QUOTIENT POSETS AND THE CHARACTERISTIC POLYNOMIAL

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics - Doctor of Philosophy

2015

ABSTRACT

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In this dissertation we consider the generating function for the Möbius function of finite partially ordered sets. This generating function is called the characteristic polynomial of the partially ordered set. We are primarily interested in explaining why certain families of partially ordered sets have characteristic polynomials where all the roots are nonnegative integers. To this end we introduce the concept of a homogeneous quotient. These quotients allow us to give new proofs of some well-known results in the literature as well as give generalizations of them. We finish by showing how to use homogeneous quotients to give unified proofs of some classic results about the Möbius function.

ACKNOWLEDGMENTS

First and foremost I would like to thank my advisor, Dr. Bruce Sagan, for all the support and encouragement he has given me. It has truly been an honor to learn from him both about mathematics as well as how to be a mathematician.

I would also like to thank all the graduate students at Michigan State, past and present. In particular, I want to thank Dave Bramer for all the encouragement over the years.

Additionally, I would like to thank my parents, William and Kelly Hallam, who always encouraged learning and supported me in all my endeavors. Finally, I would like to thank my wife, Mobashera Hallam. Without her constant support, I would have never been able to complete this work.

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Introduction

1.1 Poset Basics

We will use the following standard notation for certain sets of numbers.

- The nonnegative integers will be denoted by $\mathbb{N}$.
- The integers will be denoted by $\mathbb{Z}$.
- The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

Let us now review some background material on partially ordered sets. See [14, Chapter 3] for a more complete overview.

Definition 1.1.1. *A* partially ordered set *or* poset *is a set P and binary relation $\leq$ on P such that the following three properties hold for all $x, y, z \in P$:*

1. *$x \leq x$ (reflexivity),*
2. *if $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry),*
3. *if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).*

It is common to represent the poset P with binary relation $\leq$ as either $(P, \leq)$ or just P if the binary relation is clear from context. Additionally, it will sometimes be useful to write $\leq$ as $\leq_P$, especially in the case when several posets are being considered.

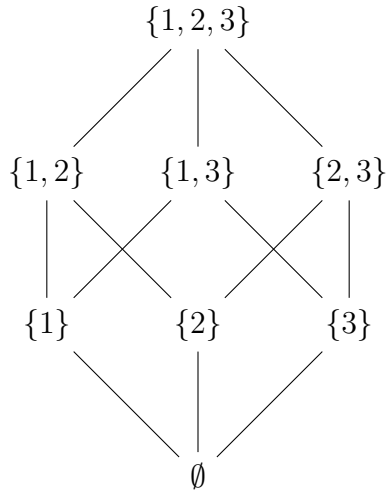

Figure 1.1: The Boolean algebra, B_3

We now consider an example of a poset. Let B_n be the set of subsets of $\{1, 2, \ldots, n\}$. For $S, T \in B_n$, define a binary relation $\leq$ on B_n as $S \leq T$ if and only if $S \subseteq T$. It is easy to verify that all three properties of a poset are satisfied and so $(B_n, \subseteq)$ is a poset. It is called the *Boolean algebra*.

It is often useful to consider a graphical representation of a poset. We do this by using Hasse diagrams. Before we can define such a diagram, we will need to develop some more terms associated with posets. If P is a poset and $x, y \in P$, we say y *covers* x, and write $x \lessdot y$, if $x < y$ and there is no z with $x < z < y$. We call a poset *finite* if the underlying set is finite. Given a finite poset, P, the *Hasse diagram* of P is a directed graph such that the vertices of the graph are the elements of P and there is a directed edge from x to y whenever $x \lessdot y$ in P. We typically do not write arrows on the directed edges, instead we draw the graph so that the edges are directed upwards. Figure 1.1 depicts the Hasse diagram of B_3.

Let P and Q be posets. We say a map $\varphi : P \to Q$ is *order preserving* if $x \leq_P y$ implies that $\varphi(x) \leq_Q \varphi(y)$ for all $x, y \in P$. We say P and Q are isomorphic if there is a bijection $\varphi : P \to Q$ such that φ and φ^{-1} are order preserving. In other words, we must have $x \leq_P y$

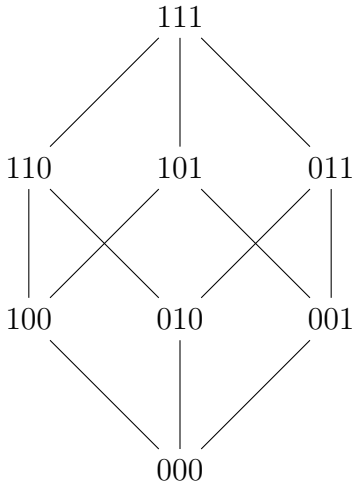

Figure 1.2: The poset of binary words, P_3

if and only if $\varphi(x) \leq_Q \varphi(y)$ for all $x, y \in P$. If P and Q are isomorphic, then we write $P \cong Q$. If P and Q are finite posets, then checking if P and Q are isomorphic is equivalent to checking that their Hasse diagrams are isomorphic as directed graphs.

We now give an example of a poset isomorphism. Let P_n be the set of all binary sequences of length n ordered by saying $a_1a_2 \dots a_n \leq b_1b_2 \dots b_n$ if and only if $a_i \leq b_i$ for all i. We claim that P_n and B_n are isomorphic. To see why, define $\varphi : P_n \to B_n$ so that $\varphi(a_1a_2 \dots a_n)$ is the subset of $\{1, 2 \dots, n\}$ obtained by including i if and only if $a_i = 1$. So for example, $\varphi(10100) = \{1, 3\}$. First, it is obvious that φ is a bijection. Now suppose that $a_1a_2 \dots a_n \leq b_1b_2 \dots b_n$, then $b_i = 0$ implies that $a_i = 0$. Therefore, $\varphi(a_1a_2 \dots a_n)$ only contains elements that are also in $\varphi(b_1b_2 \dots b_n)$ and so φ is order preserving. Finally, suppose that $S \subseteq T$ and let $\varphi^{-1}(S) = a_1a_2 \dots a_n$ and $\varphi^{-1}(T) = b_1b_2 \dots b_n$. Since $S \subseteq T$ it must be that $b_i = 0$ implies that $a_i = 0$ for all i. This implies that $a_1a_2 \dots a_n \leq b_1b_2 \dots b_n$. It follows that $P_n \cong B_n$. Figure 1.2 is the Hasse diagram of P_3. Comparing this with Figure 1.1 gives a graphical verification that the two posets are indeed isomorphic when $n = 3$.

It will be useful to have names for some special elements of a poset. If a poset, P, has a minimum element then we denote it by $\hat{0}_P$ or just $\hat{0}$ if P is clear from context. Similarly if P has a maximum element we denote it by $\hat{1}_P$ or $\hat{1}$. In the example of B_n, $\hat{0}$ is $\emptyset$ since every set contains the empty set. Also, the $\hat{1}$ of B_n is $\{1, 2, \ldots, n\}$ because every set is contained in this set. For the rest of the paper, unless otherwise noted, we will assume that all of our posets are finite and have a $\hat{0}$. We, however, do not assume that all of our posets have a $\hat{1}$.

Another set of important elements of a poset are the atoms. Let P be a poset, we say $a \in P$ is atom if $a \gtrdot \hat{0}$. The atoms of B_n are exactly the singleton subsets of $\{1, 2, \ldots, n\}$. The set of atoms of a poset P will be denoted by $A(P)$.

Let P be a poset and let S be a subset of P. The *lower order ideal generated by* S is

$$L(S) = \{x \in P \mid x \leq s \text{ for some } s \in S\}.$$

Similarly, we have the upper order ideal generated by S which is defined by

$$U(S) = \{x \in P \mid x \geq s \text{ for some } s \in S\}.$$

We say a set, S, is *totally ordered* if $x, y \in S$ implies that either $x \leq y$ or $y \leq x$. A totally ordered subset of a poset is called a *chain* of the poset. If $C : x_0 < x_1 < \cdots < x_n$ is a chain, we say C is *saturated* if $x_i \lessdot x_{i+1}$ for all $i = 0, 1, \ldots, n-1$. An example of a saturated chain in B_n is $\emptyset \lessdot \{1\} \lessdot \{1, 2\} \lessdot \ldots \lessdot \{1, 2, \ldots, n-1\} \lessdot \{1, 2, \ldots, n-1, n\}$. In terms of a Hasse diagram, a saturated chain of a poset is a directed path. The *length* of a chain, C, is $|C| - 1$ where $|\cdot|$ denotes cardinality. The previous example of the saturated chain in B_n has length n. We will also need the notion of a multichain. A *multichain* of a poset is like

a chain where we allow repetition of the elements. We say a multichain is *saturated* if the underlying set forms a saturated chain.

If P has a $\hat{0}$ then we say P is *ranked* if, for each $x \in P$, every saturated $\hat{0}$–x chain has the same length. Given a ranked poset, we get a *rank function* $\rho : P \to \mathbb{N}$ defined by setting $\rho(x)$ to be the length of a $\hat{0}$–x saturated chain. We then define

$$\rho(P) = \max_{x \in P} \rho(x).$$

In the case when P has a $\hat{1}$, we have that $\rho(P) = \rho(\hat{1})$. Returning to our example of B_n, one can see that $\rho(S) = |S|$ for all $S \in B_n$ and that $\rho(B_n) = n$.

If P and Q are posets, we define the *product* of P and Q written as $P \times Q$ as the set $P \times Q$ with the binary relation $\leq_{P \times Q}$ such that $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq_p p_2$ and $q_1 \leq_Q q_2$. It is not hard to prove the following lemma.

Lemma 1.1.2. *Let P and Q be posets, then $P \times Q$ is a poset. Moreover, if both P and Q are ranked, then $P \times Q$ is ranked as well. In this case, the rank function is given by*

$$\rho_{P \times Q}(p, q) = \rho_P(p) + \rho_Q(q).$$

Our running example, B_n, is in fact a product of smaller posets. To see why, define C_k to be the poset whose elements are $\{0, 1, \ldots, k\}$ and whose order is the normal ordering of the integers. We call C_k the *chain* of length k. For the Boolean algebra we have that $B_n \cong \underbrace{C_1 \times C_1 \times \cdots \times C_1}_{n \text{ times}}$. The isomorphism is essentially the isomorphism we gave before between B_n and the binary sequences of length n.

Given two elements x, y of a poset P, the *meet* or *greatest lower bound* of x and y (if it exists) is the element z such that $z \leq x, y$ with the property that if $w \leq x, y$, then $w \leq z$. It is denoted by $x \wedge y$. The *join* of x and y or *least upper bound* (if it exists) is the element z such that $z \geq x, y$ with the property that if $w \geq x, y$, then $w \geq z$. It is denoted by $x \vee y$. A poset is called a *lattice* if every pair of elements has a meet and a join. Note that in the case of finite posets this is equivalent to saying that every subset of elements has a meet and a join. Moreover, note that every lattice has a $\hat{0}$ and a $\hat{1}$. The poset B_n is a lattice with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

Let L be a lattice. We say an element $x \in L$ is *atomic* if there exists atoms $a_1, a_2, \ldots, a_n$ where $x = a_1 \vee a_2 \vee \cdots \vee a_n$. As a convention, we will let $\hat{0}$ be the join of the empty set. We say a lattice, L, is *atomic* if every element of L is atomic. The lattice B_n is atomic since the empty set is the empty union and since every nonempty set is the union of singleton sets. On the other hand, we claim that for $k > 1$, the chain C_k is not atomic. Recall, that there is at most one atom in any C_k. It follows that there are at most 2 elements of C_k which are atomic. Therefore if $k > 1$, then C_k is not atomic.

Let L be a ranked lattice. We say L is *(upper) semimodular* if for all $x, y \in L$ we have

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y).$$

Note that we will suppress the adjective "upper" and just call such lattices semimodular as is commonly done in the literature. We claim that B_n is semimodular. To see why, let S and T be elements of B_n. Then the Principle of Inclusion-Exclusion implies that

$$|S| + |T| = |S \cup T| + |S \cap T|.$$

Recall that if $R \in B_n$, then $\rho(R) = |R|$. Also recall that $S \cup T = S \vee T$ and $S \cap T = S \wedge T$. The previous equation now implies that for all $S, T \in B_n$ we have

$$\rho(S) + \rho(T) \geq \rho(S \wedge T) + \rho(S \vee T).$$

We conclude that B_n is semimodular.

Let L be a lattice and let $x \in L$. We say x is *left-modular* if for all $y, z \in L$ with $y \leq z$ we have

$$y \vee (x \wedge z) = (y \vee x) \wedge z.$$

If C is a multichain of L, then we say C is *left-modular* if all the elements of C are left-modular. It is not hard to see that every element of B_n is a left-modular element and so any multichain of B_n is left-modular.

If L is semimodular and contains a saturated $\hat{0}$–$\hat{1}$ left-modular multichain, then L is called *supersolvable*. Note that it is typical to define supersolvable using saturated left-modular chains as opposed to saturated left-modular multichains. Since saturated multichains always contain an underlying saturated chain, this will not lead to any problems. Recalling that B_n is semimodular and that every multichain is left-modular, we have that B_n is supersolvable.

1.2 The Möbius Function

One of the most important functions on a finite poset is its Möbius function, μ. It is a far-reaching generalization of the Möbius function from number theory. While there is a two-variable version of the Möbius function, we will only need the one-variable version.

Definition 1.2.1. *The* (one-variable) Möbius function *of P is a map $\mu : P \to \mathbb{Z}$ defined*

recursively so that

$$\sum_{x \leq y} \mu(x) = \delta_{\hat{0},y}. \tag{1.1}$$

It is sometimes convenient to use the following equivalent formulation of μ.

$$\mu(y) = \begin{cases} 1 & \text{if } y = \hat{0}, \\ -\sum_{x<y} \mu(x) & \text{otherwise.} \end{cases} \tag{1.2}$$

If we are using more than one poset, we may also use the notation μ_P to denote the Möbius function of P.

Let us consider the Möbius function of B_n. We claim that $\mu(S) = (-1)^{|S|}$ for every $S \in B_n$. This can be seen by inducting on $|S|$. The base case is true by definition of μ. Using the inductive hypothesis, we know that if $T \subsetneq S$, then $\mu(T) = (-1)^{|T|}$. Therefore, equation (1.2) implies that

$$\mu(S) = -\sum_{T \subsetneq S} (-1)^{|T|}.$$

The number of subsets of S with size k is $\binom{|S|}{k}$ and so

$$\mu(S) = -\sum_{k=0}^{|S|-1} (-1)^k \binom{|S|}{k}.$$

The binomial theorem now implies that $\mu(S) = (-1)^{|S|}$.

One of the most important results about the Möbius function is the Möbius Inversion Theorem (see [14, Proposition 3.7.1]). We do not discuss it here since we will not need it, but rather just mention of few well-known corollaries of the theorem. The Principle of Inclusion-Exclusion, the Fundamental Theorem of Difference Calculus and the Möbius

Inversion Theorem in number theory are all consequences of the poset Möbius Inversion Theorem.

The Möbius function behaves nicely with respect to products of posets and isomorphisms. More specifically, we have the next lemma. We will use these facts often in the next chapter of the paper.

Lemma 1.2.2. *Let P and Q be posets. Then we have the following.*

1. *If $P \cong Q$, then $\mu_P = \mu_Q$.*

2. *The product $P \times Q$ has Möbius function*

$$\mu_{P\times Q}(p, q) = \mu_P(p)\mu_Q(q).$$

It is not hard to see that the Möbius function for the chain, C_n, is given by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ -1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the previous lemma and the fact that $B_n \cong \underbrace{C_1 \times C_1 \times \cdots \times C_1}_{n \text{ times}}$ implies that $\mu(S) = (-1)^{|S|}$ for any $S \in B_n$, giving a second and more conceptual proof of this result.

1.3 The Characteristic polynomial

We are primarily interested in the Möbius function because of its generating function. To define it, let P be a poset and let $\rho : P \to \mathbb{N}$ be any function. Given ρ, define

$$\rho(P) = \max_{x \in P} \rho(x).$$

(The choice of ρ to denote this function is done because it replaces the rank function which is typically used in the definition of the characteristic polynomial.) Additionally, let m be any integer such that $m \geq \rho(P)$. The *generalized characteristic polynomial* of P with respect to ρ and m is

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{m - \rho(x)}. \tag{1.3}$$

In the case when ρ is the rank function of P and $m = \rho(P)$, then we call the polynomial the *characteristic polynomial* to distinguish it from the more general definition. Most of the literature on characteristic polynomials is about this case, however, many of the results we prove in this dissertation do not need this assumption.

The main goal of the first half of this dissertation is to identify posets with generalized characteristic polynomials having only nonnegative integer roots. Additionally, we wish to explain this factorization in these cases. Before we continue, let us mention some previous work done by others on the factorization of the characteristic polynomial. For a more complete overview, we suggest reading the survey paper by Sagan [12]. In [15], Stanley showed that the characteristic polynomial of a semimodular supersolvable lattice always has nonnegative integer roots. Additionally, he showed these roots were given by the sizes of blocks in a partition of the atom set of the lattice. Blass and Sagan [1] extended this

result to LL lattices. In [19], Zaslavsky generalized the concept of coloring of graphs to coloring of signed graphs and showed how these colorings were related to the characteristic polynomial of certain hyperplane arrangements. This permits one to factor characteristic polynomials using techniques for chromatic polynomials of signed graphs. Saito [13] and Terao [17] studied a module of derivations associated with a hyperplane arrangement. When this module is free, the characteristic polynomial has roots which are the degrees of its basis elements.

Let us return to our example of B_n. We have already shown that $\mu(S) = (-1)^{|S|}$ and the rank function is given by $\rho(S) = |S|$. Thus, if $\chi(B_n, t)$ is the characteristic polynomial we have

$$\chi(B_n, t) = \sum_{S \in B_n} (-1)^{|S|} t^{n-|S|}.$$

Since there are $\binom{n}{k}$ subsets of $\{1, 2 \ldots, n\}$ with size k, we get

$$\chi(B_n, t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{n-k}.$$

The binomial theorem now implies that

$$\chi(B_n, t) = (t-1)^n.$$

If we calculate the characteristic polynomial of the 1-chain, C_1, we get

$$\chi(C_1, t) = t - 1.$$

The fact that B_n is isomorphic to the n-fold product of C_1 suggests that the characteristic

polynomial of the product of posets is the product of the individual characteristic polynomials. Indeed, this is true and is the basis for our approach to explaining factorization of the characteristic polynomial. Consider the following well-known lemma which can be shown using Lemma 1.1.2 and Lemma 1.2.2.

Lemma 1.3.1. *Let P and Q be posets. If χ is the characteristic polynomial then we have the following.*

1. *If $P \cong Q$, then $\chi(P,t) = \chi(Q,t)$.*

2. *$\chi(P \times Q, t) = \chi(P,t)\chi(Q,t)$.*

This lemma explains the factorization of the characteristic polynomial of B_n. Let us now consider a different family of lattices whose characteristic polynomials have only nonnegative integer roots. We will often refer back to this example in the sequel. A *set partition* of $[n]$ is family of nonempty disjoint sets $B_1, B_2, \ldots, B_k$ whose union is $[n]$. The subsets $B_1, B_2, \ldots, B_k$ are called the *blocks* of the partition. We will denote the partition with blocks $B_1, B_2, \ldots, B_k$ by $\pi = B_1/B_2/\ldots/B_k$. The *partition lattice*, Π_n, is the lattice whose elements are the set partitions $\pi = B_1/B_2/\ldots/B_k$ of $[n]$ under the refinement ordering. In other words, we say

$$B_1/B_2/\ldots/B_k \leq C_1/C_2/\ldots/C_m$$

provided each C_i is a union of some of the B_j's. It is well-known that the characteristic polynomial of Π_n is given by

$$\chi(\Pi_n, t) = (t-1)(t-2)\cdots(t-n+1).$$

All the roots of the characteristic polynomial of B_n are 1 which is reflected in the fact

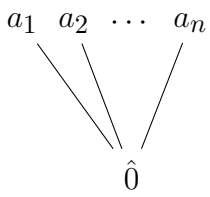

Figure 1.3: The claw with n atoms

that the characteristic polynomial of C_1 is $\chi(C_1, t) = t - 1$. However, as we have seen in the case of Π_n we need to consider other posets whose characteristic polynomial are single linear factors.

Definition 1.3.2. *The* claw *with n atoms is the poset with a $\hat{0}$, n atoms and no other elements. It will be denoted CL_n and is the poset which has Hasse diagram depicted in Figure 1.3.*

Clearly,

$$\chi(CL_n, t) = t - n$$

where χ is the characteristic polynomial.

Now let us look at the special case of Π_3. We wish to show that

$$\chi(\Pi_3, t) = (t - 1)(t - 2).$$

in a way which mimics the second proof for B_n. Since the roots of $\chi(\Pi_3, t)$ are 1 and 2, we consider $CL_1 \times CL_2$ which, by the first part of Lemma 1.3.1, has the same characteristic polynomial. Figure 1.4 contains the Hasse diagrams of Π_3 and $CL_1 \times CL_2$. Unfortunately, these two posets are not isomorphic since one contains a maximum element and the other does not. We now wish to modify $CL_1 \times CL_2$ without changing its characteristic polynomial

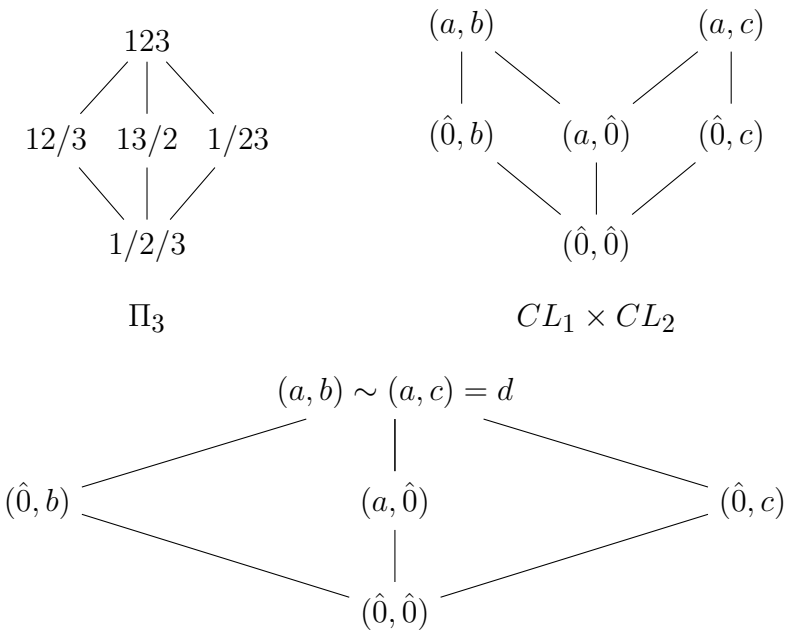

Figure 1.4: Hasse diagrams for the partition lattice example

and in such a way that the resulting poset will be isomorphic to Π_3. It will then follow from the second part of Lemma 1.3.1 that

$$\chi(\Pi_3, t) = \chi(CL_1 \times CL_2) = (t-1)(t-2).$$

Let CL_1 have its atom labeled by a and let CL_2 have its two atoms labeled by b and c. Now suppose that we identify (a,b) and (a,c) in $CL_1 \times CL_2$ and call this new element d. After this collapse, we get a poset isomorphic to Π_3 as can be seen in Figure 1.4. Note that performing this collapse did not change the characteristic polynomial since $\mu(d) = \mu((a,b)) + \mu((a,c))$ and $\rho(d) = \rho((a,b)) = \rho((a,c))$. Thus we have fulfilled our goal.

It turns out that we can use this technique of collapsing elements to find the roots of a characteristic polynomial in a wide array of posets, P. The basic idea is that it is trivial

to calculate the characteristic polynomial of a product of claws. Moreover, under certain conditions which we will see later, we are able to identify elements of the product and form a new poset without changing the characteristic polynomial. If we can show the product with identifications made is isomorphic to P, then we will have succeeded in showing that $\chi(P, t)$ has only nonnegative integer roots.

Chapter 2

Quotient Posets

In this chapter we consider how to use quotient posets to prove factorization theorems for the characteristic polynomial. In Sections 2.1–2.4we will consider the case when our posets are ranked and we are dealing with the characteristic polynomial. Unless otherwise noted, in Sections 2.1–2.4 whenever we use ρ we mean the rank function of the poset. After considering the ranked case, we will deal with the more general case where our posets need not be ranked. This material is contained in Sections 2.5–2.8.

2.1 The Basics

We begin this section by defining, in a rigorous way, what we mean by collapsing elements in a Hasse diagram of a poset. We do so by putting an equivalence relation on the poset and then ordering the equivalence classes.

Definition 2.1.1. *Let P be a poset and let $\sim$ be an equivalence relation on P. We define the* quotient $P/\sim$ *to be the set of equivalence classes with the binary relation $\leq$ defined by $X \leq Y$ in $P/\sim$ if and only if $x \leq y$ in P for some $x \in X$ and some $y \in Y$.*

Note that this binary relation on $P/\sim$ is reflexive and transitive, but is not necessarily antisymmetric. For example, let P be the chain with elements $0 < 1 < 2$ and take $X = \{0, 2\}$ and $Y = \{1\}$. Then in $P/\sim$ we have that $X \leq Y$ and $Y \leq X$, but $X \neq Y$. Since we want

the quotient to be a poset, it is necessary to require two more properties of our equivalence relation.

Definition 2.1.2. *Let P be a poset and let $\sim$ be an equivalence relation on P. Order the equivalence classes as in the previous definition. We say the poset $P/\sim$ is a* homogeneous quotient *if*

(1) $\hat{0}$ is in an equivalence class by itself, and

(2) if $X \leq Y$ in $P/\sim$, then for all $x \in X$ there is a $y \in Y$ such that $x \leq y$.

Lemma 2.1.3. *If P is a poset and $P/\sim$ is a homogeneous quotient, then $P/\sim$ is a poset.*

Proof. As previously mentioned, the fact that $\leq$ in $P/\sim$ is reflexive and transitive is clear. To see why it is antisymmetric, suppose that $X \leq Y$ and $Y \leq X$. By definition, there is a $x \in X$ and $y \in Y$ with $x \leq y$. Since $Y \leq X$ there is a $x' \in X$ with $x \leq y \leq x'$. Since $X \leq Y$ there is a $y' \in Y$ with $x \leq y \leq x' \leq y'$. Continuing, we get a chain

$$x \leq y \leq x' \leq y' \leq \dots$$

If any of the inequalities are equalities then we are done since the equivalence classes partition P. If all are strict, then we would have an infinite chain in P, but this contradicts the fact that P is finite. Therefore it must be that $X = Y$. □

Since we would like to use quotient posets to find characteristic polynomials, it would be quite helpful if the Möbius value of an equivalence class was the sum of the Möbius values of the elements of the equivalence class. This is not always the case when using homogeneous quotients, however we only need one simple requirement on the equivalence classes so that

this does occur. Note the similarity of the hypothesis in the next result to the definition of the Möbius function (equation (1.1)).

Lemma 2.1.4 ([6]). *Let $P/\sim$ be a homogeneous quotient poset. Suppose that for all nonzero $X \in P/\sim$,*

$$\sum_{y \in L(X)} \mu(y) = 0 \tag{2.1}$$

where $L(X)$ is the lower order ideal generated by X in P. Then, for all equivalence classes X

$$\mu(X) = \sum_{x \in X} \mu(x).$$

Proof. We induct on the length of the longest $\hat{0}$–X chain to prove the result. If the length is zero, then $X = \hat{0}$. Since $P/\sim$ is a homogeneous quotient, there is only one element in X and it is $\hat{0}$. The Möbius value of the minimum of any poset is 1 and so the base case holds.

Now suppose that the length is positive. Then $X \neq \hat{0}$ and so by assumption,

$$\sum_{y \in L(X)} \mu(y) = 0.$$

Breaking this sum into two parts and moving one to the other side of the equation gives

$$\sum_{x \in X} \mu(x) = - \sum_{y \in L(X) \setminus X} \mu(y). \tag{2.2}$$

Using the definition of μ and the induction hypothesis, we have that

$$\mu(X) = - \sum_{Y < X} \mu(Y) = - \sum_{Y < X} \left(\sum_{y \in Y} \mu(y) \right).$$

Since $P/\sim$ is a homogeneous quotient poset, we have that if $Y < X$ then for every $y \in Y$ there is an $x \in X$ with $y < x$. Therefore the previous sum ranges over all y such that there is an $x \in X$ with $y < x$. Thus $y \in L(X) \setminus X$. Conversely, for each $y \in L(X) \setminus X$ there is an $x \in X$ with $y < x$. By the definition of $\leq$ in $P/\sim$, we have that this implies $Y < X$ where Y is the equivalence class of y. It follows that

$$\mu(X) = -\sum_{y \in L(X) \setminus X} \mu(y). \tag{2.3}$$

Combining this equation with (2.2) completes the proof. □

For the remainder of the paper, we shall refer to the condition given by equation (2.1) as the *summation condition*. From the previous lemma, we know how the Möbius values behave when taking quotients under certain circumstances. We also need to know how the rank behaves under quotients.

Lemma 2.1.5. *Let P be a ranked poset and let $P/\sim$ be a homogeneous quotient poset. Suppose that for all $x, y \in P$, $x \sim y$ implies $\rho(x) = \rho(y)$. Then $P/\sim$ is ranked and $\rho(X) = \rho(x)$ for all $x \in P$.*

Proof. We actually prove a stronger result. We show that $X \lessdot Y$ (where $\lessdot$ denotes a covering relation) implies there is a $x \in X$ and a $y \in Y$ such that $x \lessdot y$. To see why this implies the lemma, suppose that there were two chains $\hat{0} = X_1 \lessdot X_2 \lessdot \ldots \lessdot X_n$ and $\hat{0} = Y_1 \lessdot Y_2 \lessdot \ldots \lessdot Y_m$ with $X_n = Y_m$. Then for the corresponding chains $\hat{0} = x_1 \lessdot x_2 \lessdot \ldots \lessdot x_n$ and $\hat{0} = y_1 \lessdot y_2 \lessdot \ldots \lessdot y_m$ we have that $\rho(x_n) = \rho(y_m)$ since elements in the same equivalence class have the same rank. This forces $n = m$ and so $P/\sim$ must be ranked. Additionally, it is easy to see that this implies that $\rho(X) = \rho(x)$ for all $x \in X$.

By the definition of a homogeneous quotient, if $X \lessdot Y$ then there is a $x \in X$ and $y \in Y$ with $x < y$. Suppose that there was some $z \in P$ with $x < z < y$. Then $\rho(x) < \rho(z) < \rho(y)$ and $X \leq Z \leq Y$ where Z is the equivalence class of z. Since all elements in an equivalence class have the same rank this implies that $X < Z < Y$ in $P/\sim$, which contradicts the fact that Y covered X. □

Applying Lemma 2.1.4, Lemma 2.1.5 and the definition of the characteristic polynomial we immediately get the following corollary.

Corollary 2.1.6 ([6])**.** *Let P be a ranked poset and let $P/\sim$ be a homogeneous quotient. If the summation condition (2.1) holds for all nonzero $X \in P/\sim$, and $x \sim y$ implies $\rho(x) = \rho(y)$ and χ is the characteristic polynomial, then*

$$\chi(P/\sim, t) = \chi(P, t).$$

We now have conditions under which the characteristic polynomial does not change when taking a quotient. However, the previous results do not tell us how to choose an appropriate equivalence relation for a given poset. It turns out that when the poset is a lattice, there is a canonical choice for $\sim$, as we will see in the next section.

2.2 The Standard Equivalence Relation

Let us look at the partition lattice example again and give new labelings to $CL_1 \times CL_2$ which will be helpful in determining an equivalence relation. First, we set up some notation for the atoms of the partition lattice. For $i < j$, let (i, j) denote the atom which has i and j in one block and all other elements in singleton blocks. Let CL_1 have its atom labeled by

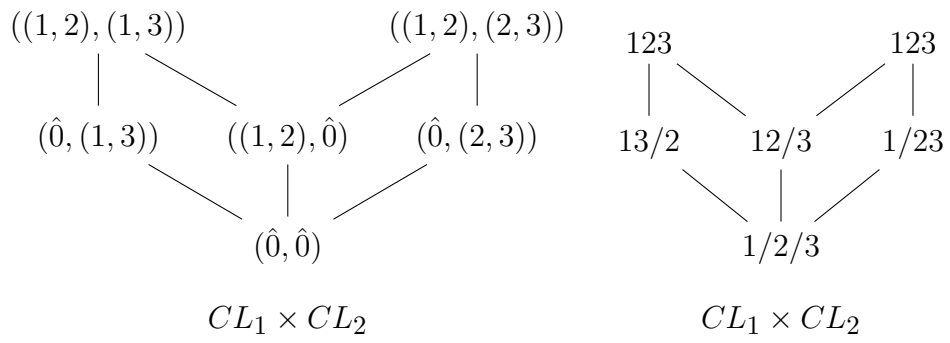

Figure 2.1: Hasse diagrams for partition lattice example with new labelings

$(1, 2)$ and CL_2 have its atoms labeled by $(1, 3)$ and $(2, 3)$. In both of the claws, label the minimum element by $\hat{0}$. The poset on the left in Figure 2.1 shows the induced labeling on $CL_1 \times CL_2$.

Now relabel $CL_1 \times CL_2$ by taking the join in Π_3 of the two elements in each pair. The poset on the right in Figure 2.1 shows this step. Finally, identify elements which have the same label. In this case, this means identifying the top two elements as we did before. Upon doing this, we get a poset which is isomorphic to Π_3 and has the same labeling as Π_3.

In order to generalize the previous example, we will be putting an equivalence relation on the product of claws whose atom sets come from partitioning the atoms of the original lattice. We need some terminology before we can define our equivalence relation.

An *ordered partition* of a set S is a set partition where we put an order on the blocks of the partition. If $B_1, B_2, \ldots, B_n$ are the blocks of the partition where we order them so $B_1 < B_2 < \cdots < B_n$, then we write the ordered partition as $(B_1, B_2, \ldots, B_n)$.

Suppose that L is a lattice and $(A_1, A_2, \ldots, A_n)$ is an ordered partition of the atoms of L. We will use CL_{A_i} to denote the claw whose atom set is A_i and whose minimum element is labeled by $\hat{0}_L$ (or just $\hat{0}$ if L is clear from context). The elements of $\prod_{i=1}^{n} CL_{A_i}$ will be called *atomic transversals* and written in boldface. (The reason for the adjective "atomic"

is because we will be considering more general transversals in Section 2.3.) Since the rank of an element in the product of claws is just the number of nonzero elements in the tuple, it will be useful to have a name for the elements of a transversal which are nonzero. For $\boldsymbol{t} \in \prod_{i=1}^{n} CL_{A_i}$ define the *support* of $\boldsymbol{t}$ as the set of nonzero elements in the tuple $\boldsymbol{t}$. We will denote it by $\operatorname{supp} \boldsymbol{t}$.

We will use the notation $\boldsymbol{t}(e^i)$ to denote the ordered tuple obtained by replacing the i^{th} coordinate of $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ with an element e. That is,

$$\boldsymbol{t}(e^i) = (t_1, t_2, \ldots, t_{i-1}, e, t_{i+1}, \ldots, t_n).$$

We will also need a notation for the join of the elements of $\boldsymbol{t}$ which will be

$$\bigvee \boldsymbol{t} = t_1 \vee t_2 \vee \cdots \vee t_n.$$

With this new terminology we are now in a position to define a natural equivalence relation on the product of the claws. Since we are trying to show that the characteristic polynomial of a lattice has certain roots, we will need to show that the quotient of the product of claws is isomorphic to the lattice. Therefore it is reasonable to define the equivalence relation by identifying two elements of the product of claws if their joins are the same in L.

Definition 2.2.1. *Let L be a lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. The* standard equivalence relation *on $\prod_{i=1}^{n} CL_{A_i}$ is defined by*

$$\boldsymbol{s} \sim \boldsymbol{t} \text{ in } \prod_{i=1}^{n} CL_{A_i} \iff \bigvee \boldsymbol{s} = \bigvee \boldsymbol{t} \text{ in } L.$$

Note that in the previous definition, we did not assume that L is ranked. We will use the

standard equivalence relation later when we will not make assumptions about if the lattice ranked or not.

We will use the notation

$$\mathcal{T}_x^a = \left\{ \boldsymbol{t} \in \prod_{i=1}^{n} CL_{A_i} : \bigvee \boldsymbol{t} = x \right\}$$

and call the elements of this set *atomic transversals of* x. Therefore, the equivalence classes of the quotient $\left(\prod_{i=1}^{n} CL_{A_i}\right) / \sim$ are of the form $\mathcal{T}_x^a$ for some $x \in L$. It is obvious that the standard equivalence relation is an equivalence relation. To be able to use any of the theorems from the previous section, we need to make sure that taking the quotient with respect to the standard equivalence relation gives us a homogeneous quotient. Moreover, we will need a way to determine if the summation condition (2.1) holds for all nonzero elements of the quotient. We do this in the next lemma. For the rest the paper we will use the notation A_x for the set of atoms below x.

Lemma 2.2.2. *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. Let $\sim$ be the standard equivalence relation on $\prod_{i=1}^{n} CL_{A_i}$. Suppose that the following hold.*

(1) For all $x \in L$, $\mathcal{T}_x^a \neq \emptyset$.

(2) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp} \boldsymbol{t}| = \rho(x)$.

Under these conditions,

(a) The lower order ideal generated by the set $\mathcal{T}_x^a$ in $\prod_{i=1}^{n} CL_{A_i}$ is given by

$$L(\mathcal{T}_x^a) = \{\boldsymbol{t} : t_i \leq x \text{ for all } i\}.$$

(b) The quotient $\left(\prod_{i=1}^{n} CL_{A_i}\right) / \sim$ *is homogeneous.*

(c) For all nonzero $\mathcal{T}_x^a \in \left(\prod_{i=1}^{n} CL_{A_i}\right) / \sim$ *the summation condition (2.1) holds if and only if for each nonzero* $x \in L$ *there is an* i *such that* $|A_i \cap A_x| = 1$.

Proof. First, we show (a). We claim that assumptions (1) and (2) imply that if $a \in A_x$ then there is an atomic transversal for x which contains a. To verify the claim, use assumption (1) to pick $\boldsymbol{t} \in \mathcal{T}_x^a$ and let $\boldsymbol{r} = \boldsymbol{t}(a^i)$. By construction and assumption (2), $\rho(\bigvee \boldsymbol{r}) = |\operatorname{supp} \boldsymbol{r}| \geq |\operatorname{supp} \boldsymbol{t}| = \rho(x)$. But also $\bigvee \boldsymbol{r} \leq x$ which forces $\bigvee \boldsymbol{r} = x$. Thus a is in the atomic transversal $\boldsymbol{r}$ for x.

The definition of $\mathcal{T}_x^a$ gives us the inclusion $L(\mathcal{T}_x^a) \subseteq \{\boldsymbol{t} : t_i \leq x \text{ for all } i\}$. The reverse inclusion holds by the previous claim.

Next, we verify (b). It is clear that $\boldsymbol{t} \in \mathcal{T}_{\hat{0}}^a$ if and only if $\boldsymbol{t} = (\hat{0}, \hat{0}, \ldots, \hat{0})$ and so part (1) of Definition 2.1.2 is satisfied. To show part (2), suppose that $\mathcal{T}_x^a \leq \mathcal{T}_y^a$ as in Definition 2.1.1. Then there is some $\boldsymbol{t} \in \mathcal{T}_x^a$ and $\boldsymbol{s} \in \mathcal{T}_y^a$ with $\boldsymbol{t} \leq \boldsymbol{s}$. It follows that $\bigvee \boldsymbol{t} \leq \bigvee \boldsymbol{s}$ and so $x \leq y$. Let $\boldsymbol{t} \in \mathcal{T}_x^a$. Using the fact that $t_i \leq x \leq y$ and part (a), we have that $\boldsymbol{t} \in L(\mathcal{T}_y^a)$. It follows that there is some $\boldsymbol{s} \in \mathcal{T}_y^a$ with $\boldsymbol{t} \leq \boldsymbol{s}$ and so part (2) of Definition 2.1.2 holds.

Finally, we demonstrate (c). Fix $x \in L$ and let N_i be the number of atoms below x in A_i. Let I be the set of indices i such that $N_i > 0$. By relabeling, if necessary, we may assume that $I = \{1, 2, \ldots, k\}$. It follows from part (a) that the number of atomic transversals in $L(\mathcal{T}_x^a)$ with support size i is $e_i(N_1, N_2, \ldots, N_k)$ where e_i is the i^{th} elementary symmetric function. For each atomic transversal $\boldsymbol{t} \in L(\mathcal{T}_x^a)$ we have that $\mu(\boldsymbol{t}) = (-1)^{|\operatorname{supp} \boldsymbol{t}|}$. Therefore,

$$\sum_{\boldsymbol{t} \in L(\mathcal{T}_x^a)} \mu(\boldsymbol{t}) = \sum_{i=0}^{k} (-1)^i e_i(N_1, N_2, \ldots, N_k) = \prod_{i=1}^{k} (1 - N_i).$$

Thus the summation condition (2.1) holds for each nonzero element in the quotient if and only if for each nonzero $x \in L$ there is an index i such that $|A_i \cap A_x| = 1$. $\square$

Combining the previous result with Corollary 2.1.6 gives conditions under which the product of claws and its quotient have the same characteristic polynomial. We also need to show that there is an isomorphism between L and this quotient. This will give us the desired factorization.

Theorem 2.2.3 ([6]). *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. Let $\sim$ be the standard equivalence relation on $\prod_{i=1}^{n} CL_{A_i}$ and let $\chi(L,t)$ be the characteristic polynomial. Suppose the following hold.*

(1) For all $x \in L$, $\mathcal{T}_x^a \neq \emptyset$.

(2) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp} \boldsymbol{t}| = \rho(x)$.

(3) For each nonzero $x \in L$ there is some i with $|A_i \cap A_x| = 1$.

Then we can conclude the following.

(a) For all $x \in L$, $\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|$.

(b) $\chi(L,t) = \prod_{i=1}^{n} (t - |A_i|)$.

Proof. Let $P = \prod_{i=1}^{n} CL_{A_i}$. First, we show that $L \cong P/\sim$. Define a map $\varphi : (P/\sim) \to L$ by $\varphi(\mathcal{T}_x^a) = x$. It is easy to see that φ is well-defined. Define $\psi : L \to (P/\sim)$ by $\psi(x) = \mathcal{T}_x^a$. By assumption $\mathcal{T}_x^a \neq \emptyset$ and so ψ is well-defined. Moreover, it is clear that φ and ψ are inverses of each other.

To show that φ is order preserving, suppose that $\mathcal{T}_x^a \leq \mathcal{T}_y^a$. Then just as in the proof of Lemma 2.2.2 part (b), we have that $x \leq y$ and so φ is order preserving.

To show that ψ is order preserving, suppose that $x \leq y$. Then applying the same technique in the proof of Lemma 2.2.2 part (b) we get that there is a $\boldsymbol{t} \in \mathcal{T}_x^a$ and $\boldsymbol{s} \in \mathcal{T}_y^a$ with $\boldsymbol{t} \leq \boldsymbol{s}$. By the definition of $\leq$ in $P/\sim$ we get that $\mathcal{T}_x^a \leq \mathcal{T}_y^a$ and so ψ is order preserving.

To obtain (a), note that the Möbius value of an element in the product of claws is $\mu(\boldsymbol{t}) = (-1)^{|\operatorname{supp} \boldsymbol{t}|}$. Therefore, using Lemma 2.1.4, we get

$$\mu(\mathcal{T}_x^a) = \sum_{\boldsymbol{t} \in \mathcal{T}_x^a} \mu(\boldsymbol{t}) = \sum_{\boldsymbol{t} \in \mathcal{T}_x^a} (-1)^{|\operatorname{supp} \boldsymbol{t}|}.$$

Using the isomorphism between L and the quotient as well as the fact that, by assumption (2), all the atomic transversals for x have size $\rho(x)$, we have

$$\mu(x) = \mu(\mathcal{T}_x^a) = (-1)^{\rho(x)} |\mathcal{T}_x^a|$$

as desired.

Finally, to verify (b) apply Corollary 2.1.6 and Lemma 2.2.2 to get that

$$\prod_{i=1}^{n} (t - |A_i|) = \chi(P, t) = \chi(P/\sim, t).$$

Now part (b) follows immediately since $L \cong P/\sim$. □

Let us return to the partition lattice and see how we can apply Theorem 2.2.3 Label the atoms (i, j) as before. Partition the atoms as $(A_1, A_2, \ldots, A_{n-1})$ where

$$A_j = \{(i, j+1) \mid i < j+1\}$$

With each atomic transversal $\boldsymbol{t}$ we will associate a graph, $G_{\boldsymbol{t}}$ on n vertices such that there

is an edge between vertex i and vertex j if and only if (i, j) is in $\boldsymbol{t}$. We will use the graph to verify the assumptions of Theorem 2.2.3 are satisfied for Π_n.

First, let us show assumption (1) holds. In the case when there is a single block $B = \{b_1 < b_2 < \cdots < b_m\}$, the elements $(b_1, b_2), (b_2, b_3), \ldots, (b_{m-1}, b_m)$ form an atomic transversal and their join is B. Now to get the elements which have more than one nontrivial block, follow the same procedure for each block and take the union of the atomic transversals. It follows every element has an atomic transversal.

Next, we prove that assumption (2) holds. We claim that if $\boldsymbol{t} \in \mathcal{T}_\pi^a$ then $G_{\boldsymbol{t}}$ is a forest. To see why, suppose that there was a cycle and let c be the largest vertex in the cycle. Then c must be adjacent to two smaller vertices a and b which implies that both (a, c) and (b, c) must be in $\boldsymbol{t}$. This is impossible since both come from A_{c-1}.

Since $G_{\boldsymbol{t}}$ is forest, if $G_{\boldsymbol{t}}$ has k components then the number of edges in $G_{\boldsymbol{t}}$ is $n - k$. It is not hard to see that i and j are in the same block in $\bigvee \boldsymbol{t}$ if and only if i and j are in the same component of $G_{\boldsymbol{t}}$. Moreover, it is well known that if $\pi \in \Pi_n$ and π has k blocks then $\rho(\pi) = n - k$. It follows that if $\boldsymbol{t} \in \mathcal{T}_\pi^a$ and π has k blocks then $|\operatorname{supp} \boldsymbol{t}| = |E(G_{\boldsymbol{t}})| = n - k = \rho(\pi)$. We conclude that assumption (2) holds.

Finally, to verify assumption (3), let $\pi \in \Pi_n$ with $\pi \neq \hat{0}$. Then π contains a nontrivial block. Let i be the second smallest number in this block. We claim that there is only one atom in A_{i-1} below π. First note that there is some atom below π in A_{i-1} namely (a, i) where a is the smallest element of the block. Suppose there was more than one atom below π in A_{i-1} and let $(a, i), (b, i) \in A_{i-1}$ with $(a, i), (b, i) \leq \pi$. Then $(a, i) \vee (b, i) \leq \pi$ and so a, b and i are all in the same block in π which is impossible since $a, b < i$ but i was chosen to be the second smallest in its block.

Now applying the theorem we get that

$$\chi(\Pi_n, t) = (t-1)(t-2)\cdots(t-n+1)$$

since $|A_i| = i$ for $1 \leq i \leq n-1$.

More generally, Theorem 2.2.3 can be used to prove Terao's result [16] about the characteristic polynomial of a hyperplane arrangement with a nice partition. In fact the notion of nice partition is the combination of assumptions (2) and (3) of Theorem 2.2.3 in the special case of a central hyperplane arrangement.

2.3 Rooted Trees

One of the drawbacks of Theorem 2.2.3 is that assumption (1) requires that every element of the lattice has an atomic transversal. This forces L to be atomic. However, by generalizing the notion of a claw to that of a rooted tree, we will be able to remove this assumption and derive Theorem 2.3.2 below which applies to a wider class of lattices.

Definition 2.3.1. *Let P be a poset and S be a subset of P containing $\hat{0}$. Let $\mathcal{C}$ be the collection of saturated chains of P which start at $\hat{0}$ and use only elements of S. The* rooted tree with respect to S *is the poset obtained by ordering $\mathcal{C}$ by inclusion and will be denoted by RT_S.*

It is easy to see that given any subset S of a poset containing $\hat{0}$, the Hasse diagram of RT_S always contains a $\hat{0}$ and has no cycles. This explains the choice of rooted tree for the name of the poset.

Strictly speaking the elements of RT_S are chains of L. However, it will be useful to think

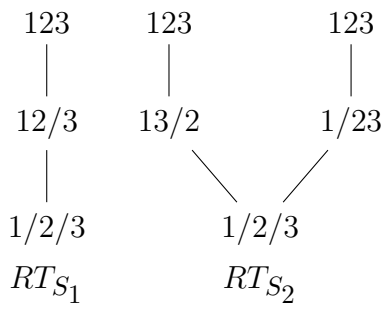

Figure 2.2: Hasse diagrams for rooted trees

of the elements of RT_S as elements of L where we associate a chain C with its top element. One can still recover the full chain by considering the unique path from $\hat{0}$ to C in RT_S. Let us consider an example in Π_3. As before, partition the atom set as $A_1 = \{12/3\}$ and $A_2 = \{13/2, 1/23\}$. Let S_1, S_2 be the upper order ideals generated by A_1, A_2, respectively, together with $\hat{0}$. Then we we get RT_{S_1} and RT_{S_2} as in Figure 2.2. Note that we label the chains $\hat{0} < 12/3 < 123$, $\hat{0} < 13/2 < 123$ and $\hat{0} < 1/23 < 123$ in S_1 and S_2 all by 123 in RT_{S_1} and RT_{S_2} since each of these chains terminates at 123.

In the previous sections, we used a partition of the atom set to form claws. In this section, we will use the partition of the atom set to form rooted trees. Given an ordered partition of the atoms of a lattice $(A_1, A_2, \ldots, A_n)$, for each i we form the rooted tree $RT_{\hat{U}(A_i)}$ where $\hat{U}(A_i)$ is the upper order ideal generated by A_i together with $\hat{0}$. We will call rooted trees of the form $RT_{\hat{U}(A_i)}$ *complete rooted trees*. We will study them in more detail in Section 2.6. Note that since $(A_1, A_2, \ldots, A_n)$ is a partition of the atoms, every element of the lattice appears in an $RT_{\hat{U}(A_i)}$ for some i.

Given $(A_1, A_2, \ldots, A_n)$, we call $\boldsymbol{t} \in \prod_{i=1}^{n} RT_{\hat{U}(A_i)}$ a *transversal*. We will use the notation,

$$\mathcal{T}_x = \left\{ \boldsymbol{t} \in \prod_{i=1}^{n} RT_{\hat{U}(A_i)} : \bigvee \boldsymbol{t} = x \right\}$$

and call such elements *transversals of* x. If $\boldsymbol{t}$ consists of only atoms of L or $\hat{0}$ then $\boldsymbol{t}$ is called an *atomic transversal*. This agrees with the terminology we used for claws. The set of atomic transversals for x will be denoted $\mathcal{T}_x^a$ as before.

There is very little change in the approach using rooted trees as opposed to claws. As before, given a partition $(A_1, A_2, \ldots, A_n)$ of the atom set of L, we will put the standard equivalence relation on $\prod_{i=1}^n RT_{\hat{U}(A_i)}$. Note that one can take the join using all the elements of a chain or just the top element as the results will be equal. Since we are using rooted trees, the natural map from $\left(\prod_{i=1}^n RT_{\hat{U}(A_i)}\right) / \sim$ to L is automatically surjective. In other words, we can remove the condition that every element of L has an atomic transversal. Additionally, since each Hasse diagram is a tree, when we take the product of the trees, the Möbius value of any transversal which is not atomic is zero and so does not affect χ. Therefore, we get the following improvement on Theorem 2.2.3.

Theorem 2.2.2 ([6])**.** *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. Let $\sim$ be the standard equivalence relation on $\prod_{i=1}^n RT_{\hat{U}(A_i)}$ and let χ be the characteristic polynomial. Suppose the following hold.*

(1) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp} \boldsymbol{t}| = \rho(x)$.

(2) For each nonzero $x \in L$ there is some i with $|A_i \cap A_x| = 1$.

Then we can conclude the following.

(a) For all $x \in L$, $\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|$.

(b) $\chi(L, t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|)$.

Proof. Let $P = \prod_{i=1}^n RT_{\hat{U}(A_i)}$. We need to show that $P/\sim$ is homogeneous. The first condition of the definition is obvious. For the second, suppose that $\mathcal{T}_x \leq \mathcal{T}_y$ and $\boldsymbol{t} \in \mathcal{T}_x$. It

follows that $x \leq y$. Let i be an index such that $A_i \cap A_y \neq \emptyset$ so that $y \in \hat{U}(A_i)$. If $\boldsymbol{t} \in \mathcal{T}_x$, then $t_j \leq x \leq y$ for all j. Therefore, $\boldsymbol{t}(y^i) \in \mathcal{T}_y$ and $\boldsymbol{t} \leq \boldsymbol{t}(y^i)$. It follows that $P/\sim$ is homogeneous.

In the proof of Theorem 2.2.3, we showed that the lattice and the quotient of the product of claws were isomorphic. The proof that L and $P/\sim$ are isomorphic is essentially the same. If we define φ and ψ analogously, then the only difference is showing ψ is order preserving in which case one can use the same ideas as in the previous paragraph to complete the demonstration.

Now we verify that the summation condition (2.1) holds for all nonzero elements of $P/\sim$. We only need to modify the proof that we gave in Lemma 2.2.2 part (c) slightly. Analogously to the proof of part (a) of that lemma, one sees that $L(\mathcal{T}_x) = \{\boldsymbol{t} : t_i \leq x \text{ for all } i\}$. Using this and the fact that only atomic transversals have nonzero Möbius values, the proof of Lemma 2.2.2 part (c) goes through as before with $\mathcal{T}_x^a$ replaced by $\mathcal{T}_x$.

Now applying Lemma 2.1.4 and the fact that $\mu(\boldsymbol{t}) = 0$ if $\boldsymbol{t}$ is not atomic, we get

$$\mu(\mathcal{T}_x) = \sum_{\boldsymbol{t} \in \mathcal{T}_x} \mu(\boldsymbol{t}) = \sum_{\boldsymbol{t} \in \mathcal{T}_x^a} \mu(\boldsymbol{t}). \tag{2.4}$$

Then applying the same proof as in Theorem 2.2.3 gives us (a).

To finish the proof we define a modification of the characteristic polynomial for any ranked poset P,

$$\bar{\chi}(P, t) = \sum_{x \in P} \mu(x) t^{-\rho(x)}.$$

We claim that $\bar{\chi}(P, t) = \bar{\chi}(P/\sim, t)$. Applying assumption (1) and the isomorphism $L \cong$

$P/\sim$, we get that for every $\boldsymbol{t} \in \mathcal{T}_x^a$ we have

$$\rho(\boldsymbol{t}) = |\operatorname{supp} \boldsymbol{t}| = \rho(x) = \rho(\mathcal{T}_x).$$

This combined with equation (2.4), proves the claim.

Now if RT is a rooted tree with k atoms then $\bar{\chi}(RT) = t^{-1}(t-k)$. It follows that

$$\bar{\chi}(P,t) = t^{-n} \prod_{i=1}^{n} (t - |A_i|).$$

Since $\bar{\chi}$ is preserved by isomorphism,

$$\bar{\chi}(L,t) = \bar{\chi}(P/\sim, t) = \bar{\chi}(P,t) = t^{-n} \prod_{i=1}^{n} (t - |A_i|).$$

Multiplying by $t^{\rho(L)}$ gives us part (b). □

2.4 Partitions Induced by a Multichain

It turns out that under certain circumstances we can show that assumption (2) of Theorem 2.3.2 and factorization of the characteristic polynomial are equivalent. To be able to prove this equivalence, we will not be able to take an arbitrary partition of the atoms, but rather we will need the partition to be induced by a multichain in the lattice.

If L is a lattice and $C : \hat{0} = x_0 \leq x_1 \leq \cdots \leq x_n = \hat{1}$ is a $\hat{0}$–$\hat{1}$ multichain of L we get an ordered partition $(A_1, A_2, \ldots, A_n)$ of the set $A(L)$ of atoms of L by defining the set A_i as

$$A_i = \{a \in A(L) \mid a \leq x_i \text{ and } a \not\leq x_{i-1}\}.$$

In this case we say $(A_1, A_2, \ldots, A_n)$ is *induced* by the multichain C. Note that we do not insist that our multichain be a chain nor does it need to be saturated as is usually done in the literature. This also implies that some of the A_i's can be empty. Partitions induced by multichains have several nice properties. The first property will apply to any lattice (Lemma 2.4.2), but for the second we will need the lattice to be semimodular (Lemma 2.4.5). Before we get to these properties, we need a modification of Lemma 2.1.4.

Lemma 2.4.1. *Suppose that $P/\sim$ is a homogeneous quotient and that for all non-maximal, nonzero $X \in P/\sim$ we have that*

$$\sum_{y \in L(X)} \mu(y) = 0$$

Then for all $X \in P/\sim$

$$\mu(X) = \begin{cases} \displaystyle\sum_{x \in X} \mu(x) & \text{if } X \text{ is not maximal,} \\ \\ \displaystyle\sum_{x \in X} \mu(x) - \sum_{y \in L(X)} \mu(y) & \text{if } X \text{ is maximal.} \end{cases}$$

Proof. If X is not maximal, then the proof of Lemma 2.1.4 goes through as before.

Now suppose that X is maximal. If $X = \hat{0}$ then the result holds because the quotient is homogeneous. So suppose $X \neq \hat{0}$. In the proof of Lemma 2.1.4, we derived equation (2.3) without using the summation condition (2.1) and so it still holds. Moreover, it is easy to see that this equation is equivalent to the one for maximal X in the statement of the current result. □

Given a lattice and a partition of the atoms, it will be useful to know when elements

of a lattice do not satisfy condition (2) of Theorem 2.3.2. This is possible to do when the partition of the atoms is induced by a multichain.

Lemma 2.4.2. *Let L be a lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by a multichain C : $\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_n = \hat{1}$. Let N_i be the number of atoms below an element $x \in L$ in A_i. If $N_i \neq 1$ for all i and $x \neq \hat{0}$ is minimal with respect to this property, then for all but one i, $N_i = 0$.*

Proof. Suppose that x is minimal, but that $N_i > 1$ for at least two i. Let k be the smallest index with $N_k \neq 0$, and $B \subseteq A_k$ be the atoms below x in A_k so $|B| \geq 2$. Let $y = \bigvee B$. So, by the choice of B, $y \leq x_k$ which implies that the atoms below y are in A_i for $i \leq k$. So the choice of A_k forces the set of atoms below y to be B which is a proper subset of the set of atoms below x, and thus $y < x$. Since $|B| \geq 2$, this contradicts the choice of x. □

The next definition gives one of the conditions equivalent to factorization when the atom partition is induced by a multichain.

Definition 2.4.3. *Let L be a lattice and let $C : \hat{0} = x_0 \leq x_1 \leq \cdots \leq x_n = \hat{1}$ be a $\hat{0}$–$\hat{1}$ multichain. For atomic $x \in L$, x neither $\hat{0}$ nor an atom, let i be the index such that $x \leq x_i$ but $x \not\leq x_{i-1}$. We say that C satisfies the* meet condition *if, for each such x, we have $x \wedge x_{i-1} \neq \hat{0}$.*

We are now in a position to give a list of equivalent conditions to factorization.

Theorem 2.4.4. *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by a $\hat{0}$–$\hat{1}$ multichain, C. Suppose that if $\boldsymbol{t} \in \mathcal{T}_x^a$, then*

$$|\operatorname{supp} \boldsymbol{t}| = \rho(x).$$

Under these conditions the following are equivalent.

1. *For every nonzero $x \in L$, there is an index i such that $|A_i \cap A_x| = 1$.*

2. *For every element which is the join of two elements from the same A_j, there is an index i such that $|A_i \cap A_x| = 1$.*

3. *The multichain C satisfies the meet condition.*

4. *We have that*

$$\chi(L,t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|)$$

where χ is the characteristic polynomial.

We note that it is possible to have $n > \rho(L)$ in part 4 of the previous theorem. In this case the exponent on the outside of the factorization will be negative. This is possible since we are using multichains and so repeating elements in the chain will give rise to as many empty blocks in the partition of the atom set as we wish. However, for each such block, we get a corresponding factor $(t-0)$. Thus $\chi(L,t)$ is still a polynomial since the negative power of t on the outside of the product will be canceled by the positive powers of t on the inside of the product.

Proof. (1) $\Rightarrow$ (4) This is Theorem 2.3.2.

(4) $\Rightarrow$ (2) We actually show that (4) $\Rightarrow$ (1) (the fact that (1) $\Rightarrow$ (2) is trivial). We do so by proving the contrapositive. By assumption, there must be a nonzero $x \in L$ such that for each i the number of atoms below x in A_i is different from one. Let k be the smallest value of $\rho(x)$ for which elements of L have this property. We show that the coefficients of $t^{\rho(L)-k}$ in $\chi(L,t)$ and in $\chi(P,t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|)$ are different, where $P = \prod_{i=1}^{n} RT_{\hat{U}(A_i)}$.

Using the same proof as we did in Theorem 2.3.2, we can show that $L \cong P/\sim$. So it suffices to show that the coefficient of $t^{\rho(L)-k}$ in $\chi(P/\sim, t)$ is different from the coefficient in $\chi(P, t)$.

Let Q be the poset obtained by removing all the elements of $P/\sim$ which have rank more than k. Let $x_1, x_2, \ldots, x_l$ be the elements of L at rank k such that the number of atoms below x_i in each block of the partition is different from one. Then by Lemma 2.4.2, each x_i has atoms above exactly one block. Now let $S = \{\mathcal{T}_{x_1}, \mathcal{T}_{x_2}, \ldots, \mathcal{T}_{x_l}\}$ be the set of the corresponding transversals. In Q, the elements of S are maximal and all the other non-maximal elements in Q satisfy the hypothesis of Lemma 2.4.1 which can be verified as in the proof of Theorem 2.3.2. Therefore we can calculate the Möbius values of the elements of rank k in Q using Lemma 2.4.1. Once we know these values we can find the coefficient of $t^{\rho(L)-k}$ in $\chi(P/\sim, t)$.

Each x_i is above at least two atoms and is above only atoms in one block. Therefore the only atomic transversals which are in $L(\mathcal{T}_{x_i})$ are transversals with single atoms and the transversal with only zeros. Since only atomic transversals have nonzero Möbius values we get that for all elements of S,

$$c_i \stackrel{\text{def}}{=} \sum_{\boldsymbol{t} \in L(\mathcal{T}_{x_i})} \mu(\boldsymbol{t}) = 1 - |A_{x_i}| < 0.$$

We know that $c_i < 0$ since the number of atoms below each x_i is at least two. Let Q_k be the set of elements of Q at rank k. Using Lemma 2.4.1, we see that the sum of the Möbius

values of Q_k is

$$\sum_{\mathcal{T}_x \in Q_k} \mu(\mathcal{T}_x) = \sum_{i=1}^{l} \mu(\mathcal{T}_{x_i}) + \sum_{\mathcal{T}_x \in Q_k \setminus S} \mu(\mathcal{T}_x)$$
$$= \sum_{i=1}^{l} \left(\sum_{\boldsymbol{t} \in \mathcal{T}_{x_i}} \mu(\boldsymbol{t}) - c_i \right) + \sum_{\mathcal{T}_x \in Q_k \setminus S} \left(\sum_{\boldsymbol{t} \in \mathcal{T}_x} \mu(\boldsymbol{t}) \right).$$

As recently noted, only elements of L which have atomic transversals have nonzero Möbius values. Using this and the assumption that $|\operatorname{supp} \boldsymbol{t}| = \rho(x) = \rho(\mathcal{T}_x)$, we get that the coefficient of of $t^{\rho(L)-k}$ in $\chi(P/\sim, t)$ is

$$\sum_{|\operatorname{supp} \boldsymbol{t}|=k} \mu(\boldsymbol{t}) - \sum_{i=1}^{l} c_i$$

where the first sum is over atomic $\boldsymbol{t}$. As we saw before, each c_i is negative and all are nonzero and so the coefficient of $t^{\rho(L)-k}$ is different from

$$\sum_{|\operatorname{supp} \boldsymbol{t}|=k} \mu(\boldsymbol{t})$$

which is the coefficient of $t^{\rho(L)-k}$ in $\chi(P, t)$. This completes the proof that (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) We show the contrapositive holds. Suppose that C does not satisfy the meet condition. Then there is some atomic x which is neither an atom nor $\hat{0}$ such that $x \leq x_i$, $x \not\leq x_{i-1}$, and $x \wedge x_{i-1} = \hat{0}$. It follows that x is only above atoms in A_i. Since x is atomic, but not an atom, there are at least two atoms, a, b below x in A_i. Let $y = a \vee b$. Since $y \leq x$, y can only be above atoms in A_i. Therefore, for all indices j, $|A_j \cap A_y| \neq 1$ and y is the join of two atoms.

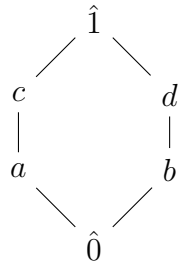

Figure 2.3: A lattice

(3) $\Rightarrow$ (1) First let us note that if x is an atom then the result is obvious. For $x \in L$ let i be the index such that $x \leq x_i$ and $x \not\leq x_{i-1}$. We now induct on i. If $i = 1$ then it suffices to show that $|A_1| = 1$ since then every nonzero $x \leq x_1$ is only above the unique element of A_1. However if a, b are distinct atoms in A_1 then $x = a \vee b$ is atomic but not an atom or zero. Further $x \leq x_1$ but $x \wedge x_{i-1} = x \wedge \hat{0} = \hat{0}$ which contradicts the meet condition. This finishes the $i = 1$ case.

Now suppose that $i > 1$ and x is not an atom. Let $z = \bigvee A_x$. Then z is atomic and $A_z = A_x$. Let $y = z \wedge x_{i-1}$. Since C satisfies the meet condition, $y \neq \hat{0}$. By construction $y < x_{i-1}$ and so by induction, there is some index $j \leq i - 1$ with $A_j \cap A_y = \{a\}$. Suppose that there was some other atom $b \in A_j \cap A_z$. Then $y \vee b$ is less than or equal to both z and x_{i-1} and so $y \vee b \leq z \wedge x_{i-1} = y$. However, this is impossible since then $A_j \cap A_y \supseteq \{a, b\}$. It follows that $1 = |A_j \cap A_z| = |A_j \cap A_x|$ and so (1) holds. $\square$

It would be nice if all atomic transversals had the correct support size when using a partition induced by a multichain since then we could remove this assumption from the previous theorem. Unfortunately this does not always occur. To see why, consider the lattice in Figure 2.3. The left-most saturated $\hat{0}$–$\hat{1}$ chain induces the ordered partition

$$(\{a\}, \{b\}).$$

It is easy to see that the support size of the transversal with both elements is not the rank of their join. Note, however, that if we had the relation $a < d$, then the support size would be the rank of the join. Moreover, note that this would also make the lattice semimodular. We see in the next lemma that semimodularity always implies transversals induced by a multichain have the correct support size.

Lemma 2.4.5. *Let L be a semimodular lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by the multichain $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = \hat{1}$. If $\sim$ is the standard equivalence, then for all $x \in L$ we have that $\boldsymbol{t} \in \mathcal{T}_x^a$ implies*

$$|\operatorname{supp} \boldsymbol{t}| = \rho(x).$$

Proof. Given an atomic $\boldsymbol{t} \in \mathcal{T}_x^a$ we induct on $|\operatorname{supp} \boldsymbol{t}|$. If $|\operatorname{supp} \boldsymbol{t}| = 0$ the result is obvious.

Now suppose that $|\operatorname{supp} \boldsymbol{t}| = k > 0$. Let i be the largest index in $\operatorname{supp} \boldsymbol{t}$. Let $\boldsymbol{s} = \boldsymbol{t}(\hat{0}^i)$, then $|\operatorname{supp} \boldsymbol{s}| = k - 1$. Suppose that $\boldsymbol{s} \in \mathcal{T}_y^a$, then $\rho(y) = k - 1$ by induction. Let j be the largest index such that $j \in \operatorname{supp} \boldsymbol{s}$. Then $y = \bigvee \boldsymbol{s} \leq x_j$ by definition of j and $t_i \not\leq x_j$ since $i > j$. Thus $x = \bigvee \boldsymbol{t} = (\bigvee \boldsymbol{s}) \vee t_i > y$. Therefore $\rho(x) > \rho(y) = k - 1$ and so $\rho(x) \geq k$. Since $|\operatorname{supp} \boldsymbol{t}| = k$, $\rho(x) \leq k$ as L is semimodular. We conclude that $\rho(x) = k = |\operatorname{supp} \boldsymbol{t}|$ and so our result holds by induction. □

Let us now consider supersolvable semimodular lattices. Recall that every supersolvable semimodular lattice contains a saturated $\hat{0}$–$\hat{1}$ left-modular chain. It turns out that saturated $\hat{0}$–$\hat{1}$ left-modular chains satisfy the meet condition as we see in the next lemma.

Lemma 2.4.6. *Let L be a lattice. If $C : \hat{0} = x_0 \leq x_1 \leq x_2 < \cdots \leq x_n = \hat{1}$ is a left-modular saturated $\hat{0}$–$\hat{1}$ multichain then C satisfies the meet condition.*

Proof. We claim that it is without loss of generality to assume that C is a chain as opposed to a multichain. To see why, suppose D is a multichain and that $x_j = x_{j+1}$ in D. Moreover, suppose that there was some x which was atomic and not an atom with $x \leq x_{j+2}$, but $x \not\leq x_{j+1}$. Then $x \wedge x_j = x \wedge x_{j+1}$ and so $x \wedge x_j \neq \hat{0}$ if and only if $x \wedge x_{j+1} \neq \hat{0}$. It follows that we can remove the repeated element without changing whether or not D satisfies the meet condition.

Let $x \in L$ be atomic and neither an atom nor $\hat{0}$. Let i be such that $x \leq x_i$ and $x \not\leq x_{i-1}$. Then we claim that there is some atom a with $a < x$ and $a \not\leq x_{i-1}$. To verify the claim, suppose that no such a existed. Since x is not an atom, it must be that all the atoms below x are also below x_{i-1}. However, x being atomic implies that $x = \bigvee A_x$ and so $x \leq x_{i-1}$ which is impossible.

By the claim, $x_{i-1} < a \vee x_{i-1} \leq x_i$. Since $x_{i-1} \lessdot x_i$ we have that $a \vee x_{i-1} = x_i$. Now (x_{i-1}, x) is a modular pair and $a < x$ so, by the definition of a modular pair,

$$a \vee (x_{i-1} \wedge x) = (a \vee x_{i-1}) \wedge x = x_i \wedge x = x.$$

But $a < x$ so $x_{i-1} \wedge x \neq \hat{0}$ and thus C satisfies the meet condition. □

We now get Stanley's Supersolvability Theorem as a corollary of Theorem 2.4.4, Lemma 2.4.5, and Lemma 2.4.6.

Theorem 2.4.7 (Stanley's Supersolvability Theorem [15])**.** *Let L be a semimodular lattice with partition of the atoms $(A_1, A_2, \ldots, A_n)$ induced by a saturated $\hat{0}$–$\hat{1}$ left-modular multichain. Then*

$$\chi(L, t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|).$$

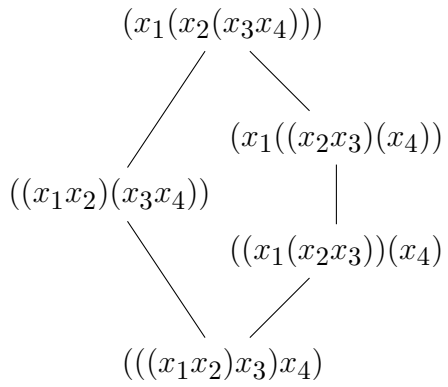

Figure 2.4: The Tamari lattice T_3

2.5 Transversal Functions

In the previous sections we had to make the assumption that our poset was a lattice and that it was ranked. In this section we develop the tools to deal with the general case when P is not necessarily a lattice nor ranked.

To start, let us do an example and calculate the generalized characteristic polynomial of an unranked poset. We will consider the *Tamari lattices* [3, 8], which will be denoted by T_n. One way to define T_n is as the set of parenthesizations of the word $x_1x_2\cdots x_{n+1}$ with ordering given by saying π is covered by σ if there exists subwords A, B, and C such that

$$\pi = \ldots((AB)C)\ldots \quad \text{and} \quad \sigma = \ldots(A(BC))\ldots$$

Figure 2.4 displays the Hasse diagrams for T_3.

As one can see from the Hasse diagram, T_3 is not ranked. In order to calculate the generalized characteristic polynomial for T_3 we need a function, ρ. We will use generalized rank which was introduced in [1]. To define generalized rank, let us set up some notation. As done previously, let A_x be the set of atoms below x in P. If $(A_1, A_2, \ldots, A_n)$ is an ordered

partition of the atoms of P, the *generalized rank* of an element x is given by

$$\rho(x) = |\{i \ : \ A_i \cap A_x \neq \emptyset\}|. \tag{2.5}$$

In other words, $\rho(x)$ counts the number of blocks in the partition that x is above.

Returning to the T_3 example, let us partition the atoms into

$$A_1 = \{((x_1x_2)(x_3x_4))\}, A_2 = \{((x_1(x_2x_3))(x_4)\}).$$

Given this partition, we see that the generalized rank of the bottom element is 0, the three middle elements all have generalized rank 1 and the top element has generalized rank 2. We take $m = 3$ which is the the length of the longest chain in T_3. Using the definition of the generalized characteristic polynomial (equation (1.3)), we get

$$\chi(T_3, t) = t^3 - 2t^2 + t = t(t-1)^2.$$

We see that $\chi(T_3, t)$ factors with roots 0 and 1. One might ask if we can decompose T_3 into the product of two smaller posets. Using this reasoning, we might guess that T_3 is the product of two chains since chains have characteristic polynomials with roots 0 and 1. Of course, this cannot be the case since chains are ranked and so their products are too, but T_3 is not ranked. However, it is possible to take the product of the chains, collapse elements in the Hasse diagram without changing the characteristic polynomial and also get a poset isomorphic to T_3. However, unlike the method we explained when our posets were ranked, we will need to collapse elements of different ranks. In order to accomplish, we introduce the notion of a transversal function.

As we did in Section 2.3 we will be using rooted trees. When the poset we are interested in is a lattice, the standard equivalence relation (Definition 2.2.1) is the canonical choice for taking quotients. Since we are interested in posets which are not necessarily lattices we need to generalize this idea. To do this, we quotient out by the kernel of a special type of map from the product of rooted trees to the poset.

Definition 2.5.1. *Let P be a poset and let $(S_1, S_2, \ldots, S_n)$ be an ordered collection of subsets of P each containing $\hat{0}$. We say $f : \prod_{i=1}^{n} RT_{S_i} \to P$ is a transversal function if it has the following properties:*

1. *The function f is order preserving.*
2. *The function f is surjective.*
3. *If $f(\boldsymbol{t}) = \hat{0}$, then $t_i = \hat{0}$ for all i.*

If f is a transversal function, the *kernel of* f, denoted $\ker f$, is the equivalence relation $\sim$ given by $\boldsymbol{s} \sim \boldsymbol{t}$ if and only if $f(\boldsymbol{s}) = f(\boldsymbol{t})$. Since we will often be referring to equivalence classes and the elements of these classes we need names for these objects. Some of the names we use were defined in earlier sections, but we use the same names since they are generalizations.

Definition 2.5.2. *Let P be a poset and let $(S_1, S_2, \ldots, S_n)$ be an ordered collection of subsets of P. Let $f : \prod_{i=1}^{n} RT_{S_i} \to P$ be a transversal function. If $\boldsymbol{t} \in \prod_{i=1}^{n} RT_{S_i}$ then we say $\boldsymbol{t}$ is a transversal for x if $f(\boldsymbol{t}) = x$. We say $\boldsymbol{t}$ is atomic or an atomic transversal if all the elements of $\boldsymbol{t}$ are atoms of RT_{S_i} or $\hat{0}$. The set of all transversals for x will be denoted by $\mathcal{T}_x$ and the set of all atomic transversals will be denoted by $\mathcal{T}_x^a$. We also define the* support *of a transversal, $\boldsymbol{t}$, as*

$$\operatorname{supp} \boldsymbol{t} = \{i : t_i \neq \hat{0}\}.$$

From the definitions it is evident that the set of equivalence classes of $\prod_{i=1}^{n} RT_{S_i} / \ker f$ is $\{\mathcal{T}_x : x \in P\}$. Moreover, it is clear that the size of the support of an atomic transversal for x is also its rank in the product of the rooted trees.

We are now in a position to give another factorization theorem. The factorization result we provide in Section 2.6 will be a special case of this one. Recall that we are using the notation $A(P)$ to denote the atom set of a poset P. In particular, $A(RT_S)$ denotes the atoms in the rooted tree RT_S.

Theorem 2.5.3. *Let P be a poset with $\rho : P \to \mathbb{N}$ and let $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Moreover, let $(S_1, S_2, \ldots, S_n)$ be an ordered collection of subsets of P which contain $\hat{0}$ and let f be a transversal function. Suppose the following hold.*

(1) If $x \leq y$ and $\boldsymbol{s} \in \mathcal{T}_x$, there exists $\boldsymbol{t} \in \mathcal{T}_y$ with $\boldsymbol{s} \leq \boldsymbol{t}$.

(2) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp} \boldsymbol{t}| = \rho(x)$.

(3) The summation condition (2.1) holds for all $\mathcal{T}_x$.

We can conclude the following.

(a) We have an isomorphism

$$P \cong \left(\prod_{i=1}^{n} RT_{S_i} \right) / \ker f.$$

(b) For each $x \in P$,

$$\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|.$$

(c) The generalized characteristic polynomial of P with respect to ρ and m (equation (1.3)) is given by

$$\chi(P, t) = t^{m-n} \prod_{i=1}^{n} (t - |A(RT_{S_i})|).$$

Proof. First, we need to show the quotient is a homogeneous quotient. Conditions (2) and (3) in the definition of a transversal function (Definition 2.5.1) imply condition (1) of a homogeneous quotient (Definition 2.1.2). To show condition (2) holds, suppose that $\mathcal{T}_x \leq \mathcal{T}_y$. Then there is a $\boldsymbol{q} \in \mathcal{T}_x$ and a $\boldsymbol{r} \in \mathcal{T}_y$ with $\boldsymbol{q} \leq \boldsymbol{r}$. Since f is order preserving, $x = f(\boldsymbol{q}) \leq f(\boldsymbol{r}) = y$. By assumption (1) of the theorem, given a $\boldsymbol{s} \in \mathcal{T}_x$ there is a $\boldsymbol{t} \in \mathcal{T}_y$ with $\boldsymbol{s} \leq \boldsymbol{t}$ and so condition (2) of a homogeneous quotient is satisfied.

Now we show (a). Let $\bar{f} : \left(\prod_{i=1}^{n} RT_{S_i}\right) / \ker f \to P$ be the induced quotient map sending $\mathcal{T}_x$ to x. Since f is surjective, it follows easily that $\bar{f}$ is a bijection and so has an inverse say g.

Next we show that $\bar{f}$ is order preserving. Recall that the elements of the quotient, $(\prod_{i=1}^{n} RT_{S_i})/\ker f$, are of the form $\mathcal{T}_x$ for some $x \in P$. Suppose that $\mathcal{T}_x \leq \mathcal{T}_y$. Then again, since f is order preserving, $x \leq y$ and so $\bar{f}$ is order preserving.

To finish the proof of (a), we show g is order preserving. Suppose that $x \leq y$. Since f is surjective, $\mathcal{T}_x \neq \emptyset$. Therefore, by assumption (1), there are $\boldsymbol{s} \in \mathcal{T}_x$ and $\boldsymbol{t} \in \mathcal{T}_y$ with $\boldsymbol{s} \leq \boldsymbol{t}$. Using the definition of a quotient poset, we get that that $\mathcal{T}_x \leq \mathcal{T}_y$ and so $g(x) \leq g(y)$.

Now we verify (b). By Lemma 2.1.4, assumption (3), and the fact that isomorphisms preserve Möbius values, we have that

$$\mu(x) = \sum_{\boldsymbol{t} \in \mathcal{T}_x} \mu(\boldsymbol{t}).$$

Since only atomic transversals have nonzero Möbius value we have

$$\mu(x) = \sum_{\boldsymbol{t} \in \mathcal{T}_x^a} \mu(\boldsymbol{t}).$$

By assumption (2), all the atomic transversals have the same support size which is the rank of x. It follows that each atomic transversal for x has Möbius value $(-1)^{\rho(x)}$. Therefore we have that

$$\mu(x) = (-1)^{\rho(x)}|\mathcal{T}_x^a|.$$

Finally we show (c). By definition,

$$\chi(P,t) = \sum_{x \in P} \mu(x) t^{m-\rho(x)}.$$

Using part (b), we get

$$\chi(P,t) = \sum_{x \in P} (-1)^{\rho(x)} |\mathcal{T}_x^a| t^{m-\rho(x)}.$$

We can break this sum into parts, depending on the rank of x. Note that by assumption (2) and part (b), every element with rank larger than n has Möbius value zero. Thus we have,

$$\chi(P,t) = \sum_{k=0}^{n} \left(\sum_{\rho(x)=k} (-1)^k |\mathcal{T}_x^a| t^{m-k} \right).$$

Neither $(-1)^k$ nor t^{m-k} depend on x so we can pull them out to get,

$$\chi(P,t) = \sum_{k=0}^{n} (-1)^k t^{m-k} \left(\sum_{\rho(x)=k} |\mathcal{T}_x^a| \right).$$

Using assumption (2) and denoting the k^{th} elementary symmetric function as e_k, we have the inner sum is exactly $e_k(|A(RT_{S_1})|, |A(RT_{S_2})|, \ldots, |A(RT_{S_n})|)$. It follows that,

$$\chi(P,t) = \sum_{k=0}^{n} (-1)^k e_k(|A(RT_{S_1})|, |A(RT_{S_2})|, \ldots, |A(RT_{S_n})|) t^{m-k}.$$

Pulling out a factor of t^{m-n} permits us to rewrite the sum as a product

$$\chi(P,t) = t^{m-n} \prod_{i=1}^{n} (t - |A(RT_{S_i})|)$$

completing the proof. □

2.6 Complete Transversal Functions

By definition, a transversal function must be surjective. However, if we impose more structure on the choice of subsets used to build the rooted trees, we can remove this assumption. In order to show this, we begin with a definition.

Definition 2.6.1. *Let P be a poset and let A be a set of atoms. The complete tree (with respect to A) is the rooted tree $RT_{\hat{U}(A)}$ where $\hat{U}(A)$ is the upper order ideal generated by the set A together with $\hat{0}$.*

Along with this new definition, we have a new type of function.

Definition 2.6.2. *Let P be a poset and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of $A(P)$. We say $f : \prod_{i=1}^{n} RT_{\hat{U}(A_i)} \to P$ is a complete transversal function if it is order preserving and has the property that if in $\boldsymbol{t}$ we have $t_i = \hat{0}$ or $t_i = x$ for all i, then $f(\boldsymbol{t}) = x$.*

Note that it may appear that complete transversal functions are not transversal functions because we dropped the condition that they are surjective. However, we will see in the next lemma that, among other nice properties, the surjectivity of the function is a consequence of the definition. We also note that if we have a lattice, then $f(\boldsymbol{t}) = \vee\boldsymbol{t}$ is a complete transversal function where $\vee\boldsymbol{t} = t_1 \vee \ldots \cdots \vee t_n$.

Lemma 2.6.3. *Let P be a poset and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of $A(P)$. Let f be a complete transversal function. Then we can conclude the following.*

(a) The function f is surjective and f is a transversal function.

(b) For all j, $t_j \leq f(t_1, t_2, \ldots, t_n)$.

(c) If $x \leq y$ and $\boldsymbol{s} \in \mathcal{T}_x$, there exists $\boldsymbol{t} \in \mathcal{T}_y$ with $\boldsymbol{s} \leq \boldsymbol{t}$.

(d) For $x \in P$, let N_i be the number of atoms below x in A_i, then

$$\sum_{\boldsymbol{s} \in L(\mathcal{T}_x)} \mu(\boldsymbol{s}) = \prod_{i=1}^{n} (1 - N_i). \tag{2.6}$$

(e) The summation condition (2.1) holds for all $\mathcal{T}_x$ if and only if for all nonzero $x \in P$, there is an index i such that $|A_i \cap A_x| = 1$.

Proof. First we show (a). Let $\hat{\mathbf{0}}$ be the transversal having all components equal to $\hat{0}$. Since we are using complete trees and a partition of the atom set, for every $x \in P$ there exists some i such that $\hat{\mathbf{0}}(x^i)$ is a transversal. It follows from the definition of a complete transversal function that $f(\hat{\mathbf{0}}(x^i)) = x$ and so f is surjective.

To show that the third condition for a transversal function holds, suppose that $f(\boldsymbol{t}) = \hat{0}$. By definition of a complete transversal function, $f(\hat{\mathbf{0}}(t_i^i)) = t_i$. Since f is order preserving and $\hat{\mathbf{0}}(t_i^i) \leq \boldsymbol{t}$ we get that $t_i = f(\hat{\mathbf{0}}(t_i^i)) \leq f(\boldsymbol{t}) = \hat{0}$. Therefore, if $f(\boldsymbol{t}) = \hat{0}$, then $\boldsymbol{t} = \hat{\mathbf{0}}$. This completes the proof that f is a transversal function.

For (b), we noted in the previous paragraph that

$$f(\hat{\mathbf{0}}(t_j^j)) = t_j.$$

Using the fact that f is order preserving, we get that

$$t_j = f(\hat{\mathbf{0}}(t_j^j)) \leq f(t_1, t_2, \ldots, t_n).$$

Next we prove (c). This is trivial if $x = \hat{0}$ so assume x is nonzero. Let $\boldsymbol{s} \in \mathcal{T}_x$. Then by by part (b), $s_i \leq x$ for all i. Let $\boldsymbol{t}$ be given by $t_i = y$ for all i with $i \in \operatorname{supp} \boldsymbol{s}$ and $t_i = \hat{0}$ for all other i. Such a $\boldsymbol{t}$ is a valid transversal since $s_i \leq x \leq y$ and we are using complete trees. Note also that since $x \neq \hat{0}$ it must be that $\boldsymbol{t}$ has at least one nonzero coordinate. It follows that $\boldsymbol{t} \in \mathcal{T}_y$ and $\boldsymbol{s} \leq \boldsymbol{t}$.

Next, let us show (d). We start by showing that

$$L(\mathcal{T}_x) = \{\boldsymbol{t} \text{ a transversal } : t_i \leq x \text{ for all } i\}. \tag{2.7}$$

To see that $L(\mathcal{T}_x)$ is contained in the other set, let $\boldsymbol{t} \in L(\mathcal{T}_x)$. Then for each i we have $\hat{\mathbf{0}}(t_i^i) \in L(\mathcal{T}_x)$. By definition of a complete transversal function, $f(\hat{\mathbf{0}}(t_i^i)) = t_i$. Since f is order preserving, $t_i = f(\hat{\mathbf{0}}(t_i^i)) \leq f(\boldsymbol{t}) = x$.

For the reverse inclusion, suppose that $\boldsymbol{t}$ is a transversal with $t_i \leq x$ for all i. Let $\boldsymbol{s}$ be the transversal obtained from $\boldsymbol{t}$ by replacing all the nonzero t_i with x. We know that $\boldsymbol{s}$ is a valid transversal because we are using complete trees. Since f is a complete transversal function, $f(\boldsymbol{s}) = x$ and so $\boldsymbol{s} \in L(\mathcal{T}_x)$. By construction, $\boldsymbol{t} \leq \boldsymbol{s}$ and therefore $\boldsymbol{t} \in L(\mathcal{T}_x)$.

Let I be the set of indices, i, such that there is an atom below x in A_i. By relabeling, if necessary, we may assume that $I = \{1, 2, \ldots, j\}$. Since $N_i = 0$ implies that $1 - N_i = 1$,

$$\prod_{i=1}^{j}(1 - N_i) = \prod_{i=1}^{n}(1 - N_i).$$

From equation (2.7) we can conclude that the number of atomic transversals in $L(\mathcal{T}_x)$ with support size i is $e_i(N_1, N_2, \ldots, N_j)$ where e_i is the i^{th} elementary symmetric function. Now for each atomic transversal $\boldsymbol{s} \in L(\mathcal{T}_x)$ we have that $\mu(\boldsymbol{s}) = (-1)^{|\operatorname{supp}(\boldsymbol{s})|}$ and all other transversals have Möbius value zero. Therefore,

$$\sum_{\boldsymbol{s} \in L(\mathcal{T}_x)} \mu(\boldsymbol{s}) = \sum_{i=0}^{j} (-1)^i e_i(N_1, N_2, \ldots, N_j) = \prod_{i=1}^{j} (1 - N_i)$$

which completes part (d).

Finally, (e) follows immediately from (d) and the definition of the summation condition.

□

Given this lemma, we can use Theorem 2.5.3 to immediately obtain the following.

Theorem 2.6.4. *Let P be a poset with $\rho : P \to \mathbb{N}$ and let $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Moreover, let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of $A(P)$ and let f be a complete transversal function. Suppose the following hold.*

(1) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp} \boldsymbol{t}| = \rho(x)$.

(2) For all nonzero $x \in P$, there is an index i such that $|A_i \cap A_x| = 1$.

We can conclude the following.

(a) We have an isomorphism

$$P \cong \left(\prod_{i=1}^{n} RT_{\hat{U}(A_i)} \right) / \ker f.$$

(b) For each $x \in P$,

$$\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|.$$

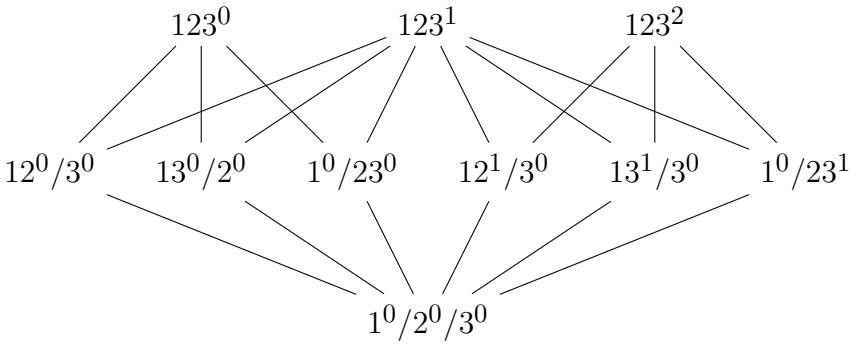

Figure 2.5: The weighted partition poset Π_3^w

(c) The generalized characteristic polynomial of P with respect to ρ and m is given by

$$\chi(P,t) = t^{m-n} \prod_{i=1}^{n} (t - |A_i|).$$

The reader may be wondering why we did not just assume from the start that we were using complete transversal functions. By doing so, we reduce the number of things we need to check and we still get the same conclusions as in Theorem 2.5.3. However, there are situations where the first theorem applies but the second does not.

Let us give an example were the summation condition (2.1) for $\mathcal{T}_x$ needed in Theorem 2.5.3 holds, but the second condition of Theorem 2.6.4 does not. We will consider the weighted partition poset, Π_n^w introduced in [2]. The elements of Π_n^w are set partitions of $[n]$ where each block B_i has one of the following weights $\{0, 1, \ldots, |B_i| - 1\}$. The weighted partitions will be denoted by $B_1^{w1}/B_2^{w2}/\ldots/B_n^{wn}$ where w_i is the weight of block B_i. The ordering is given by

$$A_1^{v1}/A_2^{v2}/\ldots/A_k^{vk} \leq B_1^{w1}/B_2^{w2}/\ldots/B_n^{wn}$$

if and only if

1. We have

$$A_1/A_2/\ldots/A_k \leq B_1/B_2/\ldots/B_n$$

in the (unweighted) partition lattice Π_n.

2. If $B_l = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m}$, then

$$v_l - (w_{i_1} + w_{i_2} + \cdots + w_{i_m}) \in \{1, 2, \ldots, m-1\}.$$

The weighted partition poset Π_3^w is shown in Figure 2.5. It is easy to check that the characteristic polynomial of this poset factors as

$$\chi(\Pi_3^w, t) = (t-3)^2.$$

Consider the sets

$$A_1 = \{12^0/3^0, 13^0/2^0, 12^1/3^0, \hat{0}\}$$

and

$$A_2 = \{1^0/23^0, 13^1/3^0, 1^0/23^1, \hat{0}\}.$$

Additionally, consider the transversal function $f : \prod_{i=1}^{2} RT_{A_i} \to \Pi_3^w$ which sends any pair which contains $\hat{0}$ to the other element in the pair and sends any pair with two non-zero elements to 123^i where i is the sum of their exponents. It is easy to check that f is a transversal function and that the summation condition (2.1) is satisfied. However, the element 123^1 is above every atom so it is impossible that it is above only one atom of either A_1 or A_2. One can also check that all the conditions of Theorem 2.5.3 are satisfied and so we have verified that the characteristic polynomial does factor using our method.

We should also point out that, as was shown in [4], the characteristic polynomial of the weighted partition poset Π_n^w factors as $\chi(\Pi_n^w, t) = (t-n)^{n-1}$. This was shown using different methods than presented here. As of now, we do not have a transversal function which gives us the factorization.

2.7 Crosscut-simplicial Lattices

As defined in [9], a *crosscut-simplicial lattice* is a lattice, L, such that if $[x, y]$ is an interval of L then any proper subset of the atoms of $[x, y]$ has a join different from y. Examples of crosscut-simplicial lattices include the Tamari lattices, the weak Bruhat order on Coxeter groups and more generally the Cambrian lattices [10]. In this section we show that a lattice is crosscut-simplicial if and only if the generalized characteristic polynomial, with respect to a function we define below, of every interval only has roots 0 and 1.

In addition to using Theorem 2.6.4 to prove this result, we will also make use of a special case of Rota's Crosscut Theorem [11]. We will give a proof of the full theorem using quotient posets in Chapter 3.

Theorem 2.7.1 (Rota's Crosscut Theorem [11])**.** *Let L be a lattice. For $x \in L$, let $a_i(x)$ be the number of subsets of $A(L)$ of size i whose join is x. Then*

$$\mu(x) = \sum_{i \geq 0} (-1)^i a_i(x).$$

Theorem 2.7.2. *Let L be lattice and let I be any interval in L. Let $\rho_I(x)$ be the number of atoms below x in I and let $m(I)$ be the length of the longest chain in I. Suppose that $\chi(I, t)$ is the generalized characteristic polynomial with respect to ρ_I and $m(I)$. The lattice*

L is crosscut-simplicial if and only if

$$\chi(I,t) = t^{m(I)-|A(I)|}(t-1)^{|A(I)|}$$

for every interval I of L.

Note that if we partition the atoms of an interval I into singleton blocks, then ρ_I is just the generalized rank of I. However, since "generalized" is ambiguous without knowing the partition of the atom set, we have decided to define it differently.

Proof. ($\Rightarrow$) Given an interval I, partition the atoms of I as $(A_1, A_2, \ldots, A_k)$ where each A_k has exactly one atom. Let f be the complete transversal function defined as $f(\boldsymbol{t}) = \vee\boldsymbol{t}$. With this partition we trivially get condition (2) of Theorem 2.6.4. By the definition of ρ_I, we must show that the join of any j atoms is above exactly j atoms in order to show condition (1).

Suppose there was some subset of j atoms of I whose join is above at least $j+1$ atoms. Let x be the meet of these $j+1$ atoms and y be the join of the $j+1$ atoms. Then L cannot be crosscut-simplicial since there is a proper subset of atoms in $[x,y]$ whose join is y. Thus we have a contradiction and so condition (1) of Theorem 2.6.4 is satisfied.

Finally, we must show that $m(I) \geq \rho_I(I)$ where $m(I)$ is the length of the longest chain in I since this was required in the definition of the generalized characteristic polynomial. Let x_0 be the $\hat{0}$ element of I and for each $1 \leq i \leq k$ define

$$x_i = \bigvee_{l=1}^{i} a_l$$

where a_i is the unique element of A_i. Since the join of j atoms is above exactly those j

atoms, we know that all the x_i's are distinct. It follows that I contains a chain of length k, namely the chain $x_0 < x_1 < \cdots < x_k$. Using the definition of ρ_I, we see that if $m(I)$ is the length of the longest chain in I then $\rho_I(I) = k \leq m(I)$. Applying Theorem 2.6.4 now yields this direction.

($\Leftarrow$) We prove the contrapositive. Suppose that L is not crosscut-simplicial. Then there must be some interval I of L where there is a set of atoms whose join is above more than just those atoms.

Let j be the minimum number of atoms needed to form a set whose join is above more than just those j atoms. We claim that the coefficient of $t^{m(I)-j}$ in $\chi(I,t)$ cannot be the same as the coefficient of $t^{m(I)-j}$ in $t^{m(I)-|A(I)|}(t-1)^{|A(I)|}$. Suppose that $\{a_1, a_2, \ldots, a_l\} \subseteq A(I)$ with $\rho_I(a_1 \vee a_2 \vee \cdots \vee a_l) = j$. By the definition of ρ_I, we have that $l \leq j$ since $a_i \vee a_2 \vee \cdots \vee a_l$ is above at least $a_1, a_2, \ldots, a_l$. If $l < j$, then there would be a subset of $A(I)$ with $l < j$ elements whose join was above more than l elements. This contradicts the definition of j. It follows that if $\rho_I(a_1 \vee a_2 \vee \cdots \vee a_l) = j$, then $l = j$. In other words, any subset of $A(I)$ whose join has rank j has exactly j atoms in it. Applying Theorem 2.7.1, we get that the coefficient of $t^{m(I)-j}$ in $\chi(I,t)$ is $(-1)^j$ multiplied by the number of j-element subsets of $A(I)$ which are above exactly those j atoms.

Since there is at least one j-element subset whose join is above more than j atoms, it must be that the coefficient of $t^{m(I)-j}$ in $\chi(I,t)$ is not $(-1)^j\binom{|A(I)|}{j}$. However, $(-1)^j\binom{|A(I)|}{j}$ is the coefficient of $t^{m(I)-j}$ in $t^{m(I)-|A(I)|}(t-1)^{|A(I)|}$ and so $\chi(I,t) \neq t^{m(I)-|A(I)|}(t-1)^{|A(I)|}$. $\square$

While the Cambrian lattices provide a large family of crosscut-simplicial lattices, we now discuss another way to find such lattices. To do so, we will consider the notion of an edge labeling of a poset. Let P be a poset, let H be its Hasse diagram and let $E(H)$ be the set

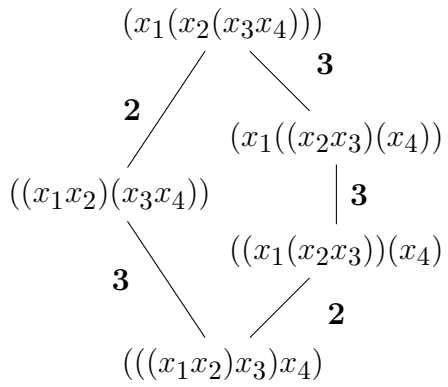

Figure 2.6: The Tamari lattice T_3 with an SB-labeling

of edges of H. An edge labeling of P is a map $\lambda : E(H) \to S$, where S is some set. In other words, we label the edges of the Hasse diagram of the poset. Introduced in [7, Definition 3.4], an *SB-labeling* of a lattice L is an edge labeling of L such that for all $u, v, w \in L$ with $u \lessdot v, w$ then we have the following.

1. If $\lambda(u, v)$ and $\lambda(u, w)$ are the labels of the edges from u to v and u and w respectively, then $\lambda(u, v) \neq \lambda(u, w)$.

2. Every inclusion-maximal chain in the interval $[u, v \vee w]$ uses only the labels $\lambda(u, v)$ and $\lambda(u, w)$ with each label occurring at least once.

An example of an SB-labeling of the Tamari Lattice T_3 is given in Figure 2.6 where the labels are in boldface. This labeling is the labeling explained in [7, Theorem 5.5]. The authors in [7, Theorem 3.7] showed that if L has an SB-labeling, then it is crosscut-simplicial. As pointed out in [9], the converse of this theorem is not true. One way to see this is to let P be the poset formed by starting with B_2 and and replacing one of its atoms with copy of B_3. Figure 2.7 contains the Hasse diagram of P. It is not hard to check that P is crosscut-simplicial. Since P only has two atoms whose join is $\hat{1}$, only 2 labels are allowed to be used to label the entire poset. However, the B_3 that we inserted needs at least 3 different

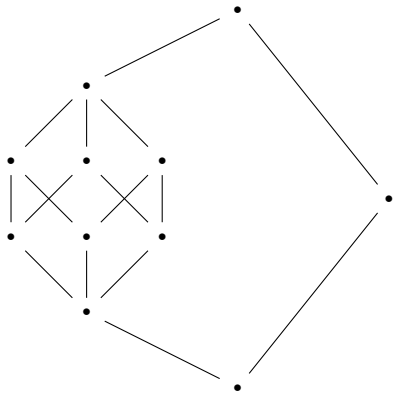

Figure 2.7: A crosscut-simplicial lattice which does not have an SB-labeling

labels and so there is no SB-labeling of P. The author of [9] suggested a generalization of an SB-labeling which avoids this type of problem. He then goes on to ask the question if a lattice being crosscut-simplicial is equivalent to having one of these generalizations of SB-labeling. It would interesting if one could use the equivalent formulation of crosscut-simplicial we described earlier using generalized characteristic polynomials to investigate this question.

2.8 LL lattices

Having shown what Theorem 2.6.4 can say about crosscut-simplicial lattices, we now turn our attention to seeing how it implies a theorem of [1]. Earlier we showed how to use the fact that partitions induced by saturated $\hat{0}$–$\hat{1}$ left-modular multichains imply assumption (2) of Theorem 2.6.4 to prove Stanley's Supersolvability Theorem [15]. We will use this fact to prove Blass and Sagan's result about LL lattices [1] which is a generalization of the supersolvability result.

In order to explain this result, we need to define the level condition. Let $(A_1, A_2, \ldots, A_n)$

be induced by $C : \hat{0} = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = \hat{1}$. This multichain also induces a partial ordering on the atoms denoted by $\trianglelefteq$. It is defined by saying $a \lhd b$ if $a \in A_i$ and $b \in A_j$ with $i < j$. We say that a lattice L with multichain C satisfies the *level condition* if

$$a \lhd b_1 \lhd b_2 \lhd \cdots \lhd b_k$$

implies that

$$a \not\leq \bigvee_{i=1}^{k} b_i.$$

The lattice L is called an *LL lattice* if it contains a left-modular multichain C and L together with C satisfy the level condition. We are now in a position to state Blass and Sagan's result.

Theorem 2.8.1 ([1])**.** *Let L be a lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by a left-modular saturated multichain such that L is an LL lattice. Let ρ be generalized rank and let m be the length of the longest $\hat{0}$–$\hat{1}$ chain. Then*

$$\chi(L, t) = t^{m-n} \prod_{i=1}^{n} (t - |A_i|).$$

Proof. We wish to use Theorem 2.6.4. First, note that since we are using generalized rank we have that $\rho(P)$ is at most the number of nonempty blocks in the partition. Since our partition is induced by a multichain and since m is the length of the largest chain in the lattice, we have that $\rho(P) \leq m$.

Define the complete transversal function to be $f(\boldsymbol{t}) = \vee \boldsymbol{t}$. Although it is not worded in the same way, the authors in [1, Theorem 6.3 and Lemma 6.4] proved assumption (1) of Theorem 2.6.4 holds. Finally, as noted before, it was shown in Lemma 2.4.6 that satu-

rated left-modular multichains satisfy the meet condition and so satisfy assumption (2) of Theorem 2.6.4. □

The theorems presented so far have provided conditions which imply factorization. We would like to finish this section with a theorem where we provide a condition which is equivalent to factorization.

Theorem 2.8.2. *Let P be a poset and let $\rho : P \to \mathbb{N}$ with $m \in \mathbb{N}$ such that $\rho(P) \leq m$. Let $\chi(P,t)$ be the generalized characteristic polynomial with respect to ρ and m. Let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of $A(P)$ and let $f : \prod_{i=1}^{n} RT_{\hat{U}(A_i)} \to P$ be a complete transversal function. Finally, define*

$$T = \{x \in P \setminus \hat{0} : \ |A_i \cap A_x| \neq 1 \text{ for all } i\}.$$

Suppose that the following hold.

1. *If $\boldsymbol{t} \in \mathcal{T}_x^a$ then $|\operatorname{supp}(\boldsymbol{t})| = \rho(x)$.*

2. *If $x, y \in P$ and $x < y$, then $\rho(x) < \rho(y)$.*

3. *For all minimal elements $x, y \in T$, the cardinality of the sets*

$$\{i : \ |A_i \cap A_x| \neq 0\} \text{ and } \{i : \ |A_i \cap A_y| \neq 0\}$$

have the same parity.

Under these conditions,

$$\chi(P,t) = t^{m-n} \prod_{i=1}^{n} (t - |A_i|)$$

if and only if for every nonzero $x \in P$ there is an index i such that $|A_i \cap A_x| = 1$.

This theorem is a generalization of Theorem 2.4.4. The two proofs are quite similar so we only provide a sketch below.

Sketch of proof. First, note that the backwards direction is Theorem 2.6.4. For the forward direction, we will prove the contrapositive. Note that the assumption in this direction implies that $T \neq \emptyset$. Let k be the smallest value of ρ applied to the elements of T. We show that the coefficient of t^{m-k} in $\chi(P,t)$ and in $t^{m-n}\prod_{i=1}^{n}(t-|A_i|)$ are different.

Define $R = \left(\prod_{i=1}^{n} RT_{\hat{U}(A_i)}\right) / \ker f$. We claim that R is a homogeneous quotient and that $P \cong R$. Since f is a complete transversal function, Lemma 2.6.3 part (c) implies that assumption (1) of Theorem 2.5.3 is satisfied. Note that the proof of part (a) of Theorem 2.5.3 only requires assumption (1). Therefore, R is homogeneous and $P \cong R$. Since $P \cong R$, it is enough to show that the coefficient of t^{m-k} in $\chi(R,t)$ and in $t^{m-n}\prod_{i=1}^{n}(t-|A_i|)$ are not the same.

Let $x_1, x_2, \ldots, x_l$ be the set of elements of T with $\rho(x_i) = k$ for all i and let $S = \{\mathcal{T}_{x_1}, \mathcal{T}_{x_2}, \ldots, \mathcal{T}_{x_l}\}$ be the corresponding equivalence classes. Moreover, define Q to be the poset obtained from R by removing all elements of R with ρ value larger than k. Using assumption (2), we can see that the Möbius value of elements with ρ at most k in R and Q are the same. In Q all elements with ρ value k are maximal. By assumption (2) and the assumption on k any element of Q which is not maximal cannot be in the set T. Then Lemma 2.6.3 part (e) implies that every non-maximal element satisfies the summation condition (2.1). Thus, we can apply Lemma 2.4.1 to conclude that

$$\mu(\mathcal{T}_{x_i}) = \sum_{\boldsymbol{t} \in \mathcal{T}_{x_i}} \mu(\boldsymbol{t}) - \sum_{\boldsymbol{s} \in L(\mathcal{T}_{x_i})} \mu(\boldsymbol{s}).$$

Let

$$c_i = \sum_{\boldsymbol{s} \in L(\mathcal{T}_{x_i})} \mu(\boldsymbol{s}).$$

We claim that all the c_i's are either 0 or have the same sign. By equation (2.6), if $c_i \neq 0$, then the sign of c_i is $(-1)^{k_i}$ where k_i is the number of blocks with atoms below x_i. By assumption (3), for the c_i's which are not equal to 0, the corresponding k_i's have the same parity. Therefore, the signs of the nonzero c_i's are the same. Since there is at least one element of $T \neq \emptyset$ in Q, there is at least one $c_i \neq 0$.

Using the same argument as in the proof of Theorem 2.4.4, we get the coefficient of t^{m-k} in $\chi(R,t)$ is

$$\sum_{|\operatorname{supp} \boldsymbol{t}|=k} \mu(\boldsymbol{t}) - \sum_{i=1}^{l} c_i$$

in which the first sum ranges over atomic transversals. Since there is at least one c_i which is nonzero and all the nonzero c_i's have the same sign, we see that this coefficient is not the same as

$$\sum_{|\operatorname{supp} \boldsymbol{t}|=k} \mu(\boldsymbol{t}).$$

However, the previous expression is the coefficient of t^{m-k} in $t^{m-n} \prod_{i=1}^{n} (t - |A_i|)$. It follows that

$$\chi(P,t) \neq t^{m-n} \prod_{i=1}^{n} (t - |A_i|)$$

which is what we wished to show. □

Chapter 3

Classic Results About the Möbius Function

In this chapter, we will give a new method to prove an array of classic results about the Möbius function. To explain the idea, we need a definition. Let P be a poset with a $\hat{1}$. A *coatom* of L is an element $c \in P$ such that $c \lessdot \hat{1}$. The idea of this new method is to use induction on the size of the poset. In order to do this, we will collapse a coatom and the $\hat{1}$ of the poset.

We begin with a lemma that explains the simple nature of the values of μ for the original poset and the poset obtained by collapsing a coatom and $\hat{1}$. In the lemma and throughout the rest of the section, we will use $[x]$ to denote the equivalence class which contains x.

Lemma 3.0.3 ([6])**.** *Let P be a poset with a $\hat{0}$ and $\hat{1}$ and at least 3 elements. Suppose c is a coatom and let $\sim$ be the equivalence relation identifying c and $\hat{1}$. Then $P/\sim$ is homogeneous and*

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1}).$$

Moreover, if P is a lattice, then $P/\sim$ is a lattice with $[x] \vee [y] = [x \vee y]$ for all $x, y \in P$ and $[x] \wedge [y] = [x \wedge y]$ provided $[x], [y] \neq [\hat{1}]$.

Proof. First, let us show that $P/\sim$ is homogeneous. Since there are at least 3 elements and we are collapsing a coatom and $\hat{1}$, we have that $\hat{0}$ is in its own equivalence class. Now

suppose that $[x] < [y]$. It follows that $[x] \neq \{c, \hat{1}\}$ since $[x] < [y]$ and $[c] = [\hat{1}]$ is the $\hat{1}$ of the quotient. Therefore, $[x] = \{x\}$ and so it is obvious that $P/\sim$ is a homogeneous quotient.

To show that $\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$ note that since every element of P is below $\hat{1}$ and every other equivalence class has only one element, we get

$$\sum_{y \in L([x])} \mu(y) = \sum_{y \leq x} \mu(y) = 0$$

for all nonzero $x \neq c$. By Lemma 2.1.4 this implies that

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$$

which is what we wished to prove.

Now suppose that P is a lattice. It is not hard to see that $(P/\sim) \cong (P \setminus \{c\})$. Therefore, if $x \vee y \neq c$, we immediately get that $[x] \vee [y]$ exists and $[x] \vee [y] = [x \vee y]$. If $x \vee y = c$, then $\hat{1}$ is the only element in $P \setminus \{c\}$ which is an upper bound for both x and y. It follows that $[x] \vee [y] = [\hat{1}] = [c] = [x \vee y]$. Since $P \setminus \{c\}$ clearly has a $\hat{0}$, we conclude $P/\sim$ is a lattice. Finally, if $[x], [y] \neq [\hat{1}]$ then $[x] = \{x\}$ and $[y] = \{y\}$ and so $[x] \wedge [y] = [x \wedge y]$. □

Let us now use Lemma 3.0.3 to prove some classic results.

Corollary 3.0.4 (Hall's Theorem [5])**.** *Let P be a finite poset, then*

$$\mu(x, y) = \sum_{i \geq 0} (-1)^i c_i$$

where c_i is the number of chains of length i which start at x and terminate at y.

Proof. Without loss of generality we may assume that $x = \hat{0}$ and $y = \hat{1}$ since all chains which

start at x and terminate at y are in the interval $[x, y]$. We prove the theorem by inducting on $|P|$. If $|P| = 1$ or $|P| = 2$ then the result is obvious.

Now suppose that $|P| > 2$. Let $P/\sim$ be obtained by identifying a coatom c and $\hat{1}$. Consider the sum

$$\sum_{i \geq 0} (-1)^i c_i$$

where c_i is the number of $\hat{0}$–$\hat{1}$ chains of length i in P. Let a_i be the number chains of length i which do not contain c and let b_i be the number chains of length i containing c. Then

$$\sum_{i \geq 0} (-1)^i c_i = \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i.$$

There exists a bijection between $\hat{0}$–$\hat{1}$ chains in P not containing c and $[\hat{0}]$–$[\hat{1}]$ chains in $P/\sim$ which preserves length. Moreover, there is a bijection between $\hat{0}$–$\hat{1}$ chains in P containing c and $\hat{0}$–c chains in $[0, c]$. Note that in this bijection, the chains decrease by one in length.

Since $|P/\sim| < |P|$, using induction we get that

$$\mu([\hat{1}]) = \sum_{i \geq 0} (-1)^i a_i.$$

Similarly since $|[0, c]| < |P|$ we get that

$$\mu(c) = -\sum_{i \geq 0} (-1)^i b_i$$

where we have multiplied the sum by -1 since the chains have decreased by one in length.

By Lemma 3.0.3, we have that

$$\mu([\hat{1}]) = \mu(c) + \mu(\hat{1})$$

or equivalently

$$\mu(\hat{1}) = \mu([\hat{1}]) - \mu(c).$$

Therefore,

$$\mu(\hat{1}) = \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i = \sum_{i \geq 0} (-1)^i c_i$$

which is what we wished to prove. □

Next, we prove a theorem of Weisner.

Corollary 3.0.5 (Weisner's Theorem [18])**.** *Let L be a lattice and let $\hat{0} \neq a \in L$. If $|L| \geq 2$, then*

$$\mu(\hat{1}) = - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Proof. Let us note that if $a = \hat{1}$, then the result is just restating the definition of μ, so we assume that $a \neq \hat{1}$ for the rest of the proof. We prove the result by induction. We have already covered the case $|L| = 2$, since then a must be $\hat{1}$.

Now suppose that $|L| > 2$. Let c be a coatom such that $a \leq c$. Consider, the lattice $L/\sim$ obtained by identifying c and $\hat{1}$. Since $|L/\sim| < |L|$, we get that

$$\mu([\hat{1}]) = - \sum_{[x] \neq [\hat{1}], [x] \vee [a] = [\hat{1}]} \mu([x]).$$

Using the facts that $[\hat{1}] = \{c, 1\}$, $[x] \vee [a] = [x \vee a]$, and $\mu([x]) = \mu(x)$ for $[x] \neq [\hat{1}]$, we obtain,

$$\mu([\hat{1}]) = - \sum_{x \neq c, \hat{1}, x \vee a = c, \hat{1}} \mu(x).$$

Since joins are unique, we can break the sum into two parts as,

$$\mu([\hat{1}]) = - \sum_{x \neq c, \hat{1}, x \vee a = c} \mu(x) - \sum_{x \neq c, \hat{1}, x \vee a = \hat{1}} \mu(x).$$

If $x \vee a = c$, then it is clear that $x \in [0, c]$. Moreover, since $a \leq c$, it is clear that $c \vee a \neq \hat{1}$. Thus, we can remove the $x \neq \hat{1}$ condition in the first sum and remove the $x \neq c$ condition in the second. This gives,

$$\mu([\hat{1}]) = - \sum_{x \neq c, x \vee a = c} \mu(x) - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Now the first sum is only over $[0, c]$ and $|[0, c]| < |L|$ so by induction,

$$\mu([\hat{1}]) = \mu(c) - \sum_{x \neq \hat{1}, x \vee a = \hat{1}} \mu(x).$$

Using the fact that $\mu([\hat{1}]) = \mu(\hat{1}) + \mu(c)$, we immediately obtain the result. □

Our next corollary will make use of crosscuts. We remind the reader of a few definitions here. Let P be a poset, a subset $C \in P$ is called an *antichain* if whenever $x, y \in C$, then $x \not\leq y$ and $x \not\geq y$.

Definition 3.0.6. *Let L be a lattice. A* crosscut *of L is a set C with the following properties:*

1. $\hat{0}, \hat{1} \notin C$.

2. C is an antichain.

3. Every maximal $\hat{0}$–$\hat{1}$ chain intersects C.

Although we will not need it to state the next theorem, we will need the notion of the dual of a poset in the proof. Let P be a poset. The *dual* of P, written as P^*, is the poset with the same underlying set and binary relation given by saying $x \leq_{P^*} y$ if and only if $y \leq_P x$.

Theorem 3.0.7 (Rota's Crosscut Theorem [11])**.** *Let L be a lattice and let C be a crosscut. Then*

$$\mu(\hat{1}) = \sum_{\vee B=\hat{1}, \wedge B=\hat{0}} (-1)^{|B|}$$

where the sum ranges over all $B \subseteq C$ such that $\vee B = \hat{1}$ and $\wedge B = \hat{0}$.

Proof. We first consider the special case when every coatom is also an atom. In this case, the crosscut must be the atom set. Moreover, a subset of the crosscut has meet $\hat{0}$ and join $\hat{1}$ if and only if it has at least two elements. Therefore, if L has n atoms we obtain the following

$$\sum_{\vee B=\hat{1}, \wedge B=\hat{0}} (-1)^{|B|} = \sum_{|B|\geq 2} (-1)^{|B|} = \sum_{k=2}^{n} (-1)^k \binom{n}{k} = n - 1.$$

This agrees with the value of $\mu(\hat{1})$ when L has n atoms and every coatom is an atom. Thus, the result holds in this special case.

Recall that if L^* is the dual lattice of L, then $\mu_L(\hat{1}) = \mu_{L^*}(\hat{1})$. Moreover, in L^*, joins and meets reverse roles. Therefore, if we have a crosscut consisting of only coatoms, then we can consider the dual lattice. As a result, we may now assume that there is always at least one coatom in the lattice which is not in the crosscut. With this in mind we proceed

by induction on $|L|$.

If $|L| \leq 3$, then it must be that $|L| = 3$ since smaller lattices do not have crosscuts. We have already done the case when $|L| = 3$. Suppose that $|L| > 3$ and let c be a coatom that is not in the crosscut. Consider the lattice $L/\sim$ where we collapse c and $\hat{1}$. Since c was not in the crosscut we still have the same crosscut. By induction, we know that

$$\mu([\hat{1}]) = \sum_{\vee B=[\hat{1}],\wedge B=[\hat{0}]} (-1)^{|B|}.$$

Lemma 3.0.3 implies that $\vee B = [\hat{1}]$ in $L/\sim$ if and only if $\vee B = c$ or $\vee B = \hat{1}$ in L. Additionally, since C does not contain c nor $\hat{1}$, Lemma 3.0.3 also implies that $\wedge B = [\hat{0}]$ in $L/\sim$ if and only if $\wedge B = \hat{0}$ in L. Therefore, we can break the previous sum as follows

$$\mu([\hat{1}]) = \sum_{\vee B=c,\wedge B=\hat{0}} (-1)^{|B|} + \sum_{\vee B=\hat{1},\wedge B=\hat{0}} (-1)^{|B|}.$$

Note that if $\vee B = c$, then B must only have elements in $[\hat{0}, c]$. Thus the first sum in the previous equation is over B contained in $[\hat{0}, c] \cap C$ such that $\vee B = c$ and $\wedge B = \hat{0}$. Since $|[\hat{0}, c]| < |L|$, induction implies that

$$\mu([\hat{1}]) = \mu(c) + \sum_{\vee B=\hat{1},\wedge B=\hat{0}} (-1)^{|B|}.$$

Subtracting $\mu(c)$ from both sides and applying Lemma 3.0.3 we see that

$$\mu(\hat{1}) = \sum_{\vee B=\hat{1},\wedge B=\hat{0}} (-1)^{|B|}$$

which completes the proof. □

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