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THEORETICAL AND NUMERICAL STUDIES ON A PENALTY-PERTURBATION FINITE ELEMENT METHOD FOR THE BIHARMONIC PLATE PROBLEMS

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ABSTRACT

THEORETICAL AND NUMERICAL STUDIES ON
A PENALTY-PERTURBATION FINITE ELEMENT
METHOD FOR THE BIHARMONIC PLATE PROBLEMS

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A penalty-perturbation finite element method for the biharmonic plate problems is analyzed. The penalty-perturbation theory leads to a new system of partial differential equations which is singularly perturbed with respect to a small parameter ε. Finite element solutions of the perturbed problems, for small ε, provide approximations to solutions of the original problems. The role of the small parameter ε in the Reissner-Mindlin plate theory is clarified. It is also shown that the present method covers a previous nonconforming finite element method of Nitsche as a special case. Efforts are taken to derive error estimates of the finite element solutions in various Sobolev norms. Numerical experiments for square and circular plates, under both axisymmetric and nonsymmetric loadings, are conducted. Results obtained using quadratic and isoparametric elements are presented and discussed in detail.

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Chapter 1 Introduction

Section 1.1: Motivation and objectives of the dissertation

This dissertation concerns a penalty-perturbation method for a clamped plate of uniform thickness and constant material properties occupying an open bounded region in the xy-plane. In [34] Westbrook proposed to approximate the plate deflection and its first partial derivatives separately and used a penalty parameter $1/\varepsilon$ to control the closeness of the first partial derivatives of the plate deflection and the new dependent variables in the perturbed energy integral. This perturbed problem was studied by T. C. Assiff and D. H. Y. Yen in [1, 2], where a proof of the existence of the weak solution of this perturbed problem by using the Lax-Milgram theorem was given and error estimates for the difference between the solution of the classical plate problem P_0 and those of the perturbed problem P_ε were derived in the $\|\ \|_1$ norm. Also in [1, 2] finite element approximate solutions for the perturbed problem P_ε were studied and error estimates for the difference between them and the classical solutions in $\|\ \|_1$ were derived in terms of the mesh size h and the parameter ε. One primary objective in this dissertation is to extend the above results by deriving new sharper error estimates in various Sobolev norms. In [24] a so-called nonconforming finite element method was introduced by Nitsche. That one version of this nonconforming method for the biharmonic plate problem is in fact related to the works [1, 2] mentioned above is established here. In particular, it will be shown that the perturbed energy integral in [24] corresponds to that in [1, 2] when Poisson's ratio in the latter is taken to be $\mu = -1$.

Finite element implementations of this penalty-perturbation method are carried out. Extensive numerical studies for both square and circular plates under various loading conditions and using different approximating finite element spaces are obtained to substantiate the theoretical error estimates derived.

Section 1.2 : Organization of the dissertation

Chapter 1 contains the introduction. Notations and nomenclature for various function spaces are given there.

In Chapter 2 the boundary value problems P_0 for the classical plate theory and P_ε for the improved plate theory are introduced as formulated in [1, 2]. The coercivity of the bilinear functional $B_\varepsilon(V, V)$ in P_ε will be shown to hold for $-1 \le \mu < 1$ and $0 < \varepsilon < 1$. This gives the existence of the weak solution of the problem P_ε when $\mu = -1$, which is the case Nitsche considered. As ε tends to zero, the solutions of the problem P_ε converge in $\| \ \|_1$ to those of the problem P_0. and this was shown in [1, 2] in the presence of $\varepsilon^{1/2}$ in the error bounds. Some improvements of the error estimates will be given in this chapter. New error bounds containing ε in $\| \ \|_1$ and $\| \ \|_0$ will be derived.

Chapter 3 establishes the convergence of the finite element approximations to the solutions of the problems P_ε and P_0. For piecewise linear elements we may allow ε to be proportional to the mesh size h. The error bounds then contain h instead of $h^{1/2}$ as in [1, 2]. For piecewise quadratic elements we may allow ε to be proportional to h^2 and have the factor h^2 in the error bounds. This means that quadratic finite elements solutions converge much faster to the solutions of the problems P_ε and P_0. An example of this comparison is given in Chapter 4.

Chapter 4 presents the construction of the finite element stiffness matrix associated with piecewise quadratic elements. The global stiffness matrix is assembled by the element stifness matrices. The element stiffness matrices for piecewise linear elements are only 9×9 matrices, but the element stiffness matrices for piecewise quadratic elements are 18×18 matrices. Although the construction of the quadratic case is much more complicated, numerical results show that we have more superior approximations. Numerical results for the clamped unit plate under polynomial loads with different Poisson's ratios $\mu = -1$, 0 , and 1/2 are given with mesh sizes $h = 1/4$, 1/8, 1/16, and 1/32.

In Chapter 5 finite element solutions for the clamped circular plate under a constant load and a non-axisymmetric load are obtained. The elements with one curved side will be mapped into a unit triangle under an isoparametric transformation. The area coordinates and the basis functions of the quadratic maps are chosen to illustrate the isoparametric transformations. The element stiffness matrix is constructed by computing the perturbed energy integral under the isoparametric transformations. The global stiffness matrix is then assembled. Numerical results show that we have excellent approximations for the constant load with mesh sizes h=1/4 ,and h=1/8. For the non-symmetric load the approximations are also very good when mesh sizes h=1/8, and h=1/16.

Chapter 6 contains discussions and conclusions of the dissertation

Section 1.3: Notations and function spaces

Let Ω be an open bounded connected region in the xy-plane with a Lipschitz boundary $\partial\Omega$. Let $L_2(\Omega)$ be the space of integrable functions on Ω, with the inner product

$$(u, v)_0 = \iint_\Omega uv\, dA,$$

and the norm $\|\ \|_0$ defined by

$$\|u\|_0^2 = (u, u)_0 = \iint_\Omega u^2\, dA.$$

The partial derivatives of u are denoted by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} = u_{,1} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_2} = u_{,2},$$

the Laplacian Δ is denoted by

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and biharmonic operator Δ^2 is denoted by

$$\Delta^2 u = \nabla^4 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Let $\alpha = (\alpha_1, \alpha_2)$ be an ordered pair of non-negative integers. Let $|\alpha| = \alpha_1 + \alpha_2$ and let $D^\alpha u$ be the αth derivatives of u defined by

$$D^\alpha u = \frac{\partial^{\alpha_1 + \alpha_2} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

Let m be a positive integer and $H^m(\Omega)$ be the standard Sobolev spaces with the norms

$$\|u\|_m = \Big(\sum_{0 \le |\alpha| \le m} \iint_\Omega |D^\alpha u|^2\, dA \Big)^{1/2},$$

and the seminorms

$$|u|_m = \Big(\sum_{|\alpha| = m} \iint_\Omega |D^\alpha u|^2 \, dA \Big)^{1/2}.$$

It is well known that $H^0(\Omega) = L_2(\Omega)$.

Let $C^\infty(\Omega)$ be the linear space of functions infinitely differentiable on Ω and $C_0^\infty(\Omega)$ be the linear subspace of $C^\infty(\Omega)$, consisting of those functions that have compact support in Ω.

Let $H_0^m(\Omega)$ be the closure of the $C_0^\infty(\Omega)$ in $H^m(\Omega)$ and define the negative spaces $H^{-m}(\Omega)$ as duals of the spaces $H_0^m(\Omega)$ with the norm

$$||v||_{-m} = \sup_{\substack{u \in H_0^m(\Omega) \\ u \neq 0}} \frac{|(v, u)_0|}{||u||_m}.$$

Let $(H_0^m(\Omega))^3 = H_0^m(\Omega) \times H_0^m(\Omega) \times H_0^m(\Omega)$ be the product space with the norm

$$||U||_m^2 = ||u_1||_m^2 + ||u_2||_m^2 + ||u_3||_m^2$$

and the seminorm

$$|U|_m^2 = |u_1|_m^2 + |u_2|_m^2 + |u_3|_m^2,$$

where $U = (u_1, u_2, u_3)$ is in $(H_0^m(\Omega))^3$.

Similar definitions will hold for $(H^m(\Omega))^3$, $(H^m(\Omega))^2$, and $(H_0^m(\Omega))^2$.

For U in $(H_0^1(\Omega))^3$ and $\varepsilon > 0$, define

$$\|U\|_\varepsilon^2 = \sum_{i=1}^{3} |u_i|_1^2 + \frac{1}{\varepsilon} \iint_\Omega \left(\frac{\partial u_3}{\partial x} + u_1\right)^2 + \left(\frac{\partial u_3}{\partial y} + u_2\right)^2 dA$$

$$= |U|_1^2 + \frac{1}{\varepsilon} \iint_\Omega \left(\frac{\partial u_3}{\partial x} + u_1\right)^2 + \left(\frac{\partial u_3}{\partial y} + u_2\right)^2 dA .$$

It was shown in [1,2] that $\| \ \|_1$ and $\| \ \|_\varepsilon$ are equivalent on $(H_0^1(\Omega))^3$. See Lemma 2.4.

Throughout this dissertation c will denote a generic constant, not necessary the same in any two places.

Chapter 2 Solutions of the perturbed boundary value problems and their error estimates

Section 2.1 : Weak formulations of the problems P_0 and P_ε.

Let Ω be an open bounded and connected region in the xy-plane with its boundary $\partial\Omega$ sufficiently smooth or polygonal. According to the classical plate theory, the plate deflection w_0 is governed by

$$P_0: \qquad \nabla^4 w_0 = f \qquad \text{in } \Omega,$$
$$w_0 = \frac{\partial w_0}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{2.1}$$

where $f = \frac{P}{D}$, P being the transverse load and D the plate bending stiffness. f is assumed in $H^0(\Omega)$. If f is in $H^{-1}(\Omega)$, then the above problem will be assumed in the sense of distributions..

The weak formulation of this problem is to find $w_0 \in H_0^2(\Omega)$ such that

$$\iint_\Omega \nabla^2 w_0 \ \nabla^2 v \, dA = \iint_\Omega f v \ \ dA \quad \text{for all } v \in H_0^2(\Omega). \tag{2.2}$$

Let $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$ be in $(H_0^1(\Omega))^3$ and let $F = (0, 0, -f)$. Define the following bilinear functionals

$$P_B(U,V)=\frac{1}{2}\iint_{\Omega}[(1+\mu)(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y})(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y})+$$

$$(1-\mu)(\frac{\partial u_1}{\partial x}-\frac{\partial u_2}{\partial y})(\frac{\partial v_1}{\partial x}-\frac{\partial v_2}{\partial y})+$$

$$(1-\mu)(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x})(\frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x})]\ dA, \tag{2.3}$$

$$P_S(U,V)=\iint_{\Omega}[(\frac{\partial u_3}{\partial x}+u_1)(\frac{\partial v_3}{\partial x}+v_1)+$$

$$(\frac{\partial u_3}{\partial y}+u_2)(\frac{\partial v_3}{\partial y}+v_2)]\ dA, \tag{2.4}$$

$$P_L(F,U)=-\iint_{\Omega}F\bullet U\ dA=\iint_{\Omega}f\,u_3\ dA, \tag{2.5}$$

and

$$B_\varepsilon(U,V)=P_B(U,V)+\frac{1}{\varepsilon}P_S(U,V), \tag{2.6}$$

where $\varepsilon > 0$.

Letting $U_0=(-\frac{\partial w_0}{\partial x},-\frac{\partial w_0}{\partial y},w_0)$ and integrating by parts, we can show that

$$P_B(U_0,V)=\iint_{\Omega}(\nabla^2 w_0)(\nabla^2 v)\ dA, \tag{2.7}$$

where $V=(-\frac{\partial v}{\partial x},-\frac{\partial v}{\partial y},v)$ and $v\in H_0^2(\Omega)$.

The problem in (2.2) can be expressed as the following problem P_0' for U_0.

P_0': Find $U_0 = (-\frac{\partial w_0}{\partial x}, -\frac{\partial w_0}{\partial y}, w_0), w_0 \in H_0^2(\Omega)$ such that

$$P_B(U_0 , V) = P_L(F , V) \qquad (2.8)$$

for all $V = (-\frac{\partial v}{\partial x}, -\frac{\partial v}{\partial y}, v)$ and $v \in H_0^2(\Omega)$.

The solution to the problem (2.1) may be characterized as the function that minimizes the energy integral

$$I(w) = \iint_\Omega (\nabla^2 w)^2 \, dA - 2 \iint_\Omega f \, w \, dA \qquad (2.9)$$

$$= P_B (U , U) - 2 P_L (F , U) , \qquad (2.10)$$

where $w \in H_0^2(\Omega)$ and $U = (-\frac{\partial w}{\partial x}, -\frac{\partial w}{\partial y}, w)$. The Euler-Lagrange equation of this variational problem leads to (2.2). Thus it follows that

$$\min_{w \in H_0^2(\Omega)} I(w) = I(w_0) . \qquad (2.11)$$

Consider the problem of minimizing the perturbed energy integral

$$J_\varepsilon(U) = B_\varepsilon(U, U) - 2 P_L (F, U) , \qquad (2.12)$$

where U is in $(H_0^1(\Omega))^3$. The Euler-Lagrange equations of the variational problem in (2.12) above lead to the problem P_ε below.

$$P_\varepsilon: \quad \text{Find } U_\varepsilon = (\psi_x, \psi_y, w_\varepsilon) \in (H_0^1(\Omega))^3 \text{ such that}$$

$$B_\varepsilon(U_\varepsilon, V) = P_L(F, V) \tag{2.13}$$

$$\text{for all } V = (v_1, v_2, v_3) \in (H_0^1(\Omega))^3 .$$

Equations (2.13) are the weak form of the following system of second order partial differential equations.

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_x + (1+\mu)\frac{\partial}{\partial x}(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y})] - \frac{1}{\varepsilon}(\psi_x + \frac{\partial w_\varepsilon}{\partial x}) = 0$$

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_y + (1+\mu)\frac{\partial}{\partial y}(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y})] - \frac{1}{\varepsilon}(\psi_y + \frac{\partial w_\varepsilon}{\partial y}) = 0 \quad \text{in } \Omega,$$

$$\frac{1}{\varepsilon}(\nabla^2 w_\varepsilon + \frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}) = -f ,$$

$$\text{and } \psi_x = \psi_y = w_\varepsilon = 0 \qquad \text{on } \partial\Omega$$

(2.14)

Section 2.2 : Existence of solutions to P_ε for $-1 \le \mu < 1$ and $0 < \varepsilon < 1$.

We establish in this section the existence of the problem P_ε for the Poisson ratio in the range $-1 \le \mu < 1$.

Lemma 2.1: (Poincare's inequality)

For any $u \in H_0^1(\Omega)$,

$\|u\|_0 \le c\,|u|_1$.

Proof :

A proof may be found in [18] and is omitted here.

Remark :

In fact for any $u \in H_0^1(\Omega)$,

$$\|u\|_1 \le c\,|u|_1 . \qquad (2.15)$$

The following Lemma, proved in [1, 2], will be needed in the proof of Theorem 2.3 .

Lemma 2.2 : For $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$ and $u_3 \in H_0^1(\Omega)$, and for all $0 < p < 1$,

$$P_S(U, U) \ge (1-p)\,|u_3|_1^2 - \frac{1}{2p}\left(|u_1|_1^2 + |u_2|_1^2\right) \qquad (2.16)$$

where $U = (u_1, u_2, u_3)$.

Theorem 2.3 :

For $\partial\Omega$ sufficiently smooth or polygonal and $f \in H^{-1}(\Omega)$, the problem P_ε has a unique solution $U_\varepsilon \in (H_0^1(\Omega))^3$, for $0 < \varepsilon < 1$ and $-1 \le \mu < 1$.

Proof:

We shall apply the Lax-Milgram theorem [18] to show that the existence of a unique solution $U_\varepsilon \in (H_0^1(\Omega))^3$. It is sufficient to show that $B_\varepsilon(U, V)$ is continuous in U and V and $B_\varepsilon(V, V)$ is coercive.

The proof for continuity of $B_\varepsilon(U, V)$:

$$
\begin{aligned}
|B_\varepsilon(U, V)| &= |P_B(U, V) + \frac{1}{\varepsilon} P_S(U, V)| \\
&\le c \iint_\Omega \left|\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right| \left|\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right| + \\
&\qquad \left|\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y}\right| \left|\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}\right| + \\
&\qquad \left|\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right| \left|\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right| \, dA
\end{aligned}
$$

$$+\frac{1}{\varepsilon}\iint_\Omega \left|\frac{\partial u_3}{\partial x}+u_1\right|\left|\frac{\partial v_3}{\partial x}+v_1\right| + \left|\frac{\partial u_3}{\partial y}+u_2\right|\left|\frac{\partial v_3}{\partial y}+v_2\right| \, dA$$

$$\le c|U|_1|V|_1 + c\frac{1}{\varepsilon}[P_s(U,U)]^{1/2}\,[P_s(V,V)]^{1/2}$$

$$\le c\|U\|_\varepsilon\|V\|_\varepsilon .$$

c is independent of ε and μ, as well as of U and V.

The proof of coercivity of $B_\varepsilon(V, V)$ is given below:

$$B_\varepsilon(V,V) = \frac{1}{2}\iint_\Omega [\,(1+\mu)(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y})^2 + (1-\mu)(\frac{\partial v_1}{\partial x}-\frac{\partial v_2}{\partial y})^2$$

$$+(1-\mu)(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x})^2\,]\,dA + \frac{1}{\varepsilon}P_s(V,V)$$

$$\ge \frac{1}{2}(1-\mu)\iint_\Omega (\frac{\partial v_1}{\partial x}-\frac{\partial v_2}{\partial y})^2 + (\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x})^2\,dA$$

$$+\frac{1}{\varepsilon}P_s(V,V) \qquad (2.17)$$

$$=\frac{1}{2}(1-\mu)\iint_\Omega [\,(\frac{\partial v_1}{\partial x})^2+(\frac{\partial v_2}{\partial y})^2+(\frac{\partial v_1}{\partial y})^2+(\frac{\partial v_2}{\partial x})^2$$

$$-2\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y}+2\frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}\,]\,dA$$

$$+\frac{1}{\varepsilon}P_s(V,V)$$

(By integration by parts)

$$= \frac{1}{2}(1-\mu)\iint_{\Omega}\left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial y}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 + \left(\frac{\partial v_2}{\partial x}\right)^2 dA$$

$$+ \frac{1}{\varepsilon} P_S(V, V)$$

$$= \frac{1}{2}(1-\mu)(|v_1|_1^2 + |v_2|_1^2) + \frac{1}{\varepsilon} P_S(V, V)$$

(Let $0 < \delta < 1$, $\delta < 1-\mu$, and $0 < p < 1$)

$$= \frac{1}{2}(1-\mu)(|v_1|_1^2 + |v_2|_1^2) + \left(\frac{1}{\varepsilon} - \delta p\right) P_S(V, V)$$

$$+ \delta p\, P_S(V, V)$$

(From Lemma 2.2 we have)

$$\geq \frac{1}{2}(1-\mu)(|v_1|_1^2 + |v_2|_1^2) + \left(\frac{1}{\varepsilon} - \delta p\right)\varepsilon \frac{1}{\varepsilon} P_S(V, V)$$

$$+ \delta p(1-p)|v_3|_1^2 - \frac{\delta}{2}(|v_1|_1^2 + |v_2|_1^2)$$

$$= \frac{1}{2}(1-\mu-\delta)(|v_1|_1^2 + |v_2|_1^2) + \delta p(1-p)|v_3|_1^2$$

$$+ (1-\varepsilon\delta p)\frac{1}{\varepsilon} P_S(V, V)$$

$$\geq M\left[|v_1|_1^2 + |v_2|_1^2 + |v_3|_1^2 + \frac{1}{\varepsilon} P_S(V, V)\right],$$

where $M = \min\{(1/2)(1-\mu-\delta), \delta p(1-p), 1-\varepsilon\delta p\}$. Clearly $M > 0$. Thus

$$B_\varepsilon(V, V) \geq M\, \|V\|_\varepsilon^2$$

and $B_\varepsilon(V, V)$ is coercive.

Remark :

(a) A result similar to that in Theorem 2.3 was proved in [1, 2] for $0 \le \mu \le 1/2$. Here the range of μ is extended to $-1 \le \mu < 1$. Note that $\mu = 1$ is not included since it is questionable whether $B_\varepsilon(V, V)$ is coercive for $\mu=1$. An example in Chapter 5 shows that a classical solution for P_ε need not exist for $\mu=1$.

(b) Taking $\mu=-1$ in $B_\varepsilon(V, V)$, we have equality hold in (2.17). Then

$$B_\varepsilon(V, V) = \iint_\Omega \left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial x}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 + \left(\frac{\partial v_2}{\partial y}\right)^2 \, dA$$

$$+ \frac{1}{\varepsilon} \iint_\Omega \left(v_1 + \frac{\partial v_3}{\partial x}\right)^2 + \left(v_2 + \frac{\partial v_3}{\partial y}\right)^2 dA .$$

The above perturbed energy integral was introduced by Nitsche in [24] in a rather unnatural manner.

Lemma 2.4 and Theorem 2.5 below were obtained in [1, 2]. They give error estimates for $U_\varepsilon - U_0$ in the norms $\|\ \|_1$ and $\|\ \|_\varepsilon$.

Lemma 2.4 : The norms $\|\ \|_1$ and $\|\ \|_\varepsilon$ on $(H_0^1(\Omega))^3$ are equivalent. In fact, for a domain Ω with largest dimension unity one has for any $U \in (H_0^1(\Omega))^3$,

$$\frac{4}{5} \|U\|_1^2 \le \|U\|_\varepsilon^2 \le \left(1 + \frac{2}{\varepsilon}\right) \|U\|_1^2 . \qquad (2.18)$$

Remark : When the domains above are not normalized, only the constants in (2.18) need be changed.

Theorem 2.5 : Let $U_0 = (-\frac{\partial w_0}{\partial x}, -\frac{\partial w_0}{\partial y}, w_0)$, $w_0 \in H_0^2(\Omega) \cap H^3(\Omega)$, be the solution of the problem P_0, and let $U_\varepsilon = (\psi_x, \psi_y, w_\varepsilon)$ be the solution of P_ε, $0 < \varepsilon < 1$. Then as $\varepsilon \to 0$ we have

$$\|U_\varepsilon - U_0\|_\varepsilon \le c_1 \varepsilon^{1/2} \|\nabla(\nabla^2 w_0)\|_0 , \qquad (2.19)$$

$$\|U_\varepsilon - U_0\|_1 \le c_2 \varepsilon^{1/2} \|\nabla(\nabla^2 w_0)\|_0 , \qquad (2.20)$$

where the constants c_1 and c_2 are independent of ε and the functions involved.

Remark :

(a) Theorem 2.5 gives the error estimates of w_0 and w_ε in $\| \ \|_1$. The error estimates of w_0 and w_ε in $\| \ \|_0$ will be given in Theorem 2.6.

(b) If $f \in L_2(\Omega)$, we have $w_0 \in H_0^2(\Omega) \cap H^4(\Omega)$, when $\partial\Omega$ is sufficiently smooth.

In the next several sections we present some new error estimates between the solutions of P_0 and P_ε.

Section 2.3 : Error estimates for $w_0 - w_\varepsilon$ in $\| \ \|_0$.

Theorem 2.6 :

$$\|w_0 - w_\varepsilon\|_0 \le c\,\varepsilon \|w_0\|_3 . \qquad (2.21)$$

Proof:

Let $e = w_0 - w_\varepsilon$ and consider the following problem :

$$\Delta^2 \phi = e \quad \text{in } \Omega,$$
$$\phi = \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad (2.22)$$

for ϕ.

Since $e \in L_2(\Omega)$, from the regularity property of solutions of elliptic partial differential equations [21, 26] we have

$$\phi \in H_0^2(\Omega) \cap H^4(\Omega),$$
and
$$\|\phi\|_4 \leq c\, \|e\|_0 \qquad (2.23)$$

Let $E = (0, 0, -e)$. Then for the same ε there exists a unique E_ε in

$(H_0^1(\Omega))^3$ such that

$$B_\varepsilon(E_\varepsilon, V) = P_L(E, V)$$
$$\text{for all } V \in (H_0^1(\Omega))^3 \qquad (2.24)$$

Let $E_0 = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, \phi\right)$. From (2.8) and (2.13) we have

$$P_B(U_0, E_0) = P_L(F, E_0) \quad \text{and} \quad B_\varepsilon(U_\varepsilon, E_0) = P_L(F, E_0).$$

Since $P_S(U_0, E_0) = 0$, we have

$$B_\varepsilon(U_0, E_0) = P_B(U_0, E_0) + \frac{1}{\varepsilon} P_S(U_0, E_0)$$
$$= P_B(U_0, E_0)$$
$$= P_L(F, E_0)$$

It follows that

$$B_\varepsilon(U_\varepsilon, E_0) = B_\varepsilon(U_0, E_0) \text{ and } B_\varepsilon(U_\varepsilon - U_0, E_0) = 0. \qquad (2.25)$$

From (2.5) and $e = w_0 - w_\varepsilon$ we have

$$\|e\|_0^2 = P_L(E, U_\varepsilon - U_0)$$
(with $V = U_\varepsilon - U_0$ in (2.24))
$$= B_\varepsilon(E_\varepsilon, U_\varepsilon - U_0)$$

(from (2.25))

$$= B_\varepsilon(E_\varepsilon , U_\varepsilon - U_0) - B_\varepsilon(E_0 , U_\varepsilon - U_0)$$

$$= B_\varepsilon(E_\varepsilon - E_0 , U_\varepsilon - U_0)$$

(from continuity of $B_\varepsilon(U , V)$, there exists a constant $M > 0$ such that)

$$\le M \, \| E_\varepsilon - E_0 \|_\varepsilon \, \| U_\varepsilon - U_0 \|_\varepsilon$$

$$\le M \, c_1 \, \varepsilon^{1/2} \, \| \nabla (\nabla^2 \phi) \|_0 \, c_2 \, \varepsilon^{1/2} \, \| \nabla (\nabla^2 w_0) \|_0$$

$$\le c \, \varepsilon \, \| \phi \|_3 \, \| w_0 \|_3$$

(from (2.23))

$$\le c \, \varepsilon \, \| e \|_0 \, \| w_0 \|_3 .$$

If both sides above are divided by $\| e \|_0$, we then have

$$\| e \|_0 \le c \, \varepsilon \, \| w_0 \|_3 .$$

Corollary 2.7: Let $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ be the solution of the problem P_ε, then for $i, j = 1, 2$

$$\left\| \frac{\partial}{\partial x_j} \left(- \frac{\partial w_0}{\partial x_i} \right) - \frac{\partial}{\partial x_j} (u_i^\varepsilon) \right\|_0 \le c \, \varepsilon^{1/2} \, \| w_0 \|_3 \qquad (2.26)$$

Proof:

$$\left\| \frac{\partial}{\partial x_j} \left(- \frac{\partial w_0}{\partial x_i} \right) - \frac{\partial}{\partial x_j} (u_i^\varepsilon) \right\|_0$$

$$= \left\| \frac{\partial}{\partial x_j} \left(- \frac{\partial w_0}{\partial x_i} - u_i^\varepsilon \right) \right\|_0$$

$$\le \left\| - \frac{\partial w_0}{\partial x_i} - u_i^\varepsilon \right\|_1$$

$$\le \| U_0 - U_\varepsilon \|_1$$

(from Theorem 2.5)

$$\le c \, \varepsilon^{1/2} \, \| w_0 \|_3 .$$

Thus (2.26) holds.

Section 2.4 : Error estimate for $w_0 - w_\varepsilon$ in $\| \|_1$.

Theorem 2.8 : If $U_\varepsilon - U_0 = (e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon)$, then

$$\| e_3^\varepsilon \|_1 = \| w_0 - w_\varepsilon \|_1 \leq c\,\varepsilon \| w_0 \|_3 . \tag{2.27}$$

Proof :

Since $\Delta e_3^\varepsilon \in H^{-1}(\Omega)$, let us consider the following problem

$$\Delta^2 \phi = \Delta e_3^\varepsilon \qquad \text{in } \Omega,$$
$$\phi = \frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

in the sense of distributions, with the solution ϕ such that

$$\phi \in H_0^2(\Omega) \cap H^3(\Omega), \text{ and } \| \phi \|_3 \leq c \; \| e_3^\varepsilon \|_1 . \tag{2.28}$$

From (2.15), there exists a constant $c > 0$ such that

$$\| e_3^\varepsilon \|_1^2 \leq c \; | e_3^\varepsilon |_1^2$$
$$= c \iint_\Omega \nabla e_3^\varepsilon \bullet \nabla e_3^\varepsilon \; d\Omega \tag{2.29}$$

(Since $e_3^\varepsilon = 0$ on $\partial\Omega$, then)

$$= c \iint_\Omega (- \Delta e_3^\varepsilon)\, e_3^\varepsilon \;\; d\Omega . \tag{2.30}$$

Let $E = (0, 0, -\Delta e_3^\varepsilon)$. Then for the same ε there exists a unique E_ε in $(H_0^1 (\Omega))^3$ such that

$$B_\varepsilon(E_\varepsilon, V) = P_L(E, V) \qquad (2.31)$$
$$\text{for all } V \in (H_0^1(\Omega))^3.$$

Let $E_0 = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, \phi)$. From (2.8) and (2.13) we have

$$B_\varepsilon(U_\varepsilon - U_0, E_0) = 0. \qquad (2.32)$$

From (2.29), (2.30), and (2.31), it follows that

$$\|e_3^\varepsilon\|_1^2 \le c \iint_\Omega (-\Delta e_3^\varepsilon)\, e_3^\varepsilon \, d\Omega$$

$$= c\, P_L(E, U_\varepsilon - U_0)$$

$$= c\, B_\varepsilon(E_\varepsilon, U_\varepsilon - U_0)$$

(from (2.32))

$$= c\, B_\varepsilon(E_\varepsilon - E_0, U_\varepsilon - U_0)$$

(for some M > 0)

$$\le M \|E_\varepsilon - E_0\|_\varepsilon \|U_\varepsilon - U_0\|_\varepsilon$$

(by Theorem 2.5)

$$\le M c_1 \varepsilon^{1/2} \|\phi\|_3 \; c_2 \varepsilon^{1/2} \|w_0\|_3$$

(from (2.28))

$$\le c\varepsilon \|e_3^\varepsilon\|_1 \|w_0\|_3 .$$

If both sides above are divided by $\|e_3^\varepsilon\|_1$, then we have

$$\|e_3^\varepsilon\|_0 \le c\varepsilon \|w_0\|_3 .$$

Section 2.5 : Error estimate for $U_0 - U_\varepsilon$ in $\| \ \|_0$.

Theorem 2.9 : If $U_\varepsilon - U_0 = (e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon)$, then

$$\| e_i^\varepsilon \|_0 \leq c\,\varepsilon \| w_0 \|_3 \tag{2.33}$$

for all $i = 1, 2$.

Proof:

$$\| e_i^\varepsilon \|_0 \leq \ \| \frac{\partial e_3^\varepsilon}{\partial x_i} \|_0 + \| \frac{\partial e_3^\varepsilon}{\partial x_i} + e_i^\varepsilon \|_0$$

$$\leq \| e_3^\varepsilon \|_1 + [\, P_S(U_\varepsilon - U_0 \,,\, U_\varepsilon - U_0)\,]^{1/2}$$

(by Theorem 2.8 we have)

$$\leq c\,\varepsilon \| w_0 \|_3 + c\,\varepsilon^{1/2} [\, B_\varepsilon(U_\varepsilon - U_0 \,,\, U_\varepsilon - U_0)\,]^{1/2}$$

$$\leq c\,\varepsilon \| w_0 \|_3 + c\,\varepsilon^{1/2} \| U_\varepsilon - U_0 \|_\varepsilon$$

(from Theorem 2.5)

$$\leq c\,\varepsilon \| w_0 \|_3 + c\,\varepsilon \| w_0 \|_3$$

$$\leq c\,\varepsilon \| w_0 \|_3 \ .$$

Thus

$$\| e_i^\varepsilon \|_0 \leq c\,\varepsilon \, \| w_0 \|_3 \,, \text{ for } i = 1, 2.$$

Theorem 2.10 :

$$\| U_\varepsilon - U_0 \|_0 \leq c\,\varepsilon \| w_0 \|_3 \ . \tag{2.34}$$

Proof:

It is clear that from Theorems 2.6 and 2.9, (2.34) holds.

Remark:

Theorem 2.9 give the error estimate

$$\| -\frac{\partial w_0}{\partial x} - \psi_x \|_0 \le c\varepsilon \|w_0\|_3 \qquad (2.35)$$

and

$$\| -\frac{\partial w_0}{\partial y} - \psi_y \|_0 \le c\varepsilon \|w_0\|_3 \,. \qquad (2.36)$$

From Theorem 2.8 we have

$$\|w_0 - w_\varepsilon\|_1 \le c\varepsilon \|w_0\|_3 \,.$$

One might guess that the following inequalities are true for $\alpha = 1$.

$$\| -\frac{\partial w_0}{\partial x} - \psi_x \|_1 \le c\varepsilon^\alpha \|w_0\|_3 \qquad (2.37)$$

and

$$\| -\frac{\partial w_0}{\partial y} - \psi_y \|_1 \le c\varepsilon^\alpha \|w_0\|_3 \,. \qquad (2.38)$$

However, as discussed in [1, 2], (2.37) and (2.38) are not true for $\alpha = 1$. In fact, an example given in the above references showed that α cannot be greater than 3/4.

Chapter 3 Finite element approximations

Section 3.1: Error estimates between U_ε and its finite element approximations U_h in $\|\ \|_1$

In this section we consider finite element approximations U_h for U_ε. Let $S_h^{t,k}$ be a linear system of functions as defined in [7] with the following properties: For $t, k \geq 0$,

(i) $S_h^{t,k}(\Omega)$ is contained in $H^k(\Omega)$.

(ii) For any $u \in H^m(\Omega)$, $m \geq 0$ and $0 \leq s \leq \min(m, k)$, there exists $\phi \in S_h^{t,k}(\Omega)$ such that

$$\|u - \phi\|_s \leq c\, h^\mu \|u\|_m \,, \qquad (3.1)$$

where $\mu = \min(t-s, m-s)$. The constant c is independent of u and h.

The above system will be considered a subspace of $H_0^1(\Omega)$ in the following theorems. For $t = 2$ and $t = 3$ this system corresponds to piecewise linear and piecewise quadratic elements respectively. Let

$$S_h = S_h^{t,k} \times S_h^{t,k} \times S_h^{t,k},$$

so that S_h is a subspace of $(H_0^1(\Omega))^3$.

We wish to find an approximation for the solution U_ε of the problem P_ε over S_h by the finite element method. The following problem is denoted by P_h.

P_h: Find $U_h \in S_h$ such that

$$B_\varepsilon(U_h, V_h) = P_L(F, V_h) \qquad (3.2)$$

for all $V_h \in S_h$.

Theorem 3.1 : There is a unique solution $U_h \in S_h$ of the problem P_h.

Theorem 3.2 : The solution U_h of the problem P_h has the projection property :

$$B_\varepsilon(U_\varepsilon - U_h, U_\varepsilon - U_h) \le B_\varepsilon(U_\varepsilon - V_h, U_\varepsilon - V_h) \qquad (3.3)$$

for all $V_h \in S_h$.

The proofs of Theorems 3.1 and 3.2 were given in [1, 2].

The following lemma will be used in proving error estimates involving U_0, U_ε, and U_h.

Lemma 3.3 : If $V = (v_1, v_2, v_3) \in (H_0^1(\Omega))^3$, then

$$B_\varepsilon(V, V) \le c\,\Big(\sum_{i=1}^{2}\sum_{j=1}^{2} \|\frac{\partial}{\partial x_j}(v_i)\|_0^2 + \frac{1}{\varepsilon}\sum_{i=1}^{2}\|v_i\|_0^2 + \frac{1}{\varepsilon}\sum_{i=1}^{2}\|\frac{\partial v_3}{\partial x_i}\|_0^2 \Big)$$

where c is a constant independent of ε and V.

(3.4)

Proof:

$$B_\varepsilon(V, V) = \frac{1}{2}\iint_\Omega (1+\mu)\Big(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\Big)^2 + (1-\mu)\Big(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}\Big)^2$$

$$+ (1-\mu)\Big(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\Big)^2 \, d\Omega + \frac{1}{\varepsilon}\iint_\Omega \Big(\frac{\partial v_3}{\partial x} + v_1\Big)^2 + \Big(\frac{\partial v_3}{\partial y} + v_2\Big)^2 \, d\Omega$$

(by integration by parts)

$$= \frac{1}{2}\iint_\Omega (1+\mu)\Big(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\Big)^2 + (1-\mu)\Big[\Big(\frac{\partial v_1}{\partial x}\Big)^2 + \Big(\frac{\partial v_2}{\partial y}\Big)^2$$

$$+ \Big(\frac{\partial v_1}{\partial y}\Big)^2 + \Big(\frac{\partial v_2}{\partial y}\Big)^2\Big] \, d\Omega + \frac{1}{\varepsilon}\|\frac{\partial v_3}{\partial x} + v_1\|_0^2 + \frac{1}{\varepsilon}\|\frac{\partial v_3}{\partial y} + v_2\|_0^2$$

(since $\left| 2\iint_\Omega \frac{\partial v_1}{\partial x} \frac{\partial v_2}{\partial y} \, d\Omega \right| \le \|\frac{\partial v_1}{\partial x}\|_0^2 + \|\frac{\partial v_2}{\partial y}\|_0^2$ and $-1 \le \mu < 1$, we have)

$$\le c\,(\sum_{i=1}^{2}\sum_{j=1}^{2} \|\frac{\partial}{\partial x_j}(v_i)\|_0^2 + \frac{1}{\varepsilon}\sum_{i=1}^{2} \|v_i\|_0^2 + \frac{1}{\varepsilon}\sum_{i=1}^{2} \|\frac{\partial v_3}{\partial x_i}\|_0^2)$$

The Lemma is thus proved.

By the approximate properties of $S_h^{t,k}(\Omega)$ there exists $Z_h = (z_1, z_2, z_3) \in S_h$ satisfying the following inequalities.

(a) For $t = 2$, the piecewise linear elements case,

(i) $\|w_0 - z_3\|_1 \le c\,h\,\|w_0\|_3$. (3.5)

(ii) $\|-\frac{\partial w_0}{\partial x_i} - z_i\|_0 \le c\,h^2\,\|\frac{\partial w_0}{\partial x_i}\|_2 \le c\,h^2\,\|w_0\|_3$, for $i=1, 2$, (3.6)

(iii) $\|-\frac{\partial w_0}{\partial x_i} - z_i\|_1 \le c\,h\,\|\frac{\partial w_0}{\partial x_i}\|_2 \le c\,h\,\|w_0\|_3$, for $i=1, 2$. (3.7)

(b) For $t = 3$, the piecewise quadratic element case,

(iv) $\|w_0 - z_3\|_1 \le c\,h^2\,\|w_0\|_3$. (3.8)

(v) $\|-\frac{\partial w_0}{\partial x_i} - z_i\|_0 \le c\,h^2\,\|\frac{\partial w_0}{\partial x_i}\|_2 \le c\,h^2\,\|w_0\|_3$, for $i=1, 2$, (3.9)

(vi) $\|-\frac{\partial w_0}{\partial x_i} - z_i\|_1 \le c\,h\,\|\frac{\partial w_0}{\partial x_i}\|_2 \le c\,h\,\|w_0\|_3$, for $i=1, 2$. (3.10)

With the aid of the above inequalities error estimates between U_0 and Z_h can be derived using Lemma 3.3. Let

$$U_0 - Z_h = \left(-\frac{\partial w_0}{\partial x} - z_1 ,\ -\frac{\partial w_0}{\partial y} - z_2 ,\ w_0 - z_3 \right)$$
$$= (e_1, e_2, e_3). \qquad (3.11)$$

Then

$$\| U_0 - Z_h \|_\varepsilon$$
$$\le c\, B_\varepsilon (U_0 - Z_h ,\, U_0 - Z_h)^{1/2}$$

(from Lemma 2.3)

$$\le c\, \{ \sum_{i=1}^{2} \sum_{j=1}^{2} \| \frac{\partial}{\partial x_j} (e_i) \|_0^2 + \frac{1}{\varepsilon} \sum_{i=1}^{2} \| e_i \|_0^2 + \frac{1}{\varepsilon} \sum_{i=1}^{2} \| \frac{\partial e_3}{\partial x_i} \|_0^2 \}^{1/2}$$

$$\le c\, \{ \sum_{i=1}^{2} \sum_{j=1}^{2} \| \frac{\partial}{\partial x_j} (e_i) \|_0 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| e_i \|_0 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| \frac{\partial e_3}{\partial x_i} \|_0 \}$$

$$\le c\, \{ 2 \sum_{i=1}^{2} \| e_i \|_1 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| e_i \|_0 + 2\, \varepsilon^{-1/2} \| e_3 \|_1 \}$$

$$\le c\, \{ \sum_{i=1}^{2} \| e_i \|_1 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| e_i \|_0 + \varepsilon^{-1/2} \| e_3 \|_1 \} .$$

Thus we have

$$\| U_0 - Z_h \|_\varepsilon \le c\, \{ \sum_{i=1}^{2} \| e_i \|_1 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| e_i \|_0 + \varepsilon^{-1/2} \| e_3 \|_1 \} . \qquad (3.12)$$

We distinguish between the following cases :

(i) For $t = 2$, from (3.12) and (3.5)-(3.7) we have

$$\| U_0 - Z_h \|_\varepsilon \le c\, \{ \sum_{i=1}^{2} \| e_i \|_1 + \varepsilon^{-1/2} \sum_{i=1}^{2} \| e_i \|_0 + \varepsilon^{-1/2} \| e_3 \|_1 \}$$
$$\le c\, \{ h + \varepsilon^{-1/2} h^2 + \varepsilon^{-1/2} h \} \| w_0 \|_3$$

(for $0 < h < 1$)

$$\le c\,(h + \varepsilon^{-1/2} h)\, \|w_0\|_3 . \qquad (3.13)$$

(ii) For $t = 3$, from (3.12) and (3.8)- (3.10) we have

$$\|U_0 - Z_h\|_\varepsilon \le c\,(\sum_{i=1}^{2} \|e_i\|_1 + \varepsilon^{-1/2} \sum_{i=1}^{2} \|e_i\|_0 + \varepsilon^{-1/2} \|e_3\|_1)$$

$$\le c\,(h + \varepsilon^{-1/2} h^2 + \varepsilon^{-1/2} h^2)\, \|w_0\|_3$$

$$\le c\,(h + \varepsilon^{-1/2} h^2)\, \|w_0\|_3 . \qquad (3.14)$$

Combining (3.13) and (3.14) for $t = 2$ and $t = 3$ respectively, we have

$$\|U_0 - Z_h\|_\varepsilon \le c\,(h + \varepsilon^{-1/2} h^{t-1})\, \|w_0\|_3 . \qquad (3.15)$$

Now error estimates between U_ε and Z_h can be derived as follows:

$$\|U_\varepsilon - Z_h\|_\varepsilon \le \|U_\varepsilon - U_0\|_\varepsilon + \|U_0 - Z_h\|_\varepsilon$$

(from Theorem 2.5)

$$\le c\, \varepsilon^{1/2} \|w_0\|_3 + c\,(h + \varepsilon^{-1/2} h^{t-1})\, \|w_0\|_3$$

$$\le c\,(\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})\, \|w_0\|_3 .$$

Hence we have

$$\|U_\varepsilon - Z_h\|_\varepsilon \le c\,(\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})\, \|w_0\|_3 . \qquad (3.16)$$

From (3.16) we can obtain error estimates between U_ε and U_h.

$$||U_\varepsilon - U_h||_\varepsilon \le c\,B_\varepsilon(U_\varepsilon - U_h, U_\varepsilon - U_h)$$

(by Theorem 3.2)

$$\le c\,B_\varepsilon(U_\varepsilon - Z_h, U_\varepsilon - Z_h)$$

(by (3.16))

$$\le c\,\{\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\,||w_0||_3 .$$

The following theorem has been proved.

Theorem 3.4 : For $t = 2$ and $t = 3$ the following inequalities hold corresponding to piecewise linear and piecewise quadratic elements respectively,

$$||U_\varepsilon - U_h||_\varepsilon \le c\,\{\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\,||w_0||_3, \tag{3.17}$$

and

$$||U_\varepsilon - U_h||_1 \le c\,\{\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\,||w_0||_3. \tag{3.18}$$

Theorem 3.5 :

$$||U_0 - U_h||_\varepsilon \le c\,\{\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\,||w_0||_3 , \tag{3.19}$$

and

$$||U_0 - U_h||_1 \le c\,\{\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\,||w_0||_3 . \tag{3.20}$$

Proof:

We have

$$||U_0 - U_h||_\varepsilon \le ||U_0 - U_\varepsilon||_\varepsilon + ||U_\varepsilon - U_h||_\varepsilon ,$$

and

$$||U_0 - U_h||_1 \le ||U_0 - U_\varepsilon||_1 + ||U_\varepsilon - U_h||_1 .$$

From Theorems 2.5 and 3.4, Theorem 3.5 now follows.

Remark : If $\|w_0\|_3$ is replaced by $\|U_0\|_2$ in the Theorems 3.4 and 3.5, then for $t = 2$ and $0 < \varepsilon < 1$ we have following results:

$$\|U_\varepsilon - U_h\|_\varepsilon \le c(\varepsilon^{1/2} + \varepsilon^{-1/2}h)\ \|U_0\|_2 ,$$

$$\|U_\varepsilon - U_h\|_1 \le c(\varepsilon^{1/2} + \varepsilon^{-1/2}h)\ \|U_0\|_2 ,$$

and

$$\|U_0 - U_h\|_\varepsilon \le c(\varepsilon^{1/2} + \varepsilon^{-1/2}h)\ \|U_0\|_2 ,$$

$$\|U_0 - U_h\|_1 \le c(\varepsilon^{1/2} + \varepsilon^{-1/2}h)\ \|U_0\|_2 .$$

These results were obtained in [2, 4].

Corollary 3.6 : If $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and $U_h = (u_1^h, u_2^h, u_3^h)$, then for $i, j = 1, 2$,

$$\left\|\frac{\partial}{\partial x_j}(u_i^\varepsilon) - \frac{\partial}{\partial x_j}(u_i^h)\right\|_0 \le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2}h^{t-1}\,]\ \|w_0\|_3 , \qquad (3.21)$$

and

$$\left\|\frac{\partial}{\partial x_j}\left(-\frac{\partial w_0}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(u_i^h)\right\|_0 \le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2}h^{t-1}\,]\ \|w_0\|_3 . \qquad (3.22)$$

Proof:

$$\left\|\frac{\partial}{\partial x_j}(u_i^\varepsilon - u_i^h)\right\|_0 \le \|u_i^\varepsilon - u_i^h\|_1$$

$$\le \|U_\varepsilon - U_h\|_1$$

(by (3.18))

$$\le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2}h^{t-1}\,]\ \|w_0\|_3 .$$

Thus (3.21) holds. For (3.22) we have

$$\left\| \frac{\partial}{\partial x_j}\left(-\frac{\partial w_0}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(u_i^h) \right\|_0 \le \left\| -\frac{\partial w_0}{\partial x_i} - u_i^h \right\|_1$$

$$\le \| U_0 - U_h \|_1$$

(by (3.20))

$$\le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\,]\, \| w_0 \|_3 \,.$$

Thus (3.22) holds.

Section 3.2 : Error estimates for $U_0 - U_h$ in $\| \ \|_0$.

Theorem 3.7 : If $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and $U_h = (u_1^h, u_2^h, u_3^h)$, then

$$\| u_3^\varepsilon - u_3^h \|_0 \le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\,]\, \| w_0 \|_3 \,, \qquad (3.23)$$

and

$$\| w_0 - u_3^h \|_0 \le c\,[\,\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\,]\, \| w_0 \|_3 \,. \qquad (3.24)$$

Proof:

Let $e = u_3^\varepsilon - u_3^h$, and consider the following problem

$$\Delta^2 \phi = e \quad \text{in } \Omega,$$

$$\phi = \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

for ϕ. We have

$$\phi \in H_0^2(\Omega) \cap H^4(\Omega) \quad \text{and} \quad \| \phi \|_4 \le c\, \| e \|_0 \qquad (3.25)$$

Let $E = (0, 0, -e)$. For the same ε and h, there exist unique $E_\varepsilon \in (H_0^1(\Omega))^3$ and $E_h \in S_h$ such that

$$B_\varepsilon(E_\varepsilon, V) = P_L(E, V), \text{ for all } V \in (H_0^1(\Omega))^3, \qquad (3.26)$$

and

$$B_\varepsilon(E_h, V) = P_L(E, V), \text{ for all } V \in S_h.$$

From (2.13) and (3.2) we have

$$B_\varepsilon(U_\varepsilon , E_h) = P_L(F , E_h) \quad \text{and} \quad B_\varepsilon(U_h , E_h) = P_L(F , E_h).$$

And then

$$B_\varepsilon(U_\varepsilon - U_h , E_h) = 0. \qquad (3.27)$$

From (2.5) and $e = u_3^\varepsilon - u_3^h$, we have

$$\|e\|_0^2 = P_L(E , U_\varepsilon - U_h)$$

(by (3.26))

$$= B_\varepsilon(E_\varepsilon , U_\varepsilon - U_h)$$

(by (3.27))

$$= B_\varepsilon(E_\varepsilon - E_h , U_\varepsilon - U_h)$$

$$\le c \, \|E_\varepsilon - E_h\|_\varepsilon \, \|U_\varepsilon - U_h\|_\varepsilon$$

(From Theorem 3.4)

$$\le c \, [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}] \, \|\phi\|_3 \, [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}] \, \|w_0\|_3$$

(by (3.25))

$$\le c \, [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \, \|e\|_0 \, \|w_0\|_3 .$$

If both sides above are divided by $\|e\|_0$, then we have

$$\|e\|_0 \le c \, [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \, \|w_0\|_3 ,$$

that is,

$$\|u_3^\varepsilon - u_3^h\|_0 \le c \, [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \, \|w_0\|_3 .$$

Thus (3.23) is proved. For (3.24) we have

$$\|w_0 - u_3^h\|_0 \le \|w_0 - u_3^\varepsilon\|_0 + \|u_3^\varepsilon - u_3^h\|_0$$

(from (2.21) and (3.23))

$$\le c_1 \varepsilon \|w_0\|_3 + c_2 [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \|w_0\|_3$$

$$\le c [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \|w_0\|_3$$

Thus (3.24) is true.

Theorem 3.8 : If $U_\varepsilon - U_h = (e_1, e_2, e_3)$, $U_\varepsilon - U_0 = (e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon)$, and $U_0 - U_h = (e_1^h, e_2^h, e_3^h)$, then we have

$$\|e_3\|_1 \le c [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \|w_0\|_3 \qquad (3.28)$$

and

$$\|e_3^h\|_1 \le c [\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}]^2 \|w_0\|_3 \qquad (3.29)$$

Proof:

Consider the following problem

$$\Delta^2 \phi = \Delta e_3 \qquad \text{in } \Omega,$$

$$\phi = \frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

in the sense of distributions, for ϕ such that

$$\phi \in H_0^2(\Omega) \cap H^3(\Omega) \quad \text{and} \quad \|\phi\|_3 \le c \|e_3\|_1 \qquad (3.30)$$

Let $E = (0, 0, -\Delta e_3)$. For the same ε and h, there exist unique $E_\varepsilon \in (H_0^1(\Omega))^3$ and $E_h \in S_h$ such that

$$B_\varepsilon(E_\varepsilon , V) = P_L (E , V), \text{ for all } V \in (H_0^1(\Omega))^3, \qquad (3.31)$$

and

$$B_\varepsilon(E_h , V) = P_L (E , V), \text{ for all } V \in S_h .$$

From (2.13) and (3.2) we have

$$B_\varepsilon(U_\varepsilon, E_h) = P_L(F, E_h) \quad \text{and} \quad B_\varepsilon(U_h, E_h) = P_L(F, E_h),$$

and then

$$B_\varepsilon(U_\varepsilon - U_h, E_h) = 0. \qquad (3.32)$$

From (2.15), there exists a constant $c > 0$ such that

$$\|e_3\|_1^2 \le c\, |e_3|_1^2$$

$$= c \iint_\Omega \nabla e_3 \bullet \nabla e_3 \, d\Omega$$

(since $e_3 = 0$ on $\partial\Omega$, then)

$$= c \iint_\Omega (-\Delta e_3)\, e_3 \, d\Omega$$

$$= c \; P_L(E, U_\varepsilon - U_h)$$

(by (3.31))

$$= c\, B_\varepsilon(E_\varepsilon, U_\varepsilon - U_h)$$

(by (3.32))

$$= c\, B_\varepsilon(E_\varepsilon - E_h, U_\varepsilon - U_h)$$

$$\le c\, \|E_\varepsilon - E_h\|_\varepsilon \, \|U_\varepsilon - U_h\|_\varepsilon$$

(from Theorem 3.4 we have)

$$\le c\, (\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})\, \|\phi\|_3 \, (\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})\, \|w_0\|_3$$

(by (3.30))

$$\le c\, (\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})^2 \, \|e_3\|_1 \, \|w_0\|_3 .$$

If both sides above are divided by $\|e_3\|_1$, then we have

$$\|e_3\|_1 \le c\, (\varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1})^2 \, \|w_0\|_3 .$$

Thus (3.28) is proved. For (3.29) we have

$$\|e_3^h\|_1 = \|w_0 - u_3^h\|_1$$

$$\leq \|w_0 - u_3^\varepsilon\|_1 + \|u_3^\varepsilon - u_3^h\|_1$$

(from (2.27) and (3.28))

$$\leq c_1 \varepsilon \|w_0\|_3 + c_2 \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3$$

$$\leq c \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3 \, .$$

Thus (3.29) holds.

Theorem 3.9: If $U_\varepsilon - U_h = (e_1, e_2, e_3)$, $U_\varepsilon - U_0 = (e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon)$, and $U_0 - U_h =$ (e_1^h, e_2^h, e_3^h), then we have

$$\|e_i\|_0 \leq c \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3 , \qquad (3.33)$$

and

$$\|e_i^h\|_0 \leq c \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3 , \qquad (3.34)$$

for $i = 1, 2$.

Proof:

$$\|e_i\|_0 \leq \|\frac{\partial e_3}{\partial x}\|_0 + \|\frac{\partial e_3}{\partial x} + e_i\|_0$$

$$\leq \|e_3\|_1 + P_S(U_\varepsilon - U_h \, , \, U_\varepsilon - U_h)^{1/2}$$

(from (3.28))

$$\leq c \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3$$

$$+ c \, \varepsilon^{1/2} B_\varepsilon(U_\varepsilon - U_h \, , \, U_\varepsilon - U_h)^{1/2}$$

$$\leq c \, [\, \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1} \,]^2 \, \|w_0\|_3$$

$$+ c \, \varepsilon^{1/2} \|U_\varepsilon - U_h\|_\varepsilon$$

$$\le c\ \{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}^2\ ||w_0||_3$$

$$+ c\,\varepsilon^{1/2}\ \{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}\ ||w_0||_3$$

$$\le c\ \{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}^2\ ||w_0||_3\ .$$

Hence we have (3.33). For (3.34) we have

$$||e_i^h||_0 \le ||e_i||_0 + ||e_i^\varepsilon||_0$$

(from (3.33) and (3.30))

$$\le c\,\varepsilon\,||w_0||_3 + c\{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}^2\ ||w_0||_3$$

$$\le c\ \{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}^2\ ||w_0||_3\ .$$

Thus (3.34) is proved.

Theorem 3.10:

$$||U_0 - U_h||_0 \le c\{\ \varepsilon^{1/2} + h + \varepsilon^{-1/2} h^{t-1}\}^2\ ||w_0||_3\ . \qquad (3.35)$$

Proof:

The result in (3.35) follows from (3.24) and (3.34).

Section 3.3: Error estimates for linear and quadratic elements with $\varepsilon = ch$ and $\varepsilon^{1/2} = ch$.

Remark:

Let $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and $U_h = (u_1^h, u_2^h, u_3^h)$.

(a) In the linear elements case (i.e. $t = 2$) if we let $\varepsilon = ch$, then we have the following results:

(i) $||w_0 - u_3^h||_0 \le c\ h\ ||w_0||_3$, by (3.24).

(ii) $\|w_0 - u_3^h\|_0 \le c\, h\, \|w_0\|_3$, by (3.29).

(iii) $\|-\frac{\partial w_0}{\partial x_i} - u_i^h\|_0 \le c\, h\, \|w_0\|_3$, for $i = 1, 2$, by (3.34).

(iv) $\|\frac{\partial}{\partial x_j}(-\frac{\partial w_0}{\partial x_i}) - \frac{\partial}{\partial x_j}(u_i^h)\|_0 \le c\, h^{1/2}\, \|w_0\|_3$, for $i, j = 1, 2$, by (3.22).

(v) $\|U_0 - U_h\|_1 \le c\, h^{1/2}\, \|w_0\|_3$, by (3.18).

(vi) $\|U_0 - U_h\|_0 \le c\, h\, \|w_0\|_3$. by (3.35).

(b) In quadratic elements case (i.e. $t = 3$) if we let $\varepsilon^{1/2} = c\, h$, then we have following results:

(i) $\|w_0 - u_3^h\|_0 \le c\, h^2\, \|w_0\|_3$, by (3.24).

(ii) $\|w_0 - u_3^h\|_0 \le c\, h^2\, \|w_0\|_3$, by (3.29).

(iii) $\|-\frac{\partial w_0}{\partial x_i} - u_i^h\|_0 \le c\, h^2\, \|w_0\|_3$, for $i = 1, 2$, by (3.34).

(iv) $\|\frac{\partial}{\partial x_j}(-\frac{\partial w_0}{\partial x_i}) - \frac{\partial}{\partial x_j}(u_i^h)\|_0 \le c\, h\, \|w_0\|_3$, for $i, j = 1, 2$, by (3.22).

(v) $\|U_0 - U_h\|_1 \le c\, h\, \|w_0\|_3$, by (3.18).

(vi) $\|U_0 - U_h\|_0 \le c\, h^2\, \|w_0\|_3$. by (3.35).

(c) In (3.8) - (3.10) we can choose quadratic elements for w_0 and linear elements for its first derivatives and have the same error estimates listed in the above part (b).

Chapter 4 Piecewise quadratic finite elements for the square plate

Section 4.1 : Construction of the element stiffness matrix

If the domain is subdivided into isosceles right triangles of two types (type 1 and type 2 as given by Figure 4.1 and Figure 4.2, respectively), the construction of the stiffness matrix of the quadratic finite elements for the clamped plate is similar to the construction of the linear elements in [2]. However, each quadratic element now contains six nodes.

Type 1 elements are as shown below :

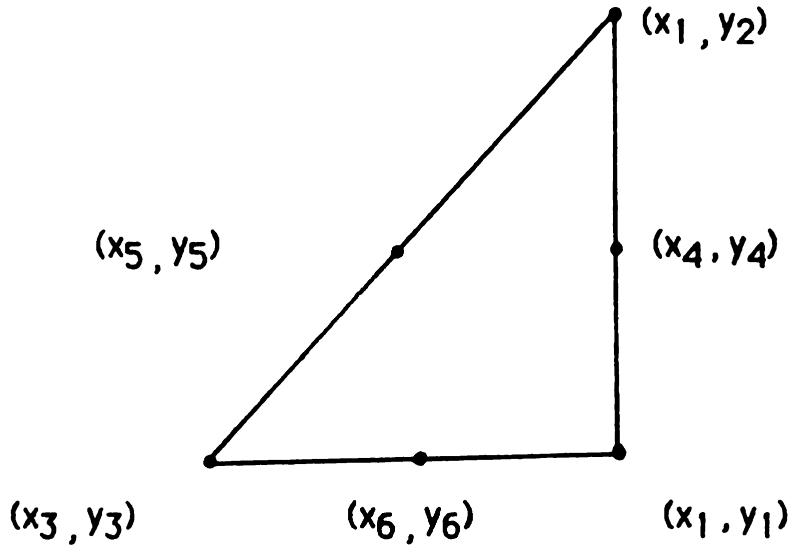


Figure 4.1

where

$$(x_1, y_1) = (\frac{h}{3}, \frac{-h}{3}), \qquad (x_4, y_4) = (\frac{h}{3}, \frac{h}{6}),$$

$$(x_2, y_2) = (\frac{h}{3}, \frac{2h}{3}), \qquad (x_5, y_5) = (\frac{-h}{6}, \frac{h}{6}), \qquad (4.1)$$

$$(x_3, y_3) = (\frac{-2h}{3}, \frac{-h}{3}), \qquad (x_6, y_6) = (\frac{-h}{6}, \frac{-h}{3}).$$

Replacing h by -h , we have the following type 2 elements.

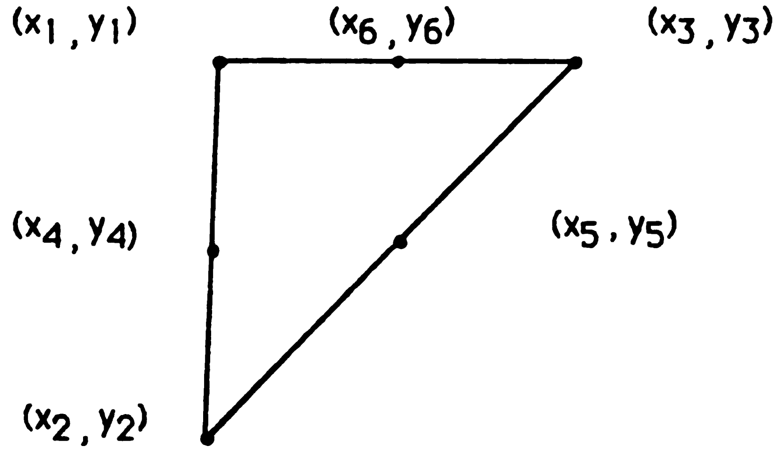


Figure 4.2

where

$$(x_1, y_1) = (\frac{-h}{3}, \frac{h}{3}), \qquad (x_4, y_4) = (\frac{-h}{3}, \frac{-h}{6}),$$

$$(x_2, y_2) = (\frac{-h}{3}, \frac{-2h}{3}), \qquad (x_5, y_5) = (\frac{h}{6}, \frac{-h}{6}), \qquad (4.2)$$

$$(x_3, y_3) = (\frac{2h}{3}, \frac{h}{3}), \qquad (x_6, y_6) = (\frac{h}{6}, \frac{h}{3}).$$

For the type 1 elements, let

$$U^{(e)} = \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} \\ a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix} \tag{4.3}$$

and let $\varphi_1, \varphi_2, \ldots,$ and φ_6 be the quadratic functions which are equal to unity at (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), and (x_6, y_6), respectively and zero at other nodes. Let $q_1, q_2, q_3, \ldots, q_{18}$ be the corresponding coefficients and

$$U^{(e)} = \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} q_1 & q_4 & q_7 & q_{10} & q_{13} & q_{16} \\ q_2 & q_5 & q_8 & q_{11} & q_{14} & q_{17} \\ q_3 & q_6 & q_9 & q_{12} & q_{15} & q_{18} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{bmatrix} . \tag{4.4}$$

From (4.1), (4.3), and (4.4) we have

$$\mathbf{A}\,\mathbf{X} = \mathbf{Q}\,, \qquad (4.5)$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} \\ a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_4 & q_7 & q_{10} & q_{13} & q_{16} \\ q_2 & q_5 & q_8 & q_{11} & q_{14} & q_{17} \\ q_3 & q_6 & q_9 & q_{12} & q_{15} & q_{18} \end{bmatrix}.$$

By inverting the second matrix of the equation (4.5) we have

$$\mathbf{A} = \mathbf{Q}\,\mathbf{H}\,, \qquad (4.6)$$

$$\mathbf{H} = \begin{bmatrix} \frac{2}{h^2} & \frac{-4}{h^2} & \frac{2}{h^2} & \frac{1}{3h} & \frac{-1}{3h} & \frac{-1}{9} \\ 0 & 0 & \frac{2}{h^2} & 0 & \frac{1}{3h} & \frac{-1}{9} \\ \frac{2}{h^2} & 0 & 0 & \frac{-1}{3h} & 0 & \frac{-1}{9} \\ 0 & \frac{4}{h^2} & \frac{-4}{h^2} & \frac{4}{3h} & 0 & \frac{4}{9} \\ 0 & \frac{-4}{h^2} & 0 & \frac{-4}{3h} & \frac{4}{3h} & \frac{4}{9} \\ \frac{-4}{h^2} & \frac{4}{h^2} & 0 & 0 & \frac{-4}{3h} & \frac{4}{9} \end{bmatrix}$$

which can be expressed as

$$\mathbf{a} = \mathbf{P}\,\mathbf{q}\,, \qquad (4.7)$$

where

$$\mathbf{a} = \left[\, a_1 \;\; a_2 \;\; a_3 \;\cdots\; a_{18} \,\right]^T,$$

$$\mathbf{q} = \left[\, q_1 \;\; q_2 \;\; q_3 \cdots\; q_{18} \,\right]^T,$$

and

$$\mathbf{P} = \begin{bmatrix} \frac{2}{h^2}I & 0 & \frac{2}{h^2}I & 0 & 0 & \frac{-4}{h^2}I \\ \frac{-4}{h^2}I & 0 & 0 & \frac{4}{h^2}I & \frac{-4}{h^2}I & \frac{4}{h^2}I \\ \frac{2}{h^2}I & \frac{2}{h^2}I & 0 & \frac{-4}{h^2}I & 0 & 0 \\ \frac{1}{3h}I & 0 & \frac{-1}{3h}I & \frac{4}{3h}I & \frac{-4}{3h}I & 0 \\ \frac{-1}{3h}I & \frac{1}{3h}I & 0 & 0 & \frac{4}{3h}I & \frac{-4}{3h}I \\ \frac{-1}{9}I & \frac{-1}{9}I & \frac{-1}{9}I & \frac{4}{9}I & \frac{4}{9}I & \frac{4}{9}I \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.8}$$

The element stiffness matrix $\mathbf{K}^{(e)}$ is introduced through

$$B_\varepsilon(U^{(e)}, U^{(e)}) = \mathbf{q}^T \mathbf{K}^{(e)} \mathbf{q} . \tag{4.9}$$

In terms of matrix $\mathbf{a}$, we define a matrix $\mathbf{N}$ by

$$B_\varepsilon(U^{(e)}, U^{(e)}) = \mathbf{a}^T \mathbf{N} \mathbf{a} . \tag{4.10}$$

Then

$$\mathbf{a}^T \mathbf{N} \mathbf{a} = \mathbf{q}^T \mathbf{K}^{(e)} \mathbf{q} . \tag{4.11}$$

From (4.7)

$$\mathbf{q}^T \mathbf{P}^T \mathbf{N} \mathbf{P} \mathbf{q} = \mathbf{q}^T \mathbf{K} \mathbf{q} \,, \tag{4.12}$$

thus

$$\mathbf{K}^{(e)} = \mathbf{P}^T \mathbf{N} \mathbf{P} \,. \tag{4.13}$$

We need the following integrals

$$I_{rs} = \iint_{(e)} x^r y^s \, dxdy \,. \tag{4.14}$$

From [19] Holland and Bell the integrals above are easily computed.

$$\begin{aligned}
I_{40} &= \frac{h^6}{270}, & I_{04} &= \frac{h^6}{270}, & I_{31} &= \frac{h^6}{540}, \\
I_{13} &= \frac{h^6}{540}, & I_{22} &= \frac{h^6}{540}, & I_{30} &= \frac{-h^5}{270}, \\
I_{03} &= \frac{h^5}{270}, & I_{21} &= \frac{-h^5}{540}, & I_{12} &= \frac{h^5}{540}, \\
I_{20} &= \frac{h^4}{36}, & I_{02} &= \frac{h^4}{36}, & I_{11} &= \frac{h^4}{72}, \\
I_{01} &= 0 \,, & I_{10} &= 0 \,, & I_{00} &= \frac{h^2}{2} \,.
\end{aligned} \tag{4.15}$$

Using the above results and (4.3) we can derive (4.10)

$B_\varepsilon(U^{(e)}, U^{(e)})$

$$= \frac{1+\mu}{2} \iint_\Omega \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right)^2 dA + \frac{1-\mu}{2} \iint_\Omega \left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y} \right)^2 dA$$

$$+ \frac{1-\mu}{2} \iint_\Omega \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 dA + \frac{1}{\varepsilon} \iint_\Omega \left(\frac{\partial u_3}{\partial x} + u_1 \right)^2 + \left(\frac{\partial u_3}{\partial y} + u_2 \right)^2 dA$$

$$= \frac{1+\mu}{2} \frac{h^4}{36} \{ (4 a_1^2 + 4 a_1 a_5 + a_5^2) + (4 a_8^2 + 4 a_8 a_4 + a_4^2)$$

$$+ (4 a_1 a_8 + 2 a_8 a_5 + 2 a_1 a_4 + a_5 a_4) \}$$

$$+ \frac{1+\mu}{2} \frac{h^2}{2} (a_{10}^2 + 2 a_{10} a_{14} + a_{14}^2)$$

$$+ \frac{1-\mu}{2} \frac{h^4}{36} \{ (4 a_1^2 - 4 a_1 a_5 + a_5^2) + (a_4^2 - 4 a_4 a_8 + 4 a_8^2)$$

$$+ (2 a_1 a_4 - a_4 a_5 - 4 a_1 a_8 + 2 a_5 a_8) \}$$

$$+ \frac{1-\mu}{2} \frac{h^2}{2} (a_{10}^2 - 2 a_{10} a_{14} + a_{14}^2)$$

$$+ \frac{1-\mu}{2} \frac{h^4}{36} \{ (a_4^2 + 4 a_4 a_2 + 4 a_2^2) + (4 a_7^2 + 4 a_7 a_5 + a_5^2)$$

$$+ (2 a_4 a_7 + 4 a_2 a_7 + a_4 a_5 + 2 a_2 a_5) \}$$

$$+ \frac{1-\mu}{2} \frac{h^2}{2} (a_{13}^2 + 2 a_{13} a_{11} + a_{11}^2)$$

$$+ \frac{1}{\varepsilon} [\frac{h^6}{270} [a_1^2 + a_7^2 + \frac{1}{2} a_4^2 + a_1 a_7 + a_1 a_4 + a_4 a_7]$$

$$+ \frac{h^5}{270} [-2 a_1 (a_{10} + 2 a_3) + 2 a_7 (a_6 + a_{13}) - a_4 (a_{10} + 2 a_3)$$

$$+a_4(a_6+a_{13})+a_7(a_{10}+2a_3)-a_1(a_6+a_{13})]$$

$$+\frac{h^4}{36}[(a_{10}+2a_3)^2+2a_1(a_{16}+a_{12})+(a_6+a_{13})^2+2a_7(a_{16}+a_{12})$$

$$+a_4(a_{16}+a_{12})+(a_{10}+2a_3)(a_6+a_{13})]$$

$$+\frac{h^2}{2}(a_{16}+a_{12})^2)$$

$$+\frac{1}{\varepsilon}(\frac{h^6}{270}[a_2^2+a_8^2+\frac{1}{2}a_5^2+a_2a_5+a_2a_8+a_5a_8]$$

$$+\frac{h^5}{270}[-2a_2(a_6+a_{11})+2a_8(2a_9+a_{14})-a_2(2a_9+a_{14})$$

$$-a_5(a_6+a_{11})+a_5(2a_9+a_{14})+a_8(a_6+a_{11})]$$

$$+\frac{h^4}{36}[(a_6+a_{11})^2+(2a_9+a_{14})^2+2a_2(a_{15}+a_{17})$$

$$+2a_2(a_{15}+a_{17})+2a_8(a_{15}+a_{17})$$

$$+(a_6+a_{11})(2a_9+a_{14})+a_5(a_{15}+a_{17})]$$

$$+\frac{h^2}{2}(a_{15}+a_{17})^2] .$$

Then

$$B_\varepsilon(U^{(e)},U^{(e)})=\mathbf{a}^T\,\mathbf{N}\,\mathbf{a}\ ,$$

where

$$\mathbf{N}=\mathbf{N}_1+\mathbf{N}_2+\mathbf{N}_3+\mathbf{N}_4+\mathbf{N}_5\ , \qquad (4.16)$$

and

$$N_1 = \frac{h^4}{36}\left[\begin{array}{cccccccc|c}
4 & & & & & & & & \\
0 & 2-2\mu & & & & \text{symmetric} & & & \\
0 & 0 & 0 & & & & & & \\
1 & 1-\mu & 0 & \frac{3-\mu}{2} & & & & & \\
2\mu & \frac{1-\mu}{2} & 0 & \frac{1+\mu}{4} & \frac{3-\mu}{2} & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 1-\mu & 0 & \frac{1-\mu}{2} & 1-\mu & 0 & 2-2\mu & & \\
2\mu & 0 & 0 & 2\mu & 1 & 0 & 0 & 4 & \\
\hline
 & & & & 0 & & & & 0
\end{array}\right],$$

$$N_2 = \frac{1}{\epsilon}\frac{h^6}{270}\left[\begin{array}{cccccccc|c}
1 & & & & & & & & \\
0 & 1 & & & \text{symmetric} & & & & \\
0 & 0 & 0 & & & & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & & & & & \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & & \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \\
\hline
 & & & & 0 & & & & 0
\end{array}\right],$$

$$N_3 = \frac{1}{\epsilon}\frac{h^5}{270}\left[\begin{array}{cccccccccccccc|c}
0 & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & \text{symmetric} & & & & & \\
-2 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & -1 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
-\frac{1}{2} & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & & & & & & & & & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & & & & & & & \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & & & & & & \\
-1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & & & & & \\
0 & -1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
 & & & & & & & 0 & & & & & & & 0
\end{array}\right],$$

$$
N_4 = \frac{1}{\varepsilon}\frac{h^4}{36}
\begin{bmatrix}
0 & & & & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 4 & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & \text{symmetric} & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 2 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & & & & & & & & & \\
0 & 0 & 2 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & & & & & & & \\
1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 2 & 0 & \frac{1}{2} & 0 & 0 & 1 & & & & \\
0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$N_5=\frac{h^2}{2}\left[\begin{array}{c|ccccccccc}
0 & & & & & 0 & & & & \\
\hline
 & 1 & & & & & & & & \\
 & 0 & \frac{1-\mu}{2} & & & & \text{symmetric} & & & \\
 & 0 & 0 & \frac{1}{\varepsilon} & & & & & & \\
 & 0 & \frac{1-\mu}{2} & 0 & \frac{1-\mu}{2} & & & & & \\
0 & \mu & 0 & 0 & 0 & 1 & & & & \\
 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon} & & & \\
 & 0 & 0 & \frac{1}{\varepsilon} & 0 & 0 & 0 & \frac{1}{\varepsilon} & & \\
 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon} & 0 & \frac{1}{\varepsilon} & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

The element stiffness matrix $K^{(e)}$ is then obtained from(4.13), (4.16), and (4.8).

Each type 1 element stiffness matrix is the same as the above stiffness matrix $K^{(e)}$.

Replacing h by -h in $K^{(e)}$ of the type 1 elements, we have the stiffness matrix of each type 2 elements. Similarly, the formulas derived for type 1 elements will be true for type 2 elements by negating h.

Section 4.2 : Construction of the element load vector

The element load vector $\mathbf{f}^{(e)}$ will be computed in the following ways.

$$P_L(F, U^{(e)})$$

$$= \iint_{(e)} f\, u_3^{(e)}\, dA$$

$$= \begin{bmatrix} a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} \end{bmatrix} \iint_{(e)} f(x, y) \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix} dA$$

(from (4.6))

$$= \begin{bmatrix} q_3 & q_6 & q_9 & q_{12} & q_{15} & q_{18} \end{bmatrix} \mathbf{H} \iint_{(e)} f(x, y) \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix} dA \,. \quad (4.17)$$

Let (x_c , y_c) be the centroid of the other elemental triangle relative to the global coordinates (X , Y). Then

$$f(X, Y) = f(x + x_c, y + y_c).$$

If

$$\mathbf{f}^{(e)} = \left[f_1 \;\; f_2 \;\; f_3 \cdots f_{18} \right]^T ,$$

then

$$f_m = \mathbf{H} \iint_{(e)} f(x + x_c,\ y + y_c) \begin{bmatrix} (x + x_c)^2 \\ (x + x_c)(y + y_c) \\ (y + y_c)^2 \\ x + x_c \\ y + y_c \\ 1 \end{bmatrix} dA$$

$$= \mathbf{H} \iint_{(e)} f(x + x_c,\ y + y_c) \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix} dA \qquad (4.18)$$

for m=3, 6, 9, 12, 15, and 18 and

$$f_m = 0 , \qquad \text{otherwise.}$$

The numerical integrations for f_m may be carried out by the standard Gaussian quadrature.

Section 4.3 : Finite element solutions

The energy integral

$$J(U) = B_{\varepsilon}(U, U) - 2P_L(F, U),$$

will be summed over the individual elements.

$$\begin{aligned} J(U) &= \sum_e J(U^{(e)}) \\ &= \sum_e [B_{\varepsilon}(U^{(e)}, U^{(e)}) - 2P_L(F, U^{(e)})] \\ &= \sum_e [\mathbf{q}^T \mathbf{K}^{(e)} \mathbf{q} - 2\mathbf{q}^T \mathbf{f}^{(e)}] \\ &= \hat{\mathbf{q}}^T \mathbf{K} \hat{\mathbf{q}} - 2\hat{\mathbf{q}}^T \mathbf{f}, \end{aligned}$$

where

$\mathbf{K}$ is the global stiffness matrix,

$\hat{\mathbf{q}}$ is the global nodal matrix,

and $\hat{\mathbf{f}}$ is the global load vector.

The finite element solutions are determined by finding the q_i's which minimize the energy integral J(U). This gives

$$\mathbf{K}\hat{\mathbf{q}} = \hat{\mathbf{f}}. \qquad (4.19)$$

Section 4.4: Examples

Example 4.1 :

Consider a clamped square plate in $-1/2 \le x \le 1/2$, $-1/2 \le y \le 1/2$ under the polynomial load

$$f(x,y) = 24(x^4 + 12x^2y^2 + y^4) - 36(x^2 + y^2) + 5.$$

The exact solution for $w_0(x, y)$ is

$$w_0(x, y) = \frac{1}{256}(4x^2 - 1)^2 (4y^2 - 1)^2,$$

from which we have

$$w_0(0, 0) = 1/256 = 0.00390625,$$

$$w_0\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{81}{256^2} = 0.00123596,$$

$$-\frac{\partial w_0}{\partial x}\left(\frac{1}{4}, \frac{1}{4}\right) = -\frac{\partial w_0}{\partial y}\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{27}{4096} = 0.0065918.$$

Since the load function is symmetric in x and y the problem can be solved over the first quadrant. The boundary conditions $u_1 = u_2 = u_3 = 0$ at $x = 1/2$ and $y = 1/2$ should be imposed. Because u_1 must be odd in x and even in y, the boundary condition at $x = 0$ is $u_1 = 0$. Similarly the boundary condition at $y = 0$ is $u_2 = 0$.

Numerical results are given in Tables 4.1-4.6. We mention that the same example was also considered in [2] using piecewise linear finite elements with mesh sizes of h = 1/4, 1/8, 1/16, and 1/32. In Tables 4.13A, 4.13B, and 4.13C numerical results are added for h = 1/64 in the linear element case. In the quadratic element case, due to the limitation of computer memories, numerical results are not obtained for $h = 1/64$. The results in Tables 4.13A, 4.13B, and 4.13C show that the quadratic element solutions yield much better approximations than the linear element solutions. It has been indicated in Chapter 3 the error bounds contain the factor $\varepsilon^{-1/2}h^{l-1}$. This implies that accuracy for small ε may require excessive fine mesh. In the linear element case when ε is less than 2^{-10}, numerical results are not reliable even for $h = 1/64$. Numerical values of $\varepsilon = 2^{-15}$ and $h = 1/32$ in the quadratic element case are, however, acceptable. In references [1-4] Poisson's ratio μ was taken in the range of [0, 0.5]. $\mu = 0.3$ was used in the present numerical computations. As we mentioned before Nitsche's method corresponds to the particular case $\mu = -1$. Tables 4.1-4.6 list numerical results for $\mu = 0.3$, 0.0, and -1, showing that the solutions are insensitive to μ.

The convergence to the solution w_0 and its first derivatives occurs only when ε and h both tend to zero. In Chapter 3, letting $\varepsilon = ch$ and $\varepsilon^{-1/2} = ch$ in linear and quadratic element cases respectively, we have the convergences in terms of h discussed at the end of Chapter 3. Figures 4.3 and 4.4 are approximations of $w_0(0, 0)$ with constants $c = 1/8$ and $c = 1$ in linear and quadratic element cases, respectively. For small ε both graphs tend to be linear. The choice of the value for c suffers no particular restriction. Figure 4.5 shows the appproximations of $w_0(0, 0)$ for $h = 1/32$ of linear and quadratic elements. In the linear element case the approximations for $\varepsilon = 2^{-7}$, 2^{-8}, and 2^{-9} are reliable. We can use extrapolations to find better approximations of $w_0(0, 0)$. The points for ε larger than 2^{-9} are not reliable. Because h is fixed ($h = 1/32$), these points tend to the origin (See [2]). The points of quadratic approximations in Figere 4.5 are all reliable and all are almost on a straight line. This suggests that w_ε tends to be linear when ε approaches to zero. Thus in the quadratic element case we can use extrapolation to obtain better approximations of $w_0(0, 0)$. For example, when $\varepsilon = 2^{-10}$ one has the approximation $w_1 = 0.00414588$ and when $\varepsilon = 2^{-12}$ one has the approximation $w_2 = 0.00396578$. By extrapolation one obtains

$$w = \frac{2^2 w_2 - w_1}{2^2 - 1} = 0.003905736.$$

which is very close to the exact value of $w_0(0, 0) = 0.00390625$. Extrapolations are commonly used to obtain improved results in penalty methods [2, 16, 17, 34].

Example 4.2 :

For the same clamped square plate we now consider the cosine load

$$f(x, y) = 4 \cos 2\pi x \cos 2\pi y + \cos 2\pi x + \cos 2\pi y.$$

The exact solution is

$$W_0(x, y) = (1/16\pi^4)(\cos 2\pi x + 1)(\cos 2\pi y + 1),$$

from which we have

$$w_0(0, 0) = 1/(4\pi^4) = 0.0025665,$$

$$w_0\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{16\pi^4} = 0.0006416,$$

$$-\frac{\partial w_0}{\partial x}\left(\frac{1}{4}, \frac{1}{4}\right) = -\frac{\partial w_0}{\partial y}\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{8\pi^3} = 0.0040314.$$

Numerical results are given in tables 4.7-4.12. These results are similar to those in Example 4.1.

Linear finite element approximations of $w_0(0, 0)$ of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.11311639	0.11305337	0.11288071
	1/8	0.11337059	0.11332004	0.11320171
2^{-1}	1/16	0.11228648	0.11224184	0.11215120
	1/32	0.11170684	0.11166374	0.11158058
	1/4	0.05785581	0.05779472	0.05762614
	1/8	0.05858934	0.05854034	0.05842447
2^{-2}	1/16	0.05819833	0.05815520	0.05806679
	1/32	0.05793962	0.05789801	0.05781707
	1/4	0.03018412	0.03012659	0.02996569
	1/8	0.03117510	0.03112893	0.03101769
2^{-3}	1/16	0.03114190	0.03110152	0.03101729
	1/32	0.03104707	0.03100818	0.03093139
	1/4	0.01627208	0.01622063	0.01607344
	1/8	0.01742362	0.01738224	0.01727916
2^{-4}	1/16	0.01759112	0.01755536	0.01747847
	1/32	0.01758495	0.01755068	0.01748114
	1/4	0.00918538	0.00914313	0.00901814
	1/8	0.01046832	0.01043401	0.01034393
2^{-5}	1/16	0.01077701	0.01074809	0.01068270
	1/32	0.01082831	0.01080085	0.01074270
	1/4	0.00544297	0.00541229	0.00531803
	1/8	0.00685689	0.00683113	0.00675869
2^{-6}	1/16	0.00730850	0.00728779	0.00723754
	1/32	0.00741368	0.00739445	0.00735129

Table 4.1A

Linear finite element approximations of $w_0(0, 0)$ of the square plate with polynomial load function.

h	ε	μ = 0.3	μ = 0.0	μ = −1.0
	1/4	0.00331845	0.00329941	0.00323919
	1/8	0.00484342	0.00482581	0.00477284
2^{-7}	1/16	0.00548202	0.00546893	0.00543429
	1/32	0.00566086	0.00564930	0.00562156
	1/4	0.00200441	0.00199461	0.00196314
	1/8	0.00354541	0.00353424	0.00349906
2^{-8}	1/16	0.00443066	0.00442294	0.00440064
	1/32	0.00472872	0.00472260	0.00470683
	1/4	0.00116201	0.00115790	0.00114458
	1/8	0.00254705	0.00254059	0.00251979
2^{-9}	1/16	0.00369543	0.00369083	0.00367654
	1/32	0.00418499	0.00418189	0.00417325
	1/4	0.00064098	0.00063955	0.00063488
	1/8	0.00171520	0.00171197	0.00170147
2^{-10}	1/16	0.00302782	0.00302497	0.00301576
	1/32	0.00378846	0.00378677	0.00378169
	1/4	0.00033961	0.00033918	0.00033775
	1/8	0.00106125	0.00105990	0.00105548
2^{-11}	1/16	0.00232820	0.00232649	0.00232088
	1/32	0.00338670	0.00338565	0.00338230
	1/4	0.00017527	0.00017515	0.00017476
	1/8	0.00060747	0.00060699	0.00060543
2^{-12}	1/16	0.00162880	0.00162789	0.00162489
	1/32	0.00288510	0.00288440	0.00288209

Table 4.1B

Linear finite element approximations of $w_0(1/4, 1/4)$ of the square plate with polynomial load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.05034891	0.05033349	0.05028546
	1/8	0.05056031	0.05054344	0.05050684
2^{-1}	1/16	0.05062559	0.05060917	0.05057759
	1/32	0.05064272	0.05062641	0.05059611
	1/4	0.02557536	0.02556030	0.02551326
	1/8	0.02587203	0.02585565	0.02581981
2^{-2}	1/16	0.02596130	0.02594542	0.02591459
	1/32	0.02598450	0.02596868	0.02593914
	1/4	0.01317655	0.01316217	0.01311701
	1/8	0.01352108	0.01350561	0.01347119
2^{-3}	1/16	0.01362528	0.01361037	0.01358097
	1/32	0.01365235	0.01363756	0.01360945
	1/4	0.00695484	0.00694162	0.00689987
	1/8	0.00733278	0.00731888	0.00728698
2^{-4}	1/16	0.00745020	0.00743694	0.00741006
	1/32	0.00748106	0.00746790	0.00744232
	1/4	0.00380525	0.00379390	0.00375777
	1/8	0.00421556	0.00420403	0.00417619
2^{-5}	1/16	0.00435059	0.00433981	0.00431693
	1/32	0.00438689	0.00437623	0.00435466
	1/4	0.00217040	0.00216162	0.00213353
	1/8	0.00261802	0.00260944	0.00258721
2^{-6}	1/16	0.00278186	0.00277415	0.00275663
	1/32	0.00282780	0.00282023	0.00280402

Table 4.2A

Linear finite element approximations of $w_0(1/4, 1/4)$ of the square plate with polynomial load function.

h	ε	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00127494	0.00126909	0.00125041
	1/8	0.00175838	0.00175270	0.00173676
2^{-7}	1/16	0.00196969	0.00196491	0.00195307
	1/32	0.00203347	0.00202888	0.00201835
	1/4	0.00074811	0.00074491	0.00073474
	1/8	0.00124210	0.00123870	0.00122841
2^{-8}	1/16	0.00152286	0.00152021	0.00151297
	1/32	0.00161895	0.00161654	0.00161060
	1/4	0.00042559	0.00042420	0.00041975
	1/8	0.00087763	0.00087575	0.00086976
2^{-9}	1/16	0.00123804	0.00123601	0.00123232
	1/32	0.00138865	0.00138749	0.00138441
	1/4	0.00023221	0.00023172	0.00023012
	1/8	0.00059008	0.00058914	0.00058610
2^{-10}	1/16	0.00100626	0.00100544	0.00100285
	1/32	0.00123736	0.00123680	0.00123517
	1/4	0.00012231	0.00012216	0.00012167
	1/8	0.00036725	0.00036684	0.00036552
2^{-11}	1/16	0.00077751	0.00077704	0.00077552
	1/32	0.00110235	0.00110204	0.00110108
	1/4	0.00006293	0.00006289	0.00006275
	1/8	0.00021169	0.00021154	0.00021106
2^{-12}	1/16	0.00055028	0.00055004	0.00054923
	1/32	0.00094487	0.00094468	0.00094407

Table 4.2B

Linear finite element approximations of $-\partial w_0/\partial x$ (1/4, 1/4) of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00722320	0.00682630	0.00606333
	1/8	0.00812437	0.00781862	0.00722232
2^{-1}	1/16	0.00839584	0.00812028	0.00758315
	1/32	0.00846726	0.00820024	0.00768079
	1/4	0.00711314	0.00673471	0.00599743
	1/8	0.00805113	0.00776098	0.00718552
2^{-2}	1/16	0.00833993	0.00807884	0.00756127
	1/32	0.00841660	0.00816372	0.00766354
	1/4	0.00690411	0.00655921	0.00586981
	1/8	0.00791251	0.00765050	0.00711368
2^{-3}	1/16	0.00823542	0.00800025	0.00751879
	1/32	0.00832249	0.00809490	0.00763028
	1/4	0.00652510	0.00623600	0.00563015
	1/8	0.00766242	0.00744653	0.00697654
2^{-4}	1/16	0.00805123	0.00785805	0.00743860
	1/32	0.00815870	0.00797194	0.00756827
	1/4	0.00589104	0.00568137	0.00520494
	1/8	0.00724537	0.00709338	0.00672517
2^{-5}	1/16	0.00775662	0.00762058	0.00729444
	1/32	0.00790308	0.00777166	0.00745956
	1/4	0.00495551	0.00483390	0.00452149
	1/8	0.00662210	0.00653739	0.00629457
2^{-6}	1/16	0.00734488	0.00726777	0.00705463
	1/32	0.00756304	0.00748859	0.00728722

Table 4.3A

Linear finite element approximations of $-\partial w_0/\partial x$ (1/4, 1/4) of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00379147	0.00373865	0.00358030
	1/8	0.00579340	0.00575643	0.00562639
2^{-7}	1/16	0.00684277	0.00680714	0.00669352
	1/32	0.00718722	0.00715332	0.00704973
	1/4	0.00260376	0.00258661	0.00252701
	1/8	0.00478691	0.00477121	0.00471036
2^{-8}	1/16	0.00626554	0.00624918	0.00619487
	1/32	0.00682426	0.00680994	0.00676510
	1/4	0.00161207	0.00160755	0.00159033
	1/8	0.00365688	0.00364861	0.00361936
2^{-9}	1/16	0.00556740	0.00555847	0.00552990
	1/32	0.00646659	0.00645987	0.00643978
	1/4	0.00091833	0.00091725	0.00091304
	1/8	0.00253240	0.00252794	0.00251357
2^{-10}	1/16	0.00467917	0.00467411	0.00465767
	1/32	0.00604466	0.00604109	0.00603019
	1/4	0.00049422	0.00049396	0.00049300
	1/8	0.00158522	0.00158321	0.00157685
2^{-11}	1/16	0.00361203	0.00360942	0.00360072
	1/32	0.00545852	0.00545643	0.00544975
	1/4	0.00025703	0.00025697	0.00025675
	1/8	0.00091172	0.00091097	0.00090859
2^{-12}	1/16	0.00251190	0.00251072	0.00250676
	1/32	0.00463212	0.00463089	0.00462685

Table 4.3B

Quadratic finite element approximations of $w_0(0, 0)$ of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.11279490	0.11275023	0.11265934
	1/8	0.11149476	0.11145222	0.11137136
2^{-1}	1/16	0.11139156	0.11134898	0.11126836
	1/32	0.11138400	0.11134140	0.11126076
	1/4	0.05851380	0.05847063	0.05838197
	1/8	0.05784318	0.05780214	0.05772349
2^{-2}	1/16	0.05778936	0.05774827	0.05766986
	1/32	0.05778540	0.05774428	0.05766586
	1/4	0.03136158	0.03132119	0.03123668
	1/8	0.03100934	0.03097130	0.03089652
2^{-3}	1/16	0.03098049	0.03094212	0.03086785
	1/32	0.03097835	0.03093995	0.03086567
	1/4	0.01776429	0.01772858	0.01765137
	1/8	0.01757835	0.01754467	0.01747740
2^{-4}	1/16	0.01756254	0.01752878	0.01746175
	1/32	0.01756134	0.01752756	0.01746052
	1/4	0.01092983	0.01090101	0.01083525
	1/8	0.01084079	0.01081395	0.01075807
2^{-5}	1/16	0.01083259	0.01080564	0.01074999
	1/32	0.01083194	0.01080496	0.01074930
	1/4	0.00745726	0.00743669	0.00738600
	1/8	0.00744258	0.00742403	0.00738315
2^{-6}	1/16	0.00744032	0.00742162	0.00738095
	1/32	0.00744008	0.00742135	0.00738067

Table 4.4A

Quadratic finite element approximations of $w_0(0, 0)$ of the square plate with polynomial load function.

h	ε	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00564180	0.00562879	0.00559369
	1/8	0.00571144	0.00570063	0.00567522
2^{-7}	1/16	0.00571619	0.00570522	0.00568002
	1/32	0.00571645	0.00570543	0.00568022
	1/4	0.00462317	0.00461547	0.00459288
	1/8	0.00481626	0.00481095	0.00479758
2^{-8}	1/16	0.00483216	0.00482671	0.00481359
	1/32	0.00483322	0.00482772	0.00481458
	1/4	0.00395746	0.00395292	0.00393878
	1/8	0.00434091	0.00433861	0.00433243
2^{-9}	1/16	0.00437647	0.00437412	0.00436823
	1/32	0.00437899	0.00437659	0.00437067
	1/4	0.00341223	0.00340946	0.00340054
	1/8	0.00407047	0.00406952	0.00406680
2^{-10}	1/16	0.00414062	0.00413971	0.00413734
	1/32	0.00414588	0.00414493	0.00414253
	1/4	0.00287314	0.00287126	0.00286518
	1/8	0.00388894	0.00388851	0.00388724
2^{-11}	1/16	0.00401639	0.00401606	0.00401517
	1/32	0.00402673	0.00402638	0.00402547
	1/4	0.00229520	0.00229382	0.00228935
	1/8	0.00373183	0.00373155	0.00373073
2^{-12}	1/16	0.00394642	0.00394630	0.00394598
	1/32	0.00396578	0.00396566	0.00396533

Table 4.4B

Quadratic finite element approximations of $w_0(1/4, 1/4)$ of the square plate with polynomial load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.04997769	0.04995958	0.04992647
	1/8	0.05061618	0.05059978	0.05056964
2^{-1}	1/16	0.05064667	0.05063034	0.05060043
	1/32	0.05064849	0.05063214	0.05060225
	1/4	0.02563617	0.02561866	0.02558635
	1/8	0.02597481	0.02595895	0.02592960
2^{-2}	1/16	0.02599128	0.02597548	0.02594634
	1/32	0.02599227	0.02597646	0.02594733
	1/4	0.01346181	0.01344539	0.01341459
	1/8	0.01365137	0.01363651	0.01360860
2^{-3}	1/16	0.01366091	0.01364610	0.01361840
	1/32	0.01366150	0.01364668	0.01361899
	1/4	0.00736812	0.00735357	0.00732545
	1/8	0.00748483	0.00747166	0.00744631
2^{-4}	1/16	0.00749108	0.00747794	0.00745278
	1/32	0.00749148	0.00747833	0.00745317
	1/4	0.00431043	0.00429868	0.00427483
	1/8	0.00439397	0.00438334	0.00436205
2^{-5}	1/16	0.00439891	0.00438828	0.00436713
	1/32	0.00439923	0.00438859	0.00436745
	1/4	0.00276529	0.00275700	0.00273892
	1/8	0.00283840	0.00283092	0.00281505
2^{-6}	1/16	0.00284329	0.00283576	0.00281999
	1/32	0.00284362	0.00283608	0.00282031

Table 4.5A

Quadratic finite element approximations of $w_0(1/4, 1/4)$ of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00197042	0.00196546	0.00195358
	1/8	0.00204963	0.00204517	0.00203506
2^{-7}	1/16	0.00205556	0.00205103	0.00204096
	1/32	0.00205598	0.00205143	0.00204136
	1/4	0.00154339	0.00154083	0.00153405
	1/8	0.00164541	0.00164316	0.00163770
2^{-8}	1/16	0.00165378	0.00165146	0.00164603
	1/32	0.00165439	0.00165205	0.00164660
	1/4	0.00128938	0.00128815	0.00128461
	1/8	0.00143483	0.00143384	0.00143127
2^{-9}	1/16	0.00144791	0.00144687	0.00144434
	1/32	0.00144888	0.00144783	0.00144529
	1/4	0.00110643	0.00110580	0.00110382
	1/8	0.00132059	0.00132018	0.00131906
2^{-10}	1/16	0.00134213	0.00134172	0.00134067
	1/32	0.00134381	0.00134338	0.00134232
	1/4	0.00093987	0.00093942	0.00093797
	1/8	0.00125161	0.00125143	0.00125091
2^{-11}	1/16	0.00128727	0.00128712	0.00128671
	1/32	0.00129026	0.00129010	0.00128968
	1/4	0.00076149	0.00076111	0.00075986
	1/8	0.00120033	0.00120022	0.00119990
2^{-12}	1/16	0.00125768	0.00125762	0.00125747
	1/32	0.00126299	0.00126299	0.00126278

Table 4.5B

Quadratic finite element approximations of $-\partial w_0/\partial x$ (1/4, 1/4) of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
2^{-1}	1/4	0.00851318	0.00826684	0.00777805
	1/8	0.00849647	0.00823233	0.00771902
	1/16	0.00849173	0.00822769	0.00771443
	1/32	0.00849146	0.00822740	0.00771415
2^{-2}	1/4	0.00845915	0.00822711	0.00775785
	1/8	0.00844731	0.00819722	0.00770313
	1/16	0.00844290	0.00819290	0.00769882
	1/32	0.00844265	0.00819266	0.00769855
2^{-3}	1/4	0.00835838	0.00815199	0.00771883
	1/8	0.00835620	0.00813122	0.00767258
	1/16	0.00835244	0.00812750	0.00766881
	1/32	0.00835223	0.00812726	0.00766857
2^{-4}	1/4	0.00818176	0.00801699	0.00764577
	1/8	0.00819834	0.00801387	0.00761599
	1/16	0.00819581	0.00801130	0.00761326
	1/32	0.00819567	0.00801114	0.00761310
2^{-5}	1/4	0.00790324	0.00779499	0.00751672
	1/8	0.00795437	0.00782471	0.00751799
	1/16	0.00795402	0.00782425	0.00751727
	1/32	0.00795403	0.00782423	0.00751724
2^{-6}	1/4	0.00752780	0.00747672	0.00730968
	1/8	0.00763675	0.00756338	0.00736670
	1/16	0.00763997	0.00756653	0.00736971
	1/32	0.00764022	0.00756676	0.00736993

Table 4.6A

Quadratic finite element approximations of $-\partial w_0/\partial x$ (1/4, 1/4) of the square plate with polynomial load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00710690	0.00709292	0.00701990
	1/8	0.00730319	0.00726978	0.00717001
2^{-7}	1/16	0.00731208	0.00727896	0.00717969
	1/32	0.00731276	0.00727963	0.00718037
	1/4	0.00669345	0.00669208	0.00666829
	1/8	0.00701871	0.00700478	0.00696310
2^{-8}	1/16	0.00703797	0.00702471	0.00698463
	1/32	0.00703944	0.00702617	0.00698616
	1/4	0.00627998	0.00627954	0.00626988
	1/8	0.00680546	0.00679977	0.00678278
2^{-9}	1/16	0.00684005	0.00684072	0.00682600
	1/32	0.00684910	0.00684375	0.00682912
	1/4	0.00580103	0.00579934	0.00579115
	1/8	0.00664383	0.00664159	0.00663415
2^{-10}	1/16	0.00672573	0.00672362	0.00671807
	1/32	0.00673187	0.00672975	0.00672430
	1/4	0.00516873	0.00516599	0.00515676
	1/8	0.00649532	0.00649459	0.00649140
2^{-11}	1/16	0.00665292	0.00665210	0.00664995
	1/32	0.00666499	0.00666420	0.00666216
	1/4	0.00431599	0.00431314	0.00430401
	1/8	0.00631741	0.00631715	0.00631574
2^{-12}	1/16	0.00660501	0.00660469	0.00660381
	1/32	0.00662828	0.00662800	0.00662725

Table 4.6B

Linear finite element approximations of $w_0(0, 0)$ of the square plate with cosine load function.

h	ε	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.06815443	0.06812177	0.06803193
	1/8	0.07178877	0.07176351	0.07170065
2^{-1}	1/16	0.07142266	0.07140290	0.07136104
	1/32	0.07095456	0.07093634	0.07090060
	1/4	0.03475508	0.03472342	0.03463568
	1/8	0.03705899	0.03703444	0.03697276
2^{-2}	1/16	0.03701545	0.03699632	0.03695543
	1/32	0.03681180	0.03679411	0.03675940
	1/4	0.01803383	0.01800400	0.01792024
	1/8	0.01968076	0.01965751	0.01959803
2^{-3}	1/16	0.01980554	0.01978758	0.01974849
	1/32	0.01973636	0.01971990	0.01968685
	1/4	0.00963347	0.00960677	0.00953013
	1/8	0.01096604	0.01094535	0.01088976
2^{-4}	1/16	0.01118893	0.01117294	0.01113700
	1/32	0.01119141	0.01117689	0.01114690
	1/4	0.00536513	0.00534317	0.00527805
	1/8	0.00656339	0.00654559	0.00649624
2^{-5}	1/16	0.00686014	0.00684706	0.00681606
	1/32	0.00690706	0.00689539	0.00687019
	1/4	0.00312705	0.00311108	0.00306192
	1/8	0.00428345	0.00426959	0.00422880
2^{-6}	1/16	0.00466190	0.00465229	0.00462777
	1/32	0.00474744	0.00473921	0.00472030

Table 4.7A

Linear finite element approximations of $w_0(0, 0)$ of the square plate with cosine load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.00187570	0.00186576	0.00183431
	1/8	0.00301900	0.00300899	0.00297794
2^{-7}	1/16	0.00350901	0.00350260	0.00348477
	1/32	0.00364427	0.00363922	0.00362677
	1/4	0.00111842	0.00111329	0.00109683
	1/8	0.00220890	0.00220215	0.00218054
2^{-8}	1/16	0.00284737	0.00284323	0.00283078
	1/32	0.00306107	0.00305826	0.00305079
	1/4	0.00064282	0.00064067	0.00063369
	1/8	0.00158782	0.00158372	0.00157042
2^{-9}	1/16	0.00238271	0.00237995	0.00237121
	1/32	0.00272122	0.00271967	0.00271517
	1/4	0.00035280	0.00035205	0.00034960
	1/8	0.00107022	0.00106810	0.00106118
2^{-10}	1/16	0.00195615	0.00195430	0.00194825
	1/32	0.00247043	0.00246947	0.00246650
	1/4	0.00018640	0.00018618	0.00018543
	1/8	0.00066275	0.00066185	0.00065888
2^{-11}	1/16	0.00150548	0.00150431	0.00150046
	1/32	0.00221146	0.00221079	0.00220864
	1/4	0.00009606	0.00009600	0.00009570
	1/8	0.00037962	0.00037930	0.00037824
2^{-12}	1/16	0.00105357	0.00105293	0.00105082
	1/32	0.00188442	0.00188394	0.00188238

Table 4.7B

Linear finite element approximations of $w_0(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.01769967	0.01769160	0.01766647
	1/8	0.01656582	0.01655850	0.01654210
2^{-1}	1/16	0.01627200	0.01626527	0.01625226
	1/32	0.01619770	0.01619103	0.01617865
	1/4	0.00905948	0.00905159	0.00902698
	1/8	0.00859350	0.00858640	0.00857034
2^{-2}	1/16	0.00847023	0.00846372	0.00845103
	1/32	0.00843842	0.00843197	0.00841991
	1/4	0.00473309	0.00472556	0.00470193
	1/8	0.00460395	0.00459725	0.00458182
2^{-3}	1/16	0.00456759	0.00456150	0.00454941
	1/32	0.00455751	0.00455147	0.00454000
	1/4	0.00255825	0.00255133	0.00252947
	1/8	0.00260277	0.00259675	0.00258245
2^{-4}	1/16	0.00261307	0.00260766	0.00259664
	1/32	0.00261479	0.00260944	0.00259903
	1/4	0.00145059	0.00144464	0.00142572
	1/8	0.00159061	0.00158561	0.00157311
2^{-5}	1/16	0.00163029	0.00162592	0.00161657
	1/32	0.00163977	0.00163547	0.00162673
	1/4	0.00086538	0.00086077	0.00084604
	1/8	0.00106485	0.00106110	0.00105104
2^{-6}	1/16	0.00113013	0.00112703	0.00111991
	1/32	0.00114707	0.00114403	0.00113752

Table 4.8A

Linear finite element approximations of $w_0(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.00053197	0.00052889	0.00051909
	1/8	0.00077084	0.00076830	0.00076094
2^{-7}	1/16	0.00086683	0.00086492	0.00086013
	1/32	0.00089419	0.00089237	0.00088813
	1/4	0.00032389	0.00032221	0.00031686
	1/8	0.00057907	0.00057747	0.00057254
2^{-8}	1/16	0.00071525	0.00071419	0.00071123
	1/32	0.00075980	0.00075886	0.00075653
	1/4	0.00018892	0.00018819	0.00018585
	1/8	0.00042747	0.00042652	0.00042350
2^{-9}	1/16	0.00060865	0.00060806	0.00060624
	1/32	0.00068157	0.00068112	0.00067992
	1/4	0.00010462	0.00010436	0.00010353
	1/8	0.00029581	0.00029530	0.00029367
2^{-10}	1/16	0.00050975	0.00050939	0.00050822
	1/32	0.00062459	0.00062437	0.00062373
	1/4	0.00005556	0.00005548	0.00005522
	1/8	0.00018749	0.00018726	0.00018652
2^{-11}	1/16	0.00040188	0.00040166	0.00040092
	1/32	0.00056630	0.00056618	0.00056578
	1/4	0.00002871	0.00002869	0.00002861
	1/8	0.00010927	0.00010919	0.00010891
2^{-12}	1/16	0.00028858	0.00028846	0.00028804
	1/32	0.00049146	0.00049138	0.00049111

Table 4.8B

Linear finite element approximations of $-\partial w_0/\partial x(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.00374882	0.00354773	0.00315812
	1/8	0.00455461	0.00441342	0.00413684
2^{-1}	1/16	0.00476598	0.00464645	0.00441339
	1/32	0.00481934	0.00470616	0.00448536
	1/4	0.00369189	0.00350021	0.00312379
	1/8	0.00451764	0.00438376	0.00411697
2^{-2}	1/16	0.00474051	0.00462731	0.00440278
	1/32	0.00479745	0.00469029	0.00447769
	1/4	0.00358375	0.00340916	0.00305731
	1/8	0.00444750	0.00432678	0.00407814
2^{-3}	1/16	0.00469279	0.00459092	0.00438215
	1/32	0.00475675	0.00466034	0.00446289
	1/4	0.00338762	0.00324145	0.00293247
	1/8	0.00432035	0.00422117	0.00400386
2^{-4}	1/16	0.00460832	0.00452479	0.00434304
	1/32	0.00468577	0.00460673	0.00443522
	1/4	0.00305937	0.00295391	0.00271098
	1/8	0.00410637	0.00403693	0.00386718
2^{-5}	1/16	0.00447193	0.00441331	0.00427217
	1/32	0.00457458	0.00451904	0.00438648
	1/4	0.00257470	0.00251364	0.00235498
	1/8	0.00378138	0.00374299	0.00363143
2^{-6}	1/16	0.00427764	0.00424459	0.00415244
	1/32	0.00442537	0.00439397	0.00430844

Table 4.9A

Linear finite element approximations of $-\partial w_0/\partial x(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00197105	0.00194474	0.00186472
	1/8	0.00333830	0.00332149	0.00326157
2^{-7}	1/16	0.00403175	0.00401650	0.00396715
	1/32	0.00425707	0.00424275	0.00419862
	1/4	0.00135439	0.00134595	0.00131611
	1/8	0.00278376	0.00277609	0.00274699
2^{-8}	1/16	0.00373196	0.00372476	0.00370051
	1/32	0.00408694	0.00408078	0.00406137
	1/4	0.00083893	0.00083675	0.00082825
	1/8	0.00214454	0.00213988	0.00212449
2^{-9}	1/16	0.00334590	0.00334169	0.00332807
	1/32	0.00390548	0.00390246	0.00389336
	1/4	0.00047806	0.00047754	0.00047550
	1/8	0.00149618	0.00149338	0.00148487
2^{-10}	1/16	0.00283282	0.00283023	0.00282177
	1/32	0.00367273	0.00367102	0.00366574
	1/4	0.00025732	0.00025720	0.00025675
	1/8	0.00094238	0.00094102	0.00093688
2^{-11}	1/16	0.00220094	0.00219949	0.00219467
	1/32	0.00333219	0.00333113	0.00332768
	1/4	0.00013384	0.00013381	0.00013371
	1/8	0.00054445	0.00054392	0.00054228
2^{-12}	1/16	0.00153949	0.00153878	0.00153643
	1/32	0.00284015	0.00283948	0.00283728

Table 4.9B

Quadratic finite element approximations of $w_0(0, 0)$ of the square plate with cosine load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.07355522	0.07353480	0.07349130
	1/8	0.07089513	0.07087735	0.07084338
2^{-1}	1/16	0.07066009	0.07064237	0.07060870
	1/32	0.07064233	0.07062461	0.07059093
	1/4	0.03813380	0.03811405	0.03807154
	1/8	0.03679209	0.03677495	0.03674190
2^{-2}	1/16	0.03667236	0.03665526	0.03662252
	1/32	0.03666327	0.03664617	0.03661342
	1/4	0.02041684	0.02039831	0.02035767
	1/8	0.01973709	0.01972109	0.01968979
2^{-3}	1/16	0.01967522	0.01965927	0.01962826
	1/32	0.01967048	0.01965452	0.01962351
	1/4	0.01154686	0.01153038	0.01149301
	1/8	0.01120348	0.01118942	0.01116118
2^{-4}	1/16	0.01117096	0.01115693	0.01112897
	1/32	0.01116843	0.01115459	0.01112641
	1/4	0.00709191	0.00707845	0.00704620
	1/8	0.00692702	0.00691583	0.00689239
2^{-5}	1/16	0.00691000	0.00689882	0.00687562
	1/32	0.00690862	0.00689742	0.00687422
	1/4	0.00483219	0.00482232	0.00479681
	1/8	0.00477572	0.00476800	0.00475086
2^{-6}	1/16	0.00476802	0.00476027	0.00474334
	1/32	0.00476733	0.00475956	0.00474262

Table 4.10A

Quadratic finite element approximations of $w_0(0, 0)$ of the square plate with cosine load function.

ε	h	$\mu = 0.3$	$\mu = 0.0$	$\mu = -1.0$
	1/4	0.00365317	0.00364660	0.00362807
	1/8	0.00368528	0.00368079	0.00367014
2^{-7}	1/16	0.00368520	0.00368066	0.00367020
	1/32	0.00368505	0.00368049	0.00367001
	1/4	0.00299042	0.00298619	0.00297339
	1/8	0.00312513	0.00312291	0.00311729
2^{-8}	1/16	0.00313441	0.00313216	0.00312672
	1/32	0.00313493	0.00313265	0.00312720
	1/4	0.00255240	0.00254965	0.00254097
	1/8	0.00282876	0.00282779	0.00282516
2^{-9}	1/16	0.00285295	0.00285198	0.00284955
	1/32	0.00285456	0.00285357	0.00285112
	1/4	0.00218881	0.00218700	0.00218117
	1/8	0.00265869	0.00265829	0.00265711
2^{-10}	1/16	0.00270822	0.00270785	0.00270687
	1/32	0.00271182	0.00271143	0.00271043
	1/4	0.00182964	0.00182839	0.00182436
	1/8	0.00254123	0.00254104	0.00254047
2^{-11}	1/16	0.00263202	0.00263188	0.00263152
	1/32	0.00263928	0.00263913	0.00263875
	1/4	0.00145080	0.00144991	0.00144703
	1/8	0.00243599	0.00243584	0.00243544
2^{-12}	1/16	0.00258854	0.00258849	0.00258837
	1/32	0.00260227	0.00260221	0.00260208

Table 4.10B

Quadratic finite element approximations of $w_0(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.01528080	0.01527376	0.01526153
	1/8	0.01609705	0.01609039	0.01607823
2^{-1}	1/16	0.01616784	0.01616118	0.01614897
	1/32	0.01617255	0.01616589	0.01615367
	1/4	0.00797161	0.00796482	0.00795294
	1/8	0.00838906	0.00838263	0.00837079
2^{-2}	1/16	0.00842515	0.00841871	0.00840682
	1/32	0.00842756	0.00842111	0.00840921
	1/4	0.00431563	0.00430931	0.00429808
	1/8	0.00453392	0.00452791	0.00451668
2^{-3}	1/16	0.00455270	0.00454667	0.00453537
	1/32	0.00455395	0.00454791	0.00453661
	1/4	0.00248519	0.00247966	0.00246957
	1/8	0.00260438	0.00259907	0.00258890
2^{-4}	1/16	0.00261454	0.00260920	0.00259896
	1/32	0.00261522	0.00260987	0.00259963
	1/4	0.00156599	0.00156165	0.00155336
	1/8	0.00163650	0.00163225	0.00162378
2^{-5}	1/16	0.00164244	0.00163815	0.00162957
	1/32	0.00164284	0.00163854	0.00162996
	1/4	0.00110073	0.00109783	0.00109191
	1/8	0.00114850	0.00114554	0.00113932
2^{-6}	1/16	0.00115246	0.00114944	0.00114309
	1/32	0.00115273	0.00114970	0.00114334

Table 4.11A

Quadratic finite element approximations of $w_0(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00086093	0.00085935	0.00085587
	1/8	0.00090024	0.00089852	0.00089467
2^{-7}	1/16	0.00090340	0.00090161	0.00089761
	1/32	0.00090362	0.00090182	0.00089780
	1/4	0.00073193	0.00073124	0.00072959
	1/8	0.00077261	0.00077179	0.00076982
2^{-8}	1/16	0.00077570	0.00077480	0.00077268
	1/32	0.00077592	0.00077500	0.00077287
	1/4	0.00065399	0.00065372	0.00065300
	1/8	0.00070631	0.00070598	0.00070515
2^{-9}	1/16	0.00070993	0.00070954	0.00070858
	1/32	0.00071019	0.00070979	0.00070881
	1/4	0.00059257	0.00059241	0.00059194
	1/8	0.00067102	0.00067091	0.00067061
2^{-10}	1/16	0.00067609	0.00067594	0.00067556
	1/32	0.00067644	0.00067628	0.00067588
	1/4	0.00052514	0.00052496	0.00052440
	1/8	0.00065049	0.00065045	0.00065034
2^{-11}	1/16	0.00065867	0.00065861	0.00065847
	1/32	0.00065920	0.00065914	0.00065899
	1/4	0.00043838	0.00043818	0.00043755
	1/8	0.00063511	0.00063508	0.00063499
2^{-12}	1/16	0.00064952	0.00064951	0.00064946
	1/32	0.00065043	0.00065041	0.00065035

Table 4.11B

Quadratic finite element approximations of $-\partial w_0/\partial x(1/4, 01/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00487528	0.00475287	0.00451431
	1/8	0.00484236	0.00472927	0.00450941
2^{-1}	1/16	0.00483753	0.00472646	0.00450964
	1/32	0.00483719	0.00472627	0.00450967
	1/4	0.00485034	0.00473442	0.00450478
	1/8	0.00482136	0.00471426	0.00450257
2^{-2}	1/16	0.00481689	0.00471173	0.00450300
	1/32	0.00481656	0.00471155	0.00450305
	1/4	0.00480381	0.00469952	0.00448647
	1/8	0.00478244	0.00468602	0.00448943
2^{-3}	1/16	0.00477863	0.00468403	0.00449024
	1/32	0.00477836	0.00468389	0.00449032
	1/4	0.00472213	0.00463673	0.00445190
	1/8	0.00471498	0.00463579	0.00446507
2^{-4}	1/16	0.00471239	0.00463481	0.00446663
	1/32	0.00471220	0.00463474	0.00446676
	1/4	0.00459298	0.00453328	0.00439095
	1/8	0.00461066	0.00455477	0.00442285
2^{-5}	1/16	0.00461012	0.00455556	0.00442582
	1/32	0.00461007	0.00455561	0.00442606
	1/4	0.00441785	0.00438435	0.00429297
	1/8	0.00447463	0.00444265	0.00435756
2^{-6}	1/16	0.00447721	0.00444632	0.00436306
	1/32	0.00447739	0.00444658	0.00436347

Table 4.12A

Quadratic finite element approximations of $-\partial w_0/\partial x(1/4, 1/4)$ of the square plate with cosine load function.

ε	h	μ = 0.3	μ = 0.0	μ = -1.0
	1/4	0.00421888	0.00420298	0.00415507
	1/8	0.00433121	0.00431620	0.00427234
2^{-7}	1/16	0.00433830	0.00432430	0.00428217
	1/32	0.00433882	0.00432489	0.00428290
	1/4	0.00401751	0.00400897	0.00398496
	1/8	0.00420768	0.00420097	0.00418186
2^{-8}	1/16	0.00422187	0.00421616	0.00419897
	1/32	0.00422295	0.00421730	0.00420025
	1/4	0.00380435	0.00379865	0.00378377
	1/8	0.00411285	0.00410967	0.00410119
2^{-9}	1/16	0.00413987	0.00413749	0.00413101
	1/32	0.00414196	0.00413964	0.00413329
	1/4	0.00353867	0.00353473	0.00352391
	1/8	0.00403700	0.00403551	0.00403132
2^{-10}	1/16	0.00408779	0.00408680	0.00408425
	1/32	0.00409180	0.00409086	0.00408844
	1/4	0.00316675	0.00316387	0.00315542
	1/8	0.00396201	0.00396131	0.00395916
2^{-11}	1/16	0.00405534	0.00405493	0.00405388
	1/32	0.00406297	0.00406261	0.00406168
	1/4	0.00265001	0.00264787	0.00265127
	1/8	0.00386651	0.00386610	0.00386488
2^{-12}	1/16	0.00403263	0.00403245	0.00403200
	1/32	0.00404695	0.00404682	0.00404647

Table 4.12B

Linear and quadratic element approximations of $w_0(0, 0)$ in polynomial load.

ε	h	linear	quadratic
	1/4	0.11311639	0.11279490
	1/8	0.11337059	0.11149476
2^{-1}	1/16	0.11228648	0.11139156
	1/32	0.11170684	0.11138400
	1/64	0.11148817	
	1/4	0.05785581	0.05851380
	1/8	0.05858934	0.05784318
2^{-2}	1/16	0.05819838	0.05778936
	1/32	0.05793962	0.05778540
	1/64	0.05783626	
	1/4	0.03018412	0.03136158
	1/8	0.03117510	0.03100934
2^{-3}	1/16	0.03114190	0.03098049
	1/32	0.03104707	0.03097835
	1/64	0.03100225	
	1/4	0.01627208	0.01776429
	1/8	0.01742362	0.01757835
2^{-4}	1/16	0.01759112	0.01756254
	1/32	0.01758495	0.01756134
	1/64	0.01757118	
	1/4	0.00918538	0.01092983
	1/8	0.01046832	0.01084079
2^{-5}	1/16	0.01077701	0.01083259
	1/32	0.01082831	0.01083194
	1/64	0.01083355	
	1/4	0.00544297	0.00745726
	1/8	0.00685689	0.00744258
2^{-6}	1/16	0.00730850	0.00744032
	1/32	0.00741368	0.00744008
	1/64	0.00743527	

Table 4.13A

Linear and quadratic element approximations of $w_0(0, 0)$ in polynomial load.

ε	h	linear	quadratic
	1/4	0.00331845	0.00564180
	1/8	0.00484342	0.00571144
2^{-7}	1/16	0.00548202	0.00571619
	1/32	0.00566086	0.00571645
	1/64	0.00570386	
	1/4	0.00200441	0.00462317
	1/8	0.00354541	0.00481626
2^{-8}	1/16	0.00443066	0.00483216
	1/32	0.00472872	0.00483322
	1/64	0.00480781	
	1/4	0.00116201	0.00395746
	1/8	0.00254705	0.00434091
2^{-9}	1/16	0.00369543	0.00437647
	1/32	0.00418499	0.00437899
	1/64	0.00432965	
	1/4	0.00064098	0.00287314
	1/8	0.00171520	0.00407047
2^{-10}	1/16	0.00302782	0.00414062
	1/32	0.00378846	0.00414588
	1/64	0.00405046	
	1/4	0.00033961	0.00287314
	1/8	0.00106125	0.00388894
2^{-11}	1/16	0.00232820	0.00401639
	1/32	0.00338670	0.00402678
	1/64	0.00384319	
	1/4	0.00017527	0.00229520
	1/8	0.00060747	0.00373183
2^{-12}	1/16	0.00162880	0.00394642
	1/32	0.00288510	0.00396578
	1/64	0.00361921	

Table 4.13B

Linear and quadratic element approximations of $w_0(0, 0)$ in polynomial load.

ε	h	linear	quadratic
	1/4	0.00008911	0.00168887
	1/8	0.00032846	0.00356000
2^{-13}	1/16	0.00102902	0.00389979
	1/32	0.00226508	0.00393401
	1/64	0.00330459	
	1/4	0.00004493	0.00112230
	1/8	0.00017136	0.00333816
2^{-14}	1/16	0.00059571	0.00385951
	1/32	0.00160231	0.00391625
	1/64	0.00284599	
	1/4	0.00002256	0.00067569
	1/8	0.00008761	0.00302265
2^{-15}	1/16	0.00032404	0.00381476
	1/32	0.00101854	0.00390448
	1/64	0.00224677	

Table 4.13C

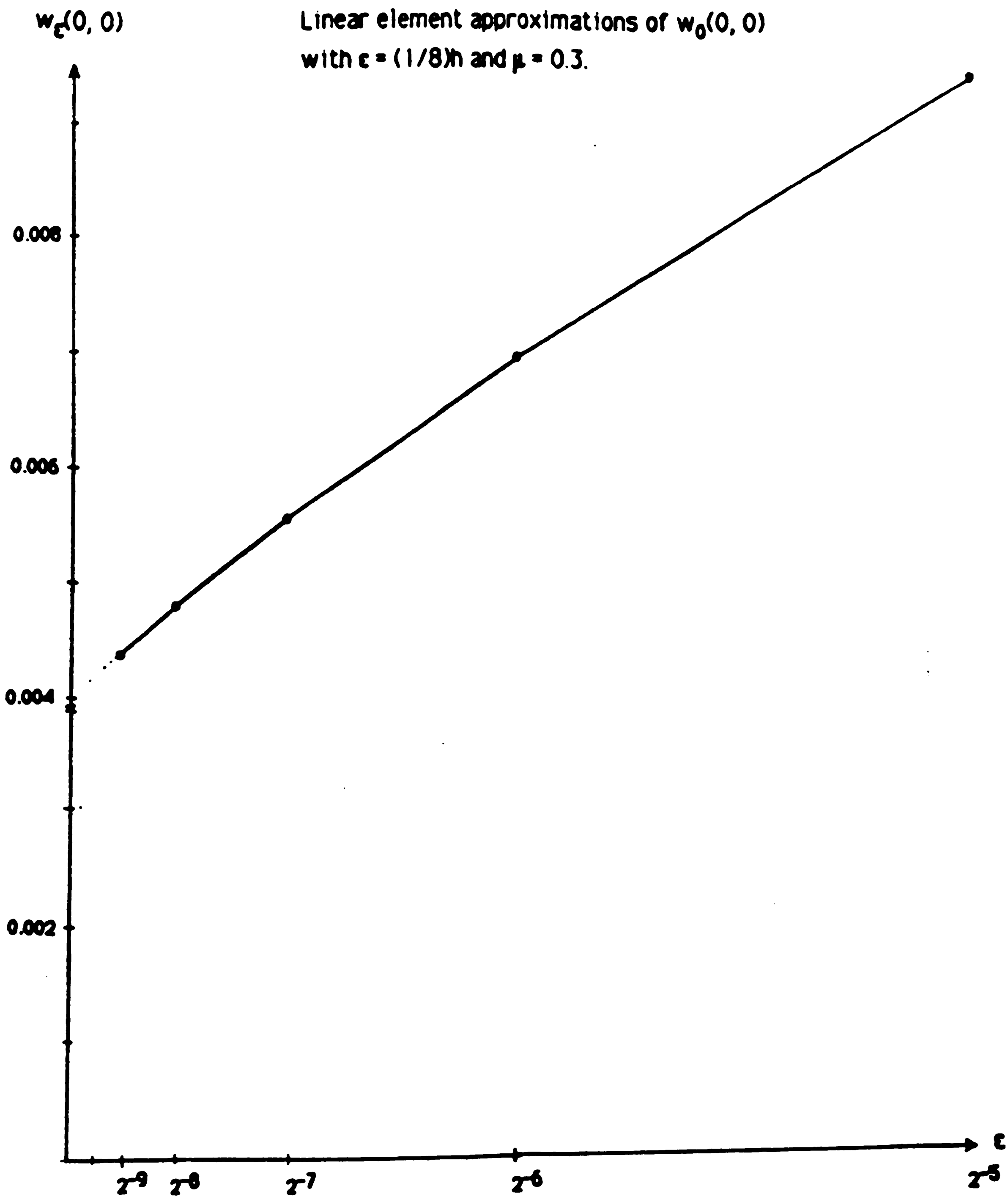

$w_\varepsilon(0, 0)$
Linear element approximations of $w_0(0, 0)$
with $\varepsilon = (1/8)h$ and $\mu = 0.3$.
0.008
0.006
0.004
0.002
ε
2^{-9}
2^{-8}
2^{-7}
2^{-6}
2^{-5}

Figure 4.3

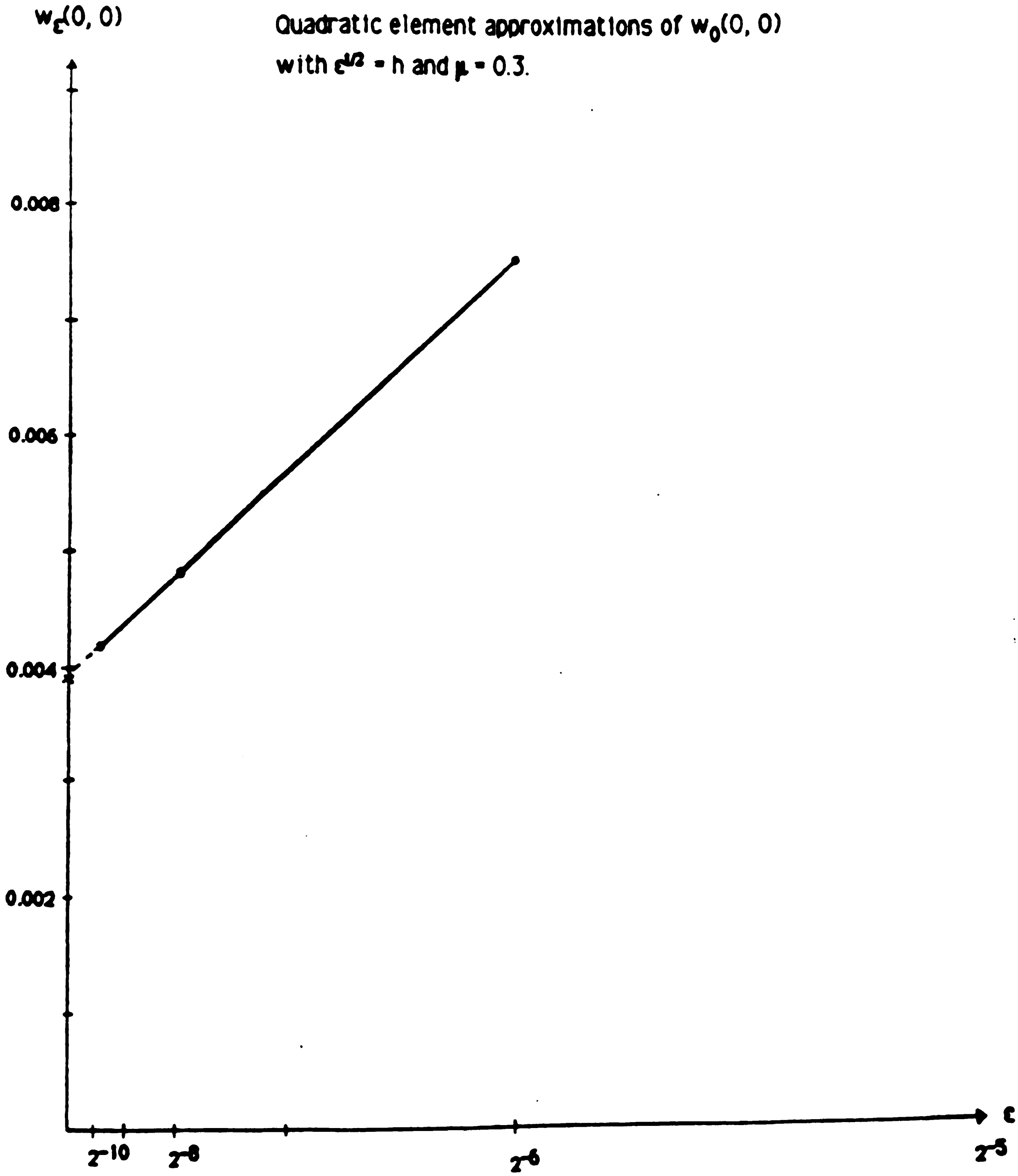
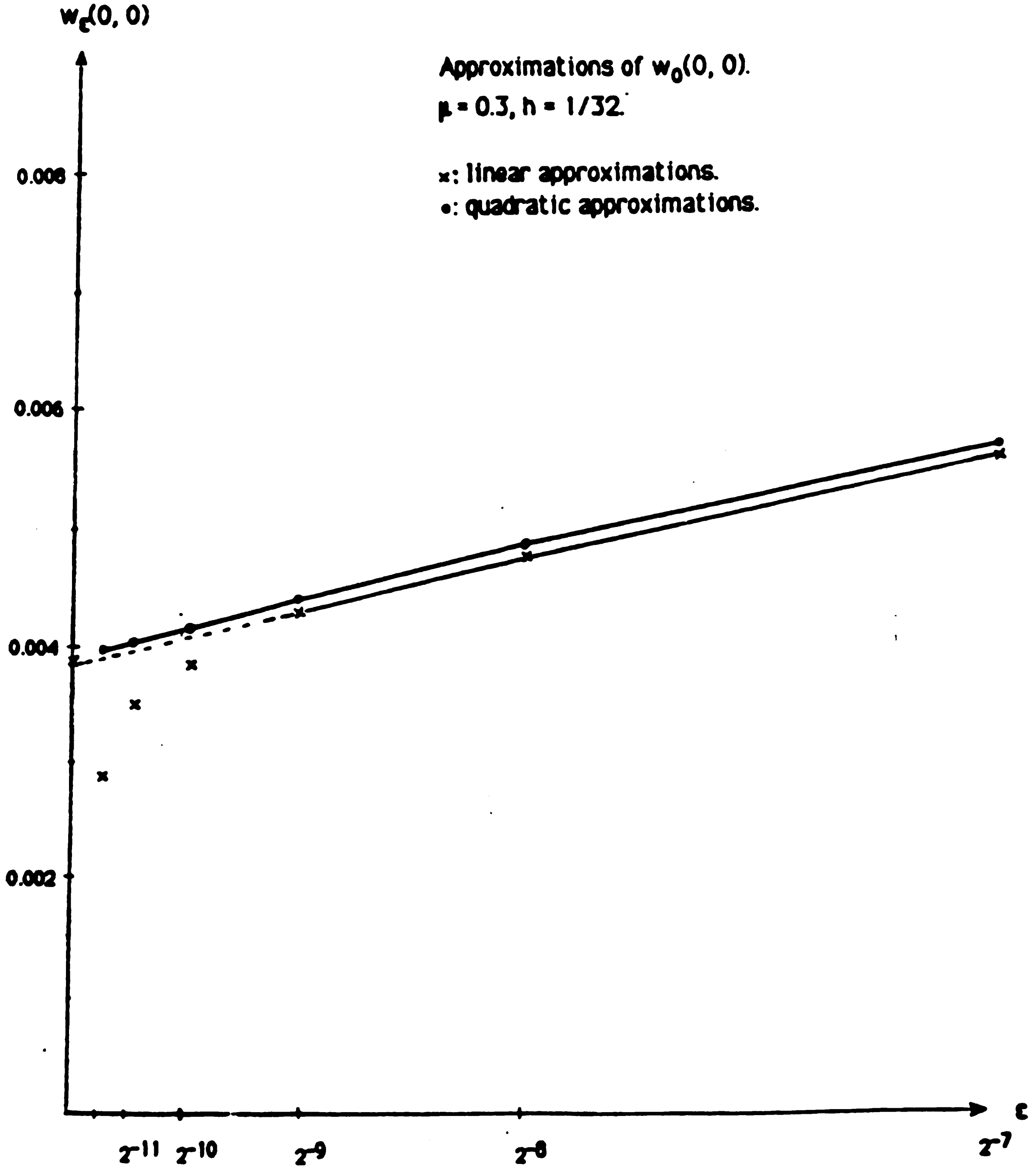

$w_\varepsilon(0, 0)$
Quadratic element approximations of $w_0(0, 0)$
with $\varepsilon^{1/2} = h$ and $\mu = 0.3$.
0.008
0.006
0.004
0.002
ε
2^{-10}
2^{-8}
2^{-6}
2^{-5}

Figure 4.4



$w_\varepsilon(0,0)$
Approximations of $w_0(0,0)$.
$\mu = 0.3$, $h = 1/32$
×: linear approximations.
•: quadratic approximations.
0.008
0.006
0.004
0.002
ε
2^{-11}
2^{-10}
2^{-9}
2^{-8}
2^{-7}

Figure 4.5

Chapter 5 Finite element solutions under the isoparametric transformations for the circular plate

Section 5.1 : Isoparametric transformations involving one curved side

Suppose that L_1, L_2, and L_3 are area coordinates with

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}, \quad \text{and}$$

$$L_1 + L_2 + L_3 = 1 \tag{5.1}$$

where A is the area of the triangle and A_1, A_2, and A_3 are those of the three smaller triangles respectively.

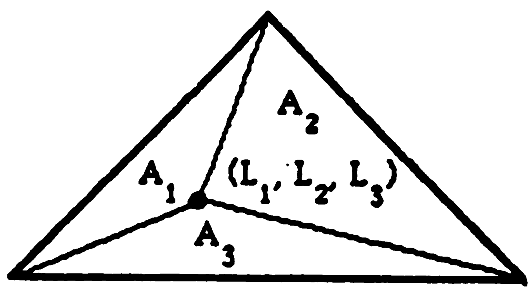


Figure 5.1

The basis functions for the quadratic maps are

$N_i = L_i (2 L_i - 1)$ corner nodes $i = 1, 2,$ and 3

$N_m = 4 L_i L_j$ node m of the midside i-j, m=4, 5, and 6. (5.2)

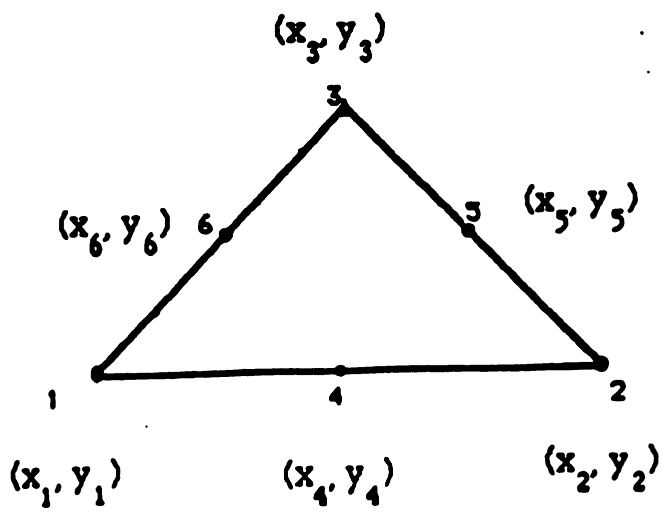


Figure 5.2

Consider a triangle (e) with one curved side and the quadratic map from the master element (E) to (e),

$$x = \sum_{i=1}^{6} x_i N_i \quad , \quad y = \sum_{i=1}^{6} x_i N_i \qquad (5.3)$$

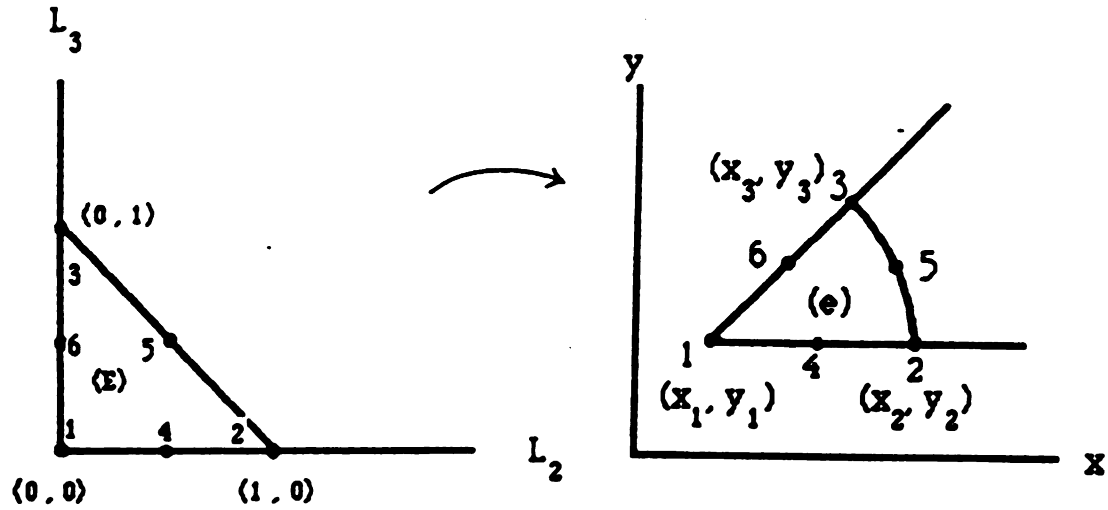


Figure 5.3

Since $L_1 + L_2 + L_3 = 1$, we have $L_1 = 1 - L_2 - L_3$ and

$$N_1 = (1 - L_2 - L_3)(1 - 2L_2 - 2L_3)$$

$$N_2 = 2L_2^2 - L_2$$

$$N_3 = 2L_3^2 - L_3$$

$$N_4 = 4L_2(1 - L_2 - L_3) \qquad (5.4)$$

$$N_5 = 4L_2L_3$$

$$N_6 = 4L_3(1 - L_2 - L_3)$$

and

$$\frac{\partial N_1}{\partial L_2} = 4L_2 + 4L_3 - 3$$

$$\frac{\partial N_2}{\partial L_2} = 4L_2 - 1$$

$$\frac{\partial N_3}{\partial L_2} = 0$$

$$\frac{\partial N_4}{\partial L_2} = 4 - 8L_2 - 4L_3 \qquad (5.5)$$

$$\frac{\partial N_5}{\partial L_2} = 4L_3$$

$$\frac{\partial N_6}{\partial L_2} = -4L_3$$

and

$$\frac{\partial N_1}{\partial L_3} = 4L_2 + 4L_3 - 3$$

$$\frac{\partial N_2}{\partial L_3} = 0$$

$$\frac{\partial N_3}{\partial L_3} = 4L_3 - 1$$

$$\frac{\partial N_4}{\partial L_3} = -4L_2 \qquad (5.6)$$

$$\frac{\partial N_5}{\partial L_3} = 4L_2$$

$$\frac{\partial N_6}{\partial L_3} = 4 - 4L_2 - 8L_3$$

From the transformations (5.3) and (5.4) we have

$$x = x(L_2, L_3) \quad \text{and} \quad y = y(L_2, L_3)$$

$$\begin{bmatrix} \frac{\partial}{\partial L_2} \\ \frac{\partial}{\partial L_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \\ \frac{\partial x}{\partial L_3} & \frac{\partial y}{\partial L_3} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{5.7}$$

and

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial L_2}{\partial x} & \frac{\partial L_3}{\partial x} \\ \frac{\partial L_2}{\partial y} & \frac{\partial L_3}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial L_2} \\ \frac{\partial}{\partial L_3} \end{bmatrix}, \tag{5.8}$$

Let

$$J = \begin{bmatrix} \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \\ \frac{\partial x}{\partial L_3} & \frac{\partial y}{\partial L_3} \end{bmatrix}. \tag{5.9}$$

then

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial L_3} & -\frac{\partial y}{\partial L_2} \\ -\frac{\partial x}{\partial L_3} & \frac{\partial x}{\partial L_2} \end{bmatrix} \tag{5.10}$$

$$= \begin{bmatrix} \frac{\partial L_2}{\partial x} & \frac{\partial L_3}{\partial x} \\ \frac{\partial L_2}{\partial y} & \frac{\partial L_3}{\partial y} \end{bmatrix}. \tag{5.10 a}$$

The det J is the Jacobian of the transformation (5.3) and

$$\det J = \det \begin{bmatrix} \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \\[2ex] \dfrac{\partial x}{\partial L_3} & \dfrac{\partial y}{\partial L_3} \end{bmatrix}$$

$$= \det \begin{bmatrix} \sum_{j=1}^{6} x_j \dfrac{\partial N_j}{\partial L_2} & \sum_{j=1}^{6} y_j \dfrac{\partial N_j}{\partial L_2} \\[2ex] \sum_{j=1}^{6} x_j \dfrac{\partial N_j}{\partial L_3} & \sum_{j=1}^{6} y_j \dfrac{\partial N_j}{\partial L_3} \end{bmatrix}$$

$$= \det (AB) , \qquad (5.11)$$

where

$$A = \begin{bmatrix} 4L_2+4L_3-3 & 4L_2-1 & 0 & -8L_2-4L_3+4 & 4L_3 & -4L_3 \\ 4L_2+4L_3-3 & 0 & 4L_3-1 & -4L_2 & 4L_2 & -4L_2-8L_3+4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix} \qquad (5.12)$$

On the element (e), let

$$N_i(L_2, L_3) = \varphi_i^{(e)}(L_2(x,y), L_3(x,y))$$

$$= \varphi_i^{(e)}(x,y) \qquad i = 1, 2, \ldots, 6.$$

$$= \varphi_i(x,y) .$$

Then

$$\frac{\partial \varphi_i}{\partial x} = \frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial x}$$

$$\frac{\partial \varphi_i}{\partial y} = \frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial y} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial y} \qquad i = 1, 2, \ldots, 6. \qquad (5.13)$$

and

$$u_k^{(e)} = \sum_{i=1}^{6} q_{ik}\, \varphi_i(x,y) = \sum_{i=1}^{6} q_{ik}\, N_i(L_2, L_3), \quad k = 1, 2, 3, \qquad (5.14)$$

where

$$q_{i1} = q_{3i-2}\ , \quad q_{i2} = q_{3i-1}\ , \quad q_{i3} = q_{3i}\ ,$$

$$i = 1, 2, \ldots, 6 .$$

Since

$$\frac{\partial u_1^{(e)}}{\partial x} = \frac{\partial}{\partial x}\sum_{i=1}^{6} q_{i1}\, \varphi_i$$

$$= \sum_{i=1}^{6} q_{i1}\, \frac{\partial \varphi_i}{\partial x} \qquad (5.15)$$

$$= \sum_{i=1}^{6} q_{i1}\left(\frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial x}\right)$$

it follows that

$$\iint_{(e)} \left(\frac{\partial u_1^{(e)}}{\partial x}\right)^2 dx\,dy$$

$$= \iint_{(E)} \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j1} \left(\frac{\partial \varphi_i}{\partial x}\right) \left(\frac{\partial \varphi_j}{\partial x}\right) |\det J|\; dL_2\, dL_3$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j1} \iint_{(E)} \left(\frac{\partial \varphi_i}{\partial x}\right) \left(\frac{\partial \varphi_j}{\partial x}\right) |\det J|\; dL_2\, dL_3 \; . \quad (5.16)$$

Similarly we have the following integrals

$$\iint_{(e)} \left(\frac{\partial u_2}{\partial y}\right)^2 dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i2}\, q_{j2} \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} |\det J|\; dL_2\, dL_3 \; ,$$

$$\iint_{(e)} \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y}\, dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j2} \iint_{(E)} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} |\det J|\; dL_2\, dL_3 \; ,$$

$$\iint_{(e)} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x}\, dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j2} \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} |\det J|\; dL_2\, dL_3 \; .$$

$$\iint_{(e)} \left(\frac{\partial u_2}{\partial x}\right)^2 dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i2}\, q_{j2} \iint_{(E)} \frac{\partial \phi_i}{\partial x}\, \frac{\partial \phi_j}{\partial y}\, |\det J|\; dL_2\, dL_3\,,$$

$$\iint_{(e)} (u_1)^2\, dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j1} \iint_{(E)} N_i\, N_j\, |\det J|\; dL_2\, dL_3\,,$$

$$\iint_{(E)} u_1\, \frac{\partial u_3}{\partial x}\, dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i1}\, q_{j3} \iint_{(E)} N_i\, \frac{\partial \phi_j}{\partial x}\, |\det J|\; dL_2\, dL_3\,,$$

$$\iint_{(e)} \left(\frac{\partial u_3}{\partial x}\right)^2 dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i3}\, q_{j3} \iint_{(E)} \frac{\partial \phi_i}{\partial x}\, \frac{\partial \phi_j}{\partial x}\, |\det J|\; dL_2\, dL_3\,,$$

$$\iint_{(e)} (u_2)^2\, dx\,dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i2}\, q_{j2} \iint_{(E)} N_i\, N_j\, |\det J|\; dL_2\, dL_3\,,$$

$$\iint_{(e)} u_2 \frac{\partial u_3}{\partial y}\, dx\, dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i2}\, q_{j3} \iint_{(E)} N_i \frac{\partial \varphi_j}{\partial y} \,|\det J|\, dL_2\, dL_3 ,$$

$$\iint_{(e)} \left(\frac{\partial u_3}{\partial y}\right)^2 dx\, dy$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} q_{i3}\, q_{j3} \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \,|\det J|\, dL_2\, dL_3 .$$

Rewriting the bilinear integral $B_\varepsilon(U, U)$ and using the above equations, we can construct the element stiffness matrix in the next section.

$$B_\varepsilon(U,U) = \iint_\Omega \{ (1-\mu)[(\frac{\partial u_1}{\partial x})^2 + (\frac{\partial u_2}{\partial y})^2 + \frac{1}{2}(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x})^2]$$

$$+ \mu (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y})^2 \}\, dx\, dy$$

$$+ \frac{1}{\varepsilon} \iint_\Omega [(u_1 + \frac{\partial u_3}{\partial x})^2 + (u_2 + \frac{\partial u_3}{\partial y})^2]\, dx\, dy$$

$$= \iint_\Omega [(\frac{\partial u_1}{\partial x})^2 + (\frac{\partial u_2}{\partial y})^2 + 2\mu \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} + \frac{1-\mu}{2} (\frac{\partial u_1}{\partial y})^2$$

$$+ (1-\mu) \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} + \frac{1-\mu}{2} (\frac{\partial u_2}{\partial x})^2 + \frac{1}{\varepsilon} (u_1)^2 + \frac{2}{\varepsilon} u_1 \frac{\partial u_3}{\partial x}$$

$$+ \frac{1}{\varepsilon} (\frac{\partial u_3}{\partial x})^2 + \frac{1}{\varepsilon} (u_2)^2 + \frac{2}{\varepsilon} u_2 \frac{\partial u_3}{\partial y} + \frac{1}{\varepsilon} (\frac{\partial u_3}{\partial y})^2]\, dx\, dy .$$

Section 5.2 : Construction of the element stiffness matrix

To construct the element matrix $\mathbf{K}^{(e)}$ from

$$B_\varepsilon(U^{(e)}, U^{(e)}) = \mathbf{q}^T \ \mathbf{K}^{(e)} \ \mathbf{q}$$

when $\mathbf{q} = [\, q_1 \ \ q_2 \ \ q_3 \ \dots \ q_{18} \,]^T$, we can construct a matrix $\mathbf{R}$ and let $\mathbf{K}^{(e)} = 1/2\,(\mathbf{R} + \mathbf{R}^T)$.

We first assume that iniatially all elements in the matrix $\mathbf{R}$ are zero and then proceed to assemble the matrix elements $\mathbf{R}(m,n)$ by

$$\mathbf{R}(m,n) = \text{previously defined } \mathbf{R}(m,n) + \text{Integral},$$

or

$$\Delta\mathbf{R}(m,n) = \mathbf{R}(m,n) - \text{previously defined } \mathbf{R}(m,n) = \text{Integral}.$$

The changes in the matrix elements $\Delta\mathbf{R}(m,n)$ are given below:

Let $m = 3i - 2$ and $n = 3j - 2$, where $i, j = 1, 2, 3, \dots, 6$. Then

$$\Delta\mathbf{R}(m,n) = \iint_{(E)} \frac{\partial\varphi_i}{\partial x} \frac{\partial\varphi_j}{\partial x} \, |\det J| \ dL_2\, dL_3 .$$

Let $m = 3i - 1$ and $n = 3j - 1$, where $i, j = 1, 2, 3, \dots, 6$. Then

$$\Delta\mathbf{R}(m,n) = \iint_{(E)} \frac{\partial\varphi_i}{\partial y} \frac{\partial\varphi_j}{\partial y} \, |\det J| \ dL_2\, dL_3 .$$

Let $m = 3i - 2$ and $n = 3j - 1$, where $i, j = 1, 2, 3, \dots, 6$. Then

$$\Delta\mathbf{R}(m,n) = 2\mu \iint_{(E)} \frac{\partial\varphi_i}{\partial x} \frac{\partial\varphi_j}{\partial y} \, |\det J| \ dL_2\, dL_3 .$$

Let m= 3i -2 and n= 3j - 2, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta R(m,n) = \frac{1-\mu}{2} \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \, |\det J| \; dL_2 \, dL_3 \; .$$

Let m= 3i - 2 and n= 3j - 1, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta R(m,n) = (1-\mu) \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \, |\det J| \; dL_2 \, dL_3 \; .$$

Let m= 3i - 1 and n= 3j - 1, where i, j= 1, 2, 3, . . . , 6. Then

$$\Delta R(m,n) = \frac{1-\mu}{2} \iint_{(E)} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} \, |\det J| \; dL_2 \, dL_3 \; .$$

Let m= 3i - 2 and n= 3j - 2, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta R(m,n) = \frac{1}{\varepsilon} \iint_{(E)} N_i N_j \, |\det J| \; dL_2 \, dL_3 \; .$$

Let m= 3i - 2 and n= 3j, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta R(m,n) = \frac{2}{\varepsilon} \iint_{(E)} N_i \frac{\partial \varphi_i}{\partial x} \, |\det J| \; dL_2 \, dL_3 \; .$$

Let m= 3i and n= 3j, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta \mathbf{R}(m,n) = \frac{1}{\varepsilon} \iint_{(E)} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} \,|\det J|\, dL_2\, dL_3 .$$

Let m= 3i - 1 and n=3j - 1, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta \mathbf{R}(m,n) = \frac{1}{\varepsilon} \iint_{(E)} N_i N_j \,|\det J|\, dL_2\, dL_3 .$$

Let m= 3i -1 and n= 3j, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta \mathbf{R}(m,n) = \frac{2}{\varepsilon} \iint_{(E)} N_i \frac{\partial \varphi_i}{\partial y} \,|\det J|\, dL_2\, dL_3 .$$

Let m= 3i and n=3j, where i, j = 1, 2, 3, . . . , 6. Then

$$\Delta \mathbf{R}(m,n) = \frac{1}{\varepsilon} \iint_{(E)} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \,|\det J|\, dL_2\, dL_3 .$$

Section 5.3 : Construction of the load vector

The element load vector $\mathbf{f}^{(e)}$ is defined by

$$\mathbf{q}^T \mathbf{f}^{(e)} = p_L(F, U^{(e)})$$

$$= \iint_{(e)} f\, u_3^{(e)}\, dx\, dy$$

$$= [\, q_3\ q_6\ q_9\ q_{12}\ q_{15}\ q_{18}\,] \iint_{(E)} f \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} |\det J|\ dL_2\, dL_3 , \qquad (5.17)$$

where

$$\mathbf{f}^{(e)} = \left[\, f_1\ \ f_2\ \ f_3 \cdots f_{18} \right]^T ,$$

$$\mathbf{q} = \left[\, q_1\ q_2\ q_3 \cdots\ q_{18} \right]^T ,$$

and

$$f_m = \iint_{(E)} f\, N_i\ |\det J|\ \ dL_2\, dL_3 \ , \quad \text{where } i = 1, 2, 3, \ldots, 6 \text{ and } m = 3i \qquad (5.18)$$

$$f_m = 0 \qquad\qquad , \quad \text{otherwise.}$$

An algorithm can be outlined as follows:

1. Place 18 indices in each element. Each node has three indices. Work horizontally to the right along each row to reduce bandwidth in the stiffness matrix. Then give x and y coordinates for each node.

2. Compute the Jacobian of (5.12) for each element.

3. Use Gaussian quadrature to calculate the elements of matrix $\mathbf{R}$ and then the element stiffness matrix $\mathbf{K}^{(e)} = 1/2\,(\mathbf{R} + \mathbf{R}^T)$.

4. Use Gaussian quadrature to compute the element load vector $\mathbf{f}^{(e)}$.

5. Assemble the global stiffness matrix $\mathbf{K}$ and global load vector $\hat{\mathbf{f}}$ as mentioned in the Section 4.3 .

6. Solve the matrix equation $\mathbf{K}\,\hat{\mathbf{q}} = \hat{\mathbf{f}}$.

Section 5.4: Examples

Let us consider a clamped circular plate with unit radius and use the methods in the previous section to construct the finite element solutions.

In [2, 4] the equations (2.14) have been written in the form

$$\frac{1}{2}[(1-\mu)\nabla^2\vec{\psi}+(1+\mu)\nabla(\nabla\bullet\vec{\psi})-\frac{1}{\varepsilon}(\vec{\psi}+\nabla w)=0,$$

$$\frac{1}{\varepsilon}(\nabla^2 w+\nabla\bullet\vec{\psi})=-f,$$

$$\text{in } \Omega, \tag{5.19}$$

with the boundary conditions

$$W=\vec{\psi}=0, \quad \text{on } \partial\Omega,$$

where $\vec{\psi}=(\psi_x,\psi_y)$.

Let $\vec{\psi}=\psi_r\vec{e}_r+\psi_\theta\vec{e}_\theta$ in polar coordinates (r,θ).

Since

$$\vec{e}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j},$$

$$\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j},$$

we have

$$\begin{bmatrix}\psi_r\\ \psi_\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\psi_x\\ \psi_y\end{bmatrix}. \tag{5.20}$$

Equations (5.19) in polar coordinates become

$$\frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} + \frac{1}{2r^2}\frac{\partial^2 \psi_r}{\partial \theta^2} - \frac{1}{r^2}\psi_r - \frac{3}{2r^2}\frac{\partial \psi_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_\theta}{\partial r \partial \theta}$$

$$+\mu\left(-\frac{1}{2r^2}\frac{\partial^2 \psi_r}{\partial \theta^2} + \frac{1}{2r^2}\frac{\partial \psi_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_\theta}{\partial r \partial \theta}\right) - \frac{1}{\varepsilon}\left(\psi_r + \frac{\partial w}{\partial r}\right) = 0 \,,$$

(5.21 a)

$$\frac{1}{2}\frac{\partial^2 \psi_\theta}{\partial r^2} + \frac{1}{2r}\frac{\partial \psi_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_\theta}{\partial \theta^2} - \frac{1}{2r^2}\psi_\theta + \frac{3}{2r^2}\frac{\partial \psi_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_r}{\partial r \partial \theta}$$

$$+\mu\left(-\frac{1}{2}\frac{\partial^2 \psi_\theta}{\partial \theta^2} - \frac{1}{2r}\frac{\partial \psi_\theta}{\partial r} + \frac{1}{2r^2}\psi_\theta - \frac{1}{2r^2}\frac{\partial \psi_r}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_r}{\partial r \partial \theta}\right)$$

$$-\frac{1}{\varepsilon}\left(\psi_\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = 0 \,,$$

(5.21 b)

$$\frac{1}{\varepsilon}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \psi_r}{\partial r} + \frac{1}{r}\psi_r + \frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta}\right) = -f \,,$$

(5.21 c)

and the boundary conditions become

$$w = \psi_r = \psi_\theta = 0 \quad \text{at} \quad r=1. \qquad (5.21\ d)$$

Example 5.1

For axisymmetric solutions $\psi_\theta = 0$, and the functions ψ_r and w_ε are functions of r only. The equations degenerate into

$$\frac{d^2\psi_r}{dr^2} + \frac{1}{r}\frac{d\psi_r}{dr} - \frac{1}{r^2}\psi_r - \frac{1}{\varepsilon}\left(\psi_r + \frac{dw}{dr}\right) = 0 ,$$

(5.22)

$$\frac{1}{\varepsilon}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{d\psi_r}{dr} + \frac{1}{r}\psi_r\right) = -f .$$

Taking f= 1, we have the following solutions.

$$w_\varepsilon = \frac{1}{64}(1-r^2)^2 + \frac{\varepsilon}{4}(1-r^2) ,$$

$$\psi_r = \frac{r}{16}(1-r^2).$$

(5.23)

The corresponding problem in the classical plate theory is

$$\nabla^4 w_0 = 1, \qquad r < 1$$

$$w_0 = \frac{dw_0}{dr} = 0, \quad \text{at } r = 1, \tag{5.24}$$

with the solution

$$w_0 = \frac{1}{64}(1-r^2)^2. \tag{5.25}$$

Thus in this example we have

$$w_\varepsilon = w_0 + \frac{\varepsilon}{4}(1-r^2) , \quad \psi_r = -\frac{dw_0}{dr} , \quad \text{and } \psi_\theta = 0 . \tag{5.26}$$

Example 5.2 :

In [2, 4] , with $f = \cos\theta$, we have the following solutions for the equations (5.21).

$$w_\varepsilon = \{ \frac{r}{90}(1-r)^2(2r+1) + \varepsilon r(1-r)[\tilde{a}(r+1) - \frac{r}{3}]\}\cos\theta$$

(5.27 a)

$$\psi_r + \frac{\partial w}{\partial r} = \{ \varepsilon(\frac{11-30\tilde{a}}{15r} \frac{I_1(\alpha r)}{I_1(\alpha)} - \frac{2r}{3} + \frac{4}{15})$$

$$+ \frac{8\varepsilon^2}{3}(\frac{-3\tilde{a}+1}{r} \frac{I_1(\alpha r)}{I_1(\alpha)} - 1 + 3\tilde{a})\}\cos\theta ,$$

(5.27 b)

$$\psi_\theta + \frac{\partial w}{\partial \theta} = \{ \varepsilon^{\frac{1}{2}} \frac{\sqrt{2}}{\sqrt{1-\mu}} \frac{30\tilde{a}-11}{15} \frac{I_0(\alpha r)}{I_1(\alpha)}$$

$$- \varepsilon(\frac{30\tilde{a}-11}{15r} \frac{I_1(\alpha r)}{I_1(\alpha)} - \frac{5r-4}{15})$$

$$- \frac{8}{3}(1-3\tilde{a})\varepsilon^{\frac{3}{2}} \frac{\sqrt{2}}{\sqrt{1-\mu}} \frac{I_0(\alpha r)}{I_1(\alpha)}$$

$$+ \frac{8}{3}(1-3\tilde{a})\varepsilon^2 (\frac{I_1(\alpha r)}{r I_1(\alpha)} + 1)\}\sin\theta ,$$

(5.27 c)

where

$$\alpha = \frac{\sqrt{2}}{\sqrt{1-\mu}} \varepsilon^{-\frac{1}{2}} ,$$

and

$$\tilde{a} = \frac{1}{\varepsilon} \frac{-\frac{4}{5}\alpha^{-1}\varepsilon - \frac{16}{3}\alpha^{-1}\varepsilon^2 + \frac{I_0(\alpha)}{I_1(\alpha)}\left(\frac{11}{15}\varepsilon + \frac{8}{3}\varepsilon^2\right)}{(2+8\varepsilon)\frac{I_0(\alpha)}{I_1(\alpha)} - 2\alpha^{-1} - 16\alpha^{-1}\varepsilon} , \qquad (5.28)$$

where I_0 and I_1 are modified Bessel functions of the first kind of order zero and one respectively.

The following expansions will be used to compute the values of $I_0(x)$ and $I_1(x)$.

$$I_0(x) = 1 + \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \frac{x^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} + \ldots \qquad (5.29\text{ a})$$

and

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^3\, 1!\, 2!} + \frac{x^5}{2^5\, 2!\, 3!} + \frac{x^7}{2^7\, 3!\, 4!} + \frac{x^9}{2^9\, 4!\, 5!} + \ldots \qquad (5.29\text{ b})$$

The corresponding problem in the classical plate is

$$\nabla^4 w_0 = \cos\theta, \qquad r < 1,$$
$$w_0 = \frac{\partial w_0}{\partial r} = \frac{\partial w_0}{\partial \theta} = 0, \quad \text{at } r = 1, \qquad (5.30)$$

and the solutions are

$$w_0 = \frac{r}{90}(1-r)^2(2r+1)\cos\theta ,$$
$$\frac{\partial w_0}{\partial r} = -\frac{1}{90}(1-r)(8r^2-r-1)\cos\theta , \qquad (5.31)$$
$$\frac{\partial w_0}{\partial \theta} = -\frac{r}{90}(1-r)^2(2r+1)\sin\theta .$$

From these solutions we have

$$w_0\,(0.5, 0.75) = 0.00015142,$$

$$w_0\,(0.5, 0.5) = 0.00115059,$$

$$w_0\,(0.125, 0.125) = 0.001274025,$$

$$-\frac{\partial w_0}{\partial r}\,(0.5, 0.75) = 0.00279494,$$

$$-\frac{\partial w_0}{\partial r}\,(0.5, 0.5) = 0.00527638,$$

$$-\frac{\partial w_0}{\partial r}\,(0.125, 0.125) = -0.00599426,$$

$$-\frac{1}{r}\frac{\partial w_0}{\partial \theta}\,(0.5, 0.75) = 0.0002519,$$

$$-\frac{1}{r}\frac{\partial w_0}{\partial \theta}\,(0.5, 0.5) = 0.0016272,$$

$$-\frac{1}{r}\frac{\partial w_0}{\partial \theta}\,(0.125, 0.125) = 0.0072068.$$

In Example 5.1 the load function is symmetric in x and y. So the solutions w_ε, ψ_r, and w_0 are also expected to be symmetric in x and y. Numerical solutions can be obtained over the first quadrant. Using the results in Sections 5.1, 5.2, and 5.3, we can obtain numerical results of u_1, u_2, and u_3 in rectangular coordinates. We can then use the equation (5.20) to convert the solutions to polar coordinates. Since u_1 must be odd in x and even in y, the boundary condition at $x = 0$ is $u_1 = 0$. Similarly the boundary condition at $y = 0$ is $u_2 = 0$.

In Example 5.2 the load function $f = \cos\theta$ is symmetric with respect to the x-axis and anti-symmetric with respect to the y-axis. The solutions w_ε, ψ_r, w_0, and $-\partial w_0/\partial r$ are symmetric with respect to the x-axis and anti-symmetric with respect to the y-axis. The solutions ψ_θ and $-(1/r)\partial w_0/\partial\theta$ are symmetric with respect to the y-axis and anti-symmetric with respect to the x-axis. Numerical solutions can be obtained over the first quadrant. The boundary conditions $u_1 = u_2 = u_3 = 0$ are imposed at $r = 1$, and u_3 = "zero" at $x = 0$. Because $-\partial w_\varepsilon/\partial y$ does not exist at the origin, the boundary condition $u_2 = 0$ on the x-axis and the y-axis should be imposed except at the origin.

We note the solutions given in (5.27) for example 5.2 are not valid for $\mu = 1$ since α then becomes undefined. An examination of the method of solution reveals there is a reduction of order in the govering differential equations when $\mu = 1$ and hence the resulting solutions can not satisfy all the boundary conditions. This nonexistence of a classical solution when $\mu = 1$ appeas to be related to the loss coercivity discussed in Chapter 2.

Finite element solutions of these two examples are obtained over the first quadrant of the unit circle for mesh sizes $h = 1/4$ and $h = 1/8$ and are given in Figures 5.4 and 5.5. Most of elements taken are similar to the type 1 and type 2 elements in Chapter 4. These element stiffness matrices are the same matrices derived before. Numerical results are given in Tables 5.1A-5.5B. For ε larger than 2^{-6} both finite element solutions of h = 1/4 and h = 1/8 are very close to the solutions of w_ε, ψ_r, and ψ_θ . But some sharper angles in one curved side elements of h = 1/8 produce relatively larger errors. When ε becomes small, the finite element solutions of h = 1/4 are not close to the solutions of w_ε, ψ_r, and ψ_θ. This is due to the fact that the error bounds contain a factor of $\varepsilon^{-1/2}h^{l-1}$ mentioned in Chapter 3.

In Example 5.2 $-\partial w_\varepsilon/\partial x$ and $-\partial w_\varepsilon/\partial y$ do not exist at the origin. According to the equations (5.27b) and (5.27c) ψ_r and ψ_θ do not exist at the origin. The finite element solutions at the origin can be regarded as approximations of $(-\partial w_0/\partial x)(0, 0) = (-\partial w_0/\partial r)(0, 0) = -1/90 = -0.0111111$ and $(-\partial w_0/\partial y)(0, 0) = (-\partial w_0/\partial \theta)(0, 0) = 0$. In fact we have $u_1 = -0.0109444$ and $u_2 = -0.0000739$ at the origin when $\varepsilon = 2^{-8}$ and h = 1/4 . The values of u_1 and u_2 at the point (0.125, 0.125) have similar approximate properties. In polar coordinates we have $(-\partial w_0/\partial r)(0.125, 0.125) = -0.0059943$ and $(1/r)\,(-\partial w_0/\partial \theta)(0.125, 0.125) = 0.00720904$ and the finite element solutions of ψ_r and ψ_θ are -0.00609793 and 0.0070264, respectively. when $\varepsilon = 2^{-8}$ and h = 1/4 . All finite element solusions of w_ε, ψ_r, and ψ_θ at the points of (0.125, 0.125), (0.5, 0.5) and (0.5, 0.75) are given in tables 5.1A-5.3C.

In Example 5.1 the finite element solutions at the points of (0.5, 0.5) and (0.5, 0.75) are given in Tables 5.4A-5.5B. For function f = 1 the solutions w_ε and ψ_r have simple expressions in (5.23) and $\psi_\theta = 0$. Also ψ_r and ψ_θ are independent of ε . Thus the finite element solutions are very close to ψ_r and ψ_θ even when $\varepsilon = 1/2$ for both h = 1/4 and h = 1/8. These finite element approximations are reliable.

Examples 5.1 and 5.2 show how the solutions w_ε, ψ_r, and ψ_θ approach the classical plate solutions w_0, $-\partial w_0/\partial r$, and $(-1/r)\,\partial w_0/\partial\theta$ and finite element solutions give approximations to w_ε, ψ_r, and ψ_θ. Numerical results in the tables show that we have excellent approximations for each ε when the mesh size h is as large as 1/8.

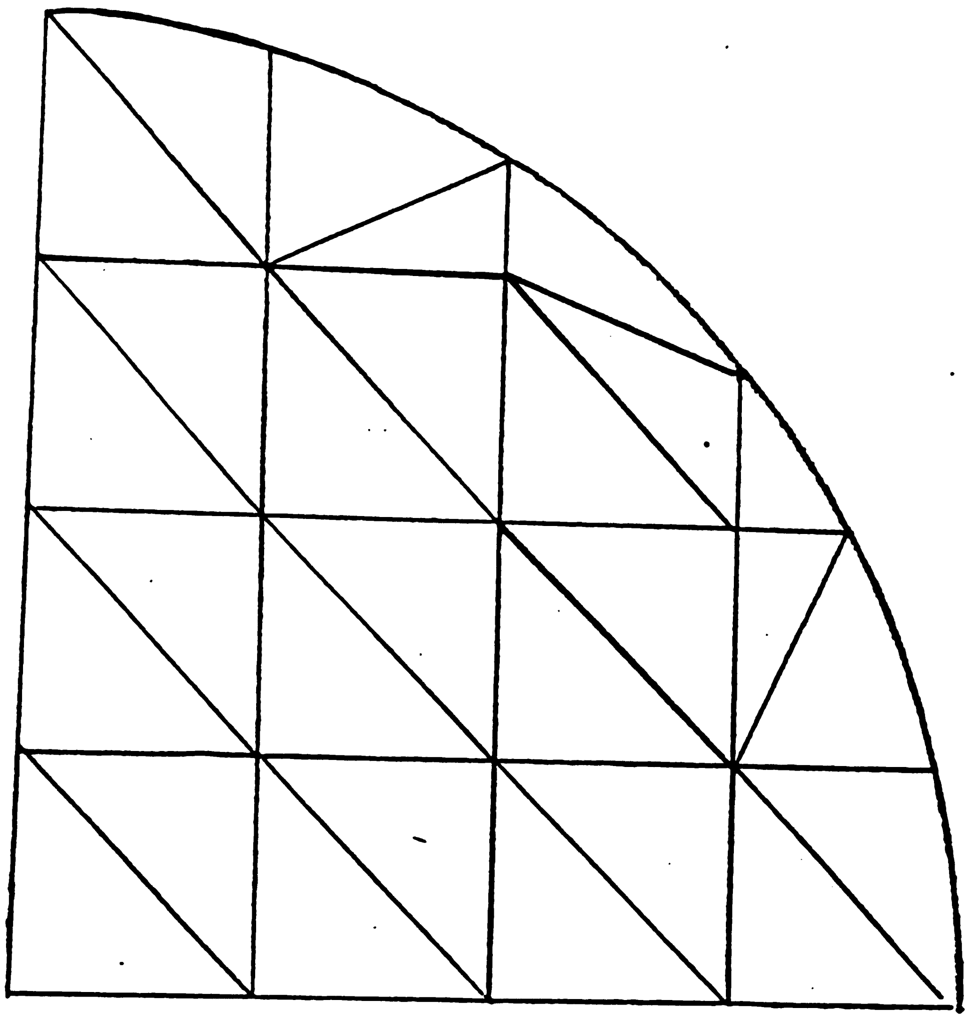
In these two examples the extrapolation method mentioned in Chapter 4 can be used. In Example 5.1 with h = 1/4. We have

$$w_1 = 0.00348261 \;, \varepsilon = 2^{-4} \;\text{ and }\; w_2 = 0.00128379 \;, \varepsilon = 2^{-6}.$$

By extrapolation we obtain

$$w = \frac{2^2 w_2 - w_1}{2^2 - 1} = 0.0005509.$$

which is very close to $w_0(0.5, 0.75) = 0.0005493$.



$h = \frac{1}{4}$

Figure 5.4

Isoparametric finite element approximations of the circular plate at the point (0.125, 0.125) with load function $f(x, y) = \cos\theta$.

ε	W_ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.01894870	0.01882121	0.01896802
2^{-2}	0.01025469	0.01018327	0.01026548
2^{-3}	0.00584171	0.00579844	0.00584877
2^{-4}	0.00359111	0.00356161	0.00359710
2^{-5}	0.00244403	0.00241906	0.00244943
2^{-6}	0.00186238	0.00183422	0.00186646
2^{-7}	0.00156910	0.00152988	0.00157119
2^{-8}	0.00142179	0.00136148	0.00142121
2^{-9}	0.00134797	0.00125327	0.00134346
2^{-10}	0.00131101	0.00116630	0.00130031
2^{-11}	0.00129252	0.00108152	0.00127230
2^{-12}	0.00128328	0.00098852	0.00124955
2^{-13}	0.00127865	0.00087923	0.00122695
2^{-14}	0.00127634	0.00074841	0.00120023
2^{-15}	0.00127518	0.00059819	0.00116315

Table 5.1A

Isoparametric finite element approximations of the circular plate at the point (0.125, 0.125) with load function $f(x, y) = \cos\theta$.

ε	Ψ_r	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	-0.01730272	-0.01728923	-0.01733225
2^{-2}	-0.01445521	-0.01444185	-0.01447946
2^{-3}	-0.01160284	-0.01158944	-0.01162450
2^{-4}	-0.00933695	-0.00932309	-0.00936016
2^{-5}	-0.00784221	-0.00782269	-0.00786812
2^{-6}	-0.00696981	-0.00693075	-0.00699493
2^{-7}	-0.00649605	-0.00641630	-0.00651709
2^{-8}	-0.00624881	-0.00609793	-0.00626473
2^{-9}	-0.00612246	-0.00585611	-0.00613285
2^{-10}	-0.000605860	-0.00561690	-0.00606206
2^{-11}	-0.00602649	-0.00533137	-0.00601976
2^{-12}	-0.00601039	-0.00495466	-0.00598762
2^{-13}	-0.00600232	-0.00444284	-0.00595376
2^{-14}	-0.00599829	-0.00377916	-0.00590506
2^{-15}	-0.00599627	-0.00300172	-0.00581741

Table 5.1B

Isoparametric finite element approximations of the circular plate at the point (0.125, 0.125) with load function $f(x, y) = \cos\theta$.

ε	Ψ_θ	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.01858303	0.01851573	0.01861731
2^{-2}	0.01575124	0.01568154	0.01578038
2^{-3}	0.01289510	0.01282119	0.01292221
2^{-4}	0.01060798	0.01052471	0.01063709
2^{-5}	0.00909002	0.00898253	0.00912139
2^{-6}	0.00820158	0.00803963	0.00823095
2^{-7}	0.00771865	0.00745363	0.00774179
2^{-8}	0.00746656	0.00702640	0.00748062
2^{-9}	0.00733772	0.00662840	0.00733875
2^{-10}	0.00727259	0.00619646	0.00725289
2^{-11}	0.00723985	0.00571501	0.00718613
2^{-12}	0.00722343	0.00517546	0.00711559
2^{-13}	0.00721521	0.00455281	0.00702296
2^{-14}	0.00721109	0.00382768	0.00688435
2^{-15}	0.00720904	0.00302217	0.00666247

Table 5.1C

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.5) with load function $f(x, y) = \cos\theta$.

ε	W_ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.02664056	0.02665895	0.02667031
2^{-2}	0.01419147	0.01420201	0.01420858
2^{-3}	0.00783073	0.00783811	0.00784258
2^{-4}	0.00455929	0.00456584	0.00457019
2^{-5}	0.00287860	0.00288275	0.00288907
2^{-6}	0.00202152	0.00201877	0.00203010
2^{-7}	0.00158791	0.00157238	0.00159375
2^{-8}	0.00136973	0.00133298	0.00137259
2^{-9}	0.00126028	0.00118975	0.00125971
2^{-10}	0.00120547	0.00108479	0.00120024
2^{-11}	0.00117804	0.00098927	0.00116612
2^{-12}	0.00116432	0.00089021	0.00114337
2^{-13}	0.00115746	0.00078210	0.00112548
2^{-14}	0.00115402	0.00066252	0.00110912
2^{-15}	0.00115231	0.00053254	0.00109085

Table 5.2A

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.5) with load function $f(x, y) = \cos\theta$.

ε	Ψ_r	$h = \frac{1}{4}$	$h = \frac{1}{8}$
2^{-1}	-0.00069060	-0.00069533	-0.00069441
2^{-2}	0.00074963	0.00074395	0.00074827
2^{-3}	0.00223101	0.00222278	0.00223137
2^{-4}	0.00344329	0.00343242	0.00344627
2^{-5}	0.00425958	0.00424693	0.00426725
2^{-6}	0.00473953	0.00472200	0.00475085
2^{-7}	0.00500039	0.00497054	0.00501253
2^{-8}	0.00513643	0.00508338	0.00514760
2^{-9}	0.00520591	0.00511426	0.00521526
2^{-10}	0.00524102	0.00508884	0.00524777
2^{-11}	0.00525866	0.00501224	0.00526154
2^{-12}	0.00526751	0.00487056	0.00526375
2^{-13}	0.00527194	0.00463233	0.00525474
2^{-14}	0.00527416	0.00425321	0.00522784
2^{-15}	0.00527527	0.00368881	0.00516900

Table 5.2B

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.5) with load function $f(x, y) = \cos\theta$.

ε	Ψ_0	$h = \frac{1}{4}$	$h = \frac{1}{8}$
2^{-1}	0.00819676	0.00820638	0.00822171
2^{-2}	0.00694528	0.00696058	0.00697237
2^{-3}	0.00550204	0.00553712	0.00554577
2^{-4}	0.00413592	0.00420148	0.00420860
2^{-5}	0.00308704	0.00316091	0.00316921
2^{-6}	0.00241726	0.00246716	0.00248124
2^{-7}	0.00203810	0.00205017	0.00207879
2^{-8}	0.00183668	0.00180303	0.00186120
2^{-9}	0.00173295	0.00163890	0.00174724
2^{-10}	0.00168032	0.00150392	0.00168647
2^{-11}	0.00165382	0.00136424	0.00165058
2^{-12}	0.00164052	0.00119196	0.00162354
2^{-13}	0.00163385	0.00096699	0.00159455
2^{-14}	0.00163052	0.00069366	0.00155367
2^{-15}	0.00162885	0.00041387	0.00148929

Table 5.2C

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.75) with load function $f(x, y) = \cos\theta$.

ε	W_ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.00877493	0.00871188	0.00877948
2^{-2}	0.00457413	0.00454255	0.00457701
2^{-3}	0.00242266	0.00240765	0.00242513
2^{-4}	0.00131278	0.00130689	0.00131568
2^{-5}	0.00074097	0.00073963	0.00074412
2^{-6}	0.00044879	0.00044876	0.00045148
2^{-7}	0.00030080	0.00030000	0.00030264
2^{-8}	0.00022629	0.00022305	0.00022721
2^{-9}	0.00018890	0.00018112	0.00018879
2^{-10}	0.00017017	0.00015482	0.00016872
2^{-11}	0.00016080	0.00013410	0.00015748
2^{-12}	0.00015611	0.00011453	0.00015038
2^{-13}	0.00015376	0.00009531	0.00014545
2^{-14}	0.00015259	0.00007699	0.00014209
2^{-15}	0.00015200	0.00005999	0.00013990

Table 5.3A

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.75) with load function $f(x, y) = \cos\theta$.

ε	Ψ_r	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.00101174	0.00098112	0.00101022
2^{-2}	0.00142650	0.00139304	0.00142550
2^{-3}	0.00186024	0.00182212	0.00185890
2^{-4}	0.00222385	0.00217982	0.00222177
2^{-5}	0.00247483	0.00242691	0.00247354
2^{-6}	0.00262498	0.00256844	0.00262565
2^{-7}	0.00270733	0.00263675	0.00270960
2^{-8}	0.00275046	0.00265002	0.00275370
2^{-9}	0.00277253	0.00261389	0.00277678
2^{-10}	0.00278370	0.00252270	0.00278976
2^{-11}	0.00278931	0.00236576	0.00279868
2^{-12}	0.00279212	0.00213722	0.00280725
2^{-13}	0.00279353	0.00184647	0.00281775
2^{-14}	0.00279424	0.00151735	0.00283002
2^{-15}	0.00279459	0.00118444	0.00283973

Table 5.3B

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.75) with load function $f(x, y) = \cos\theta$.

ε	Ψ_θ	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.00338160	0.00340160	0.00339152
2^{-2}	0.00292723	0.00295447	0.00294291
2^{-3}	0.00234684	0.00240386	0.00239126
2^{-4}	0.00171144	0.00183101	0.00181881
2^{-5}	0.00114928	0.00132104	0.00131179
2^{-6}	0.00075366	0.00092092	0.00091819
2^{-7}	0.00051738	0.00063471	0.00064286
2^{-8}	0.00038845	0.00044258	0.00046587
2^{-9}	0.00032117	0.00031850	0.00035928
2^{-10}	0.00028681	0.00023867	0.00029651
2^{-11}	0.00026945	0.00018245	0.00025691
2^{-12}	0.00026073	0.00013076	0.00022801
2^{-13}	0.00025636	0.00006910	0.00020472
2^{-14}	0.00025417	-0.00000273	0.00018539
2^{-15}	0.00025307	-0.00007211	0.00016775

Table 5.3C

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.75) with load function f(x, y) = 1.

ε	W_ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.02398682	0.02398235	0.02400185
2^{-2}	0.01226807	0.01226862	0.01227741
2^{-3}	0.00640868	0.00641154	0.00641508
2^{-4}	0.00347900	0.00348261	0.00348378
2^{-5}	0.00201416	0.00201749	0.00201796
2^{-6}	0.00128174	0.00128379	0.00128483
2^{-7}	0.00091553	0.00091494	0.00091790
2^{-8}	0.00073242	0.00072692	0.00073377
2^{-9}	0.00064087	0.00062656	0.00064046
2^{-10}	0.00059509	0.00056593	0.00059165
2^{-11}	0.00057220	0.00052011	0.00056396
2^{-12}	0.00056076	0.00047652	0.00054594
2^{-13}	0.00055504	0.00042914	0.00053267
2^{-14}	0.00055218	0.00037434	0.00052233
2^{-15}	0.00055075	0.00030986	0.00051327

Table 5.4A

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.75) with load function f(x, y) = 1.

	$\psi_r = 0.01056314$		$\psi_\theta = 0.0$	
ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.01055426	0.01056760	0.00004867	0.00000513
2^{-2}	0.01055342	0.01056827	0.00004797	0.00000529
2^{-3}	0.01055163	0.01056926	0.00004645	0.00000548
2^{-4}	0.01054764	0.01057049	0.00004313	0.00000561
2^{-5}	0.01053907	0.01057169	0.00003624	0.00000548
2^{-6}	0.01052135	0.01057273	0.00002284	0.00000508
2^{-7}	0.01048786	0.01057383	-0.00000169	0.00000463
2^{-8}	0.01042952	0.01057556	-0.00004315	0.00000412
2^{-9}	0.01033152	0.01057820	-0.00010514	0.00000280
2^{-10}	0.01016553	0.01058121	-0.00018656	-0.00000060
2^{-11}	0.00988111	0.01058288	-0.00028514	-0.00000767
2^{-12}	0.00940784	0.01058026	-0.00040183	-0.00001982
2^{-13}	0.00867611	0.01056808	-0.00053483	-0.00003698
2^{-14}	0.00764197	0.01053653	-0.00066205	-0.00005693
2^{-15}	0.00631681	0.01046936	-0.00073213	-0.00007718

Table 5.4B

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.5) with load function f(x, y) = 1.

ε	W_ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.06640625	0.06641678	0.06646851
2^{-2}	0.03515625	0.03516582	0.03519169
2^{-3}	0.01953125	0.01953892	0.01955312
2^{-4}	0.01171875	0.01172271	0.01173357
2^{-5}	0.00781250	0.00780924	0.00782338
2^{-6}	0.00585938	0.00584231	0.00586761
2^{-7}	0.00488281	0.00484003	0.00488858
2^{-8}	0.00439453	0.00430597	0.00439702
2^{-9}	0.00415039	0.00398609	0.00414755
2^{-10}	0.00402832	0.00374998	0.00401655
2^{-11}	0.00396729	0.00353175	0.00394157
2^{-12}	0.00393677	0.00329506	0.00389181
2^{-13}	0.00392151	0.00301100	0.00385300
2^{-14}	0.00391388	0.00265203	0.00381764
2^{-15}	0.00391006	0.00220443	0.00377767

Table 5.5A

Isoparametric finite element approximations of the circular plate at the point (0.5, 0.5) with load function f(x, y) = 1.

Ψ_r = 0.02209709 Ψ_θ = 0.0

ε	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{4}$	$h=\frac{1}{8}$
2^{-1}	0.02209481	0.02211521	-0.00000775	0.00001337
2^{-2}	0.02209451	0.02211662	-0.00000679	0.00001452
2^{-3}	0.02209387	0.02211892	-0.00000545	0.00001611
2^{-4}	0.02209242	0.02212212	-0.00000423	0.00001781
2^{-5}	0.02208903	0.02212536	-0.00000460	0.00001939
2^{-6}	0.02208109	0.02212886	-0.00000941	0.00001935
2^{-7}	0.02206288	0.02213091	-0.00002344	0.00001915
2^{-8}	0.02202380	0.02213194	-0.00005436	0.00001912
2^{-9}	0.02194522	0.02213222	-0.00011416	0.00001999
2^{-10}	0.02179272	0.02213162	-0.00022018	0.00002240
2^{-11}	0.02150275	0.02212855	-0.00039263	0.00002709
2^{-12}	0.02096521	0.02211786	-0.00064347	0.00003482
2^{-13}	0.02000483	0.02208736	-0.00095827	0.00004644
2^{-14}	0.01838980	0.02201301	-0.00127511	0.00006302
2^{-15}	0.01591966	0.02185068	-0.00147264	0.00008565

Table 5.5B

Chapter 6 Discussions and conclusions

Plate bending problems in engineering mechanics are governed in the classical plate theory by the well known nonhomogeneous biharmonic equations. When finite element methods are used to obtain numerical solutions to such fourth order equations, globally C^1 functions must be employed. This excessive smoothness requirement on the trial functions may be eased by treating the plate deflection and its two first partial derivatives as separate unknowns via a penalty-function argument. In this new formulation one works with trial functions in the space $(C^0)^3$, and this was the idea proposed in [34] by Westbrook.

In [34] the perturbed energy integral is constructed from the classical bending energy integral, and this perturbed energy integral corresponds to the energy integral in the improved plate theory that incorporates the effect of shear deformation. The new problem, which consists of a set of three second order partial differential equations, is singularly perturbed with respect to the penalty perturbation ε in that as ε tends to zero one recovers the single fourth order equation in the classical plate theory. Some consequences of this singular perturbation nature of the problem such as the nonuniformity of convergence and appearance of boundary layers solutions have recently been studied [1-4].

The energy integrals above contain Poisson's ratio μ as a general parameter, though the range of μ must be restricted for the perturbed problem to remain elliptic or coercive. In his work [24] on nonconforming finite element methods mentioned before Nitsche also arrived at one version of our present formulation. Nitsche started out with a simplified form of the classical elastic bending energy which when perturbed is not coercive. To circumvent this difficulty he performed integration by parts to arrive at an alternative form of the classical energy before the penalty term was added. Although Poissson's ratio is not present in Nitsche's work it can be seen that it corresponds to a special case of the present formulation with $\mu = -1$ in the latter. We can also show that as $\mu \rightarrow 1$ coercivity of the perturbed energy is lost.

Behaviors of the perturbed problems as $\varepsilon \to 0$ have been investigated in this dissertation in terms of various Sobolev norms. Numerical results have been obtained that serve to verify these error bounds. The method has been applied to both square and circular plates using linear and quadratic shape functions and in the case of circular plates isoparametric transformations are made to treat curved boundaries. We find that the method is easy to use, gives good results and is not sensitive to changes in μ. The errors follow closely those predicted in the dissertation for the type of shape functions used. we plan to conduct further test cases using higher order shape elements such as the cubic and apply the method to plates with other boundary geometries.

It was pointed out in [1-4] that the class of mathematical problems arising from the penalty function approach is usually ill-conditioned as $\varepsilon \to 0$. As a result a small interval about $\varepsilon = 0$ must be avoided and some form of extrapolation is necessary. The numerical results reported here suggest that this difficulty is not severe.

The method presented here can be applied to higher dimensional problems. For example for three-dimensional problems we let

$$U = (u_1, u_2, u_3, u_4) = (\psi_x, \psi_y, \psi_z, w_\varepsilon)$$

and U is in $(H_0^1(\Omega))^4$. The perturbed energy integral is

$$J_\varepsilon(U) = B_\varepsilon(U, U) - 2\, P_L(U, U)$$

where

$$B_\varepsilon(V, V) = \frac{1}{2} \iint_\Omega [\, (1+\mu) \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right)^2$$

$$+ (1-\mu)\, [\, \left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_3}{\partial z} \right)^2$$

$$-2\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial u_2}{\partial y}\right)-2\left(\frac{\partial u_2}{\partial y}\right)\left(\frac{\partial u_3}{\partial z}\right)$$

$$-2\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial u_3}{\partial z}\right)]$$

$$+(1-\mu)\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2+(1-\mu)\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)^2$$

$$+(1-\mu)\left(\frac{\partial u_2}{\partial z}+\frac{\partial u_3}{\partial y}\right)^2 \}\, dA$$

$$+\frac{1}{\varepsilon}\iint_\Omega \left(\frac{\partial u_4}{\partial x}+u_1\right)^2+\left(\frac{\partial u_4}{\partial y}+u_2\right)^2+\left(\frac{\partial u_4}{\partial z}+u_3\right)^2 dA$$

and

$$F(U)=\iint_\Omega f(x,y,z)\,u_4\; d\Omega .$$

The corresponding system of second order partial differential equations is

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_x+(1+\mu)\frac{\partial}{\partial x}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\frac{\partial\psi_z}{\partial z}\right)]-\frac{1}{\varepsilon}\left(\psi_x+\frac{\partial w_\varepsilon}{\partial x}\right)=0$$

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_y+(1+\mu)\frac{\partial}{\partial y}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\frac{\partial\psi_z}{\partial z}\right)]-\frac{1}{\varepsilon}\left(\psi_y+\frac{\partial w_\varepsilon}{\partial y}\right)=0$$

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_z+(1+\mu)\frac{\partial}{\partial z}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\frac{\partial\psi_z}{\partial z}\right)]-\frac{1}{\varepsilon}\left(\psi_z+\frac{\partial w_\varepsilon}{\partial z}\right)=0$$

$$\frac{1}{\varepsilon}\left(\nabla^2 w+\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\frac{\partial\psi_z}{\partial z}\right)=-f$$

in Ω,

and

$$\psi_x=\psi_y=\psi_z=w_\varepsilon=0 \text{ on } \partial\Omega.$$

Possible relationships to other works such as those by King [20], Falk [17] and Scholz [30] also deserve to be explored.

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