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ENERGY DECAY ESTIMATES FOR THE VON KARMAN PLATE EQUATIONS IN NONLINEAR ELASTICITY

By

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ABSTRACT

ENERGY DECAY ESTIMATES FOR THE VON KARMAN PLATE EQUATIONS IN NONLINEAR ELASTICITY

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This dissertation is concerned with the analysis of Saint-Venant edge effects for nonlinear elastic plates. The model used is based on the von Kármán plate equations: a coupled system of two nonlinear elliptic partial differential equations with the biharmonic operator as the principal part. Energy methods are used to establish a nonlinear integro-differential inequality for a quadratic functional. Arguments based on comparison theorems are then used to establish exponential decay of end effects. The results constitute a version of Saint-Venant's principle for nonlinear elastic plates.

Στην Καίτη Δακή

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CHAPTER 1

INTRODUCTION

Saint-Venant's principle in elasticity theory has had a long history. For long prismatic bars with traction-free lateral surface and subjected to end loads only, SAINT-VENANT suggested that the detailed mode of application and distribution of forces over the end of the cylinder is immaterial so that it is always possible, to some degree of approximation, to replace the applied system of forces by a statically equivalent one having the same resultant force and moment.

Much work has been done to justify this engineering principle, which lies at the foundations of elasticity theory and strength of materials. For an account of the major developments, up to 1972, concerning this issue for the *linear* theory of elasticity, see GURTIN [1].

In 1965 renewed efforts began on establishing a principle of Saint-Venant type for the original cylinder as well as for thin elastic bodies. In the framework of the linear theory of elasticity, the problem of comparing two statically equivalent systems of traction is reduced, by applying superposition, to the study of the stresses in the same body subject to arbitrary *self-equilibrated* tractions on part of the boundary with the remainder of the boundary traction-free. TOUPIN [2] and KNOWLES [3] made use of the connection between Saint-Venant's principle and the decay of strain-energy away from the loaded end of the body to provide estimates of the rate of stress decay. For a comprehensive review of work on Saint-Venant's principle up to 1983 see the survey by HORGAN and KNOWLES

[4].

As is discussed in [4], and also is clear from earlier work, the issues underlying the validity of a Saint-Venant principle are mathematical questions for the elliptic partial differential equations involved. Thus the *exponential* stress decay established in [2], [3], for example, demonstrate clearly the boundary-layer character of the solutions corresponding to a *self-equilibrated* load.

Compared to the amount of research carried out on the analysis of Saint-Venant's principle for *linear* elasticity, very little has been done to examine such questions for *nonlinear* elasticity. Clearly the issues are much more complicated here. A major difficulty arises since superposition no longer holds and so considering *self-equilibrated* loads is not sufficient. Furthermore, instabilities might have to be taken into account. Also the decay rate for end effects, even if exponential, might depend on the overall loading as well as on geometry. Early work on the nonlinear elasticity problem was carried out by ROSEMAN [5] and BREUER and ROSEMAN [6]. A review of this research and of other work up to 1983 on second-order nonlinear partial differential equations is given in [4]. HORGAN and KNOWLES [7] considered a Saint-Venant principle for *finite anti-plane shear* and obtained exponential decay rates depending on loading and on material. Since then BREUER and ROSEMAN [8] have considered the plane problem and KNOPS and PAYNE [9] the three-dimensional cylinder problem. See also [10] for results on small deformations superimposed on large for plane incompressible elasticity. Results for second-order quasilinear partial differential equations in two and three dimensions have been obtained recently in [11 - 14].

In this thesis, we are concerned with a nonlinear elasticity problem for plates modeled by the v. KARMAN [15] equations. The exponential decay of an energy-like quadratic functional involving the stresses and second derivatives of the deflections due to self-equilibrated tractions and prescribed deflections on part of its boundary can be regarded as a Saint-Venant principle type of result. The only previous work on questions like this for the von Kármán equations is that of KALANTAROV [16]. He established an upper bound for such an energy functional which is inversely proportional to the square of the distance from the loaded end for sufficiently large distances. We obtain stronger results involving an *exponential* decay of energy here.

Here in Chapter 2 we formulate the boundary value problem to be studied. The stresses and deflections in a semi-infinite thin elastic plate are described by the von Kármán system. The plate is clamped and is rendered traction-free along its long sides while self-equilibrated tractions and prescribed displacements are assigned to its left end. In Chapter 3 we discuss the linearized version of the problem posed, the biharmonic problem for a semi-infinite strip with traction-free lateral sides and self-equilibrated end loads. We review the technique of KNOWLES [3] to derive exponentially decaying upper bounds for an energy functional. These results are used by us in Chapter 4. In Chapter 4 we present the main results of this thesis. We establish an integro-differential inequality for an energy-like quadratic functional for the von Kármán system. By means of a comparison theorem for such inequalities a series of upper bounds demonstrating exponential decay of this energy functional are established. In Chapter 5 these results are compared to one another and discussed.

CHAPTER 2

STATEMENT OF THE BOUNDARY VALUE PROBLEM

Consider the rectangular semi-infinite isotropic elastic plate of Figure 2.1. The plate has thickness 2h, width 2H and extends to infinity in the positive ξ direction. The two semi-infinite lateral sides described by the planes η = -H and η = H, are clamped, that is, no displacement is allowed along these sides and the plate's slope there is equal to zero. An arbitrary self-equilibrated traction is applied at the end $\xi = 0$, while the stresses are assumed to vanish as $\xi \to \infty$.

In 1910, v. KARMAN [15] introduced a coupled system of two fourth-order quasilinear elliptic partial differential equations to describe the large deflections and stresses in a thin elastic plate. We use v. KARMAN's model for a geometrically nonlinear elastic plate here.

Let $\hat{\phi}$ be the Airy stress function, whose second derivatives yield the stresses in the plate. Then for fixed ς in [-h,h] $\hat{\phi}(\xi,\eta)$ satisfies the following inhomogeneous boundary conditions

$$\left\{\begin{array}{lll}\hat{\phi}(\xi,-H) = g_1(\xi), & \hat{\phi}_\eta(\xi,-H) = g_2(\xi), & \\ \hat{\phi}(\xi,H) = g_3(\xi), & \hat{\phi}_\eta(\xi,H) = g_4(\xi), & 0<\xi<\infty, \\ \hat{\phi}(0,\eta) = h_1(\eta), & \hat{\phi}_\xi(0,\eta) = h_2(\eta), & -H<\eta<H, \\ \text{and} & & \\ \multicolumn{3}{l}{\hat{\phi}_{\xi\xi}(\xi,\eta),\ \hat{\phi}_{\xi\eta}(\xi,\eta),\ \hat{\phi}_{\eta\eta}(\xi,\eta) \to 0 \ (uniformly\ in\ \eta)\ as\ \xi \to \infty,}\end{array}\right\} \tag{2.1}$$

where the subscript on $\hat{\phi}$ indicates partial differentiation and g_1, g_2, g_3, g_4, h_1, h_2

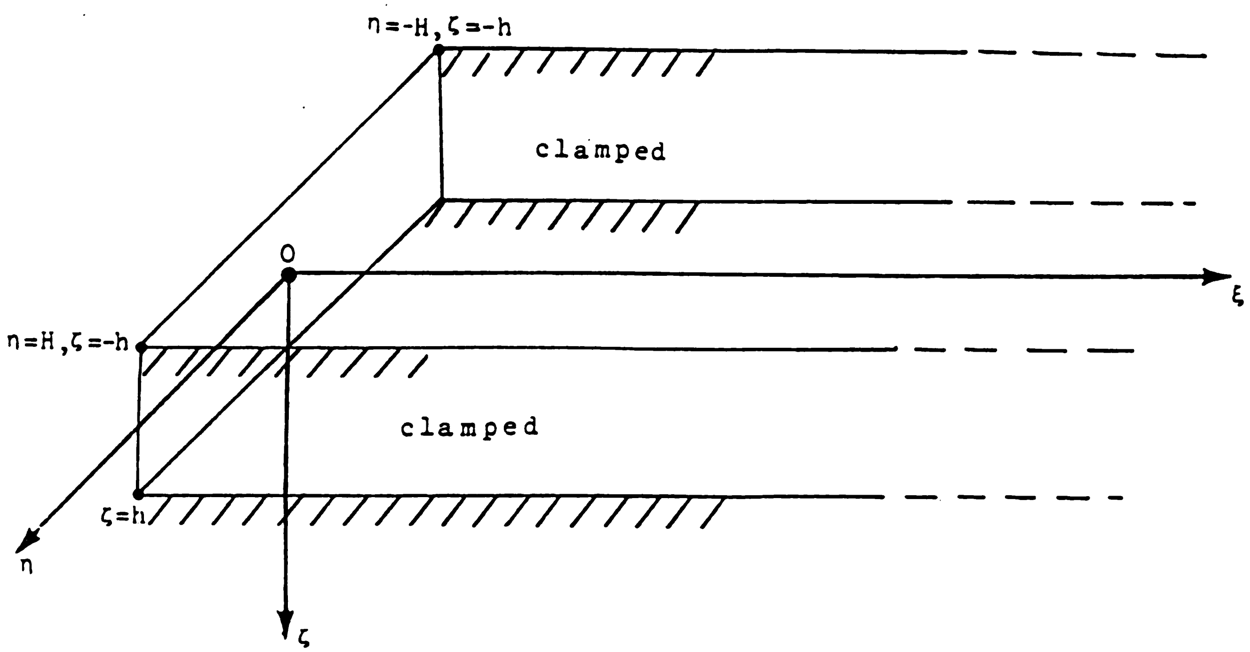


FIGURE 2.1 SEMI-INFINITE ISOTROPIC ELASTIC PLATE

are prescribed functions. Let $\tilde{\phi}$ be the Airy stress function associated with the stresses necessary to cancel out any deformation of the plate ($\tilde{\phi}$ is the Airy stress function for the so-called bending stresses, see STOKER [17] and KNIGHTLY [18]); then $\tilde{\phi}$ satisfies the same inhomogeneous boundary conditions as $\hat{\phi}$ along the clamped sides and homogeneous boundary conditions elsewhere. Thus

$$\left.\begin{cases} \tilde{\phi}(\xi,-H) = \hat{\phi}(\xi,-H) = g_1(\xi), \\ \tilde{\phi}_\eta(\xi,-H) = \hat{\phi}_\eta(\xi,-H) = g_2(\xi), \\ \tilde{\phi}(\xi,H) = \hat{\phi}(\xi,H) = g_3(\xi), \\ \tilde{\phi}_\eta(\xi,H) = \hat{\phi}_\eta(\xi,H) = g_4(\xi), \qquad 0<\xi<\infty, \\ \tilde{\phi}(0,\eta) = 0, \quad \tilde{\phi}_\xi(0,\eta) = 0, \qquad -H<\eta<H, \\ \tilde{\phi}_{\xi\xi}(\xi,\eta)\ ,\tilde{\phi}_{\xi\eta}(\xi,\eta)\ ,\tilde{\phi}_{\eta\eta}(\xi,\eta) \ \longrightarrow 0 \ \ (uniformly\ in\ \eta)\ as\ \xi\longrightarrow\infty. \end{cases}\right\} \tag{2.2}$$

We now define

$$\phi = \hat{\phi}-\tilde{\phi}, \tag{2.3}$$

the Airy stress function for the so-called membrane stresses. Note that ϕ satisfies the original inhomogeneous boundary conditions on $\xi = 0$, and homogeneous boundary conditions on the lateral clamped sides

$$\left.\begin{cases} \phi(0,\eta) = h_1(\eta),\ \phi_\xi(0,\eta) = h_2(\eta), \qquad -H<\eta<H, \\ \phi(\xi,-H) = \phi_\eta(\xi,-H) = 0, \\ \phi(\xi,H) = \phi_\eta(\xi,H) = 0, \qquad 0<\xi<\infty, \end{cases}\right\} \tag{2.4}$$

and that the corresponding stress field tends to zero as $\xi\longrightarrow\infty$, that is,

$$\phi_{\xi\xi}(\xi,\eta),\ \phi_{\xi\eta}(\xi,\eta),\ \phi_{\eta\eta}(\xi,\eta) \ \longrightarrow 0 \ \ (uniformly\ in\ \eta)\ as\ \xi\longrightarrow\infty. \tag{2.5}$$

The von Kármán system of equations for the midplane reads [15]:

$$\begin{aligned} \Delta^2_{(\xi,\eta)}\phi &= E(\omega^2_{\xi\eta}-\omega_{\xi\xi}\omega_{\eta\eta}) \\ \Delta^2_{(\xi,\eta)}\omega &= (h/D)(\phi_{\eta\eta}\omega_{\xi\xi}-2\phi_{\xi\eta}\omega_{\xi\eta}+\phi_{\xi\xi}\omega_{\eta\eta}) \end{aligned} \tag{2.6}$$

where $\Delta^2_{(\xi,\eta)}$ is the biharmonic operator in terms of the (ξ,η) variables, $\omega(\xi,\eta)$ is the midplane deflection, E is Young's modulus, $D = Eh^3/12(1-\nu^2)$ is the plate's modulus and ν is Poisson's ratio. The functions $\phi(\xi,\eta)$ and $\omega(\xi,\eta)$ are assumed to be four times continuously differentiable on the open semi-infinite strip and twice continuously differentiable on its boundary.

The boundary conditions on ϕ are given in (2.4), (2.5). The functions h_1 and h_2 must satisfy some additional conditions in view of the fact that the tractions on the $\xi=0$ face are *self-equilibrated*. If the tractions on the $\xi=0$ face are $\mathbf{T} =(T^{\xi},T^{\eta})$, where the superscript indicates the corresponding component, we have

$$\int_{-H}^{H} \mathbf{T}\, d\eta = \mathbf{O}, \tag{2.7a}$$

$$\int_{-H}^{H} \eta T^{\xi} d\eta = 0. \tag{2.7b}$$

In view of the fact that on the $\xi=0$ face the unit outward normal is $\mathbf{n} =$ (-1,0) we have

$$T^{\xi} = -\phi_{\eta\eta} \qquad \text{and} \quad T^{\eta} = -\phi_{\xi\eta} \quad on \ \xi=0. \tag{2.8}$$

On integration by parts, using the boundary conditions and the assumed smoothness of ϕ we deduce that

$$h_2(H) = h_2(-H) = 0, \tag{2.9}$$

and

$$h_1(-H) = h_1(H) = h_1'(-H) = h_1'(H) = 0. \tag{2.10}$$

By virtue of the smoothness assumptions on ϕ and the boundary condition (2.4), the boundary condition (2.5) may be integrated to yield

$$\phi(\xi,\eta),\ \phi_{\xi}(\xi,\eta),\ \phi_{\eta}(\xi,\eta) \ \rightarrow 0 \ (uniformly \ in \ \eta) \ as \ \xi \rightarrow \infty. \tag{2.11}$$

The boundary conditions on ω are:

$$\left\{\begin{array}{l} \omega(\xi,-H) = \omega_{\eta}(\xi,-H) = 0, \\ \omega(\xi,H) = \omega_{\eta}(\xi,H) = 0, \qquad 0<\xi<\infty, \\ \omega(\xi,\eta),\ \omega_{\xi}(\xi,\eta),\ \omega_{\eta}(\xi,\eta) \rightarrow 0 \\ (uniformly \ in \ \eta) \ as \ \xi \rightarrow \infty \text{ for } any \ \varsigma \ in \ [-h,h], \end{array}\right\} \tag{2.12}$$

and

$$\omega(0,\eta) = j_1(\eta) \qquad \omega_{\xi}(0,\eta) = j_2(\eta) \tag{2.13}$$

on the $\xi=0$ face where the prescribed functions $j_1(\eta)$, $j_2(\eta)$are such that

$$j_1(\pm H) = j_1'(\pm H) = j_2(\pm H) = 0. \tag{2.14}$$

We now introduce dimensionless spatial variables by setting

$$x = \frac{\xi}{H}, \qquad y = \frac{\eta}{H}. \tag{2.15}$$

We also introduce a dimensionless midplane deflection

$$z = \frac{\omega}{h}. \tag{2.16}$$

In terms of these dimensionless quantities (2.6) becomes:

$$\Delta^2_{(x,y)} \frac{\phi}{Eh^2} = z_{xy}^2 - z_{xx} z_{yy},$$

$$\Delta^2_{(x,y)} z = 12(1-\nu^2)\left(\frac{\phi_{yy}}{Eh^2} z_{xx} - 2\frac{\phi_{xy}}{Eh^2} z_{xy} + \frac{\phi_{xx}}{Eh^2} z_{yy}\right), \tag{2.17}$$

where $\Delta^2_{(x,y)}$ is the biharmonic operator in terms of the (x,y) variables. Henceforth, we drop the subscript notation on the biharmonic operator and so $\Delta^2 = \Delta^2_{(x,y)}$.

We now introduce the dimensionless Airy stress function $\psi(x,y)$ by

$$\psi(x,y) = \frac{\phi(x,y)}{Eh^2} \tag{2.18}$$

and employ the following scale change in z and ψ:

$$z = \frac{1}{\sqrt{6(1-\nu^2)}} u,$$

$$\psi = \frac{1}{12(1-\nu^2)} v. \tag{2.19}$$

Thus we obtain

$$\Delta^2 v = 2(u_{xy}^2 - u_{xx} u_{yy}) = -[u,u],$$

$$\Delta^2 u = (u_{xx} v_{yy} - 2u_{xy} v_{xy} + u_{yy} v_{xx}) = [u,v] \quad on\ R, \tag{2.20}$$

where the bilinear form [.,.] is defined as

$$[f,g] = f_{xx} g_{yy} - 2f_{xy} g_{xy} + f_{yy} g_{xx} \tag{2.21}$$

and R is the semi-infinite strip $0 < x < \infty$, $-1 < y < 1$.

The boundary conditions on v, the dimensionless Airy stress function, are

$$\left\{\begin{array}{l} v(0,y) = \tilde{h}_1(y), \quad v_x(0,y) = \tilde{h}_2(y), \quad -1<y<1, \\ v(x,\pm 1) = v_y(x,\pm 1)=0, \quad 0<x<\infty, \\ v(x,y),\ v_x(x,y),\ v_y(x,y) \rightarrow 0 \ (uniformly\ in\ y)\ as\ x \rightarrow \infty, \end{array}\right\} \tag{2.22}$$

where the functions $\tilde{h}_1, \tilde{h}_2$ are such that

$$\tilde{h}_1(\pm 1) = \tilde{h}_1'(\pm 1) = \tilde{h}_2(\pm 1) = 0. \tag{2.23}$$

The boundary conditions on u, the dimensionless displacement out of the plane, are

$$\left\{\begin{array}{l} u(0,y) = \tilde{j}_1(y), \quad u_x(0,y) = \tilde{j}_2(y), \quad -1<y<1, \\ u(x,\pm 1) = u_y(x,\pm 1) = 0, \quad 0<x<\infty, \\ u(x,y),\ u_x(x,y),\ u_y(x,y) \rightarrow 0 \ (uniformly\ in\ y)\ as\ x \rightarrow \infty. \end{array}\right\} \tag{2.24}$$

The functions $\tilde{j}_1, \tilde{j}_2$ satisfy

$$\tilde{j}_1(\pm 1) = \tilde{j}_1'(\pm 1) = \tilde{j}_2(\pm 1) = 0. \tag{2.25}$$

By a classical solution to the boundary value problem posed above we mean a pair of functions (u, v) that are four times continuously differentiable in the interior of R and twice continuously differentiable on its boundary, that are solutions to the system (2.20) and satisfy the boundary conditions (2.22) and (2.24) for prescribed functions $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{j}_1$, $\tilde{j}_2$, assumed to be sufficiently smooth.

CHAPTER 3

THE LINEARIZED PROBLEM: THE BIHARMONIC PROBLEM

In 1966, J. K. KNOWLES [3] formulated and proved a version of Saint-Venant's principle appropriate to the plane strain and generalized plane stress solutions of the equations of the linear theory of isotropic elastic equilibrium in bounded simply-connected plane domains of general shape. In [3] an explicit estimate (lower bound) is obtained for the rate of exponential decay of the energy with distance from a portion of the domain boundary carrying a self-equilibrated load. This result for biharmonic functions was established in [3] using differential inequality arguments. The special case of the biharmonic equation in a semi-infinite strip subject to self-equilibrated loads on the near end only constitutes a linearized version of the problem described in Chapter 2 here. In the present chapter we provide a brief description of the methods and results of [3] for the semi-infinite strip (see also KNOWLES [19]). Subsequent improvements on these results obtained by other authors will also be summarized.† Our treatment in Chapter 4 of the *nonlinear* problem described in Chapter 2 will be seen to make use of energy decay estimates of the type used in [3].

Thus we consider a semi-infinite strip R of width 2 in the (x,y) plane whose

† A comprehensive review of work on Saint-Venant's principle was given in 1983 by HORGAN and KNOWLES [4].

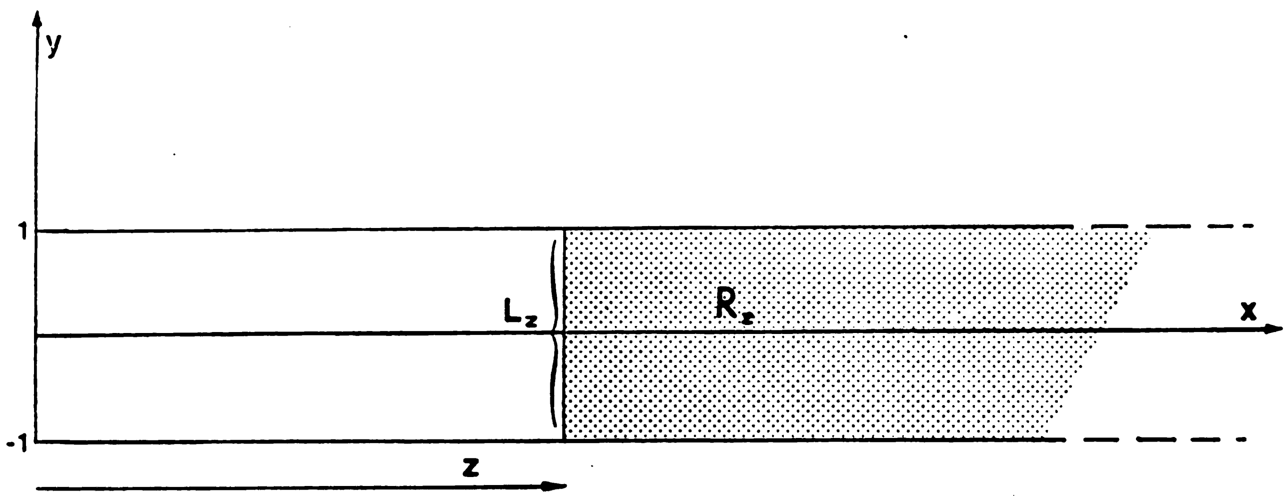


FIGURE 3.1: SEMI-INFINITE STRIP R

long sides are traction free and whose end $x=0$ carries a self-equilibrated load (see Fig. 3.1). The stress field is assumed to vanish at infinity. We are concerned with solutions ϕ of the biharmonic equation

$$\Delta^2\phi = 0 \qquad on\ R, \tag{3.1}$$

(ϕ is the Airy stress function), subject to the boundary conditions:

$$\left.\begin{cases} \phi(0,y) = k(y),\ \ \phi_x(0,y) = l(y), \quad -1<y<1, \\ \phi(x,\pm 1) = \phi_y(x,\pm 1) = 0, \quad 0<x<\infty, \\ \phi_{xx}(x,y),\ \phi_{xy}(x,y),\ \phi_{yy}(x,y) \to 0\ \ (uniformly\ in\ y)\ as\ x\to\infty, \end{cases}\right\} \tag{3.2}$$

where k and l are prescribed functions such that

$$k(\pm 1) = k'(\pm 1) = l(\pm 1) = 0. \tag{3.3}$$

We introduce the notation:

$$R_z = \left\{(x,y) in R\ |x\geq z\right\},$$
$$L_z = \left\{(x,y) in R\ |x=z\right\}, \tag{3.4}$$

so that z is a running variable along the x-axis. Clearly, $R_0 \equiv R$.

Following [3], [19] we define the function e(ϕ) by

$$e(\phi) = \phi_{xx}^2 + 2\phi_{xy}^2 + \phi_{yy}^2 \tag{3.5}$$

so that the energy stored in R_z is given by‡

$$E(z) = \int_{R_z} e(\phi)dA. \tag{3.6}$$

Thus, the total energy is:

$$E(0) = \int_{R} e(\phi)dA. \tag{3.7}$$

By repeatedly applying Green's theorem and using the boundary conditions (3.2) it is shown in [3] that

$$E(z) = -\int_{L_z} (\phi_x\phi_{xx} - \phi\phi_{xxx} + 2\phi_y\phi_{xy})dy. \tag{3.8}$$

‡ It is shown in [19] that E(z) is finite.

We recall from [3] the following differentiation formula for functions f continuous on the closure of R,

$$\frac{d}{dz}\int_{R_z} f\,dA = -\int_{L_z} f\,dy. \tag{3.9}$$

Thus we may write

$$E(z) = \frac{d}{dz}\int_{R_z}(\phi_z\phi_{zz} - \phi\phi_{zzz} + 2\phi_y\phi_{zy})dA \tag{3.10}$$

which implies that

$$\int_z^\infty E(s)ds = -\int_{R_z}(\phi_z\phi_{zz} - \phi\phi_{zzz} + 2\phi_y\phi_{zy})dA. \tag{3.11a}$$

It is convenient for subsequent purposes in Chapter 4 to introduce the Volterra integral operator

$$FU \equiv \int_0^z U(s)ds, \tag{3.11b}$$

on continuous scalar valued functions. Thus the left-hand side of (3.11a) can be written as $\int_0^\infty E(s)ds - FE$. Applying Green's theorem and the boundary conditions (3.2) we can now write

$$\int_z^\infty E(s)ds = \int_0^\infty E(s)ds - FE = \int_{L_z}(\phi_z^2 + \phi_y^2 - \phi\phi_{zz})dy, \quad 0<z<\infty. \tag{3.12}$$

We can now construct as in [3] the integro-differential functional

$$f(z,E',FE) \equiv E'(z) + 4k^2\left(\int_0^\infty E(s)ds - FE\right) = E'(z) + 4k^2\int_z^\infty E(s)ds, \tag{3.13}$$

for any real constant k. The result (3.12) and the expression for $E'(z)$ given by (3.5),(3.6), (3.9) may be combined to yield

$$f(z,E',FE) = -\int_{L_z}[\phi_{zz}^2 + \phi_{yy}^2 + 2\phi_{zy}^2 - 4k^2(\phi_z^2 + \phi_y^2 - \phi\phi_{zz})]dy, \quad 0<z<\infty. \tag{3.14}$$

We now seek, as in [4], to find a positive value of k for which

$$f(z, E', FE) \leq 0, \qquad 0 \leq z < \infty. \tag{3.15}$$

In [3] Knowles made use of Wirtinger-type inequalities (see Appendix A here) to establish (3.15) with the value of k given by

$$k = 0.70. \tag{3.16}$$

In 1974, FLAVIN [20], by using a sharper Wirtinger inequality, established (3.15) with the larger value of k given by

$$k = 1.11. \tag{3.17}$$

Using arguments based on first-order differential inequalities Knowles established in [3] that (3.15) yields the exponential decay estimate

$$E(z) \leq 2E(0)e^{-2kz}, \qquad 0 \leq z < \infty, \tag{3.18}$$

with the value of k given by (3.16). Such a value of k, which provides a *lower bound* for the actual decay rate we shall call an "estimated decay rate". FLAVIN [20] employed the same arguments as Knowles to proceed from (3.15) to (3.18) and so obtained (3.18) with the improved value given by (3.17) for the estimated decay rate.

We shall show in Chapter 4 (see Section 4.3) that direct employment of a comparison theorem for integro-differential inequalities (see Section 4.2) applied to (3.15) yields the result

$$E(z) \leq E(0)e^{-2kz}, \qquad z \geq 0, \tag{3.19}$$

for any value of k for which (3.15) holds. Thus we obtain an improvement on (3.18) by eliminating the multiplicative factor of two in that estimate.

By using a different argument, OLEINIK and YOSIFIAN [21], [22] established in 1978 that

$$E(z) \leq \frac{E(0)}{\cosh(2kz)} = \frac{2E(0)e^{-2kz}}{1+e^{-4kz}}, \tag{3.20}$$

with the value of k given by (3.17). Earlier, in 1975, MIETH [23] had obtained (3.20) with the value of k given by (3.16). Observe that (3.19) is a sharper result than (3.20).

Recently, KNOWLES [19] has used arguments based on higher-order energies to obtain exponential decay estimates of a more elaborate nature than (3.18). His results do yield an exponential decay of the energy $E(z)$ with estimated decay rate given by

$$k = 1.35. \tag{3.21}$$

As is discussed in [4], (see also [19]) all of the preceding results ***underestimate the exact decay rate*** which is given by

$$k = 2.10. \tag{3.22}$$

This result may by arrrived at through consideration of eigenfunction expansions for biharmonic functions in the semi-infinite strip ([4],[19]), the completeness of which has been established by GREGORY [24].

In the sequel, we shall invoke directly the inequality (3.15) for the integro-differential functional (3.13), with the best constant k obtained to date, namely (3.17).

CHAPTER 4

ENERGY DECAY ESTIMATES

4.1 DERIVATION OF AN INTEGRO-DIFFERENTIAL INEQUALITY FOR THE ENERGY FOR THE VON KARMAN EQUATIONS

On the semi-infinite strip R we have the system of partial differential equations (2.20):

$$\begin{aligned} \Delta^2 v &= 2(u_{xy}^2 - u_{xx}u_{yy}) = -[u\,,u], \\ \Delta^2 u &= (u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}) = [u\,,v], \end{aligned} \tag{4.1}$$

subject to the boundary conditions (2.22) and (2.24). It is convenient to rewrite the system (4.1) in divergence form (BERGER [25], p. 692) and so

$$\Delta^2 v = (u_x^2)_{yy} - 2(u_x u_y)_{xy} + (u_y^2)_{xx}, \tag{4.2}$$

$$\Delta^2 u = (u_{yy}v)_{xx} - 2(u_{xy}v)_{xy} + (u_{xx}v)_{yy} \qquad on\ R. \tag{4.3}$$

We now introduce an energy for the von Kármán system. Observing that the principal part of the von Kármán system of equations is the biharmonic operator, we define, for solutions $(u\,,\,v)$ of (4.1),

$$\hat{e}(u,v) = e(u) + e(v) \tag{4.4}$$

where

$$e(u) = u_{xx}^2 + 2\,u_{xy}^2 + u_{yy}^2. \tag{4.5}$$

We note that the quadratic function (4.5) has exactly the same form as the energy density for the biharmonic problem (see (3.5)).

Using the same notation as that introduced in Chapter 3, we define the energy functional E(z) by

$$E(z) = \int_{R_z} \hat{e}(u,v)dA, \qquad 0 \leq z < \infty. \tag{4.6}$$

Adopting the arguments of [19] it can be shown that E(z) is finite for z in [0,∞). The total energy then is

$$E(0) = \int_{R} \hat{e}(u,v)dA. \tag{4.7}$$

Clearly, $E(z)$ can be decomposed into two parts

$$E(z) = \int_{R_z} \hat{e}(u,v)dA = \int_{R_z} e(u)dA + \int_{R_z} e(v)dA = E_u(z) + E_v(z). \tag{4.8}$$

We now seek to establish an integro-differential inequality for $E(z)$. First, we multiply (4.3) by u and integrate over R_z to get

$$\int_{R_z} u\,\Delta^2 u dA = \int_{R_z} u[u,\, v]dA. \tag{4.9}$$

Applying Green's theorem to the left-hand side of (4.9) and using the boundary conditions (2.24) we obtain

$$\int_{R_z} u\,\Delta^2 u dA = -\int_{L_z} u u_{zzz}\,dy + \int_{L_z} (u_z u_{zz} + 2u_y u_{zy})dy + E_u(z). \tag{4.10}$$

Applying Green's theorem to the right-hand side of (4.9) using the boundary conditions (2.22), (2.24) and equation (4.2) we get

$$\begin{aligned}\int_{R_z} u[u,\, v]dA &= \int_{L_z} [-(u_{yy} v)_z + u_z u_{yy} v - 2u_y u_{zy} v]dy \\ &\quad + \int_{L_z} (v v_{zzz} - v_z v_{zz} - 2v_y v_{zy})dy \\ &\quad - E_v(z).\end{aligned} \tag{4.11}$$

On equating (4.10) and (4.11), combining terms and integrating by parts, making use of the boundary conditions (2.22), (2.24) we get

$$E(z) = \int_{L_z} -[(u_z^2+u_y^2-uu_{zz})+(v_z^2+v_y^2-vv_{zz})]_z \, dy.$$
$$+ \int_{L_z} (u_y^2 v_z + 2u_z u_{yy} v + uu_y v_{zy} + u_z u_y v_y + uu_{zy} v_y) dy. \tag{4.12}$$

We are now ready to construct the functional

$$E'(z) + 4k^2 \int_z^\infty E(s) ds. \tag{4.13}$$

Using the differentiation result (3.9), and the boundary conditions (2.22), (2.24) we obtain, for any constant k, the identity

$$E'(z) + 4k^2 \int_z^\infty E(s) ds$$
$$= \left\{ -\int_{L_z} [u_{zz}^2 + u_{yy}^2 + 2u_{zy}^2 - 4k^2(u_z^2+u_y^2-uu_{zz})] dy \right\}$$
$$+ \left\{ -\int_{L_z} [v_{zz}^2 + v_{yy}^2 + 2v_{zy}^2 - 4k^2(v_z^2+v_y^2-vv_{zz})] dy \right\}$$
$$+ 4k^2 \int_{R_z} (u_y^2 v_z + 2u_z u_{yy} v + uu_y v_{zy} + u_z u_y v_y + uu_{zy} v_y) dA. \tag{4.14}$$

The first two integrals in (4.14) have exactly the same form as their counterparts for solutions of the biharmonic equation (see equation (3.14)). Thus, if we choose

$$k = 1.11 \tag{4.15}$$

it follows from (4.14) and the results of Chapter 3 that

$$E'(z) + 4k^2 \int_z^\infty E(s) ds$$
$$\leq 4k^2 \int_{R_z} (u_y^2 v_z + 2u_z u_{yy} v + uu_y v_{zy} + u_z u_y v_y + uu_{zy} v_y) dA$$
$$\equiv 4k^2 [\, I_1 + I_2 + I_3 + I_4 + I_5 \,]. \tag{4.16}$$

It is shown in Appendix A that

$$I_1 + I_2 + I_3 + I_4 + I_5 \leq \mu E^{3/2}(z) \tag{4.17}$$

where the constant μ has the value

$$\mu = 0.619. \tag{4.18}$$

Thus, on using (4.17) and (4.16) we obtain the integro-differential inequality

$$E'(z) - 4k^2\mu E^{3/2}(z) + 4k^2\int_z^\infty E(s)ds \leq 0, \tag{4.19}$$

where k and μ are given by (4.15) and (4.18) respectively. The inequality (4.19) allows us to establish several different exponential decay estimates for the energy $E(z)$ defined in (4.8). These results follow from a comparison theorem for integro-differential inequalities which we now describe.

4.2 A COMPARISON THEOREM FOR INTEGRO-DIFFERENTIAL INEQUALITIES

We state here a comparison theorem for integro-differential inequalities of which we will make repeated use in the sequel. For a complete discussion and proof see LAKSHMIKANTHAM and LEELA [26], pp. 350-1, and WALTER [27] p. 122.

THEOREM 1

Let $F:C[J,\mathbf{R}]\rightarrow C[J,\mathbf{R}]$ be an integral operator mapping continuous scalar valued functions on J, a subset of $\mathbf{R}$, into continuous scalar valued functions on J.

Assume that

(*i*) $\hat{f}(z,U',U,FU)$ *maps continuously* $C(J\times\mathbf{R}^3)$ *into* $\mathbf{R}$ *and that* $\hat{f}$ *is nondecreasing in* U' for *fixed* (z,U,FU) *and nonincreasing in* FU for *fixed* (z,U',U); (4.20)

(*ii*) *For any* u_1, u_2, *in* $C(J,\mathbf{R})$

$u_1(z)\leq u_2(z)$ *implies that* $Fu_1\leq Fu_2$, for *all* z *in* $J=(0,\infty)$; (4.21)

(*iii*) $\hat{f}(z,V',V,FV)\leq 0$ and (4.22)

$\hat{f}(z,W',W,FW)\geq 0$, *for all* z *in* $(0,\infty)$, *where* V, W *are* $C^1(J,\mathbf{R})$. (4.23)

Then

$V(0)\leq W(0)$ *implies that* (4.24)

$V(z)\leq W(z)$, $\quad z\geq 0$. (4.25)

We introduce the notation

$$\hat{f}(z,E',E,FE)\equiv E'(z)-4k^2\mu E^{3/2}(z)+4k^2\left(\int_0^\infty E(s)ds-FE\right)$$

$$=E'(z)-4k^2\mu E^{3/2}(z)+4k^2\int_z^\infty E(s)ds \tag{4.26}$$

for the left-hand side of our inequality (4.19), where the Volterra integral operator

F has been defined in (3.11b). It may be readily verified that $\hat{f}$ satisfies the hypotheses (4.20), (4.21), and (4.22) of Theorem 1. Thus it remains for us to determine a comparison function $H(z) \geq 0$, such that

$$\hat{f}(z, H', H, FH) \geq 0, \tag{4.27}$$

and

$$H(0) \geq E(0), \tag{4.28}$$

in order to conclude from (4.25) that

$$E(z) \leq H(z), \qquad z \geq 0. \tag{4.29}$$

In what follows we show that several such choices of $H(z)$ lead to exponentially decaying estimates for $E(z)$.

4.3 AN IMPROVED ESTIMATE FOR THE BIHARMONIC PROBLEM

First we consider the application of Theorem 1 to the integro-differential inequality (3.15) obtained for the biharmonic problem discussed in Chapter 3. We recall that (3.15) reads:

$$f(z,E',FE) = E'(z) + 4k^2\left(\int_0^\infty E(s)ds - FE\right) = E'(z) + 4k^2\int_z^\infty E(s)ds \leq 0. \tag{4.30}$$

Since E does not appear explicitly on the left-hand sides of (4.30), we have used the notation $f(z,E',FE)$ instead of $f(z,E',E,FE)$. It may again be readily verified that f satisfies the hypotheses (4.20), (4.21) of Theorem 1. Furthermore, the hypothesis (4.22) is satisfied by virtue of (4.30) if the constant k is chosen as in (3.17). Now the function $H(z)$ defined by

$$H(z) = E(0)e^{-2kz} \tag{4.31}$$

is such that

$$f(z,H',FH) = H'(z) + 4k^2\int_z^\infty H(s)ds = 0. \tag{4.32}$$

Furthermore, E(0) = H(0) and so Theorem 1 yields the result

$$E(z) \leq E(0)e^{-2kz}, \qquad z \geq 0, \tag{4.33}$$

where k is given by (3.17). This provides a sharper estimate than (3.18) or (3.20).

4.4 A FIRST RESULT FOR THE VON KARMAN EQUATIONS

Returning to the nonlinear problem, we seek a comparison function $H_1(z)$ satisfying (4.27), (4.28), where $\hat{f}$ is given by (4.26). Our first choice is to consider a function similar to (4.31), which was employed for the biharmonic problem. Thus we consider

$$H_1(z) = E(0)e^{-\kappa z} \tag{4.34}$$

where κ is a positive constant, as yet undetermined. Clearly,

$$H_1(0) = E(0) \tag{4.35}$$

and so (4.28) is satisfied. We now seek the largest value of κ such that (4.27) holds. We have

$$H_1'(z) = -\kappa E(0)e^{-\kappa z} = -\kappa H_1(z), \tag{4.36}$$

and

$$\int_0^\infty H_1(s)ds - FH_1 = \int_z^\infty H_1(s)ds = \frac{E(0)e^{-\kappa z}}{\kappa} = \frac{H_1(z)}{\kappa}, \tag{4.37}$$

and so recalling the definition of $\hat{f}$ in (4.26) we obtain

$$\begin{aligned} \hat{f}(z,H_1',H_1,FH_1) &= -\kappa H_1(z) - 4k^2\mu H_1^{3/2}(z) + \frac{4k^2}{\kappa}H_1(z) \\ &= H_1(z)\left\{-\kappa - 4k^2\mu H_1^{1/2}(z) + \frac{4k^2}{\kappa}\right\} \end{aligned} \tag{4.38}$$

Since $0 \le H_1^{1/2}(z) \le H_1^{1/2}(0)$ for all z in $(0,\infty)$, and $H_1^{1/2}(0) = E^{1/2}(0)$ by virtue of (4.34) it follows that $\hat{f}(z,H_1',H_1,FH_1)$ is non-negative for all z in $(0,\infty)$ if $\kappa>0$ is such that

$$-\kappa - 4k^2\mu E^{1/2}(0) + \frac{4k^2}{\kappa} \ge 0. \tag{4.39}$$

The largest value of κ is obtained by taking the equality sign in (4.39) and so we find

$$\kappa = 2k(\sqrt{M^2+1}-M) \tag{4.40}$$

where

$$M = k\mu\sqrt{E(0)} \tag{4.41}$$

and k, μ are given by (4.15) and (4.18) respectively. Thus (4.27), (4.28) are satisfied and so from (4.29) and (4.34) we obtain the exponential decay estimate

$$E(z) \leq E(0)e^{-\kappa z}, \qquad z \geq 0. \tag{4.42}$$

A decay estimate similar in form to (4.40), (4.41), (4.42) was obtained by HORGAN [28] in his investigation of plane entry flows for the Navier-Stokes equations.

We observe that the decay rate κ given by (4.40) is a monotonically decreasing function of M for $0<M<\infty$ and so $\kappa<2k$. Thus the estimated decay rate for the von Kármán equations is slower than that predicted by (4.33) for the biharmonic problem. Moreover, as the total energy E(0) increases, M increases and so κ decreases. The dependence of κ on E(0) is shown graphically in Figure 4.1. In the sequel, we shall refer to the estimate (4.42) as result I.

We should also note that in deducing the condition (4.39) from (4.38), we used the inequality $H_1^{1/2}(z) \leq E^{1/2}(0)$ for $0 \leq z < \infty$. Recalling the definition of $H_1(z)$ in (4.34), we see that this bound deteriorates as z increases. In the next section, we present an argument which does not have this shortcoming and which leads to a sharper decay estimate than (4.42) for sufficiently large values of z.

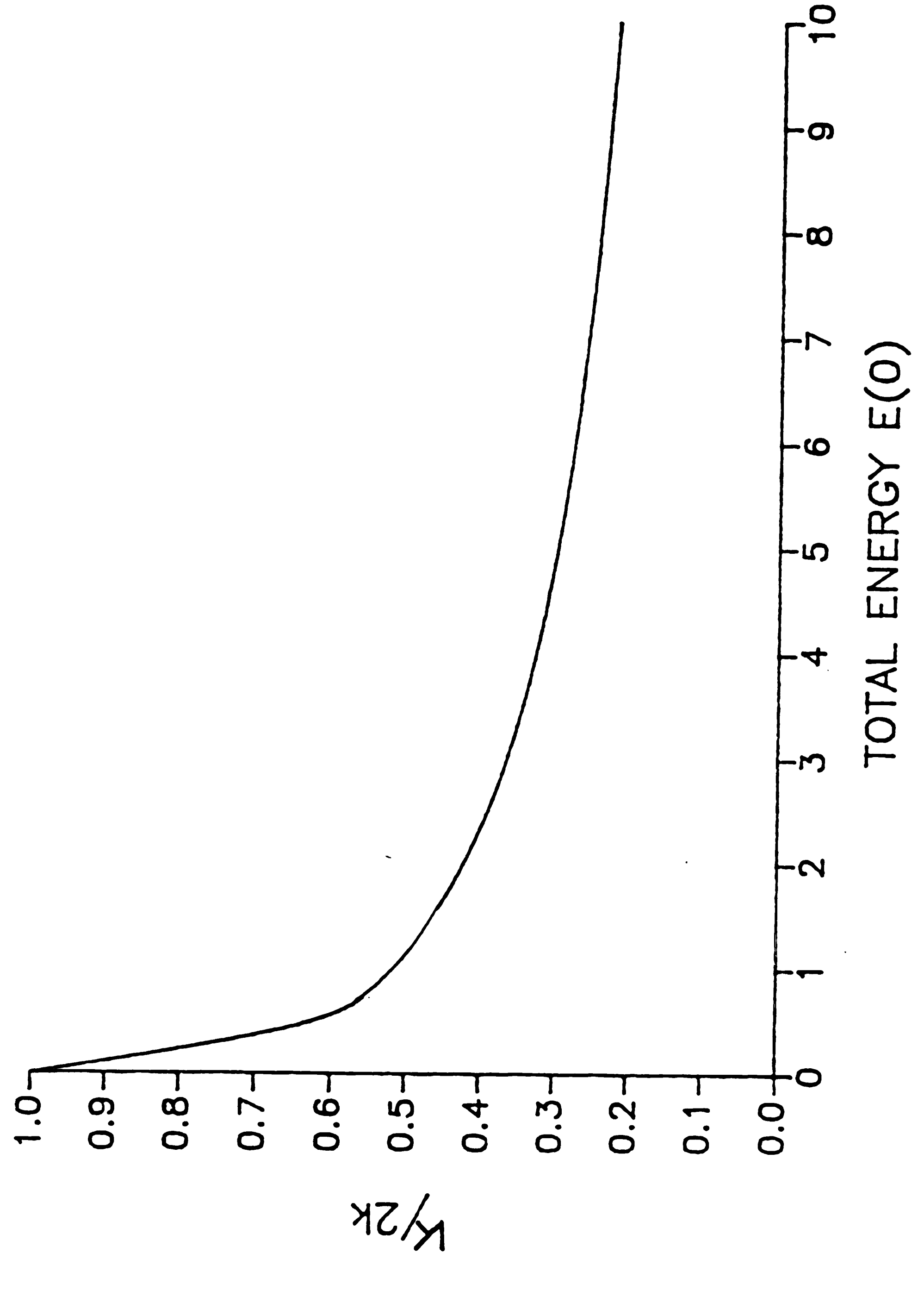


FIG 4.1 $K/2k$ VS. TOTAL ENERGY

4.5 AN IMPROVEMENT ON THE RESULT (4.42) FOR E(0) > 0.209

In Section 4.4 we have established the decay estimate (4.40)-(4.42). On using the result (4.42) in our basic integro-differential inequality (4.19), we obtain the weaker integro-differential inequality

$$E'(z) - 4k^2\mu E^{3/2}(0)e^{\frac{-3\kappa z}{2}} + 4k^2\int_z^\infty E(s)$$

$$= E'(z) - 4k^2\mu E^{3/2}(0)e^{\frac{-3\kappa z}{2}} + 4k^2(\int_0^\infty E(s)ds - FE) \leq 0. \tag{4.43}$$

Denoting the middle term of (4.43) by $\hat{g}(z,E',FE)$ it may be readily verified that $\hat{g}$ satisfies the hypotheses (4.20), (4.21) and (4.22) of Theorem 1. Thus, it remains for us to determine a comparison function $H_2(z)$ such that

$$\hat{g}(z,H_2',FH_2) \geq 0 \text{ and } H_2(0) \geq E(0), \tag{4.44}$$

in order to conclude that

$$E(z) \leq H_2(z), \qquad z \geq 0. \tag{4.45}$$

Consider the function

$$H_2(z) = C_1 E^{3/2}(0)\mu e^{\frac{-3\kappa z}{2}} - De^{-2kz} \tag{4.46}$$

where C_1 and D are constants to be determined. We have

$$H_2'(z) = -\frac{3\kappa}{2}C_1E^{3/2}(0)\mu e^{\frac{-3\kappa z}{2}} + 2kDe^{-2kz}, \tag{4.47}$$

and

$$\int_0^\infty H_2(s)ds - FH_2 = \int_z^\infty H_2(s)ds$$

$$= \frac{C_1 E^{3/2}(0)\mu e^{\frac{-3\kappa z}{2}}}{\frac{3\kappa}{2}} - \frac{De^{-2kz}}{2k}. \tag{4.48}$$

Thus,

$$\hat{g}(z, H_2', FH_2) = E^{3/2}(0)\mu e^{\frac{-3\kappa z}{2}}\left[-\frac{3\kappa C_1}{2} - 4k^2 + \frac{8k^2 C_1}{3\kappa}\right]. \tag{4.49}$$

To ensure that $\hat{g}(z, H_2', FH_2) \geq 0$, the constant C_1 is chosen to satisfy

$$C_1\left(\frac{8\kappa^2}{3\kappa} - \frac{3\kappa}{2}\right) \geq 4k^2 \tag{4.50}$$

where κ, k are given by (4.40), (4.15) respectively. The smallest constant C_1 is obtained by taking the equality sign in (4.50), and so we get

$$C_1 = \frac{24k^2\kappa}{16k^2 - 9\kappa^2}, \tag{4.51}$$

provided that $16k^2 - 9\kappa^2 > 0$. On using (4.40), (4.41) we see that this condition is satisfied if

$$E(0) > \frac{4}{45k\mu^2}, \tag{4.52}$$

that is, if

$$E(0) > 0.209. \tag{4.53}$$

It remains to satisfy the second inequality in (4.44). Thus we set

$$C_1 E^{3/2}(0)\mu - D = E(0), \tag{4.54}$$

thereby satisfying the second of (4.44) with equality, and so

$$D = C_1 E^{3/2}(0)\mu - E(0). \tag{4.55}$$

By virtue of (4.51) and (4.55), the result (4.45) reads:

$$E(z) \leq E(0)e^{-2kz} + C_1\mu E^{(3/2)}(0)[e^{\frac{-3\kappa z}{2}} - e^{-2kz}], \qquad z \geq 0, \tag{4.56}$$

provided that E(0) > 0.209, where

$$C_1 = \frac{24k^2\kappa}{16k^2 - 9\kappa^2}, \tag{4.57}$$

$$\kappa = 2k(\sqrt{M^2+1} - M), \tag{4.58}$$

$$M = k\mu\sqrt{E(0)}, \tag{4.59}$$

where k=1.11 and μ=0.619.

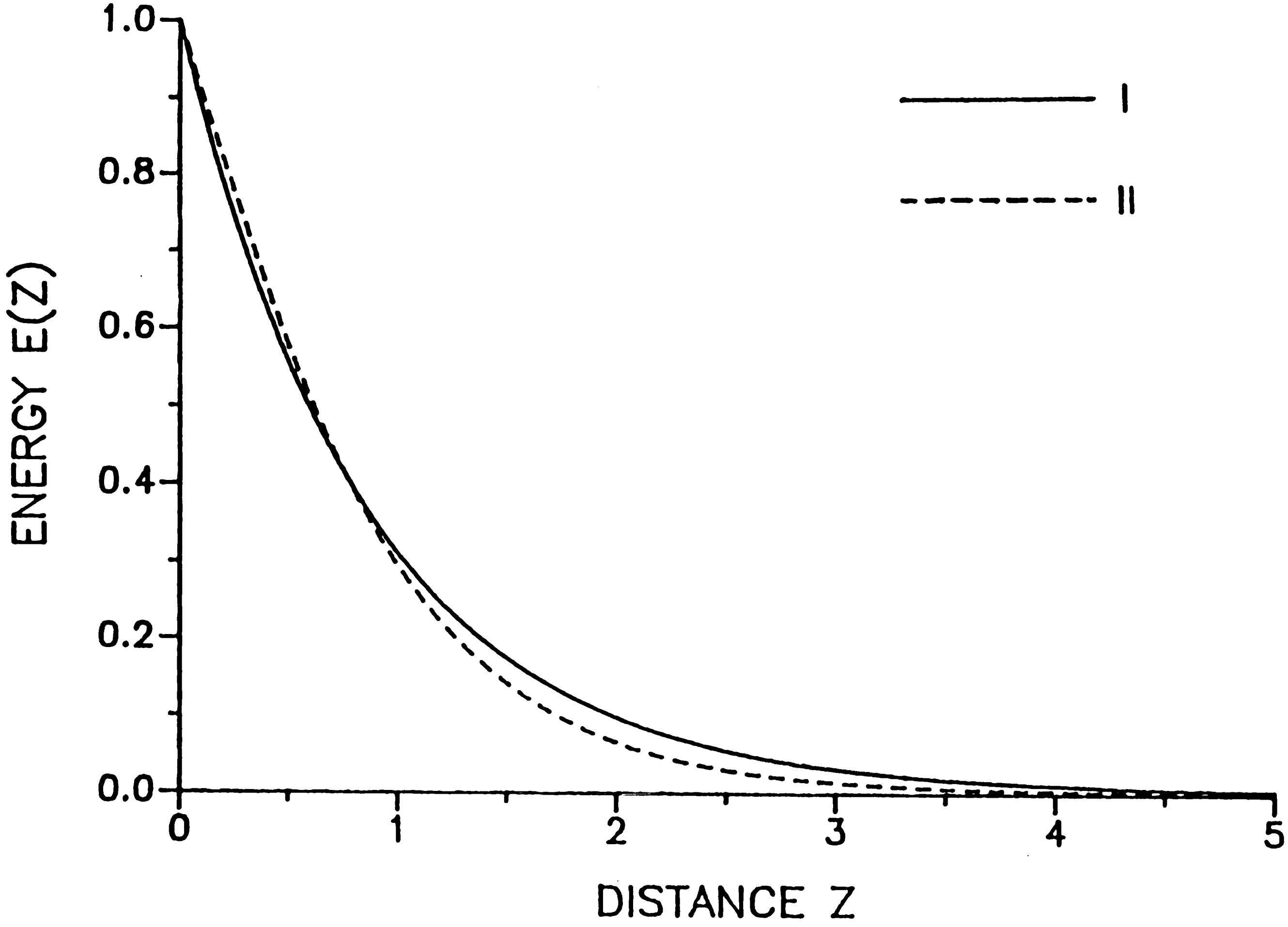
Note that the estimate is made up of two terms: the first is the estimate for the linear problem, i.e. the biharmonic problem, while the second is an exponentially decreasing correction which depends on the magnitude of E(0). In the sequel, we shall refer to (4.56) as result II.

The result is not valid for E(0)<0.209 but for such E(0) the estimated decay rate κ as given by (4.40) is quite close to the estimated decay rate 2k for the biharmonic problem. In fact for E(0)=0.209 we get κ= 0.734(2k). Moreover, κ is even closer to 2k for smaller energies (see Figure 4.1).

The result II provides a sharper estimate than I for values of z greater than $\tilde{z}$, where $\tilde{z}$ depends on the total energy E(0) as indicated in Table 4.1 (see also Fig. 4.2).

Table 4.1: Values of $\tilde{z}$

E(0)	$\tilde{z}$
0.25	0.54
0.50	0.63
1.00	0.77
1.50	0.89
2.00	0.98
2.50	1.06
3.00	1.14
3.50	1.21
4.00	1.27
4.50	1.33
5.00	1.39
6.00	1.49
7.00	1.59
8.00	1.68
9.00	1.76
10.0	1.84



TOTAL ENERGY E(0)=1.0

FIG 4.2 COMPARISON OF RESULTS I AND II

4.6 AN IMPROVEMENT ON RESULT (4.42) FOR E(0) < 0.529

The approach we used in the previous section was to simplify our basic integro-differential inequality

$$\hat{f}(z,E',E,FE) = E'(z) - 4k^2\mu E^{3/2}(z) + 4k^2\int_z^\infty E(s)ds \leq 0, \tag{4.60}$$

by using the estimate (4.42) for the nonlinear term $E^{3/2}(z)$. Here we consider a different method which also involves simplification of (4.60).

Recall from the end of Section 4.2 that the estimate

$$E(z) \leq H(z) \tag{4.61}$$

holds for comparison functions $H(z)$ such that

$$\hat{f}(z,H',H,FH) \geq 0, \qquad 0 \leq z < \infty, \tag{4.62}$$

and

$$H(0) \geq E(0). \tag{4.63}$$

We seek a comparison function $H_3(z)>0$ which satisfies

$$H_3'(z) - 4k^2\mu H_3^{3/2}(z) = -2kH_3(z), \qquad z \geq 0, \tag{4.64}$$

so that $\hat{f}(z,H_3',H_3,FH_3)$ reduces to

$$\hat{f}(z,H_3',H_3,FH_3) = -2kH_3(z) + 4k^2\int_z^\infty H_3(s)ds$$

$$= -2kH_3(z) + 4k^2\left(\int_0^\infty H_3(s)ds - FH_3\right) \equiv \tilde{f}(z,H_3,FH_3). \tag{4.65}$$

The first order ordinary differential equation (4.64) has the solution

$$H_3(z) = (C_2e^{kz} + 2k\mu)^{-2}, \tag{4.66}$$

where $C_2>0$ is a constant yet to be determined. It remains to establish that the choice $H(z) = H_3(z)$ satisfies (4.62) and (4.63).

To show that (4.62) is satisfied, it suffices, by virtue of (4.65), to show that

$$\tilde{f}(z,H_3,FH_3) \geq 0, \qquad 0 \leq z < \infty. \tag{4.67}$$

A direct verification of (4.67) is difficult due to the complexity in integration of the function $H_3(z)$ defined in (4.66). We will establish (4.67) by using a contradiction argument. Thus, suppose that

$$\tilde{f}(z,H_3,FH_3) < 0, \qquad 0 \leq z < \infty. \tag{4.68}$$

Consider the function $H_4(z)$ defined by

$$H_4(z) = [(C_2+2k\mu)e^{kz}]^{-2}. \tag{4.69}$$

It may by readily verified that

$$\tilde{f}(z,H_4,FH_4) = 0, \qquad z \geq 0, \tag{4.70}$$

and that

$$H_3(0+) < H_4(0+). \tag{4.71}$$

We now employ a stricter version of Theorem 1 (WALTER [27] p. 122) wherein (4.22) is strict and (4.24) is replaced by $V(0+) < W(0+)$. Now the operator $\tilde{f}$ defined in (4.65) satisfies the hypotheses (4.20), (4.21) of Theorem 1. Furthermore, in view of (4.68), (4.70) and (4.71) the remaining hypotheses of the stricter version of Theorem 1 are satisfied for the choices $V = H_3$, $W = H_4$. Thus, we conclude that

$$H_3(z) < H_4(z), \qquad z \geq 0, \tag{4.72}$$

which is a contradiction by virtue of the definitions of H_3, H_4 in (4.66), (4.69) respectively. Thus, (4.68) cannot hold and we deduce that (4.67) holds.

It remains for us to choose the constant $C_2 > 0$ such that (4.63) is satisfied with $H \equiv H_3$. The largest such constant is obtained by taking the equality sign in (4.63) and so

$$C_2 = E(0)^{-1/2} - 2k\mu. \tag{4.73}$$

To ensure that C_2 is positive, we require that

$$E(0) < \frac{1}{4k^2\mu^2} \approx 0.529. \tag{4.74}$$

Thus for sufficiently small total energies E(0) satisfying (4.74), we conclude from (4.61), (4.66), and (4.73) the decay estimate

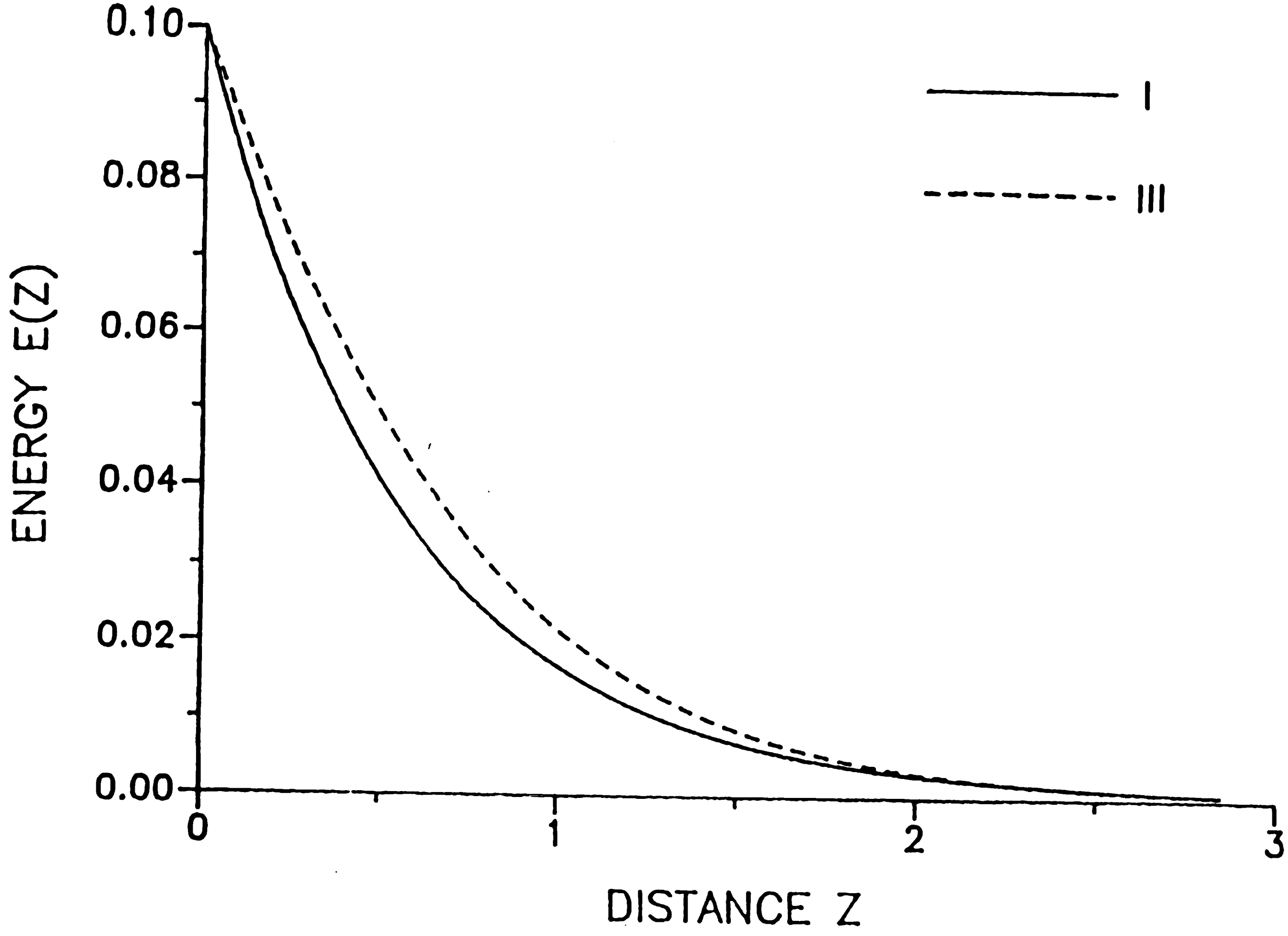
$$E(z) \leq \frac{1}{[\frac{1}{\sqrt{E(0)}}e^{kz} - 2k\mu(e^{kz} - 1)]^2}, \tag{4.75}$$

where k, μ are given by (4.15) and (4.18) respectively. In what follows we refer to (4.75) as result III.

The result III is an improvement over the estimate I of Section 4.4 for $z \geq z_0$ (see Fig. 4.3). The value of z_0 depends on the total energy E(0) as indicated in Table 4.2 below.

Table 4.2: Values of z_0

E(0)	z_0
0.10	2.41
0.20	3.13
0.30	3.99
0.40	5.28
0.50	8.58



TOTAL ENERGY E(0)=0.10

FIG 4.3 COMPARISON OF RESULTS I AND III

4.7 AN ESTIMATE WITH Z-DEPENDENT DECAY RATE

We now describe an argument which is more elaborate than the preceding and which leads to a decay estimate with a z-dependent decay rate. We seek comparison functions $H(z)$ of the form

$$H_5(z) = E(0)e^{-\chi(z)}\frac{\chi'(z)}{\chi'(0)}, \tag{4.76}$$

where $\chi(z)\geq 0$ is a function to be determined. The function $\chi(z)$ must be such that $H_5(z)$ satisfies (4.27), (4.28). The particular form of the right-hand side of (4.76) has been chosen so that $H_5(z)$ may be readily integrated. This integration is necessary in the evaluation of $\hat{f}(z, H_5', H_5, FH_5)$. Furthermore, the results obtained already suggest that we seek a decay estimate with decay rate κ (given by (4.40)) for small values of z and decay rate 2k (k is the decay rate for the biharmonic problem) for large values of z. Thus, the function $\chi(z)$ appearing in (4.76) will be required to satisfy

$$\chi(0) = 0, \qquad \chi'(0) = \kappa, \tag{4.77}$$

$$\chi(z)\rightarrow\infty, \qquad \chi'(z)\rightarrow 2k \;\; as \;\; z\rightarrow\infty, \tag{4.78}$$

and

$$\chi''(z) \geq 0, \qquad z \geq 0. \tag{4.79}$$

Observe that the first of (4.77) ensures that $H_5(0) = E(0)$ and so (4.28) holds. By direct calculation from (4.76) we obtain

$$\hat{f}(z, H_5', H_5, FH_5) =$$

$$\frac{E(0)e^{-\chi(z)}}{\kappa}[\chi''(z) - \chi'^2(z) - \frac{4k^2\mu\sqrt{E(0)}e^{\frac{-\chi(z)}{2}}}{\sqrt{\kappa}}\chi'^{3/2}(z) + 4k^2]. \tag{4.80}$$

We now seek to find a $\chi(z)$ satisfying conditions $(4.77)_2$-(4.79) and which renders the right-hand side of (4.80) nonnegative.

Consider the choice

$$\chi'(z) = \kappa + \frac{(2k-\kappa)z}{z+C_3}, \tag{4.81}$$

where $C_3>0$ is a constant to be determined. It is readily verified that $(4.77)_2$, $(4.78)_2$ and (4.79) are satisfied by $\chi'(z)$ for arbitrary values of C_3. Upon integration and use of $(4.77)_1$ we obtain

$$\chi(z) = \int_0^z \chi'(s)ds = 2k[z - C_3(1-\lambda)ln(\frac{z+C_3}{C_3})], \tag{4.82}$$

where

$$\lambda = \frac{\kappa}{2k} < 1. \tag{4.83}$$

Clearly χ satisfies $(4.78)_1$ for arbitrary values of C_3. It remains to satisfy (4.27), that is $\hat{f} \geq 0$, in order to conclude that

$$E(z) \leq H_5(z), \qquad z \geq 0. \tag{4.84}$$

Thus by virtue of (4.80), we choose the constant C_3 in (4.81) such that

$$\chi''(z) - \chi'^2(z) - \frac{4k^2\mu\sqrt{E(0)}}{\sqrt{\kappa}} e^{\frac{-\chi(z)}{2}} \chi'^{3/2}(z) + 4k^2 \geq 0, \qquad z \geq 0. \tag{4.85}$$

The largest such value of C_3, depending on the value of E(0), is determined numerically and the results are shown in Table 4.3.

Thus we have established the estimate

$$E(z) \leq E(0)e^{-2kz'} \frac{\lambda^{-1}z + C_3}{z + C_3}, \qquad z \geq 0, \tag{4.86}$$

where

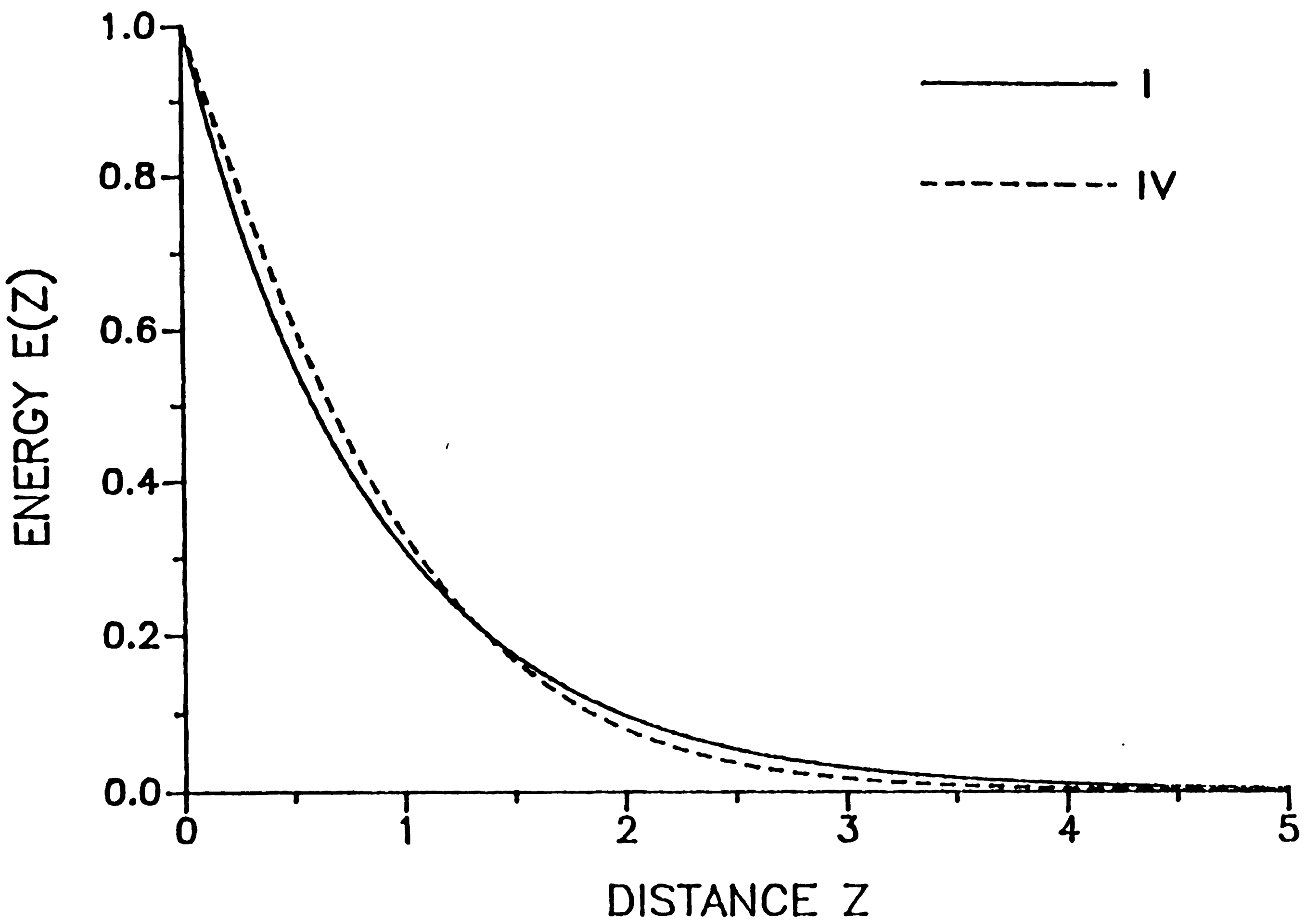
$$z' = z - C_3(1-\lambda)\ln(\frac{z+C_3}{C_3}), \qquad \lambda = \frac{\kappa}{2k} < 1. \tag{4.87}$$

This result will be called result IV.

The result IV is sharper than estimate I for sufficiently large z(see Fig. 4.4). The values of z beyond which (4.86) provides the sharper estimate, $z \geq \bar{z}$, are shown in Table 4.3.

Table 4.3: Values of C_3 and $\bar{z}$

E(0)	C_3	$\bar{z}$
0.05	1.00	0.85
0.10	1.13	0.90
0.15	1.25	0.94
0.20	1.35	0.97
0.25	1.45	1.00
0.30	1.55	1.03
0.35	1.64	1.06
0.40	1.73	1.08
0.45	1.82	1.11
0.50	1.91	1.13
1.00	2.75	1.33
1.50	3.59	1.50
2.00	4.43	1.65
2.50	5.26	1.78
3.00	6.10	1.90
3.50	6.95	2.02
4.00	7.81	2.13
5.00	9.54	2.33
6.00	11.27	2.51
7.00	13.03	2.69
8.00	14.81	2.85
10.00	18.38	3.15



TOTAL ENERGY E(0)=1.0

FIG 4.4 COMPARISON OF RESULTS I AND IV

CHAPTER 5

DISCUSSION OF RESULTS

5.1 A SUMMARY OF THE RESULTS OF CHAPTER 4 FOR THE VON KARMAN EQUATIONS

In Chapter 4 we have established four upper bound results on the energy E(z) associated with the the von Kármán system for our problem. Of these, result I of Section 4.4 and result IV of Section 4.7 hold for all total energies E(0), whereas result II of Section 4.5 holds for sufficiently large total energies (E(0) > 0.209) and result III of Section 4.6 holds for sufficiently small total energies (E(0) < 0.529). We now present a summary of these results:

Result I: For any total energy E(0):

$$E(z) \leq E(0)e^{-\kappa z}, \qquad z \geq 0, \tag{5.1}$$

where

$$\kappa = 2k(\sqrt{M^2+1}-M), \tag{5.2}$$

$$M = k\mu\sqrt{E(0)}, \tag{5.3}$$

and

$$k = 1.11, \qquad \mu = 0.619. \tag{5.4}$$

Result II: For sufficiently large energies, E(0) > 0.209:

$$E(z) \leq E(0)e^{-2kz} + C_1\mu E^{3/2}(0)[e^{\frac{-3\kappa z}{2}} - e^{-2kz}], \qquad z \geq 0, \tag{5.5}$$

where

$$C_1 = \frac{24k^2\kappa}{16k^2 - 9\kappa^2} > 0, \tag{5.6}$$

and κ, k are still given by (5.2)-(5.4).

Result III: For sufficiently small total energies, E(0) < 0.529:

$$E(z) \leq \frac{1}{[\frac{1}{\sqrt{E(0)}}e^{kz} - 2k\mu(e^{kz}-1)]^2}, \qquad z \geq 0, \tag{5.7}$$

where k, μ are given by (5.4).

Result IV: For any total energy E(0):

$$E(z) \leq E(0)e^{-2kz'}\frac{\lambda^{-1}z + C_3}{z + C_3}, \qquad z \geq 0, \tag{5.8}$$

where

$$z' = z - C_3(1-\lambda)ln(\frac{z + C_3}{C_3}), \tag{5.9}$$

$$\lambda = \frac{\kappa}{2k} < 1, \tag{5.10}$$

where κ, k are given by (5.2)-(5.4) and C_3 is a constant that depends on E(0) (see Table 4.3).

5.2 A COMPARISON OF ALL RESULTS

In Chapter 4 after establishing result I of Section 4.4 we proceeded to establish a series of improvements on this first result. Once presented, all new results were compared to result I. We now compare all results to one another and construct the best upper bound for $E(z)$ for different total energies E(0). Due to the fact that results II and III are not valid for all energies we break up the discussion into three parts:

A. For E(0) $<$ 0.209 where results I, III and IV are valid.

B. For 0.209 $<$ E(0) $<$ 0.529 where all results are valid.

C. For E(0) $>$ 0.529 where the results I, II, and IV are valid.

A. E(0) $<$ 0.209:

As can be seen from Figures 5.1-5.4 the best upper bound for $E(z)$ is provided by the function:

$$U_1(z) = \begin{Bmatrix} E(0)e^{-\kappa z}, & \text{for } 0 \le z < \bar{z} \\ E(0)e^{-2kz'}\dfrac{\lambda^{-1}z + C_3}{z + C_3}, & \text{for } \bar{z} < z < \infty \end{Bmatrix} \tag{5.11}$$

where z' is as defined in (5.9) and $\bar{z}$ depends on the total energy E(0) (see Table 4.3).

Note that the estimate III does not enter into the composition of $U_1(z)$, and that for this range of total energies result I is the sharpest available for small z, whereas IV is the sharpest for larger z.

B. 0.209 $<$ E(0) $<$ 0.529:

As can be seen from Figures 5.5-5.8 the best upper bound for $E(z)$ is provided by the function:

$$U_2(z) = \begin{cases} E(0)e^{-\kappa z}, & \text{for } 0 < z < \tilde{z} \\ E(0)e^{-2kz} + C_1\mu E^{3/2}(0)(e^{\frac{-3\kappa z}{2}} - e^{-2kz}), & \text{for } z > \tilde{z} \end{cases} \tag{5.12}$$

where $\tilde{z}$ depends on the total energy E(0) (see Table 4.1).

Note that results III and IV do not enter into the composition of $U_2(z)$, and that for this range of energies result I is the sharpest available for small z, whereas II is the sharpest for larger z.

C. E(0) > 0.529:

As can be seen from Figures 5.9-5.12 the best upper bound for $E(z)$ is provided by the function $U_2(z)$ defined in (5.12). Qualitatively, for this range of E(0) we have exactly what held for the range of energies in B with the only exception being that the estimate III is not valid here.

Another way to compare the bounds obtained is to determine the distance z_{95} from the left end of the plate $z = 0$, at which 95% of the total energy E(0) has dissipated (see Table 5.1) and the distance z_{99} from the end, at which 99% of the total energy E(0) has dissipated (see Table 5.2). These distances are called "characteristic decay lengths". Note that a characteristic decay length of 2.00 corresponds to one width of the plate.

Table 5.1: z_{95} values

E(0)	I	II	III	IV
0.05	1.58	-	1.62	*1.53*
0.10	1.68	-	1.78	*1.61*
0.15	1.76	-	1.92	*1.67*
0.20	1.83	-	2.08	*1.73*
0.25	1.90	*1.70*	2.25	1.78
0.30	1.95	*1.74*	2.45	1.82
0.40	2.06	*1.82*	2.99	1.91
0.50	2.16	*1.88*	4.35	1.98
1.00	2.57	*2.16*	-	2.30
2.00	3.20	*2.60*	-	2.79
3.00	3.71	*2.96*	-	3.20
4.00	4.15	*3.28*	-	3.55
5.00	4.55	*3.57*	-	3.87
6.00	4.92	*3.84*	-	4.16
7.00	5.26	*4.09*	-	4.44
10.00	6.16	*4.76*	-	5.17

Table 5.2: z_{90} values

E(0)	I	II	III	IV
0.05	2.42	-	2.38	*2.32*
0.10	2.58	-	2.55	*2.40*
0.15	2.70	-	2.71	*2.48*
0.20	2.81	-	2.88	*2.55*
0.25	2.91	*2.52*	3.06	2.61
0.30	3.00	*2.57*	3.27	2.67
0.40	3.17	*2.67*	3.83	2.77
0.50	3.32	*2.76*	4.35	2.87
1.00	3.95	*3.14*	-	3.28
5.00	6.99	*5.20*	-	5.33
10.00	9.47	*6.96*	-	7.04

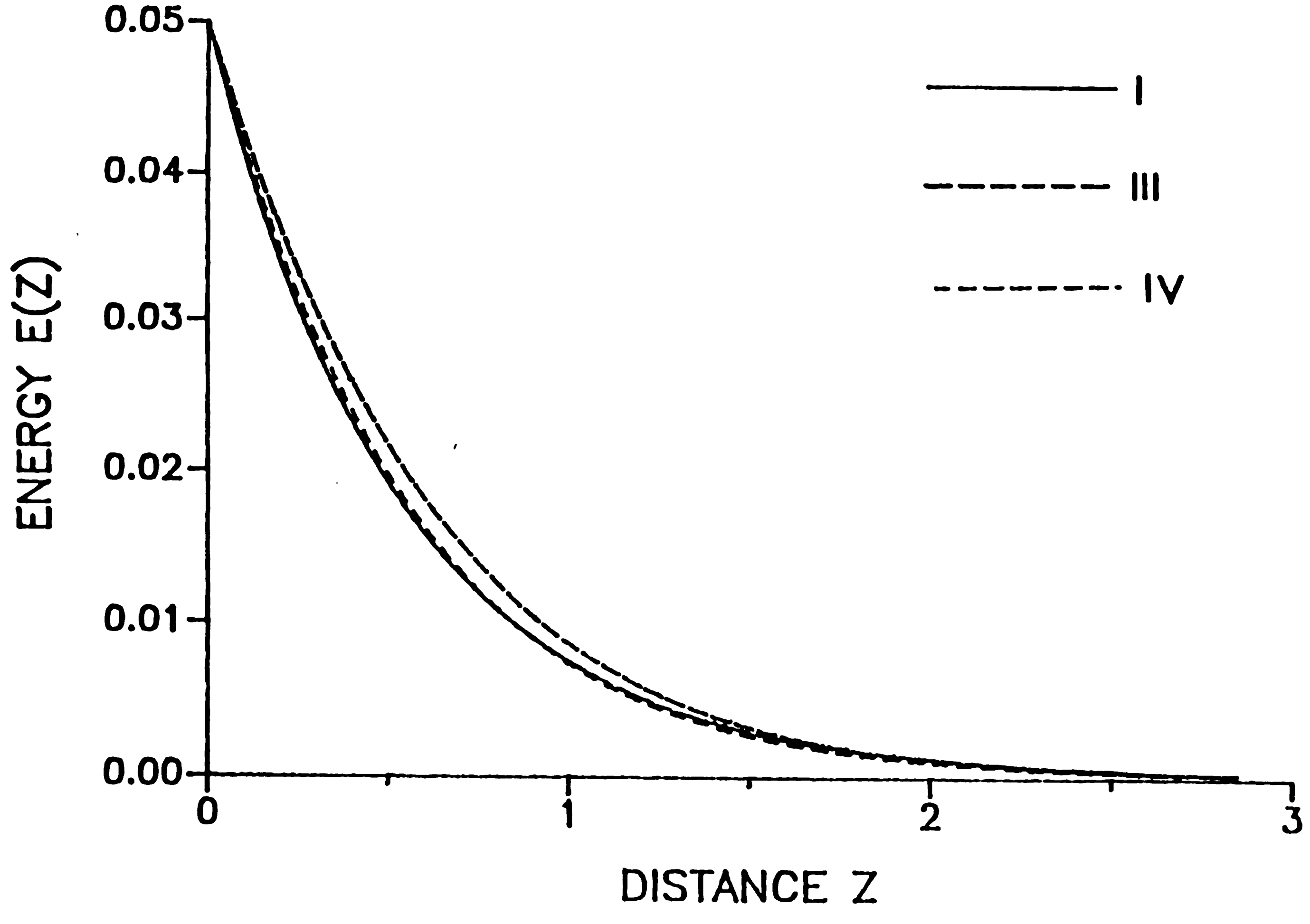


FIG. 5.1 TOTAL ENERGY E(0)=0.05 (CASE A)

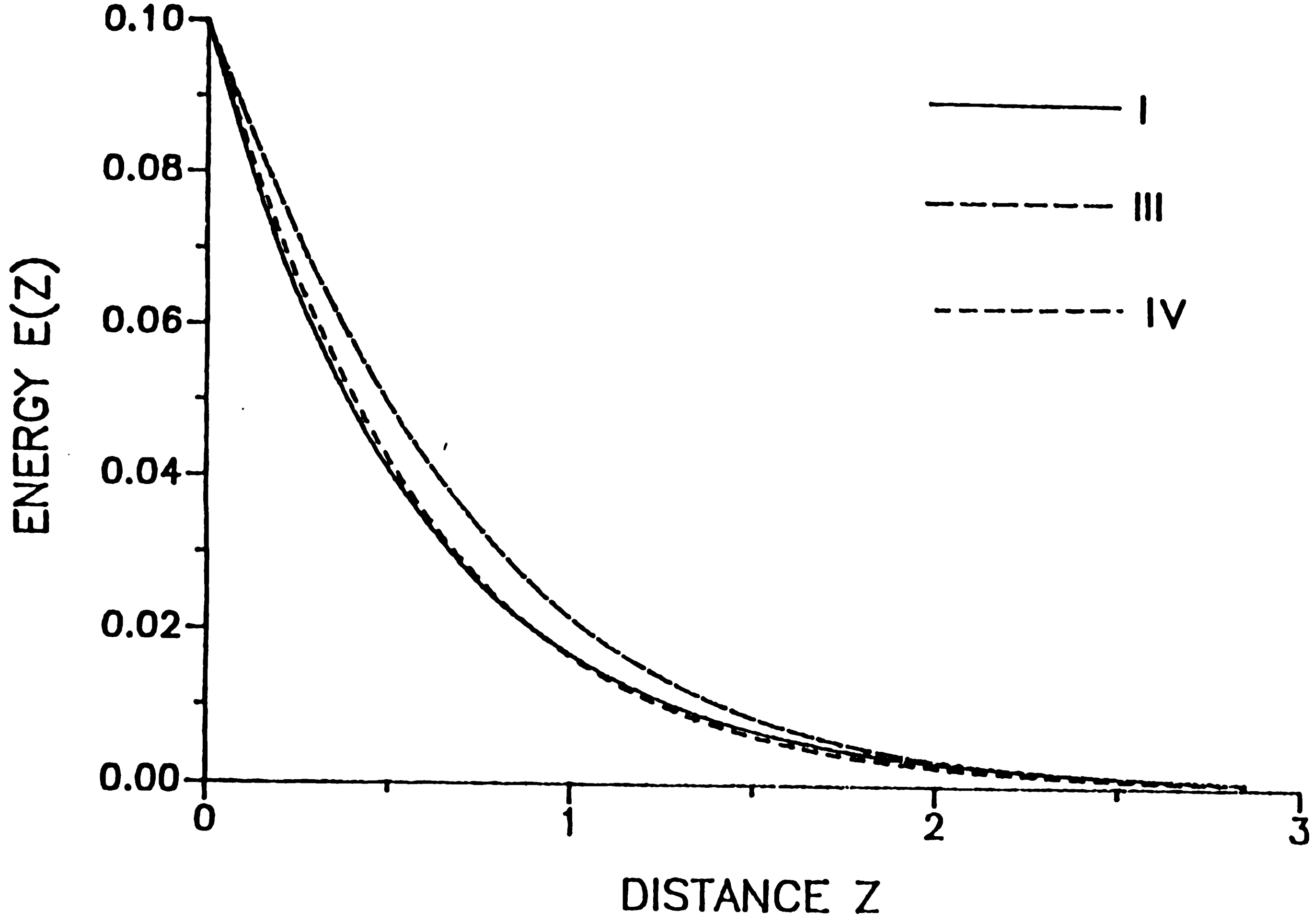


FIG. 5.2 TOTAL ENERGY E(0)=0.10 (CASE A)

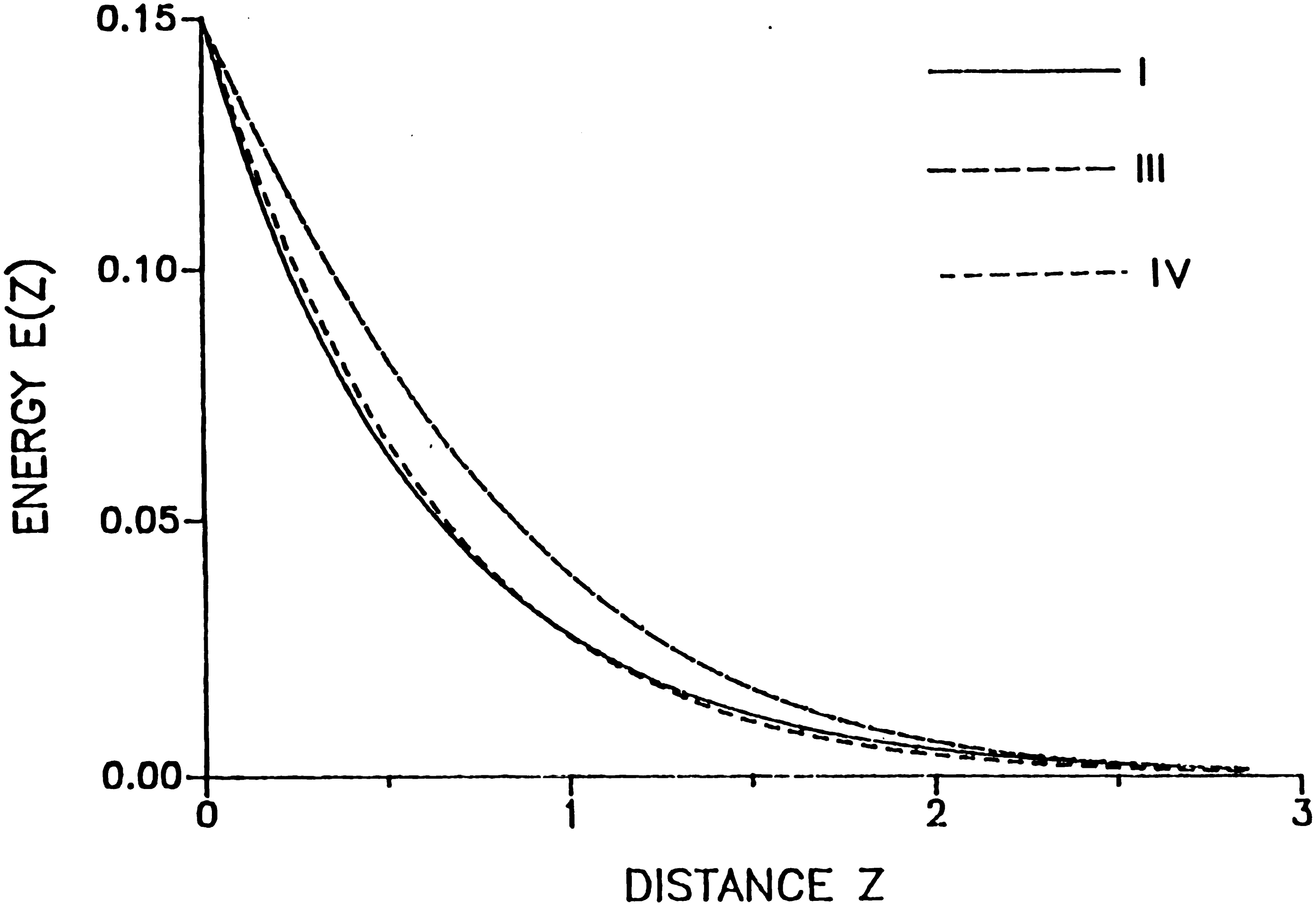


FIG. 5.3 TOTAL ENERGY E(0)=0.15 (CASE A)

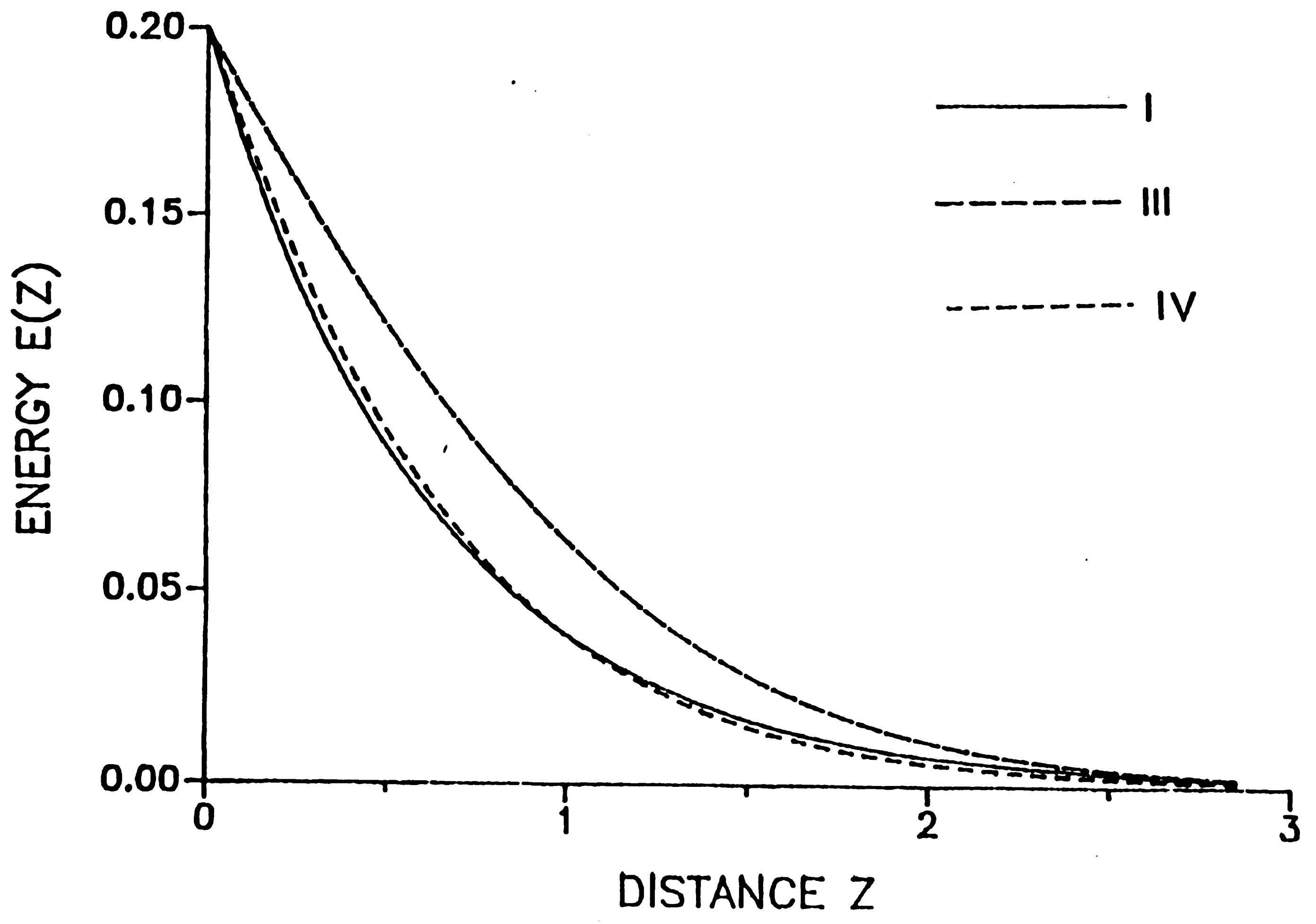


FIG. 5.4 TOTAL ENERGY E(0)=0.20 (CASE A)

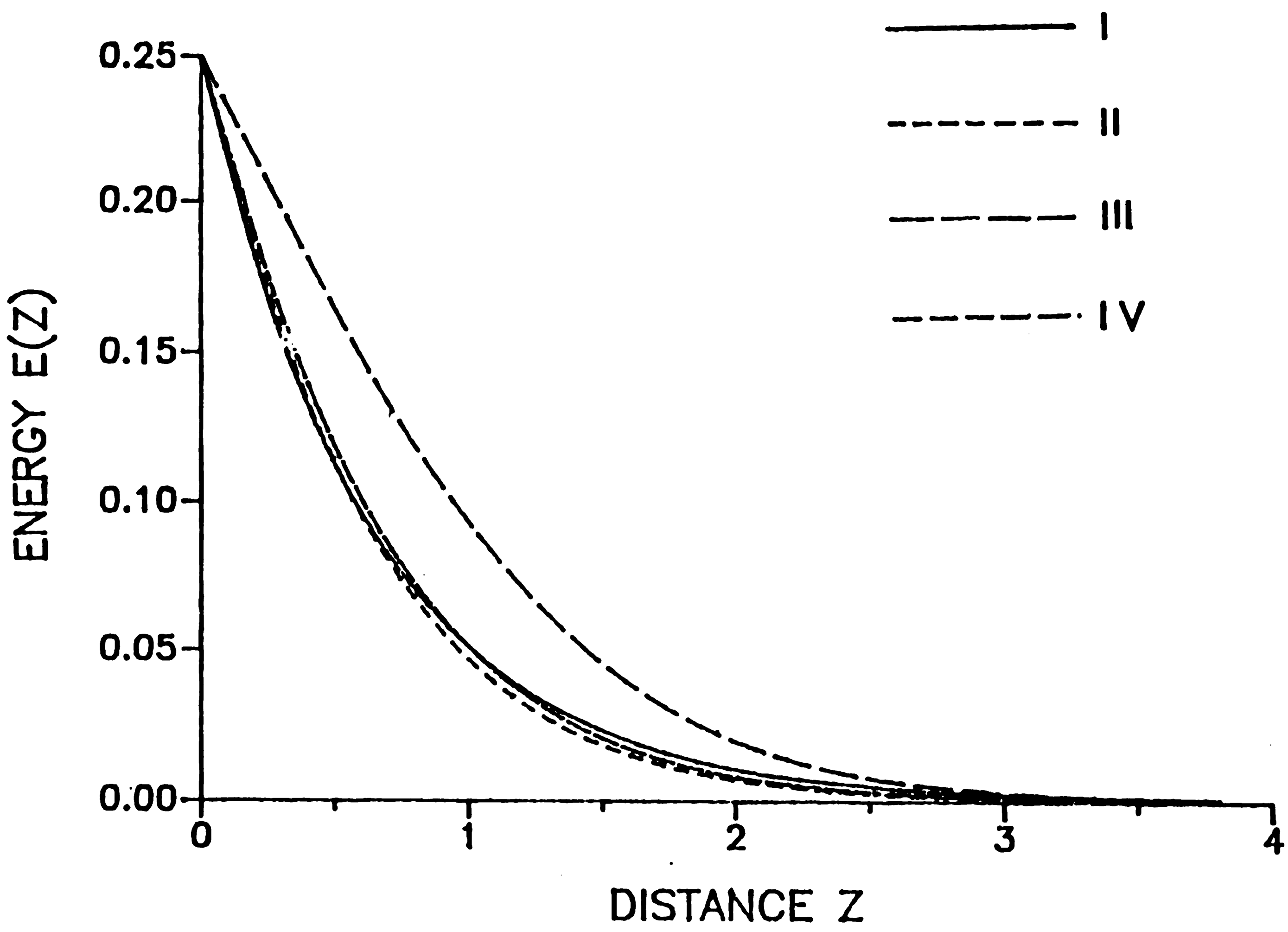


FIG. 5.5 TOTAL ENERGY E(0)=0.25 (CASE B)

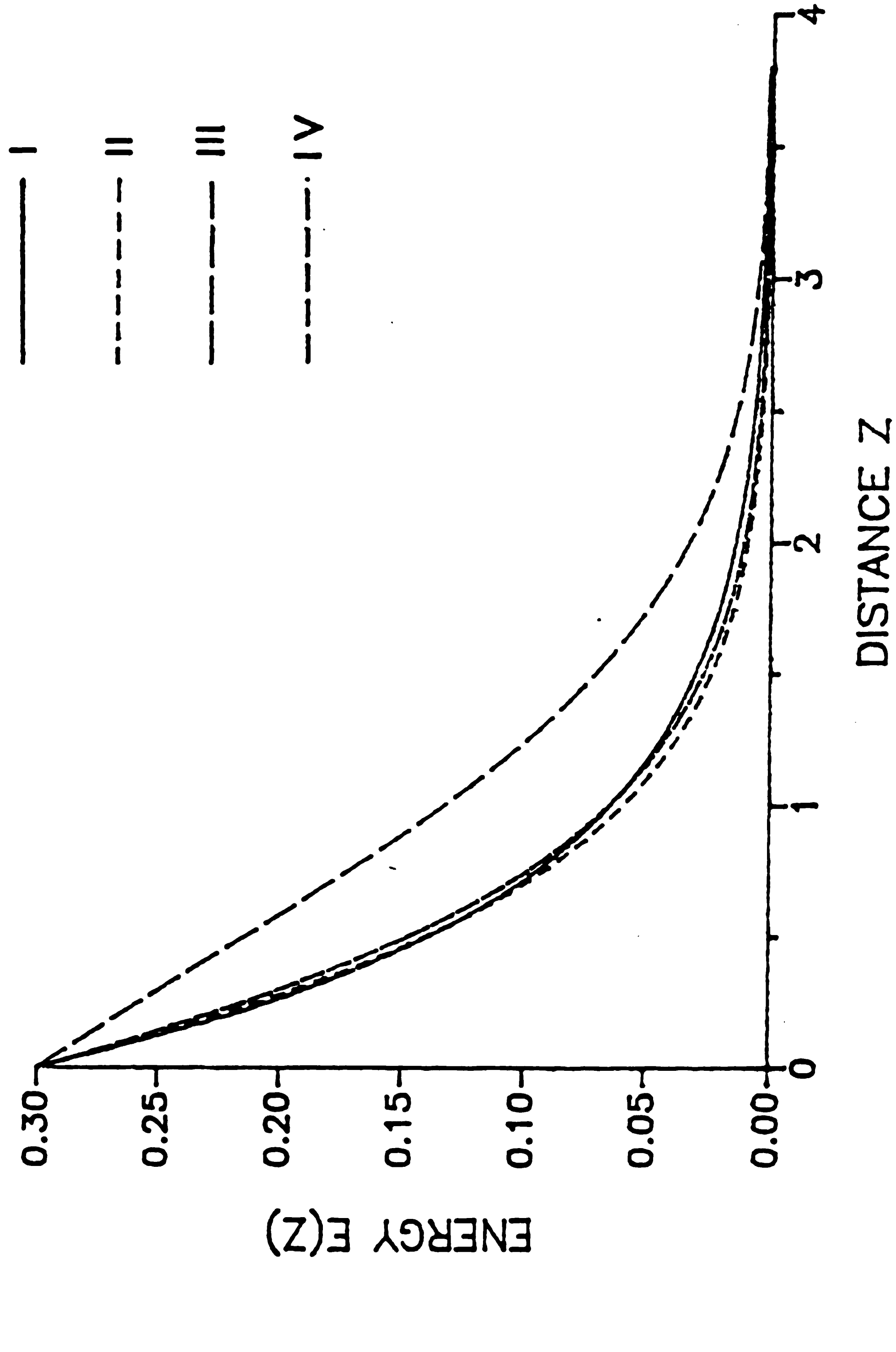


FIG. 5.6 TOTAL ENERGY E(O)=0.30 (CASE B)

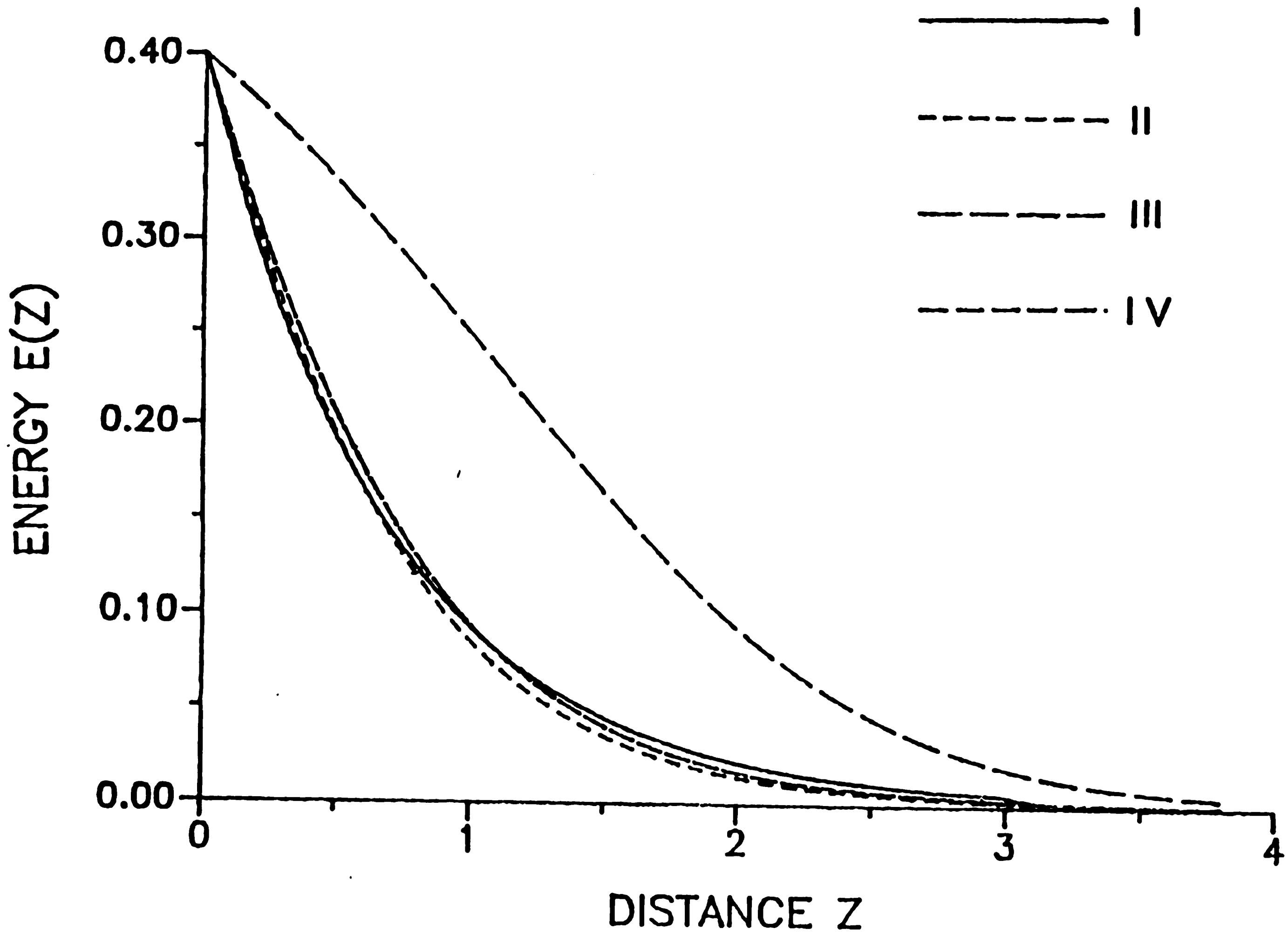


FIG. 5.7 TOTAL ENERGY E(0)=0.40 (CASE B)

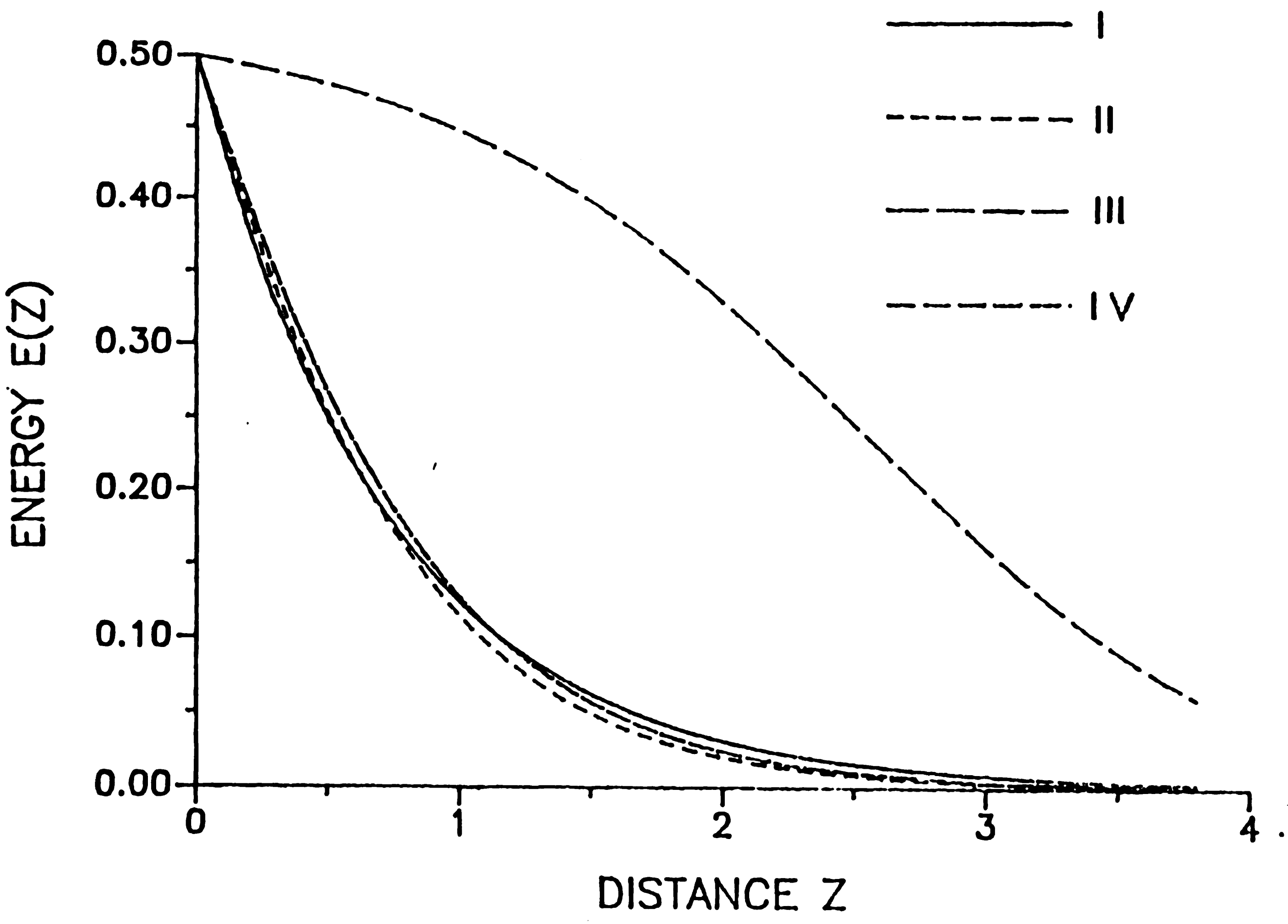


FIG. 5.8 TOTAL ENERGY E(0)=0.50 (CASE B)

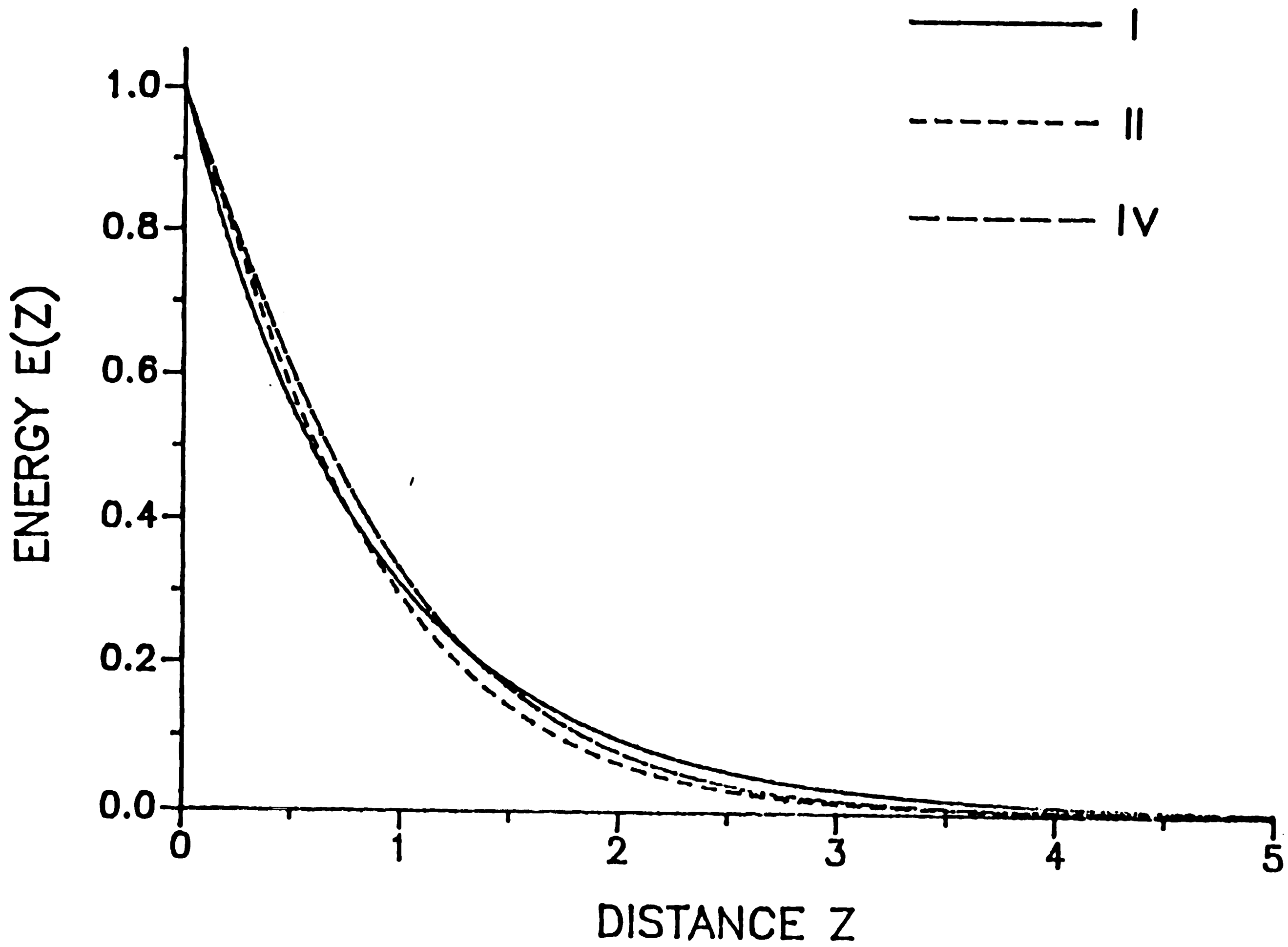


FIG. 5.9 TOTAL ENERGY E(0)=1.0 (CASE C)

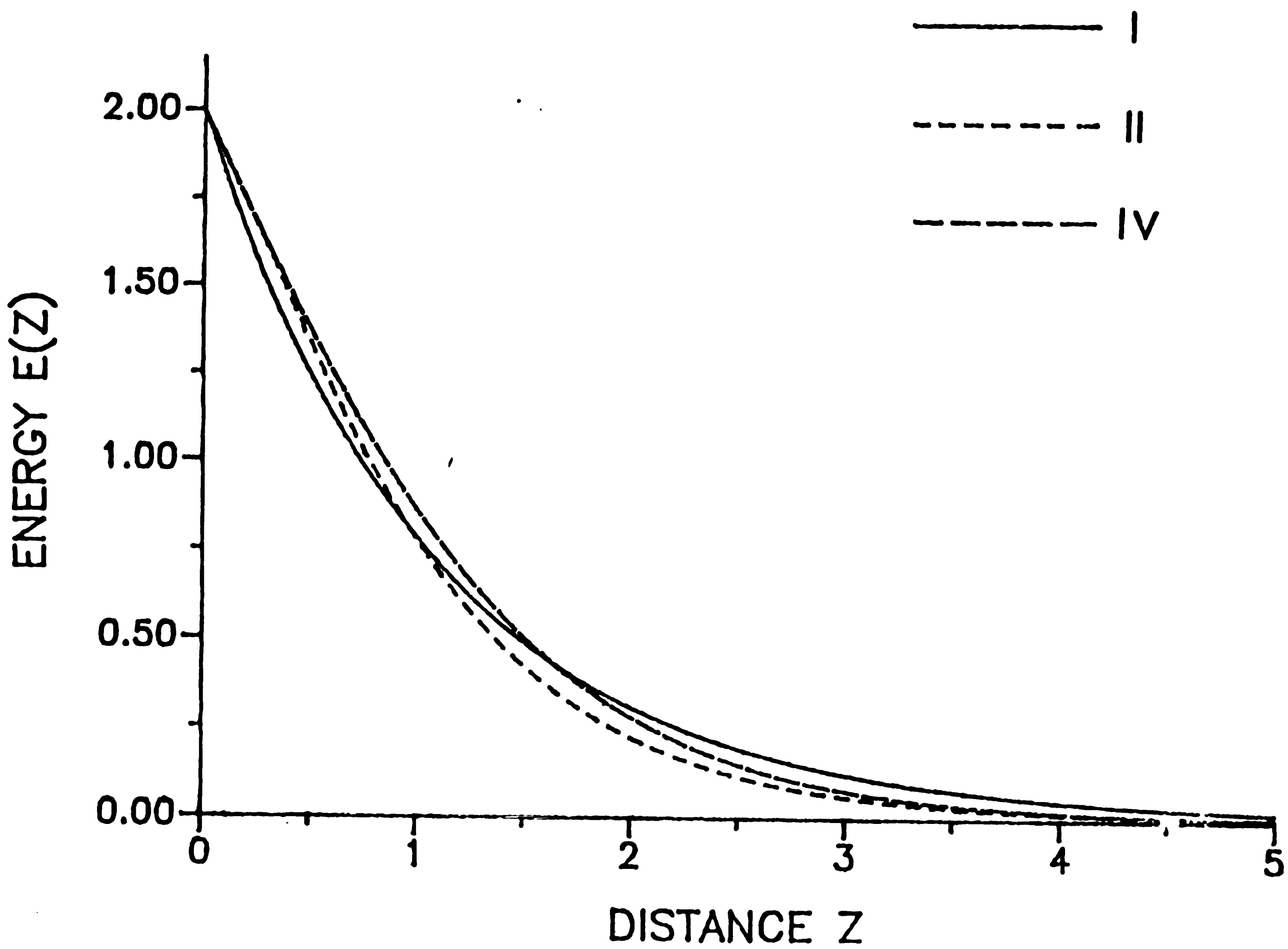


FIG. 5.10 TOTAL ENERGY E(0)=2.0 (CASE C)

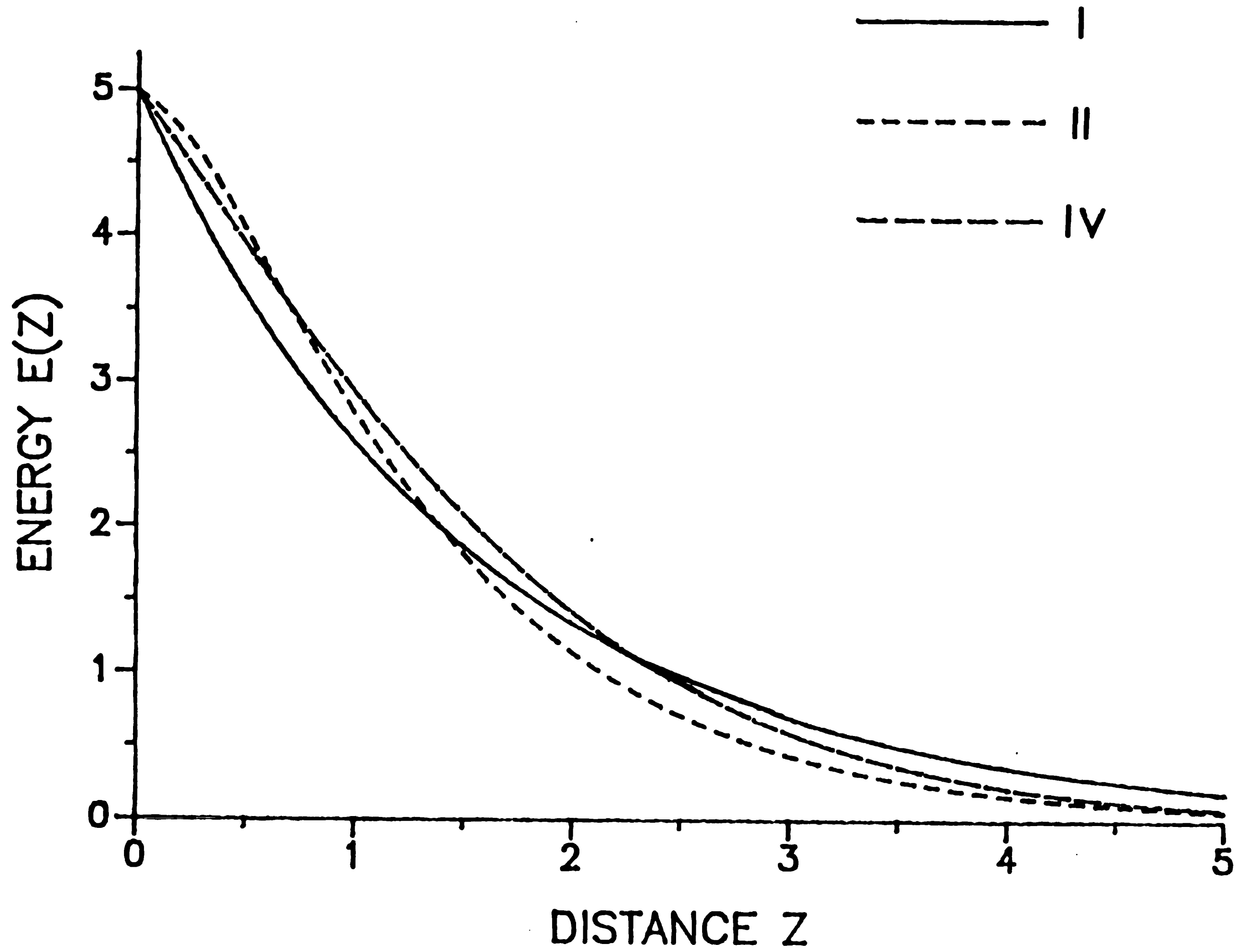


FIG. 5.11 TOTAL ENERGY E(0)=5.0 (CASE C)

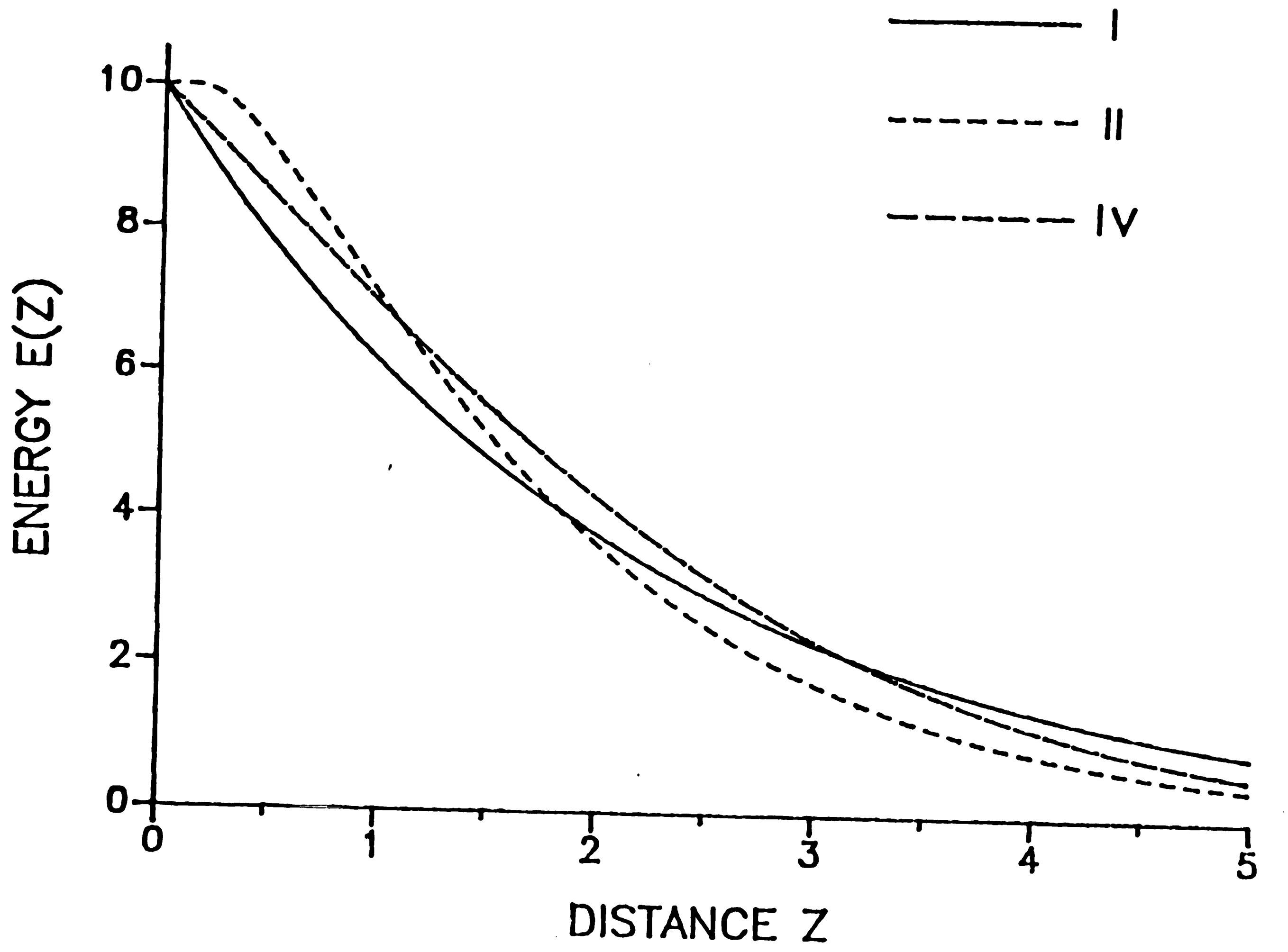


FIG. 5.12 TOTAL ENERGY E(0)=10.0 (CASE C)

5.3 REMARKS ON THE CONSTANTS k, μ AND THE TOTAL ENERGY E(0)

As already discussed in Chapter 3, we have used the largest available value of k for which (3.15) holds, k = 1.11, which underestimates the exact decay rate, k = 2.10. If one could establish (3.15) with a value of k larger than 1.11, the results of Chapter 4 would still hold and would become sharper. One would only need to re-evaluate the values of $\tilde{z}$, $\bar{z}$, z_0 and the constant C_3.

The constant μ in (4.17) is probably greatly overestimated because of the repeated use of some weak inequalities in its derivation (see Appendix A). Any improvement in the value of μ would result in an immediate improvement of all results in Chapter 4.

All results presented in Chapter 4 involve the total energy E(0) which depends on the geometry and the boundary data at the $z = 0$ end. In [3] Knowles established an upper bound for the total energy E(0) of the biharmonic problem in terms of the applied traction for a certain class of finite domains with the help of variational arguments (see also [4]). The results of Chapter 3 remain valid when E(0) is replaced by an upper bound. It is reasonable to anticipate that by using variational arguments an upper bound for the total energy E(0) of our problem can be established in terms of the traction applied at the end and the displacement there. We shall not pursue this issue here.

5.4 QUESTIONS FOR FURTHER INVESTIGATION

We can now raise a number of interesting questions for further investigation. Firstly, in the context of the semi-infinite strip, one could ask how our decay estimates depend on the particular boundary conditions used. Secondly, one could introduce a distributed lateral load on the plate and investigate its effect. Thirdly, interesting bifurcation and associated stability questions arise when non self-equilibrated compressive loads are introduced. Such stability questions have already been investigated by many authors for specific sets of boundary conditions. It would be very interesting to attempt to extend these already existing results by admitting wider classes of boundary conditions through the application of a Saint-Venant principle type of argument. Finally, our problem as well as any of the above mentioned open questions can be posed for a "long" thin plate of finite size.

APPENDIX A

VERIFICATION OF (4.17)

To verify (4.17) we follow an approach used by HORGAN [28]. In this development we make use of the following Wirtinger-type inequalities for sufficiently smooth functions $w(y)$ defined on the interval (-1,1) of length 2:

(i)If $w(y)$ is C^1(-1,1) and $w(-1) = w(1) = 0$, then[1]

$$\int_{-1}^{1} w_y^2 dy \geq \frac{\pi^2}{2^2} \int_{-1}^{1} w^2 dy. \tag{A.1}$$

(ii) If $w(y)$ is C^2(-1,1) and $w(-1) = w'(-1) = w(1) = w'(1) = 0$, then

$$\int_{-1}^{1} w_{yy}^2 dy \geq \frac{4\pi^2}{2^2} \int_{-1}^{1} w_y^2 dy, \tag{A.2}$$

and

$$\int_{-1}^{1} w_{yy}^2 dy \geq \frac{\mu_0^4}{2^4} \int_{-1}^{1} w^2 dy, \tag{A.3}$$

where μ_0 is the smallest positive root of the transcendental equation $\cos\mu\cosh\mu = 1$ and so $\mu_0 = 4.73$ which is slightly larger than $3\pi/2$. For convenience, we use the latter value in (A.3) and thus we obtain

$$\int_{-1}^{1} w_{yy}^2 dy > \frac{3^4\pi^4}{2^8} \int_{-1}^{1} w^2 dy. \tag{A.4}$$

For a more detailed discussion of the inequalities (A.1) - (A.3), see [28] and the

[1] The multiplicative constant is written in a manner displaying the interval length explicitly.

references cited therein.

We also make use of the following one-dimensional Sobolev inequality:

$$\int_{-1}^{1} w^4 dy \leq \frac{\sigma}{2}\left(\int_{-1}^{1} w^2 dy\right)\left(\int_{-1}^{1} w_y^2 dy\right) \tag{A.5}$$

for sufficiently smooth functions $w(y)$ such that $w(-1) = w(1) = 0$. Inequalities of the form (A.5) in two and three dimensions have been widely used in investigations of uniqueness and stability for the Navier-Stokes equations. A direct proof of (A.5) is given by HORGAN [28], where it is shown that (A.5) holds with σ=4. A modification of the argument given in [28] may be used to show that (A.5) holds with σ=1. This value of σ is taken in (A.25) below.

Finally here, we state the following simple algebraic inequality

$$(a + b)^{3/2} \geq \sqrt{3}a\sqrt{b} \qquad \text{for } a, b > 0, \tag{A.6}$$

which we also use in the sequel.

We now employ some of the above inequalities to establish a series of intermediate results:

A. We first establish an upper bound for

$$\int_{R_z} m^4(x,y)\, dA \tag{A.7}$$

where $m(x,y)$ is a function that vanishes along the semi-infinite clamped sides of the plate and that tends uniformly to zero as $x \rightarrow \infty$. In view of (2.22) and (2.24) the functions u, v as well as their first partial derivatives u_x, u_y, v_x, v_y can be substituted for m. On applying the Sobolev inequality (A.5) we get

$$\int_{R_z} m^4(x,y)\, dA \leq \frac{\sigma}{2}\int_z^{\infty}\left[\int_{L_s} m^2(s,y)\, dy\right]\left[\int_{L_s} m_y^2(s,y)\, dy\right] ds. \tag{A.8}$$

Green's theorem, the regularity conditions and the Cauchy-Schwarz inequality allow us to write, for such $s \geq z$,

$$\int_{L_s} m^2(s,y)dy = -2\int_{R_s} m(s,y)m_s(s,y)dA$$
$$\leq 2\Big(\int_{R_s} m^2 dA\Big)^{1/2}\Big(\int_{R_s} m_s^2 dA\Big)^{1/2}$$
$$\leq 2\Big(\int_{R_s} m^2 dA\Big)^{1/2}\Big(\int_{R_s} m_z^2 dA\Big)^{1/2}. \tag{A.9}$$

On combining (A.8) and (A.9) we get

$$\int_{R_s} m^4 dA \leq \sigma\Big(\int_{R_s} m^2 dA\Big)^{1/2}\Big(\int_{R_s} m_z^2 dA\Big)^{1/2}\Big(\int_{R_s} m_y^2 dA\Big). \tag{A.10}$$

B. Suppose now that the function m in (A.10) is chosen to be u (or equivalently v). Then we have

$$\int_{R_s} u^4 dA \leq \sigma\Big(\int_{R_s} u^2 dA\Big)^{1/2}\Big(\int_{R_s} u_z^2 dA\Big)^{1/2}\Big(\int_{R_s} u_y^2 dA\Big). \tag{A.11}$$

We now apply inequality (A.3) with $w = u$, in an obvious way in (A.11) to get

$$\int_{R_s} u^4 dA \leq \frac{\sigma 2^4}{3^2\pi^2}\Big(\int_{R_s} u_{yy}^2 dA\Big)^{1/2}\Big(\frac{2}{\pi^2}\int_{R_s} 2u_{zy}^2 dA\Big)^{1/2}\Big(\frac{1}{\pi^2}\int_{R_s} u_{yy}^2 dA\Big)$$
$$\leq \frac{\sigma 2^{9/2}}{9\pi^5}[E_u(z)]^2. \tag{A.12}$$

Note that (A.12) is also valid when we write v in place of u.

C. Now suppose that the function m in (A.10) is chosen to be u_y (or equivalently v_y). We have

$$\int_{R_s} u_y^4 dA \leq \sigma\Big(\int_{R_s} u_y^2 dA\Big)^{1/2}\Big(\int_{R_s} u_{zy}^2 dA\Big)^{1/2}\Big(\int_{R_s} u_{yy}^2 dA\Big). \tag{A.13}$$

We now apply inequality (A.2) with $w=u$ to the first integral on the right-hand side of (A.13) to get

$$\int_{R_s} u_y^4 dA \leq \frac{\sigma}{\pi}\Big(\int_{R_s} u_{yy}^2 dA\Big)^{3/2}\Big(\int_{R_s} u_{zy}^2 dA\Big)^{1/2} \leq \frac{\sigma}{2^{1/2}\pi}E_u^2(z). \tag{A.14}$$

Again, we note that (A.14) is also valid when we write v in place of u.

D. Now, suppose that the function m in (A.10) is chosen to be u_z (or equivalently v_z). We have

$$\int_{R_s} u_z^4 dA \leq \sigma\Big(\int_{R_s} u_z^2 dA\Big)^{1/2}\Big(\int_{R_s} u_{zz}^2 dA\Big)^{1/2}\Big(\int_{R_s} u_{zy}^2 dA\Big). \tag{A.15}$$

Now apply inequality (A.1) with $w = u_x$ to the first integral on the right-hand side of (A.15) to get

$$\int_{R_z} u_x^4 dA \le \frac{2\sigma}{\pi}\Big(\int_{R_z} u_{xy}^2 dA\Big)^{3/2}\Big(\int_{R_z} u_{xx}^2 dA\Big)^{1/2}$$
$$\le \frac{\sigma}{2^{1/2}\pi} E_u^2(z). \tag{A.16}$$

Again, we note that (A.16) is also valid when we write v in place of u.

E. In view of inequality (A.2) we have

$$\int_{R_z} u_y^2 dA \le \frac{1}{\pi^2}\int_{R_z} u_{yy}^2 dA$$
$$\le \frac{E_u(z)}{\pi^2}, \tag{A.17}$$

which is also valid when we write v in place of u.

F. In view of inequality (A.1) we have

$$\int_{R_z} u_x^2 dA \le \frac{2^2}{\pi^2}\int_{R_z} u_{xy}^2 dA$$
$$\le \frac{2E_u(z)}{\pi^2}, \tag{A.18}$$

which is also valid when we write v in place of u.

We now proceed to obtain upper bounds for the five integrals $I_1 - I_5$ defined in (4.16) and thereby establish (4.17). The Cauchy-Schwarz inequality, (A.14), (A.18) for v_x and (A.6) yield:

$$I_1 = \int_{R_z} u_y^2 v_x dA$$
$$\le \Big(\int_{R_z} u_y^4 dA\Big)^{1/2}\Big(\int_{R_z} v_x^2 dA\Big)^{1/2}$$
$$\le \frac{\sigma^{1/2} 2^{1/4}}{3^{1/2}\pi^{3/2}} E^{3/2}(z). \tag{A.19}$$

The Cauchy-Schwarz inequality, (A.16), (A.12) for v and (A.6) yield:

$$
\begin{aligned}
I_2 &= 2\int_{R_s} u_x u_{yy} v\, dA \\
&\le 2\Big(\int_{R_s} u_{yy}^2 dA\Big)^{1/2}\Big(\int_{R_s} u_x^4 dA\Big)^{1/4}\Big(\int_{R_s} v^4 dA\Big)^{1/4} \\
&\le 2\Big(\frac{2\sigma^{1/2}}{3\pi^{3/2}}\Big)E^{3/2}(z).
\end{aligned} \tag{A.20}
$$

Similarly, the Cauchy-Schwarz inequality, (A.12), (A.14) for v and (A.6) yield:

$$
\begin{aligned}
I_3 &= \int_{R_s} u u_y v_{xy}\, dA \\
&\le \Big(\int_{R_s} v_{xy}^2 dA\Big)^{1/2}\Big(\int_{R_s} u^4 dA\Big)^{1/4}\Big(\int_{R_s} u_y^4 dA\Big)^{1/4} \\
&\le \frac{\sigma^{1/2}2^{1/2}}{3\pi^{3/2}}E^{3/2}(z).
\end{aligned} \tag{A.21}
$$

The Cauchy Schwarz inequality, (A.17), (A.16), (A.14) for v and (A.6) yield:

$$
\begin{aligned}
I_4 &= \int_{R_s} u_x u_y v_y\, dA \\
&\le \Big(\int_{R_s} u_y^2 dA\Big)^{1/2}\Big(\int_{R_s} u_x^4 dA\Big)^{1/4}\Big(\int_{R_s} v_y^4 dA\Big)^{1/4} \\
&\le \frac{\sigma^{1/2}}{2^{1/4}3^{1/2}\pi^{3/2}}E^{3/2}(z).
\end{aligned} \tag{A.22}
$$

The Cauchy-Schwarz inequality, (A.12), (A.14) for v and (A.16) yield:

$$
\begin{aligned}
I_5 &= \int_{R_s} u u_{xy} v_y\, dA \\
&\le \Big(\int_{R_s} u_{xy}^2 dA\Big)^{1/2}\Big(\int_{R_s} u^4 dA\Big)^{1/4}\Big(\int_{R_s} v_y^4 dA\Big)^{1/4} \\
&\le \frac{2^{1/2}\sigma^{1/2}}{3\pi^{3/2}}E^{3/2}(z).
\end{aligned} \tag{A.23}
$$

Combining the estimates (A.19)-(A.23) we obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 \le \mu E^{3/2}(z) \tag{A.24}$$

where

$$\mu = \Big(\frac{2^{1/4}}{3^{1/2}} + \frac{4}{3} + \frac{2^{1/2}}{3} + \frac{1}{2^{1/4}3^{1/2}} + \frac{2^{1/2}}{3}\Big)\frac{\sigma^{1/2}}{\pi^{3/2}} \approx 0.619, \tag{A.25}$$

and the numerical value for μ in (A.25) has been obtained on taking σ=1. This establishes (4.17) as desired.

REFERENCES

[1] GURTIN, M. E., The linear theory of elasticity, Handbuch der Physik, C. TRUESDELL ed., Vol. VIa/2, pp. 1-295. Springer-Verlag, Berlin, 1972.

[2] TOUPIN, R. A., Saint-Venant's principle. Arch. Rational Mech. Anal. **18**, 83-96 (1965).

[3] KNOWLES, J. K., On Saint-Venant's principle in the two-dimensional theory of elasticity. Arch. Rational Mech. Anal. **21**, 1-22 (1966).

[4] HORGAN, C. O., & KNOWLES, J. K., Recent developments concerning Saint-Venant's principle, Advances in Applied Mechanics, J. W. HUTCHINSON ed., Vol 23, pp. 179-269. Academic Press, New York, 1983.

[5] ROSEMAN, J. J., The principle of Saint-Venant in linear and nonlinear plane elasticity. Arch. Rational Mech. Anal. **26**, 142-162 (1967).

[6] BREUER, S., & ROSEMAN, J. J., On Saint-Venant's principle in three-dimensional nonlinear elasticity. Arch. Rational Mech. Anal. **63**, 191-203 (1977).

[7] HORGAN, C. O., & KNOWLES, J. K., The effect of nonlinearity on a principle of Saint-Venant type. Journal of Elasticity **11**, 271-291 (1981).

[8] BREUER, S., & ROSEMAN, J. J., Saint-Venant's principle in nonlinear plane elasticity with sufficiently small strains. Arch. Rational Mech. Anal. **80**, 19-37

(1982).

[9] KNOPS, R. J. & PAYNE, L. E., A Saint-Venant principle for nonlinear elasticity. Arch. Rational Mech. Anal. **81**, 1-12 (1983).

[10] ABEYARATNE, R., HORGAN, C. O., & CHUNG, D.-T., Saint-Venant end effects for incremental plane deformations of incompressible nonlinearly elastic materials. J. Appl. Mech. **52**, 847-852 (1985).

[11] HORGAN, C. O., & PAYNE, L. E., Decay estimates for second order quasilinear partial differential equations. Advances in Appl. Math. **5**, 309-332 (1984).

[12] HORGAN, C. O., & PAYNE, L. E., Decay estimates for a class of second-order quasilinear equations in three dimensions. Arch. Rational Mech. Anal. **86**, 279-289 (1984).

[13] BREUER, S., & ROSEMAN, J. J., Phragmen-Lindelöf decay theorems for classes of nonlinear Dirichlet problems in a circular cylinder. J. Math. Anal. Appl. **113**, 59-77 (1986).

[14] BREUER, S., & ROSEMAN, J. J., Decay theorems for nonlinear Dirichlet problems in semi-infinite cylinders. Arch. Rational Mech. Anal. **94**, 363-371 (1986).

[15] KARMAN, T. von, Festigkeitsprobleme im maschinbau, Encyl. der Math. Wissenschaften, Vol. IV-4, pp. 348-352, Leipzig, 1907-1914.

[16] KALANTAROV, V., Determination of the solutions of the first boundary value problem for a system of Kármán equations having an unbounded energy integral. J. Soviet Math. **21**, 711-714 (1983).

[17] STOKER, J. J., Nonlinear Elasticity, Gordon and Breach, New York, 1968.

[18] KNIGHTLY, G. H., An existence theorem for the von Kármán equations. Arch. Rational Mech. Anal. **27**, 233-242 (1967).

[19] KNOWLES, J. K., An energy estimate for the biharmonic equation and its application to Saint-Venant's principle in plane elastostatics. Indian J. Pure Appl. Math. **14**, 791-805 (1983).

[20] FLAVIN, J. N., On Knowles' version of Saint-Venant's principle in two-dimensional elastostatics. Arch. Rational Mech. Anal. **53**, 366-375 (1974).

[21] OLEINIK, O. A., & YOSIFIAN, G. A., On Saint-Venant's principle in plane elasticity theory. Soviet Math. Dokl. **19**, 364-368 (1978).

[22] OLEINIK, O. A., & YOSIFIAN, G. A., The Saint- Venant principle in the two-dimensional theory of elasticity and boundary problems for a biharmonic equation in unbounded domains. Siberian Math. J. **19**, 813-822 (1978).

[23] MIETH, H. J., Über abklingende Lösungen elliptischer Randwertprobleme (Prinzip von Saint-Venant). Dissertation, Technische Hochschule Darmstadt (1975).

[24] GREGORY, R. D., The traction boundary-value problem for the elastostatic semi-infinite strip; existence of solution and completeness of the Papkovich-Fadle eigenfunctions. J. Elasticity **10**, 295-327 (1980).

[25] BERGER, M. S., On von Kármán's equations and the buckling of a thin elastic plate I. Comm. Pure Appl. Math. **20**, 687-719 (1967).

[26] LAKSHMIKANTHAM, V., & LEELA, S., Differential and Integral Inequali-

ties, Vol. 1, Academic Press, New York, 1969.

[27] WALTER, W., Differential and Integral Inequalities, Springer-Verlag, New York, 1970.

[28] HORGAN, C. O., Plane entry flows and energy estimates for the Navier-Stokes equations. Arch. Rational Mech. Anal. **68**, 359-381 (1978).