ABSTRACT

GENERALIZATION OF GRAD'S THIRTEEN-MOMENT
METHOD TO MAGNETOGASDYNAMICS

By

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The present work is primarily concerned with the generalization of thirteen-moment method developed by Grad valid in neutral gases to one-component charged gases.

A distribution function is defined as the mass density of particles in a one-component, homogeneous, uniformly charged gas. Several moments of distribution function are defined as symmetrical tensors. The Boltzmann equation is multiplied by a summational invariant and integrated over the velocity space. By setting the summational invariant to equal to unity, velocity and velocity square, the Boltzmann equation yields the continuity, momentum and energy equations respectively. Using Hermite polynomial approximation, the equations of time, physical and velocity space variation of the second and third order tensors are obtained from the Boltzmann equation. These equations, along with the conservation equations constitute the system of thirteen-moment equations. By considering a one-dimensional heat flow in a gas at rest and a plane Couette flow, the thermal conductivity and the coefficient of viscosity are deduced from the system of thirteen-moment equations.

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To Agnes, my wife

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NOMENCLATURE

$a_i^{(n)}$	Hermite coefficients
B_i	Magnetic induction
$\bar{B}_1$	Geometric collision parameter
$\vec{C}$	Intrinsic velocity $= \vec{\xi} - \vec{u}$
$\vec{d}$	Intrinsic velocity $= \vec{\xi}_1 - \vec{u}$
e	Electric charge
E_i	Electric field
f	Distribution function (mass density)
$f^{(0)}$	Maxwellian distribution
F	Distribution function (number density)
G	Mass ratio $= 2 \frac{m_e}{m_N}$
$\vec{H}$	Magnetic flux density
$\mathcal{H}_i^{(n)}$	Hermite polynomials
$\vec{J}$	Electric current density
$J^{(n)}$	Collision integral
k	Boltzmann's constant
L	Characteristic dimension
L_i	Particle acceleration $= \frac{\partial \xi_i}{\partial t}$
m	Mass of charged particle
m_e	Mass of electron
m_N	Mass of neutral particle
N_N	Number density of neutral particles

p

p_{ij}

P_{ij}

$Q_{ijk\ell}$

r

R

S_i

S_{ijk}

t

T

$\vec{u}$

U

$\vec{v}$

$\vec{x}$

$\vec{\alpha}$

γ

δ_{ij}

ϵ

$\bar{\epsilon}$

ϵ_{ijk}

θ

λ

$\bar{\lambda}$

λ^*

μ

μ^*

μ_e

p	Pressure
p_{ij}	Stress tensor
P_{ij}	Second moment of d.f. f
$Q_{ijk\ell}$	Fourth moment of d.f. f
r	Radius in spherical coordinate system
R	Gas constant
S_i	Heat flow
S_{ijk}	Third moment of d.f. f
t	Time
T	Temperature
$\vec{u}$	Mean velocity
U	Relative velocity $= \xi - \xi_1$
$\vec{v}$	Dimensionless velocity $= \frac{\vec{c}}{\sqrt{RT}}$
$\vec{x}$	Position vector
$\vec{\alpha}$	Unit vector
γ	Collision frequency
δ_{ij}	Kronecker delta
ϵ	Azimuthal angle
$\bar{\epsilon}$	Dielectric coefficient
ϵ_{ijk}	Alternating unit tensor
θ	Polar angle
λ	Thermal conductivity
$\bar{\lambda}$	Mean free path
λ^*	"Apparent" thermal conductivity
μ	Coefficient of viscosity
μ^*	"Apparent" coefficient of viscosity
μ_e	Magnetic permeability

ν	Kinematic viscosity $= \frac{\mu}{\rho}$
$\vec{\xi}$	Particle velocity
ρ	Density
ρ_e	Excess electric charge
dx	$= dx_1 dx_2 dx_3$
$d\xi$	$= d\xi_1 d\xi_2 d\xi_3$
$d\xi_1$	$= (d\xi_1 d\xi_2 d\xi_3)_1$
$d\omega$	Element area $= r dr d\epsilon$

INTRODUCTION

In neutral gases, the methods of approximate solution of the Boltzmann equation had been developed by many authors, namely, Maxwell [35], Enskog [12], Hilbert [19], Chapman [9], Wang Chang and Uhlenbeck [8]. Grad [13] developed the thirteen-moment method with the aim of obtaining phenomenological equations. He derived the equations for the successive moments of the distribution function expressed as a series of Hermitean tensors.

In the present work, we extend the use of the thirteen-moment method to an one-component charged gas (i.e., all particles are either electrons or identical ions). Two transport coefficients are obtained from the thirteen-moment equations.

Chapter I gives a brief review of approximate solution of Boltzmann's equation in both neutral and ionized gases. In Chapter II, various distribution functions which include continuous distributions, discrete distributions and some distribution functions in plasma dynamics have been collected. Chapter III contains the derivation of the system of twenty-moment equations. With the use of the distribution function expressed in terms of Hermitean tensors, we reduce the system of twenty-moment equations to the system of thirteen-moment equations. In Chapter IV, the thermal conductivity and coefficient of viscosity of the gas are obtained by considering a one-dimensional heat flow and a plane Couette flow respectively. In addition, the transport coefficients are calculated and plotted for an electron gas.

CHAPTER I

BRIEF REVIEW OF BOLTZMANN'S EQUATION

In most gases, where the departures from the local thermodynamic equilibrium are not too large and the flow speed does not exceed Mach number three approximately, the Navier-Stokes equations prove to be valid, according to our present knowledge in gas-dynamics. The Navier-Stokes equations are derivable from the Boltzmann equation, although they are generally derived by considering the elastic deformation in continuum mechanics. In many cases, the Navier-Stokes equations are no longer valid and one must return to the Boltzmann equation to obtain a general solution for the distribution function.

1.1 Solutions to Boltzmann's Equation in Neutral Gases:

Before Boltzmann established his integro-differential equation satisfied by the particle velocity distribution function, Maxwell [35] established transport equations with the assumption of Maxwellian molecules. He obtained approximate solutions to his equations by means of a method of successive iterations. It was Boltzmann's [2] merit to establish an integro-differential equation which describes the variation with time of the distribution function f, the state of gas, the molecular interactions and the external force. However, Boltzmann did not find a general solution to his equation. Lorentz [33] associated a nonhomogeneous gas with the theory of electrons in a metal and sought a solution of the form $f = f_o + v_x \varphi(v)$ to Boltzmann's equation. This method also failed to reach the general solution of the integro-differential equation. Hilbert [19] proposed a solution of Boltzmann's equation by solving a linear integral equation of second kind. His approximate method was obtained purely from mathematical viewpoint. Chapman [9] calculated the coefficient of viscosity and thermal conductivity by means of second approximation to f. Enskog [12] modified Hilbert's method and obtained general formulas for the viscosity and thermal conductivity in gases. The Enskog-Chapman method is valid only if the mean free paths $\bar{\lambda}$ are small with respect to the characteristic dimension L. Distribution function f was expanded in the form of a power series of $\bar{\lambda}/L$. Burnett [5] expressed the distribution function in the form of expansions with respect to the product of Sonine polynomials and spherical tensors. He successfully calcluated the complete second-order approximation.

Grad [13] obtained the celebrated thirteen-moment equations for the successive moments of the distribution function which was expressed as a series in Hermitian tensors. It is a very effective method and probably more general than Enskog-Chapman method. Wang Chang and Uhlenbeck [8] treated the case of rarefied gases $(\frac{\bar{\lambda}}{L} \approx 1)$ by means of a linearization of Boltzmann's equation. Jaffé [26], Ikenberry and Truesdell [25] developed the theory for highly rarefied gases $(\frac{\bar{\lambda}}{L} >> 1)$ by the method of linearizing the Boltzmann's equation.

1.2 Solutions to Boltzmann's Equation in Ionized Gases:

Spitzer and Härm [43] solved the Boltzmann equation by direct numerical solution in the absence of magnetic field. Gross [17] considered the problem of plasma oscillations in a static magnetic field. He linearized the Boltzmann equation by neglecting the collision term and assuming $f = f_o + f_1$ with $f_1 \ll f_o$.

Howard [24] attacked the problem of hydrodynamic properties in electron gas. He obtained two coefficients of viscosity for both shearing and normal stresses by assuming Lorentz forces existed between individual particles. Unfortunately, there are no known experimental method to verify his results. Krzywoblocki [31] applied Howard's results to the problem of boundary layer in electron gas.

A different approach to the ionized gases is provided by the Fokker-Planck equation which is derived from the Boltzmann equation. Chandrasekhar [7], Rosenbluth, MacDonald and Judd [39] employed this equation to obtain the transport coefficients. Krzywoblocki and Wadhwa [32] proposed to extend Grad's method to magnetogasdynamics. Kolodner [30], Burgers [4], Herdan and Liley [18], and Yen [48] applied Grad's method for the Boltzmann equation to ionized gases. Kelleher and Everett [29] extended the Grad-Everett method to partially ionized gases. Hochstim [21] expressed the distribution function in the form of Laguerre polynomial expansion.

Meador and Staton [36] applied the concept of simultaneous many-body interactions in solving the Boltzmann equation. Shkarofsky

[41] used a different approach to solve the Boltzmann collision integral. He employed the concept of the supposition of many successive binary encounters. The results from both models proved to be in good agreement.

Marshall [34] used the variational principle for ionized gases. He considered the trial function of two polynomials. Robinson and Bernstein [38] employed a different variational method with a trial function of six polynomials.

CHAPTER II

SOME DISTRIBUTION FUNCTIONS

2.1 General Remarks on Distribution Function:

A random variable (denoted by r.v.) is defined in the usual manner. Namely, a r.v. X is a real valued function whose domain is S, and whose range is a set of real numbers. For every real number x, the set of elementary events s for which $X(s) \leq x$ is an event [11]. The event s belongs to a probability set S (i.e., s is an element of S).

The distribution function (denoted by d.f.) of a r.v. X is defined by [47]

$$F_X(x) = P[X \leqq x] \ , \tag{2.1.1}$$

for every real number x. This d.f. $F_X(x)$ satisfies the following conditions:

$$\text{(i)} \quad F_X(x_1) \leq F_X(x_2), \text{ if } x_1 < x_2 \ ; \tag{2.1.2}$$

$$\text{(ii)} \quad \lim_{x \to \infty} F_X(x) = 1 \ ; \tag{2.1.3}$$

and

$$\text{(iii)} \quad \lim_{x \to -\infty} F_X(x) = 0 \ . \tag{2.1.4}$$

There are two kinds of distributions known as the discrete and continuous kinds. For the discrete kind, the total mass of the distribution is concentrated in discrete mass points. Thus, the discrete d.f. is given by

$$F_X(x) = \sum_{x_i \leq x} p_i \quad . \tag{2.1.5}$$

If the distribution is continuous, the ratio $\frac{F_X(x+h)-F_X(x)}{h}$ represents the mean density within the interval $(x, x+h)$. The derivative

$$F_X'(x) = f(x) = \lim_{h \to 0} \frac{F_X(x,x+h)-F_X(h)}{h} , \tag{2.1.6}$$

if it exist, gives the probability density or frequency function $f(x)$, and the continuous d.f. is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)dt \quad . \tag{2.1.7}$$

Absolutely continuous $F_X(x)$ is a necessary and sufficient condition for the existence of the frequency function $f(x)$. Since $F_X(x)$ is monotone non-decreasing, the frequency function is non-negative and

$$\int_{-\infty}^{\infty} f(x)dx = 1 \quad . \tag{2.1.8}$$

We may also be interested in studying simultaneously several r.v.'s $X_1, X_2, \ldots, X_n$ of the distribution. In such a circumstance, we can choose points of the n-dimensional space to represent all possible values of the r.v.'s. The definitions of distribution and frequency functions are defined in the same manner.

2.2 Discrete Distribution Functions:

The Binomial Distribution. A trial of a random experiment with the outcome either "success" or "failure" is called a Bernoulli trial. Let us consider a sequence of n Bernoulli trials, where the probability of S (success) in each trial is p, and $0 < p < 1$.

There are $\binom{n}{k}$ ways of selecting k trials at which S occurs. Thus the discrete distribution $F_X(x)$ is written as

$$F_X = \sum_{k=0}^{[x]} \binom{n}{k} p^k (1-p)^{n-k} \quad , \tag{2.2.1}$$

for every real number k.

The Poisson Distribution. The Poisson distribution may be obtained as a limit of the binomial distribution. Let X_n denote the number of successes that occur in the n Bernoulli trials, and let p_n be the probability of success. Then the d.f. is

$$F_{X_n}(x) = \sum_{x=0}^{[x]} \frac{e^{-\lambda}\lambda^x}{x!} \; , \; x = 0,1,2,\ldots, \tag{2.2.2}$$

where $\lambda = np_n$.

2.3 Continuous Distribution Functions:

The Gamma Function. Its density function has the following property

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{\alpha} e^{-x/\beta} & \text{for } x > 0 \; ; \\ 0 & \text{for } x \leqq 0 \; , \end{cases} \tag{2.3.1}$$

provided $\alpha > -1$ and $\beta > 0$.

The Normal Distribution. Many references cite this distribution as the Gaussian, Laplace or bell-shaped distribution. The d.f. is defined by the relation:

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})dt \; . \tag{2.3.2}$$

The corresponding normal frequency function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . \tag{2.3.3}$$

The χ^2 Distribution. The χ^2 frequency is

$$f_{\chi^2}(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & \text{for } x > 0 ; \\ 0 & \text{for } x \leqq 0 . \end{cases} \tag{2.3.4}$$

The corresponding χ^2 d.f. is

$$F_{\chi^2}(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \int_o^x t^{\frac{n}{2}-1} e^{-\frac{t}{2}} dt & \text{for } x > 0 ; \\ 0 & \text{for } x \leqq 0 . \end{cases} \tag{2.3.5}$$

The parameter n is often denoted as the number of degrees of freedom in the distribution.

Fisher's z-Distribution. Two independent r.v.'s X and Y have a χ^2-distribution with m and n degrees of freedom, respectively. The z-distribution with (m,n) degrees of freedom is

$$F_{X,Y}(x) = \begin{cases} \dfrac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \int_o^x \dfrac{t^{\frac{m}{2}-1}}{(t+1)^{\frac{m+n}{2}}} dt & \text{for } x > 0 ; \\ 0 & \text{for } x \leqq 0 . \end{cases} \tag{2.3.6}$$

Student's Distribution. Again we consider two independent r.v.'s. X and Y. The distribution of X being $N(0,1)$ and

the distribution of Y being χ^2-distribution with n degrees of freedom. Thus the Student's d.f. with n degrees of freedom is expressed as

$$F_{X,Y}(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \int_{-\infty}^{x} \frac{dt}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}} , \qquad (2.3.7)$$

for all x.

2.4 Distribution Functions in Neutral Gases:

Maxwell [35] proposed that the d.f. f be expressed as

$$f = f_o(1+F) \qquad , \qquad (2.4.1)$$

where f_o is the solution of equilibrium state and F is a rational function of particle velocity.

Hilbert [19] and Enskog [12] used an asymptotic expansion in a power series of a small parameter ϵ for d.f. f

$$f = \frac{f^{(o)}}{\epsilon} + f^{(1)} + \epsilon f^{(2)} + \ldots.. \qquad (2.4.2)$$

where $f^{(o)}$ is locally Maxwellian, and $f^{(1)}, f^{(2)}, \ldots$ are obtained by solving a series of integral equations derived from the Boltzmann equation.

Burnett [5] introduced the d.f. f expanded in an infinite sum of Sonine polynomials S's. The d.f. f takes the form

$$f = f_m^{(o)}\{\sum_{\rho} A_\rho S_{\frac{1}{2}}^{(\rho)}(\xi^2) + \sum_{\ell\rho} Y_{\ell\rho} S_{\ell+\frac{1}{2}}^{(\rho)}(\xi^2)\} \qquad (2.4.3)$$

where $Y_{\ell\rho}$ are general spherical functions.

2.5 Distribution Functions in Ionized Gases:

Considering the forces acting on electrons by electric and magnetic fields, the Boltzmann equation is

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \frac{e}{m}(E_i + \epsilon_{ijk}\xi_j B_k)\frac{\partial f}{\partial \xi_i} = \left(\frac{\partial f}{\partial t}\right)_{coll.} \quad . \qquad (2.5.1)$$

The d.f. f could be expressed as

$$f = f^{(o)} + \phi(\xi_i) \quad , \qquad (2.5.2)$$

where $f^{(o)}$ is an isotropic distribution, and $\phi(\xi_i)$ is a small perturbation which causes f to be anisotropic.

The Maxwellian Distribution. For a weakly ionized plasma, the d.f. $f^{(o)}$ is often assumed as Maxwellian:

$$f^{(o)} = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{c^2}{2RT}\right) \quad . \qquad (2.5.3)$$

The Margenau Distribution. For a plasma in an alternating electric field, the d.f. $f^{(o)}$ is known as the Margenau distribution [23]:

$$f^{(o)} = C \exp\left\{-\int_o^{\xi} \frac{m\xi d\xi}{\left[kT + \frac{e^2A^2}{3Gm(r^2+\omega^2)}\right]}\right\} \quad . \qquad (2.5.4)$$

The Druyvesteyn Distribution. For a plasma with a constant cross section under the influence of a strong direct current or low frequency electric field, the d.f. $f^{(o)}$ is known as the Druyvesteyn Distribution [23]:

$$f^{(o)} = C \exp(-3Gm^2N_N^2Q^2C^4/4e^2A^2) \quad . \qquad (2.5.5)$$

The approach to the distribution function in terms of Hermite polynomials, used by Grad, will be thoroughly discussed and used in the present work.

CHAPTER III

THE THIRTEEN-MOMENT EQUATIONS FOR ONE-COMPONENT, CHARGED GASES

3.1 Moments of Distribution Function and Boltzmann's Equation:

In a one-component, homogeneous, uniformly charged gas, the d.f. F which is defined as the number density of charged particle, is a function of seven variables, namely, velocity ξ_i, position x_i and time t. The d.f. f is defined as the mass density

$$f(\vec{\xi},\vec{x},t) = mF(\vec{\xi},\vec{x},t) \quad , \tag{3.1.1}$$

where m is the constant mass of charged particle. The moments of d.f. f with respect to velocity ξ_i and the intrinsic velocity are defined as follows:

Zeroth moment: $$\rho(\vec{x},t) = \int f(\vec{\xi},\vec{x},t)d\xi \ . \tag{3.1.2}$$

Mean velocity: $$\vec{u}(\vec{x},t) = \frac{1}{\rho}\int \vec{\xi} f \ d\xi \ . \tag{3.1.3}$$

Intrinsic velocity: $$\vec{c}(\vec{\xi},\vec{x},t) = \vec{\xi} - \vec{u}(\vec{x},t). \tag{3.1.4}$$

First moment: $$\int \vec{c} \ f \ d\xi = 0 \ . \tag{3.1.5}$$

Second moment: $$P_{ij}(\vec{x},t) = \int c_i c_j f \ d\xi \ . \tag{3.1.6}$$

Third moment: $$S_{ijk}(\vec{x},t) = \int c_i c_j c_k f \ d\xi \ . \tag{3.1.7}$$

Fourth moment: $$Q_{ijk\ell}(\vec{x},t) = \int c_i c_j c_k c_\ell f \ d\xi \ . \tag{3.1.8}$$

The second, third and fourth moments are symmetrical tensors. Contraction of tensors by using a dummy index and summing over all yields

$$P_{ii} = 3p \qquad ; \qquad (3.1.9)$$

$$S_{ijj} = S_i \qquad ; \qquad (3.1.10)$$

$$Q_{ijkk} = Q_{ij} \qquad ; \qquad (3.1.11)$$

and

$$Q_{ii} = 3q \qquad . \qquad (3.1.12)$$

The divergenceless tensors can be formed as

$$p_{ij} = P_{ij} - p\delta_{ij} \qquad ; \qquad (3.1.13)$$

$$q_{ij} = Q_{ij} - q\delta_{ij} \qquad ; \qquad (3.1.14)$$

together with

$$p_{ii} = q_{ii} = 0 \qquad , \qquad (3.1.15)$$

where δ_{ij} is the Kronecker delta.

The Boltzmann equation is taken to be

$$\frac{df(\vec{\xi},\vec{x},t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial f}{\partial \xi_i}\frac{\partial \xi_i}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}} , \qquad (3.1.16)$$

where $\frac{\partial x_i}{\partial t}$ is the particle velocity ξ_i and $\frac{\partial \xi_i}{\partial t}$ is the particle acceleration. Equation (3.1.6) can be written as (B.1)

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + L_i \frac{\partial f}{\partial \xi_i} = \left(\frac{\partial f}{\partial t}\right)_c \qquad . \qquad (3.1.17)$$

The symbol L_i denotes the sum of all electromagnetic forces:

$$\vec{L} = \frac{\partial \vec{\xi}}{\partial t} = \frac{e}{m}(\vec{E} + \vec{\xi} \times \vec{B}) \qquad , \qquad (3.1.18)$$

where $\vec{E}$ and $\vec{B}$ are independent of $\vec{\xi}$.

Since the gas is assumed to consist of only one kind of particle, it is fully justified that the governing Maxwell's equations are taken in their simplest forms:

$$\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \epsilon \vec{B}}{\partial t} \quad ; \qquad (3.1.19)$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad ; \qquad (3.1.20)$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \cdot \mu_e \vec{H} = 0 \quad ; \qquad (3.1.21)$$

and

$$\vec{\nabla} \cdot \overrightarrow{\epsilon E} = \rho_e \quad . \qquad (3.1.22)$$

The equation of conservation of electric charge is

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad . \qquad (3.1.23)$$

The Ohm's law gives the equation of electric current as

$$\vec{J} = \sigma[\vec{E} + \mu_e (\vec{u} \times \vec{B})] + \rho_e u \quad . \qquad (3.1.24)$$

3.2 The Maxwell Transport Equation:

Let U denote the relative velocity of the two colliding identical particles. Here $\vec{\xi}$, $\vec{\xi}'$ represent the velocities of the first particle before and after the collision, respectively, and $\vec{\xi}_1$, $\vec{\xi}_1'$ the velocities of the second particle. The symbol $d\omega$ is given as the element of area in the plane passed through the fixed first particle and perpendicular to the relative velocity U [Appendix A]. The notations $f = f(\vec{\xi})$, $f' = f(\vec{\xi}')$, $f_1 = f(\vec{\xi}_1)$, $f_1' = f(\vec{\xi}_1')$ are used here. The collision term [Appendix B] is taken as

$$(\frac{\partial f}{\partial t})_c = \frac{1}{m}\int U(f'f_1' - ff_1)d\omega d\xi_1 \quad . \tag{3.2.1}$$

Thus, equation (3.1.17) becomes

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + L_i \frac{\partial f}{\partial \xi_i} = \frac{1}{m}\int U(f'f'_1 - ff_1)d\omega d\xi_1 \ . \tag{3.2.2}$$

Multiplying (3.2.2) by a summational invariant $\varphi(\vec{\xi})$ [Appendix A] and integrating over the entire velocity space with the integration limit from $-\infty$ to $+\infty$, we have

$$\int\varphi \frac{\partial f}{\partial t} d\xi + \int\varphi\xi_i \frac{\partial f}{\partial x_i} d\xi + \int\varphi L_i \frac{\partial f}{\partial \xi_i} d\xi = 0 \quad , \tag{3.2.3}$$

where the collision integral vanishes [Appendix B].

The first term of equation (3.2.3) could be written as

$$\int\varphi \frac{\partial f}{\partial t} d\xi = \int\frac{\partial}{\partial t}(\varphi f) d\xi - \int f \frac{\partial \varphi}{\partial t} d\xi \quad . \tag{3.2.4}$$

Similarly, the second term of equation (3.2.3) is

$$\int\varphi\xi_i \frac{\partial f}{\partial x_i} d\xi = \int\frac{\partial}{\partial x_i}(\varphi\xi_i f) d\xi - \int f \frac{\partial}{\partial x_i}(\varphi\xi_i) d\xi \quad . \tag{3.2.5}$$

Since $\varphi(\vec{\xi})$ and ξ_i are independent of x_i, equation (3.2.5) becomes

$$\int\varphi\xi_i \frac{\partial f}{\partial x_i} d\xi = \int\frac{\partial}{\partial x_i} (\varphi\xi_i f) d\xi \quad . \tag{3.2.6}$$

The third term of equation (3.2.3) could also be written as

$$\int\varphi L_i \frac{\partial f}{\partial \xi_i} d\xi = \int\frac{\partial}{\partial \xi_i}(\varphi L_i f) d\xi - \int f \frac{\partial}{\partial \xi_i}(\varphi L_i) d\xi \ . \tag{3.2.7}$$

The first term of the right hand side of (3.2.7) is

$$\int\frac{\partial}{\partial \xi_i} (\varphi L_i f) d\xi = \varphi L_i f\Big|_{-\infty}^{+\infty} \quad . \tag{3.2.8}$$

As the velocity tends to infinity, d.f. f tends to zero faster than any velocity-dependent function, because of the exponential character of d.f. f. Hence

$$\varphi L_i f\Big|_{-\infty}^{+\infty} = 0 \quad . \tag{3.2.9}$$

The second term of the right hand side of (3.2.7) could be split into two terms:

$$-\int f \frac{\partial}{\partial \xi_i}(\varphi L_i)d\xi = -\int f L_i \frac{\partial \varphi}{\partial \xi_i} d\xi - \int f\varphi \frac{\partial L_i}{\partial \xi_i} d\xi \quad . \tag{3.2.10}$$

The gradient $\dfrac{\partial L_i}{\partial \xi_i}$ could be written in vector form as $\vec{\nabla}_\xi \cdot \vec{L}$. We recall equation (3.1.18), where $\vec{E}$ and $\vec{B}$ are independent of $\vec{\xi}$. Thus

$$\vec{\nabla}_\xi \cdot \vec{L} = \frac{e}{m} \vec{\nabla}_\xi \cdot (\vec{\xi} \times \vec{B}) \quad . \tag{3.2.11}$$

Since

$$\vec{\nabla}_\xi \cdot (\vec{\xi} \times \vec{B}) = 0 \quad , \tag{3.2.12}$$

we get

$$\vec{\nabla}_\xi \cdot \vec{L} = 0 \quad ; \tag{3.2.13}$$

or

$$\frac{\partial L_i}{\partial \xi_i} = 0 \quad . \tag{3.2.14}$$

Then, equation (3.2.10) gives

$$-\int f \frac{\partial}{\partial \xi_i} (\varphi L_i)d\xi = -\int f L_i \frac{\partial \varphi}{\partial \xi_i} d\xi \quad ; \tag{3.2.15}$$

or

$$\int \varphi L_i \frac{\partial f}{\partial \xi_i} d\xi = -\int f L_i \frac{\partial \varphi}{\partial \xi_i} d\xi \quad . \tag{3.2.16}$$

Using the relations (3.2.4), (3.2.6) and (3.2.16), equation (3.2.3) yields

$$\int\frac{\partial}{\partial t}(\varphi f)d\xi - \int f\,\frac{\partial\varphi}{\partial t}\,d\xi + \int\frac{\partial}{\partial x_i}(\varphi\xi_i f)d\xi - \int fL_i\frac{\partial\varphi}{\partial\xi_i}\,d\xi = 0. \quad (3.2.17)$$

If the functions (φf) and $(\varphi\xi_i f)$ are uniformly convergent, we can write (3.2.17) as

$$\frac{\partial}{\partial t}\int\varphi f d\xi - \int f\,\frac{\partial\varphi}{\partial t}\,d\xi + \frac{\partial}{\partial x_i}\int\varphi\xi_i f d\xi - \int fL_i\,\frac{\partial\varphi}{\partial\xi_i}\,d\xi = 0\ . \quad (3.2.18)$$

This is the Maxwell transport equation.

3.3 Conservation Equations:

Letting $\varphi(\vec{\xi}) = 1$, equation (3.2.18) becomes

$$\frac{\partial}{\partial t}\int f d\xi + \frac{\partial}{\partial x_i}\int\xi_i f d\xi = 0 \quad . \quad (3.3.1)$$

By use of (3.1.2) and (3.1.3), equation (3.3.1) yields the equation of conservation of mass

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}\,(\rho u_i) = 0 \quad . \quad (3.3.2)$$

Choosing $\varphi(\vec{\xi}) = \vec{\xi}$, equation (3.2.18) gives

$$\frac{\partial}{\partial t}\int\xi_i f d\xi - \int f\,\frac{\partial\xi_i}{\partial t}\,d\xi + \frac{\partial}{\partial x_j}\int\xi_i\xi_j f d\xi - \int fL_i d\xi = 0\ . \quad (3.3.3)$$

Using the identity $\xi_i\xi_j = c_i c_j + c_i u_j + c_j u_i + u_i u_j$ and (3.1.18), we get

$$\frac{\partial}{\partial t}\int\xi_i f d\xi + \frac{\partial}{\partial x_j}\int(c_i c_j + c_i u_j + c_j u_i + u_i u_j)f d\xi - 2\int fL_i d\xi = 0. \quad (3.3.4)$$

Use of relations (3.1.3), (3.1.5) and (3.1.6) yields

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(P_{ij} + \rho u_i u_j) - 2\int fL_i d\xi = 0\ . \quad (3.3.5)$$

Recalling equations (3.1.4) and (3.1.18), we may write

$$L_i = \frac{e}{m}(E_i + \epsilon_{iyz} u_y B_z + \epsilon_{iyz} c_y B_z) \ , \qquad (3.3.6)$$

where ϵ_{iyz} is the alternating unit tensor. Substituting (3.3.6) for L_i in (3.3.5), we get

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(P_{ij} + \rho u_i u_j) = 2\,\frac{e\rho}{m}(E_i + \epsilon_{iyz} u_y B_z) \ . \qquad (3.3.7)$$

Eliminating $\frac{\partial \rho}{\partial t}$ by the use of equation of conservation of mass (3.3.2), we obtain

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_j} = 2\,\frac{e}{m}(E_i + \epsilon_{iyz} u_y B_z) \ . \qquad (3.3.8)$$

This is the equation of conservation of momentum.

Taking $\varphi(\vec{\xi}) = \xi^2$, and using the identity $\xi_i \xi_i = u_i u_i + 2u_i c_i + c_i c_i$, equation (3.2.18) becomes

$$\frac{\partial}{\partial t}\int (u_i u_i + 2u_i c_i + c_i c_i) f d\xi - \int f \frac{\partial(\xi_i^2)}{\partial t} d\xi$$

$$+ \frac{\partial}{\partial x_j}\int (u_i u_i + 2u_i c_i + c_i c_i)(c_j + u_j) f d\xi$$

$$- \int f L_j \frac{\partial \xi_i^2}{\partial \xi_j} d\xi = 0 \ , \qquad (3.3.9)$$

where $$\frac{\partial(\xi_i^2)}{\partial t} = 2\xi_i \frac{\partial \xi_i}{\partial t} = 2\xi_j \frac{\partial \xi_j}{\partial t} \ ; \qquad (3.3.10)$$

and

$$\frac{\partial(\xi_i^2)}{\partial \xi_j} = 2\xi_j \ . \qquad (3.3.11)$$

Hence, equation (3.3.9) can be written as

$$\frac{\partial}{\partial t}\int (u_i u_i + 2u_i c_i + c_i c_i) f d\xi + \frac{\partial}{\partial x_j}\int (u_i u_i + 2u_i c_i + c_i c_i)(c_j + u_j) f d\xi$$

$$= 4\int f L_j \xi_j d\xi \ . \qquad (3.3.12)$$

By use of relations (3.1.2), (3.1.4), (3.1.5), (3.1.6), (3.1.7), (3.1.9), (3.1.10) and (3.3.6), equation (3.3.12) becomes

$$\frac{\partial}{\partial t}(\rho u^2 + 3p) + \frac{\partial}{\partial x_j}(2u_i P_{ij} + S_j + \rho u^2 u_j + 3u_j p)$$

$$= 4\ \frac{e}{m}\{\rho u_j E_j + \epsilon_{jyz} B_z(\rho u_j u_y + P_{jy})\}\ ; \tag{3.3.13}$$

or

$$u_i \frac{\partial}{\partial t}(\rho u_i) + (\rho u_i)\frac{\partial u_i}{\partial t} + 3\ \frac{\partial p}{\partial t} + 2\ \frac{\partial}{\partial x_j}(u_i P_{ij}) + \frac{\partial S_j}{\partial x_j}$$

$$+ \frac{\partial}{\partial x_j}(\rho u^2 u_j) + 3\ \frac{\partial}{\partial x_j}(u_j p) = \frac{4e}{m}\ \rho u_j E_j$$

$$+ \frac{4e}{m}\ \epsilon_{jyz} B_z(\rho u_j u_y + P_{jy}) \quad . \tag{3.3.14}$$

Eliminating $\frac{\partial}{\partial t}(\rho u_i)$ and $\frac{\partial u_i}{\partial t}$ by use of (3.3.7) and (3.3.8) respectively, we have

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(u_j p) + \frac{2}{3}\ P_{ij}\ \frac{\partial u_i}{\partial x_j} + \frac{1}{3}\ \frac{\partial S_j}{\partial x_j} = \frac{4e}{m}\epsilon_{jyz} B_z P_{jy} \quad . \tag{3.3.15}$$

This is the equation of conservation of energy. By use of (3.1.13), we get

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(u_j p) + \frac{2}{3}(p_{ij} + p\delta_{ij})\frac{\partial u_i}{\partial x_j} + \frac{1}{3}\ \frac{\partial S_j}{\partial x_j} =$$

$$\frac{4e}{m}\ \epsilon_{jyz} B_z(p_{jy} + p\delta_{jy}) \quad , \tag{3.3.16}$$

where

$$p\delta_{ij}\ \frac{\partial u_i}{\partial x_j} = p\ \frac{\partial u_j}{\partial x_j} \quad . \tag{3.3.17}$$

Hence,

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(u_j p) + \frac{2}{3}\ p_{ij}\ \frac{\partial u_i}{\partial x_j} + \frac{2}{3}\ p\ \frac{\partial u_j}{\partial x_j} + \frac{1}{3}\ \frac{\partial S_j}{\partial x_j} =$$

$$\frac{4e}{m}\ \epsilon_{jyz} B_z(p_{jy} + p\delta_{jy}) \quad . \tag{3.3.18}$$

3.4 Generalized n'th Moment Equation:

Let us define

$$\bar{c}^n = c_{i_1} \cdot c_{i_2} \cdot c_{i_3} \cdots\cdots c_{i_n} = c_1 c_2 c_3 \cdots\cdots c_n \ , \tag{3.4.1}$$

and

$$f^{(n)} = \int \bar{c}^n f d\xi \quad . \tag{3.4.2}$$

Multiplying the Boltzmann equation (3.2.2) by $\bar{c}^n$ and integrating, we get

$$\int \bar{c}^n \left(\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + L_i \frac{\partial f}{\partial \xi_i}\right) d\xi = J^{(n)} \quad . \tag{3.4.3}$$

From (B.5), we write

$$J^{(n)} = \frac{1}{m} \int \bar{c}^n \bar{B}(\theta, U)(f' f_1' - f f_1) d\theta d\varepsilon d\xi d\xi_1 \quad . \tag{3.4.4}$$

Evaluating the first term of the left hand side of (3.4.3), we have

$$\int \bar{c}^n \frac{\partial f}{\partial t} d\xi = \int \frac{\partial}{\partial t}(\bar{c}^n f) d\xi - \int f \frac{\partial \bar{c}^n}{\partial t} d\xi \quad . \tag{3.4.5}$$

If the function $(\bar{c}^n f)$ is uniformly convergent, equation (3.4.5) takes the form

$$\int \bar{c}^n \frac{\partial f}{\partial t} d\xi = \frac{\partial}{\partial t} \int \bar{c}^n f \, d\xi - \int f \frac{\partial \bar{c}^n}{\partial t} d\xi \quad . \tag{3.4.6}$$

The term $\frac{\partial \bar{c}^n}{\partial t}$ could be written as

$$\frac{\partial \bar{c}^n}{\partial t} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s} \frac{\partial c_s}{\partial t} \quad ; \tag{3.4.7}$$

or

$$\frac{\partial \bar{c}^n}{\partial t} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s}\left(\frac{\partial \xi_s}{\partial t} - \frac{\partial u_s}{\partial t}\right) \quad . \tag{3.4.8}$$

Following (3.1.18), we write

$$\frac{\partial \xi_s}{\partial t} = L_s = \frac{e}{m} E_s + \frac{e}{m} \epsilon_{syz}\xi_y B_z \quad . \tag{3.4.9}$$

Also from (3.3.8), we have

$$\frac{\partial u_s}{\partial t} = -u_i \frac{\partial u_s}{\partial x_i} - \frac{1}{\rho}\frac{\partial P_{si}}{\partial x_i} + \frac{2e}{m} E_s + \frac{2e}{m} \epsilon_{syz} u_y B_z \quad . \tag{3.4.10}$$

Using the values obtained from (3.4.9), (3.4.10) for $\frac{\partial \xi_s}{\partial t}$ and $\frac{\partial u_s}{\partial t}$ respectively, equation (3.4.8) gives

$$\frac{\partial \bar{c}^n}{\partial t} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s}\Big\{u_i \frac{\partial u_s}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{si}}{\partial x_i} - \frac{e}{m} E_s - \frac{2e}{m} \epsilon_{syz} u_y B_z$$

$$+ \frac{e}{m} \epsilon_{syz}\xi_y B_z\Big\} \quad ; \tag{3.4.11}$$

or

$$\frac{\partial \bar{c}^n}{\partial t} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s}\Big\{u_i \frac{\partial u_s}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{si}}{\partial x_i} - \frac{e}{m} E_s - \frac{e}{m} \epsilon_{syz} u_y B_z$$

$$+ \frac{e}{m} \epsilon_{syz} c_y B_z\Big\} \quad . \tag{3.4.12}$$

By use of (3.4.2) and (3.4.12), equation (3.4.6) becomes

$$\int \bar{c}^n \frac{\partial f}{\partial t} d\xi = \frac{\partial f^{(n)}}{\partial t} - \sum_{s=1}^{n} \int \frac{\bar{c}^n}{c_s} \Big\{u_i \frac{\partial u_s}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{si}}{\partial x_i} - \frac{e}{m} E_s$$

$$- \frac{e}{m} \epsilon_{syz} u_y B_z + \frac{e}{m} \epsilon_{syz} c_y B_z\Big\} f d\xi \quad ; \tag{3.4.13}$$

or

$$\int \bar{c}^n \frac{\partial f}{\partial t} d\xi = \frac{\partial f^{(n)}}{\partial t} - \sum_{s=1}^{n} \{u_i \frac{\partial u_s}{\partial x_i} + \frac{1}{\rho} \frac{\partial P_{si}}{\partial x_i} - \frac{e}{m} E_s$$

$$- \frac{e}{m} \epsilon_{syz} u_y B_z\} \int \frac{\bar{c}^n}{c_s} f d\xi - \sum_{s=1}^{n} \frac{e}{m} \epsilon_{syz} B_z \int \frac{\bar{c}^n}{c_s} c_y f d\xi . \quad (3.4.14)$$

Let

$$f^{(n/s)} = \int \frac{\bar{c}^n}{c_s} f d\xi \quad ; \quad (3.4.15)$$

and

$$f_y^{(n+1)/s} = \int \frac{\bar{c}^n}{c_s} c_y f d\xi \quad . \quad (3.4.16)$$

Hence, equation (3.4.14) can be written as

$$\int \bar{c}^n \frac{\partial f}{\partial t} d\xi = \frac{\partial f^{(n)}}{\partial t} - \sum_{s=1}^{n} \{u_i \frac{\partial u_s}{\partial x_i} + \frac{1}{\rho} \frac{\partial P_{si}}{\partial x_i} - \frac{e}{m} E_s - \frac{e}{m} \epsilon_{syz} u_y B_z\} f^{(n/s)}$$

$$- \sum_{s=1}^{n} \frac{e}{m} \epsilon_{syz} B_z f_y^{(n+1)/s} \quad . \quad (3.4.17)$$

The second term of the left hand side of (3.4.3) is

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i} \int \bar{c}^n \xi_i f d\xi - \int f \frac{\partial}{\partial x_i} (\bar{c}^n \xi_i) d\xi \; , \quad (3.4.18)$$

Use of (3.1.4) gives

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i} \int \bar{c}^n (c_i + u_i) f d\xi - \int f \bar{c}^n \frac{\partial \xi_i}{\partial x_i} d\xi - \int f \xi_i \frac{\partial \bar{c}^n}{\partial x_i} d\xi \; ; \quad (3.4.19)$$

or

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i} \int \bar{c}^n (c_i + u_i) f d\xi - \int f \bar{c}^n \frac{\partial \xi_i}{\partial x_i} d\xi$$

$$- \int f (c_i + u_i) \frac{\partial \bar{c}^n}{\partial x_i} d\xi \quad . \quad (3.4.20)$$

Since $\vec{\xi}$ is independent of $\vec{x}$, thus $\frac{\partial \xi_i}{\partial x_i} = 0$. We denote

$$f_i^{(n+1)} = \int \bar{c}^n c_i f d\xi \quad . \quad (3.4.21)$$

Equation (3.4.20) becomes

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i}\{f_i^{(n+1)} + u_i f^{(n)}\} - \int f(c_i + u_i)\frac{\partial \bar{c}^n}{\partial x_i} d\xi \ . \quad (3.4.22)$$

Following (3.4.8), we can write

$$\frac{\partial \bar{c}^n}{\partial x_i} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s}\left(\frac{\partial \xi_s}{\partial x_i} - \frac{\partial u_s}{\partial x_i}\right) \quad ; \quad (3.4.23)$$

or

$$\frac{\partial \bar{c}^n}{\partial x_i} = - \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s} \frac{\partial u_s}{\partial x_i} \quad . \quad (3.4.24)$$

Replacing $\frac{\partial \bar{c}^n}{\partial x_i}$ with the right hand side of (3.4.24), equation (3.4.22) gives

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i}\{f_i^{(n+1)} + u_i f^{(n)}\} + \sum_{s=1}^{n} \frac{\partial u_s}{\partial x_i}\{\int \frac{\bar{c}^n}{c_s} c_i f d\xi$$

$$+ u_i \int \frac{\bar{c}^n}{c_s} f d\xi\} \quad ; \quad (3.4.25)$$

or

$$\int \bar{c}^n \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i}\{f_i^{(n+1)} + u_i f^{(n)}\} + \sum_{s=1}^{n} \frac{\partial u_s}{\partial x_i}\{f_i^{(n+1)/s} + u_i f^{(n/s)}\}. \quad (3.4.26)$$

The third term of the left hand side of (3.4.3) gives

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = \int \frac{\partial}{\partial \xi_i}(\bar{c}^n L_i f) d\xi - \int f \frac{\partial}{\partial \xi_i}(L_i \bar{c}^n) d\xi \quad ; \quad (3.4.27)$$

or

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = \bar{c}^n L_i f \Big|_{-\infty}^{+\infty} - \int f \frac{\partial}{\partial \xi_i}(L_i \bar{c}^n) d\xi \quad . \quad (3.4.28)$$

Following (3.2.9), we write

$$\bar{c}^n L_i f \Big|_{-\infty}^{+\infty} = 0 \quad . \tag{3.4.29}$$

We can also write

$$\int f \frac{\partial}{\partial \xi_i}(L_i \bar{c}^n) d\xi = \int f\ L_i \frac{\partial \bar{c}^n}{\partial \xi_i} d\xi + \int f\ \bar{c}^n \frac{\partial L_i}{\partial \xi_i} d\xi \quad . \tag{3.4.30}$$

Using (3.2.14), we have

$$\int f \frac{\partial}{\partial \xi_i}(L_i \bar{c}^n) d\xi = \int f\ L_i \frac{\partial \bar{c}^n}{\partial \xi_i} d\xi \quad . \tag{3.4.31}$$

Equation (3.4.28) becomes

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = -\int f\ L_i \frac{\partial \bar{c}^n}{\partial \xi_i} d\xi \quad , \tag{3.4.32}$$

with the use of (3.4.29) and (3.4.31). Following (3.4.8), we have

$$\frac{\partial \bar{c}^n}{\partial \xi_i} = \sum_{s=1}^{n} \frac{\bar{c}^n}{c_s} \left(\frac{\partial \xi_s}{\partial \xi_i} - \frac{\partial u_s}{\partial \xi_i}\right) \quad , \tag{3.4.33}$$

where

$$\sum_{s=1}^{n} \frac{\partial \xi_s}{\partial \xi_i} = \sum_{s=1}^{n} \delta_{si} \quad . \tag{3.4.34}$$

Since $\vec{u}$ is independent of $\vec{\xi}$, thus

$$\frac{\partial u_s}{\partial \xi_i} = 0 \quad . \tag{3.4.35}$$

Using (3.4.33), (3.4.34) and (3.4.35), equation (3.4.32) takes the form:

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = -\sum_{s=1}^{n} \delta_{si} \int f\ L_i \frac{\bar{c}^n}{c_s} d\xi \quad . \tag{3.4.36}$$

Substituting the value of L_i obtained from (3.3.6), equation (3.4.36) gives

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = -\sum_{s=1}^{n} \delta_{si} \int f\{\frac{e}{m} E_i + \frac{e}{m} \epsilon_{iyz} u_y B_z + \frac{e}{m} \epsilon_{iyz} c_y B_z\} \frac{\bar{c}^n}{c_s} d\xi ; \quad (3.4.37)$$

or

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = -\sum_{s=1}^{n} \delta_{si} \{\frac{e}{m} E_i + \frac{e}{m} \epsilon_{iyz} u_y B_z\} f^{(n/s)}$$

$$- \sum_{s=1}^{n} \delta_{si} \frac{e}{m} \epsilon_{iyz} B_z f_y^{(n+1)/s} \quad , \quad (3.4.38)$$

where

$$\delta_{si} \epsilon_{iyz} = \epsilon_{syz} \quad ; \quad (3.4.39)$$

and

$$\delta_{si} E_i = E_s \quad . \quad (3.4.40)$$

Thus,

$$\int \bar{c}^n L_i \frac{\partial f}{\partial \xi_i} d\xi = -\sum_{s=1}^{n} \{\frac{e}{m} E_s + \frac{e}{m} \epsilon_{syz} u_y B_z\} f^{(n/s)}$$

$$- \sum_{s=1}^{n} \frac{e}{m} \epsilon_{syz} B_z f_y^{(n+1)/s} \quad . \quad (3.4.41)$$

Using (3.4.17), (3.4.26) and (3.4.41) and changing the subscript from i to r, equation (3.4.3) gives the n'th moment equation:

$$\frac{\partial f^{(n)}}{\partial t} + \frac{\partial}{\partial x_r}\{f_r^{(n+1)} + u_r f^{(n)}\} + \sum_{s=1}^{n} \frac{\partial u_s}{\partial x_r} f_r^{(n+1)/s} - \sum_{s=1}^{n} \frac{1}{\rho} \frac{\partial P_{sr}}{\partial x_r} f^{(n/s)}$$

$$- \sum_{s=1}^{n} \frac{2e}{m} \epsilon_{syz} B_z f_y^{(n+1)/s} = J^{(n)} \quad . \quad (3.4.42)$$

3.5 The Second and Third Moment Equations:

For $n = 2$, equation (3.4.42) gives

$$\frac{\partial f^{(2)}}{\partial t} + \frac{\partial}{\partial x_r}\{f_r^{(3)} + u_r f^{(2)}\} + \sum_{s=1}^{2} \frac{\partial u_s}{\partial x_r} f_r^{(3/s)} - \sum_{s=1}^{2} \frac{1}{\rho} \frac{\partial P_{sr}}{\partial x_r} f^{(2/s)}$$

$$- \sum_{s=1}^{2} \frac{2e}{m} \epsilon_{syz} B_z f_y^{(3/s)} = J^{(2)} \quad . \qquad (3.5.1)$$

Recalling the definitions (3.4.1), (3.4.2) and (3.4.4) we have

$$f^{(2)} = \int c_{i_1} c_{i_2} f d\xi \quad ; \qquad (3.5.2)$$

or

$$f^{(2)} = \int c_i c_j f d\xi \quad ; \qquad (3.5.3)$$

and

$$J^{(2)} = \int c_i c_j \bar{B}(\theta,U)(f'f_1' - ff_1) d\theta d\epsilon d\xi d\xi_1 = J_{ij}^{(2)} \quad . \qquad (3.5.4)$$

Similarly, we write

$$f_r^{(3/s)} = \int \frac{c_i c_j c_r}{c_s} f d\xi \quad ; \qquad (3.5.5)$$

$$f_r^{(2/s)} = \int \frac{c_i c_r}{c_s} f d\xi \quad ; \qquad (3.5.6)$$

and

$$f_y^{(3/s)} = \int \frac{c_i c_j c_y}{c_s} f d\xi \quad . \qquad (3.5.7)$$

By use of (3.1.5), (3.1.6) and (3.1.7), equation (3.5.1) furnishes

$$\frac{\partial P_{ij}}{\partial t} + \frac{\partial}{\partial x_r} (S_{ijr} + u_r P_{ij}) + P_{ir} \frac{\partial u_j}{\partial x_r} + P_{jr} \frac{\partial u_i}{\partial x_r}$$

$$- \frac{2e}{m} B_z (\epsilon_{iyz} P_{jy} + \epsilon_{jyz} P_{iy}) = J_{ij}^{(2)} \quad . \qquad (3.5.8)$$

For $n = 3$, equation (3.4.42) yields

$$\frac{\partial f^{(3)}}{\partial t} + \frac{\partial}{\partial x_r}\{f_r^{(4)} + u_r f^{(3)}\} + \sum_{s=1}^{n} \frac{\partial u_s}{\partial x_r} f_r^{(4/s)} - \sum_{s=1}^{n} \frac{1}{\rho} \frac{\partial P_{sr}}{\partial x_r} f^{(3/s)}$$

$$- \sum_{s=1}^{n} \frac{2e}{m} \epsilon_{syz} B_z f_y^{(4/s)} = J^{(3)} \quad . \tag{3.5.9}$$

Using definitions (3.4.1) and (3.4.2) and also the relations (3.4.4),(3.1.5), (3.1.6), (3.1.7) and (3.1.8), equation (3.5.9) gives

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}\{Q_{ijkr} + u_r S_{ijk}\} + S_{ijr} \frac{\partial u_k}{\partial x_r} + S_{irk} \frac{\partial u_j}{\partial x_r} + S_{rjk} \frac{\partial u_i}{\partial x_r}$$

$$- \frac{1}{\rho}(P_{ij} \frac{\partial P_{kr}}{\partial x_r} + P_{ik} \frac{\partial P_{jr}}{\partial x_r} + P_{jk} \frac{\partial P_{ir}}{\partial x_r})$$

$$- \frac{2e}{m} B_z(\epsilon_{iyz} S_{yjk} + \epsilon_{jyz} S_{iyk} + \epsilon_{kyz} S_{ijy})$$

$$= J_{ijk}^{(3)} \quad . \tag{3.5.10}$$

3.6 Third-Order Approximation in Hermite Polynomials:

We assume that the d.f. f is not too far from the state of equilibrium. We can express the d.f. f in the neighborhood of Maxwellian distribution $f^{(0)}$ by use of Hermite polynomials. Thus, the d.f. f is taken in the form

$$f = f^{(0)} \sum_{n=0}^{\infty} \frac{1}{n!} a_i^{(n)}(\vec{x},t) \mathcal{H}_i^{(n)}(\vec{v}) \quad . \tag{3.6.1}$$

From (C.13), we have

$$f^{(0)} = \frac{\rho}{(2\pi RT)^{3/2}} \exp(- \frac{c^2}{2RT}) \quad . \tag{3.6.2}$$

The Hermite polynomial $\mathscr{H}_i^{(n)}$ is introduced as a tensor with n subscripts, $i = i_1, i_2, \ldots, i_n$, as well as a polynomial of n'th degree

$$\mathscr{H}^{(n)} = \sum_{s=0}^{[n/2]} \frac{(-1)^s n!}{2^s (n-2s)! s!} v^{n-2s} \quad , \tag{3.6.3}$$

where v is the dimensionless velocity

$$\vec{v} = \frac{\vec{c}}{\sqrt{RT}} \quad . \tag{3.6.4}$$

The first few polynomials are

$$\begin{aligned}
\mathscr{H}^{(0)} &= 1 \quad ; \\
\mathscr{H}_i^{(1)} &= v_i \quad ; \\
\mathscr{H}_{ij}^{(2)} &= v_i v_j - \delta_{ij} \quad ; \\
\mathscr{H}_{ijk}^{(3)} &= v_i v_j v_k - (v_i \delta_{jk} + v_j \delta_{ik} + v_k \delta_{ij}) \quad ; \text{ and} \\
\mathscr{H}_{ijk\ell}^{(4)} &= v_i v_j v_k v_\ell - (v_i v_j \delta_{k\ell} + v_i v_k \delta_{j\ell} + v_i v_\ell \delta_{jk} + v_j v_k \delta_{i\ell} \\
&\quad + v_j v_\ell \delta_{ik} + v_k v_\ell \delta_{ij}) + (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \quad .
\end{aligned} \tag{3.6.5}$$

The formula for the coefficients $a_i^{(n)}$ is

$$a_i^{(n)} = \frac{1}{\rho} \int f \, \mathscr{H}_i^{(n)} d\xi \quad . \tag{3.6.6}$$

These are dimensionless polynomials. The first few coefficients are

$$\begin{aligned}
a^{(0)} &= 1 \quad ; \\
a_i^{(1)} &= 0 \quad ; \\
a_{ij}^{(2)} &= p_{ij}/p \quad ; \\
a_{ijk}^{(3)} &= S_{ijk}/p\sqrt{RT} \quad ; \text{ and}
\end{aligned} \tag{3.6.7}$$

$$a^{(4)}_{ijk\ell} = Q_{ijk\ell}/pRT - \frac{1}{p}(P_{ij}\delta_{k\ell} + P_{ik}\delta_{j\ell} + P_{i\ell}\delta_{jk} + P_{jk}\delta_{i\ell}$$

$$+ P_{j\ell}\delta_{ik} + P_{k\ell}\delta_{ij}) + (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}).$$

Taking only three terms in Hermite polynomial expansion, we have

$$f = f^{(0)}(1 + \frac{1}{2} a^{(2)}_{ij}\mathcal{H}^{(2)}_{ij} + \frac{1}{6} a^{(3)}_{ijk}\mathcal{H}^{(3)}_{ijk}) \quad ; \qquad (3.6.8)$$

or

$$f = f^{(0)}(1 + \frac{P_{ij}}{2pRT} c_i c_j + \frac{S_{ijk}}{6pR^2T^2} c_i c_j c_k - \frac{S_i}{2pRT} c_i). \qquad (3.6.9)$$

The variables which define the state of gas are twenty in number, namely, $\rho(1)$, $\vec{u}(3)$, $p_{ij}(6)$ and $S_{ijk}(10)$.

Following Grad [13], we use the contracted Hermite coefficients instead of the full set $a^{(3)}_{ijk}$. We write

$$a^{(3)}_{ijj} = \frac{1}{\rho}\int f\, \mathcal{H}^{(3)}_{ijj}\, d\xi \quad . \qquad (3.6.10)$$

Using Grad's derivation, the corresponding contracted Hermite polynomials are introduced as

$$\mathcal{H}^{(3)}_{i} = v_i(v^2 - 5) \quad . \qquad (3.6.11)$$

The contraction is obtained by letting $j = k$ and summing over all j's in $\mathcal{H}^{(3)}_{ijk}$ to obtain $\mathcal{H}^{(3)}_{i}$. We write

$$f = f^{(0)}(1 + \frac{1}{2} a^{(2)}_{ij}\mathcal{H}^{(2)}_{ij} + b_i\mathcal{H}^{(3)}_{i}) \quad . \qquad (3.6.12)$$

Multiplying (3.6.12) by $\mathcal{H}^{(3)}_{i}$ and integrating, we find

$$b_i = \frac{1}{10} a^{(3)}_{i} \quad . \qquad (3.6.13)$$

Hence, (3.6.10) becomes

$$f = f^{(0)}(1 + \frac{1}{2} a_{ij}^{(2)} \mathcal{H}_{ij}^{(2)} + \frac{1}{10} a_i^{(3)} \mathcal{H}_i^{(3)}) \quad ; \tag{3.6.14}$$

or

$$f = f^{(0)}\{1 + \frac{p_{ij}}{2pRT} c_i c_j - \frac{S_i c_i}{2pRT}(1 - \frac{c^2}{5RT})\} \quad . \tag{3.6.15}$$

The number of variables in the expansion of d.f. f is reduced to thirteen - namely, $\rho(1)$, $\vec{u}(3)$, $p_{ij}(6)$ and $S_i(3)$. Integrating (3.6.14) and taking account of (3.6.8), we obtain

$$a_{ijk}^{(3)} = \frac{1}{5}(a_i^{(3)} \delta_{jk} + a_j^{(3)} \delta_{ik} + a_k^{(3)} \delta_{ij}) \quad , \tag{3.6.16}$$

i.e.,

$$S_{ijk}^{(3)} = \frac{1}{5}(S_i \delta_{jk} + S_j \delta_{ik} + S_k \delta_{ij}) \quad . \tag{3.6.17}$$

3.7 The Twenty-Moment Equations:

Letting

$$a_{ijk\ell}^{(4)} = 0 \quad , \tag{3.7.1}$$

equation (3.6.7) gives

$$Q_{ijkr} = RT(P_{ij}\delta_{kr} + P_{ik}\delta_{jr} + P_{ir}\delta_{jk} + P_{jk}\delta_{ir} + P_{jr}\delta_{ik} + P_{kr}\delta_{ij}) - pRT(\delta_{ij}\delta_{kr} + \delta_{ik}\delta_{jr} + \delta_{ir}\delta_{jk}) \quad . \tag{3.7.2}$$

Use of (3.1.13) yields

$$Q_{ijkr} = RT(p_{ij}\delta_{kr} + p_{ik}\delta_{jr} + p_{ir}\delta_{jk} + p_{jk}\delta_{ir} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij}) + pRT(\delta_{ij}\delta_{kr} + \delta_{ik}\delta_{jr} + \delta_{ir}\delta_{jk}) \quad . \tag{3.7.3}$$

Substituting the value of Q_{ijkr} obtained from (3.7.3), equation

(3.5.10) gives

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}\{u_r S_{ijk} + RT(p_{ij}\delta_{kr} + p_{ik}\delta_{jr} + p_{ir}\delta_{jk} + p_{jk}\delta_{ir}$$

$$+ p_{jr}\delta_{ik} + p_{kr}\delta_{ij}) + pRT(\delta_{ij}\delta_{kr} + \delta_{ik}\delta_{jr} + \delta_{ir}\delta_{jk})\}$$

$$+ S_{ijr}\frac{\partial u_k}{\partial x_r} + S_{irk}\frac{\partial u_j}{\partial x_r} + S_{rjk}\frac{\partial u_i}{\partial x_r} - \frac{1}{\rho}(P_{ij}\frac{\partial P_{kr}}{\partial x_r} + P_{ik}\frac{\partial P_{jr}}{\partial x_r}$$

$$+ P_{jk}\frac{\partial P_{ir}}{\partial x_r}) - \frac{2e}{m} B_z(\epsilon_{iyz}S_{yjk} + \epsilon_{jyz}S_{iyk} + \epsilon_{kyz}S_{ijy})$$

$$= J^{(3)}_{ijk} \qquad ; \qquad (3.7.4)$$

or

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}(u_r S_{ijk}) + (\frac{\partial u_i}{\partial x_r} S_{rjk} + \frac{\partial u_j}{\partial x_r} S_{irk} + \frac{\partial u_k}{\partial x_r} S_{ijr})$$

$$+ \frac{\partial}{\partial x_i}(p_{jk}RT) + \frac{\partial}{\partial x_j}(p_{ik}RT) + \frac{\partial}{\partial x_k}(p_{ij}RT)$$

$$+ (p_{ir}\delta_{jk} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})\frac{\partial RT}{\partial x_r} + RT\frac{\partial}{\partial x_r}(p_{ir}\delta_{jk} + p_{jr}\delta_{ik}$$

$$+ p_{kr}\delta_{ij}) + p(\frac{\partial RT}{\partial x_i}\delta_{jk} + \frac{\partial RT}{\partial x_j}\delta_{ik} + \frac{\partial RT}{\partial x_k}\delta_{ij})$$

$$+ RT(\frac{\partial p}{\partial x_i}\delta_{jk} + \frac{\partial p}{\partial x_j}\delta_{ik} + \frac{\partial p}{\partial x_k}\delta_{ij}) - \frac{1}{\rho}\{(p_{ij} + p\delta_{ij})\frac{\partial P_{kr}}{\partial x_r}$$

$$+ (p_{ik} + p\delta_{ik})\frac{\partial P_{jr}}{\partial x_r} + (p_{jk} + p\delta_{jk})\frac{\partial P_{ir}}{\partial x_r}\}$$

$$- \frac{2e}{m} B_z(\epsilon_{iyz}S_{yjk} + \epsilon_{jyz}S_{iyk} + \epsilon_{kyz}S_{ijy}) = J^{(3)}_{ijk} \quad . \qquad (3.7.5)$$

Rearranging it, we have

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}(u_r S_{ijk}) + (\frac{\partial u_i}{\partial x_r} S_{rjk} + \frac{\partial u_j}{\partial x_r} S_{irk} + \frac{\partial u_k}{\partial x_r} S_{ijr})$$

$$+ \frac{\partial}{\partial x_i}(p_{jk}RT) + \frac{\partial}{\partial x_j}(p_{ik}RT) + \frac{\partial}{\partial x_k}(p_{ij}RT)$$

$$+ (p_{ir}\delta_{jk} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})\frac{\partial RT}{\partial x_r} + RT\{\frac{\partial(p_{ir} + p\delta_{ir})}{\partial x_r}\delta_{jk}$$

$$+ \frac{\partial(p_{jr} + p\delta_{jr})}{\partial x_r}\delta_{ik} + \frac{\partial(p_{kr} + p\delta_{kr})}{\partial x_r}\delta_{ij}\} - \frac{1}{\rho}\{p_{ij}\frac{\partial P_{kr}}{\partial x_r} + p_{ik}\frac{\partial P_{jr}}{\partial x_r}$$

$$+ p_{jk}\frac{\partial P_{ir}}{\partial x_r}\} - RT\{\frac{\partial P_{ir}}{\partial x_r}\delta_{jk} + \frac{\partial P_{jr}}{\partial x_r}\delta_{ik} + \frac{\partial P_{kr}}{\partial x_r}\delta_{ij}\}$$

$$+ p(\frac{\partial RT}{\partial x_i}\delta_{jk} + \frac{\partial RT}{\partial x_j}\delta_{ik} + \frac{\partial RT}{\partial x_k}\delta_{ij})$$

$$- \frac{2e}{m}B_z(\epsilon_{iyz}S_{yjk} + \epsilon_{jyz}S_{iyk} + \epsilon_{kyz}S_{ijy}) = J^{(3)}_{ijk} \quad ; \qquad (3.7.6)$$

or

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}(u_r S_{ijk}) + (\frac{\partial u_i}{\partial x_r} S_{rjk} + \frac{\partial u_j}{\partial x_r} S_{irk} + \frac{\partial u_k}{\partial x_r} S_{ijr})$$

$$+ \frac{\partial}{\partial x_i}(p_{jk}RT) + \frac{\partial}{\partial x_j}(p_{ik}RT) + \frac{\partial}{\partial x_k}(p_{ij}RT)$$

$$+ (p_{ir}\delta_{jk} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})\frac{\partial RT}{\partial x_r} - \frac{1}{\rho}\{p_{ij}\frac{\partial(p_{kr} + p\delta_{kr})}{\partial x_r} + p_{ik}\frac{\partial(p_{jr} + p\delta_{jr})}{\partial x_r}$$

$$+ p_{jk}\frac{\partial(p_{ir} + p\delta_{ir})}{\partial x_r}\} + p(\frac{\partial RT}{\partial x_i}\delta_{jk} + \frac{\partial RT}{\partial x_j}\delta_{ik} + \frac{\partial RT}{\partial x_k}\delta_{ij})$$

$$- \frac{2e}{m}B_z(\epsilon_{iyz}S_{yjk} + \epsilon_{jyz}S_{iyk} + \epsilon_{kyz}S_{ijy}) = J^{(3)}_{ijk} \quad . \qquad (3.7.7)$$

By use of (3.1.13), we can write (3.5.8) in the following form:

$$\frac{\partial(p_{ij} + p\delta_{ij})}{\partial t} + \frac{\partial}{\partial x_r}\{S_{ijr} + u_r(p_{ij} + p\delta_{ij})\} + (p_{ir} + p\delta_{ir})\frac{\partial u_j}{\partial x_r}$$

$$+ (p_{jr} + p\delta_{jr})\frac{\partial u_i}{\partial x_r} - \frac{2e}{m} B_z\{\epsilon_{iyz}(p_{jy} + p\delta_{jy}) + \epsilon_{jyz}(p_{iy} + p\delta_{iy})\}$$

$$= J_{ij}^{(2)} \qquad ; \qquad (3.7.8)$$

or

$$\frac{\partial p_{ij}}{\partial t} + \delta_{ij} \frac{\partial p}{\partial t} + u_r \frac{\partial p_{ij}}{\partial x_r} + p_{ij} \frac{\partial u_r}{\partial x_r} + u_r\delta_{ij} \frac{\partial p}{\partial x_r} + p\delta_{ij} \frac{\partial u_r}{\partial x_r} + \frac{\partial S_{ijr}}{\partial x_r}$$

$$p_{ir} \frac{\partial u_j}{\partial x_r} + p\delta_{ir} \frac{\partial u_j}{\partial x_r} + p_{jr} \frac{\partial u_i}{\partial x_r} + p\delta_{jr} \frac{\partial u_i}{\partial x_r} - \frac{2e}{m} B_z\{\epsilon_{iyz}p_{jy}$$

$$+ \epsilon_{iyz}p\delta_{jy} + \epsilon_{jyz}p_{iy} + \epsilon_{jyz}p\delta_{iy}\} = J_{ij}^{(2)} \qquad . \qquad (3.7.9)$$

Substituting the value of $\frac{\partial p}{\partial t}$ obtained from (3.3.16) and using the following tensor notations:

$$p_{ir} \frac{\partial u_i}{\partial x_r} \rightarrow p_{rs} \frac{\partial u_r}{\partial x_s} \quad ;$$

$$\delta_{jr} \frac{\partial u_i}{\partial x_r} \rightarrow \frac{\partial u_i}{\partial x_j} \quad ; \quad \text{and} \qquad (3.7.10)$$

$$\delta_{ir} \frac{\partial u_j}{\partial x_r} \rightarrow \frac{\partial u_j}{\partial x_i} \quad ,$$

equation (3.7.9) gives

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{\partial S_{ijr}}{\partial x_r} - \frac{1}{3} \delta_{ij} \frac{\partial S_r}{\partial x_r} + p_{ir} \frac{\partial u_j}{\partial x_r} + p_{jr} \frac{\partial u_i}{\partial x_r}$$

$$- \frac{2}{3} \delta_{ij}p_{rs} \frac{\partial u_r}{\partial x_s} + p \frac{\partial u_i}{\partial x_j} + p \frac{\partial u_j}{\partial x_i} - \frac{2}{3} p\delta_{ij} \frac{\partial u_r}{\partial x_r}$$

$$+ \frac{4e}{m} \delta_{ij}\epsilon_{ryz}B_z(p_{ry} + p\delta_{ry}) - \frac{2e}{m} B_z(\epsilon_{iyz}p_{jy} + \epsilon_{iyz}p\delta_{jy}$$

$$+ \epsilon_{jyz}p_{iy} + \epsilon_{jyz}p\delta_{iy}) = J_{ij}^{(2)} \qquad . \qquad (3.7.11)$$

We add the conservation equations (3.3.2), (3.3.8) and (3.3.16) to (3.7.11) and (3.7.7), and obtain the system of twenty-moment equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) = 0 \qquad ;$$

$$\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial (p_{ir} + p\delta_{ir})}{\partial x_r} = 2\,\frac{e}{m}(E_i + \epsilon_{iyz}u_y B_z) \quad ;$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3}(p_{ir} + p\delta_{ir})\frac{\partial u_i}{\partial x_r} + \frac{1}{3}\frac{\partial S_r}{\partial x_r} = \frac{4e}{m}\,\epsilon_{ryz}B_z(p_{ry} + p\delta_{ry}) \quad ;$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{\partial S_{ijr}}{\partial x_r} - \frac{1}{3}\,\delta_{ij}\frac{\partial S_r}{\partial x_r} + p_{ir}\frac{\partial u_j}{\partial x_r} + p_{jr}\frac{\partial u_i}{\partial x_r}$$

$$- \frac{2}{3}\,\delta_{ij}p_{rs}\frac{\partial u_r}{\partial x_s} + p\,\frac{\partial u_i}{\partial x_j} + p\,\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\,p\delta_{ij}\frac{\partial u_r}{\partial x_r} + \frac{4e}{m}\,\delta_{ij}\epsilon_{ryz}B_z(p_{ry} + p\delta_{ry})$$

$$- \frac{2e}{m}\,B_z(\epsilon_{iyz}p_{jy} + \epsilon_{iyz}p\delta_{jy} + \epsilon_{jyz}p_{iy} + \epsilon_{jyz}p\delta_{iy}) \tag{3.7.12}$$

$$= J_{ij}^{(2)} \qquad ; \qquad \text{and}$$

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}(u_r S_{ijk}) + \left(\frac{\partial u_i}{\partial x_r}S_{rjk} + \frac{\partial u_j}{\partial x_r}S_{irk} + \frac{\partial u_k}{\partial x_r}S_{ijr}\right)$$

$$+ \frac{\partial}{\partial x_i}(p_{jk}RT) + \frac{\partial}{\partial x_j}(p_{ik}RT) + \frac{\partial}{\partial x_k}(p_{ij}RT)$$

$$+ (p_{ir}\delta_{jk} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})\frac{\partial RT}{\partial x_r} - \frac{1}{\rho}\Big\{p_{ij}\frac{\partial (p_{kr} + p\delta_{kr})}{\partial x_r}$$

$$+ p_{ik}\frac{\partial (p_{jr} + p\delta_{jr})}{\partial x_r} + p_{jk}\frac{\partial (p_{ir} + p\delta_{ir})}{\partial x_r}\Big\} + p\Big(\frac{\partial RT}{\partial x_i}\delta_{jk} + \frac{\partial RT}{\partial x_j}\delta_{ik}$$

$$+ \frac{\partial RT}{\partial x_k}\delta_{ij}\Big) - \frac{2e}{m}\,B_z(\epsilon_{iyz}S_{yjk} + \epsilon_{jyz}S_{iyk} + \epsilon_{kyz}S_{ijy}) = J_{ijk}^{(3)}\,.$$

3.8 The Thirteen-Moment Equations:

Replacing S_{ijr} with its value obtained from (3.6.17), the second order tensor equation of (3.7.12) gives

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{1}{5}\left(\frac{\partial S_i}{\partial x_r}\delta_{jr} + \frac{\partial S_j}{\partial x_r}\delta_{ir} + \frac{\partial S_r}{\partial x_r}\delta_{ij}\right) - \frac{1}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r}$$

$$+ p_{ir}\frac{\partial u_j}{\partial x_r} + p_{jr}\frac{\partial u_i}{\partial x_r} - \frac{2}{3}\delta_{ij}p_{rs}\frac{\partial u_r}{\partial x_s} + p\frac{\partial u_i}{\partial x_j} + p\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}p\frac{\partial u_r}{\partial x_r}$$

$$+ \frac{4e}{m}\delta_{ij}\epsilon_{ryz}B_z(p_{ry} + p\delta_{ry}) - \frac{2e}{m}B_z\{\epsilon_{iyz}p_{jy} + \epsilon_{iyz}p\delta_{jy}$$

$$+ \epsilon_{jyz}p_{iy} + \epsilon_{jyz}p\delta_{iy}\} = J_{ij}^{(2)} \quad . \tag{3.8.1}$$

From (B.34), the collision integral $J_{ij}^{(2)}$ is taken to be

$$J_{ij}^{(2)} = -\frac{6\overline{B}_1}{m}\rho\, p_{ij} \quad . \tag{3.8.2}$$

Hence, equation (3.8.1) can be written as

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{1}{5}\left(\frac{\partial S_i}{\partial x_j} + \frac{\partial S_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r}\right.$$

$$+ p_{ir}\frac{\partial u_j}{\partial x_r} + p_{jr}\frac{\partial u_i}{\partial x_r} - \frac{2}{3}\delta_{ij}p_{rs}\frac{\partial u_r}{\partial x_s} + p\frac{\partial u_i}{\partial x_j} + p\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}p\frac{\partial u_r}{\partial x_r}$$

$$+ \frac{4e}{m}\delta_{ij}\epsilon_{ryz}B_z(p_{ry} + p\delta_{ry}) - \frac{2e}{m}B_z\{\epsilon_{iyz}p_{jy} + \epsilon_{iyz}p\delta_{jy}$$

$$+ \epsilon_{jyz}p_{iy} + \epsilon_{jyz}p\delta_{iy}\} + \frac{6\overline{B}_1}{m}\rho\, p_{ij} = 0 \quad . \tag{3.8.3}$$

Similarly, we replace S_{ijk} with its value obtained from (3.6.17), the third order tensor equation of (3.7.12) yields

$$
\begin{aligned}
&\overbrace{\frac{1}{5}\left(\frac{\partial S_i}{\partial t}\,\delta_{jk} + \frac{\partial S_j}{\partial t}\,\delta_{ik} + \frac{\partial S_k}{\partial t}\,\delta_{ij}\right)}^{1} + \overbrace{\frac{1}{5}\,\frac{\partial}{\partial x_r}(u_r S_i \delta_{jk} + u_r S_j \delta_{ik} + u_r S_k \delta_{ij})}^{2} \\
&+ \overbrace{\frac{1}{5}\,\frac{\partial u_i}{\partial x_r}(S_r \delta_{jk} + S_j \delta_{rk} + S_k \delta_{rj})}^{3} + \overbrace{\frac{1}{5}\,\frac{\partial u_j}{\partial x_r}(S_i \delta_{rk} + S_r \delta_{ik} + S_k \delta_{ir})}^{4} \\
&+ \overbrace{\frac{1}{5}\,\frac{\partial u_k}{\partial x_r}(S_i \delta_{jr} + S_j \delta_{ir} + S_r \delta_{ij})}^{5} + \overbrace{\frac{\partial}{\partial x_i}(p_{jk} RT)}^{6} + \overbrace{\frac{\partial}{\partial x_j}(p_{ik} RT)}^{7} \\
&+ \overbrace{\frac{\partial}{\partial x_k}(p_{ij} RT)}^{8} + \overbrace{(p_{ir}\delta_{jk} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})\frac{\partial RT}{\partial x_r}}^{9} - \overbrace{\frac{1}{\rho}\Big\{p_{ij}\,\frac{\partial(p_{kr} + p\delta_{kr})}{\partial x_r}}^{} \\
&\overbrace{+ p_{ik}\,\frac{\partial(p_{jr} + p\delta_{jr})}{\partial x_r} + p_{jk}\,\frac{\partial(p_{ir} + p\delta_{ir})}{\partial x_r}\Big\}}^{10} + \overbrace{p\left(\frac{\partial RT}{\partial x_i}\,\delta_{jk} + \frac{\partial RT}{\partial x_j}\,\delta_{ik}\right.}^{11} \\
&\overbrace{\left.+ \frac{\partial RT}{\partial x_k}\,\delta_{ij}\right)} - \overbrace{\frac{2}{5}\,\frac{e}{m}\,B_z\{\epsilon_{iyz}(S_y\delta_{jk} + S_j\delta_{yk} + S_k\delta_{yj}) + \epsilon_{jyz}(S_i\delta_{yk} + S_y\delta_{ik}}^{12} \\
&\overbrace{+ S_k\delta_{iy}) + \epsilon_{kyz}(S_i\delta_{jy} + S_j\delta_{iy} + S_y\delta_{ij})\}} = \overbrace{J^{(3)}_{ijk}}^{13} \;. \qquad (3.8.4)
\end{aligned}
$$

The terms are numbered here, so reader may follow through easily. Taking 1, 2, 3, 4 and 5 of (3.8.4), we have

$$
\begin{aligned}
1 + 2 + 3 + 4 + 5 &= \frac{1}{5}\left\{\frac{\partial S_i}{\partial t} + \frac{\partial(u_r S_i)}{\partial x_r} + S_r\,\frac{\partial u_i}{\partial x_r}\right\}\delta_{jk} \\
&+ \frac{1}{5}\left\{\frac{\partial S_j}{\partial t} + \frac{\partial(u_r S_j)}{\partial x_r} + S_r\,\frac{\partial u_j}{\partial x_r}\right\}\delta_{ik} + \frac{1}{5}\left\{\frac{\partial S_k}{\partial t} + \frac{\partial(u_r S_k)}{\partial x_r} + S_r\,\frac{\partial u_k}{\partial x_r}\right\}\delta_{ij} \\
&+ \frac{1}{5}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)S_i + \frac{1}{5}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)S_j + \frac{1}{5}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)S_k \;. \qquad (3.8.5)
\end{aligned}
$$

Contraction of tensors is performed by letting $j = k$. Equation (3.8.5) yields

$$(1) + (2) + (3) + (4) + (5) = \frac{3}{5}\{\frac{\partial S_i}{\partial t} + \frac{\partial (u_r S_i)}{\partial x_r} + S_r \frac{\partial u_i}{\partial x_r}\}$$

$$+ \frac{1}{5}\{\frac{\partial S_j}{\partial t} + \frac{\partial (u_r S_j)}{\partial x_r} + S_r \frac{\partial u_j}{\partial x_r}\}\delta_{ij} + \frac{1}{5}\{\frac{\partial S_j}{\partial t} + \frac{\partial (u_r S_j)}{\partial x_r} + S_r \frac{\partial u_j}{\partial x_r}\}\delta_{ij}$$

$$+ \frac{2}{5} S_i \frac{\partial u_r}{\partial x_r} + \frac{2}{5} S_j \frac{\partial u_i}{\partial x_j} + \frac{2}{5} S_j \frac{\partial u_j}{\partial x_i} \quad . \tag{3.8.6}$$

Now we replace dummy subscript j with r and get

$$(1) + (2) + (3) + (4) + (5) = \frac{\partial S_i}{\partial t} + \frac{\partial (u_r S_i)}{\partial x_r} + \frac{7}{5} S_r \frac{\partial u_i}{\partial x_r}$$

$$+ \frac{2}{5} S_r \frac{\partial u_r}{\partial x_i} + \frac{2}{5} S_i \frac{\partial u_r}{\partial x_r} \quad . \tag{3.8.7}$$

Similarly, we let $j = k$ and obtain

$$(6) + (7) + (8) + (9) + (10) + (11) = \frac{\partial}{\partial x_i}(p_{jj}RT) + \frac{\partial}{\partial x_j}(p_{ij}RT)$$

$$+ \frac{\partial}{\partial x_j}(p_{ij}RT) + (p_{ir}\delta_{jj} + p_{jr}\delta_{ij} + p_{jr}\delta_{ij})\frac{\partial RT}{\partial x_r}$$

$$- \frac{1}{\rho}\{p_{ij} \frac{\partial (p_{jr} + p\delta_{jr})}{\partial x_r} + p_{ij} \frac{\partial (p_{jr} + p\delta_{jr})}{\partial x_r}$$

$$+ p_{jj} \frac{\partial (p_{ir} + p\delta_{ir})}{\partial x_r}\} + p(\frac{\partial RT}{\partial x_i}\delta_{jj} + \frac{\partial RT}{\partial x_j}\delta_{ij} + \frac{\partial RT}{\partial x_j}\delta_{ij}) \quad . \tag{3.8.8}$$

From (3.1.15), we write

$$p_{jj} = 0 \quad . \tag{3.8.9}$$

Summing the Kronecker delta with repeated subscripts and rearranging, we have

$$(6) + (7) + (8) + (9) + (10) + (11) = 2RT \frac{\partial p_{ir}}{\partial x_r} + 7\, p_{ir} \frac{\partial RT}{\partial x_r}$$

$$- \frac{2p_{ir}}{\rho} \frac{\partial(p_{rs} + p\delta_{rs})}{\partial x_s} + 5p \frac{\partial RT}{\partial x_i} \quad . \tag{3.8.10}$$

Letting $j = k$, we also have

$$\begin{aligned} (12) = & - \frac{2}{5} \frac{e}{m} B_z \{ \epsilon_{iyz}(S_y\delta_{jj} + S_j\delta_{yj} + S_j\delta_{yj}) \\ & + \epsilon_{jyz}(S_i\delta_{yj} + S_y\delta_{ij} + S_j\delta_{iy}) \\ & + \epsilon_{jyz}(S_i\delta_{jy} + S_j\delta_{iy} + S_y\delta_{ij})\} \quad ; \end{aligned} \tag{3.8.11}$$

or

$$\begin{aligned} (12) = & - \frac{2}{5} \frac{e}{m} B_z \{ \epsilon_{iyz}(3S_y + 2S_j\delta_{yj}) + 2\epsilon_{jyz}(S_i\delta_{yj} \\ & + S_y\delta_{ij} + S_j\delta_{iy})\} \quad . \end{aligned} \tag{3.8.12}$$

Using the following relations of alternating unit tensor:

$$\epsilon_{iyz}\delta_{yj} = \epsilon_{ijz} \quad ; \tag{3.8.13}$$

$$\epsilon_{ijz} = - \epsilon_{jiz} \quad ; \tag{3.8.14}$$

and

$$\epsilon_{jjz} = 0 \quad , \tag{3.8.15}$$

equation (3.8.12) furnishes

$$(12) = - \frac{2e}{m} B_z \epsilon_{iyz} S_y \quad . \tag{3.8.16}$$

Again, letting $j = k$, (13) takes the form

$$J_{ijj}^{(3)} = J_i^{(3)} \quad . \tag{3.8.17}$$

From (B.33), the collision integral is given as

$$J_i^{(3)} = -\frac{4\bar{B}_1}{m}\rho S_i \tag{3.8.18}$$

Using relations (3.8.7), (3.8.10), (3.8.12), (3.8.16) and (3.8.18), equation (3.8.4) gives

$$\frac{\partial S_i}{\partial t} + \frac{\partial(u_r S_i)}{\partial x_r} + \frac{7}{5} S_r \frac{\partial u_i}{\partial x_r} + \frac{2}{5} S_r \frac{\partial u_r}{\partial x_i} + \frac{2}{5} S_i \frac{\partial u_r}{\partial x_r} + 2RT \frac{\partial p_{ir}}{\partial x_r}$$
$$+ 7\, p_{ir} \frac{\partial RT}{\partial x_r} - \frac{2p_{ir}}{\rho} \frac{\partial(p_{rs} + p\delta_{rs})}{\partial x_s} + 5p \frac{\partial RT}{\partial x_i}$$
$$- \frac{2e}{m} B_z \epsilon_{iyz} S_y + \frac{4\bar{B}_1}{m}\rho S_i = 0 \quad . \tag{3.8.19}$$

Adding the conservation equations (3.3.2), (3.3.8) and (3.3.16) to (3.8.3) and (3.8.19), we obtain the system of thirteen-moment equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) = 0 \qquad ;$$

$$\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial(p_{ir} + p\delta_{ir})}{\partial x_r} = \frac{2e}{m}(E_i + \epsilon_{iyz} u_y B_z) \quad ;$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3}(p_{ir} + p\delta_{ir})\frac{\partial u_i}{\partial x_r} + \frac{1}{3}\frac{\partial S_r}{\partial x_r} = \frac{4e}{m}\epsilon_{ryz} B_z (p_{ry} + p\delta_{ry}) \; ;$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{1}{5}\left(\frac{\partial S_i}{\partial x_j} + \frac{\partial S_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r}\right) + p_{ir}\frac{\partial u_j}{\partial x_r}$$
$$+ p_{jr}\frac{\partial u_i}{\partial x_r} - \frac{2}{3}\delta_{ij} p_{rs}\frac{\partial u_r}{\partial x_s} + p\frac{\partial u_i}{\partial x_j} + p\frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} p \frac{\partial u_r}{\partial x_r} \tag{3.8.20}$$
$$+ \frac{4e}{m}\delta_{ij}\epsilon_{ryz} B_z (p_{ry} + p\delta_{ry}) - \frac{2e}{m} B_z\{\epsilon_{iyz} p_{jy} + \epsilon_{iyz} p\delta_{jy}$$
$$+ \epsilon_{jyz} p_{iy} + \epsilon_{jyz} p\delta_{iy}\} + \frac{6\bar{B}_1}{m}\rho\, p_{ij} = 0 \quad ; \quad \text{and}$$

$$\frac{\partial S_i}{\partial t} + \frac{\partial (u_r S_i)}{\partial x_r} + \frac{7}{5} S_r \frac{\partial u_i}{\partial x_r} + \frac{2}{5} S_r \frac{\partial u_r}{\partial x_r} + \frac{2}{5} S_i \frac{\partial u_r}{\partial x_r} + 2RT \frac{\partial p_{ir}}{\partial x_r}$$

$$+ 7p_{ir} \frac{\partial RT}{\partial x_r} - \frac{2p_{ir}}{\rho} \frac{\partial (p_{rs} + p\delta_{rs})}{\partial x_s} + 5p \frac{\partial RT}{\partial x_i}$$

$$- \frac{2e}{m} B_z \epsilon_{iyz} S_y + \frac{4\bar{B}_1}{m} \rho \, S_i = 0 \qquad .$$

This system of thirteen differential equations governs the thirteen state variables - namely the velocity $\vec{u}(3)$, stress tensor $p_{ij}(6)$, heat flow vector $\frac{1}{2} S_i(3)$, density ρ or pressure $p(1)$. The equation of state:

$$RT = \frac{p}{\rho} \qquad (3.8.21)$$

is also used to describe temperature of the gas.

CHAPTER IV

TRANSPORT COEFFICIENTS

4.1 Thermal Conductivity:

Let us consider a one-dimensional heat flow in a gas at rest (i.e., $\frac{\partial}{\partial t} = 0$; $S_2 = S_3 = 0$; $\vec{u} = 0$). The second order tensor equation of (3.8.20) gives

$$-\frac{2e}{m} B_z\{\epsilon_{1yz}(p_{2y} + p\delta_{2y}) + \epsilon_{2yz}(p_{1y} + p\delta_{1y})\} + \frac{6\bar{B}_1}{m} \rho\, p_{12} = 0, \quad (4.1.1)$$

with $i \neq j$; $i = 1$; and $j = 2$. Having $y = 2$, $z = 3$ and using (3.8.15), equation (4.1.1) reduces to

$$-\frac{2e}{m} B_3(p_{22} + p) + \frac{6\bar{B}_1}{m} \rho\, p_{12} = 0 \quad . \quad (4.1.2)$$

Neglecting p_{22} as defined in (3.1.15), equation (4.1.2) becomes

$$-\frac{2e}{m} B_3 p + \frac{6\bar{B}_1}{m} \rho\, p_{12} = 0 \quad . \quad (4.1.3)$$

Using the equation of state (3.8.21), equation (4.1.3) yields

$$-\frac{2e}{m} B_3 \rho\, RT + \frac{6\bar{B}_1}{m} \rho\, p_{12} = 0 \quad ; \quad (4.1.4)$$

or

$$p_{12} = \frac{eB_3RT}{3\bar{B}_1} \quad . \quad (4.1.5)$$

Applying the conditions $\frac{\partial}{\partial t} = 0$; $S_2 = S_3 = 0$; and $\vec{u} = 0$ to the third order tensor equation of (3.8.20), we obtain

$$2RT \frac{\partial p_{1r}}{\partial x_r} + 7p_{1r} \frac{\partial RT}{\partial x_r} - \frac{2p_{1r}}{\rho} \frac{\partial (p_{rs} + p\delta_{rs})}{\partial x_s} + 5p \frac{\partial RT}{\partial x_1}$$

$$- \frac{2e}{m} \epsilon_{1yz} B_z S_y + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad . \qquad (4.1.6)$$

Letting $r = 2$; $y = 2$; and $z = 3$, equation (4.1.6) gives

$$- \frac{2p_{12}}{\rho} \frac{\partial (p_{2s} + p\delta_{2s})}{\partial x_s} + 5p \frac{\partial RT}{\partial x_1} + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad . \qquad (4.1.7)$$

Having $s = 1$, equation (4.1.7) yields

$$- \frac{2p_{12}}{\rho} \frac{\partial p_{21}}{\partial x_1} + 5p \frac{\partial RT}{\partial x_1} + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad . \qquad (4.1.8)$$

Due to the symmetry of second order tensor, we have

$$p_{12} = p_{21} \quad . \qquad (4.1.9)$$

By use of (4.1.5), equation (4.1.8) takes the form

$$- \frac{2}{\rho} \frac{eB_3RT}{3\bar{B}_1} \frac{\partial}{\partial x_1}\left(\frac{eB_3RT}{3\bar{B}_1}\right) + 5p \frac{\partial RT}{\partial x_1} + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad . \qquad (4.1.10)$$

For uniform B_3 and $\bar{B}_1$, we can write

$$- \frac{2}{\rho} \frac{e^2B_3^2R^2T}{9\bar{B}_1^{\,2}} \frac{\partial T}{\partial x_1} + 5pR \frac{\partial T}{\partial x_1} + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad . \qquad (4.1.11)$$

The equation of state (3.8.21) is used. Thus, we have

$$- \frac{2}{\rho} \frac{e^2B_3^2R^2T}{9\bar{B}_1^{\,2}} \frac{\partial T}{\partial x_1} + 5\rho R^2T \frac{\partial T}{\partial x_1} + \frac{4\bar{B}_1}{m} \rho \, S_1 = 0 \quad ; \qquad (4.1.12)$$

or

$$S_1 = -\left(\frac{5mR^2T}{4\bar{B}_1} - \frac{me^2B_3^2R^2T}{18\,\bar{B}_1^{\,3}\rho^2}\right)\frac{\partial T}{\partial x_1} \quad . \qquad (4.1.13)$$

Letting

$$\lambda = \frac{5mR^2T}{8\,\bar{B}_1} \quad , \tag{4.1.14}$$

we obtain

$$S_1 = -2\lambda\left(1 - \frac{2e^2B_3^2}{45\,\bar{B}_1^2\rho^2}\right)\frac{\partial T}{\partial x_1} \quad , \tag{4.1.15}$$

where λ is the thermal conductivity for a hypothetical gas in the absence of electric and magnetic fields but having identical geometric collision properties as the charged electron gas.

We may introduce an "apparent" thermal conductivity λ^* by writing

$$S_1 = -2\lambda^* \frac{\partial T}{\partial x_1} \quad , \tag{4.1.16}$$

where

$$\lambda^* = \left(1 - \frac{2e^2B_3^2}{45\,\bar{B}_1^2\rho^2}\right)\lambda \quad . \tag{4.1.17}$$

The equation (4.1.16) can be generalized as

$$S_i = -2\lambda^* \frac{\partial T}{\partial x_i} \quad , \tag{4.1.18}$$

where

$$\lambda^* = \left(1 - \frac{2e^2B_k^2}{45\,\bar{B}_1^2\rho^2}\right)\lambda \quad . \tag{4.1.19}$$

4.2 Coefficient of Viscosity:

Considering a plane Couette flow under the conditions

$$u_1 \neq 0;\; u_2 = u_3 = 0;\; \frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x_3} = 0 \quad ;$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x_1} = S_1 = 0 \quad ,$$

the second order tensor equation in (3.8.20) gives

$$p_{2r}\frac{\partial u_1}{\partial x_r} + p\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_z\{\epsilon_{1yz}p_{2y} + \epsilon_{1yz}p\delta_{2y} + \epsilon_{2yz}p_{1y}$$

$$+ \epsilon_{2yz}p\delta_{1y}\} + \frac{6\bar{B}_1}{m}\rho\, p_{12} = 0 \quad , \tag{4.2.1}$$

with $i \neq j$; $i = 1$; and $j = 2$. Letting $r = 2$, $y = 2$ and $z = 3$, equation (4.2.1) reduces to

$$(p_{22} + p)\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_3\{\epsilon_{123}(p_{22} + p)\} + \frac{6\bar{B}_1}{m}\rho\, p_{12} = 0 \quad , \tag{4.2.2}$$

or

$$(p_{22} + p)(\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_3) + \frac{6\bar{B}_1}{m}\rho\, p_{12} = 0 \quad . \tag{4.2.3}$$

Omitting p_{22} as defined in (3.1.15), we have

$$p(\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_3) + \frac{6\bar{B}_1}{m}\rho\, p_{12} = 0 \quad . \tag{4.2.4}$$

Using equation of state (3.8.21), we obtain

$$\rho RT(\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_3) + \frac{6\bar{B}_1}{m}\rho\, p_{12} = 0 \; . \tag{4.2.5}$$

After rearranging, we may write

$$p_{12} = -\frac{mRT}{6\bar{B}_1}(\frac{\partial u_1}{\partial x_2} - \frac{2e}{m}B_3) \quad . \tag{4.2.6}$$

Letting

$$\mu = \frac{mRT}{6\bar{B}_1} \quad , \tag{4.2.7}$$

we get

$$p_{12} = -\mu\left(\frac{\partial u_1}{\partial x_2} - \frac{2e}{m} B_3\right) \quad , \qquad (4.2.8)$$

where μ is the coefficient of viscosity for a hypothetical gas in the absence of electric and magnetic fields but having identical geometric collision properties as the charged electron gas.

The "apparent" coefficient of viscosity μ^* is introduced

$$\mu^* = \mu\left(1 - \frac{\frac{2e}{m} B_3}{\frac{\partial u_1}{\partial x_2}}\right) \quad . \qquad (4.2.9)$$

Hence, equation (4.2.8) is written as

$$p_{12} = -\mu^* \frac{\partial u_1}{\partial x_2} \quad . \qquad (4.2.10)$$

Equation (4.2.10) can be generalized as

$$p_{ij} = -\mu^* \frac{\partial u_i}{\partial x_j} \quad , \qquad (4.2.11)$$

where

$$\mu^* = \mu\left(1 - \frac{\frac{2e}{m} \epsilon_{ijk} B_k}{\frac{\partial u_i}{\partial x_j}}\right) \quad . \qquad (4.2.12)$$

4.3 Tables and Graphs of Transport Coefficients:

Coefficient of Viscosity: We can rewrite (4.2.9) as

$$\frac{\mu^*}{\mu} = 1 - \frac{\frac{2e}{m} B_3}{\frac{\partial u_1}{\partial x_2}} \quad , \qquad (4.3.1)$$

where $\frac{e}{m} = 1.76 \times 10^{11}$ coul/kg for electron. The table of ratios between the "apparent" coefficient of viscosity and the coefficient of viscosity for electron gas is listed as follows:

Table 1. Ratios Between the "Apparent" Coefficient of Viscosity and the Coefficient of Viscosity for Electron Gas.

B_3 Weber/m^2					$\frac{B_3}{\partial u_1/\partial x_2}$ $\frac{\text{Weber sec}}{m^2}$	$\frac{\frac{2e}{m} B_3}{\partial u_1/\partial x_2}$	$\frac{u^*}{u}$
$\frac{\partial u_1}{\partial x_2}$ = .2 1/sec	.4	.6	.8	1.0			
5.68×10^{-14}	1.136×10^{-13}	1.7×10^{-13}	2.27×10^{-13}	2.84×10^{-13}	2.84×10^{-13}	.1	.9
1.136×10^{-13}	2.27×10^{-13}	3.41×10^{-13}	4.5×10^{-13}	5.68×10^{-13}	5.68×10^{-13}	.2	.8
1.7×10^{-13}	3.41×10^{-13}	5.1×10^{-13}	6.82×10^{-13}	8.52×10^{-13}	8.52×10^{-13}	.3	.7
2.27×10^{-13}	4.55×10^{-13}	6.82×10^{-13}	9.1×10^{-13}	1.136×10^{-12}	1.136×10^{-12}	.4	.6
2.84×10^{-13}	5.68×10^{-13}	8.5×10^{-13}	1.14×10^{-12}	1.42×10^{-12}	1.42×10^{-12}	.5	.5
3.41×10^{-13}	6.82×10^{-13}	1.02×10^{-12}	1.36×10^{-12}	1.704×10^{-12}	1.704×10^{-12}	.6	.4
3.98×10^{-13}	7.95×10^{-13}	1.19×10^{-12}	1.59×10^{-12}	1.988×10^{-12}	1.988×10^{-12}	.7	.3
4.55×10^{-13}	9.08×10^{-13}	1.36×10^{-12}	1.82×10^{-12}	2.272×10^{-12}	2.272×10^{-12}	.8	.2
5.11×10^{-13}	1.02×10^{-12}	1.53×10^{-12}	2.05×10^{-12}	2.556×10^{-12}	2.556×10^{-12}	.9	.1
5.68×10^{-13}	1.14×10^{-12}	1.7×10^{-12}	2.27×10^{-12}	2.84×10^{-12}	2.84×10^{-12}	1.0	0

The graph of ratios between the "apparent" coefficient of viscosity and the coefficient of viscosity for electron gas is plotted.

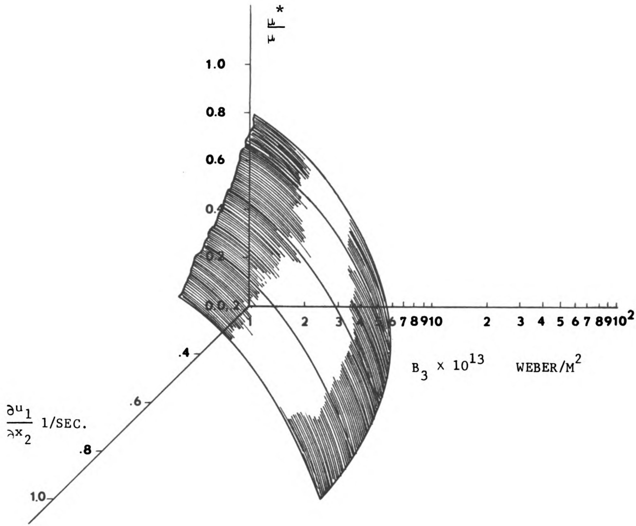

Figure 1. Ratios Between the "Apparent" Coefficient of Viscosity and the Coefficient of Viscosity for Electron Gas.

We can write (4.2.11) in the following ways:

(a). $$p_{ij} = -\mu\left(\frac{\partial u_i^*}{\partial x_j}\right) \quad , \tag{4.3.2}$$

where

$$u_i^* = u_i - \bar{u}_i \quad ; \tag{4.3.3}$$

and

$$\bar{u}_i = \frac{2e}{m}\,\epsilon_{ijk}B_k x_j \quad . \tag{4.3.4}$$

The symbol u_i^* denotes a new velocity coordinate due to the magnetic induction effect.

(b). $$p_{ij} = -\mu\left(\frac{\partial u_i}{\partial x_j}\right)^* \quad , \tag{4.3.5}$$

where

$$\left(\frac{\partial u_i}{\partial x_j}\right)^* = \frac{1}{g(x_j)}\left(\frac{\partial u_i}{\partial x_j}\right) \quad ; \tag{4.3.6}$$

and

$$\frac{1}{g(x_j)} = \left(1 - \frac{\frac{2e}{m}\,\epsilon_{ijk}B_k}{\frac{\partial u_i}{\partial x_j}}\right) \quad . \tag{4.3.7}$$

The function $\frac{1}{g(x_j)}$ may be considered as a coordinate shift due to the magnetic induction effect.

Thermal Conductivity: We can write (4.1.17) as

$$\frac{\lambda^*}{\lambda} = 1 - \frac{2e^2 B_3^2}{45\,\bar{B}_1^2 \rho^2} \quad . \tag{4.3.8}$$

Using (4.2.7), we have

$$\frac{\lambda^*}{\lambda} = 1 - \frac{2e^2B_3^2}{45\,\rho^2}\left(\frac{6\mu}{mRT}\right)^2 \quad . \tag{4.3.9}$$

We define the kinematic viscosity ν as

$$\nu = \frac{\mu}{\rho} \quad . \tag{4.3.10}$$

Thus, equation (4.3.9) becomes

$$\frac{\lambda^*}{\lambda} = 1 - \frac{72}{45}\left(\frac{eB_3}{m}\right)^2\left(\frac{\nu}{RT}\right)^2 \quad . \tag{4.3.11}$$

The table of ratios between the "apparent" thermal conductivity and thermal conductivity for electron gas is listed as follows:

Table 2. Ratios Between the"Apparent"Thermal Conductivity and the Thermal Conductivity for Electron Gas.

$\frac{\nu}{RT}$ sec.						$\frac{72}{45}\left(\frac{eB_3}{m}\right)^2\left(\frac{\nu}{RT}\right)^2$	$\frac{\lambda^*}{\lambda}$
$B_3 = 2.0\text{x}10^{-13}$ Weber/m^2	$5.0\text{x}10^{-13}$	$1.0\text{x}10^{-12}$	$3.0\text{x}10^{-12}$	$6.0\text{x}10^{-12}$	$1.0\text{x}10^{-12}$		
1.01x10	4.01	2.02	$6.7\text{x}10^{-1}$	$3.35\text{x}10^{-1}$	$2.02\text{x}10^{-1}$	.2	.8
1.42x10	5.68	2.85	$9.5\text{x}10^{-1}$	$4.73\text{x}10^{-1}$	$2.85\text{x}10^{-1}$	.4	.6
1.74x10	6.95	3.48	1.16	$5.80\text{x}10^{-1}$	$3.48\text{x}10^{-1}$	.6	.4
2.01x10	8.05	4.03	1.34	$6.70\text{x}10^{-1}$	$4.03\text{x}10^{-1}$	.8	.2
2.25x10	8.95	4.05	1.5	$7.50\text{x}10^{-1}$	$4.5\text{x}10^{-1}$	1.0	0

The graph of ratios between the"apparent"thermal conductivity and the thermal conductivity for electron gas is plotted.

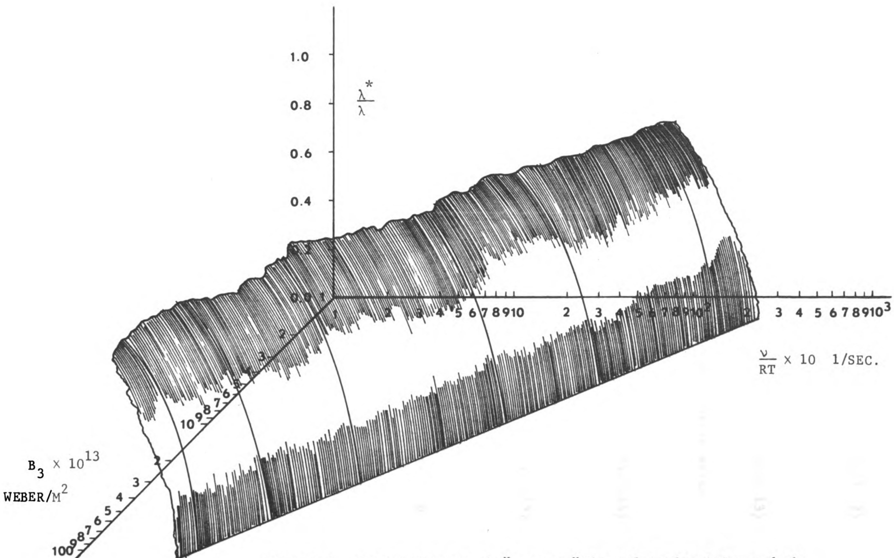

Figure 2. Ratios Between the "Apparent" Thermal Conductivity and the Thermal Conductivity for Electron Gas.

We may also rewrite (4.1.18) in the following ways:

(a) $$S_i = -2\lambda \frac{\partial T^*}{\partial x_i} \quad , \tag{4.3.12}$$

where

$$T^* = \left(1 - \frac{2e^2 B_k^2}{45\, \bar{B}_1^2 \rho^2}\right) T \quad . \tag{4.3.13}$$

The symbol T^* may be treated as a new temperature scale which is caused by the magnetic induction effect.

(b) $$S_i = -2\lambda \left(\frac{\partial T}{\partial x_i}\right)^* \quad , \tag{4.3.14}$$

where

$$\left(\frac{\partial T}{\partial x_i}\right)^* = \frac{1}{h(B_k)} \frac{\partial T}{\partial x_i} \quad ; \tag{4.3.15}$$

and

$$\frac{1}{h(B_k)} = \left(1 - \frac{2e^2 B_k^2}{45\, \bar{B}_1^2 \rho^2}\right) \quad . \tag{4.3.16}$$

The function $\frac{1}{h(B_k)}$ may be considered as the effect of magnetic induction on the temperature gradient of the gas.

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APPENDICES

APPENDIX A

SUMMATIONAL INVARIANT

By the conservation of momentum and energy, one gets

$$\left.\begin{aligned} &\vec{\xi} + \vec{\xi}_1 = \vec{\xi}\,' + \vec{\xi}_1{}' \quad ; \quad \text{and} \\ &\xi^2 + \xi_1^2 = \xi'^2 + \xi_1'^2 \ . \end{aligned}\right\} \tag{A.1}$$

Grad [15] stated that a collisional invariant is a point function φ defined in the six-dimensional space $(\vec{\xi},\vec{\xi}_1)$ as

$$\varphi(\vec{\xi}\,',\vec{\xi}_1{}') = \varphi(\vec{\xi},\vec{\xi}_1) \quad . \tag{A.2}$$

A collisional invariant which split into a sum of functions of $\vec{\xi}$ and of $\vec{\xi}_1$ is called a summational invariant. Thus we have

$$\varphi(\vec{\xi},\vec{\xi}_1) = \Psi(\vec{\xi}) + \Psi(\vec{\xi}_1) \quad . \tag{A.3}$$

If a continuous function satisfies the relation

$$\varphi(\vec{\xi} + \vec{\xi}_1, \xi^2 + \xi_1^2) = \Psi(\vec{\xi}) + \Psi(\vec{\xi}_1) \quad , \tag{A.4}$$

then it follows that

$$\Psi(\vec{\xi}) = a'\xi^2 + \vec{b}\,'\cdot\vec{\xi} + c' \quad . \tag{A.5}$$

APPENDIX B

COLLISION INTEGRALS

The rate of change of d.f. $f(\vec{\xi},\vec{x},t)$ is given as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + L_i \frac{\partial f}{\partial \xi_i} = \left(\frac{\partial f}{\partial t}\right)_c \quad , \tag{B.1}$$

where $\left(\frac{\partial f}{\partial t}\right)_c$ is the rate of change of $f(\vec{\xi},\vec{x},t)$ due to forces between particles.

The evaluation of the Boltzmann collision term is taken from Grad's paper [15]. The following assumptions are used in evaluating the collision term $\left(\frac{\partial f}{\partial t}\right)_c$:

(a) Point Particles: This assumption provides the justification in writing equation (B.1).

(b) Complete Collision: This assumption states that the time of collision is small. Hence, the energy of the gas is almost entirely translational kinetic energy.

(c) Slowly Varying f: This assumption reveals that $f(\vec{\xi},\vec{x},t + dt)$, $f(\vec{\xi},\vec{x} + d\vec{x},t)$ and $f(\vec{\xi} + d\vec{\xi},x,t)$ do not differ appreciably from $f(\vec{\xi},\vec{x},t)$.

(d) Molecular Chaos: This assumption reduces to the fact that the joint distribution function of two particles which are exerting forces on each other is equal to the product of the two individual distribution functions.

We can also write

$$\frac{dF}{dt} = \left(\frac{\partial F}{\partial t}\right)_c \quad , \tag{B.2}$$

where $F = f/m$ is the number density of particles. The symbols θ, ϵ and r are polar angle, azimuthal angle and radius respectively in a spherical coordinate system with axis along the relative velocity U between the two particles (0) and (1). Let $d\omega = r dr d\epsilon$ as the element of area. We use symbols $d\xi$ and dx to denote $d\xi_1 d\xi_2 d\xi_3$ and $dx_1 dx_2 dx_3$ respectively. The probability that particle (1) approaching $d\omega$ collides with particle (0) in the time dt is the probability that particle (1) lies in a cylinder of volume $d\omega U dt$. By the assumption (d), this probability is $[F(\vec{\xi})d\xi dx][F(\vec{\xi}_1)d\xi_1 U d\omega dt]$, where $\vec{U} = \vec{\xi} - \vec{\xi}_1$. The probability that in time dt particles $\vec{\xi}'$ and $\vec{\xi}_1'$ will collide and become $\vec{\xi}$ and $\vec{\xi}_1$ respectively, is $[F(\vec{\xi}')d\xi' dx][F(\vec{\xi}_1')d\xi_1' U' d\omega dt]$. The notations $(\vec{\xi}, \vec{\xi}_1)$ and $(\vec{\xi}', \vec{\xi}_1')$ designate the velocities of particles before and after collision respectively. The Jacobian of transformation from $(\vec{\xi}, \vec{\xi}_1)$ to $(\vec{\xi}', \vec{\xi}_1')$ is proved to be unity, i.e. $\partial(\vec{\xi}', \vec{\xi}_1')/\partial(\vec{\xi}, \vec{\xi}_1) = 1$. Thus $d\xi' d\xi_1' = d\xi d\xi_1$ and $U' = U$. For fixed $d\omega$, the net increase due to collisions of the number of particles (0) in the product space $dx d\xi$ during the time dt is $(F'F_1' - FF_1) U d\omega d\xi_1 dx d\xi dt$. Here the notations F', F_1', F, and F_1 imply $F(\vec{\xi}')$, $F(\vec{\xi}_1')$, $F(\vec{\xi})$ and $F(\vec{\xi}_1)$ respectively. Integrating over all orientation, $d\omega$, and over all colliding particles $d\xi_1$, we obtain the rate of change of the particle density F due to collision,

$$\frac{dF}{dt} = \int U(F'F_1' - FF_1)d\omega d\xi_1 \qquad ; \qquad (B.3)$$

or

$$\frac{df}{dt} = \frac{1}{m}\int U(f'f_1' - ff_1)d\omega d\xi_1 \qquad . \qquad (B.4)$$

Equating (B.1) and (B.4), we get

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{1}{m}\int U(f'f_1' - ff_1)d\omega d\xi_1 \qquad ; \qquad (B.5)$$

or

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{1}{m}\int \overline{B}(\theta,U)(f'f_1' - ff_1)d\theta d\varepsilon d\xi_1 \qquad ; \qquad (B.6)$$

where

$$\overline{B}(\theta,U) = U\ r(\theta,V)\frac{\partial r}{\partial \theta} \qquad . \qquad (B.7)$$

We define

$$J_\varphi^{(n)} = \frac{1}{m}\int \varphi(\vec{\xi})\,U(f'f_1' - ff_1)d\omega d\xi_1 d\xi \qquad , \qquad (B.8)$$

where $\varphi(\vec{\xi})$ is an arbitrary function of velocity. A change of variables from $(\vec{\xi},\vec{\xi}_1)$ to $(\vec{\xi}',\vec{\xi}_1')$ has Jacobian unity, implies that $(\vec{\xi}',\vec{\xi}_1')$ becomes $(\vec{\xi},\vec{\xi}_1)$, so

$$J_\varphi^{(n)} = -\frac{1}{m}\int \varphi(\vec{\xi}')\,U(f'f_1' - ff_1)d\omega d\xi d\xi_1 \qquad ; \qquad (B.9)$$

or

$$J_\varphi^{(n)} = -\frac{1}{m}\int \varphi(\vec{\xi}_1')\,U(f'f_1' - ff_1)d\omega d\xi d\xi_1 \qquad . \qquad (B.10)$$

From these relations, we write

$$J_\varphi^{(n)} = \frac{1}{2m}\int (\varphi - \varphi')\,U(f'f_1' - ff_1)d\omega d\xi d\xi_1 \qquad ; \qquad (B.11)$$

or

$$J_\varphi^{(n)} = \frac{1}{4m}\int(\varphi + \varphi_1 - \varphi' - \varphi_1')U(f'f_1' - ff_1)d\omega d\xi d\xi_1 . \quad (B.12)$$

The term $J_\varphi^{(n)} = 0$, if φ is a summational invariant, since by definition $\varphi + \varphi_1 = \varphi' + \varphi_1'$. Applying the same transformation of variables, equation (B.12) gives

$$J_\varphi^{(n)} = \frac{1}{2m}\int(\varphi' + \varphi_1' - \varphi - \varphi_1)Uff_1 d\omega d\xi d\xi_1 \quad ; \quad (B.13)$$

or

$$J_\varphi^{(n)} = \frac{1}{2m}\int I_\varphi ff_1 d\xi d\xi_1 \quad , \quad (B.14)$$

where

$$I_\varphi = \int(\varphi' + \varphi_1' - \varphi - \varphi_1)\bar{B}(\theta,U)d\theta d\epsilon \quad . \quad (B.15)$$

Using notations $c_i c_j = \vec{c}\,\vec{c}$ and $c_i \vec{c}^{\,2} = \vec{c}\,\vec{c}^{\,2}$, we obtain from (B.14)

$$\left.\begin{aligned} J_i^{(3)} &= \frac{1}{2m}\int I_i ff_1 d\xi d\xi_1 \quad ; \quad \text{and} \\ J_{ij}^{(2)} &= \frac{1}{2m}\int I_{ij} ff_1 d\xi d\xi_1 \quad , \end{aligned}\right\} \quad (B.16)$$

where

$$\left.\begin{aligned} I_i &= \int[c_i \vec{c}^{\,2}]\bar{B}(\theta,U)d\theta d\epsilon \quad ; \quad \text{and} \\ I_{ij} &= \int[c_i c_j]\bar{B}(\theta,U)d\theta d\epsilon \quad . \end{aligned}\right\} \quad (B.17)$$

The symbols $[\vec{c}\,\vec{c}]$ and $[\vec{c}\,\vec{c}^{\,2}]$ represent $\vec{c}'\vec{c}' + \vec{d}'\vec{d}' - \vec{c}\,\vec{c} - \vec{d}\,\vec{d}$ and $\vec{c}'\vec{c}'^{2} + \vec{d}'\vec{d}'^{2} - \vec{c}\,\vec{c}^{\,2} - \vec{d}\,\vec{d}^{\,2}$ respectively. The following notations are introduced

$$\left.\begin{aligned}
\vec{c} &= \vec{\xi} - \vec{u} && ; \\
\vec{c}\,' &= \vec{\xi}\,' - \vec{u} = \vec{c} + \vec{\alpha}(\vec{\alpha}\cdot\vec{U}) && ; \\
\vec{d} &= \vec{\xi}_1 - \vec{u} && ; \\
\vec{d}\,' &= \vec{\xi}_1' - \vec{u} = \vec{d} - \vec{\alpha}(\vec{\alpha}\cdot\vec{U}) && ; \\
\vec{U} &= \vec{\xi}_1 - \vec{\xi} = \vec{d} - \vec{c} && ; \quad \text{and} \\
\vec{V} &= \vec{\xi}_1 + \vec{\xi} = \vec{d} + \vec{c} && .
\end{aligned}\right\} \quad (B.18)$$

Using notations (B.18), we have

$$[\vec{c}\ \vec{c}] = 2(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U}) - (\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})\vec{U} - \vec{U}(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U}) \quad ; \qquad (B.19)$$

and

$$[\vec{c}\ \vec{c}^{\,2}] = -(\vec{V}\cdot\vec{U})(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U}) - (\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})\cdot(\vec{V}\ \vec{U} + \vec{V}\ \vec{U})$$
$$+ 2(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})\cdot\vec{V} + (\vec{\alpha}\cdot\vec{U})^2\vec{V} \quad . \qquad (B.20)$$

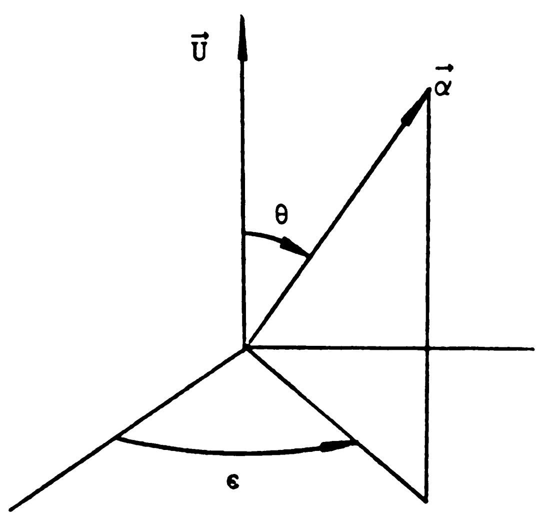

Figure B.1 Unit Vector $\vec{\alpha}$ in Spherical Coordinates

Defining the unit vector $\vec{\alpha}$ (Fig. B.1) with respect to the polar direction U as $\alpha\cdot U = U \cos\theta$, we obtain

$$\int_o^{2\pi} (\vec{\alpha}\cdot\vec{U})^2 d\epsilon = 2\pi\ U^2 \mathrm{Cos}^2\theta \qquad ; \qquad (B.21)$$

$$\int_o^{2\pi} (\vec{\alpha}\ \vec{\alpha}\cdot\vec{U}) d\epsilon = 2\pi\ \mathrm{Cos}^2\theta\ \vec{U} \qquad ; \qquad (B.22)$$

and

$$\int_o^{2\pi} (\vec{\alpha}\ \vec{\alpha}\cdot\vec{U})(\vec{\alpha}\ \vec{\alpha}\cdot\vec{U}) d\epsilon = \pi\ U^2 \mathrm{Cos}^2\theta\ \sin^2\theta\ \delta_{ij}$$
$$+ \pi\ \mathrm{Cos}^2\theta (2\ \mathrm{Cos}^2\theta - \sin^2\theta) U_i U_j \quad . \qquad (B.23)$$

Using relations (B.19), (B.20), (B.21), (B.22) and (B.23), we obtain

$$\int_o^{2\pi} [c_i c_j] d\epsilon = 2\pi\ \mathrm{Cos}^2\theta\ \sin^2\theta (U^2\delta_{ij} - 3U_iU_j) \quad ; \qquad (B.24)$$

and

$$\int_o^{2\pi} [c_i \vec{c}^{\,2}] d\epsilon = 2\pi\ \mathrm{Cos}^2\theta\ \sin^2\theta (U^2\delta_{ij} - 3U_iU_j) V_j \quad . \qquad (B.25)$$

Let

$$\overline{B}_1(U) = \pi \int \overline{B}(\theta, U) \mathrm{Cos}^2\theta\ \sin^2\theta\ d\theta \quad . \qquad (B.26)$$

Equations (B.17) can be written as

$$I_i = 2\overline{B}_1(U)(U^2\delta_{ij} - 3U_iU_j)V_j \quad ; \qquad (B.27)$$

and

$$I_{ij} = 2\overline{B}_1(U)(U^2\delta_{ij} - 3U_iU_j) \quad . \qquad (B.28)$$

Assuming that all particles are Maxwellian (i.e. the potential energy function varies as $\frac{1}{r^4}$, also the cross section for momentum transfer varies as $\frac{1}{g}$, where g is the gravitational acceleration, and the collision frequency for momentum transfer is independent of the particle energy.) and using the universe fifth power law, we retain only the highest order terms in $\vec{c}$ and $\vec{d}$ in the following expansions:

$$U^2\delta_{ij} - 3U_iU_j = \vec{c}^{\,2}\delta_{ij} - 3c_ic_j + \vec{d}^{\,2}\delta_{ij} - 3d_id_j$$

$$+ \text{ terms linear in } c_i \text{ or } d_i \quad ; \tag{B.29}$$

and

$$(U^2\delta_{ij} - 3U_iU_j)V_j = -2c_i\vec{c}^{\,2} - 2d_i\vec{d}^{\,2} + \text{ terms linear in}$$

$$c_i \text{ or } d_i \quad . \tag{B.30}$$

Using relations (B.27), (B.28), (B.29) and (B.30), equations (B.16) become:

$$J_i^{(3)} = \frac{\overline{B}_1}{m}\int(-2c_i\vec{c}^{\,2} - 2d_i\vec{d}^{\,2})ff_1 d\xi d\xi_1 \quad ; \tag{B.31}$$

and

$$J_{ij}^{(2)} = \frac{\overline{B}_1}{m}\int(\vec{c}^{\,2}\delta_{ij} - 3c_ic_j + \vec{d}^{\,2}\delta_{ij} - 3d_id_j)ff_1 d\xi d\xi_1 . \tag{B.32}$$

Integrating $d\xi$ and $d\xi_1$ separately and using definitions (3.1.6), (3.1.7), (3.1.9) and (3.1.10), we obtain

$$J_i^{(3)} = -\frac{4\overline{B}_1}{m}\rho\, S_i \quad ; \tag{B.33}$$

and

$$J_{ij}^{(2)} = -\frac{6\overline{B}_1}{m}\rho\, p_{ij} \quad . \tag{B.34}$$

APPENDIX C

THE EQUILIBRIUM SOLUTION OF THE BOLTZMANN EQUATION

Following Grad [15], we define that state of equilibrium is a state in which f is independent of $\vec{x}$ and t, and $L_i = 0$. Thus, the Boltzmann equation reduces to

$$\left(\frac{\partial f}{\partial t}\right)_c = 0 \qquad ; \tag{C.1}$$

or

$$\int U(f'f_1' - ff_1)d\omega d\xi_1 = 0 \qquad , \tag{C.2}$$

where f is a function of $\vec{\xi}$ only. We can also write

$$\int \varphi\, U(f'f_1' - ff_1)d\omega d\xi d\xi_1 = 0 \qquad ; \tag{C.3}$$

or

$$\int (\varphi + \varphi_1 - \varphi' - \varphi_1')U(f'f_1' - ff_1)d\omega d\xi d\xi_1 = 0 \qquad . \tag{C.4}$$

Letting $\varphi = \log f$ and inserting this in (C.4), we get

$$\int (\log f + \log f_1 - \log f' - \log f_1')U(f'f_1' - ff_1)d\omega d\xi d\xi_1 = 0; \tag{C.5}$$

or

$$\int U \log \frac{ff_1}{f'f_1'}(f'f_1' - ff_1)d\omega d\xi d\xi_1 = 0 \qquad . \tag{C.6}$$

The integrand of (C.6) is never positive, since $U \geq 0$ and $\log \frac{ff_1}{f'f_1'}$ has the opposite sign as $(f'f_1' - ff_1)$. If f is continuous, the integrand must vanish identically. Hence

$$\log \frac{ff_1}{f'f_1'} = 0 \qquad ; \qquad (C.7)$$

or

$$\log f + \log f_1 = \log f' + \log f_1' \qquad . \qquad (C.8)$$

We see that $\log f$ is a summational invariant. By (A.5), it follows that

$$\log f = a'\xi^2 + \vec{b}'\cdot\vec{\xi} + c' \qquad ; \qquad (C.9)$$

or

$$f = a \exp\{-b(\vec{\xi} - \vec{u})^2\} \qquad . \qquad (C.10)$$

The parameters a, b, and $\bar{u}$ are introduced instead of a', b', and c'. Integrating (C.10) and using relations (3.1.2), (3.1.3) and (3.1.6), we obtain

$$\left.\begin{aligned} \rho &= a\left(\frac{\pi}{b}\right)^{3/2} && ; \\ \vec{u} &= \bar{u} && ; \quad \text{and} \\ p &= \frac{a}{2b}\left(\frac{\pi}{b}\right)^{3/2} && . \end{aligned}\right\} \qquad (C.11)$$

After rearranging and using the equation of state (3.8.21), we get

$$\left.\begin{aligned} a &= \frac{\rho}{(2\pi RT)^{3/2}} && ; \\ \bar{u} &= \vec{u} && ; \quad \text{and} \\ b &= \frac{1}{2RT} && . \end{aligned}\right\} \qquad (C.12)$$

Hence, equation (C.10) can be written as

$$f^{(0)} = \frac{\rho}{(2\pi RT)^{3/2}} \exp\{-c^2/2RT\} \qquad . \qquad (C.13)$$

This is the well known Maxwellian distribution.