomial finite approximation schemes. These schemes are for the computation of invariant measures of nonsingular measurable transformations on the unit interval, and fall into two groups. The first one is based on the Galerkin projection method for L^1-spaces. The second one uses the idea of Markov approximations of finite rank to the Frobenius-Perron operator. These methods are proved to converge for a class of transformations satisfying the condition of the Lasota-Yorke theorem. Moreover the computational experiments show that these schemes converge faster than Ulam-Li's method for most problems.

To my mother Wang Liu-Feng

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Chapter 1

Introduction

Let I be the unit interval $[0,1]$ and $S : I \to I$ be a mapping. Given $x_0 \in I$, we recursively define a sequence $\{x_n\}$ by letting $x_{n+1} = S(x_n)$ for $n = 0, 1, 2, \ldots$.

Suppose for a particular sequence of iterates $\{x_n\}$, we happen to have $S(x_{2000}) = x_{1000}$. Then eventually, the iterates will be in the set of finitely many elements $\{x_{1000}, x_{1001}, \ldots, x_{2000}\}$. Thus we have a predictable behavior for the iterates.

The study of chaotic dynamical systems has become very popular in sciences and engineering. The term "chaos" was first introduced by Li and Yorke in their seminal paper [6]. Various definitions have been given for "chaos" since then, but basically a common fundamental feature of "chaos" is "unpredictability". A typical example of chaotic dynamical systems is given by the "logistic model" $S(x) = 4x(1-x)$. It is well-known that for almost all $x_0 \in [0,1]$, the iterates are dense in I. Thus we cannot predict the limiting behavior of the iterates.

We may look at a dynamical system from a different point of view. Given a set A, we determine the probability of the iterates entering A. For this purpose let χ_A be the characteristic function of A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Starting at x_0, if the nth iterate $S^n(x_0)$ is in A, then $\chi_A(S^n(x_0)) = 1$. Otherwise $\chi_A(S^n(x_0)) = 0$. Thus $\frac{1}{N}\sum_{n=0}^{N-1} \chi_A(S^n(x_0))$ gives the ratio of the points among the first N iterates in A. In ergodic theory

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_A(S^n(x_0))$$

is called the *time average.* A classic theorem of ergodic theory basically says: The time average coincides with the *space average*, that is

$$\frac{\mu(A)}{\mu(I)} = \frac{1}{\mu(I)} \int_I \chi_A d\mu,$$

where μ is a measure on I.

Now the question arises: Is the measure invariant with respect to time? A more careful statement of the classic ergodic theorem should be: The time average equals the space average which is invariant with respect to time. Using mathematical terms, it simply says that the measure μ is "invariant" with respect to S, or S "preserves" the measure μ. That is, for any measurable set A, $\mu(S^{-1}(A)) = \mu(A)$. In this case, μ is called an invariant measure. If in addition $\mu(I) = 1$, we call it an invariant probability measure.

Now the problem is: Is there any probability measure which is invariant with respect to S? This leads to the concept of the Frobenius-Perron operator. This operator gives the way in which the probability distribution changes according to the transformation S.

Now let m be Lebesgue measure and $L^1(m)$ the set of m-integrable functions. Suppose a probability measure μ is given by a nonnegative $L^1(m)$-function f, that is, $\mu(A) = \int_A f dm$ for every m-measurable subset A of I. Given a nonsingular measurable transformation $S : I \to I$, we examine how the probability distribution is converted by S.

For any measurable set A we would like this set A to have the probability of the set it comes from under S. Thus, A should have the probability $\tilde{\mu}(A) = \int_{S^{-1}(A)} f dm$. Since S is nonsingular, $\tilde{\mu}$ is absolutely continuous with respect to m. Thus by the Radon-Nikodym theorem, there exists a unique density $\tilde{f} \in L^1(m)$ such that $\int_A \tilde{f} dm = \int_{S^{-1}(A)} f dm$ for any measurable subset A. The correspondence between f

and $\tilde{f}$ defines the Frobenius-Perron operator $P_S : L^1(m) \to L^1(m)$ associated with S:

$$\int_A P_S f dm = \int_{S^{-1}(A)} f dm \tag{1.1}$$

It is apparent that if $f \geq 0$ is a fixed point of P_S, then the measure μ_f defined by

$$\mu_f(A) = \int_A f dm$$

is invariant with respect to S. In this case we call f an invariant density. Thus to find an invariant measure we may instead find a fixed point of the corresponding Frobenius-Perron operator.

To calculate fixed points of the Frobenius-Perron operator numerically, it is important to make finite approximations of this operator. For this purpose divide I into n subintervals $I_1, I_2, \ldots, I_n$. Suppose a piecewise constant density f gives I_i the probability a_i for $i = 1, 2, \ldots, n$. For $S : I \to I$, it is easy to see that the probability of I_i induced by S is

$$b_i = \sum_{j=1}^{n} \frac{m(I_j \cap S^{-1}(I_i))}{m(I_j)} a_j$$

for $i = 1, 2, \ldots, n$. Let $P_n = [p_{ij}]$ with $p_{ij} = \frac{m(I_j \cap S^{-1}(I_i))}{m(I_j)}$, $a = [a_1, a_2, \ldots, a_n]^T$, and $b = [b_1, b_2, \ldots, b_n]^T$. Then the relation

$$b = P_n a$$

gives a finite approximation of the Frobenius-Perron operator.

Since P_n is an $n \times n$ nonnegative matrix and the sum of each column of P_n is 1, P_n is a stochastic matrix. It is well known that a stochastic matrix has 1 as an eigenvalue with a nonnegative eigenvector. Therefore P_n has a fixed point which gives a nonnegative piecewise constant function f_n.

In 1960 S. Ulam conjectured [10]: The piecewise constant functions f_n converge to an invariant density f of the Frobenius-Perron operator P_S as n approaches infinity under the stretching condition, i. e., $\inf_{x \in I} | S'(x) | > 1$. In 1976, Li proved this conjecture [5].

Numerical experiments show that the Ulam-Li's method converges very slowly for most problems. This situation motivates the investigation of higher order approximations of the Frobenius-Perron operator. It seems difficult to generalize the previous argument based on probability analysis. However, we may consider this problem from a totally different point of view.

To solve $Pf = g$ for f and g in a Banach space X, we may employ Galerkin's projection method [9]. That is, we project this equation into a finite dimensional subspace of X and solve the resulting finite dimensional problem in this subspace. To find the projection, we need an *inner product* between X and its adjoint space. For our fixed point problem $P_S f - f = 0$, we define the inner product of two functions $f \in L^1(m)$ and $g \in L^\infty(m) = L^1(m)^*$ to be

$$< f, g > \;=\; \int_0^1 f(x)g(x)dx. \tag{1.2}$$

To construct a finite dimensional subspace, we divide the interval I into n subintervals $I_1, I_2, \ldots, I_n$. For $i = 1, 2, \ldots, n$, let

$$1_i \;=\; \frac{\chi_{I_i}}{m(I_i)} \tag{1.3}$$

and $\Delta_n = \{\sum_{i=1}^n a_i 1_i : a_i \in \Re, i = 1, \ldots, n\}$. Then Δ_n is the n-dimensional subspace of L^1 which is made up of piecewise constant functions.

Using Galerkin's projection method to solve the equation $P_S f - f = 0$ in Δ_n, the function

$$P_S(\sum_{j=1}^n a_j 1_j) - \sum_{j=1}^n a_j 1_j$$

should be orthogonal to each basis function, that is,

$$< P_S(\sum_{j=1}^n a_j 1_j) - \sum_{j=1}^n a_j 1_j, 1_i >= 0,$$

or,

$$\sum_{j=1}^n a_j < P_S 1_j, 1_i > \;=\; \sum_{j=1}^n a_j < 1_j, 1_i > . \tag{1.4}$$

From (1.1), (1.2), and (1.3),

$$\begin{aligned} < P_S 1_j, 1_i > &= \int_I (P_S 1_j) 1_i dm = \frac{1}{m(I_i)} \int_{I_i} P_S 1_j dm \\ &= \frac{1}{m(I_i)} \int_{S^{-1}(I_i)} 1_j dm = \frac{1}{m(I_i)m(I_j)} \int_{S^{-1}(I_i)} \chi_{I_j} dm \\ &= \frac{m(I_j \cap S^{-1}(I_i))}{m(I_i)m(I_j)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{n} a_j < 1_j, 1_i > &= a_i < 1_i, 1_i >= \frac{a_i}{m(I_i)^2}\int_{I_i}\chi_{I_i}\chi_{I_i}dm \\ &= \frac{a_i}{m(I_i)^2}\int_{I_i} dm = \frac{a_i}{m(I_i)}.\end{aligned}$$

Thus (1.4) becomes

$$\sum_{j=1}^{n}\frac{m(I_j\cap S^{-1}(I_i))}{m(I_i)m(I_j)}a_j = \frac{a_i}{m(I_i)},\quad i = 1,\ldots,n,$$

or,

$$\sum_{j=1}^{n}\frac{m(I_j\cap S^{-1}(I_i))}{m(I_j)}a_j = a_i,\quad i = 1,\ldots,n.$$

This gives exactly Ulam-Li's method. Hence Ulam-Li method is essentially an application of Galerkin's method on the subspace of piecewise constant functions.

From this point of view, we may generalize the method by choosing higher order basis functions to improve the convergence rate. In the following we introduce a first order and a second order piecewise polynomial approximation scheme for the computation of fixed points of the Frobenius-Perron operator, based on Galerkin's projection method. In [2] a general piecewise polynomial projection procedure is proposed. But in order to prove the convergence of the method, it is assumed that the invariant density of the Frobenius-Perron operator is bounded as well as unique. This makes the analysis easier, because Hilbert space techniques may be used in this case. Without the assumption of boundedness of invariant densities, we show that our schemes are convergent for a general class of nonsingular measurable transformations.

Chapter II discusses the Frobenius-Perron operator, its basic properties, and the general framework of Galerkin's projection method. Chapter III and Chapter IV are devoted to the piecewise linear and piecewise quadratic polynomial projection approximation methods, respectively. In Chapter V, we develop the Markov finite approximation schemes. Numerical results are presented in Chapter VI, and compared with Li's original method in [5]. The last chapter gives some comments and conclusions.

Chapter 2

Frobenius-Perron Operators and Projection Methods

The purpose of this chapter is to provide the background material for the subsequent chapters. In Section 2.1, we define the Frobenius-Perron operator and list those properties it has which are useful to us. Section 2.2 is a brief introduction to the Galerkin projection method.

2.1 Frobenius-Perron Operators

Let $I = [0,1]$ and S be a transformation from I into itself. For $A \subset [0,1]$ we write $S^{-1}(A)$ for $\{x : S(x) \in A\}$. Lebesgue measure on $[0,1]$ will be denoted by m and the Borel-algebra of subsets of $[0,1]$, the minimal σ-algebra of $[0,1]$ containing all the open sets of $[0,1]$, will be denoted by $\mathcal{B}$. For any measure μ on $\mathcal{B}$, the triple $(I, \mathcal{B}, \mu)$ is called a Borel measure space. Let $L^1(0,1)$ be the space of all Lebesgue integrable functions defined on $[0,1]$. $L^1(0,1)$ is a Banach space with norm $\| f \| = \int_0^1 |f(x)|\, dx$.

Definition 2.1.1. Let $(I, \mathcal{B}, \mu)$ be a Borel measure space.

(1) A transformation $S : I \to I$ is measurable if $S^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}$.

(2) A measurable transformation $S : I \to I$ is said to be nonsingular if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ satisfying $\mu(A) = 0$.

In the sequel we are interested only in nonsingular measurable transformations. For the purpose of defining the Frobenius-Perron operator $P_S : L^1(0,1) \to L^1(0,1)$ associated with a nonsingular measurable transformation S, we first state the Radon-Nikodym Theorem. (For a proof, see [1].)

Theorem 2.1.1. (Radon-Nikodym)

Let $(X, \mathcal{A})$ be a measurable space, μ be a (positive) measure, and ν be a finite signed measure. Suppose ν is absolutely continuous with respect to μ, i.e., $\nu(A) = 0$ whenever $\mu(A) = 0$ for any $A \in \mathcal{A}$. Then there exists a unique function $f \in L^1(\mu)$ such that

$$\nu(A) = \int_A f d\mu.$$

For $f \in L^1(0,1)$,

$$\nu(A) = \int_{S^{-1}(A)} f dm$$

defines a finite signed measure. Since $S : I \to I$ is nonsingular, the measure ν is absolutely continuous with respect to m. By the Radon-Nikodym Theorem there exists a unique L^1-function, which we denote by $P_S f$, such that

$$\int_A P_S f dm = \nu(A) = \int_{S^{-1}(A)} f dm.$$

Definition 2.1.2. The operator $P_S : L^1(0,1) \to L^1(0,1)$ defined by

$$\int_A P_S f(x) dx \quad = \quad \int_{S^{-1}(A)} f(x) dx \tag{2.1}$$

is called the Frobenius-Perron operator associated with S. If there is no ambiguity, we shall write P for P_S.

Proposition 2.1.1. [Properties of the Frobenius-Perron operator]

(1) P is linear, i. e., for $f_1, f_2 \in L^1(0,1)$ and $\lambda_1, \lambda_2 \in \Re$

$$P(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 P f_1 + \lambda_2 P f_2.$$

(2) If $f \geq 0$, then $Pf \geq 0$.

(3) $\int_0^1 Pf(x)dx = \int_0^1 f(x)dx$.

(4) For the nth power $S^n, P_{S^n} = (P_S)^n$.

Proof. See [3].

It follows from (2) and (3) that the Frobenius-Perron operator P_S not only preserves nonnegative functions, but also preserves their norms. Thus P_S is a Markov operator. Hence $\| P_S \| = 1$.

Definition 2.1.3. Let $(I, \mathcal{B}, \mu)$ be a Borel measure space and $S : I \to I$ be a measurable transformation. We say that μ is invariant under S, or S is measure-preserving with respect to μ, if $\mu(S^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.

Theorem 2.1.2. For $f \in L^1(0,1)$ and $f \geq 0$, the measure

$$\mu_f(A) \quad = \quad \int_A f(x)dx \tag{2.2}$$

is invariant under S if and only if $P_S f = f$.

Proof. Since

$$\mu_f(S^{-1}(A)) = \int_{S^{-1}(A)} f(x)dx = \int_A P_S f(x)dx,$$

$\mu_f(A) = \mu_f(S^{-1}(A))$ for any A implies $P_S f = f$ and vise verse. **Q.E.D.**

The function f in (2.2) is usually called the density function of the measure μ_f. It is apparent that a measure can be calculated when its density is known. The density of an invariant measure is characterized by Theorem 2.1.2 as a nonnegative fixed point of the Frobenius-Perron operator. In this case it is called an invariant density. Therefore to calculate invariant measures for S, we may calculate instead invariant densities of the corresponding Frobenius-Perron operator.

However, to find a fixed point of the Frobenius-Perron operator is, unfortunately, not so simple in general. First of all, the space $L^1(0,1)$ is not reflexive. Moreover, the operator P_S is not compact. Let $A = [0, x]$. Then from the definition of the

Frobenius-Perron operator,

$$\int_0^x Pf(t)dt = \int_{S^{-1}(0,x)} f(t)dt.$$

Differentiating it, we obtain the Frobenius-Perron operator explicitly:

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}(0,x)} f(t)dt. \tag{2.3}$$

For the logistic model $S(x) = 4x(1-x)$, (2.3) becomes

$$Pf(x) = \frac{1}{4\sqrt{1-x}}\{f(\frac{1}{2}(1-\sqrt{1-x})) + f(\frac{1}{2}(1+\sqrt{1-x}))\}.$$

For a class of stretching transformations from I into itself, Lasota and Yorke [3] established the existence of invariant densities. In [7], Li and Yorke gave a sufficient condition for the uniqueness of the invariant density and thus the ergodicity of the mapping.

Definition 2.1.4. A mapping $S : [0,1] \to [0,1]$ is called piecewise C^2, if there exists a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of the unit interval such that for each integer $k = 1, \ldots, r$, the restriction S_k of S to the open interval (a_{k-1}, a_k) is a C^2-function which can be extended to the closed interval $[a_{k-1}, a_k]$ as a C^2-function. S need not be continuous at the points a_k.

Theorem 2.1.3. (Lasota-Yorke)

Let $S : [0,1] \to [0,1]$ be a piecewise C^2-mapping satisfying the stretching condition: there exists a constant $\lambda > 1$ such that

$$\mid S'(x) \mid \geq \lambda, x \neq a_i (i = 0, 1, \ldots, r).$$

Then for any function $f \in L^1(0,1)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} P_S^k f$$

converges uniformly in $L^1(0,1)$ to some f^* of bounded variation with $P_S f^* = f^*$.

Proof. See [4].

Theorem 2.1.4. (Li-Yorke)

Under the condition of Theorem 2.1.3, if the mapping S has a single point of discontinuity, then the invariant density f^* of the Frobenius-Perron operator P_S associated with S is unique, and S is ergodic with respect to the measure μ^* defined by

$$\mu^*(A) = \int_A f^* dm.$$

Proof. See [7].

A straightforward numerical way to calculate invariant measures can be obtained from the classical Birkhoff Individual Ergodic Theorem which uses the Koopman operator instead of the Frobenius-Perron operator. By Birkhoff's theorem if μ is an ergodic invariant probability measure for S, then for any measurable set $A \subset [0,1]$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(S^k(x)),$$

which measures the "average time" spent in A under iterations of S, exists and is $\mu(A)$ for μ-almost all x. Hence, to obtain $\mu(A)$ one might choose almost any x in $[0,1]$ and calculate the average time for iterates $S^k(x)$ to be in A. However, computer round-off error can completely dominate the calculation and make the implementation difficult. A typical example is given in [5]. For the purpose of overcoming this difficulty, Li proposed in [5] a rigorous numerical procedure which can be implemented on a computer with negligible round-off error. Piecewise constant approximations are used to reduce the original infinite-dimensional fixed point problem to a fixed point problem of a stochastic matrix, thus solving a conjecture of Ulam's [10].

The numerical procedure proposed by Li is actually a Galerkin projection method with piecewise constant function approximations. We shall give a brief introduction to Galerkin's projection method in the next section.

2.2 Galerkin's Projection Method

Let X be a Banach space. Suppose M and N are both closed subspaces of X. If $X = M + N$ and $M \cap N = \{0\}$, then we say X is a direct sum of M and N, or M and N are complementary to each other. In this case we may define a linear operator $Q : X \to X$ by

$$Qx = u \quad \text{if} \quad x = u + v, \;\; u \in M, \;\; v \in N.$$

This operator is continuous and satisfies $Q^2 = Q$ [1]. We call Q the projection of X onto M along N.

Now let X and Y be two Banach spaces, $T : X \to Y$ be a bounded linear operator, and $y \in Y$. We want to solve the operator equation

$$Tx = y.$$

The general principle of projection methods is as follows. Choose two sequences of finite-dimensional subspaces X_n and Y_n of X and Y, respectively. Let $\{Q_n\}$ be a sequence of projections of Y onto Y_n. We want to find $x^{(n)}$ in X_n such that $Q_n(Tx^{(n)} - y) = 0$, or

$$Q_n T x^{(n)} = Q_n y.$$

If we choose a basis of X_n and a basis of Y_n, then the above approximate operator equation of finite rank can be written as a system of linear algebraic equations. Thus we can use the usual numerical algorithms to solve the algebraic system and obtain approximate solutions to the original problem. This procedure is referred to as the projection method. In particular, if $X = Y$ and if we choose $X_n = Y_n$ and the same basis in Y_n as in X_n, then the corresponding projection method is called Galerkin's method.

Chapter 3

Piecewise Linear Projection Approximations

Assume $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $\inf |S'(x)| > 1$. In this chapter we look for approximate solutions of the Frobenius-Perron operator equation $P_S f = f$ in the space of piecewise linear functions. In Section 3.1 we define a sequence of projection operators from $L^1(0,1)$ to subspaces consisting of piecewise linear functions. Section 3.2 establishes the uniform boundedness of the variation of the projected functions. The convergence theorem is proved in Section 3.3.

3.1 Projection Operators

Divide $I = [0,1]$ into n subintervals $I_1, I_2, \ldots, I_n$. For $i = 1, \ldots, n$, let $I_i = (x_{i-1}, x_i)$ and $1_i = \chi_{I_i}/m(I_i)$. Denote by Δ_n the $2n$-dimensional subspace of $L^1(0,1)$ spanned by the basis $\{1_i, x1_i\}_{i=1}^n$; i. e., $\Delta_n \subset L^1(0,1)$ is the set of all functions which are linear on each subinterval I_i.

To define the projection $Q_n : L^1(0,1) \to \Delta_n$ we require that, for $i = 1, \ldots, n$,

$$< f - Q_n f, 1_i > = 0$$

and

$$< f - Q_n f, x1_i > = 0.$$

Here for $g \in L^1(0,1)$ and $h \in L^\infty(0,1) = [L^1(0,1)]^*$, $< g,h >= \int_0^1 g(x)h(x)dx$. The following lemma shows that these requirements uniquely define Q_n and make Q_n a projection from $L^1(0,1)$ to Δ_n along ${}^\perp\Delta_n \equiv \{g \in L^1(0,1) : < g,h > = 0$ for all $h \in \Delta_n\}$. Because of the similarity in the "orthogonality condition" with the L^2-space case, we may call $Q_n : L^1(0,1) \to \Delta_n$ the orthogonal projection, even though its norm may not be 1.

Lemma 3.1.1. Let $\tilde{x}_i = (x_{i-1} + x_i)/2, \quad i = 1,\ldots,n$. For any $f \in L^1(0,1)$, we have

$$Q_n f = \sum_{i=1}^{n}(c_i + d_i x)1_i$$

where for $i = 1,\ldots,n$,

$$\begin{cases} c_i = \int_{I_i} f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^2} \int_{I_i}(x - \tilde{x}_i)f(x)dx \\ d_i = \frac{12}{m(I_i)^2} \int_{I_i}(x - \tilde{x}_i)f(x)dx. \end{cases} \tag{3.1}$$

Proof. Let $Q_n f = \sum_{i=1}^{n} (c_i + d_i x)1_i$. Then

$$\begin{aligned} < Q_n f, 1_i > &= c_i < 1_i, 1_i > + d_i < x1_i, 1_i > = \frac{1}{m(I_i)} c_i + \frac{\tilde{x}_i}{m(I_i)} d_i, \\ < Q_n f, x1_i > &= c_i < 1_i, x1_i > + d_i < x1_i, x1_i > \\ &= \frac{\tilde{x}_i}{m(I_i)} c_i + \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)} d_i. \end{aligned}$$

From the condition of the orthogonal projection, we have

$$\begin{cases} \frac{1}{m(I_i)} c_i + \frac{\tilde{x}_i}{m(I_i)} d_i = \frac{1}{m(I_i)} \int_{I_i} f(x)dx \\ \frac{\tilde{x}_i}{m(I_i)} c_i + \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)} d_i = \frac{1}{m(I_i)} \int_{I_i} x f(x)dx. \end{cases} \tag{3.2}$$

The equation (3.2) has a unique solution

$$\begin{cases} c_i = \int_{I_i} f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i)f(x)dx \\ d_i = \frac{12}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i)f(x)dx. \end{cases}$$

Q.E.D.

The next Lemma establishes the uniform boundedness of the sequence Q_n.

Lemma 3.1.2. For all n, $\| Q_n \| \leq 2$.

Proof. Given n and $f \in L^1(0,1)$,

$$\begin{aligned}\| Q_n f \| &= \int_0^1 |(Q_n f)(x)|\,dx = \int_0^1 \sum_{i=1}^n |(c_i + d_i x) 1_i(x)|\,dx \\ &= \sum_{i=1}^n \int_{I_i} \frac{1}{m(I_i)}\,|c_i + d_i x|\,dx.\end{aligned}$$

By (3.1), in the subinterval I_i, $Q_n f$ only depends on the values of f on I_i. Hence it is enough to estimate a typical $\frac{1}{m(I_i)} \int_{I_i} |c_i + d_i x|\,dx$. Without loss of generality, we may assume $d_i \neq 0$. For simplicity, let $I = I_i = [a,b]$, $\tilde{x} = \tilde{x}_i$, $c = c_i$, $d = d_i$ and f be defined on I. Let $\varphi(x) = (c + dx)/m(I)$.

First, assume $f \geq 0$. If $\varphi \geq 0$, then from the first equality of (2),

$$\begin{aligned}\int_I |\varphi(x)|\,dx &= \int_I \varphi(x)dx = \frac{1}{m(I)} \int_a^b (c + dx)dx \\ &= \frac{1}{m(I)}\, \frac{(c+dx)^2}{2d}\bigg|_a^b = \frac{1}{2dm(I)}\,[(c+db)^2 - (c+da)^2] \\ &= \frac{1}{2dm(I)}\,[2cdm(I) + d^2 m(I)\cdot 2\tilde{x}] = c + d\tilde{x} \\ &= \int_I f(x)dx = \int_I |f(x)|\,dx.\end{aligned}$$

If $\varphi \not\geq 0$, then from the fact that φ is the best approximation to f among all linear functions on $[a,b]$ under L^2-norm if $f \in L^2(0,1)$, we see that φ cannot be non-positive. Therefore φ must have a zero $z = -\frac{c}{d}$ in (a,b). We assume $\varphi(b) > 0$ and $\varphi(a) < 0$. The other case can be treated similarly. Thus we have

$$\begin{aligned}\int_I |\varphi(x)|\,dx &= \frac{1}{2}\,[(z-a)\,|\varphi(a)| \;+\; (b-z)\,|\varphi(b)|\,] \\ &= \frac{1}{2}\,[(b + \frac{c}{d})\varphi(b) + (a + \frac{c}{d})\varphi(a)],\end{aligned}$$

and,

$$b + \frac{c}{d} = \frac{m(I)^2 \int_I f(x)dx}{12 \int_I (x - \tilde{x}) f(x)dx} + \frac{m(I)}{2},$$

$$a + \frac{c}{d} = \frac{m(I)^2 \int_I f(x)dx}{12 \int_I (x - \tilde{x}) f(x)dx} - \frac{m(I)}{2},$$

$$\varphi(b) = \frac{1}{m(I)}(c + db) = \frac{1}{m(I)}\,[\int_I f(x)dx + \frac{6}{m(I)} \int_I (x - \tilde{x}) f(x)dx],$$

$$\varphi(a) = \frac{1}{m(I)}(c + da) = \frac{1}{m(I)}\,[\int_I f(x)dx - \frac{6}{m(I)} \int_I (x - \tilde{x}) f(x)dx].$$

Hence,

$$\begin{aligned}
\int_I |\varphi(x)|\,dx &= \frac{1}{2}\{(\frac{m(I)^2\int_I f(x)dx}{12\int_I(x-\tilde{x})f(x)dx}+\frac{m(I)}{2}) \\
&\quad \cdot \frac{1}{m(I)}(\int_I f(x)dx+\frac{6}{m(I)}\int_I(x-\tilde{x})f(x)dx) \\
&\quad +(\frac{m(I)^2\int_I f(x)dx}{12\int_I(x-\tilde{x})f(x)dx}-\frac{m(I)}{2}) \\
&\quad \cdot \frac{1}{m(I)}(\int_I f(x)dx-\frac{6}{m(I)}\int_I(x-\tilde{x})f(x)dx)\} \\
&= \frac{m(I)[\int_I f(x)dx]^2}{12\int_I(x-\tilde{x})f(x)dx}+\frac{3}{m(I)}\int_I(x-\tilde{x})f(x)dx \\
&\leq \frac{m(I)[\int_I f(x)dx]^2}{12\int_I(x-\tilde{x})f(x)dx}+\frac{3}{2}\int_I f(x)dx.
\end{aligned}$$

Since $z=-\frac{c}{d}\in(a,b)$, we have

$$a<\tilde{x}-\frac{m(I)^2\int_I f(x)dx}{12\int_I(x-\tilde{x})f(x)dx}<b.$$

It follows that

$$\frac{m(I)^2\int_I f(x)dx}{12\int_I(x-\tilde{x})f(x)dx}<\tilde{x}-a=\frac{m(I)}{2}.$$

Therefore

$$\int_I |\varphi(x)|\,dx<\frac{1}{2}\int_I f(x)dx+\frac{3}{2}\int_I f(x)dx=2\int_I f(x)dx.$$

For general $f\in L^1(I)$, write $f=f^+-f^-$ where $f^+=\max\{f,0\}$ and $f^-=\max\{-f,0\}$, and we have

$$\begin{aligned}
\int_I |\varphi(x)|\,dx &= \int_I |Qf|\,dx=\int_I |Qf^+-Qf^-|\,dx \\
&\leq \int_I |Qf^+|\,dx+\int_I |Qf^-|\,dx \\
&\leq 2\int_I f^+dx+2\int_I f^-dx=2\int_I |f|dx
\end{aligned}$$

where $Q: L^1(I)\to \mathrm{Span}\{1,x\}$ is the orthogonal projection mentioned above.

From the above estimate, we obtain

$$\begin{aligned}
\| Q_n f \| &= \int_0^1 |(Q_nf)(x)|dx=\sum_{i=1}^n\int_{I_i}\frac{1}{m(I_i)}|c_i+d_ix|dx \\
&\leq \sum_{i=1}^n 2\int_{I_i}|f(x)|\,dx=2\int_0^1|f(x)|dx=2\,\| f \|,
\end{aligned}$$

i. e., for all n, $\| Q_n \| \leq 2$. **Q.E.D.**

Lemma 3.1.3. When mesh $\Delta_n \equiv \max\{m(I_i) : 1 \leq i \leq n\} \to 0$, $Q_n f \to f$ in L^1 norm for all $f \in L^1(0,1)$.

Proof. Given $f \in L^1(0,1)$ and $\varepsilon > 0$, there exists a continuous function g such that $\| f - g \| < \varepsilon$. Now

$$\begin{aligned}
\| Q_n g - g \| &= \sum_{i=1}^{n} \int_{I_i} |(Q_n g)(y) - g(y)| dy \\
&= \sum_{i=1}^{n} \int_{I_i} \left| \frac{1}{m(I_i)}(c_i + d_i y) - g(y) \right| dy \\
&= \sum_{i=1}^{n} \int_{I_i} \left| \frac{1}{m(I_i)} \int_{I_i} g(x)dx - \frac{12\tilde{x}_i}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) g(x) dx \right. \\
&\quad \left. + \left(\frac{12}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) g(x) dx\right) y - g(y) \right| dy \\
&\leq \sum_{i=1}^{n} \int_{I_i} \left| \frac{1}{m(I_i)} \int_{I_i} g(x)dx - g(y) \right| dy \\
&\quad + \sum_{i=1}^{n} \int_{I_i} \left| \frac{12}{m(I_i)^3} \left(\int_{I_i} (x - \tilde{x}_i) g(x) dx\right)(y - \tilde{x}_i) \right| dy \\
&\leq \sum_{i=1}^{n} \int_{I_i} \frac{1}{m(I_i)} \left(\int_{I_i} |g(x) - g(y)| dx \right) dy \\
&\quad + \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \int_{I_i} |(x - \tilde{x}_i) g(x)| dx \cdot \int_{I_i} |y - \tilde{x}_i| dy.
\end{aligned}$$

Since g is uniformly continuous on $[0,1]$, when mesh Δ_n is sufficiently small, for any x, $y \in I_i$, $i = 1, \ldots, n$, we have $|g(x) - g(y)| < \varepsilon$. Applying Hölder's inequality, we get

$$\begin{aligned}
\| Q_n g - g \| &\leq \sum_{i=1}^{n} \int_{I_i} \frac{1}{m(I_i)} \cdot m(I_i) \frac{\varepsilon}{2} dy \\
&\quad + \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \left[\int_{I_i} (x - \tilde{x}_i)^2 dx \right]^{1/2} \\
&\quad \cdot \left[\int_{I_i} g(x)^2 dx \right]^{1/2} \cdot \int_{I_i} |y - \tilde{x}_i| dy \\
&\leq \sum_{i=1}^{n} \frac{\varepsilon}{2} m(I_i) + \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \left\{ \left[\frac{(x - \tilde{x}_i)^3}{3} \right]_{x_{i-1}}^{x_i} \right\}^{1/2} \\
&\quad \cdot \left[\int_{I_i} g(x)^2 dx \right]^{1/2} \cdot \left[\frac{(m(I_i)/2)^2}{2} + \frac{(m(I_i)/2)^2}{2} \right]
\end{aligned}$$

$$= \frac{\varepsilon}{2} + \frac{\sqrt{3}}{2}\sum_{i=1}^{n} m(I_i)^{1/2}\left[\int_{I_i} g(x)^2 dx\right]^{1/2}.$$

For n sufficiently large, $\left(\int_{I_i} g(x)^2 dx\right)^{1/2} < \frac{\varepsilon}{\sqrt{3}}$, $i = 1, \ldots, n$. Consequently,

$$\| Q_n g - g \| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\sum_{i=1}^{n} m(I_i)^{1/2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

From Lemma 3.1.2, $\| Q_n \| \leq 2$ for all n. Hence, for n sufficiently large,

$$\begin{aligned}
\| Q_n f - f \| &\leq \| Q_n f - Q_n g \| + \| Q_n g - g \| + \| g - f \| \\
&\leq 2 \| f - g \| + \varepsilon + \| f - g \| \leq 4\varepsilon.
\end{aligned}$$

This proves $\lim_{n\to\infty} Q_n f = f$. **Q.E.D.**

3.2 An Inequality for Variation

The following result is essential for our convergence analysis.

Lemma 3.2.1. For any $f \in L^1(0,1)$ of bounded variation and for all n

$$\bigvee_0^1 Q_n f \leq 13 \bigvee_0^1 f.$$

Proof. By definition $Q_n f = \sum_{i=1}^{n}(c_i + d_i x)1_i$, where $\{c_i, d_i\}$ are given by (3.1). Since $Q_n f$ is piecewise linear, its variation is given by

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \sum_{i=1}^{n} \frac{1}{m(I_i)} |(c_i + d_i x_i) - (c_i + d_i x_{i-1})| \\
&\quad + \sum_{i=1}^{n-1} \left| \frac{c_i + d_i x_i}{m(I_i)} - \frac{c_{i+1} + d_{i+1} x_i}{m(I_{i+1})} \right| \\
&= \sum_{i=1}^{n} |d_i| + \sum_{i=1}^{n-1} \left| \frac{c_i}{m(I_i)} - \frac{c_{i+1}}{m(I_{i+1})} + \left(\frac{d_i}{m(I_i)} - \frac{d_{i+1}}{m(I_{i+1})} \right) x_i \right| \\
&= \sum_{i=1}^{n} |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right. \\
&\quad + \frac{12\tilde{x}_i}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) f(x)dx \\
&\quad \left. + \frac{12x_i}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) f(x)dx - \frac{12x_i}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx \right|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}|d_i| + \sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right.\\
&\quad + \frac{12(\tilde{x}_{i+1}-x_i)}{m(I_{i+1})^3}\int_{I_{i+1}}(x-\tilde{x}_{i+1})f(x)dx\\
&\quad \left. + \frac{12(x_i-\tilde{x}_i)}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)f(x)dx\right|\\
&= \sum_{i=1}^{n}|d_i| + \sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right.\\
&\quad \left. + \frac{6}{m(I_{i+1})^2}\int_{I_{i+1}}(x-\tilde{x}_{i+1})f(x)dx + \frac{6}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)f(x)dx\right|.
\end{aligned}
$$

From the definition of d_i we have

$$
\begin{aligned}
\bigvee_0^1 Q_n f &\le \sum_{i=1}^{n}|d_i| + \sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right|\\
&\quad + \sum_{i=1}^{n-1}\left|\frac{1}{2}d_{i+1} + \frac{1}{2}d_i\right|\\
&\le \sum_{i=1}^{n}|d_i| + \sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right|\\
&\quad + \sum_{i=1}^{n}|d_i|.
\end{aligned}
$$

It is easy to see that the middle summation of the above inequality is not greater than $\bigvee_0^1 f$ (for a proof, see [5]). Hence,

$$
\bigvee_0^1 Q_n f \le 2\sum_{i=1}^{n}|d_i| + \bigvee_0^1 f.
$$

Now we estimate $\sum_{i=1}^n |d_i|$. Let $F_i(x) = \int_{x_{i-1}}^x f(t)dt$. Then the integration by parts formula for the Stieljes-Lebesgue integral [8] gives

$$
\begin{aligned}
d_i &= \frac{12}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)f(x)dx = \frac{12}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)dF_i(x)\\
&= \frac{12}{m(I_i)^2}\left[(x-\tilde{x}_i)F_i(x)\,\Big|_{x_{i-1}}^{x_i} - \int_{I_i}F_i(x)d(x-\tilde{x}_i)\right]\\
&= \frac{12}{m(I_i)^2}\left[\frac{m(I_i)}{2}F_i(x_i) - \int_{I_i}F_i(x)dx\right]\\
&= \frac{6}{m(I_i)}\int_{I_i}f(t)dt - \frac{12}{m(I_i)^2}\int_{I_i}\left(\int_{x_{i-1}}^x f(t)dt\right)dx\\
&= 6\left[\frac{1}{m(I_i)}\int_{I_i}f(t)dt - \frac{1}{A_i}\int\int_{\Omega_i}f(t)dtdx\right]
\end{aligned}
$$

where $\Omega_i = \{(x,t) : x_{i-1} \le x \le x_i,\ x_{i-1} \le t \le x\}$ is a triangular region in the (x,t)-plane and $A_i = \frac{1}{2}m(I_i)^2$ is the area of Ω_i. Using the same technique that was used in [5], we obtain

$$\sum_{i=1}^{n} |d_i| = 6\sum_{i=1}^{n} \left| \frac{1}{m(I_i)} \int_{I_i} f(t)dt - \frac{1}{A_i} \int\int_{\Omega_i} f(t)dtdx \right| \le 6 \bigvee_0^1 f.$$

Therefore,

$$\bigvee_0^1 Q_n f \le 13 \bigvee_0^1 f.$$

Q.E.D.

3.3 Convergence

Let $P_n = Q_n \circ P_S|_{\Delta_n}$. Then $P_n : \Delta_n \to \Delta_n$ is linear. We want to find the fixed points of P_n in Δ_n. For this purpose we first investigate the representation of P_n using the basis $\{1_i, x1_i\}_{i=1}^n$.

Lemma 3.3.1. For $i = 1, \ldots, n$,

$$\begin{aligned} P_n 1_i &= \sum_{j=1}^{n} c_j(1_i)1_j + \sum_{j=1}^{n} d_j(1_i)x1_j \\ P_n(x1_i) &= \sum_{j=1}^{n} c_j(x1_i)1_j + \sum_{j=1}^{n} d_j(x1_i)x1_j, \end{aligned}$$

where

$$\begin{aligned} c_j(1_i) &= \frac{m(S^{-1}(I_j) \cap I_i)}{m(I_i)} - \frac{12\tilde{x}_j}{m(I_j)^2} \int_{I_j} (x - \tilde{x}_j)(P1_i)(x)dx, \\ d_j(1_i) &= \frac{12}{m(I_j)^2} \int_{I_j} (x - \tilde{x}_j)(P1_i)(x)dx, \\ c_j(x1_i) &= \int_{I_j} (P(x1_i))(x)dx - \frac{12\tilde{x}_j}{m(I_j)^2} \int_{I_j} (x - \tilde{x}_j)(P(x1_i))(x)dx, \\ d_j(x1_i) &= \frac{12}{m(I_j)^2} \int_{I_j} (x - \tilde{x}_j)(P(x1_i))(x)dx. \end{aligned}$$

Proof. By definition $P_n 1_i = Q_n \circ P1_i$, $P_n(x1_i) = Q_n \circ P(x1_i)$. From the definition of P, $\int_{I_j} (P1_i)(x)dx = \frac{m(S^{-1}(I_j)\cap I_i)}{m(I_i)}$. Using Lemma 3.1.1, we achieve the assertion.
Q.E.D.

Lemma 3.3.2. P_n has a nontrivial fixed point f_n in Δ_n.

Proof. Let $C_1 = (c_{ji}^1) = (c_j(1_i))$, $C_2 = (c_{ji}^2) = (c_j(x1_i))$, $D_1 = (d_{ji}^1) = (d_j(1_i))$, $D_2 = (d_{ji}^2) = (d_j(x1_i))$, where $c_j(1_i)$, $d_j(1_i)$, $c_j(x1_i)$ and $d_j(x1_i)$ are as in Lemma 3.3.1. Then the function $f_n(x) = \sum_{i=1}^n c_i 1_i + \sum_{i=1}^n d_i x 1_i$ is a fixed point of P_n if and only if the column vector $(c_1, \ldots, c_n, d_1, \ldots, d_n)^T$ is a fixed point of the matrix

$$\tilde{P}_n = \begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}.$$

We first prove that the row vector $l = (1, \ldots, 1, \tilde{x}_1, \ldots, \tilde{x}_n)$ satisfies $l = l\tilde{P}_n$. In fact, from the first equality of (3.2),

$$\begin{aligned}
\sum_{j=1}^n (c_j(1_i) + \tilde{x}_j d_j(1_i)) &= \sum_{j=1}^n \int_{I_j} (P1_i)(x)dx = \sum_{j=1}^n \int_{S^{-1}(I_j)} 1_i(x)dx \\
&= \sum_{j=1}^n \frac{m(S^{-1}(I_j) \cap I_i)}{m(I_i)} = 1, \\
\sum_{j=1}^n (c_j(x1_i) + \tilde{x}_j d_j(x1_i)) &= \sum_{j=1}^n \int_{I_j} (P(x1_i))(x)dx \\
&= \sum_{j=1}^n \int_{S^{-1}(I_j)} x1_i(x)dx = \int_0^1 x1_i(x)dx \\
&= \frac{1}{m(I_i)} \int_{I_i} xdx = \frac{1}{m(I_i)} \frac{x_i^2 - x_{i-1}^2}{2} = \tilde{x}_i.
\end{aligned}$$

Hence the matrix $\tilde{P}_n$ has an eigenvalue 1 and it follows that $\tilde{P}_n \mu = \mu$ has a nontrivial solution. **Q.E.D.**

In [4], Lasota and Yorke prove that, if $S : [0,1] \to [0,1]$ is a piecewise C^2-function satisfying $M = \inf |S'| > 2$, then for any $f \in L^1(0,1)$ of bounded variation,

$$\bigvee_0^1 P_S f \;\leq\; \alpha \parallel f \parallel + \beta \bigvee_0^1 f \tag{3.3}$$

with $\alpha > 0$ and $\beta = \frac{2}{M} < 1$. With this result, we can prove the following:

Lemma 3.3.3. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 with $M = \inf |S'| > 26$. And for each n let f_n be a fixed point of P_n such that $\parallel f_n \parallel = 1$. Then $\{\bigvee_0^1 f_n\}$ is bounded.

Proof. Since f_n is piecewise linear, it has bounded variation. From (3.3), Pf_n is a function of bounded variation. From the same inequality and the fact that $f_n = P_n f_n = Q_n \circ Pf_n$, using Lemma 3.2.1, we obtain

$$\begin{aligned} \bigvee_0^1 f_n &= \bigvee_0^1 Q_n \circ Pf_n \le 13 \bigvee_0^1 Pf_n \le 13(\alpha \parallel f_n \parallel + \beta \bigvee_0^1 f_n) \\ &= 13\alpha + 13\beta \bigvee_0^1 f_n = 13\alpha + \frac{26}{M} \bigvee_0^1 f_n . \end{aligned}$$

By assumption $M > 26$. Therefore for all n

$$\bigvee_0^1 f_n \le \frac{13\alpha}{1 - 26/M} < +\infty .$$

Q.E.D.

Now we can prove our convergence theorem for the first order piecewise polynomial Galerkin approximation scheme of the Frobenius-Perron operator equations.

Theorem 3.3.1. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $M = \inf |S'| > 26$. Then for any n, P_n has a fixed point f_n with $\parallel f_n \parallel = 1$ in Δ_n and when mesh $\Delta_n \to 0$, there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$ such that f_{n_i} converges to a fixed point of P_S in L^1 norm.

Proof. By Lemma 3.3.3 and Helly's theorem [8], there is a subsequence $\{f_{n_i}\} \subset \{f_n\}$ which converges in L^1 norm to some $f \in L^1(0,1)$. Now

$$\begin{aligned} \parallel P_S f - f \parallel \ \le \ & \parallel f - f_{n_i} \parallel \ + \ \parallel f_{n_i} - Q_{n_i} \circ P_S f_{n_i} \parallel \\ & + \ \parallel Q_{n_i} \circ P_S f_{n_i} - Q_{n_i} \circ P_S f \parallel \ + \ \parallel Q_{n_i} \circ P_S f - P_S f \parallel . \end{aligned}$$

Since $\{\parallel Q_{n_i} \circ P_S \parallel\}$ is uniformly bounded and $Q_{n_i} \circ P_S f_{n_i} = f_{n_i}$, lemma 3.1.3 implies that the right hand side of the above inequality approaches zero as $i \to \infty$. Thus $P_S f = f$. **Q.E.D.**

Corollary 3.3.1. Let $S : [0,1] \to [0,1]$ be piecewise C^2 satisfying $\inf |S'| > 1$. Then a sequence g_n from the piecewise linear functions can be constructed which converges to a fixed point of P_S.

Proof. Choose $k > 0$ such that for $M = \inf |S'|$, $M^k > 26$. Let $\varphi = S^k$. Then $P_n(\varphi)$ has a fixed point $f_n^{(\varphi)}$ of unit length in Δ_n. Define

$$g_i = \frac{1}{k}\sum_{j=0}^{k-1}(P_S)^j f_{n_i}^{(\varphi)},$$

where f_{n_i} is a convergent subsequence of $\{f_n\}$ obtained by applying above theorem. Then g_i converges, by Theorem 3.3.1, to

$$g = \frac{1}{k}\sum_{j=1}^{k-1}(P_S)^j f^{(\varphi)},$$

where $f^{(\varphi)}$ is a fixed point of $P_\varphi = P_{S^k}$. This g is a fixed point of P_S. In fact, since $(P_S)^k f^{(\varphi)} = P_{S^k} f^{(\varphi)} = P_\varphi f^{(\varphi)} = f^{(\varphi)}$,

$$P_S g = \frac{1}{k}\{P_S f^{(\varphi)} + \cdots + (P_S)^k f^{(\varphi)}\} = g.$$

Q.E.D.

Chapter 4

Piecewise Quadratic Projection Approximations

In this chapter we will generalize the piecewise linear approximations of the previous section to piecewise quadratic ones, that is, we look for approximate solutions of the Frobenius-Perron operator equation in the space of piecewise quadratic functions. As in the previous chapter, we divide the discussion into three sections.

4.1 Projection Operators

Let $x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a finite partition of the interval $[0,1]$ as before. For $i = 1, \ldots, n$, $I_i = (x_{i-1}, x_i), \tilde{x}_i = \frac{x_{i-1}+x_i}{2}$. Let $\Delta_n = \text{span}\{1_i, x1_i, x^2 1_i\}_{i=1}^n$ where $1_i = \frac{1}{m(I_i)}\chi_{I_i}$ and denote $\max_i m(I_i)$ by mesh (Δ_n). Then $\Delta_n \subset L^1(0,1)$ is a subspace of dimension $3n$.

Define the projection $Q_n : L^1(0,1) \to \Delta_n$ by the orthogonal conditions for $i = 1, \ldots, n$

$$\langle f - Q_n f, 1_i \rangle = 0, \ \langle f - Q_n f, x1_i \rangle = 0, \ \langle f - Q_n f, x^2 1_i \rangle = 0$$

for $i = 1, \ldots, n$. Let $Q_n f = \sum_{j=1}^n (c_j + d_j x + e_j x^2) 1_j$. We show that $\{c_j, d_j, e_j\}_{j=1}^n$ are uniquely determined by the above conditions.

For $i = 1, \ldots, n$,

$$\begin{aligned}
\langle Q_n f, 1_i\rangle &= \frac{1}{m(I_i)}c_i + \frac{\tilde{x}_i}{m(I_i)}d_i + \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)}e_i, \\
\langle Q_n f, x1_i\rangle &= \frac{\tilde{x}_i}{m(I_i)}c_i + \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)}d_i + \frac{\tilde{x}_i(x_i^2 + x_{i-1}^2)}{2m(I_i)}e_i, \\
\langle Q_n f, x^2 1_i\rangle &= \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)}c_i + \frac{\tilde{x}_i(x_i^2 + x_{i-1}^2)}{2m(I_i)}d_i \\
&\quad + \frac{1}{5m(I_i)}(x_i^4 + x_i^3 x_{i-1} + x_i^2 x_{i-1}^2 + x_i x_{i-1}^3 + x_{i-1}^4)e_i.
\end{aligned}$$

By the orthogonal condition we have

$$\begin{cases}
c_i + \tilde{x}_i d_i + \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)e_i = \int_{I_i} f(x)dx \\
\tilde{x}_i c_i + \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)d_i + \frac{\tilde{x}_i}{2}(x_i^2 + x_{i-1}^2)e_i = \int_{I_i} x f(x)dx \\
\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)c_i + \frac{\tilde{x}_i}{2}(x_i^2 + x_{i-1}^2)d_i + \\
\qquad \frac{1}{5}(x_1^4 + x_i^3 x_{i-1} + x_i^2 x_{i-1}^2 + x_i x_{i-1}^3 + x_{i-1}^4)e_i = \int_{I_i} x^2 f(x)dx.
\end{cases} \tag{4.1}$$

Eliminating c_i from the above system yields

$$\begin{cases}
\frac{1}{12}m(I_i)^2 d_i + \frac{1}{6}m(I_i)^2 \tilde{x}_i e_i = \int_{I_i}(x - \tilde{x}_i)f(x)dx \\
\frac{1}{6}m(I_i)^2 \tilde{x}_i d_i + \frac{1}{45}m(I_i)^2[4x_i^2 + 7x_i x_{i-1} + 4x_{i-1}^2]e_i = \\
\qquad \int_{I_i} x^2 f(x)dx - \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)\int_{I_i} f(x)dx.
\end{cases} \tag{4.2}$$

The solutions are given by

$$\begin{cases}
c_i = \frac{3}{2}\int_{I_i} f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^2}\int_{I_i} x f(x)dx - \frac{3x_i x_{i-1}}{m(I_i)^2}\int_{I_i} f(x)dx + \\
\qquad \frac{60}{m(I_i)^4}[2\tilde{x}_i^2 + x_i x_{i-1}]\int_{I_i}(x - \tilde{x}_i)f(x)dx \\
d_i = \frac{12}{m(I_i)^2}\int_{I_i} x f(x)dx + \frac{18\tilde{x}_i}{m(I_i)^2}\int_{I_i} f(x)dx - \\
\qquad \frac{360}{m(I_i)^4}\tilde{x}_i \int_{I_i}(x - \tilde{x}_i)^2 f(x)dx \\
e_i = \frac{180}{m(I_i)^4}\int_{I_i}[(x - \tilde{x}_i)^2 - \frac{1}{12}m(I_i)^2]f(x)dx.
\end{cases} \tag{4.3}$$

Lemma 4.1.1. $\| Q_n \| \leq 62$ for all n.

Proof. The values of $Q_n f$ on the subinterval I_i depends only on the values of f on I_i. So we only need to estimate the integral $\int_{I_i} |(Q_n f)(x)|dx$. Let $I = I_i =$

$[a,b]$, $\tilde{x} = \frac{a+b}{2}$ and let $\varphi(x) = \frac{1}{m(I)}(c + dx + ex^2)$ be the orthogonal projection of f onto $\mathrm{Span}\{1_I, x1_I, x^2 1_I\}$. First of all assume $f \geq 0$. We consider different cases.

(i) $\varphi \geq 0$. Then from the first equality of (4.1),

$$\begin{aligned}
\int_I |\varphi(x)|dx &= \int_I \varphi(x)dx = \frac{1}{m(I)}\int_I (c + dx + ex^2)dx \\
&= \frac{1}{m(I)}[cx + \frac{d}{2}x^2 + \frac{e}{3}x^3]_a^b = c + d\tilde{x} + \frac{1}{3}(a^2 + ab + b^2)e \\
&= \int_I f(x)dx = \int_I |f(x)|dx.
\end{aligned}$$

(ii) $\varphi \ngeq 0$. Then φ has two distinct zeros on the real axis. Without loss of generality, assume $e > 0$. We will deal with the different distribution of the zeros. Let ζ_1 and ζ_2 be the zeros of φ with $\zeta_1 < \zeta_2$.

First, assume $\zeta_1 \in (a,b)$ and $\zeta_2 \in (a,b)$. Then $\zeta_1 + \zeta_2 = -\frac{d}{e}$, $\zeta_1 \cdot \zeta_2 = \frac{c}{e}$. It follows that

$$\begin{aligned}
\int_I |\varphi(x)|dx &= \frac{1}{m(I)}\left[\int_a^{\zeta_1}(c + dx + ex^2)dx - \int_{\zeta_1}^{\zeta_2}(c + dx + ex^2)dx \right. \\
&\quad \left. + \int_{\zeta_2}^b (c + dx + ex^2)dx\right] \\
&= \frac{1}{m(I)}\left\{\left[cx + \frac{d}{2}x^2 + \frac{e}{3}x^3\right]_a^{\zeta_1} - \left[cx + \frac{d}{2}x^2 + \frac{e}{3}x^3\right]_{\zeta_1}^{\zeta_2} \right. \\
&\quad \left. + \left[cx + \frac{d}{2}x^2 + \frac{e}{3}x^3\right]_{\zeta_2}^b\right\} \\
&= \frac{1}{m(I)}\left[c(b-a) + \frac{d}{2}(b^2 - a^2) + \frac{e}{3}(b^3 - a^3)\right] \\
&\quad + \frac{2}{m(I)}\left[c(\zeta_1 - \zeta_2) + \frac{d}{2}(\zeta_1^2 - \zeta_2^2) + \frac{e}{3}(\zeta_1^3 - \zeta_2^3)\right] \\
&= \left[c + d\tilde{x} + \frac{e}{3}(a^2 + ab + b^2)\right] \\
&\quad - \frac{2(\zeta_2 - \zeta_1)}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + \zeta_2) + \frac{e}{3}(\zeta_1 + \zeta_2)^2 - \zeta_1\zeta_2\right] \\
&= \int_I f(x)dx - \frac{2(\zeta_2 - \zeta_1)}{m(I)}\left[c + \frac{d}{2}\left(-\frac{d}{e}\right) + \frac{e}{3}\left(-\frac{d}{e}\right)^2 - \frac{c}{e}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int_I f(x)dx - \frac{2(\zeta_2-\zeta_1)}{m(I)}\left[\frac{2c}{3} - \frac{d^2}{6e}\right] \\
&= \int_I f(x)dx + \frac{\zeta_2-\zeta_1}{m(I)} \cdot \frac{d^2-4ec}{3e}.
\end{aligned}
$$

Since $\zeta_1 = \frac{-d-\sqrt{d^2-4ec}}{2e}$ and $\zeta_2 = \frac{-d+\sqrt{d^2-4ec}}{2e}$, $\zeta_2 - \zeta_1 = \frac{\sqrt{d^2-4ec}}{e} \leq m(I)$. Hence $(d^2-4ec)^{\frac{3}{2}} \leq e^3 m(I)^3$. So,

$$
\frac{\zeta_2-\zeta_1}{m(I)}\frac{d^2-4ec}{3e} = \frac{1}{m(I)}\frac{(d^2-4ec)^{3/2}}{3e^2} \leq \frac{1}{m(I)}\frac{e^3m(I)^3}{3e^2} = \frac{1}{3}em(I)^2.
$$

From the last equality of (4.3), it is easy to see that

$$
\begin{aligned}
\frac{1}{3}em(I)^2 &= \frac{60}{m(I)^2}\int_I (x-\tilde{x})^2 f(x)dx - 5\int_I f(x)dx \\
&\leq \frac{60}{m(I)^2}\int_I \frac{1}{4}m(I)^2 \cdot f(x)dx - 5\int_I f(x)dx \\
&= 10\int_I f(x)dx.
\end{aligned}
$$

Therefore,

$$
\int_I |\varphi(x)|dx \leq \int_I f(x)dx + 10\int_I f(x)dx = 11\int_I f(x)dx = 11\int_I |f(x)|dx.
$$

Second, assume there is only one zero of φ in (a,b). Say $\zeta_1 \in (a,b)$ without loss of generality. Now,

$$
\begin{aligned}
\int_I |\varphi(x)|dx &= \frac{1}{m(I)}\left[\int_a^{\zeta_1}(c+dx+ex^2)dx + \int_b^{\zeta_1}(c+dx+ex^2)dx\right] \\
&= \frac{1}{m(I)}\left\{\left[c(\zeta_1-a) + \frac{d}{2}(\zeta_1^2-a^2) + \frac{e}{3}(\zeta_1^3-a^3)\right]\right. \\
&\quad \left. + \left[c(\zeta_1-b) + \frac{d}{2}(\zeta_1^2-b^2) + \frac{e}{3}(\zeta_1^3-b^3)\right]\right\} \\
&= \frac{\zeta_1-a}{m(I)}\left[c + \frac{d}{2}(\zeta_1+a) + \frac{e}{3}(\zeta_1^2+\zeta_1 a+a^2)\right] \\
&\quad - \frac{(b-\zeta_1)}{m(I)}\left[c + \frac{d}{2}(\zeta_1+b) + \frac{e}{3}(\zeta_1^2+\zeta_1 b+b^2)\right] \\
&= \frac{\zeta_1-a}{m(I)}\left[c + \frac{d}{2}(\zeta_1+a) + \frac{e}{3}(\zeta_1^2+\zeta_1 a+a^2)\right] \\
&\quad + \frac{b-\zeta_1}{m(I)}\left[c + \frac{d}{2}(\zeta_1+b) + \frac{e}{3}(\zeta_1^2+\zeta_1 b+b^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{2(b-\zeta_1)}{m(I)}\left[c+\frac{d}{2}(\zeta_1+b)+\frac{e}{3}(\zeta_1^2+\zeta_1 b+b^2)\right] \\
= & \left[c+\tilde{x}d+\frac{e}{3}(a^2+ab+b^2)\right] \\
& -\frac{2(b-\zeta_1)}{m(I)}\left[c+\frac{d}{2}(\zeta_1+b)+\frac{e}{3}(\zeta_1^2+\zeta_1 b+b^2)\right] \\
= & \int_I f(x)dx+\frac{2(\zeta_1-b)}{m(I)}\left[c+\frac{d}{2}b+\frac{d}{2}\zeta_1+\frac{1}{3}e\zeta_1^2+\frac{1}{3}eb\zeta_1+\frac{1}{3}eb^2\right] \\
= & \int_I f(x)dx+\frac{2(\zeta_1-b)}{m(I)}\left[\frac{1}{3}(e\zeta_1^2+d\zeta_1+c)+\frac{1}{3}(eb-d)\zeta_1\right. \\
& \left.+\frac{2}{3}c+\frac{d}{2}b+\frac{d}{2}\zeta_1+\frac{1}{3}eb^2\right] \\
= & \int_I f(x)dx+\frac{2(\zeta_1-b)}{m(I)}\left[\frac{1}{3}eb^2+\frac{1}{2}db+\frac{1}{6}d\zeta_1+\frac{1}{3}eb\zeta_1+\frac{2}{3}c\right] \\
= & \int_I f(x)dx+\frac{(\zeta_1-b)}{3m(I)}[2eb^2+3db+d\zeta_1+2eb\zeta_1+4c] \\
= & \int_I f(x)dx+\frac{(\zeta_1-b)}{3m(I)}\left[2(e\zeta_1^2+d\zeta_1+c)+2e(b^2-\zeta_1^2)\right. \\
& \left.+2d(b-\zeta_1)+d(b+\zeta_1)+2eb\zeta_1+2c\right] \\
= & \int_I f(x)dx+\frac{\zeta_1-b}{3m(I)}[2e(b^2-\zeta_1^2)+d(b-\zeta_1)+2(db+eb\zeta_1+c)] \\
= & \int_I f(x)dx+\frac{\zeta_1-b}{3m(I)}[2e(b^2-\zeta_1^2)+d(b-\zeta_1)+2(b-\zeta_1)(e\zeta_1+d)] \\
= & \int_I f(x)dx-\frac{(b-\zeta_1)^2}{3m(I)}[2e(b+\zeta_1)+d+2(e\zeta_1+d)] \\
= & \int_I f(x)dx-\frac{(b-\zeta_1)^2}{3m(I)}[4e(\zeta_1-b)+3(2eb+d)].
\end{aligned}
$$

From (4.3) we have

$$
\begin{aligned}
2eb+d = & \frac{360b}{m(I)^4}\int_I(x-\tilde{x})^2 f(x)dx-\frac{30b}{m(I)^2}\int_I f(x)dx \\
& +\frac{12}{m(I)^2}\int_I xf(x)dx+\frac{18\tilde{x}}{m(I)^2}\int_I f(x)dx-\frac{360\tilde{x}}{m(I)^4}\int_I(x-\tilde{x})^2 f(x)dx \\
= & \frac{180}{m(I)^3}\int_I(x-\tilde{x})^2 f(x)dx-\frac{12b}{m(I)^2}\int_I f(x)dx-\frac{18b}{m(I)^2}\int_I f(x)dx \\
& +\frac{12}{m(I)^2}\int_I xf(x)dx+\frac{18\tilde{x}}{m(I)^2}\int_I f(x)dx \\
= & \frac{180}{m(I)^3}\int_I(x-\tilde{x})^2 f(x)dx-\frac{12}{m(I)^2}\int_I(b-x)f(x)dx-\frac{9}{m(I)}\int_I f(x)dx.
\end{aligned}
$$

Hence,

$$\begin{aligned}
\int_I |\varphi(x)|dx &= \int_I f(x)dx + \frac{4}{3}e\frac{(b-\zeta_1)^3}{m(I)} - \frac{(b-\zeta_1)^2}{m(I)} \\
&\quad \cdot \left[\frac{180}{m(I)^3}\int_I (x-\tilde{x})^2 f(x)dx - \frac{12}{m(I)^2}\int_I (b-x)f(x)dx \right. \\
&\qquad \left. -\frac{9}{m(I)}\int_I f(x)dx\right] \\
&\leq \int_I f(x)dx + \frac{4}{3}em(I)^2 + \frac{12}{m(I)}\int_I (b-\zeta_1)f(x)dx + 9\int_I f(x)dx \\
&\leq [1+4\cdot 10+12+9]\int_I f(x)dx = 62\int_I f(x)dx \\
&= 62\int_I |f(x)|dx.
\end{aligned}$$

Therefore for $f \geq 0$, $\| \varphi \| \leq 62 \| f \|$. For general $f \in L^1(0,1)$ applying the above to f^+ and f^- we conclude $\| \varphi \| \leq 62 \| f \|$. **Q.E.D.**

Lemma 4.1.2. For any $f \in L^1(0,1)$, if mesh$(\Delta_n) \to 0$, then $Q_n f \to f$ in L^1 norm.

Proof. First assume $f \in L^2(0,1) \subset L^1(0,1)$. From the definition of $Q_n f$ we see that $\| f - Q_n f \|_2 = \min\{\| f - g \|_2 : g \in \Delta_n\}$ where $\| \cdot \|_2$ is the L^2-norm. From the theory of the finite element method we have $\| f - Q_n f \|_2 \to 0$. By the Cauchy inequality $\| Q_n f - f \| \leq \| Q_n f - f \|_2$. Thus $\| Q_n f - f \| \to 0$.

Now for $f \in L^1(0,1)$ and $\varepsilon > 0$, there exists $g \in L^2(0,1)$ such that $\| f - g \| < \varepsilon$. From

$$\begin{aligned}
\| Q_n f - f \| &\leq \| Q_n f - Q_n g \| + \| Q_n g - g \| + \| g - f \| \\
&\leq 62 \| f - g \| + \| Q_n g - g \| + \| g - f \|
\end{aligned}$$

and since $\| Q_n g - g \| \to 0$, we obtain $\| Q_n f - f \| \to 0$. **Q.E.D.**

4.2 An Inequality for Variation

We establish the uniform boundedness of the variation of the projected functions as follows.

Lemma 4.2.1. If $f \in L^1(0,1)$ is of bounded variation, then for all n,

$$\bigvee_0^1 Q_n f \leq 121 \bigvee_0^1 f.$$

Proof. Since $Q_n f$ is piecewise quadratic, we have

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \bigvee_0^1 \sum_{i=1}^n (c_i + d_i x + e_i x^2) 1_i \\
&= \sum_{i=1}^n \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + \sum_{i=1}^{n-1} \left| \frac{c_i + d_i x_i + e_i x_i^2}{m(I_i)} - \frac{c_{i+1} + d_{i+1} x_i + e_{i+1} x_i^2}{m(I_{i+1})} \right| \\
&= \sum_{i=1}^n \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + \sum_{i=1}^{n-1} \left| \left(\frac{c_i}{m(I_i)} - \frac{c_{i+1}}{m(I_{i+1})} \right) + \left(\frac{d_i}{m(I_i)} - \frac{d_{i+1}}{m(I_{i+1})} \right) x_i \right. \\
&\quad \left. + \left(\frac{e_i}{m(I_i)} - \frac{e_{i+1}}{m(I_{i+1})} \right) x_i^2 \right| \\
&= \sum_{i=1}^n \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_{i+1})^3} \int_{I_{i+1}} \left[12\tilde{x}_{i+1} x + 3x_{i+1} x_i - 12 x_i \left(x + \frac{3}{2} \tilde{x}_{i+1} \right) \right. \right. \\
&\qquad\qquad \left. +15 x_i^2 \right] f(x) dx \\
&\quad - \frac{1}{m(I_i)^3} \int_{I_i} \left[12\tilde{x}_i x + 3x_i x_{i-1} - 12 x_i \left(x + \frac{3}{2} \tilde{x}_i \right) + 15 x_i^2 \right] f(x) dx \\
&\quad + \frac{1}{m(I_{i+1})^5} \int_{I_{i+1}} \left[360 x_i \tilde{x}_{i+1} - 60(2\tilde{x}_{i+1}^2 + x_{i+1} x_i) - 180 x_i^2 \right] \\
&\qquad\qquad \cdot (x - \tilde{x}_{i+1})^2 f(x) dx \\
&\quad - \frac{1}{m(I_i)^5} \int_{I_i} [360 x_i \tilde{x}_i - 60(2\tilde{x}_i^2 + x_i x_{i-1}) - 180 x_i^2] (x - \tilde{x}_i)^2 f(x) dx \\
&\quad \left. + \frac{3}{2} \left(\frac{1}{m(I_i)} \int_{I_i} f(x) dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x) dx \right) \right| \\
&= \sum_{i=1}^m \frac{1}{m(I_i)} |d_i + 2e_i x| + \sum_{i=1}^{n-1} \left| \frac{6}{m(I_{i+1})^2} \int_{I_{i+1}} (x - \tilde{x}_i) f(x) dx \right. \\
&\quad + \frac{6}{m(I_i)^2} \int_{I_i} (x - x_i) f(x) dx + \frac{30}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i)^2 f(x) dx \\
&\quad - \frac{30}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1})^2 f(x) dx + \frac{3}{2} \frac{1}{m(I_i)} \int_{I_i} f(x) dx
\end{aligned}$$

$$-\frac{3}{2}\frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\Bigg|. \tag{4.4}$$

Let $\Omega_i = \{(x,t) : x \in I_i,\ x_{i-1} \le t \le x\}$, $V_i = \{(x,t,s) : x \in I_i,\ x_{i-1} \le t \le x,\ x_{i-1} \le s \le t\}$. Again the integration by parts formula for functions of bounded variation yields

$$\begin{aligned}
\frac{1}{m(I_{i+1})^2}\int_{I_{i+1}}(x-x_i)f(x)dx &= \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx \\
&\quad -\frac{1}{2A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdt, \\
\frac{1}{m(I_i)^2}\int_{I_i}(x-x_i)f(x)dx &= -\frac{1}{2A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx, \\
\frac{1}{m(I_{i+1})^2}\int_{I_{i+1}}(x-\tilde{x}_{i+1})^2 f(x)dx &= \frac{1}{4m(I_{i+1})}\int_{I_{i+1}} f(x)dx \\
&\quad -\frac{1}{2A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx \\
&\quad +\frac{1}{3V(V_{i+1})}\int\int\int_{V_{i-1}} f(s)dsdtdx, \\
\frac{1}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)^2 f(x)dx &= \frac{1}{4m(I_i)}\int_{I_i} f(x)dx - \frac{1}{2A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx \\
&\quad +\frac{1}{3V(V_i)}\int\int\int_{V_i} f(s)dsdtdx,
\end{aligned}$$

where $A(\Omega_i) = \frac{1}{2}m(I_i)^2$ is the area of Ω_i and $V(V_i) = \frac{1}{6}m(I_i)^3$ is the volume of V_i. Substituting into (4.4) we have

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i + 2e_i x|dx \\
&\quad +\sum_{i=1}^{n-1}\Bigg|\frac{6}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx - \frac{3}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx \\
&\quad -\frac{3}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx + \frac{15}{2m(I_i)}\int_{I_i} f(x)dx \\
&\quad -\frac{15}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx + \frac{10}{V(V_i)}\int\int\int_{V_i} f(s)dsdtdx \\
&\quad -\frac{15}{2m(I_{i+1})}\int_{I_{i+1}} f(x)dx + \frac{15}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx \\
&\quad -\frac{10}{V(V_{i+1})}\int\int\int_{V_{i+1}} f(s)dsdtdx + \frac{3}{2m(I_i)}\int_{I_i} f(x)dx \\
&\quad -\frac{3}{2m(I_{i+1})}\int_{I_{i+1}} f(x)dx\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + \sum_{i=1}^{n-1} \left| \frac{9}{m(I_i)} \int_{I_i} f(x)dx - \frac{3}{2m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right. \\
&\quad + \frac{12}{A(\Omega_{i+1})} \int\int_{\Omega_{i+1}} f(t)dtdx - \frac{18}{A(\Omega_i)} \int\int_{\Omega_i} f(t)dtdx \\
&\quad \left. + \frac{10}{V(V_i)} \int\int\int_{V_i} f(s)dsdtdx - \frac{10}{V(V_{i+1})} \int\int\int_{V_{i+1}} f(s)dsdtdx \right| \\
&\le \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + 3\sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right| \\
&\quad + 6\sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{A(\Omega_i)} \int\int_{\Omega_i} f(t)dtdx \right| \\
&\quad + 12\sum_{i=1}^{n-1} \left| \frac{1}{A(\Omega_{i+1})} \int\int_{\Omega_{i+1}} f(t)dtdx - \frac{1}{A(\Omega_i)} \int\int_{\Omega_i} f(t)dtdx \right| \\
&\quad + 10\sum_{i=1}^{n-1} \left| \frac{1}{V(V_i)} \int\int\int_{V_i} f(s)dsdtdx \right. \\
&\quad \left. - \frac{1}{V(V_{i+1})} \int\int\int_{V_{i+1}} f(s)dsdtdx \right| \\
&\le \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx \\
&\quad + 3\bigvee_0^1 f + 6\bigvee_0^1 f + 12\bigvee_0^1 f + 10\bigvee_0^1 f \\
&= \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx + 31\bigvee_0^1 f. && (4.5)
\end{aligned}
$$

Now we estimate the first term of (4.5). For $i = 1, \ldots, n$, $\frac{1}{m(I_i)} \int_{I_i} |d_i + 2e_i x| dx$ is the variation of $Q_n f$ on I_i. For simplicity we omit the subscript. Let $\varphi(x) = \frac{1}{m(I)}(c + dx + ex^2)$, $I = [a, b]$. Without loss of generality assume $e > 0$. Then $\zeta = -\frac{d}{2e}$ is the minimal point of φ and $\varphi\left(-\frac{d}{2e}\right) = \frac{1}{m(I)}\left(c - \frac{d^2}{4e}\right)$ is the minimum value of φ. If $a < \zeta < b$, then

$$
\begin{aligned}
\bigvee_I \varphi &= \frac{1}{m(I)}(\varphi(a) + \varphi(b) - 2\varphi(\zeta)) \\
&= \frac{1}{m(I)}\left(c + da + ea^2 + c + db + eb^2 - 2c + \frac{d^2}{2e}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{m(I)}\left[d(a+b)+e(a^2+b^2)+\frac{d^2}{2e}\right] \\
&= [-2e\zeta(a+b)+e(a^2+b^2)+2e\zeta^2]/m(I) \\
&= e[2\zeta^2-2\zeta(a+b)+a^2+b^2]/m(I) \\
&= e[(a-b)^2+2ab-2\zeta(a+b)+2\zeta^2]/m(I) \\
&= \frac{1}{m(I)}[em(I)^2+2e(a-\zeta)(b-\zeta)] \leq em(I) \\
&= \frac{15}{m(I)}\left[\frac{12}{m(I)^2}\int_I (x-\tilde{x})^2 f(x)dx - \int_I f(x)dx\right] \\
&\leq 30\left|\frac{1}{m(I)}\int_I f(x)dx - \frac{3}{A(\Omega)}\int\int_\Omega f(t)dtdx + \frac{2}{V(V)}\int\int\int_V f(s)dsdtdx\right| \\
&\leq 90\bigvee_I f.
\end{aligned}
$$

If $\zeta \notin (a,b)$, then

$$
\begin{aligned}
\bigvee_I \varphi &= \frac{1}{m(I)}|\varphi(a)-\varphi(b)| = \frac{1}{m(I)}|d(b-a)+e(b^2-a^2)| \\
&= |d+2e\tilde{x}| = \frac{12}{m(I)}\left|\int_I (x-\tilde{x})f(x)dx\right| \\
&\leq 6\left|\frac{1}{m(I)}\int_I f(x)dx - \frac{1}{A(\Omega)}\int\int_\Omega f(t)dtdx\right| \\
&\leq 6\bigvee_I f < 90\bigvee_I f.
\end{aligned}
$$

Substituting into (4.5) we have

$$
\bigvee_0^1 Q_n f \leq 90\sum_{i=1}^{n}\bigvee_{I_i} f + 31\bigvee_0^1 f = 121\bigvee_0^1 f.
$$

Q.E.D.

4.3 Convergence

To prove the convergence of the method, we need the following lemma.

Lemma 4.3.1. Let $P_n = Q_n \circ P_S|_{\Delta_n}$, where P_S is the Frobenius-Perron operator associated with $S : [0,1] \to [0,1]$. Then P_n has a nontrivial fixed point in Δ_n.

Proof: Denote by $\tilde{P}_n$ the matrix representation of $P_n : \Delta_n \to \Delta_n$ using the basis $\{1_i, x1_i, x^2 1_i\}_{i=1}^n$ of Δ_n, i. e., $P_n\{1_1, x1_1, x^2 1_1, \cdots, 1_n, x1_n, x^2 1_n\} = \{1_1, x1_1, x^2 1_1, \cdots, 1_n,$

$x1_n, x^2 1_n\}\tilde{P}_n$. Let $\zeta = (1, \tilde{x}_1, \tilde{y}_1, 1, \tilde{x}_2, \tilde{y}_2, \cdots, 1, \tilde{x}_n, \tilde{y}_n)$ where $\tilde{y}_i = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$. Then for $i = 1, \ldots, n$,

$$
\begin{aligned}
(\zeta\tilde{P}_n)_{3(i-1)+1} &= \sum_{j=1}^{n}(c_j(1_i) + \tilde{x}_j d_j(1_i) + \tilde{y}_j e_j(1_i)) \\
&= \sum_{j=1}^{n}\int_{I_j}(P_S 1_i)(x)dx = \int_0^1 (P_S 1_i)(x)dx \\
&= \int_0^1 1_i(x)dx = 1, \\
(\zeta\tilde{P}_n)_{3(i-1)+2} &= \sum_{j=1}^{n}(c_j(x1_i) + \tilde{x}_j d_j(x1_i) + \tilde{y}_j e_j(x1_i)) \\
&= \sum_{j=1}^{n}\int_{I_j}(P_S(x1_i))(x)dx = \int_0^1 (P_S(x1_i))(x)dx \\
&= \int_0^1 x1_i(x)dx = \tilde{x}_i, \\
(\zeta\tilde{P}_n)_{3i} &= \sum_{j=1}^{n}(c_j(x^2 1_i) + \tilde{x}_j d_j(x^2 1_i) + \tilde{y}_j e_j(x^2 1_i)) \\
&= \sum_{j=1}^{n}\int_{I_j}(P_S(x^2 1_i))(x)dx = \int_0^1 (P_S(x^2 1_i))(x)dx \\
&= \int_0^1 x^2 1_i(x)dx = \tilde{y}_i.
\end{aligned}
$$

That is, ζ is a left eigenvector of the matrix $\tilde{P}_n$ corresponding to the eigenvalue 1. Therefore there is a nonzero $c \in \Re^{3n}$ such that $\tilde{P}_n c = c$. Thus P_n has a nonzero fixed point in Δ_n. **Q.E.D.**

Theorem 4.3.1. Let $S : [0,1] \to [0,1]$ be piecewise C^2 with $M = \inf |S'| > 242$. For each n let f_n be a fixed point of P_n in Δ_n with $\| f_n \| = 1$. Then there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$ convergent in L^1 norm to a fixed point of P_S.

Proof. By inequality (3.3) in the previous section, we have for any n

$$
\begin{aligned}
\bigvee_0^1 f_n &= \bigvee_0^1 P_n f_n = \bigvee_0^1 Q_n \circ P_S f_n \le 121 \bigvee_0^1 P_S f_n \\
&\le 121\left(\alpha \| f_n \| + \frac{2}{M}\bigvee_0^1 f_n\right) = 121\alpha + \beta \bigvee_0^1 f_n
\end{aligned}
$$

with $\beta = \frac{242}{M} < 1$. Hence

$$
\bigvee_0^1 f_n \le \frac{121\alpha}{1-\beta} < +\infty.
$$

From the Helly theorem there is a subsequence $\{f_{n_i}\} \subset \{f_n\}$ which converges in L^1 norm to some $f \in L^1(0,1)$. From

$$\begin{aligned} \| P_S f - f \| \;\leq\; & \| f - f_{n_i} \| + \| f_{n_i} - Q_{n_i} \circ P_S f_{n_i} \| \\ & + \| Q_{n_i} \circ P_S f_{n_i} - Q_{n_i} \circ P_S f \| \\ & + \| Q_{n_i} \circ P_S f - P_S f \| \end{aligned}$$

we immediately see that $P_S f = f$. **Q.E.D.**

The proof of the following corollary is the same as that of Corollary 3.3.1.

Corollary 4.3.1. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $\inf |S'| > 1$. Then a sequence g_n from the piecewise quadratic functions can be constructed which converges to a nontrivial fixed point of P_S.

Chapter 5

Markov Finite Approximations

In this chapter we take a different approach. Since the Frobenius-Perron operator is also a Markov operator, it is natural to approximate it by Markov operators of finite rank. The "orthogonal projections" of the Galerkin scheme used earlier are not Markov operators in general except in the piecewise constant case in [5]. To overcome this defect, we use continuous piecewise linear and piecewise quadratic Markov approximation schemes to find an approximate fixed point of the Frobenius-Perron operator. The convergence of these methods will be shown for a general class of nonsingular measurable transformations.

In section 5.1 we discuss piecewise linear Markov approximations and section 5.2 is devoted to piecewise quadratic ones.

5.1 Piecewise Linear Markov Approximation

Assume $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $\inf \mid S' \mid > 1$. In this section, we look for approximate solutions of the Frobenius-Perron operator equation $P_S f = f$ in the space of continuous piecewise linear functions.

For simplicity divide the interval $[0,1]$ into n equal parts. Let $I_i = [x_{i-1}, x_i], x_i = \frac{i}{n}, i = 0, \ldots, n$. Each subinterval I_i has length $\frac{1}{n}$. Denote by Δ_n the space of continuous, piecewise linear functions corresponding to the above partition. Then Δ_n is a linear subspace of $L^1(0,1)$ with dimension $n+1$. First of all, we choose a basis for

Δ_n. Let

$$\psi(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi(x)$ is the so-called hat function. For $i = 0, \ldots, n$, let

$$\varphi_i(x) = \psi(n(x - x_i)).$$

Then φ_i is a continuous piecewise linear function with $\varphi_i(x_i) = 1$ and $\varphi_i(x_j) = 0$ for $j \neq i$. It is easy to see that $\{\varphi_0, \ldots, \varphi_n\}$ is a basis of Δ_n. This basis has the following properties:

1. $\|\varphi_i\| = \frac{1}{n}$ for $i = 1, \ldots, n-1$, $\|\varphi_0\| = \|\varphi_n\| = \frac{1}{2n}$;

2. $\varphi_i \ge 0$ for $i = 0, \ldots, n$ and $\sum_{i=0}^{n} \varphi_i(x) \equiv 1$;

3. Suppose $f \in \Delta_n$. Then $f = \sum_{i=0}^{n} q_i \varphi_i$ if and only if $f(x_i) = q_i$ for $i = 0, \ldots, n$.

Now we define $Q_n : L^1(0,1) \to \Delta_n$ as follows

$$Q_n f = (n \int_{I_1} f)\varphi_0 + \frac{n}{2} \sum_{i=1}^{n-1} (\int_{I_i} f + \int_{I_{i+1}} f)\varphi_i + (n \int_{I_n} f)\varphi_n. \tag{5.1}$$

Then Q_n is a bounded linear operator.

Lemma 5.1.1. For any n $Q_n : L^1(0,1) \to \Delta_n$ is a Markov operator and hence $\|Q_n\| = 1$.

Proof. From (5.1) it is easy to see that $f \ge 0$ implies $Q_n f \ge 0$. To prove Q_n is a Markov operator, it remains to show that for $f \ge 0$, $\|Q_n f\| = \|f\|$. This can be done by the following computation.

$$\begin{aligned}
\|Q_n f\| &= \int_0^1 |Q_n f| = \int_0^1 Q_n f \\
&= n[\int_{I_1} f \int_0^1 \varphi_0 + \frac{1}{2} \sum_{i=1}^{n-1} (\int_{I_i} f + \int_{I_{i+1}} f) \int_0^1 \varphi_i + \int_{I_n} f \int_0^1 \varphi_n] \\
&= \frac{1}{2}[\int_{I_1} f + \sum_{i=1}^{n-1} (\int_{I_i} f + \int_{I_{i+1}} f) + \int_{I_n} f] \\
&= \sum_{i=1}^{n} \int_{I_i} f = \|f\|.
\end{aligned}$$

It follows that $\|Q_n\| = 1$. **Q.E.D.**

Lemma 5.1.2. For all $f \in L^1(0,1)$, $\lim_{n\to\infty} Q_n f = f$.

Proof. We first assume f is a continuous function on $[0,1]$. Notice that $\sum_{i=0}^{n} \varphi_i(x) \equiv 1$. Hence,

$$\begin{aligned}
Q_n f(x) - f(x) &= (n\int_{I_1} f)\varphi_0(x) + \frac{n}{2}\sum_{i=1}^{n-1}(\int_{I_i} f + \int_{I_{i+1}} f)\varphi_i(x) \\
&+ (n\int_{I_n} f)\varphi_n(x) - \sum_{i=0}^{n} f(x)\varphi_i(x) \\
&= (n\int_{I_1} f - f(x))\varphi_0(x) + \sum_{i=1}^{n-1}[\frac{n}{2}(\int_{I_i} f + \int_{I_{i+1}} f) - f(x)]\varphi_i(x) \\
&+ (n\int_{I_n} f - f(x))\varphi_n(x).
\end{aligned}$$

Since f is uniformly continuous on $[0,1]$, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\frac{1}{n} < \delta$ and $x, y \in I_i \cup I_{i+1}$ for any i, $|f(x) - f(y)| < \epsilon$. For $\frac{1}{n} < \delta$, $|n\int_{I_1} f - f(x)| < \epsilon$ for $x \in I_1$, $|n\int_{I_n} f - f(x)| < \epsilon$ for $x \in I_n$ and $|\frac{n}{2}(\int_{I_i} f + \int_{I_{i+1}} f) - f(x)| < \epsilon$ for $x \in I_i \cup I_{i+1}, i = 1, \ldots, n-1$. Hence,

$$|Q_n f(x) - f(x)| \le \epsilon \sum_{i=0}^{n} \varphi_i(x) = \epsilon, \forall x \in [0,1].$$

That is, for $f \in L^1(0,1)$ continuous, $Q_n f$ converges uniformly to f.

Now let $f \in L^1$ be arbitrary. Given $\epsilon > 0$, there is a continuous function g satisfying $\|f - g\| < \frac{\epsilon}{3}$. Because $\|Q_n\| = 1$, we have

$$\begin{aligned}
\|Q_n f - f\| &\le \|Q_n f - Q_n g\| + \|Q_n g - g\| + \|g - f\| \\
&\le 2\|f - g\| + \|Q_n g - g\| < \frac{2}{3}\epsilon + \|Q_n g - g\|.
\end{aligned}$$

From what we proved above, there is an $N > 0$ such that for $n \ge N$, $\|Q_n g - g\| < \frac{\epsilon}{3}$. Therefore for $n \ge N$

$$\|Q_n f - f\| < \frac{2}{3}\epsilon + \frac{\epsilon}{3} = \epsilon.$$

Q.E.D.

We show now that $Q_n f$ will not increase the variation of a function f of bounded variation.

Lemma 5.1.3. If $f \in L^1$ is of bounded variation, then for any n,

$$\bigvee_0^1 Q_n f \le \bigvee_0^1 f.$$

Proof. Let $Q_n f(x) = \sum_{i=0}^n q_i \varphi_i(x)$. Then $Q_n f(x_i) = q_i, i = 0, \ldots, n$. And

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \sum_{i=1}^n |q_i - q_{i-1}| \\
&= |\frac{n}{2}(\int_{I_1} f + \int_{I_2} f) - n\int_{I_1} f| + \sum_{i=2}^{n-1} |\frac{n}{2}(\int_{I_i} f + \int_{I_{i+1}} f) \\
&- \frac{n}{2}(\int_{I_{i-1}} f + \int_{I_i} f)| + |n\int_{I_n} f - \frac{n}{2}(\int_{I_{n-1}} f + \int_{I_n} f)| \\
&= \frac{1}{2}\{|n\int_{I_2} f - n\int_{I_1} f| + \sum_{i=2}^{n-1} |n\int_{I_{i+1}} f - n\int_{I_{i-1}} f| \\
&+ |n\int_{I_n} f - n\int_{I_{n-1}} f|\} \\
&\le \frac{1}{2}\{|n\int_{I_2} f - n\int_{I_1} f| + \sum_{i=2}^{n-1} [|n\int_{I_{i+1}} f - n\int_{I_i} f| \\
&+ |n\int_{I_i} f - n\int_{I_{i-1}} f|] + |n\int_{I_n} f - n\int_{I_{n-1}} f|\} \\
&= \sum_{i=1}^{n-1} |n\int_{I_{i+1}} f - n\int_{I_i} f| \le \bigvee_0^1 f.
\end{aligned}$$

See [5] for the proof of the last inequality. **Q.E.D.**

Let $P_n = Q_n \circ P_S|_{\Delta_n}$. Since both $P_S : L^1(0,1) \to L^1(0,1)$ and $Q_n : L^1(0,1) \to \Delta_n$ are Markov operators, $P_n : \Delta_n \to \Delta_n$ is a Markov operator of finite rank. Let $P_n\varphi_i = \sum_{j=0}^n p_{ij}\varphi_j, i = 0, \ldots, n$. Let $\tilde{P}_n = (p_{ij})$ be the corresponding $(n+1) \times (n+1)$ matrix. Let $f_n = \sum_{i=0}^n c_i\varphi_i$. Then $P_n f_n = f_n$ if and only if $c\tilde{P}_n = c$ with the row vector $c = (c_0, \ldots, c_n)$.

Lemma 5.1.4. For each n there exists a nontrivial nonnegative function $f_n \in \Delta_n$ satisfying $P_n f_n = f_n$.

Proof. Since

$$P_n\varphi_i = Q_n \circ P_S\varphi_i$$

$$= n\{(\int_{I_1} P_S\varphi_i)\varphi_0 + \frac{1}{2}\sum_{j=0}^{n-1}(\int_{I_j} P_S\varphi_i + \int_{I_{j+1}} P_S\varphi_i)\varphi_j + (\int_{I_n} P_S\varphi_i)\varphi_n\},$$

we have $p_{i0} = n\int_{I_1} P_S\varphi_i, p_{ij} = \frac{n}{2}(\int_{I_j} P_S\varphi_i + \int_{I_{j+1}} P_S\varphi_i), j = 1,\dots,n-1$, and $p_{in} = n\int_{I_n} P_S\varphi_i$ for $i = 0,\dots,n$. The matrix $\tilde{P}_n$ is nonnegative. Let $\xi = (\xi_0,\dots,\xi_n) = (\frac{1}{2},1,\dots,1,\frac{1}{2})$. Then

$$\begin{aligned}\sum_{j=0}^{n} p_{ij}\xi_j &= \frac{n}{2}\int_{I_1} P_S\varphi_i + \frac{n}{2}\sum_{j=1}^{n-1}(\int_{I_j} P_S\varphi_i + \int_{I_{j+1}} P_S\varphi_i) + \frac{n}{2}\int_{I_n} P_S\varphi_i \\ &= n\sum_{j=1}^{n}\int_{I_j} P_S\varphi_i = n\int_0^1 P_S\varphi_i = n\int_0^1 \varphi_i.\end{aligned}$$

Since $\|\varphi_0\| = \|\varphi_n\| = \frac{1}{2n}$ and $\|\varphi_i\| = \frac{1}{n}$ for $i = 1,\dots,n-1$, we have $\tilde{P}_n\xi = \xi$. Hence there exists a nonnegative row vector $c = (c_0,\dots,c_n) \neq 0$ such that $c\tilde{P}_n = c$. Let $f_n = \sum_{i=0}^{n} c_i\varphi_i$. Then $P_n f_n = f_n$. **Q.E.D.**

Lemma 5.1.5. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 with $M = \inf|S'(x)| > 2$. Then for nonnegative fixed points $f_n \in \Delta_n$ of P_n, the sequence $\{\bigvee_0^1 f_n\}$ is uniformly bounded.

Proof. By Lemma 5.1.3 and inequality (3.3),

$$\begin{aligned}\bigvee_0^1 f_n &= \bigvee_0^1 P_n f_n = \bigvee_0^1 Q_n \circ P_S f_n \leq \bigvee_0^1 P_S f_n \\ &\leq \alpha\|f_n\| + \beta\bigvee_0^1 f_n = \alpha + \beta\bigvee_0^1 f_n.\end{aligned}$$

Since $\beta = \frac{2}{M} < 1$ by assumption, we have

$$\bigvee_0^1 f_n \leq \frac{\alpha}{1-\beta} < \infty.$$

Q.E.D.

Now we can prove our main theorem.

Theorem 5.1.1. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 and $M = \inf|S'(x)| > 2$. If the corresponding Frobenius-Perron operator P_S has a unique invariant density f, then the sequence $\{f_n\}$ of nonnegative piecewise linear fixed points of P_n in Δ_n converges to f in $L^1(0,1)$.

Proof. From Lemma 5.1.5, the sequence $\{f_n\}$ is bounded in variation. Helly's theorem [8] implies that $\{f_n\}$ is precompact. Let $\{f_{n_k}\} \subset \{f_n\}$ converge in L^1 norm to some $g \in L^1(0,1)$. Then

$$\begin{aligned} \|P_S g - g\| &\leq \|g - f_{n_k}\| + \|f_{n_k} - Q_{n_k} \circ P_S f_{n_k}\| \\ &+ \|Q_{n_k} \circ P_S f_{n_k} - Q_{n_k} \circ P_S g\| + \|Q_{n_k} \circ P_S g - P_S g\|. \end{aligned}$$

Since $Q_{n_k} \circ P_S f_{n_k} = f_{n_k}$ and since $\|Q_{n_k} \circ P_S\| \leq 1$, $P_S g = g$. Obviously $\|g\| = 1$ and $g \geq 0$. By the uniqueness of the fixed density of P_S we assert that $g = f$ and that all the convergent subsequences of $\{f_n\}$ converge to f. This proves $\lim_{n\to\infty} f_n = f$. **Q.E.D.**

By the following familiar trick we can ignore the condition $M = \inf |S'(x)| > 2$.

Corollary 5.1.1. If $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $\inf |S'| > 1$ and P_S has only one invariant density, a sequence g_n from the piecewise linear functions can be constructed which converges to the fixed density of P_S.

Proof. Choose k such that $M^k = (\inf |S'(x)|)^k > 2$. Let $\phi = S^k$. Let $f_n(\phi)$ be the fixed density of $P_n(\phi)$ as in the above theorem. Define

$$g_n = \frac{1}{k} \sum_{j=0}^{k-1} (P_S)^j f_n(\phi).$$

Then g_n converges, by Theorem 2.1, to

$$g = \frac{1}{k} \sum_{j=0}^{k-1} (P_S)^j f(\phi),$$

where $f(\phi)$ is a fixed point of $\phi = S^k$. This g is the fixed density of P_S. In fact, since $(P_S)^k f(\phi) = P_{S^k} f(\phi) = P_\phi f(\phi) = f(\phi)$,

$$P_S g = \frac{1}{k}\{P_S f(\phi) + \cdots + (P_S)^k f(\phi)\} = g.$$

Q.E.D.

5.2 Piecewise Quadratic Markov Approximation

We generalize the method in the previous section to piecewise quadratic Markov approximations. As above, let $I_i = [x_{i-1}, x_i]$ with $x_i = \frac{i}{n}, i = 0, \ldots, n$. Each subinterval I_i has length $\frac{1}{n}$. Let Ω_n be the corresponding space of continuous piecewise quadratic functions on $[0,1]$ associated with that partition. Then Ω_n is a $(2n+1)$-dimensional subspace of $L^1(0,1)$. In order to construct a basis for Ω_n, we define

$$\tau(x) = \begin{cases} (x+1)^2, & -1 < x \leq 0 \\ (x-1)^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\rho(x) = \begin{cases} 2x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\phi_{2i}(x) = \tau(n(x - x_i)), i = 0, \ldots, n$, and $\phi_{2i-1}(x) = \rho(n(x - x_{i-1})), i = 1, \ldots, n$. Then it is easy to see that $\{\phi_0, \ldots, \phi_{2n}\}$ is a basis of Ω_n. This basis has the following properties:

1. $\|\phi_{2i}\| = \frac{2}{3n}$ for $i = 1, \ldots, n-1, \|\phi_{2i-1}\| = \frac{1}{3n}$ for $i = 1, \ldots, n$, and $\|\phi_0\| = \|\phi_{2n}\| = \frac{1}{3n}$;

2. $\phi_k \geq 0$ for all k and $\sum_{k=0}^{2n} \phi_k(x) \equiv 1$;

3. If $f = \sum_{k=0}^{2n} q_k \phi_k$, then $f(x_i) = q_{2i}$ for $i = 0, \ldots, n$.

Define $Q_n : L^1(0,1) \to \Omega_n$ as follows

$$Q_n f = (n \int_{I_1} f)\phi_0 + n \sum_{i=1}^{n} (\int_{I_i} f)\phi_{2i-1} + \frac{n}{2} \sum_{i=1}^{n-1} (\int_{I_i} f + \int_{I_{i+1}} f)\phi_{2i} + (n \int_{I_n} f)\phi_{2n}.$$

Lemma 5.2.1. Q_n is a Markov operator and hence $\|Q_n\| = 1$ for any n.

Proof. It is obvious that $f \geq 0$ implies $Q_n f \geq 0$. By direct computation, for $f \geq 0$,

$$\|Q_n f\| = \int_0^1 |Q_n f| = \int_0^1 Q_n f$$

$$
\begin{aligned}
&= n\int_{I_1} f\int_0^1 \phi_0 + n\sum_{i=1}^{n}\int_{I_i} f\int_0^1 \phi_{2i-1} \\
&+ \frac{n}{2}\sum_{i=1}^{n-1}(\int_{I_i} f + \int_{I_{i+1}} f)\int_0^1 \phi_{2i} + n\int_{I_n} f\int_0^1 \phi_{2n} \\
&= \frac{1}{3}\int_{I_1} f + \frac{1}{3}\sum_{i=1}^{n}\int_{I_i} f + \frac{1}{3}\sum_{i=1}^{n-1}(\int_{I_i} f + \int_{I_{i+1}} f) + \frac{1}{3}\int_{I_n} f \\
&= \int_0^1 f = \int_0^1 |f| = \|f\|.
\end{aligned}
$$

Q.E.D.

Lemma 5.2.2. $\lim_{n\to\infty} Q_n f = f$ for any $f \in L^1(0,1)$.

Proof. We First assume f is continuous. Then,

$$
\begin{aligned}
Q_n f - f &= (n\int_{I_1} f)\phi_0 + n\sum_{i=1}^{n}(\int_{I_i} f)\phi_{2i-1} + \frac{n}{2}\sum_{i=1}^{n-1}(\int_{I_i} f + \int_{I_{i+1}} f)\phi_{2i} \\
&+ (n\int_{I_n} f)\phi_{2n} - \sum_{j=0}^{2n} f\phi_j \\
&= (n\int_{I_1} f - f)\phi_0 + \sum_{i=1}^{n}(n\int_{I_i} f - f)\phi_{2i-1} \\
&+ \sum_{i=1}^{n-1}[\frac{n}{2}(\int_{I_i} f + \int_{I_{i+1}} f) - f]\phi_{2i} + (n\int_{I_n} f - f)\phi_{2n}.
\end{aligned}
$$

For any $\epsilon > 0$, there is $\delta > 0$ such that for $\frac{1}{n} < \delta$ we have $|f(x) - f(y)| < \epsilon$ for any $x, y \in I_i \cup I_{i+1}, i = 1, \ldots, n-1$. Hence $|n\int_{I_1} f - f|\chi_{I_1} < \epsilon, |\frac{n}{2}(\int_{I_i} f + \int_{I_{i+1}} f) - f|\chi_{I_i \cup I_{i+1}} < \epsilon, i = 1, \ldots, n-1$, and $|n\int_{I_n} f - f|\chi_{I_n} < \epsilon$, where χ_A is the characteristic function of A. It follows that for n sufficiently large and $x \in [0,1]$

$$
|Q_n f(x) - f(x)| < \epsilon\sum_{j=0}^{2n}\phi_j(x) = \epsilon,
$$

i. e. , $Q_n f$ converges to f uniformly as $n \to \infty$. Therefore $\|Q_n f - f\| \to 0$.

Now for arbitrary $f \in L^1(0,1)$, we can find a continuous function g such that $\|f - g\| < \frac{\epsilon}{3}$ for given $\epsilon > 0$. Since

$$
\begin{aligned}
\|Q_n f - f\| &\leq \|Q_n f - Q_n g\| + \|Q_n g - g\| + \|g - f\| \\
&< \frac{2}{3}\epsilon + \|Q_n g - g\|,
\end{aligned}
$$

from the first part of the proof, we conclude that $\lim_{n\to\infty} Q_n f = f$ in L^1-norm. **Q.E.D.**

The next lemma is crucial to our convergence theorem.

Lemma 5.2.3. If $f \in L^1(0,1)$ is of bounded variation, then

$$\bigvee_0^1 Q_n f \leq \bigvee_0^1 f.$$

Proof. On the interval $I_i = [x_{i-1}, x_i]$ we have

$$\begin{aligned}
Q_n f(x) &= q_{2i-2}\phi_{2i-2}(x) + q_{2i-1}\phi_{2i-1}(x) + q_{2i}\phi_{2i}(x) \\
&= q_{2i-2}(n(x-x_{i-1})-1)^2 + q_{2i-1}[2n(x-x_{i-1})(1-n(x-x_{i-1}))] + q_{2i}(n(x-x_i)+1)^2 \\
&= q_{2i-2}(nx-i)^2 + 2q_{2i-1}(nx-(i-1))(i-nx) + q_{2i}(nx-(i-1))^2 \\
&= (q_{2i-2}-2q_{2i-1}+q_{2i})(nx)^2 - 2[(q_{2i-2}-2q_{2i-1}+q_{2i})i + q_{2i-1} - q_{2i}]nx \\
&+ [(q_{2i-2}-2q_{2i-1}+q_{2i})i^2 + 2(q_{2i-1}-q_{2i})i + q_{2i}].
\end{aligned}$$

For simplicity, let $a = q_{2i-2}, b = q_{2i-1}, c = q_{2i}$. Then for $x \in I_i$,

$$Q_n f(x) = (a-2b+c)(nx)^2 - 2[(a-2b+c)i + b - c]nx + [(a-2b+c)i^2 + 2(b-c)i + c].$$

Denote the right hand side of the above equality by $\phi(nx)$. Then the extreme point of $\phi(nx)$ is

$$\hat{x} = \frac{-2[(a-2b+c)i+b-c]}{-2(a-2b+c)n} = \frac{i}{n} + \frac{b-c}{(a-2b+c)n},$$

and the extreme value is

$$\begin{aligned}
\phi(n\hat{x}) &= [(a-2b+c)i^2 + 2(b-c)i + c] - \frac{[(a-2b+c)i+b-c]^2}{a-2b+c} \\
&= [(a-2b+c)i^2 + 2(b-c)i + c] - [(a-2b+c)i^2 + 2(b-c)i + \frac{(b-c)^2}{a-2b+c}] \\
&= c - \frac{(b-c)^2}{a-2b+c}.
\end{aligned}$$

We consider all the possible cases as follows.

- $\hat{x} \in I_i$. Then $x_{i-1} < \frac{i}{n} + \frac{b-c}{(a-2b+c)n} < x_i$, so, $0 < \frac{c-b}{a-2b+c} < 1$.

  1. If $a-2b+c > 0$, i. e. , the graph of ϕ opens upward, then $c-b>0$. Hence

$$0 < \frac{(c-b)^2}{a-2b+c} < c-b.$$

In this case, we have

$$\begin{aligned}
\bigvee_{I_i} Q_n f &= [a - (c - \frac{(b-c)^2}{a-2b+c})] + [c - (c - \frac{(b-c)^2}{a-2b+c})] \\
&= a - c + \frac{2(b-c)^2}{a-2b+c} < a - c + 2(c-b) \\
&= a - b + c - b \leq |a-b| + |c-b|.
\end{aligned}$$

2. If $a - 2b + c < 0$, i. e. , the graph of ϕ opens downward, then $c - b < 0$. Hence

$$0 > \frac{(c-b)^2}{a-2b+c} > c - b.$$

In this case, we have

$$\begin{aligned}
\bigvee_{I_i} Q_n f &= (c - \frac{(b-c)^2}{a-2b+c} - a) + (c - \frac{(b-c)^2}{a-2b+c} - c) \\
&= c - a - \frac{2(b-c)^2}{a-2b+c} < c - a - 2(c-b) \\
&= b - a + b - c < |a-b| + |c-b|.
\end{aligned}$$

- $\hat{x} \notin I_i$. Then $Q_n f$ is monotonic on I_i. So,

$$\bigvee_{I_i} Q_n f = |a-c| \leq |a-b| + |c-b|.$$

- $a - 2b + c = 0$. Then $Q_n f$ is linear on I_i. We also have

$$\bigvee_{I_i} Q_n f = |a-c| \leq |a-b| + |c-b|.$$

In any case, we obtain the following inequality

$$\bigvee_{I_i} Q_n f \leq |a-b| + |c-b| = |q_{2i-2} - q_{2i-1}| + |q_{2i} - q_{2i-1}|.$$

It follows that

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \sum_{i=1}^n \bigvee_{I_i} Q_n f \leq \sum_{i=1}^n \{|q_{2i-2} - q_{2i-1}| + |q_{2i} - q_{2i-1}|\} \\
&= n\sum_{i=2}^{n-1} |\frac{1}{2}(\int_{I_{i-1}} f + \int_{I_i} f) - \int_{I_i} f| + n\sum_{i=2}^{n-1} |\frac{1}{2}(\int_{I_i} f + \int_{I_{i+1}} f) - \int_{I_i} f| \\
&+ n\{|\int_{I_1} f - \int_{I_1} f| + |\frac{1}{2}(\int_{I_1} f + \int_{I_2} f) - \int_{I_1} f|\}
\end{aligned}$$

$$+ \quad n\{|\frac{1}{2}\int_{I_{n-1}} f + \int_{I_n} f) - \int_{I_n} f| + |\int_{I_n} f - \int_{I_n} f|$$

$$= \quad \sum_{i=1}^{n} |n\int_{I_{i+1}} f - n\int_{I_i} f| \leq \bigvee_0^1 f.$$

Q.E.D.

Let $P_n = Q_n \circ P_S|_{\Omega_n}$. Then $P_n : \Omega_n \to \Omega_n$ is a Markov operator of finite rank. Let $P_n\phi_k = \sum_{j=0}^{2n} p_{kj}\phi_j$ for $k = 0, \ldots, 2n$. Denote the $(2n+1) \times (2n+1)$ matrix (p_{kj}) by $\tilde{P}_n$.

Lemma 5.2.4. P_n has a nontrivial nonnegative fixed point in Ω_n.

Proof. For $k = 0, \ldots, 2n$, from

$$\begin{aligned} P_n\phi_k &= Q_n \circ P_S\phi_k \\ &= n\{(\int_{I_1} P_S\phi_k)\phi_0 + \sum_{i=1}^{n}(\int_{I_i} P_S\phi_k)\phi_{2i-1} \\ &+ \frac{1}{2}\sum_{i=1}^{n-1}(\int_{I_i} P_S\phi_k + \int_{I_{i+1}} P_S\phi_k)\phi_{2i} + (\int_{I_n} P_S\phi_k)\phi_{2n}\}, \end{aligned}$$

we have $p_{k0} = n\int_{I_1} P_S\phi_k, p_{k,2i-1} = n\int_{I_i} P_S\phi_k, i = 1, \ldots, n, p_{k,2i} = \frac{n}{2}(\int_{I_i} P_S\phi_k + \int_{I_{i+1}} P_S\phi_k), i = 1, \ldots, n-1$, and $p_{k,2n} = n\int_{I_n} P_S\phi_k$ for $k = 0, \ldots, 2n$.

Let $\zeta = (\zeta_0, \ldots, \zeta_{2n})$ where $\zeta_0 = \zeta_{2n} = \frac{1}{3}, \zeta_{2i} = \frac{2}{3}$ for $i = 1, \ldots, n-1, \zeta_{2i-1} = \frac{1}{3}$ for $i = 1, \ldots, n$. Then,

$$\begin{aligned} \sum_{j=0}^{2n} p_{kj}\zeta_j &= n\{\frac{1}{3}\int_{I_1} P_S\phi_k + \frac{1}{3}\sum_{i=1}^{n}\int_{I_i} P_S\phi_k \\ &+ \frac{2}{3}\sum_{i=1}^{n-1}\frac{1}{2}(\int_{I_i} P_S\phi_k + \int_{I_{i+1}} P_S\phi_k) + \frac{1}{3}\int_{I_n} P_S\phi_k\} \\ &= n[\frac{1}{3}\int_0^1 P_S\phi_k + \frac{2}{3}\int_0^1 P_S\phi_k] = n\int_0^1 \phi_k. \end{aligned}$$

Note that $\|\phi_{2i}\| = \frac{2}{3n}$ for $i = 1, \ldots, n-1, \|\phi_{2i-1}\| = \frac{1}{3n}$ for $i = 1, \ldots, n$, and $\|\phi_0\| = \|\phi_{2n}\| = \frac{1}{3n}$, we have $\tilde{P}_n\zeta = \zeta$. Therefore we can find a nonnegative row vector $c \neq 0$ such that $c\tilde{P}_n = c$. This implies that $f_n = \sum_{j=0}^{2n} c_j\phi_j$ is a nontrivial nonnegative fixed point of P_n in Ω_n. **Q.E.D.**

The proofs of the following results follow exactly the same line of arguments as in the previous section. So we omit the corresponding proofs.

Lemma 5.2.5. If $S : [0,1] \to [0,1]$ is piecewise C^2 with $M = \inf |S'(x)| > 2$,then the sequence $\{\bigvee_0^1 f_n\}$ is uniformly bounded, where $f_n \in \Omega_n$ is a nonnegative fixed point of P_n for each n.

Theorem 5.2.1. Suppose $S : [0,1] \to [0,1]$ is piecewise C^2 and $M = \inf |S'(x)| > 2$. If the corresponding Frobenius-Perron operator P_S has unique invariant density f, then the sequence $\{f_n\}$ of nonnegative piecewise quadratic fixed points of P_n in Ω_n converges to f in L^1.

Corollary 5.2.1. If $S : [0,1] \to [0,1]$ is piecewise C^2 satisfying $M = \inf |S'(x)| > 1$, and P_S has a unique fixed density, then a sequence g_n from the piecewise quadratic functions can be constructed that converges to the unique invariant density of P_S.

Chapter 6

Numerical Results

In this chapter we present numerical results for some mappings from $[0,1]$ into itself with our new methods and compare them to Li's original one. These calculations were performed on the IBM 3090 180VF at Michigan State University, using double precision. Section 6.1 gives the numerical results with the projection methods, while Section 6.2 gives those using the Markov approximation schemes.

6.1 Numerical Results with Projection Methods

The test functions are as follows

$$
\begin{aligned}
S_1(x) &= \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1, \end{cases} \\
S_2(x) &= (\frac{1}{8} - 2|x - \frac{1}{2}|^3)^{\frac{1}{3}} + \frac{1}{2}, \\
S_3(x) &= \begin{cases} \frac{2x}{1-x^2} & 0 \le x \le \sqrt{2}-1 \\ \frac{1-x^2}{2x} & \sqrt{2}-1 \le x \le 1, \end{cases} \\
S_4(x) &= \begin{cases} \frac{2x}{1-x} & 0 \le x \le \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \le x \le 1, \end{cases} \\
S_5(x) &= 4x(1-x).
\end{aligned}
$$

The invariant densities f_i^* of S_i are given by

$$f_1^*(x) \equiv 1,$$

$$
\begin{aligned}
f_2^*(x) &= 12(x-\frac{1}{2})^2, \\
f_3^*(x) &= \frac{4}{\pi(1+x^2)}, \\
f_4^*(x) &= \frac{2}{(1+x)^2}, \\
f_5^*(x) &= \frac{1}{\pi\sqrt{x(1-x)}}.
\end{aligned}
$$

Let a_j be the nondifferentiable point of S_j. We divide the intervals $[0, a_j]$ and $[a_j, 1]$ into $n/2$ equal subintervals, respectively. On the i-th subinterval $I_i = [x_{i-1}, x_i]$, we choose $(x - x_{i-1})^k \chi_{I_i}$ as the basis functions for $k = 0, 1, \ldots, m$. Here, of course, $m = 0$ for Li's piecewise constant method, $m = 1$ and $m = 2$ for our piecewise linear and piecewise quadratic methods, respectively. Gaussian quadrature formulas using three nodes were used on each subinterval to evaluate the integrals for the matrix in each scheme. The order of each matrix is $kn \times kn$ for $k = 0, 1$, or 2, respectively. The QR decomposition subroutine was used together with backward substitution to solve the linear fixed point problem. In order to estimate the convergence of the approximate density f_n to f^*, we used the L^1-norm $\|f_n - f^*\| = \int_0^1 |f_n - f^*| dx$ for each method. Also we used the Gaussian quadrature formula on each subinterval to calculate this norm. In the following we present numerical experiments in order from example 1 to example 5.

Since the invariant density of S_1 is constant, all the methods work very well even for small n, and the largest error is less than 10^{-8}. Table 6.1, Table 6.2, and Table 6.3 give the computational results with S_2, S_3, and S_4, respectively. For these tables, the first column is the number of subintervals, and error0, error1, and error2 in the remaining columns give the corresponding errors for the piecewise constant, piecewise linear, and piecewise quadratic projection methods in succession. The symbol $*$ in the tables indicates that the dimension of the matrix is out of the limited virtual storage range.

From Tables 6.1, 6.2, and 6.3 it is clear that the rate of convergence for the piecewise linear and piecewise quadratic schemes is much better than that of the piecewise constant one. Also notice that when the mesh of the partition is reduced

n	error0	error1	error2
4	0.4516118411	0.0555555556	0.0149254220
8	0.2292409861	0.0148102010	0.0027094962
16	0.1027134038	0.0035791431	0.0003887813
32	0.0525720871	0.0010220004	0.0000496560
64	0.0256408424	0.0002415574	0.0000066700
128	0.0133770827	0.0000693106	*
256	0.0064525269	*	*

Table 6.1: Error Estimates for S_2

n	error0	error1	error2
4	0.0830968916	0.0123173042	0.0101762505
8	0.0406691860	0.0031483283	0.0025306884
16	0.0202492312	0.0007860024	0.0006211069
32	0.0097816838	0.0001869041	0.0001531915
64	0.0048954498	0.0000479155	0.0000380217
128	0.0024518719	0.0000117453	*
256	0.0012361596	*	*

Table 6.2: Error Estimates for S_3

n	error0	error1	error2
4	0.1195535492	0.0393188225	0.0370286858
8	0.0586617412	0.0118494476	0.0103420855
16	0.0279057298	0.0032289581	0.0027443529
32	0.0138824563	0.0008396322	0.0007082720
64	0.0067786733	0.0002162473	0.0001800303
128	0.0032945853	0.0000623525	*
256	0.0016371634	*	*

Table 6.3: Error Estimates for S_4

n	error0	error1	error2
4	0.3896329138	0.3075260533	0.2015446439
8	0.3101171199	0.2275786385	0.1739440930
16	0.2427747128	0.1688182194	0.1355681505
32	0.1932292835	0.1342031192	0.1048131049
64	0.1464393853	0.0969966808	0.0796186481
128	0.1121331945	0.0765348792	*
256	0.0828776585	*	*

Table 6.4: Error Estimates for S_5

in half, the error decreases with a factor about $\frac{1}{2}$ for the piecewise constant method, a factor between $\frac{1}{4}$ to $\frac{1}{6}$ for the piecewise linear and piecewise quadratic ones. The better the invariant density f^* can be approximated by quadratic polynomials, the faster the convergence rate is for the piecewise quadratic scheme comparing to the piecewise linear scheme.

If the invariant density is unbounded, it appears that all the methods converge very slowly. Table 6.4 demonstrates this defect. Here the density f_5^* of S_5 is unbounded, though it belongs to L^1.

Table 6.4 indicates that the convergence rate of projection methods depends not only on the order of the polynomial approximation but also on the regularity of the invariant density. To develop more efficient methods for transformations with unbounded invariant densities is a problem for future research.

6.2 Numerical Results for Markov Finite Approximations

In this section we present the numerical results for the invariant measures of the mappings S_1 through S_5 from section 6.2, using our piecewise linear and piecewise quadratic Markov approximations. For comparison we also include the results using the piecewise constant method of the previous section.

Unlike the projection method, numerical integration is not required in our program here because we can use the Koopman operator U_S [3] instead of the Frobenius-Perron operator P_S to calculate the matrix representation of the Markov operator P_n. This property makes the new schemes much easier to implement than the projection ones. Suppose the interval $[0, 1]$ is divided into n subintervals. Then the algorithms for the piecewise constant , piecewise linear, or piecewise quadratic Markov approximation methods need to solve an $n \times n$, $(n+1) \times (n+1)$, or $(2n+1) \times (2n+1)$ system of linear equation $c\tilde{P}_n = c$. The QR decomposition together with the backward substitution were also used to solve this algebraic equation. Again the L^1-norm $\|f_n - f^*\| = \int_0^1 |f_n - f^*| dx$ is used to estimate the convergence of the approximate

n	error0	error1	error2
4	0.4516118411	0.4641185272	0.3959913833
8	0.2292409861	0.1800796348	0.1462659757
16	0.1027134038	0.0626959897	0.0487315033
32	0.0525720871	0.0206618342	0.0157728634
64	0.0256408424	0.0069115342	0.0050447087
128	0.0133770827	0.0022033011	0.0015826892
256	0.0064525269	0.0006804461	*

Table 6.5: Error Estimates for S_2

density f_n to f^* for each method.

It is no surprise that our new methods also work very well for S_1. Table 6.5, Table 6.6, and Table 6.7 give the computational results for S_2, S_3, and S_4, respectively.

The last three columns, error0, error1, and error2, represent errors for the piecewise constant, piecewise linear, and piecewise quadratic Markov approximation schemes, respectively. It is apparent from these tables that the piecewise linear and piecewise quadratic methods are better than the piecewise constant one not only for the same partition (that is, with the same n), but also under the same dimension of the system of linear equations involved.

Table 6.8 shows the unsatisfactory computation with S_5. Thus it is necessary to investigate new approaches in this case.

We may construct continuous piecewise cubic Markov approximation or even higher order ones along the same lines. But in practice for the consideration of stability, higher order polynomial approximations are rarely used . Our numerical experiments show that, if the fixed density of P_S is smooth enough, the Markov approximations converge quickly. We believe that under some regularity condition for the invariant density of the Frobenius-Perron operator, the convergence rate can be obtained to explain our numerical phenomenon.

n	error0	error1	error2
4	0.0830968916	0.0291096553	0.0209641927
8	0.0406691860	0.0101965728	0.0070473567
16	0.0202492312	0.0031790398	0.0021596521
32	0.0097816838	0.0009425484	0.0006358728
64	0.0048954498	0.0002723532	0.0001836483
128	0.0024518719	0.0000774104	0.0000521595
256	0.0012361596	0.0000216967	*

Table 6.6: Error Estimates for S_3

n	error0	error1	error2
4	0.1195535492	0.0492295286	0.0436642029
8	0.0586617412	0.0216968214	0.0192011196
16	0.0279057298	0.0104087996	0.0082975396
32	0.0138824563	0.0041061066	0.0031177980
64	0.0067786733	0.0014008562	0.0010345743
128	0.0032945853	0.0004428463	0.0003202538
256	0.0016371634	0.0001335304	*

Table 6.7: Error Estimates for S_4

n	error0	error1	error2
4	0.3896329138	0.3691582134	0.3681119076
8	0.3101171199	0.3181004411	0.3057295967
16	0.2427747128	0.2584324400	0.2453954228
32	0.1932292835	0.1884801640	0.1801104382
64	0.1464393853	0.1438197957	0.1346727768
128	0.1121331945	0.1021434498	0.0966334472
256	0.0828776585	0.0754555981	*

Table 6.8: Error Estimates for S_5

Chapter 7

Conclusions

In this work piecewise linear and piecewise quadratic polynomial projection methods and Markov finite approximations are found for the approximation of invariant densities of the Frobenius-Perron operator. Convergence of the methods is proved for a general class of measurable nonsingular transformations of the unit interval into itself . Our proof is based on the following observation: The operators $Q_n : L^1(0,1) \to L^1(0,1)$ defined in the previous chapters satisfy

(1) $\| Q_n \| \leq C_1$, $\qquad$ C_1 is a constant.

(2) $Q_n f \to f$ uniformly for any $f \in L^1(0,1)$.

(3)For any $f \in L^1(0,1)$ of bounded variation $\bigvee_0^1 Q_n f \leq C_2 \bigvee_0^1 f$ with C_2 a constant.

(4) $Q_n \circ P$ has a nontrivial fixed point f_n for each n.

In general a numerical scheme for the Frobenius-Perron operator equation $P_S f - f = 0$ is convergent if the "discretization" operators Q_n satisfy the above four requirements, as the following theorem shows.

Theorem 7.1.1. Suppose that the sequence of operators Q_n of finite rank satisfy the conditions (1) through (4) above. Then a sequence of functions can be constructed which converge to a nontrivial fixed point of P_S if S is piecewise C^2 satisfying $\inf |S'| > 1$.

Based on the convergence analysis for the piecewise linear and piecewise quadratic polynomial approximation methods, we believe that convergence can also be established for general higher order piecewise polynomial approximation methods, though it would not have much computational practicality due to instability.

It is important to estimate the rate of convergence for a convergent numerical method. Further research will be focused on this aspect for our finite approximation methods for Frobenius-Perron operator equations or more general Markov operator equations.

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