INTEGRAL MODELS OF CERTAIN PEL SHIMURA VARIETIES WITH $\Gamma_1(p)$-TYPE LEVEL STRUCTURE

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ABSTRACT

INTEGRAL MODELS OF CERTAIN PEL SHIMURA VARIETIES WITH $\Gamma_1(p)$-TYPE LEVEL STRUCTURE

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We study p-adic integral models of certain PEL-Shimura varieties with level subgroup at p given by the pro-unipotent radical of an Iwahori. We will consider two cases: the case of Shimura varieties associated to unitary groups that split over an unramified extension of $\mathbb{Q}_p$ and the case of Siegel modular varieties. We construct local models, i.e. simpler schemes which are étale locally isomorphic to the integral models. Our integral models are defined by a moduli scheme using the notion of an Oort-Tate generator of a group scheme. We use these local models to find a resolution of the integral model in the case of the Siegel modular variety of genus 2. The resolution is regular with special fiber a nonreduced divisor with normal crossings.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Introduction

In the arithmetic study of Shimura varieties, one seeks to have a model of the Shimura variety over the ring of integers $\mathcal{O}_E$, where E is the completion of the reflex field $\mathbb{E}$ at some finite place $\mathfrak{p}$. Denote by $\mathrm{Sh}_K(\mathbb{G}, X)$ the Shimura variety given by the Shimura datum $(\mathbb{G}, X)$ and choice of an open compact subgroup $K = \prod_\ell K_\ell \subset \mathbb{G}(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite rational adèles. For Shimura varieties of PEL-type, which are moduli spaces of abelian varieties with certain (polarization, endomorphism, and level) structures, one can define such an integral model by proposing a moduli problem over $\mathcal{O}_E$. The study of such models began with modular curves by Shimura and Deligne-Rapoport. More generally, Langlands, Kottwitz, Rapoport-Zink, Chai, and others studied these models for various types of PEL Shimura varieties. The reduction modulo $\mathfrak{p}$ of these integral models is nonsingular if the factor $K_p \subset \mathbb{G}(\mathbb{Q}_p)$ is chosen to be "hyperspecial" for the rational prime p lying under $\mathfrak{p}$. However if the level subgroup K_p is not hyperspecial, usually (although not always) singularities occur. It is important to determine what kinds of singularities can occur, and this is expected to be influenced by the level subgroup K_p.

In order to study the singularities of these integral models, significant progress has been made by finding "local models". These are schemes defined in simpler terms which control the singularities of the integral model. They first appeared in [DP] for Hilbert modular varieties and in [dJ2] for Siegel modular varieties with Iwahori level subgroup. More generally in [RZ], Rapoport and Zink constructed local models for PEL Shimura varieties with parahoric level subgroup.

In [Gör1] Görtz showed that in the case of a Shimura variety of PEL-type associated with a unitary group which splits over an unramified extension of $\mathbb{Q}_p$, the Rapoport-Zink local models are flat with reduced special fiber. In [Gör2], the same is shown for the local models of Siegel modular varieties. On the other hand, Pappas has shown that these local models can fail to be flat in the case of a ramified extension [Pap2]. In [PR1], [PR2], and [PR3], Pappas and Rapoport give alternative definitions of the local models which are flat. More recently in [PZ], Pappas and Zhu have given a general group-theoretic definition of the local models which, for PEL cases, agree with Rapoport-Zink local models in the unramified case and the alternative definitions in the ramified case.

Throughout this article, K_p is assumed to be either an Iwahori subgroup of $\mathbb{G}(\mathbb{Q}_p)$ or the pro-unipotent radical of an Iwahori subgroup. There is some ambiguity in calling these $\Gamma_0(p)$-level structure and $\Gamma_1(p)$-level structure respectively; indeed one may consider more generally a parahoric subgroup. As such, we will call the former $\mathrm{Iw}_0(p)$-level structure and the latter $\mathrm{Iw}_1(p)$-level structure. In all the situations we consider, $G = \mathbb{G}_{\mathbb{Q}_p}$ extends to a reductive group over $\mathbb{Z}_p$ and one can take an Iwahori subgroup as being the inverse image of a Borel subgroup of $G(\mathbb{F}_p)$ under the reduction $G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$. We will also take $K^p = \prod_{\ell \neq p} K_\ell$ to be a sufficiently small open compact subgroup of $\mathbb{G}(\mathbb{A}_f^p)$ so that the moduli problems we consider below are represented by schemes.

In [HR] Haines and Rapoport, interested in determining the local factor of the zeta function associated with the Shimura variety, constructed affine schemes which are étale locally isomorphic to integral models of certain Shimura varieties with $\mathrm{Iw}_1(p)$-level structure. This follows the older works of Pappas [Pap1] and Harris-Taylor [HT]. Haines and Rapoport consider the case of a Shimura variety associated with a unitary group which splits locally at p given by a division algebra B defined over an imaginary quadratic extension of $\mathbb{Q}$. The cocharacter associated with the Shimura datum is assumed to be of "Drinfeld type".

In this article, we will consider $\mathrm{Iw}_1(p)$-level structure for two particular types of Shimura varieties. First the unitary case, where the division algebra B has center F, an imaginary quadratic extension of a totally real finite extension F^+ of $\mathbb{Q}$ which is unramified at p. We will make assumptions on p so that the unitary group $\mathbb{G}$ in the Shimura datum splits over an unramified extension of $\mathbb{Q}_p$ as $\mathrm{GL}_n \times \mathbb{G}_m$. The second case is that of the Siegel modular varieties where the group in the Shimura datum is $\mathbb{G} = \mathrm{GSp}_{2n}$. We will refer to this as the symplectic case.

The moduli problem defining the integral model with $\mathrm{Iw}_0(p)$-level structure is given in terms of chains of isogenies of abelian schemes with certain additional structures. We write $\mathcal{A}_0^{\mathrm{GL}}$ and $\mathcal{A}_0^{\mathrm{GSp}}$ for the scheme representing this moduli problem in the unitary and symplectic cases respectively. Then in these two cases, the moduli problem defining the integral model with $\mathrm{Iw}_1(p)$-level structure is given by also including choices of "Oort-Tate generators" for certain group schemes associated with the kernels of the isogenies (see Section A.4.2 for the notion of an Oort-Tate generator). Let $\mathcal{A}_1^{\mathrm{GL}}$ and $\mathcal{A}_1^{\mathrm{GSp}}$ denote the schemes representing these moduli problems in each case respectively.

To study the singularities of $\mathcal{A}_1^{\mathrm{GL}}$ and $\mathcal{A}_1^{\mathrm{GSp}}$ we will construct étale local models.

Definition 0.1. Let X and M be schemes. We say that M is an étale local model of X if there exists an étale cover $V \to X$ and an étale morphism $V \to M$.

In order to describe our results in the unitary case, we begin by recalling the local model of $\mathcal{A}_0^{\mathrm{GL}}$ as constructed in [RZ, Chapter 3]. In this introduction, we assume for simplicity that $F^+ = \mathbb{Q}$. The local model in the general case will be a product of such local models after an unramified base extension. As mentioned above, we also make assumptions so that $\mathbb{G}_{\mathbb{Q}_p} = \mathrm{GL}_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$. We can choose an isomorphism $B_{\overline{\mathbb{Q}}_p} \cong M_n(\overline{\mathbb{Q}}_p) \times M_n(\overline{\mathbb{Q}}_p)$ so that the minuscule cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is identified with

$$\mu(z) = \mathrm{diag}(1^{n-r}, (z^{-1})^r) \times \mathrm{diag}((z^{-1})^{n-r}, 1^r), \quad 1 \le r \le n-1,$$

which we will write concisely as $\mu = (0^{n-r}, (-1)^r)$. Then for a $\mathbb{Z}_p$-scheme S, an S-valued point of the local model $M_{\mathrm{GL}}^{\mathrm{loc}}$ of $\mathcal{A}_0^{\mathrm{GL}}$ is determined by giving a diagram

$$\begin{tikzcd}
\mathcal{O}_S^n \arrow[r, "\varphi_0"] & \mathcal{O}_S^n \arrow[r, "\varphi_1"] & \cdots \arrow[r, "\varphi_{n-2}"] & \mathcal{O}_S^n \arrow[r, "\varphi_{n-1}"] & \mathcal{O}_S^n \\
\mathcal{F}_0 \arrow[r] \arrow[u, hook] & \mathcal{F}_1 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & \mathcal{F}_{n-1} \arrow[r] \arrow[u, hook] & \mathcal{F}_n \arrow[u, hook]
\end{tikzcd}$$

where φ_i is given by the matrix $\mathrm{diag}((p^{-1})^{i+1}, 1^{n-i-1})$ with respect to the standard basis, $\mathcal{F}_i$ is an $\mathcal{O}_S$-submodule of $\mathcal{O}_S^n$, and Zariski locally on S, $\mathcal{F}_i$ is a direct summand of $\mathcal{O}_S^n$ of rank r. With $S = M_{\mathrm{GL}}^{\mathrm{loc}}$, the determinants

$$\bigwedge^{\mathrm{top}} \mathcal{F}_i \to \bigwedge^{\mathrm{top}} \mathcal{F}_{i+1} \quad \text{and} \quad \bigwedge^{\mathrm{top}} \mathcal{O}_S^n/\mathcal{F}_i \to \bigwedge^{\mathrm{top}} \mathcal{O}_S^n/\mathcal{F}_{i+1}$$

determine global sections q_i and q_i^* of the universal line bundles

$$\mathcal{Q}_i = \left(\bigwedge^{\mathrm{top}} \mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\mathrm{top}} \mathcal{F}_{i+1} \quad \text{and} \quad \mathcal{Q}_i^* = \left(\bigwedge^{\mathrm{top}} \mathcal{O}_S^n/\mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\mathrm{top}} \mathcal{O}_S^n/\mathcal{F}_{i+1}$$

respectively.

As shown in [Gör1], the special fiber of the local model can be embedded into the affine flag variety for SL_n and identified with a disjoint union of Schubert cells. Let $U \subset M^{\mathrm{loc}}_{\mathrm{GL}}$ be an affine open neighborhood of the "worst point", i.e. the unique cell which consists of a single closed point, with U sufficiently small so that each $\mathcal{Q}_i^*$ is trivial. Choosing such a trivialization, we can then identify the sections q_i^* with regular functions on U.

Theorem 0.2. *The scheme*

$$U_1 = \mathrm{Spec}_U \left(\mathcal{O}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^*) \right)$$

is an étale local model of $\mathcal{A}_1^{GL}$.

By loc. cit. we can take $U = \mathrm{Spec}(B_{\mathrm{GL}})$ where

$$B_{\mathrm{GL}} = \mathbb{Z}_p[a^i_{jk}, i = 0, \ldots, n-1,\ j = 1, \ldots, n-r,\ k = 1, \ldots, r]/I$$

and I is the ideal generated by the entries of certain matrices. In this chosen presentation, we will show that, up to a unit, $q_i^* = a^{i+1}_{n-r,r}$ for $0 \le i \le n-1$ where the upper index is taken modulo n.

For the symplectic case, the integral model $\mathcal{A}_0^{\mathrm{GSp}}$ is again given in terms of chains of isogenies of abelian schemes with certain additional structures. Our construction of the local models for $\mathcal{A}_1^{\mathrm{GSp}}$ is similar to that of the unitary case. In particular, they are explicitly defined as well.

It is also of interest to have certain resolutions of the integral model of the Shimura variety with "nice" singularities, for example one which is semi-stable or locally toroidal. This problem was considered in the case of $\mathrm{Iw}_0(p)$-level structure by Genestier [Gen], Faltings [Fal], de Jong [dJ1], and Görtz [Gör3] among others. Using the explicitly defined local model, and in particular the rather simple expression for q_i^*, we will construct a resolution of $\mathcal{A}_1^{\mathrm{GSp}}$ in the case $n = 2$. By a "nonreduced divisor with normal crossings" we mean a divisor D such that in the completion of the local ring at every closed point, D is given by $Z(f_1^{e_1} \cdots f_t^{e_t})$ where $\{f_1, \ldots, f_t\}$ are part of a regular system of parameters and the integers e_i are greater than zero.

Theorem 0.3. *Let $\mathcal{A}_1$ denote the moduli scheme for the Siegel modular variety of genus 2 with $Iw_1(p)$-level structure. There is a regular scheme $\widetilde{\mathcal{A}}_1$ with special fiber a nonreduced divisor with normal crossings that supports a birational morphism $\widetilde{\mathcal{A}}_1 \to \mathcal{A}_1$.*

Moreover, we will describe the irreducible components of $\widetilde{\mathcal{A}}_1 \otimes \mathbb{F}_p$ and how they intersect using a "dual complex", see Theorem 5.6.6 for details.

Let us outline the construction of $\widetilde{\mathcal{A}}_1$. We begin with the known semi-stable resolution $\widetilde{\mathcal{A}}_0 \to \mathcal{A}_0$ [dJ1]. This gives a modification (i.e. proper birational morphism) $\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0 \to \mathcal{A}_1$. The scheme $\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0$ is not normal. Let $\mathcal{Z}$ be the reduced closed subscheme of $\mathcal{A}_0$ whose support is the locus of closed points where all of the corresponding group schemes are infinitesimal. Take the strict transform of $\mathcal{Z}$ with respect to the morphism $\widetilde{\mathcal{A}}_0 \to \mathcal{A}_0$ followed by the reduced inverse image of this with respect to the projection $\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0 \to \widetilde{\mathcal{A}}_0$ and denote the resulting scheme by $\mathcal{Z}'$. Consider the modification given by the blowup of $\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0$ along $\mathcal{Z}'$:

$$\mathrm{Bl}_{\mathcal{Z}'}(\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0) \to \mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0.$$

We will see that $\mathrm{Bl}_{\mathcal{Z}'}(\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0)$ is normal. In $\mathcal{A}_0$, denote by $\mathcal{W}$ the unique irreducible component of the special fiber where each corresponding group scheme is generically isomorphic to μ_p. Transform $\mathcal{W}$ via the morphisms

$$\mathrm{Bl}_{\mathcal{Z}'}(\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0) \to \mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0 \to \widetilde{\mathcal{A}}_0 \to \mathcal{A}_0$$

by taking the strict transform with respect to the first and third morphisms, and the reduced inverse image with respect to the second morphism. Denote the resulting subscheme of $\mathrm{Bl}_{\mathcal{Z}'}(\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0)$ by $\mathcal{W}'$. We arrive at $\widetilde{\mathcal{A}}_1$ by first blowing up $\mathrm{Bl}_{\mathcal{Z}}(\mathcal{A}_1 \times_{\mathcal{A}_0} \widetilde{\mathcal{A}}_0)$ along $\mathcal{W}'$ and then blowing up each resulting modification along the strict transform of $\mathcal{W}'$, stopping after a total of $p-2$ blowups. Carrying out the corresponding process on the local model, by explicit computation we will show that the resulting resolution of the local model is regular with special fiber a nonreduced divisor with normal crossings. It will then follow that $\widetilde{\mathcal{A}}_1$ has these properties as well. By keeping track of how certain subschemes transform in each step of the above process, with much of this information coming from the explicit computation of the modifications of the local model, we will be able to describe certain aspects of the irreducible components of $\widetilde{\mathcal{A}}_1 \otimes \mathbb{F}_p$ as mentioned above.

In closing we mention that as this article was prepared, T. Haines and B. Stroh announced a similar construction of local models in order to prove the analogue of the Kottwitz nearby cycles conjecture. They relate their local models to "enhanced" affine flag varieties.

Finally, I would like to thank G. Pappas for introducing me to this area of mathematics and for his invaluable support. I would also like to thank M. Rapoport for a useful conversation, T. Haines and B. Stroh for communicating their results, and U. Görtz for providing the source for Figure 5.1.4 to which some modifications were made.

Chapter 1

Shimura Varieties

In this section we review the definition of a Shimura datum, the group theoretic data which is used to construct a Shimura variety. We specialize this data in the unitary and symplectic cases.

1.1 Shimura datum

Let $\mathbb{S}$ denote $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, where $\mathrm{Res}(\cdot)$ is the Weil restriction of scalars. Note that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ and we have the homomorphism

$$\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}) \quad \text{sending} \quad z \to (z, \overline{z}).$$

For a connected algebraic group H defined over $\mathbb{R}$, a Cartan involution of H is an involution θ of H as an algebraic group over $\mathbb{R}$ such that

$$H^{(\theta)}(\mathbb{R}) := \{g \in H(\mathbb{C}) : g = \theta(\bar{g})\}$$

is compact, where $\bar{g}$ denotes complex conjugation. The following definition uses the language introduced by Deligne in [Del].

Definition 1.1.1. A Shimura datum is a collection $(\mathbb{G}, \{h\}, K)$ where

- $\mathbb{G}$ is a reductive group defined over $\mathbb{Q}$;
- $\{h\}$ a $\mathbb{G}(\mathbb{R})$-conjugacy class of homomorphisms of real algebraic groups $\mathbb{S} \to \mathbb{G}_{\mathbb{R}}$; and
- K is a sufficiently small compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$

such that the following conditions hold.

(SV1) For any $h : \mathbb{S} \to \mathbb{G}_{\mathbb{R}}$, only the characters 1, $z/\bar{z}$, and $\bar{z}/z$ occur in the induced representation of $\mathbb{S}$ on $\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}})_{\mathbb{C}}$.

(SV2) The adjoint action of $h(i)$ induces a Cartan involution on the adjoint group of $\mathbb{G}_{\mathbb{R}}$.

(SV3) The adjoint group $\mathbb{G}^{\mathrm{ad}}_{\mathbb{R}}$ does not admit a factor H defined over $\mathbb{Q}$ such that the projection of h on H is trivial.

Remark 1.1.2.

- For $h \in \{h\}$, $h : \mathbb{S} \to \mathbb{G}_{\mathbb{R}}$, the action of $g \in \mathbb{G}(\mathbb{R})$ is given as follows. For an $\mathbb{R}$-algebra A, we define the homomorphism

$$(g \cdot h)(A) : \mathbb{S}(A) \to \mathbb{G}_{\mathbb{R}}(A) \qquad \text{sending} \qquad \alpha \to g \cdot h(A)(\alpha) \cdot g^{-1}$$

where we are identifying $g \in \mathbb{G}(\mathbb{R})$ with its image under $\mathbb{G}(\mathbb{R}) \to \mathbb{G}(A)$.

- Condition (SV1) means the following. Given a homomorphism $h : \mathbb{S} \to \mathbb{G}_{\mathbb{R}}$, we can compose this with the adjoint representation $\mathbb{G}_{\mathbb{R}} \to \mathrm{GL}(\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}}_{\mathbb{R}}))$ and then complexify so that we have $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}) \to \mathrm{GL}(\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}})_{\mathbb{C}})$ where the first homomorphism is as described above. Thus for $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ we have a natural action on $\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}})_{\mathbb{C}}$. The condition is that

$$\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}})_{\mathbb{C}} = V^0 \oplus V^1 \oplus V^{-1}$$

with

$$\begin{aligned} V^0 &:= \left\{ v \in V_{\mathbb{C}} : z \cdot v = v \text{ for all } z \in \mathbb{C}^\times \right\} \\ V^1 &:= \left\{ v \in V_{\mathbb{C}} : z \cdot v = z\overline{z}^{-1} v \text{ for all } z \in \mathbb{C}^\times \right\} \\ V^{-1} &:= \left\{ v \in V_{\mathbb{C}} : z \cdot v = \overline{z} z^{-1} v \text{ for all } z \in \mathbb{C}^\times \right\} \end{aligned}$$

where on the right hand side the product is given by the natural action of $\mathbb{C}$ on the complex vector space $\mathrm{Lie}(\mathbb{G}^{\mathrm{ad}})_{\mathbb{C}}$.

- The condition that K is sufficiently small will be explained in Section 2.4.

1.2 PEL Shimura varieties

We now specialize to the case of PEL Shimura varieties. Fix once and for all a choice $i = \sqrt{-1}$.

Definition 1.2.1. A PEL Shimura datum is given by a tuple $(B, \iota, V, (\cdot, \cdot), h_0, K)$ satisfying the following conditions.

- B is a finite-dimensional semi-simple $\mathbb{Q}$-algebra with positive involution ι.
- $V \neq 0$ is a finitely-generated left B-module.
- $(\cdot,\cdot)$ is a non-degenerate alternating form $V \times V \to \mathbb{Q}$ such that $(bv, w) = (v, b^\iota w)$ for all $b \in B$ and $v, w \in V$.
- h_0 is given as follows. The form $(\cdot,\cdot)$ determines an involution $*$ on $\mathrm{End}_B(V)$ where for $f \in \mathrm{End}_B(V)$, $f^* \in \mathrm{End}_B(V)$ is the unique element such that

$$(f(x), y) = (x, f^*(y)) \text{ for all } x, y \in V.$$

In particular, for $b \in B$ we have $b^* = b^\iota$. Here were are making the identification $b \in \mathrm{End}_B(V)$ by left multiplication. We require that $h_0 : \mathbb{C} \to \mathrm{End}_{B\otimes\mathbb{R}}(V \otimes \mathbb{R})$ be an $\mathbb{R}$-algebra homomorphism satisfying $h_0(\overline{z}) = h_0(z)^*$ for all $z \in \mathbb{C}$ and is such that the symmetric bilinear form

$$(\cdot, h_0(i)\cdot) : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$$

is positive definite.

- We define the $\mathbb{Q}$-group $\mathbb{G}$ on a $\mathbb{Q}$-algebra R by

$$\mathbb{G}(R) = \left\{ g \in \mathrm{GL}_{B\otimes R}(V \otimes R) : g^* g \in R^\times \right\}.$$

We require K be a sufficiently small compact open subgroup of $\mathbb{G}(\mathbb{A}_f^p)$.

Remark 1.2.2. In the above definition, given a PEL Shimura datum $(B, \iota, V, (\cdot,\cdot), h_0, \mathbb{K})$ we defined the involution $*$ on $\mathrm{End}_B(V)$ and the algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$. We will also define the objects h, μ_h, $\mathbb{E}$, and $\mathbb{E}'$ below. Henceforth we will implicitly associate these objects with a PEL Shimura datum.

We define h to be the homomorphism of real algebraic groups

$$h : \mathbb{C}^\times \xrightarrow{z \to z^{-1}} \mathbb{C}^\times \xrightarrow{h_0} \mathbb{G}(\mathbb{R})$$

and the cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_\mathbb{C}$ as follows. By base change $h_\mathbb{C} : \mathbb{S}_\mathbb{C} \to \mathbb{G}_\mathbb{C}$ where for any $\mathbb{C}$-algebra R we have

$$\mathbb{S}_\mathbb{C}(R) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)(R) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C} \otimes_\mathbb{R} R) \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}}(R) \times \mathbb{G}_{m,\mathbb{C}}(R).$$

The isomorphism above is induced by $\mathbb{C} \otimes_\mathbb{R} R \xrightarrow{\sim} R \times R$ sending $z \otimes r \to (zr, \overline{z}r)$. We thus have $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ and we define μ_h by restricting the map $h_\mathbb{C} : \mathbb{S}_\mathbb{C} \to \mathbb{G}_\mathbb{C}$ to the factor of $\mathbb{S}_\mathbb{C}$ corresponding to the identity (as opposed to complex conjugation).

Define the field $\mathbb{E}$ to be the field of definition of the $\mathbb{G}(\mathbb{R})$-conjugacy class $\{\mu_h\}$. We will postpone the definition of $\mathbb{E}'$ until after the proof of Proposition 1.2.4.

Remark 1.2.3. Let k be an algebraically closed field. For B a semisimple k-algebra with involution ι, (B, ι) is isomorphic to a product of the following three types [Kot2, Section 1].

(A) $\quad M_n(k) \times M_n(k), \quad (a,b)^* = (b^t, a^t)$

(C) $\quad M_n(k), \quad b^* = b^t$

(BD) $\quad M_n(k), \quad b^* = Jb^tJ^{-1}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

Now suppose B is a semisimple $\mathbb{Q}$-algebra with involution ι and center a field F. Set $F_0 = \{x \in F : x^* = x\}$. Then for all $\mathbb{Q}$-homomorphisms $\rho : F_0 \to \overline{\mathbb{Q}}$, the $\overline{\mathbb{Q}}$-algebra with involution $(B \otimes_{F_0,\rho} \overline{\mathbb{Q}}, \iota)$ is a product of types (A), (C), or (BD). We will refer to this multiset of types as the type of $(B \otimes_{F_0,\rho} \overline{\mathbb{Q}}, \iota)$. Since $\mathrm{Gal}(F_0/\mathbb{Q})$ acts transitively on the collection $(B \otimes_{F_0,\rho} \overline{\mathbb{Q}}, \iota)_\rho$ by isomorphisms, we define the type of (B, ι) to be the type of any extension $(B \otimes_{F_0,\rho} \overline{\mathbb{Q}}, \iota)$.

From here on we will assume that (B, ι) is a product of types (A) or (C).

Proposition 1.2.4. *Given a PEL Shimura datum* $(B, \iota, V, (\cdot,\cdot), h_0, K)$*, the induced* $(\mathbb{G}, h, K)$ *(see Remark 1.2.2) is a Shimura datum.*

Proof. To see that $\mathbb{G}$ is reductive, we consider $\mathbb{G}_{\bar{\mathbb{Q}}}$. Then (B, ι) decomposes into a product where each factor is of type (A) or (C). Hence $\mathbb{G}_{\bar{\mathbb{Q}}}$ decomposes into a product of reductive groups, each being $\mathrm{GL}_n \times \mathbb{G}_m$ or GSp_n depending on whether the corresponding factor of (B, ι) is of type (A) or (C).

(SV1) Set $J = h(i) \in \mathbb{G}(\mathbb{R}) \subset \mathrm{GL}_{B\otimes R}(V \otimes R)$. The action of J makes $V \otimes_{\mathbb{Q}} \mathbb{R}$ into a complex vector space with complex structure $h : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$. Now consider the the action of $\mathbb{C}^\times$ on the Lie algebra of $\mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ through h and the adjoint action. Note that $V_{\mathbb{C}} = V^+ \oplus V^-$ where

$$V^+ = \{v \in V_{\mathbb{C}} : Jv = iv\}$$
$$V^- = \{v \in V_{\mathbb{C}} : Jv = -iv\}$$

which induces the decomposition

$$\mathrm{Hom}(V_{\mathbb{C}}, V_{\mathbb{C}}) = \mathrm{Hom}(V^+, V^+) \oplus \mathrm{Hom}(V^+, V^-) \oplus \mathrm{Hom}(V^-, V^+) \oplus \mathrm{Hom}(V^-, V^-).$$

The adjoint action of $h(z)$ on $\mathrm{Hom}(V_{\mathbb{C}}, V_{\mathbb{C}})$ is by conjugation, i.e. for $g : V_{\mathbb{C}} \to V_{\mathbb{C}}$ and

$z = a + bi \in \mathbb{C}^\times$, we have that $h(a+bi) \cdot g = (a+bJ) \circ g \circ (a+bJ)^{-1}$. Therefore in the decomposition above, $h(z)$ acts as 1 on $\mathrm{Hom}(V^+, V^+)$, $z\overline{z}^{-1}$ on $\mathrm{Hom}(V^+, V^-)$, $z^{-1}\overline{z}$ on $\mathrm{Hom}(V^-, V^+)$, and 1 on $\mathrm{Hom}(V^-, V^-)$ as required.

(SV2) We must show that the group

$$(\mathbb{G}^{\mathrm{ad}})^{(h(i))}(\mathbb{R}) := \left\{ g \in \mathbb{G}^{\mathrm{ad}}(\mathbb{C}) : gg^* = 1,\ h(i)^{-1}\overline{g}h(i) = g \right\}$$

is compact, where $\overline{g}$ denotes complex conjugation. For $g \in \mathbb{G}^{\mathrm{ad}}(\mathbb{C})$, denote by $g \to g'$ the involution given by the tensor product of $*$ on $\mathrm{End}_B(V)$ and complex conjugation on $\mathbb{C}$, i.e. for $\alpha \otimes \lambda \in \mathrm{End}_{B\otimes\mathbb{C}}(V \otimes \mathbb{C})$ we have

$$(\alpha \otimes \lambda)' = \alpha^* \otimes \overline{\lambda}.$$

As $*$ and complex conjugation are both positive involutions, it follows that $g \to g'$ is also positive [Kot2, Lemma 2.3]. Note that $(\mathbb{G}^{\mathrm{ad}})^{(h(i))}(\mathbb{R})$ is a closed subgroup of $\left\{ g \in \mathbb{G}^{\mathrm{ad}}(\mathbb{C}) : gg' = 1 \right\}$, namely $(\mathbb{G}^{\mathrm{ad}})^{(h(i))}(\mathbb{R})$ is given by enforcing the condition $gg^* = 1$. Since $g \to g'$ is positive, by [Kot2, Lemma 2.2] there is a faithful positive definite Hermitian $\mathbb{G}^{\mathrm{ad}}(\mathbb{C})$-module W. Denote its Hermitian form by $(\cdot,\cdot)_W$. Now the transpose defined by $(\cdot,\cdot)_W$ is precisely $g \to g'$ and thus $(\mathbb{G}^{\mathrm{ad}})^{(h(i))}(\mathbb{R})$ can be viewed as a closed subgroup of the orthogonal group with respect to $(\cdot,\cdot)_W$. As this orthogonal group is compact, it therefore follows that $(\mathbb{G}^{\mathrm{ad}})^{(h(i))}(\mathbb{R})$ is compact.

(SV3) If $\mathbb{G}^{\mathrm{ad}}_{\mathbb{R}}$ has a $\mathbb{Q}$-factor on which h is trivial, then the form $(\cdot, h_0(i)\cdot)$ on $V_{\mathbb{R}} \times V_{\mathbb{R}}$ could not be positive definite since $(\cdot,\cdot)$ is alternating.

$\square$

Remark 1.2.5. From (SV1), we get that μ_h induces a decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$ where

μ_h acts as z^{-1} on V^+ and 1 on V^-. We define $\mathbb{E}'$ to be the finite extension of $\mathbb{E}$ over which this decomposition is defined. Fix an isomorphism $B_{\mathbb{C}} \cong M_n(\mathbb{C}) \times M_n(\mathbb{C})$ so that $\mu_h(z)$ is identified with

$$\mathrm{diag}(1^n - r, (z^{-1})^r) \times \mathrm{diag}(z^{-1})^{n-r}, 1^r).$$

We will write this as $\mu_h = (0^{n-r}, (-1)^r)$.

1.2.1 Unitary case

We now give a specialized set of data for which the group $\mathbb{G}$ in the induced Shimura datum is a unitary group.

Definition 1.2.6. A unitary PEL Shimura datum is a tuple $(D, *, h_0)$ where

- D is a finite dimensional division algebra with center a field F, where F is an imaginary quadratic extension of some totally real field $F^+/\mathbb{Q}$;
- $*$ is an involution of D which induces on F the nontrivial element of $\mathrm{Gal}(F/F^+)$; and
- $h_0 : \mathbb{C} \to D \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism such that $h_0(z)^* = h_0(\overline{z})$ and the involution $x \to h_0(i)^{-1} x^* h_0(i)$ is positive.

A datum $(D, *, h_0)$ induces a PEL datum $(B, \iota, V, (\cdot, \cdot), h_0, K)$, up to a choice of K, as follows. Set $B = D^{\mathrm{opp}}$ and $V = D$ where we view V as a left B-module using right multiplications. That is, for $v \in D$ and $b \in D^{\mathrm{opp}}$ we define $b \cdot v = vb$, where on the right hand side the multiplication is given by the multiplication in D. Then $\mathrm{End}_B(V)$ can be identified with D using left multiplications. It remains to define the involution ι on $B = D^{\mathrm{opp}}$ and the pairing $(\cdot, \cdot)$ on $V = D$. We will use the following lemma.

Lemma 1.2.7. *There exists $\xi \in D^\times$ such that $\xi^* = -\xi$ and the involution $x \to \xi x^* \xi^{-1}$ is positive. With such a ξ, the alternating pairing $(\cdot,\cdot) : D \times D \to \mathbb{Q}$ defined by*

$$(x, y) = \mathrm{Tr}_{D/\mathbb{Q}}(x\xi y^*)$$

is nondegenerate. We may also choose ξ, still subject to the above conditions, such that the pairing $(\cdot, h_0(i)\cdot)$ is positive definite.

Proof. Let σ be any involution of D of the second kind, meaning σ restricts to the nontrivial element of $\mathrm{Gal}(F/F^+)$. Then $* \circ \sigma$ fixes F. Thus we may apply Skolem-Noether to the F-algebra homomorphisms $* \circ \sigma : D \to D$ and $\mathrm{Id}_D : D \to D$. Hence there is a unit u such that $(* \circ \sigma)(d) = u^{-1}du$ for all $d \in D$. Applying the involution $*$ to both sides gives $\sigma(d) = u^* d^* (u^{-1})^*$ for all $d \in D$. As u^* is also a unit of D, we will replace u with u^* and write this as $\sigma(d) = ud^*u^{-1}$. Then for all $d \in D$,

$$\sigma(\sigma(d)) = u(ud^*u^{-1})^*u^{-1} = u(u^{-1})^*du^*u^{-1}.$$

The condition that σ is an involution implies that for all $d \in D$, $u(u^{-1})^*du^*u^{-1} = d$. This condition is satisfied if and only if $u(u^*)^{-1}$ lies in the center F. Conversely, for any unit u such that $u(u^*)^{-1} \in F$, we have that $d \to ud^*u^{-1}$ is an involution of the second kind.

By [Mum, pg. 201-2], positive involutions of the second kind exist. So there is a $u \in D$ such that $d \to ud^*u^{-1}$ is a positive involution. Then

$$u(u^*)^{-1} \cdot (u(u^*)^{-1})^* = u(u^*)^{-1} \cdot (u^{-1}u^*) = uu^{-1}u^*(u^*)^{-1} = 1$$

where the second to last equality is using that $u(u^*)^{-1}$ is in the center of D. Since $*$ restricted to F is the nontrivial element in $\mathrm{Gal}(F/F^+)$, this says precisely that $N_{F/F^+}(u(u^*)^{-1}) = 1$ where N_{F/F^+} is the norm of F over F^+. By Hilbert's Theorem 90 [Hil, Theorem 90], there exists $f \in F^\times$ such that $u(u^*)^{-1} = f^* f^{-1}$. Using this equation we have $(uf)^* = f^* u^* = uf$. Finally, there exists $\varepsilon \in F^\times$ such that $\varepsilon^* = -\varepsilon$ (take any nonzero element of $F \setminus F^+$ and complete the square). We (temporarily, see below) set $\xi = \varepsilon f u$, and denote the involution $d \to \xi d^* \xi^{-1}$ by ι. Note that since εf is in the center of D, this is the same involution as $d \to u d^* u^{-1}$ and hence is positive and of the second kind.

Since $\mathrm{Tr}_{D/\mathbb{Q}}$ is invariant with respect to $*$ [Kot2, Lemma 2.7], we have that for all $x, y \in D$

$$(x, y) = \mathrm{Tr}_{D/\mathbb{Q}}(x\xi y^*) = \mathrm{Tr}_{D/\mathbb{Q}}((x\xi y^*)^*) = -\mathrm{Tr}_{D/\mathbb{Q}}(y\xi x^*) = -(y, x)$$

so the claimed pairing is indeed alternating. It is also non-degenerate because the pairing $(x, y) \to \mathrm{Tr}_{D/\mathbb{Q}}(xy)$ is non-degenerate, $*$ is bijective, and ξ is a unit.

We claim that $(\cdot, h_0(i)\cdot)$ is either positive or negative definite. To see this, fix an isomorphism

$$D \otimes_{F_0} \mathbb{R} \xrightarrow{\sim} M_n(\mathbb{C})$$

such that the involution ι goes over to the standard involution $X \to \overline{X}^t$ on $M_n(\mathbb{C})$. Denote by H the image of $\xi h_0(i)^{-1}$ under this isomorphism. Since $\xi h_0(i)^{-1}$ is invertible in D, H is invertible in $M_n(\mathbb{C})$. Furthermore

$$\iota(\xi h_0(i)^{-1}) = \xi(\xi h_0(i)^{-1})^* \xi^{-1} = \xi h_0(i)(-\xi)\xi^{-1} = -\xi h_0(i) = \xi h_0(i)^{-1}$$

and thus $\overline{H}^t = H$, i.e. H is Hermitian. Unwinding the definition of the pairing $(\cdot, h_0(i)\cdot)$ we have for all $x, y \in D_{\mathbb{R}}$

$$(x, h_0(i)y) = \mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}}(x\xi y^* h_0(i)^{-1}) = \mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}}(x(\xi y^* \xi^{-1})\xi h_0(i)^{-1}) = \mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}}(x\iota(y)\xi h_0(i)^{-1})$$

and hence under the fixed isomorphism this pairing becomes

$$\langle X, Y \rangle = \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(X\overline{Y}^t H) \qquad \text{for } X, Y \in M_n(\mathbb{C}).$$

Let $U \in M_n(\mathbb{C})$ be a unitary matrix such that $U^{-1}HU = D$ where $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ is some diagonal matrix with $\lambda_i \in \mathbb{C}$. In fact, since H is Hermitian and hence has real eigenvalues, we have $\lambda_i \in \mathbb{R}$ for all i. Then since the involution $x \to h_0(i)^{-1}x^* h_0(i)$ is positive by hypothesis, we have

$$\mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}}(xh_0(i)^{-1}x^* h_0(i)) > 0 \quad \text{for all } 0 \neq x \in D_{\mathbb{R}}.$$

Now

$$xh_0(i)^{-1}x^* h_0(i) = xh_0(i)^{-1}\xi^{-1}(\xi x^* \xi^{-1})\xi h_0(i) = x(\xi h_0(i)^{-1})^{-1}\iota(x)(\xi h_0(i)^{-1})$$

where the last equality is using that $-h_0(i) = h_0(i)^{-1}$. Thus under the fixed isomorphism

$$\mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(XH^{-1}\overline{X}^t H) > 0 \quad \text{for all } 0 \neq X \in M_n(\mathbb{C}).$$

As U is unitary,

$$\begin{aligned}
\mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(XH^{-1}\overline{X}^t H) &= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}((UXU^{-1})H^{-1}(U\overline{X}^t U^{-1})H) \\
&= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(X(U^{-1}H^{-1}U)\overline{X}^t(U^{-1}HU)) \\
&= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(XD^{-1}\overline{X}^t D) \\
&> 0.
\end{aligned}$$

Finally, we calculate

$$\mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(XD^{-1}\overline{X}^t D) = 2\sum_{i,j}^{n} |x_{ij}|^2 \frac{\lambda_j}{\lambda_i} \quad \text{where } X = (x_{ij}).$$

Since this last quantity must be positive for any $X \in M_n(\mathbb{C})$, it must be that every $\lambda_i \in \mathbb{R}$ has the same sign.

We now show that $\langle \cdot, \cdot \rangle$ is either positive or negative definite.

$$\begin{aligned}
\langle X, X \rangle &= \langle UX, UX \rangle \\
&= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(UX\overline{X}^t U^{-1}H) \\
&= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(X\overline{X}^t U^{-1}HU) \\
&= \mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(X\overline{X}^t D)
\end{aligned}$$

Letting $X = (x_{ij})$, one can calculate that

$$\mathrm{Tr}_{M_n(\mathbb{C})/\mathbb{R}}(X\overline{X}^t D) = 2\sum_{i,j=1}^{n} x_{ij}\overline{x_{ij}}\lambda_r = 2\sum_{i,j=1}^{n} |x_{ij}|^2 \lambda_r,$$

and with all λ_i's possessing the same sign, the claim follows. Therefore, by possibly replacing ξ with $-\xi$, we have that $\langle \cdot, \cdot \rangle$ and hence $(\cdot, h_0(i)\cdot)$ is positive definite. $\square$

We take the involution ι and the pairing $(\cdot,\cdot)$ defined by ξ as in the lemma.

Proposition 1.2.8. *Let $(D, *, h_0)$ induce $(B, \iota, V, (\cdot,\cdot), h_0)$ as described above. This datum satisfies all the conditions of being a PEL datum, up to a choice of K.*

Proof. All claims have already been shown except that $(\cdot,\cdot)$ is a Hermitian form. It remains to see that $(bx, y) = (x, b^\iota y)$ for all $x, y \in D$ and $b \in D^{\mathrm{opp}}$. Regarding b as an element of D, we need to show $(xb, y) = (x, yb^\iota)$. Recall that ξ is chosen so that $\xi^* = -\xi$.

$$\begin{aligned}
(x, yb^\iota) &= (x, y\xi b^* \xi^{-1}) \\
&= \mathrm{Tr}_{D/\mathbb{Q}}(x\xi(y\xi b^*\xi^{-1})^*) \\
&= \mathrm{Tr}_{D/\mathbb{Q}}(x\xi(-\xi^{-1})b(-\xi)y^*) \\
&= \mathrm{Tr}_{D/\mathbb{Q}}(xb\xi y^*) \\
&= (xb, y)
\end{aligned}$$

□

1.2.2 Symplectic case

In this section we will describe the PEL datum $(B, \iota, V, (\cdot,\cdot), h_0, K)$ for the Siegel modular varieties. The data here is given by

- $B = \mathbb{Q}$;
- ι is the trivial involution on $\mathbb{Q}$;
- $V = \mathbb{Q}^{2n}$;

- $(\cdot,\cdot)$ is the alternating pairing on V given by the $2n \times 2n$ matrix

$$J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \qquad \text{where} \qquad J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

- $h_0 : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ is the unique $\mathbb{R}$-algebra homomorphism with $h_0(i) = J$; and
- K is a sufficiently small compact open subgroup of $\mathbb{G}$.

Proposition 1.2.9. *The datum $(B, \iota, V, (\cdot,\cdot), h_0, K)$ described above is a PEL datum.*

Proof. B is a finite semi-simple $\mathbb{Q}$-algebra and ι, being trivial, is positive since $\alpha^2 > 0$ for $\alpha \in \mathbb{R}$. The pairing $(\cdot,\cdot)$ induced by J is certainly non-degenerate and alternating, and the equality $(bv, w) = (v, b^\iota w)$ for $v, w \in V$ and $b \in B$ is an immediate consequence of $(\cdot,\cdot)$ being bilinear.

It remains to show that $h_0(\overline{z}) = h_0(z)^*$, where $*$ is the involution on $\mathrm{End}_B(V)$ induced by $(\cdot,\cdot)$, and $(\cdot, J\cdot)$ is positive definite. Note that $h_0(a+bi) = a+bJ$ and $h_0(\overline{a+bi}) = a-bJ$, where on the right hand side we are writing a and b as the linear map given as scalar multiplication by a and b respectively. The involution $*$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$$

where ${}^tA$ denotes the transpose of A along the anti-diagonal. Thus it follows that $(a+bJ)^* = (a-bJ)$. One can compute the pairing $(\cdot, J\cdot)$ in standard coordinates and see that it is positive definite. $\square$

Remark 1.2.10. From the definition of h_0, we have $\mu = (0^n, (-1)^n)$.

Chapter 2

Integral and local models of $\mathcal{A}_0$

In this chapter we describe the integral and local models for PEL Shimura varieties where the level subgroup at an odd rational prime p is given by a parahoric subgroup. We then specialize the description of the integral and local models to the unitary and symplectic cases where the level subgroup at p is given by an Iwahori subgroup. Finally we prove the representability of the moduli problems defining the integral models in these two special cases.

2.1 PEL case

In order to define the integral model, we first need to specify additional integral data.

Definition 2.1.1. Fix an odd rational prime p. An integral PEL Shimura datum is a tuple

$$(B, \iota, V, (\cdot,\cdot), h_0, \mathcal{O}_B, \mathcal{L}, K^p)$$

where

- $(B, \iota, V, (\cdot,\cdot), h_0, K)$ is a PEL Shimura datum with $K = K_p K^p$ (see below);
- $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$-order in B whose p-adic completion $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$ that is stable under ι;
- $\mathcal{L}$ is a self-dual multichain of $\mathcal{O}_B \otimes \mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$, where duality is with respect to the pairing induced by $(\cdot,\cdot)$;
- $K_p = \mathrm{Aut}(\mathcal{L}) \subset \mathbb{G}(\mathbb{Q}_p)$; and
- $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ is an open compact subgroup.

Furthermore, if $K_p \subset \mathbb{G}(\mathbb{Q}_p)$ is an Iwahori subgroup, we say that the integral datum is of Iwahori-type.

Remark 2.1.2. We will not recall the definition of a self-dual multichain of $\mathcal{O}_B \otimes \mathbb{Z}_p$-lattices, see [RZ, Defintions 3.1, 3.4, 3.13]. However in the two cases we consider, we will make the definition of $\mathcal{L}$ explicit.

For the remainder of this section we fix an odd rational prime p and an integral PEL Shimura datum. Recall that associated with a PEL Shimura datum is the reflex field $\mathbb{E}$ described in Remark 1.2.2. Fix once and for all embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $\mathfrak{p}$ denote the corresponding prime of $\mathcal{O}_{\mathbb{E}}$ lying over p. Set $E = \mathbb{E}_{\mathfrak{p}}$, $E' = \mathbb{E}'_{\mathfrak{p}}$, and $G = \mathbb{G}_{\mathbb{Q}_p}$. Using these fixed embeddings, we have that the conjugacy class $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{\mathbb{C}}$ induces, by abuse of notation, the conjugacy class $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$.

Definition 2.1.3. The moduli functor $\mathcal{A}_0$ is defined over $\mathrm{Spec}(\mathcal{O}_E)$ as follows. For an $\mathcal{O}_E$-scheme S, the set $\mathcal{A}_0(S)$ is given by the collection of tuples $(\{A_\Lambda\}, i, \bar{\lambda}, \bar{\eta})$ up to isomorphism where

- $\{A_\Lambda\}_{\Lambda\in\mathcal{L}}$ is an $\mathcal{L}$-set of abelian schemes with an action of $\mathcal{O}_B$

$$i : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)};$$

- $\overline{\lambda}$ is a $\mathbb{Q}$-homogeneous principal polarization of the $\mathcal{L}$-set A; and
- $\overline{\eta} : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A, \mathbb{A}_f^p) \bmod K^p$ is a K^p-level structure that respects the bilinear forms on both sides up to a constant in $(\mathbb{A}_f^p)^\times$ (see below)

such that the determinant condition of Kottwitz holds: for $b \in \mathcal{O}_B$ and $\Lambda \in \mathcal{L}$ we have

$$\det_{\mathcal{O}_S}(b|\mathrm{Lie}(A_\Lambda)) = \det_{E'}(b|V^+) \qquad \text{(see below).}$$

An isomorphism between two S-valued points $(\{A_\Lambda\}, i, \bar{\lambda}, \bar{\eta})$ and $(\{A'_\Lambda\}, i', \bar{\lambda}', \bar{\eta}')$ is an isomorphism of the $\mathcal{L}$-sets $\{A_\Lambda\} \xrightarrow{\sim} \{A'_\Lambda\}$ which carries $\bar{\lambda}$, i, and $\bar{\eta}$ to $\bar{\lambda}'$, i', and $\bar{\eta}'$ respectively.

Remark 2.1.4. We will not recall the definition of an $\mathcal{L}$-set of abelian schemes or a $\mathbb{Q}$-homogeneous principal polarization, see [RZ, Definitions 6.5,6.7]. We will make these notions explicit in the two cases we consider (Definitions 2.2.1 and 2.3.1).

K^p-level structures

Let (A, λ, i) be a polarized abelian scheme over S with $\mathcal{O}_B$-action by i as in the definition above. Fix a geometric point s of S and consider an isomorphism

$$\eta : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p).$$

The pairing $(\cdot,\cdot)$ on V induces a pairing on $V \otimes \mathbb{A}_f^p$, we again denote this by $(\cdot,\cdot)$. The polarization λ induces the pairing

$$\langle\cdot,\cdot\rangle : H_1(A_s, \mathbb{A}_f^p) \times H_1(A_s, \mathbb{A}_f^p) \to \mathbb{A}_f^p(1).$$

Noncanonically, $\mathbb{A}_f^p(1) \cong \mathbb{A}_f^p$, where the isomorphism is well-defined up to some scalar multiple in $(\hat{\mathbb{Z}}^{(p)})^\times$. We say that η respects the pairings if there exists $c_\eta \in (\mathbb{A}_f^p)^\times$ such that for all $x, y \in V \otimes \mathbb{A}_f^p$ we have

$$(x, y) = c_\eta \langle \eta(x), \eta(y) \rangle.$$

Now consider the $\mathcal{O}_B$-action on A given by i. This induces an $\mathcal{O}_B$-action on $H_1(A, \mathbb{A}_f^p)$ which we again denote by i. Then we say that η respects the $\mathcal{O}_B$-action if for all $b \in \mathcal{O}_B$ and $x \in V \otimes \mathbb{A}_f^p$ we have

$$i(b) \cdot \eta(x) = \eta(b \cdot x).$$

Proposition 2.1.5. *If $\eta : V \otimes \mathbb{A}_f^p \to H_1(A, \mathbb{A}_f^p)$ respects the pairings and $\mathcal{O}_B$-action, then so does $\eta \circ g$ for $g \in G(\mathbb{A}_f^p)$.*

Proof. This follows immediately from the definition of $G(\mathbb{A}_f^p)$ consisting of elements $g \in \mathrm{GL}_{B \otimes \mathbb{A}_f^p}(V \otimes \mathbb{A}_f^p)$ such that there exists $c \in (\mathbb{A}_f^p)^\times$ with $(gx, gy) = c \cdot (x, y)$ for all $x, y \in V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. $\square$

Definition 2.1.6. Let S be an $\mathcal{O}_E$-scheme and (A, λ, i) be a principally polarized abelian scheme over S with $\mathcal{O}_B$-action by i. Then a K^p-level structure on (A, λ, i) is a choice of geometric point s in S for each connected component of S and a K^p-orbit $\overline{\eta}$ of isomorphisms $\eta : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p)$ respecting the pairings and $\mathcal{O}_B$-action such that the orbit is fixed under the action of $\pi_1(S, s)$.

Here is what is meant by the π_1-action in the definition above. For this, we suppose S is connected and choose a geometric point s of S. View $A[\ell^n]$ as a locally constant constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf on the étale site $S_{ét}$. That is, $A[\ell^n](U \xrightarrow{\text{étale}} S) = \mathrm{Hom}_S(U, A[\ell^n])$. Denote by F_s the functor from finite étale covers of S to sets given by

$$F_s(T \xrightarrow{f} S) = \{\text{geometric points } t \text{ of } T : f(t) = s\}$$

with the monodromy action of $\pi_1(S, s)$ on $F_s(T \to S)$. Then we have the canonical identification $A[\ell^n]_s = F_s(A[\ell^n], S) = A_s[\ell^n]$.

Given $\varphi \in \pi_1(S, s)$, we have $\varphi(A[\ell^n] \to S) : F_s(A[\ell^n] \to S) \to F_s(A[\ell^n] \to S)$ and this gives the isomorphism

$$[\varphi] : A_s[\ell^n] = F_s(A[\ell^n] \to S) \xrightarrow{\varphi} F_s(A[\ell^n] \to S) = A_s[\ell^n].$$

Now let s' be another geometric point of the (connected) scheme S. Then there exists an isomorphism $\Phi : F_s \xrightarrow{\sim} F_{s'}$ of the fiber functions and hence as above, we get an isomorphism

$$[\Phi] : A_s[\ell^n] \xrightarrow{\sim} A_{s'}[\ell^n].$$

Taking the inverse limit over $n \in \mathbb{Z}_{>0}$ of these isomorphisms we have $[\varphi] : T_\ell(A_s) \to T_\ell(A_s)$ and $[\Phi] : T_\ell(A_s) \to T_\ell(A_{s'})$. Taking the product over all $\ell \neq p$ and tensoring with $\mathbb{Q}$ gives $[\varphi] : H_1(A_s, \mathbb{A}_f^p) \to H_1(A_s, \mathbb{A}_f^p)$ and $[\Phi] : H_1(A_s, \mathbb{A}_f^p) \to H_1(A_{s'}, \mathbb{A}_f^p)$.

Proposition 2.1.7. *Let (A, λ, i) a principally polarized abelian scheme over an $\mathcal{O}_E$-scheme S with $\mathcal{O}_B$-action. The collection of geometric points of S where there exists a K^p-level* *structure is a union of connected components of S.*

Proof. It suffices to show that in a connected component of S, if a K^p-level structure exists at a single geometric point, then it exists at all geometric points. So let S be connected, suppose there is a K^p-level structure at some geometric point s, and let s' be any geometric point of S. Choose an isomorphism $\Phi : F_s \xrightarrow{\sim} F_{s'}$ of the fiber functions. This induces the isomorphism $[\Phi] : H_1(A_s, \mathbb{A}_f^p) \xrightarrow{\sim} H_1(A_{s'}, \mathbb{A}_f^p)$ described above. We define $\overline{\eta}'$ to be the collection of symplectic similitudes

$$V \otimes A_f^p \xrightarrow{\eta} H_1(A_s, \mathbb{A}_f^p) \xrightarrow{[\Phi]} H_1(A_{s'}, \mathbb{A}_f^p)$$

for all $\eta \in \overline{\eta}$. That this is a K^p-orbit follows immediately from the fact that $\overline{\eta}$ is. To see that $\overline{\eta}'$ is also fixed under the action of $\pi_1(S, s')$, first note that the map

$$\pi_1(S, s) \to \pi_1(S, s') \quad \text{sending} \quad \varphi \to \Phi \circ \varphi \circ \Phi^{-1} \in \pi_1(S, s')$$

is an isomorphism. Letting $\Phi \circ \varphi \circ \Phi^{-1}$ be an arbitrary element of $\pi_1(S, s')$, its action on an element $\Phi \circ \eta \in \overline{\eta}'$ is given by

$$V \otimes \mathbb{A}_f^p \xrightarrow{\eta} H_1(A_s, \mathbb{A}_f^p) \xrightarrow{\Phi} H_1(A_{s'}, \mathbb{A}_f^p) \xrightarrow{\Phi^{-1}} H_1(A_s, \mathbb{A}_f^p) \xrightarrow{\varphi} H_1(A_s, \mathbb{A}_f^p) \xrightarrow{\Phi} H_1(A_{s'}, \mathbb{A}_f^p).$$

This is also an element of $\overline{\eta}'$ because $\varphi \circ \eta \in \overline{\eta}$. $\square$

The determinant condition of Kottwitz

Recalling the decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$, we have that V^+ is a $B_{\mathbb{C}}$-module. Thus it makes sense to consider

$$\det_{V^+} : \mathbb{V}_{B_{\mathbb{C}}} \to \mathbb{A}^1_{\mathbb{C}}$$

where $\mathbb{V}_{B_{\mathbb{C}}}$ is the functor on $\mathbb{C}$-algebras sending S to $S \otimes_{\mathbb{C}} B_{\mathbb{C}}$ (see Section A.1 with $R = \mathbb{C}$ and $A = B_{\mathbb{C}}$).

Proposition 2.1.8. $\det_{V^+}$ *is defined over* $\mathcal{O}_{\mathbb{E}}$.

Proof. The B-module structure of V^+ gives a $\mathbb{Q}$-algebra homomorphism $\varphi : B \to \mathrm{End}_{\mathbb{C}}(V^+)$. Choosing a $\mathbb{C}$-basis of V^+ gives an embedding of $\mathrm{End}_{\mathbb{C}}(V^+) \hookrightarrow M_m(\mathbb{C})$ where m is the dimension of V^+ over $\mathbb{C}$. With B finite-dimensional, we in fact have $\varphi_0 : B \hookrightarrow M_n(\widetilde{\mathbb{E}})$ for some finite extension $\widetilde{\mathbb{E}}$ of $\mathbb{Q}$. By enlarging $\widetilde{\mathbb{E}}$ if necessary, assume $\mathbb{E} \subset \widetilde{\mathbb{E}}$. Set $W = \widetilde{\mathbb{E}}^m$ with φ_0 giving W the structure of a $B_{\widetilde{\mathbb{E}}}$-module. Note that $W \otimes_{\widetilde{\mathbb{E}}} \mathbb{C}$ is isomorphic to V^+ as $B_{\mathbb{C}}$-modules.

Choose a basis of W over $\widetilde{\mathbb{E}}$ and let M be the $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_{\widetilde{\mathbb{E}}}$-module generated by this basis. Then we can recover the $B_{\widetilde{\mathbb{E}}}$-module W by $M \otimes_{\mathcal{O}_{\widetilde{\mathbb{E}}}} \widetilde{\mathbb{E}}$. Note that M is finite and locally free over $\mathcal{O}_{\widetilde{\mathbb{E}}}$, so we may consider $\det_M$. Since $M \otimes_{\mathcal{O}_{\widetilde{\mathbb{E}}}} \widetilde{\mathbb{E}} \otimes_{\widetilde{\mathbb{E}}} \mathbb{C} = V$ as $B_{\mathbb{C}}$-modules, we have

$$\det_{M \otimes_{\mathcal{O}_{\widetilde{\mathbb{E}}}} \mathbb{C}} = \det_{V^+} .$$

Therefore $\det_{V^+}$ is defined over $\mathcal{O}_{\widetilde{\mathbb{E}}}$.

It remains to show that $\det_{V^+}$ is defined over $\mathbb{E}$, as then it is defined over $\mathcal{O}_{\widetilde{\mathbb{E}}} \cap \mathbb{E} = \mathcal{O}_{\mathbb{E}}$. It is therefore sufficient to show that $\det_{\mathbb{C}}(x|V^+) \in \mathbb{E}$ for all $x \in B_{\mathbb{E}}$. Since we have

$\mathrm{Tr}_{\mathbb{C}}(x|V^+) \in \mathbb{E}$, the result will follow if we are able to show that we can express $\det_{\mathbb{C}}(x|V^+)$ as a polynomial in

$$\left\{\mathrm{Tr}_{\mathbb{C}}(x|V^+), \mathrm{Tr}_{\mathbb{C}}(x^2|V^+), \ldots, \mathrm{Tr}_{\mathbb{C}}(x^m|V^+)\right\}.$$

Let $p_i \in \mathbb{Z}[X_1, \ldots, X_m]$ denote the ith power sum, so $p_1 = X_1 + X_2 + \cdots + X_m$, $p_2 = X_1^2 + X_1X_2 + \cdots + X_{m-1}X_t + X_m^2$, etc. Let e_i denote the ith complete symmetric polynomial. With $\lambda_1, \ldots, \lambda_m$ denoting the eigenvalues of x, we have

$$\mathrm{Tr}_{\mathbb{C}}(x^i|V^+) = p_i(\lambda_1, \ldots, \lambda_m)$$

$$\det_{\mathbb{C}}(x|V^+) = e_1(\lambda_1, \ldots, \lambda_m).$$

By [Mac, I.2.12], $\mathbb{Z}[p_1, \ldots, p_m] = \mathbb{Z}[e_1, \ldots, e_m]$ and therefore e_1 can be expressed as a polynomial in $p_1, \ldots, p_m$, giving the result. $\square$

Let S be an $\mathcal{O}_E$-scheme, $b_1, \ldots, b_t$ be a set of generators of $\mathcal{O}_B$ as a $\mathbb{Z}_{(p)}$-module, and $(A, \lambda, i, \overline{\eta})$ be a principally polarized abelian scheme over S equipped with an $\mathcal{O}_B$-action and K^p-level structure. The action $i : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ induces an action of $\mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ on $\mathrm{Lie}(A)$, a locally free $\mathcal{O}_S$-module. Thus on each affine open $U \subset S$, we have $\det_{\mathrm{Lie}(A)}(U) \in \Gamma(U, \mathcal{O}_U)[X_1, \ldots, X_t]$. Since the determinant respects localization (Proposition A.1.2), these sections glue to define

$$\det_{\mathrm{Lie}A} \in \Gamma(S, \mathcal{O}_S)[X_1, \ldots, X_t].$$

Likewise, from the above proposition the $\mathcal{O}_B$-action on V^+ gives

$$\det_{V^+} \in \mathcal{O}_{\mathbb{E}}[X_1, \ldots, X_t].$$

By applying the ring homomorphism $\mathcal{O}_{\mathbb{E}} \to \Gamma(S, \mathcal{O}_S)$ to the coefficients of $\det_{V^+}$, we can compare these two determinants. Then the determinant condition is that $\det_{V^+} = \det_{\mathrm{Lie}(A)}$.

Local model

Definition 2.1.9. A local model of a scheme X is a scheme M such that there exists an étale cover $V \to X$ and an étale morphism $V \to M$.

For the integral models described above, [RZ, Chapter 3] constructs a local model diagram.

$$\begin{tikzcd} & \widetilde{\mathcal{A}_0} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{A}_0 & & M^{\mathrm{loc}} \end{tikzcd}$$

This gives, in particular, a local model of the integral model. To construct $\widetilde{\mathcal{A}_0}$, let $H^1_{dR}(A_i)^\vee$ denote the $\mathcal{O}_S$-dual of the de Rham cohomology sheaf (see Section A.5). It is a locally free $\mathcal{O}_S$-module of rank $2n^2$ [BBM, Section 2.5]. Then $(H^1_{dR}(A_i)^\vee)_i$ gives a polarized multichain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules of type $(\mathcal{L})$ in the sense of [RZ, Definition 3.14]. Define $\widetilde{\mathcal{A}_0}$ to be the $\mathcal{O}_E$-scheme that represents the functor defined as follows. We associate with an $\mathcal{O}_E$-scheme S the set of tuples $(\{A_i\}, \bar{\lambda}, \bar{\eta}, \{\gamma_i\})$ up to isomorphism where $(\{A_i\}, \bar{\lambda}, \bar{\eta}) \in \mathcal{A}_0(S)$ and

$$\gamma_i : H^1_{dR}(A_i)^\vee \xrightarrow{\sim} \Lambda_i \otimes_{\mathbb{Z}_p} \mathcal{O}_S$$

is an isomorphism of polarized multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules. Then we define the morphism

$$\Phi : \widetilde{\mathcal{A}_0} \to \mathcal{A}_0 \quad \text{by} \quad (\{A_i\}, \bar{\lambda}, \bar{\eta}, \{\gamma_i\}) \to (\{A_i\}, \bar{\lambda}, \bar{\eta}).$$

Proposition 2.1.10. *[RZ, Theorems 3.11, 3.16] Let $\mathcal{L} = \{\Lambda\}$ be a self-dual multichain of $\mathcal{O}_B \otimes \mathbb{Z}_p$ lattices in V in the sense of loc. cit. Let S be any $\mathbb{Z}_p$-scheme where p is locally nilpotent. Then any polarized multichain $\{M_\Lambda\}$ of $\mathcal{O}_B \otimes_{Z_p} \mathcal{O}_S$-modules of type $(\mathcal{L})$ is locally (for the étale topology on S) isomorphic to the polarized multichain $\left\{\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S\right\}$.*

Moreover, the functor sending

$$T \to Isom(\{M_\Lambda \otimes \mathcal{O}_T\}, \{\Lambda \otimes \mathcal{O}_T\}),$$

is represented by a smooth affine scheme over S.

The proposition holds for any $\mathbb{Z}_p$-scheme S [Pap2, Theorem 2.2]. Let $\mathcal{G}$ be the smooth affine S-group scheme given by

$$\mathcal{G}(T) = \mathrm{Aut}(\{\Lambda \otimes \mathcal{O}_T\}) \quad \text{for an } S\text{-scheme } T.$$

It follows from the proposition above that $\Phi : \widetilde{\mathcal{A}_0} \to \mathcal{A}_0$ is a smooth surjective $\mathcal{G}$-torsor.

Definition 2.1.11. [RZ, Definition 3.27] With S an $\mathcal{O}_E$-scheme, an S-valued point of M^{loc} is given by the following data.

(i). A functor from the category $\mathcal{L}$ to the category of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules on S

$$\Lambda \to \omega_\Lambda, \quad \Lambda \in \mathcal{L}.$$

(ii). A morphism of functors $\psi_\Lambda : \omega_\Lambda \to \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$.

We require the following conditions are satisfied:

(i). the morphisms ψ_Λ are inclusions;

(ii). the quotient $t_\Lambda := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S/\psi(\omega_\Lambda)$ is a locally free $\mathcal{O}_S$-module of finite rank. For the action of $\mathcal{O}_B$ on t_Λ, we have the Kottwitz condition

$$\det_{\mathcal{O}_S}(b|t_\Lambda) = \det_{E'}(b|V^+), \quad b \in \mathcal{O}_B; \text{ and}$$

(iii). for each $\Lambda \in \mathcal{L}$, $\omega_\Lambda^{\mathrm{perp}} = \omega_{\Lambda^\perp}$ where

$$\omega_\Lambda^{\mathrm{perp}} = \left\{ y \in \Lambda^\perp \otimes \mathcal{O}_S : (x, y) = 0 \text{ for all } x \in \omega_\Lambda \right\}.$$

Remark 2.1.12. The above definition of the local model is the subobject variant. Denoting $\Lambda_S = \Lambda \otimes \mathcal{O}_S$, the moduli problem remains the same if one replaces ω_Λ and the injective morphisms $\psi_\Lambda : \omega_\Lambda \to \Lambda_S$ with t_Λ and surjective morphisms $\varphi_\Lambda : \Lambda_S \to t_\Lambda$. In such a case, condition (iii) can be restated as follows. For each $\Lambda \in \mathcal{L}$ the composition

$$\hat{t}_\Lambda \xrightarrow{\hat{\varphi}_\Lambda} \widehat{\Lambda_S} \xrightarrow{(\cdot,\cdot)} \Lambda_S^\perp \xrightarrow{\psi_{\Lambda^\perp}} t_{\Lambda^\perp}$$

is zero. Here $\hat{\cdot} = \mathcal{H}om_S(\cdot, \mathcal{O}_S)$.

Note that $\mathcal{G}$ acts on the local model by acting on ψ_Λ through its natural action on $\Lambda \otimes \mathcal{O}_S$. For an abelian scheme A/S we have the Hodge filtration (Proposition A.5.1)

$$0 \to \omega_{\hat{A}} \to H^1_{dR}(A)^\vee \to \mathrm{Lie}(A) \to 0.$$

We can now associate with each point $(\{A_i\}, \bar{\lambda}, \bar{\eta}, \{\gamma_i\}) \in \widetilde{\mathcal{A}_0}(S)$ a collection of injective morphisms

$$0 \to \omega_{\hat{A}_i} \to H^1_{dR}(A_i)^\vee \underset{\gamma_i}{\xrightarrow{\sim}} \Lambda_i \otimes_{\mathbb{Z}_p} \mathcal{O}_S$$

and this defines the map Ψ in the following diagram.

$$\begin{tikzcd} & \widetilde{\mathcal{A}_0} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{A}_0 & & M^{\mathrm{loc}} \end{tikzcd}$$

Proposition 2.1.13. *[RZ, Chapter 3] The diagram above is a local model diagram. Specifically the morphisms Φ and Ψ are smooth, Φ is surjective, and étale locally $\mathcal{A}_0 \cong M^{loc}$: there exists an étale cover $V \to \mathcal{A}_0$ and a section $s : V \to \widetilde{\mathcal{A}_0}$ of Φ such that $\Psi \circ s$ is étale.*

Also the morphism Φ is a torsor for the smooth affine group scheme $\mathcal{G}$ and Ψ is $\mathcal{G}$-equivariant.

2.2 Unitary case

We now specialize the moduli problem of Definition 2.1.3 defining the integral model to the unitary case. Let $(D, *, h_0)$ be a unitary datum as in Definition 1.2.6. As described in Section 1.2.1, this induces the PEL Shimura datum $(B, \iota, V, (\cdot,\cdot), h_0, K)$ up to the choice of K. As we are considering the split unitary case, we make the following two assumptions on the odd rational prime p.

(i). (p) is unramifed in F^+ and each factor of (p) in F^+ splits in F.

(ii). $D_{\mathbb{Q}_p}$ splits.

In view of the first assumption, write $(p) = \prod_j p_j$ in F^+ and $p_j = \mathfrak{p}_j \mathfrak{p}_j^*$ in F. Then

$$F_{\mathbb{Q}_p} = \prod_j F_{\mathfrak{p}_j} \times F_{\mathfrak{p}_j^*} \quad \text{making} \quad D_{\mathbb{Q}_p} = \prod_j D_{\mathfrak{p}_j} \times D_{\mathfrak{p}_j^*}$$

where $D_{\mathfrak{p}_j}$ and $D_{\mathfrak{p}_j^*}$ are respectively a central simple $F_{\mathfrak{p}_j}$ and $F_{\mathfrak{p}_j^*}$ algebra for each j. Recalling that $*$ induces on F the nontrivial element of $\mathrm{Gal}(F/F^+)$, we have $* : D_{\mathfrak{p}_j} \xrightarrow{\sim} D_{\mathfrak{p}_j^*}^{\mathrm{opp}}$. The second assumption means $D_{\mathfrak{p}_j} \cong M_n(F_{\mathfrak{p}_j})$ for every j.

The splitting of $D_{\mathbb{Q}_p}$ makes

$$G = \prod_j G_j$$

where each factor is given on a $\mathbb{Q}_p$-algebra R as

$$G_j(R) = \left\{(x_1, x_2) \in (D_{\mathfrak{p}_j} \times R)^\times \times (D_{\mathfrak{p}_j^*} \times R)^\times : x_1 = c(x_2^*)^{-1} \text{ for some } c \in R^\times\right\}.$$

Thus $G_j \cong D_{\mathfrak{p}_j}^\times \times \mathbb{G}_{m,\mathbb{Q}_p} \cong \mathrm{GL}_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$. Finally we define $\mu_j : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{j,\overline{\mathbb{Q}}_p}$ by composing $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ with the jth projection.

With these decompositions, by taking the product over all j it suffices to describe the order $\mathcal{O}_B$ and lattice chain $\mathcal{L}$ on each factor. Set $D_j = D_{\mathfrak{p}_j} \times D_{\mathfrak{p}_j^*}$ and fix an isomorphism

$$D_{\mathfrak{p}_j} \times D_{\mathfrak{p}_j^*} \xrightarrow{\sim} M_n(F_{\mathfrak{p}_j}) \times M_n(F_{\mathfrak{p}_j})$$

such that the involution ι becomes $(X, Y) \to (Y^t, X^t)$. With ξ as in Lemma 1.2.7, set $(\chi^t, -\chi)$ to be the image of ξ under this isomorphism where $\chi \in \mathrm{GL}_n(F_{\mathfrak{p}_j})$. Then the pairing $(\cdot, \cdot)$ becomes

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle = \mathrm{Tr}_{D_j/\mathbb{Q}_p}(X_1 Y_2^t \chi^t, -X_2 Y_1^t \chi).$$

Letting π_j be a uniformizer of $\mathcal{O}_{F_{\mathfrak{p}_j}}$, the $\mathcal{O}_{F_{\mathfrak{p}_j}}$-lattice chain $\mathcal{L}_j$ in $D_{\mathfrak{p}_j} \times D_{\mathfrak{p}_j^*}$ is given by

$$\Lambda_{j,i} = \mathrm{diag}((\pi_j^{-1})^i, 1^{n-i}) M_n(\mathcal{O}_{F_{\mathfrak{p}_j}}) \quad \text{and} \quad \Lambda_{j,i}^* = \chi^{-1} \mathrm{diag}(1^{n-i}, (\pi_j^{-1})^i) M_n(\mathcal{O}_{F_{\mathfrak{p}_j}})$$

where again we are using the fixed isomorphism above. Of course the description of $\Lambda_{j,i}$ and $\Lambda^*_{j,i}$ is independent of the choice of uniformizer π_j. By definition we have

$$(\Lambda_{j,0} \oplus \Lambda^*_{j,0}) \subset (\Lambda_{j,1} \oplus \Lambda^*_{j,1}) \subset \cdots \subset (\Lambda_{j,n} \oplus \Lambda^*_{j,n}) = \pi_j^{-1}(\Lambda_{j,0} \oplus \Lambda^*_{j,0})$$

and one can compute

$$(\Lambda_{j,i} \oplus \Lambda^*_{j,i})^{\perp} = \Lambda_{j,-i} \oplus \Lambda^*_{j,-i}.$$

Recalling that $B_j = D_j^{\mathrm{opp}}$, take $\mathcal{O}_{B_j} \subset B_j$ to be the unique maximal $\mathbb{Z}_{(p)}$-order such that under the fixed isomorphism we have

$$\mathcal{O}_{B_j} \otimes \mathbb{Z}_p \xrightarrow{\sim} M_n^{\mathrm{opp}}(\mathcal{O}_{F_{\mathfrak{p}_j}}) \times M_n^{\mathrm{opp}}(\mathcal{O}_{F_{\mathfrak{p}_j}}).$$

Then it is immediate that $\Lambda_{j,i} \oplus \Lambda^*_{j,i}$ is an $\mathcal{O}_{B_j} \otimes \mathbb{Z}_p$-lattice and $\mathcal{O}_{B_j} \otimes \mathbb{Z}_p$ is invariant under ι.

Set $\mathcal{L} = (\Lambda_i)_i$ with $\Lambda_i = \prod_j \Lambda_{j,i} \oplus \Lambda^*_{j,i}$ and $\mathcal{O}_B = \prod_j \mathcal{O}_{B_j}$. Finally take $K = K^p K_p$ where $K_p = \mathrm{Aut}(\mathcal{L})$ and K^p is a sufficiently small open compact subgroup of $\mathbb{G}(\mathbb{A}_f^p)$. With this data, the moduli problem for the integral model given in Definition 2.1.3 becomes the following.

Definition 2.2.1. For an $\mathcal{O}_E$-scheme S, $\mathcal{A}_0^{\mathrm{GL}}(S) = \mathcal{A}_{0,K^p}^{\mathrm{GL}}(S)$ is the collection of tuples $(A_\bullet, \bar{\lambda}, i, \bar{\eta})$ up to isomorphism where

- $A_\bullet$ is a chain of abelian schemes

$$\cdots \to A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} A_n \to \cdots$$

  of relative dimension n^2 over S, indexed by $\mathcal{L}$ by setting $A_i = A_{\Lambda_i}$ for $i \in \mathbb{Z}$;

- Each A_i is equipped with an $\mathcal{O}_B$-action $i : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A_i) \otimes \mathbb{Z}_{(p)}$;
- $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous class of principal polarizations. That is, $\bar{\lambda}$ is a collection of isogenies $\left\{\lambda_i : A_i \to \hat{A}_{-i}\right\}$ making the diagram

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{\alpha_{-2}} & A_{-1} & \xrightarrow{\alpha_{-1}} & A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} \cdots \\
& \downarrow{\scriptstyle \lambda_{-1}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_n} & \\
\cdots \xrightarrow{\alpha_1^\vee} & \hat{A}_1 & \xrightarrow{\alpha_0^\vee} & \hat{A}_0 & \xrightarrow{\alpha_{-1}^\vee} & \hat{A}_{-1} & \xrightarrow{\alpha_{-2}^\vee} \cdots \xrightarrow{\alpha_{-n}^\vee} & \hat{A}_{-n} & \xrightarrow{\alpha_{-n-1}^\vee} \cdots
\end{array}$$

  commute and satisfying the following two conditions: up to some $\mathbb{Q}$-multiple every λ_i is an isomorphism and for each i we have $A_i \xrightarrow{\lambda_i} \hat{A}_{-i} \to \hat{A}_i$ is a rational multiple of a polarization of A_i; and
- $\bar{\eta}$ is a K^p-level structure

$$\bar{\eta} : H_1(A_0, \mathbb{A}_f^p) \xrightarrow{\sim} V \otimes \mathbb{A}_f^p \mod K^p.$$

We require that the following conditions hold:

(i). Each α_i is an isogeny of degree p^{2n};

(ii). There are "periodicity isomorphisms" $\theta_p : A_{i+n} \xrightarrow{\sim} A_i$ such that for each i the map

$$A_i \to A_{i+1} \to \cdots \to A_{i+n} \xrightarrow{\theta_p} A_i$$

is multiplication by p;

(iii). α_i commutes with the $\mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ actions;

(iv). For all i and $b \in \mathcal{O}_B$

$$\det_{\mathcal{O}_S}(b|\mathrm{Lie}(A_i)) = \det_{E'}(b|V^+).$$

An isomorphism of S-valued points $f : (\{A_i\}, \bar{\lambda}, i, \bar{\eta}) \xrightarrow{\sim} (\{A_i'\}, \bar{\lambda}', i, \bar{\eta}')$ is a collection of $\mathbb{Z}_{(p)}$-isogenies $f_i : A_i \to A_i'$ each making the diagram

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & A_{i+1} \arrow[d, "f_{i+1}"] \\ A_i' \arrow[r, "\alpha_i'"] & A_{i+1}' \end{tikzcd}$$

commute such that

- for each i there exists a locally constant function r_i with values in $\mathbb{Z}_{(p)}^\times$ such that $\lambda_i = r_i \cdot (f_i^\vee \circ \lambda_i' \circ f_i)$;
- for each i the morphism $\mathrm{End}(A_i) \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A_i') \otimes \mathbb{Z}_{(p)}$ induced by f_i, which is again denoted by f_i, is such that $f_i \circ i(b) = i'(b)$ for all $b \in \mathcal{O}_B$; and
- $H_1(f_0) \circ \overline{\eta} = \overline{\eta}'$.

We now turn our attention to the local model. Let F^{Gal} denote the Galois closure of F inside F_0^{sep}. Under our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (see Section 2.1), we will identify $F_0 \subset F \subset F^{\mathrm{Gal}} \subset \overline{\mathbb{Q}}_p$. As we have that $G_{F^{\mathrm{Gal}}}$ splits, $E \subset F^{\mathrm{Gal}}$ by [Kot1, 1.2]. We can thus consider $M^{\mathrm{loc}} \otimes_{\mathcal{O}_E} \mathcal{O}_{F^{\mathrm{Gal}}}$. An S-valued point of this scheme is given by a functor $\Lambda \to \omega_\Lambda$ from the category $\mathcal{L}$ to the category of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules, satisfying the additional conditions as in Definition 2.1.11. Now we have the decompositions

$$F \otimes_{\mathbb{Q}_p} F^{\mathrm{Gal}} = \bigoplus_{\varphi : F \to F^{\mathrm{Gal}}} F^{\mathrm{Gal}},$$

$$V \otimes_{\mathbb{Q}_p} F^{\mathrm{Gal}} = \bigoplus_{\varphi} V_\varphi, \qquad V^+ = \bigoplus_{\varphi} V_\varphi^+$$

where $\dim_{F^{\mathrm{Gal}}} V_\varphi = \dim_F V = n$ and the number of summands is $[F : \mathbb{Q}_p]$. Since $F/\mathbb{Q}_p$ is unramified, there is a ring isomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{F^{\mathrm{Gal}}} = \bigoplus_{\varphi} \mathcal{O}_{F^{\mathrm{Gal}}}.$$

We therefore have that an $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{F^{\mathrm{Gal}}}$-module M is a family $(M_\varphi)_\varphi$ of $\mathcal{O}_{F^{\mathrm{Gal}}}$-modules and likewise, homomorphisms $M \to N$ are families $(M_\varphi \to N_\varphi)_\varphi$ of homomorphisms of $\mathcal{O}_{F^{\mathrm{Gal}}}$-modules. The following proposition immediately follows.

Proposition 2.2.2. *$M^{loc} \otimes_{\mathcal{O}_E} \mathcal{O}_{F^{Gal}}$ is isomorphic to the base change of a product of local models in the case $F^+ = \mathbb{Q}$.*

For the remainder of this section, we will work on a single factor, taking the product over all such factors. We thus omit any subscript φ or j and we will assume that $F^+ = \mathbb{Q}$. Then F is an imaginary quadratic extension of $\mathbb{Q}$ and the prime p splits in F. Thus $F = F^{\mathrm{Gal}}$ and $G_{\mathbb{Q}_p}$ splits so $E = \mathbb{Q}_p$. Furthermore $F_{\mathfrak{p}} = F_{\mathfrak{p}^*} = \mathbb{Q}_p$.

Recall from earlier in this section that the assumptions on p give the splitting

$$\mathcal{O}_B \otimes \mathbb{Z}_p \cong M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p).$$

For a scheme S over $\mathrm{Spec}(\mathcal{O}_E)$, the sheaves on S induced by a S-valued point of the local model carry an $\mathcal{O}_B \otimes \mathbb{Z}_p$-action, and as such we get a corresponding splitting. Using Morita equivalence, we will be able to reduce the "size" of the local model data. We will now describe this in more detail.

An S-valued point of the local model $M^{\mathrm{loc}}_{\mathrm{GL}}$ is determined by the following commutative diagram.

$$\begin{tikzcd}
\mathscr{L}_{0,S} \oplus \mathscr{L}^*_{0,S} \arrow[r, "\varphi_0"] & \mathscr{L}_{1,S} \oplus \mathscr{L}^*_{1,S} \arrow[r, "\varphi_1"] & \cdots \arrow[r, "\varphi_{n-1}"] & \mathscr{L}_{n,S} \oplus \mathscr{L}^*_{n,S} \\
\omega_0 \arrow[u] \arrow[r] & \omega_1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \omega_n \arrow[u]
\end{tikzcd}$$

Here we are writing $\mathscr{L}_{i,S} \oplus \mathscr{L}^*_{i,S}$ for what was $\Lambda_{i,S}$ in Definition 2.1.11, where the splitting is given by $\mathcal{O}_B \otimes \mathbb{Z}_p = M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p)$. In particular, $\mathscr{L}_{i,S}$ and $\mathscr{L}^*_{i,S}$ are locally free $\mathcal{O}_S$-sheaves of rank n^2. Now ω_i is a locally free $\mathcal{O}_S$-submodule of $\mathscr{L}_{i,S} \oplus \mathscr{L}^*_{i,S}$, ω_i is Zariski-locally a direct summand of $\mathscr{L}_{i,S} \oplus \mathscr{L}^*_{i,S}$ of rank n^2, and the vertical arrows are inclusions,.

The action of $M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p)$ gives

$$\mathscr{L}_{i,S} = \bigoplus_n e_{11}\mathscr{L}_{i,S} \quad \text{and} \quad \mathscr{L}^*_{i,S} = \bigoplus_n f_{11}\mathscr{L}^*_{i,S}$$

where e_{11} and f_{11} are respectively idempotents of the first and second factors of $M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p)$.

Let W and W^* be $\mathbb{Z}_p^n$, viewed as left $\mathcal{O}_B \otimes \mathbb{Z}_p$-modules via right multiplications by elements of the first and second factor of $M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p)$ respectively. Recall the decomposition $V_{E'} = V^+ \oplus V^-$ induced by $\mu_h = (0^{n-r}, (-1)^r)$.

$$V^+ = W_{E'}^{n-r} \oplus (W^*_{E'})^r.$$

With $t_{\mathscr{L}_i} = \mathscr{L}_i/\omega_i$, the determinant condition

$$\det{}_{\mathcal{O}_S}(a; t_{\mathscr{L}_i}) = \det{}_{E'}(a; V^+), \quad a \in \mathcal{O}_B$$

is equivalent to the splitting of $t_{\mathscr{L}_i}$ into two factors of rank n^2-nr and nr, each respectively a quotient of $\mathscr{L}_i$ and $\mathscr{L}_i^*$. Thus, we have the splitting of ω_i into two summands of ranks nr and n^2-nr contained in $\mathscr{L}_i$ and $\mathscr{L}_i^*$ respectively. The action of each copy of $M_n^{\mathrm{opp}}(\mathbb{Z}_p)$ splits these into a direct sum of n copies of $\mathcal{F}_i$ of ranks r and $\mathcal{F}_i^*$ of rank $n-r$ respectively. As such, we write $\Lambda_{i,S}=e_{11}\mathscr{L}_{i,S}$, $\Lambda_{i,S}^*=f_{11}\mathscr{L}_{i,S}$, $\mathcal{F}_i=e_{11}\omega_i$, and $\mathcal{F}_i^*=f_{11}\omega_i$. Therefore, an S-valued point of the local model $M_{\mathrm{GL}}^{\mathrm{loc}}$ is determined by the commutative diagram

$$\begin{array}{ccccccc}
\Lambda_{0,S}\oplus\Lambda_{0,S}^* & \xrightarrow{\varphi_0} & \Lambda_{1,S}\oplus\Lambda_{1,S}^* & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_{n-1}} & \Lambda_{n,S}\oplus\Lambda_{n,S}^* \\
\uparrow & & \uparrow & & & & \uparrow \\
\mathcal{F}_0\oplus\mathcal{F}_0^* & \longrightarrow & \mathcal{F}_1\oplus\mathcal{F}_1^* & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_n\oplus\mathcal{F}_n^*
\end{array}$$

where $\mathcal{F}_i\oplus\mathcal{F}_i^*$ is an $\mathcal{O}_S$-submodule of $\Lambda_{i,S}\oplus\Lambda_{i,S}^*$, the vertical arrows are inclusions, and Zariski locally over S, $\mathcal{F}_i$ is a direct summand of $\Lambda_{i,S}$ of rank r and $\mathcal{F}_i^*$ is a direct summand of $\Lambda_{i,S}^*$ of rank $n-r$.

Condition (iii) of the local model

$$\omega_\Lambda^{\mathrm{perp}}=\omega_{\Lambda^\perp}$$

is simply $(\mathcal{F}_i\oplus\mathcal{F}_i^*)^{\mathrm{perp}}=\mathcal{F}_{-i}\oplus\mathcal{F}_{-i}^*$. From the explicit definition of $\langle\cdot,\cdot\rangle$, we have $(\mathcal{F}_i\oplus\mathcal{F}_i^*)^{\mathrm{perp}}=(\mathcal{F}_i^*)^{\mathrm{perp}}\oplus\mathcal{F}_i^{\mathrm{perp}}$. Therefore $\{\mathcal{F}_i\}$ (or alternatively, $\{\mathcal{F}_i^*\}$) determines $\{\mathcal{F}_i\oplus\mathcal{F}_i^*\}$. The following proposition summarizes this discussion.

Proposition 2.2.3. *In the unitary case, an S-valued point of the local model is determined by a diagram*

$$\begin{array}{ccccccc}
\Lambda_{0,S} & \xrightarrow{\varphi_0} & \Lambda_{1,S} & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_{n-1}} & \Lambda_{n,S} \\
\uparrow & & \uparrow & & & & \uparrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_n
\end{array}$$

where

- $\Lambda_{i,S} = e_{11}(\mathscr{L}_i \oplus \mathscr{L}_i^*)$;
- φ_i *is the morphism induced from the inclusions of the lattice chain;*
- $\mathcal{F}_i$ *is a locally free* $\mathcal{O}_S$*-module which is Zariski locally a direct summand of* $\Lambda_{i,S}$ *of rank* r*; and*
- *the vertical arrows are inclusions.*

Note that in the above diagram we are, by abuse of notation, labeling the restriction $\varphi_i|_{\Lambda_{i,S}}$ as φ_i.

2.3 Symplectic case

We now consider the symplectic case. With the datum $(B, \iota, V, (\cdot,\cdot), h_0, K)$ as in Section 1.2.2, we take $\mathcal{O}_B = \mathbb{Z}_{(p)}$ and $\mathcal{L}$ to be the standard $\mathbb{Z}_p$-lattice chain as follows. Let $\{e_1, \dots, e_{2n}\}$ be the standard basis of $V = \mathbb{Q}_p^{2n}$ and define the $\mathbb{Z}_p$-lattice chain

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{n-1}$$

where for $0 \leq i \leq 2n-1$,

$$\Lambda_i = \langle p^{-1}e_1, \dots, p^{-1}e_i, e_{i+1}, \dots, e_{2n}\rangle \subset \mathbb{Q}_p^{2n} \text{ as a } \mathbb{Z}_p\text{-module}$$

extended periodically by $\Lambda_{i+n} = p^{-1}\Lambda_i$ for all integers i. Note that $\Lambda_i^{\perp} = \Lambda_{-i}$. Let $K_p = \mathrm{Aut}(\mathcal{L})$ and K^p be a sufficiently small open compact subgroup of $\mathbb{G}(\mathbb{A}_f^p)$. With G split over $\mathbb{Q}_p$, we have $E = \mathbb{Q}_p$. The moduli problem in Definition 2.1.3 for the integral model $\mathcal{A}_0^{\mathrm{GSp}}$

may be described as follows.

Definition 2.3.1. For any $\mathbb{Z}_p$-scheme S, $\mathcal{A}_0^{\mathrm{GSp}}(S) = \mathcal{A}_{0,K^p}^{\mathrm{GSp}}(S)$ is the collection of tuples $(A_\bullet, \lambda_0, \lambda_n, \bar{\eta})$ up to isomorphism (defined below) where

(i). $A_\bullet$ is a chain of abelian schemes

$$\cdots \to A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} A_n \to \cdots$$

over S of relative dimension n, indexed by $\mathcal{L}$ by setting $A_i = A_{\Lambda_i}$ for $i \in \mathbb{Z}$, where each morphism $\alpha_i : A_i \to A_{i+1}$ is an isogeny of degree p;

(ii). the maps $\lambda_0 : A_0 \to \hat{A}_0$ and $\lambda_n : A_n \to \hat{A}_n$ are principal polarizations making the loop starting at any A_i or $\hat{A}_i$ in the diagram

$$\begin{tikzcd} A_0 \arrow[r, "\alpha_0"] & A_1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[d, "\lambda_n"] \\ \hat{A}_0 \arrow[u, "\lambda_0^{-1}"] & \hat{A}_1 \arrow[l, "\alpha_0^\vee"'] & \cdots \arrow[l, "\alpha_1^\vee"'] & \hat{A}_n \arrow[l, "\alpha_{n-1}^\vee"'] \end{tikzcd}$$

multiplication by p; and

(iii). $\bar{\eta}$ is a K^p-level structure on A_0.

An isomorphism of S-valued points $f : (A_\bullet, \lambda_0, \lambda_n, \bar{\eta}) \xrightarrow{\sim} (A_\bullet, \lambda_0, \lambda_n, \bar{\eta}')$ is a collection of $\mathbb{Z}_{(p)}$-isogenies $f_i : A_i \to A_i'$ making the diagrams

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & A_{i+1} \arrow[d, "f_{i+1}"] \\ A_i' \arrow[r, "\alpha_i'"] & A_{i+1}' \end{tikzcd}$$

commute and are such that

- for $i = 0$ and $i = n$ there exists a locally constant function s_i with values in $\mathbb{Z}_{(p)}^\times$ such that $\lambda_i = s_i \cdot (f_i^\vee \circ \lambda_i' \circ f)$; and
- $H_1(f_0) \circ \overline{\eta} = \overline{\eta}'$.

There is an alternative description of this moduli problem in terms of chains of finite flat group subschemes instead of chains of isogenies [dJ2, Section 1].

Turning our attention to the local model, we have that an S-valued point of $M_{\mathrm{GSp}}^{\mathrm{loc}}$ is given by a commutative diagram

$$\begin{array}{ccccccccc}
\Lambda_{0,S} & \longrightarrow & \Lambda_{1,S} & \longrightarrow & \cdots & \longrightarrow & \Lambda_{2n-1,S} & \longrightarrow & p^{-1}\Lambda_{0,S} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{2n-1} & \longrightarrow & p^{-1}\Lambda_0
\end{array}$$

satisfying the three conditions as in Definition 2.1.11. Condition (iii) is equivalent to the following condition. Set

$$\hat{\Lambda}_{i,S} = \mathcal{H}om_{\mathcal{O}_S}(\Lambda_{i,S}, \mathcal{O}_S)$$

$$\hat{\mathcal{F}}_i = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}_i, \mathcal{O}_S).$$

(iii') For any i the composition

$$\mathcal{F}_i \to \Lambda_{i,S} \xrightarrow{\sim} \hat{\Lambda}_{2n-i,S} \to \hat{\mathcal{F}}_{2n-i}$$

is zero, where the middle isomorphism is induced by the isomorphism

$$\Lambda_i \xrightarrow{\sim} \hat{\Lambda}_{2n-i} \quad \text{sending} \quad x \to (px, \cdot).$$

To see this, first assume (iii) making $\mathcal{F}_i^{\mathrm{perp}} = p\mathcal{F}_{2n-i}$. Then for a morphism $S \to M^{\mathrm{loc}}$ and an open subscheme $U \subset S$, the composition above sends a section $x \in \mathcal{F}_i(U)$ to the map

$$(px, \cdot)|_{\mathcal{F}_{2n-i}(U)} : \mathcal{F}_{2n-i}(U) \to \mathcal{O}_S(U)$$

which is clearly zero. Conversely, assume condition (iii'). Then in the same way we see that $p\mathcal{F}_{2n-i} \subset \mathcal{F}_i^{\mathrm{perp}}$. But since they have the same rank (namely, n), we get equality as required. The following proposition summarizes this discussion.

Proposition 2.3.2. *An S-valued point of M_{GSp}^{loc} is given by a commutative diagram*

$$\begin{tikzcd}
\Lambda_{0,S} \arrow[r, "\varphi_0"] & \Lambda_{1,S} \arrow[r, "\varphi_1"] & \cdots \arrow[r, "\varphi_{2n-1}"] & \Lambda_{2n-1,S} \arrow[r, "\varphi_{2n}"] & p^{-1}\Lambda_{0,S} \\
\mathcal{F}_0 \arrow[r] \arrow[u] & \mathcal{F}_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{F}_{2n-1} \arrow[r] \arrow[u] & p^{-1}\Lambda_0 \arrow[u]
\end{tikzcd}$$

where $\Lambda_{i,S} = \Lambda_i \otimes_{\mathbb{Z}_p} \mathcal{O}_S$, the φ_i are induced by the inclusions of the lattice chain, $\mathcal{F}_i$ are locally free $\mathcal{O}_S$-submodules of rank n which are Zariski-locally direct summands of $\Lambda_{i,S}$, the vertical arrows are inclusions, and the $\mathcal{F}_i$ satisfy the following duality condition. We require the map

$$\mathcal{F}_i \to \Lambda_{i,S} \xrightarrow{\sim} \hat{\Lambda}_{2n-i,S} \to \hat{\mathcal{F}}_{2n-i}$$

is zero for all i. Here $\Lambda_{i,S} \xrightarrow{\sim} \hat{\Lambda}_{2n-i,S}$ is induced from the duality $\Lambda_i \xrightarrow{\sim} \hat{\Lambda}_{2n-i}$.

2.4 Representability

Let $\mathcal{A}_{n,1,N}$ denote the moduli space of principally polarized abelian schemes of relative dimension n equipped with a full symplectic level N structure. Then for $N \geq 3$ with $p \nmid N$,

$\mathcal{A}_{n,1,N}$ is represented by a scheme (Theorem A.6.7). To prove the representability of $\mathcal{A}_0^{\mathrm{GL}}$ and $\mathcal{A}_0^{\mathrm{GSp}}$, we will show that they are relatively representable over $\mathcal{A}_{n,1,N}$. We first give an equivalent moduli problem for $\mathcal{A}_{n,1,N}$ involving a K^p-level structure.

Fix a integral PEL Shimura datum $(B, \iota, V, (\cdot,\cdot), h_0, \mathcal{O}_B, \mathcal{L}, K^p)$ induced from the data in either the unitary or symplectic case and an integer $N \geq 3$ with $p \nmid N$. We assume that K^p is the principle level N structure with respect to the integral PEL Shimura datum. That is,

$$K^p = K(N) = \left\{ g \in \mathbb{G}(\mathbb{A}_f^p) : (g-1)(\Lambda_0 \otimes \hat{\mathbb{Z}}^{(p)}) \subset N \cdot \left(\Lambda_0 \otimes \hat{\mathbb{Z}}^{(p)} \right) \right\}.$$

Note that $K(N) = \prod_{\ell \neq p} K_\ell(N)$ with $K_\ell(N) \subset \mathbb{G}(\mathbb{Q}_\ell)$.

Remark 2.4.1. Let s be a geometric point of a $\mathbb{Z}_p$-scheme S. For any $\mathbb{Z}_{(p)}$-isogeny $f : A \to A'$ of abelian schemes over S, there is an induced isomorphism $H_1(f) : H_1(A_s, \mathbb{A}_f^p) \xrightarrow{\sim} H_1(A'_s, \mathbb{A}_f^p)$, and hence isomorphisms $V_\ell(f) : V_\ell(A_s) \xrightarrow{\sim} V_\ell(A'_s)$ for all primes $\ell \neq p$. Therefore given such a $\mathbb{Z}_{(p)}$-isogeny f we will implicitly identify $H_1(A_s, \mathbb{A}_f^p)$ and $H_1(A'_s, \mathbb{A}_f^p)$ using this isomorphism, and similarly with $V_\ell(A_s)$ and $V_\ell(A'_s)$.

Proposition 2.4.2. *Let S be a connected $\mathbb{Z}_p$-scheme, s a geometric point of S, and A/S an abelian scheme. Let $\Lambda \subset H_1(A_s, \mathbb{A}_f^p)$ be a self-dual $\hat{\mathbb{Z}}^{(p)}$-lattice that is fixed by the action of $\pi_1(S, s)$. Then there exists a unique abelian scheme B, up to isomorphism, equipped with a $\mathbb{Z}_{(p)}$-isogeny $A \to B$ such that $H_1(B_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda$.*

Proof. We first show uniqueness. Suppose that there are two such abelian schemes B and B' over S. Then we have the $\mathbb{Z}_{(p)}$-isogenies $f : A \to B$ and $f' : A \to B'$ with $H_1(B_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda = H_1(B'_s, \hat{\mathbb{Z}}^{(p)})$. Therefore we have that the $\mathbb{Z}_{(p)}$-isogeny $f^{-1} \circ f' : B' \to B$ carries $H_1(B_s, \hat{\mathbb{Z}}^{(p)})$ isomorphically onto $H_1(B'_s, \hat{\mathbb{Z}}^{(p)})$, and hence it must be that $f^{-1} \circ f'$ is an isomorphism.

To show existence, we must produce an abelian scheme B and a $\mathbb{Z}_{(p)}$-isogeny $A \to B$ such that $H_1(B_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda$. Note that $\Lambda \cap H_1(A_s, \hat{\mathbb{Z}}^{(p)}) \subset H_1(A_s, \mathbb{A}_f^p)$ is a $\hat{\mathbb{Z}}^{(p)}$-lattice. Thus there exists $\alpha \in \hat{\mathbb{Z}}^{(p)}$ such that

$$\alpha H_1(A_s, \hat{\mathbb{Z}}^{(p)}) \subset \Lambda \cap H_1(A_s, \hat{\mathbb{Z}}^{(p)}) \subset H_1(A_s, \hat{\mathbb{Z}}^{(p)})$$

and writing $\alpha = (\alpha_\ell)_\ell \in \prod_{\ell \neq p} \mathbb{Z}_\ell$, we have α_ℓ is a unit in $\mathbb{Z}_\ell$ for almost every prime ℓ. In the following, we will work on a single factor where α_ℓ is not a unit in $\mathbb{Z}_\ell$, taking the product over all such factors. Denote by $\Lambda_\ell \subset V_\ell(A_s)$ the factor of Λ corresponding to ℓ.

Now for such an ℓ, there exists an integer $k > 0$ such that

$$\ell^k T_\ell(A_s) \subset \Lambda_\ell \cap T_\ell(A_s) \subset T_\ell(A_s).$$

Thus we can consider the quotients

$$\big(\Lambda_\ell \cap T_\ell(A_s)\big)/\ell^k T_\ell(A_s) \subset T_\ell(A_s)/\ell^k T_\ell(A_s) = A_s[\ell^k].$$

Set $C_\ell = (\Lambda_\ell \cap T_\ell(A_s))/\ell^k T_\ell(A_s) \subset A_s[\ell^k]$. Denote the order of C_ℓ by ℓ^m and consider the isogeny $A/C_\ell \to A$ with kernel $A[\ell^m]/C_\ell$. Then from Proposition A.6.12, we have the exact sequence

$$0 \to T_\ell(A/C_\ell) \to T_\ell(A) \to A[\ell^m]/C_\ell \to 0.$$

Since $A[\ell^m]/C_\ell = T_\ell(A)/\Lambda_\ell \cap T_\ell(A_s)$, we have $T_\ell(A/C_\ell) = \Lambda_\ell \cap T_\ell(A_s)$.

As mentioned above, set $C = \prod_{\ell \neq p} C_\ell$. Then $A \to A/C$ is a $\mathbb{Z}_{(p)}$-isogeny with

$$H_1((A/C)_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda \cap H_1(A_s, \hat{\mathbb{Z}}^{(p)}).$$

Thus we are reduced to the case where $H_1(A_s, \hat{\mathbb{Z}}^{(p)}) \subset \Lambda$.

Similar to the above, for each prime $\ell \neq p$ there exists an integer $k_\ell \leq 0$ such that $\Lambda_\ell \subset \ell^{k_\ell} T_\ell(A_s)$. Note that $k_\ell = 0$ for almost every prime ℓ. By using the multiplication by ℓ^{k_ℓ} map, we may assume that $\Lambda \subset H_1(A_s, \hat{\mathbb{Z}}^{(p)})$.

Now there exists an $\alpha \in \hat{\mathbb{Z}}^{(p)}$ such that $\alpha H_1(A_s, \hat{\mathbb{Z}}^{(p)}) \subset \Lambda \subset H_1(A_s, \hat{\mathbb{Z}}^{(p)})$. For each prime ℓ set $C_\ell = \Lambda_\ell / \alpha_\ell T_\ell(A_s)$. Then as above, $T_\ell(A_s/C_\ell) = \Lambda_\ell$ and taking $C = \prod_{\ell \neq p} C_\ell$ gives the equality $H_1(A_s/C, \hat{\mathbb{Z}}^{(p)}) = \Lambda$ as required. □

Proposition 2.4.3. *Let S be an $\mathcal{O}_E$-scheme and A/S a principally polarized abelian scheme. Fix a geometric point s of S. In the following statements, we take $\ell \neq p$ to be a rational prime.*

(i). Let $\ell \nmid N$. Then giving a $K_\ell(N)$-orbit of symplectic similitudes $\overline{\eta} : V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$, with similitude in $\mathbb{Z}_\ell^\times$, is equivalent to giving a self-dual $\mathbb{Z}_\ell$-lattice $\Lambda \subset V_\ell(A_s)$.

(ii). Let $\ell \mid N$ and set $N = a\ell^k$ with $\ell \nmid a$. Then giving a $K_\ell(N)$-orbit of symplectic similitudes $\overline{\eta} : V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$, with similitude in $\mathbb{Z}_\ell^\times$, is equivalent to giving a self-dual $\mathbb{Z}_\ell$-lattice $\Lambda \subset V_\ell(A_s)$ and an isomorphism

$$\Lambda_0 \otimes \mathbb{Z}_\ell / \big(\ell^k \cdot (\Lambda_0 \otimes \mathbb{Z}_\ell)\big) \xrightarrow{\sim} \Lambda/\ell^k \Lambda.$$

Proof.

(i). Suppose first that $\overline{\eta}$ is such a $K_\ell(N)$-orbit. Define Λ by choosing $\eta \in \overline{\eta}$ and setting $\Lambda = \eta(\Lambda_0 \otimes \mathbb{Z}_\ell)$. Since η preserves the pairing up to some $\mathbb{Z}_\ell^\times$-multiple and $\Lambda_0 \otimes \mathbb{Z}_\ell$ is self-dual, it follows that Λ is self-dual. As $g(\Lambda_0 \otimes \mathbb{Z}_\ell) = \Lambda_0 \otimes \mathbb{Z}_\ell$ for all $g \in K_\ell(N)$, we have Λ is well-defined.

Conversely, let $\Lambda \subset V_\ell(A_s)$ be a self-dual lattice. Then we may choose a symplectic basis of Λ. The choice of basis of Λ gives a symplectic isomorphism $\eta : V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$ sending $\Lambda_0 \otimes \mathbb{Z}_\ell$ onto Λ. While the isomorphism η depends on the choice of basis, the $K_\ell(N)$-orbit $\{\eta \circ g : g \in K_\ell(N)\}$ does not.

(ii). Suppose now that $\overline{\eta}$ is a $K_\ell(N)$-orbit with $N = a\ell^k$ and $\ell \nmid a$. Choose $\eta \in \overline{\eta}$ and set $\Lambda = \eta(\Lambda_0 \otimes \mathbb{Z}_\ell)$ as before. Since $(g-1)(\Lambda_0 \otimes \mathbb{Z}_\ell) \subset N \cdot (\Lambda_0 \otimes \mathbb{Z}_\ell)$, for all possible choices of $\eta \in \overline{\eta}$ the induced isomorphism $\Lambda_0 \otimes \mathbb{Z}_\ell/(\ell^k \cdot \Lambda_0 \otimes \mathbb{Z}_\ell) \xrightarrow{\sim} \Lambda/\ell^k\Lambda$ is fixed.

Conversely, fix a self-dual $\Lambda \subset V_\ell(A_s)$ and an isomorphism $\Lambda_0 \otimes \mathbb{Z}_\ell/\big(\ell^k \cdot (\Lambda_0 \otimes \mathbb{Z}_\ell)\big) \xrightarrow{\sim} \Lambda/\ell^k\Lambda$. As in (i), the choice of a symplectic basis of Λ determines an isomorphism $V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$. We choose a symplectic basis so that the induced map $V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$ extends the fixed isomorphism $\Lambda_0 \otimes \mathbb{Z}_\ell/(\ell^k \cdot \Lambda_0 \otimes \mathbb{Z}_\ell) \xrightarrow{\sim} \Lambda/\ell^k\Lambda$. This is well-defined up to an element of $K_\ell(N)$.

$\square$

Proposition 2.4.4. *The functor $\mathcal{A}_{n,1,N}$ on the category of $\mathbb{Z}_p$-schemes is isomorphic to the following functor. For a $\mathbb{Z}_p$-scheme S, let $\mathcal{A}'_{n,1,N}(S)$ be the set of all tuples $(A, \lambda, \overline{\eta})$ up to isomorphism where*

- *A is an abelian scheme of dimension n over S;*
- *$\lambda : A \to \hat{A}$ is a polarization which is also a $\mathbb{Z}_{(p)}$-isogeny; and*
- *$\overline{\eta}$ is a $K(N)$-orbit of symplectic similitudes $\eta : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p)$.*

We furthermore require that there be some representative $(A, \lambda, \overline{\eta})$ with λ a principal polarization.

Here an isomorphism $f : (A, \lambda, \overline{\eta}) \to (A', \lambda', \overline{\eta}')$ between two objects in $\mathcal{A}'_{n,1,N}(S)$ consists of *a $\mathbb{Z}_{(p)}$-isogeny $f : A \to A'$ such that*

- *there exists a locally constant function r with values in $\mathbb{Z}_{(p)}^\times$ such that $\lambda = r(f^\vee \circ \lambda' \circ f)$; and*
- *$H_1(f) \circ \overline{\eta} = \overline{\eta}'$.*

Proof. We start by defining a natural transformation $\Phi : \mathcal{A}_{n,1,N} \to \mathcal{A}'_{n,1,N}$. Let (A, λ, α) be a representative of an element of $\mathcal{A}_{n,1,N}(S)$. Then $\alpha : (\mathbb{Z}/N\mathbb{Z})^{2n} \xrightarrow{\sim} A[N]$ induces an isomorphism $\alpha_s : (\mathbb{Z}/N\mathbb{Z})^{2n} \xrightarrow{\sim} A_s[N]$ that is invariant under the action of $\pi_1(S, s)$. To define $\overline{\eta}$, we need to define it at every place $\ell \neq p$, i.e. a $K_\ell(N)$-orbit of symplectic similitudes $V \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(A_s)$ that is invariant under the action of $\pi_1(S, s)$. By Proposition 2.4.3, for $\ell \nmid N$ such a $K_\ell(N)$-orbit is given by the self-dual lattice $H_1(A_s, \mathbb{Z}_\ell)$. Note this lattice is also fixed under the action of $\pi_1(S, s)$. For $\ell \mid N$ with $N = a\ell^k$ and $\ell \nmid a$, a $K_\ell(N)$-orbit of symplectic similitudes is given by $T_\ell(A_s)$ and the isomorphism $(\mathbb{Z}/\ell^k\mathbb{Z})^{2n} \xrightarrow{\sim} A_s[\ell^k]$ which is induced from α.

Suppose that the S-valued points (A, λ, α) and (A', λ', α') of $\mathcal{A}_{n,1,N}$ are isomorphic. Then there is an isomorphism $f : A \to A'$ such that $\lambda = f^\vee \circ \lambda' \circ f$ and $\alpha' = f \circ \alpha$. Hence f also serves as an isomorphism between $(A, \lambda, \overline{\eta})$ and $(A', \lambda', \overline{\eta}')$ constructed as above.

Suppose now that two objects (A, λ, α) and (A', λ', α') of $\mathcal{A}_{n,1,N}(S)$ are sent to isomorphic objects $(A, \lambda, \overline{\eta})$ and $(A', \lambda', \overline{\eta}')$ of $\mathcal{A}'_{n,1,N}(S)$. We may assume that λ and λ' are principal polarizations. Let f be an isomorphism between these two objects, i.e. a $\mathbb{Z}_{(p)}$-isogeny respecting the polarization and K^p-level structures. Then since the objects $(A, \lambda, \overline{\eta})$ and $(A', \lambda', \overline{\eta}')$ arise from objects of $\mathcal{A}_{n,1,N}(S)$, $H_1(f)$ maps $H_1(A_s, \hat{\mathbb{Z}}^{(p)})$ isomorphically onto $H_1(A'_s, \hat{\mathbb{Z}}^{(p)})$. Therefore f must be an isomorphism. Now $\lambda = r \cdot f^\vee \circ \lambda' \circ f$ for some locally constant function r taking values in $\mathbb{Z}_{(p)}^\times$. But since λ, λ', and f are isomorphisms, it must be that

$r = 1$. That $\alpha' = f \circ \alpha$ follows from Proposition 2.4.3. Therefore f induces an isomorphism between the objects (A, λ, α) and (A', λ', α') of $\mathcal{A}_{n,1,N}(S)$.

Now let $(A, \lambda, \overline{\eta})$ be a representative of some element in $\mathcal{A}'_{n,1,N}(S)$. Then we must find an object in the same isomorphism class of $(A, \lambda, \overline{\eta})$ that arises from some object of $\mathcal{A}_{n,1,N}(S)$ via the functor constructed above. The K^p-level structure provides a self-dual $\hat{\mathbb{Z}}^{(p)}$-lattice $\Lambda \subset H_1(A_s, \mathbb{A}_f^p)$ invariant under the action of $\pi_1(S, s)$. From Proposition 2.4.2 we can find an abelian scheme B/S, unique up isomorphism (of abelian schemes), and a $\mathbb{Z}_{(p)}$-isogeny $f : A \to B$ such that $\Lambda = H_1(B_s, \hat{\mathbb{Z}}^{(p)})$. We claim that B suffices, and to show this it remains to equip B with a principal polarization λ' and K^p-level structure $\overline{\eta}'$ so that $f : A \to B$ induces an isomorphism of objects in $\mathcal{A}'_{n,1,N}(S)$ and $(B, \lambda', \overline{\eta}')$ arises from an object of $\mathcal{A}_{n,1,N}(S)$. To equip B with a principal polarization we first define the $\mathbb{Z}_{(p)}$-isogeny $\lambda' = (f^\vee)^{-1} \circ \lambda \circ f^{-1}$. By construction, $H_1(B_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda$ and hence $H_1(\hat{B}_s, \hat{\mathbb{Z}}^{(p)}) = \Lambda$ since the lattice Λ is self-dual. Therefore λ' sends $H_1(B_s, \hat{\mathbb{Z}}^{(p)})$ onto $H_1(\hat{B}_s, \hat{\mathbb{Z}}^{(p)})$ and hence λ' must be an isomorphism. We also define $\overline{\eta}' = H_1(f) \circ \overline{\eta}$. With $\Lambda = H_1(B_s, \hat{\mathbb{Z}}^{(p)})$ we have the canonical identification $\Lambda / N\Lambda = B_s[N]$. Thus $(B, \lambda', \overline{\eta}')$ induces (B, λ', α') as required. $\square$

Remark 2.4.5. We will henceforth use either description of the functor $\mathcal{A}_{n,1,N}$.

As an intermediate step in proving the representability of $\mathcal{A}_0^{\mathrm{GL}}$ and $\mathcal{A}_0^{\mathrm{GSp}}$, we will show that the moduli problem defined below is representable by a quasi-projective scheme.

Definition 2.4.6. [Kot2, Section 5] Let $\mathcal{A}_{\mathrm{hyp}}$ denote the the following functor on schemes over $\mathrm{Spec}(\mathbb{Z}_p)$. For a $\mathbb{Z}_p$-scheme S, let $\mathcal{A}_{\mathrm{hyp}}(S)$ be the set of all tuples $(A, \lambda, i, \overline{\eta})$ up to isomorphism, where

- A is an abelian scheme of dimension n over S;

- $\lambda : A \to \hat{A}$ is a $\mathbb{Z}_{(p)}$-isogeny, which at every geometric point s of S is a polarization of A;
- $i : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ a homomorphism;
- $\det_S(b|\mathrm{Lie}A) = \det_{E'}(b|V^+)$ for all $b \in \mathcal{O}_B$; and
- $\overline{\eta}$ is a $K(N)$-orbit of symplectic similitudes $V \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p)$.

We furthermore require that there be some representative $(A, \lambda, i, \overline{\eta})$ with λ a principal polarization.

An isomorphism $f : (A, \lambda, i, \overline{\eta}) \to (A', \lambda', i', \overline{\eta}')$ between two objects in $\mathcal{A}_{\mathrm{hyp}}(S)$ consists of a $\mathbb{Z}_{(p)}$-isogeny $f : A \to A'$ such that

- there exists a locally constant function r with values in $\mathbb{Z}_{(p)}^\times$ such that $\lambda = r(f^\vee \circ \lambda' \circ f)$;
- the morphism $\mathrm{End}(A) \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A') \otimes \mathbb{Z}_{(p)}$ induced by f, which is again denoted by f, is such that $f \circ i(b) = i'(b)$ for all $b \in \mathcal{O}_B$; and
- $H_1(f) \circ \overline{\eta} = \overline{\eta}'$.

Proposition 2.4.7. *The functor $\mathcal{A}_{hyp}$ is representable by a quasi-projective scheme.*

Proof. We have the forgetful functor $\mathcal{A}_{\mathrm{hyp}} \to \mathcal{A}_{n,1,N}$ by forgetting the $\mathcal{O}_B$-action. We will show that $\mathcal{A}_{\mathrm{hyp}}$ is relatively representable over $\mathcal{A}_{n,1,N}$ by a projective scheme.

Fix a $\mathbb{Z}_p$-scheme S and a morphism $S \to \mathcal{A}_{n,1,N}$ inducing $(A, \lambda, \overline{\eta})$. Consider the functor $\mathcal{A}_{\mathrm{hyp}} \times_{\mathcal{A}_{n,1,N}} S$. By [Hid, Section 6.1] the functor on S-schemes sending $T \to \mathrm{End}(A_T) \otimes \mathbb{Z}_{(p)}$ is representable by a union of projective schemes over S, denote the scheme representing this functor by $\mathcal{E}$. Now the polarization λ induces the Rosati involution on $\mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$, and this in turn gives an involution $r : \mathcal{E} \to \mathcal{E}$. Choose a set of generators $\{a_1, \ldots, a_{2m}\}$ of $\mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ as a $\mathbb{Z}_{(p)}$-algebra such that $a_{m+i} = \iota(a_i)$ for $1 \leq i \leq m$ and define the closed subscheme Z

of $\mathcal{E}^{2m}$ as follows. For an S-scheme T, a point $(x_1, \ldots, x_{2m}) \in \mathcal{E}^{2m}(T)$ is in Z if and only if any relationship satisfied by $(a_1, \ldots, a_{2m})$ is also satisfied by $(x_1, \ldots, x_{2m})$ and $r(x_i) = x_{m+i}$. Note that any morphism $T \to \mathcal{E}^{2m}$ induced by a T-valued point of $\mathcal{A}_{\text{hyp}} \times_{\mathcal{A}_{n,1,N}} S$ factors through Z.

With this, $A_Z = A \times_S Z$ is an abelian scheme over Z and the morphism $A_Z \to Z$ induces $\{x_1, \ldots, x_{2m}\} \subset \text{End}(A_Z) \otimes \mathbb{Z}_{(p)}$ giving the algebra homomorphism $\mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \text{End}(A_Z) \otimes \mathbb{Z}_{(p)}$ by sending a_i to x_i. By the conditions defining Z, this homomorphism is compatible with the Rosati involution.

By Proposition 2.1.7, the locus where there is a K^p-level structure is a union of connected components of Z. Now consider the determinant condition. With $\text{Lie}(A_Z)$ locally free, we have $\det_{\text{Lie}(A_Z)} \in \Gamma(Z, \mathcal{O}_Z)[X_1, \ldots, X_{2m}]$. The condition $\det_{\text{Lie}(A_Z)} = \det_{E'}(V^+)$ is an equality of global sections of Z. Thus enforcing both of these conditions gives a closed subscheme $X \subset Z$. Again, any morphism $T \to \mathcal{E}^{2m}$ induced by a T-valued point of $\mathcal{A}_{\text{hyp}} \times_{\mathcal{A}_{n,1,N}} S$ factors through X.

Therefore X, with the universal abelian scheme A_X, represents the functor $\mathcal{A}_{\text{hyp}} \times_{\mathcal{A}_{n,1,N}} S$. Since X is projective over S, we have that the scheme representing $\mathcal{A}_{\text{hyp}}$ is quasi-projective.

□

Lemma 2.4.8. *Let S be a $\mathbb{Z}_p$-scheme. Any object $(A, \lambda, i, \overline{\eta})$ of $\mathcal{A}_{hyp}(S)$ only has the trivial automorphism.*

Proof. An automorphism of $(A, \lambda, i, \overline{\eta})$ is given by $f : (A, \lambda, i, \overline{\eta}) \to (A', \lambda', i', \overline{\eta}')$ where $(A', \lambda', i', \overline{\eta}')$ is in the same isomorphism class as $(A, \lambda, i, \overline{\eta})$. With the forgetful morphism of functors $\mathcal{A}_{\text{hyp}} \to \mathcal{A}_{n,1,N}$, the automorphism f induces an automorphism of the induced objects of $\mathcal{A}_{n,1,N}(S)$. By [Ser2], this induced automorphism must be the identity. It follows that $f : A \to A'$ is the identity. $\square$

Proposition 2.4.9. *$\mathcal{A}_0^{GL}$ is representable by a quasi-projective scheme.*

Proof. One has the forgetful functor $\mathcal{A}_0^{\mathrm{GL}} \to \mathcal{A}_{\mathrm{hyp}}$ and hence it suffices to show that $\mathcal{A}_0^{\mathrm{GL}}$ is relatively representable over $\mathcal{A}_{\mathrm{hyp}}$.

Fix $S \to \mathcal{A}_{\mathrm{hyp}}$ inducing $(A_0, \lambda_0, \overline{\eta})$ and consider the functor $\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S$. Note that we may choose the representative $(A_0, \lambda_0, \overline{\eta})$ such that λ_0 is a principal polarization.

Let $FH = FH^n_{A_0}$ denote the Flag Hilbert scheme with respect to the projective scheme A_0 and the Hilbert polynomials $\left\{p^{2n}, p^{4n}, \ldots, p^{2n^2}\right\}$ [Ser1, Section 4.5]. This is a projective scheme representing the following functor: to give a T valued point of FH is to give a chain $H_1 \subset H_2 \subset \cdots \subset H_n$ of T-flat closed subschemes of A_T such that H_i has Hilbert polynomial p^{2ni}. Let $Z_\bullet = (Z_i)_i$ denote the universal object over FH, so in particular for each i we have $Z_i \hookrightarrow A_0 \times_S FH$.

Given an object of $(\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$, we claim that there is an induced morphism $T \to FH$. Choosing a representative $(A_\bullet, \overline{\lambda}, i, \overline{\eta})$, we have the chain of finite flat subgroup schemes $H_i = \ker(\alpha^i : A_0 \to A_i)$. We must show that this chain is well-defined, i.e. that it does not depend on the choice of representative. To see this, recall that an isomorphism of objects $f : (A_\bullet, \overline{\lambda}, i, \overline{\eta}) \to (A'_\bullet, \overline{\lambda}', i', \overline{\eta}')$ of $(\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$ is given by a collection $f_i : A_i \to A'_i$ of $\mathbb{Z}_{(p)}$-isogenies such that, in particular, the diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{\alpha^i} & A_i \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_i} \\ A_0 & \xrightarrow{(\alpha')^i} & A_i' \end{array}$$

commutes. By the lemma above, $f_0 : A_0 \to A_0$ must be the identity. Furthermore $f_i : A_i \to A_i'$, if it exists, is determined uniquely by f_0. We may identify $A_i = A_0/H_i$, and likewise $A_i' = A_0'/H_i'$. Then we see that $f_i : A_0/H_i \to A_0'/H_i'$ must descend from $f_0 : A_0 \to A_0'$. Thus $f_0(H_i) = H_i'$ and so the chain is well-defined. Therefore an object of $(\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$ induces a morphism $T \to FH$.

We define the closed subscheme $FH_1 \subset FH$ as follows. A geometric point x of FH is a point of FH_1 if and only if for all i, the morphisms $x \xrightarrow{\varepsilon} A_{0,x}$, $Z_{i,x} \times_{FH} Z_{i,x} \xrightarrow{\mu} A_{0,x}$, and $Z_{i,x} \xrightarrow{\iota} A_{0,x}$ factor through $Z_{i,x} \hookrightarrow A_{0,x}$. Here $Z_{i,x}$ and $A_{0,x}$ denote the geometric fiber with respect to x and ε, μ, and ι are the restrictions of the identity, multiplication, and inverse of $A_{0,x}$ respectively. Any morphism $T \to FH$ induced by an object of $(\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$ factors through FH_1.

Now define the closed scheme $FH_2 \subset FH_1$ as follows. A geometric point x of FH_1 is a point of FH_2 if and only if $Z_{n,x} = A_{0,x}[p]$ as closed subschemes of $A_{0,x}$. Any morphism $T \to FH_1$ induced by an object of $(\mathcal{A}_0^{\mathrm{GL}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$ factors through FH_2.

Define $Z_i' = Z_i \times_{FH} FH_2$ for $1 \leq i \leq n-1$. Then each Z_i' is a flat subgroup scheme of $A_0 \times_S FH_2$ and moreover each Z_i' is finite over FH_2. So any morphism $T \to FH_2$ induces a chain

$$A_{0,T} \xrightarrow{\alpha_0} A_{0,T}/H_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} A_{0,T}/H_{n-1} \xrightarrow{\alpha_{n-1}} A_{0,T}/A_0[p]$$

where $A_{0,T} = A_0 \times_S T$ and each morphism is the canonical quotient map and hence an isogeny of degree p^{2n}. By abuse of notation write A_0 for $A_{0,T}$ and set $A_i = A_0/H_i$ for $0 \leq i \leq n-1$.

Since $H_{n-1} \subset A_0[p]$, we can define $A_n = A_0$ with the map $A_{n-1} \xrightarrow{\alpha_{n-1}} A_n$ being induced by $[p] : A_0 \to A_0$. We then take the periodicity isomorphism $\theta_p : A_n \to A_0$ to be the identity. Now extend this chain periodically to be an infinite chain $(A_i)_i$ for $i \in \mathbb{Z}$.

It remains to enforce the condition that there exists a $\mathbb{Q}$-homogeneous class of principal polarizations. Consider the diagram

$$\begin{tikzcd}
A_{-i} \arrow[r, "\alpha^i"] & A_0 \arrow[r, "\alpha^i"] \arrow[d, "\lambda_0"] & A_i \\
\hat{A}_{-i} & \hat{A}_0 \arrow[l, "(\alpha^\vee)^i"'] &
\end{tikzcd}$$

and write $A_i = A_{-i}/H_i$ where we now denote by H_i a finite flat subgroup scheme of A_{-i}. We claim that an isomorphism $\lambda_i : A_i \xrightarrow{\sim} \hat{A}_{-i}$ making the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha^i"] \arrow[d, "\lambda_0"'] & A_i \arrow[d, "\lambda_i"] \\
\hat{A}_0 \arrow[r, "(\alpha^\vee)^i"] & \hat{A}_{-i}
\end{tikzcd}$$

commute, if it exists, must be descended from the map $A_{-i} \xrightarrow{\alpha^i} A_0 \xrightarrow{\lambda_0} \hat{A}_0 \xrightarrow{(\alpha^\vee)^i} \hat{A}_{-i}$. To see this, denoting $\theta_p : A_n \xrightarrow{\sim} A_0$ the periodicity isomorphism and $[p]$ multiplication by p, we have

$$\begin{aligned}
\left((\alpha^\vee)^i \circ \lambda_0 \circ \theta_p \circ \alpha^{n-i}\right) \circ \alpha^{2i} &= (\alpha^\vee)^i \circ \lambda_0 \circ [p] \circ \alpha^i \\
&= \lambda_i \circ \alpha^i \circ [p] \circ \alpha^i
\end{aligned}$$

Since $[p]$ is an isogeny, this implies that $(\alpha^\vee)^i \circ \lambda_0 \circ \alpha^i = \lambda_i \circ \alpha^{2i}$ which is precisely to say that λ_i descends from $(\alpha^\vee)^i \circ \lambda_0 \circ \alpha^i$.

The map $(\alpha^\vee)^i \circ \lambda_0 \circ \alpha^i$ will descend to an isomorphism $A_i \xrightarrow{\sim} \hat{A}_{-i}$ if and only if its kernel is precisely H_i. Enforcing this condition for each i, we see that there exists a closed subscheme FH_3 of FH_2 such that a morphism $T \to FH_2$ will factor through FH_3 if and only if the corresponding point of $T \to FH_2$ may be equipped with a $\mathbb{Q}$-homogeneous class of principal polarizations. Therefore FH_3 represents $\mathcal{A}_0^{\mathrm{GL}}$. With FH_3 a projective over S, $\mathcal{A}_0^{\mathrm{GL}}$ is quasi-projective. $\square$

Proposition 2.4.10. *$\mathcal{A}_0^{GSp}$ is representably by a scheme.*

Proof. As in the previous theorem, one has the forgetful functor $\mathcal{A}_0^{\mathrm{GSp}} \to \mathcal{A}_{\mathrm{hyp}}$ and hence it suffices to show that $\mathcal{A}_0^{\mathrm{GSp}}$ is relatively representable over $\mathcal{A}_{\mathrm{hyp}}$. Fix $S \to \mathcal{A}_{\mathrm{hyp}}$ inducing $(A_0, \lambda_0, \overline{\eta})$ and consider the functor $\mathcal{A}_0^{\mathrm{GSp}} \times_{\mathcal{A}_{\mathrm{hyp}}} S$. Note that we may choose the representative $(A_0, \lambda_0, \overline{\eta})$ so that λ_0 is a principal polarization. As in the above theorem, an object of $(\mathcal{A}_0^{\mathrm{GSp}} \times_{\mathcal{A}_{\mathrm{hyp}}} S)(T)$ induces a morphism $T \to FH^n_{A_0}$ where we now take $\{p, p^2, \ldots, p^n\}$ to be the collection of Hilbert polynomials. There is a closed subscheme Z of $FH^n_{A_0}$ such that $T \to FH^n_{A_0}$ factors through Z if and only if the corresponding chain of flat subschemes forms a chain of finite flat subgroup schemes of A_T contained in $A_T[p]$.

By abuse of notation denote $A_{0,T}$ by A_0 and set $A_i = A_0/H_i$ for $1 \leq i \leq n-1$. Then we have the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha"] \arrow[d, "\lambda_0"'] & A_1 \arrow[r, "\alpha"] & \ldots \arrow[r, "\alpha"] & A_{n-1} \arrow[d, dashed, "\lambda_n"] \\
\hat{A}_0 & \hat{A}_1 \arrow[l, "\alpha^\vee"'] & \ldots \arrow[l, "\alpha^\vee"'] & \hat{A}_{n-1} \arrow[l, "\alpha^\vee"']
\end{tikzcd}$$

It remains to enforce the condition that there exists principal polarizations $\lambda_n : A_n \to \hat{A}_n$ making the loop in the diagram multiplication by p. Such a polarization exists if and only if the kernel $H_{n-1} = \ker(\alpha^{n-1})$ is totally isotropic with respect to the Weil pairing induced by λ_0. Thus there exists a closed subscheme of $X \subset Z$ such that $T \to Z$ factors through X if and only if there exists such a λ_n as above.

Therefore Z represents $\mathcal{A}_0^{\mathrm{GSp}}$. $\square$

Chapter 3

Stratification of $\mathcal{A}_0$

In this chapter we will describe the following result.

Theorem 3.1 ([Gör1],[Gör2])**.** *Let $\mathcal{A}_0$ and M^{loc} respectively denote the integral model and local model in either the unitary or symplectic case. Then $M^{loc} \otimes \mathbb{F}_p$ can be embedded into an affine flag variety associated with SL_n and Sp_{2n} respectively. Each affine flag variety is stratified by Schubert cells which induces a stratification of $M^{loc} \otimes \mathbb{F}_p$. There is a unique stratum of $M^{loc} \otimes \mathbb{F}_p$ which consists of a single closed point, called the "worst point". Any open subscheme of M^{loc} containing the worst point is an étale local model for $\mathcal{A}_0$.*

Following loc. cit., we will also make a specific choice of open subscheme $U_0 \subset M^{\mathrm{loc}}$ (denoted U_τ in loc. cit.) containing the worst point and give an explicit presentation of U_0.

Consider the local model M^{loc} associated with an integral Shimura datum of Iwahori-type in either the unitary or symplectic case. As before, let $G = \mathbb{G}_{\mathbb{Q}_p}$ where $\mathbb{G}$ is the reductive group defined over $\mathbb{Q}$ in the Shimura datum. In the unitary case, recall that the assumptions on p gave the splitting of $G = \prod_j G_j$ which in turn makes the local model (after some finite base extension) isomorphic to a product of local models in the case $F^+ = \mathbb{Q}$ (see Proposition 2.2.2). We thus assume that in the unitary case, $F^+ = \mathbb{Q}$.

Throughout this chapter, we will fix notation in such a way that we may describe the stratifications in both the unitary and the symplectic cases simultaneously. In the unitary case we have $G = \mathrm{GL}_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$ with $\mu = (0^{n-r}, (-1)^r)$ and in the symplectic case we have $G = \mathrm{GSp}_{n,\mathbb{Q}_p}$ where $n = 2r$ for $r \in \mathbb{N}$ with $\mu = (0^r, (-1)^r)$. Thus in either case, $\mu = (0^{n-r}, (-1)^r)$. Note that this notation differs from slightly from Section 2.3 where we had $\mathrm{GSp}_{2n,\mathbb{Q}_p}$.

Let k be an algebraically closure of $\mathbb{F}_p$ and let $\mathcal{F}$ denote the affine flag variety over k associated with SL_n in the unitary case and Sp_n in the symplectic case. We identify $\mathcal{F}_{\mathrm{SL}}$ with the space of complete $(n-r)$-special lattice chains and likewise $\mathcal{F}_{\mathrm{Sp}}$ with the space of complete self-dual $(n-r)$-special lattice chains (see Propositions A.3.5 and A.3.7). Let R be a k-algebra and set

$$\lambda_i = (t^{-1}R[[t]])^i \oplus R[[t]]^{n-i}, \quad 0 \leq i \leq n-1$$

noting that $\lambda_i^{\perp} = \lambda_{-i}$.

Proposition 3.2. *[Gör1, Proposition 3.5],[Gör2, Proposition 3.2]*

(i). Let $(\mathcal{F}_i)_i$ be an R-valued point of $M^{loc}_{GL} \otimes k$ so that $\mathcal{F}_i$ is a subspace of $\Lambda_{i,R} = R^n \cong \lambda_i / t\lambda_i$. Let $\mathcal{L}_i$ be the inverse image of $\mathcal{F}_i$ under the canonical projection $\lambda_i \to \lambda_i / t\lambda_i$. Then $(\mathcal{L}_i)_i$ is a complete $(n-r)$-special lattice chain and the map

$$M^{loc}_{GL} \otimes k \to \mathcal{F}_{SL} \quad \textit{sending} \quad (\mathcal{F}_i)_i \to (\mathcal{L}_i)_i$$

is a closed immersion. The image is precisely the lattice chains $(\mathcal{L}_i)_i \in \mathcal{F}_{SL}$ such that $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$.

(ii). Let $(\mathcal{F}_i)_i$ be an R-valued point of $M^{loc}_{GSp} \otimes k$ inducing $(\mathcal{L}_i)_i$ as in (i). Then $(\mathcal{L}_i)_i$ is a complete self-dual r-special lattice chain and the map $M^{loc}_{GSp} \otimes k \to \mathcal{F}_{Sp}$ is a closed immersion. The image is precisely the lattice chains $(\mathcal{L}_i)_i \in \mathcal{F}_{Sp}$ such that $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$.

Proof.

(i). That $(\mathcal{L}_i)_i$ forms a complete lattice chain is immediate from the definition of $(\mathcal{F}_i)_i$. We now show that $\mathcal{L}_0$ is $(n-r)$-special. By localization, we can assume that $\mathcal{F}_0 \subset R^n$ is a direct summand. Without loss of generality, suppose $\mathcal{F}_0 = R^r \hookrightarrow R^r \oplus R^{n-r} = R^n$. Then under the map $\pi_0 : \lambda_0 \to \lambda_0/t\lambda_0 \cong R^n$ we have that $\pi_0^{-1}(\mathcal{F}_0)$ is the $R[[t]]$-lattice generated by the elements of λ_0 where the last $n-r$ coordinates are contained in $tR[[t]]$. As $\mathcal{L}_0 \to \wedge^n \mathcal{L}_0$ is given by the determinant, it follows that $\wedge^n \mathcal{L}_i = t^{n-r} R[[t]]$ as required.

By definition, for $(\mathcal{F}_i) \in M^{\mathrm{loc}}_{\mathrm{GL}} \otimes k$ the induced lattice chain $(\mathcal{L}_i)$ is such that $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$ for all i. Now suppose $(\mathcal{L}_i)_i \in \mathcal{F}_{\mathrm{SL}}$ such that $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$ for all i. Then the image of $\mathcal{L}_i$ under the map $\lambda_i \to \lambda_i/t\lambda_i \xrightarrow{\sim} R^n$ is a locally free R-module of rank r, denote it by $\mathcal{F}_i$. Thus $(\mathcal{F}_i)_i$ is a point of $M^{\mathrm{loc}}_{\mathrm{GL}} \otimes k$ that maps onto $(\mathcal{L}_i)_i$ as required.

(ii). Noting that $n - r = r$ here, all of the claims will follow from (i) by showing that enforcing the duality condition on $(\mathcal{F}_i)_i$ is equivalent to enforcing the condition that $(\mathcal{L}_i)_i$ is self-dual. Since $(\mathcal{F}_i)_i$ and hence $(\mathcal{L}_i)_i$ are periodic, it suffices to show the claim on the indices $i = 0, \ldots, n-1$. We claim that in fact the duality condition is equivalent to $\mathcal{L}_i^{\perp} = \mathcal{L}_{n-i}$ for $0 \leq i \leq n-1$.

Since $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$, certainly we have $\lambda_i^\perp \subset \mathcal{L}_i^\perp \subset t^{-1}\lambda_i^\perp$. Using $\lambda_i^\perp = t\lambda_{n-i}$, we have $t\lambda_{n-i} \subset \mathcal{L}_i^\perp \subset \lambda_{n-i}$. This implies that $\mathcal{L}_{n-i} \subset \mathcal{L}_i^\perp$ if and only if $\mathcal{L}_{n-i} = \mathcal{L}_i^\perp$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}_i \times \mathcal{L}_{-i} & \longrightarrow & R[[t]] \\ \downarrow & & \downarrow \\ \mathcal{F}_{i,R} \times \mathcal{F}_{-i,R} & \longrightarrow & R \end{array}$$

where the horizontal map on the bottom is the restriction of $\Lambda_{i,R} \times \Lambda_{-i,R} \to R$ induced by the duality $\Lambda_i^\perp = \Lambda_{-i}$.

The duality condition is equivalent to the pairing $\mathcal{L}_i \times \mathcal{L}_{-i} \to R[[t]]$ having image in $tR[[t]]$, and this in turn is equivalent to $t^{-1}\mathcal{L}_{-i} \subset \mathcal{L}_i^\perp$. Since $(\mathcal{L}_i)_i$ is periodic, $\mathcal{L}_{n-i} = \mathcal{L}_i^\perp$ and the result follows.

□

As a result of the proposition, we will frequently identify $M^{\mathrm{loc}} \otimes k$ with its image in $\mathcal{F}$. Recall the description of the stratification of $\mathcal{F}$ (see Section A.3 for details). Set

$$\omega = (\omega_0, \ldots, \omega_{n-1}) \quad \text{where} \quad \omega_i = (1^i, 0^{n-i}).$$

$$\tau = \big((1^r, 0^{n-r}), (1^{r+1}, 0^{n-r-1}), \ldots, (2^{r-2}, 1^{n-r}), (2^{r-1}, 1^{n-r+1})\big)$$

With τ as a base alcove, the extended affine Weyl group $\widetilde{W}$ acts simply transitively on the set of alcoves and hence we may identify $\widetilde{W}$ with the set of alcoves. Thus given $(\mathcal{L}_i)_i \in \mathcal{F}$, there is a uniquely determined element $w \in \widetilde{W}$ and an element b in the Iwahori subgroup I, such that $(\mathcal{L}_i)_i = bw\tau$. We say that $w\tau$ is the alcove associated to the point $(\mathcal{F}_i)_i$. This gives a stratification

$$\mathcal{F} = \bigcup_{w \in W^{\mathrm{aff}}} S_{w\tau}, \qquad \text{where } S_{w\tau} = IwI/I.$$

From the description of $M^{\mathrm{loc}} \otimes k$ in $\mathcal{F}$ in Proposition 3.2, we immediately have that $M^{\mathrm{loc}} \otimes k$ is invariant under the action of the Iwahori subgroup I, and hence that $M^{\mathrm{loc}} \otimes k$ is set-theoretically a disjoint union of Schubert cells $S_{w\tau}$.

Now let $(\mathcal{F}_i)_i \in M^{\mathrm{loc}}(k)$ and $(\mathcal{L}_i)_i \in \mathcal{F}$ the corresponding point of the affine flag variety. Let $x = (x_0, \ldots, x_{n-1})$ be an alcove and suppose $(\mathcal{L}_i)_i \in S_x$. Then

$$\mathcal{L}_i = b \cdot \begin{pmatrix} t^{-x_i(1)+1} & & & \\ & t^{-x_i(2)+1} & & \\ & & \ddots & \\ & & & t^{-x_i(n)+1} \end{pmatrix}$$

for some $b \in I$. Here we are writing the matrix on the right with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $k((t))^n$ and identifying this matrix with the $k[[t]]$-submodule of $k((t))^n$ given by its column space.

Remark 3.3. Note that for an alcove x such that $0 \leq x_i(j) - \omega_i(j) \leq 1$, the quotient $\lambda_i/\mathcal{L}_i$ is generated by the e_j^i such that $x_i(j) = \omega_i(j)$. Indeed, we certainly have that the claim holds for $b = 1$. Now let $b \in I$ be arbitrary. Since I stabilizes the standard lattice chain, we have $\lambda_i/\mathcal{L}_i$ is generated by $\left\{be_j^i : x_i(j) = \omega_i(j)\right\}$. Of course in the quotient $\lambda_i/\mathcal{L}_i$, the subspace generated by this set is the same as that generated by $\left\{e_j^i : x_i(j) = \omega_i(j)\right\}$.

Definition 3.4. Let $x = (x_0, \ldots, x_{n-1})$ be an alcove. Then

(i). The number $\sum_j x_i(j) - \sum_j \omega_i(j)$ is independent of i and is called the size of x.

(ii). We say that x is minuscule if $0 \leq x_i(j) - \omega_i(j) \leq 1$ for all $0 \leq i \leq n-1$ and $1 \leq j \leq n$.

(iii). We say that x is μ-permissible if x is minuscule of size r.

We will denote the collection of μ-permissible alcoves by $\mathrm{Perm}(\mu)$.

Remark 3.5. τ is μ-permissible.

Proposition 3.6. *$S_x \cap (M^{loc} \otimes k) \neq \emptyset$ if and only if x is a μ-permissible alcove and hence, set-theoretically,*

$$M^{loc} \otimes k = \bigcup_{x \in Perm(\mu)} S_x.$$

Proof. For all $0 \leq i \leq n-1$ and $1 \leq j \leq n$, the condition $t\lambda_i \subset \mathcal{L}_i \subset \lambda_i$ is equivalent to the condition $0 \leq x_i(j) - \omega_i(j) \leq 1$. Thus we may assume that this is indeed the case. Now $(\mathcal{L}_i)_i$ is $(n-r)$-special is equivalent to $\lambda_i/\mathcal{L}_i$ has rank $n-r$. By Remark 3.3, this is in turn equivalent to $\#\{j : x_i(j) = \omega_i(j)\} = n - r$ for every $0 \leq i \leq n-1$, and hence equivalent to

$$\sum_{j=1}^{n} x_i(j) - \omega_i(j) = r.$$

$\square$

Definition 3.7. Let $x = (x_0, \ldots, x_{n-1})$ be a μ-permissible alcove. Define U_x to be the subset of $M^{\mathrm{loc}} \otimes k$ which consists of all the points $(\mathcal{F}_i)_i$ such that for all i, the quotient $\Lambda_i/\mathcal{F}_i$ is generated by those e_j^i with $\omega_i(j) = x_i(j)$.

Proposition 3.8. *Let $x = (x_0, \ldots, x_{n-1})$ be a μ-permissible alcove and U_x be as in the definition above. Then we have the following.*

(i). U_x is an open subscheme of $M^{loc} \otimes k$.

(ii). The stratum S_x is contained in U_x.

(iii). The irreducible components of $M^{loc} \otimes k$ are the closures of the S_x where x is an extreme alcove, i.e. $x = t_{w(\mu)}$ for some $w \in W$.

(iv). The stratum S_τ consists of only one point.

(v). For any open subscheme $U \subset M^{loc} \otimes k$ containing S_τ, U intersects every stratum.

Proof.

(i). Let $x \in \mathrm{Perm}(\mu)$, $(\mathcal{F}_i)_i \in M^{\mathrm{loc}}(k)$, and $(\mathcal{L}_i)_i$ the associated point of $\mathcal{F}$. For any fixed i, as shown in the proof of Proposition 3.6, the collection of e^i_j with $\omega_i(j) = x_i(j)$ consists of $n - r$ elements, and this set is a subset of a basis of λ_i. Since $\lambda_i/\mathcal{L}_i$ is free of rank $n - r$, in general for an arbitrary set $\{s_1, \ldots, s_{n-r}\} \subset \lambda_i$, the set $\{\overline{s}_1, \ldots, \overline{s}_{n-r}\}$ forms a basis of $\lambda_i/\mathcal{L}_i$ if and only if $\{s_1, \ldots, s_{n-r}\} \notin \mathcal{L}_i$ and $\{s_1, \ldots, s_{n-r}\}$ is part of a basis of λ_i.

Let $\{j_k\}_{k=1}^{n-r}$ be a collection of distinct integers with $1 \leq j_k \leq n$. Consider the collection

$$T = \left\{e^i_{j_k} : 0 \leq i \leq n-1,\ 1 \leq k \leq n-r\right\}.$$

By the above it suffices to show that the collection $(\mathcal{L}_i)_i \in M^{\mathrm{loc}}$ such that $e^i_{j_k} \notin \mathcal{L}_i$ for all $e^i_{j_k} \in T$ is open. This is the intersection of finitely many sets, each defined by the condition that for some fixed i and k, $e^i_{j_k} \notin \mathcal{L}_i$. As each such set is open, it follows that U_x is open.

(ii). This is Remark 3.3.

(iii). From [KR], the extreme alcoves are precisely the μ-permissible alcoves x such that $x \leq y$ implies $x = y$ for y a μ-permissible alcove. Recalling that the stratification has the property $S_x \subset \overline{S_y}$ if and only if $x \leq y$, the result follows.

(iv). With I the Iwahori subgroup of $LG(k)$, consider the action of I on $M^{\mathrm{loc}} \otimes k$ given by $I \times M^{\mathrm{loc}} \otimes k \to M^{\mathrm{loc}} \otimes k$. As S_τ is an I-orbit, for any $p \in S_\tau$ we have that the restriction $I \times \{p\} \to S_\tau$ is surjective. Furthermore, since the action of I on $M^{\mathrm{loc}} \otimes k$ is continuous and I is connected, we get that S_τ is connected. Finally, S_τ is zero dimensional since $\ell(\tau) = 0$ (Proposition A.3.9). Therefore S_τ consists of a single point.

(v). Let $U \subset M^{\mathrm{loc}} \otimes k$ be an open subscheme with $S_\tau \in U$. From Proposition A.3.9, as $\tau \leq x$ for all μ-permissible x, we have $S_\tau \in \overline{S_x}$. Hence U intersects $\overline{S_x}$ and thus U meets S_x.

□

Remark 3.9. Let U be an open subscheme of $M^{\mathrm{loc}} \otimes k$ with $S_\tau \in U$ and let x be μ-permissible. Then since U meets S_x and S_x is an I-orbit, we have that the image of $I \times U \to M^{\mathrm{loc}}$ contains all of S_x. As $M^{\mathrm{loc}} \otimes k$ is the disjoint union of S_x for $x \in \mathrm{Perm}(\mu)$, we have $I \times U \to M^{\mathrm{loc}}$ is surjective. It follows immediately that any open subscheme $U \subset M^{\mathrm{loc}}$ containing the point S_τ serves as an étale local model of $\mathcal{A}_0$. In particular, U_τ is a local model of $\mathcal{A}_0$.

As in [Gör1] and [Gör2], we now write down a presentation of U_τ, which we will henceforth denote by U_0. With M^{loc} being a closed subscheme of a product of Grassmannians, we represent a point of M^{loc} by giving $(\mathcal{F}_i)_{i=0}^{n-1}$ where each $\mathcal{F}_i$ is an r-dimensional subspace, and we represent $\mathcal{F}_i$ as the column space of the $n \times r$ matrix (a^i_{jk}) with respect to the basis $\left\{e^i_j\right\}$. It is then easy to check $(\mathcal{F}_i)_i \in U_0$ implies that the $r \times r$ minor given by rows $i+1$ to $r+i$ (taken cyclically, so row $n+1$ is row 1) of $\mathcal{F}_i$ is invertible for $0 \leq i < n$. As such, we require this submatrix to be the identity matrix. For example, $\mathcal{F}_0$ and $\mathcal{F}_1$ are represented by

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a^0_{11} & a^0_{12} & \dots & a^0_{1r} \\ \vdots & \vdots & & \vdots \\ a^0_{n-r,1} & a^0_{n-r,2} & \dots & a^0_{n-r,r} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a^1_{n-r,1} & a^1_{n-r,2} & \dots & a^1_{n-r,r} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a^1_{11} & a^1_{12} & \dots & a^1_{1r} \\ \vdots & \vdots & & \vdots \\ a^1_{n-r-1,1} & a^1_{n-r-1,2} & \dots & a^1_{n-r-1,r} \end{pmatrix}.$$

Note that by requiring the matrix to have a specific $r \times r$ submatrix which is the identity, all the entries of the matrix are uniquely determined by its column space. By abuse of notation, we will use $\mathcal{F}_i$ to denote both the subspace and the matrix representing it. To express the condition $\mathcal{F}_i$ is mapped into $\mathcal{F}_{i+1}$, we must have

$$\varphi_i(\mathcal{F}_i) = \mathcal{F}_{i+1} A_i$$

for some $r \times r$ matrix A_i. However $\mathcal{F}_{i+1}$ has an $r \times r$ submatrix which is given by the identity matrix, and so A_i is determined:

$$A_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a^i_{11} & a^i_{12} & \dots & a^i_{1r} \end{pmatrix}.$$

Proposition 3.10.

(i). $U_0^{GL} \cong \mathrm{Spec}(B_{GL})$ *with*

$$B_{GL} = \mathbb{Z}_p[a^i_{jk}; i = 0, \dots, n-1,\ j = 1, \dots, n,\ k = 1, \dots, r]/I$$

where I is the ideal generated by following two collections of relations. The first collection is given for $0 \le i \le n-1$ by the entries of the matrices

$$\begin{pmatrix} a^{i+1}_{11} & \dots & a^{i+1}_{1r} \\ \ddots & & \ddots \\ a^{i+1}_{n-r,1} & \dots & a^{i+1}_{n-r,r} \end{pmatrix} A_i - \begin{pmatrix} a^i_{21} & a^i_{22} & \dots & a^i_{2r} \\ \ddots & \ddots & & \ddots \\ a^i_{n-r,1} & a^i_{n-r,2} & \dots & a^i_{n-r,r} \\ p & 0 & \dots & 0 \end{pmatrix}.$$

The second collection is given by the entries of the matrices

$$A_{n-1}A_{n-2}\cdots A_0 - p\cdot Id, \qquad A_{n-2}\cdots A_0 A_{n-1} - p\cdot Id, \qquad \dots, \qquad A_0 A_{n-1}\cdots A_1 - p\cdot Id.$$

(ii). *[Gör2, Section 5]* $U_0^{GSp} \cong \mathrm{Spec}(B_{GSp})$ *with* $B_{GSp} = B_{GL}/J$ *where J is the ideal generated by*

$$a^{2n-i}_{jk} - \varepsilon_{jk} a^i_{n-k+1,n-j+1} \quad \text{with} \quad \varepsilon_{jk} = \begin{cases} 1 & j,k \le i \text{ or } j,k \ge i+1 \\ -1 & \text{otherwise} \end{cases}$$

for each $0 \le i \le n-1$.

Proof.

(i). The first collection of equations is equivalent to the requirement that $\varphi(\mathcal{F}_i) \subset \mathcal{F}_{i+1}$

and the second collection is equivalent to the compositions

$$\varphi_{n-1}\varphi_{n-2}\cdots\varphi_0, \qquad \varphi_{n-2}\varphi_{n-3}\cdots\varphi_0\varphi_{n-1}, \qquad \dots, \qquad \varphi_0\varphi_{n-1}\cdots\varphi_1$$

are all multiplication by p.

(ii). $M^{\mathrm{loc}}_{\mathrm{GSp}}$ is a closed subscheme of $M^{\mathrm{loc}}_{\mathrm{GL}}$ given by enforcing the duality condition. That this condition is given by the equations above is computed in [Gör2, Section 5.1].

□

Remark 3.11. We mention here without proof the vastly reduced presentation from [Gör3, Section 3]: U^{GL}_{τ} is isomorphic to the spectrum of

$$\mathbb{Z}_p[a^i_{1k}; i = 0, \dots, n-1, k = 1, \dots, r]/I$$

where I is the ideal generated by the entries of the matrices

$$A_{n-1}A_{n-2}\cdots A_0 - p\cdot \mathrm{Id}, \qquad A_{n-2}\cdots A_0A_{n-1} - p\cdot \mathrm{Id}, \qquad \dots, \qquad A_0A_{n-1}\cdots A_1 - p\cdot \mathrm{Id}.$$

The following lemma will be used in Chapter 4.

Lemma 3.12. *With the presentation of U_0 as in the above proposition, let x be a closed point of U_0 associated with $\{\mathcal{F}_i \subset \Lambda_i\}$.*

(i). The map $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is an isomorphism if and only if $a^i_{11} \neq 0$.

(ii). The map $\Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ is an isomorphism if and only if $a^{i+1}_{n-r,r} \neq 0$.

Proof.

(i). The map $\mathcal{F}_i \to \mathcal{F}_{i+1}$ will be an isomorphism if and only if $\det(A_i) \neq 0$, which is if and only if $a^i_{11} \neq 0$.

(ii). Before proceeding, let us make the following remark on the indices for e^i_j and a^i_{jk}. The upper indices refer to the flag and so are taken modulo n with the standard set of representatives $\{0, \ldots, n-1\}$. The lower indices refer to the position in a vector or matrix, and as such are also taken modulo n however with the set of representatives $\{1, \ldots, n\}$.

The relations defining the quotient $\Lambda_i/\mathcal{F}_i$ are

$$e^i_{i+j} = -\sum_{k=1}^{n-r} a^i_{kj} e^i_{i+r+k} \quad \text{for} \quad 1 \leq j \leq r.$$

Thus we may take $\left\{\overline{e}^i_{i+r+1}, \ldots, \overline{e}^i_{i+n}\right\}$ as a basis of $\Lambda_i/\mathcal{F}_i$. By abuse of notation, let φ_i be the induced map $\Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ and note we have following equations.

$$\varphi_i(\overline{e}^i_{i+r+1}) = \overline{e}^{i+1}_{i+r+1} = -\sum_{k=1}^{n-r} a^{i+1}_{kr} \overline{e}^{i+1}_{i+r+k+1}$$

and

$$\varphi_i(\overline{e}^i_{i+r+j}) = \overline{e}^{i+1}_{i+r+j} \quad \text{for} \quad 2 \leq j \leq n-r.$$

Therefore the matrix representing the map $\Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ with respect to these bases is

$$\begin{pmatrix} -a^{i+1}_{1r} & 1 & & & \\ -a^{i+1}_{2r} & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ -a^{i+1}_{n-r,r} & & & & \end{pmatrix}.$$

From this, we see that the map $\Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ is an isomorphism if and only if $a^{i+1}_{n-r,r} \neq 0$.

$\square$

Chapter 4

Integral and local models of $\mathcal{A}_1$

In this section we use the theory of Oort-Tate to define the integral model of $\mathcal{A}_1$ and construct affine local models. Throughout this chapter, let S be an $\mathcal{O}_E$-scheme and k an algebraically closed field. In the unitary case we let $G = \mathrm{GL}_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$ with minuscule cocharacter $\mu = (1^r, 0^{n-r})$ and in the unitary case we let $G = \mathrm{GSp}_{2n,\mathbb{Q}_p}$ with minuscule cocharacter $\mu = (1^n, 0^n)$.

4.1 The group schemes G_i

An S-valued point $x : S \to \mathcal{A}_0$ is given by the data $(\{A_i\}, \overline{\lambda}, i, \overline{\eta})$. In this section we will associate to x a collection $\{G_i\}_{i=0}^{n-1}$ of finite flat group schemes of rank p corresponding to the kernel of $A_i \to A_{i+1}$. Furthermore, in the case where $S = \mathrm{Spec}(k)$, there exists an étale neighborhood $\varphi : V \to \mathcal{A}_0$ with a closed point p of V and a section $\sigma : V \to \widetilde{\mathcal{A}}_0$

$$\begin{tikzcd}
 & \widetilde{\mathcal{A}}_0 \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
V \arrow[r] \arrow[ur, bend left, "\sigma"] & \mathcal{A}_0 & M^{\mathrm{loc}}
\end{tikzcd}$$

such that $\varphi(p) = x$ and $\Psi \circ \sigma$ is étale at p. Let $y = \Psi \circ \sigma(p)$. In such a case, we will be able to tell the isomorphism type of G_i from the data $(\{\mathcal{F}_i \hookrightarrow \Lambda_{i,S}\})$ given by y.

We start by first restricting our attention to the unitary case. Recall that we have the splittings

$$\mathscr{L}_{i,S} \oplus \mathscr{L}_{i,S}^* = (\Lambda_{i,S})^n \oplus (\Lambda_{i,S}^*)^n \quad \text{and} \quad \omega_i = (\mathcal{F}_i)^n \oplus (\mathcal{F}_i^*)^n$$

induced by the splitting $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S \cong M_n^{\mathrm{opp}}(\mathcal{O}_{F_{\mathfrak{p}}}) \times M_n^{\mathrm{opp}}(\mathcal{O}_{F_{\overline{\mathfrak{p}}}})$. Consider the p-divisible group defined by

$$A_i(p^\infty) = \varinjlim_n A_i[p^n]$$

and note $H_i = \ker(A_i \to A_{i+1})$ is contained in $A_i(p^\infty)$. Now the action of $\mathcal{O}_B \otimes \mathbb{Z}_p$ gives

$$A_i(p^\infty) = \bigoplus_{j=1}^n e_{11} A_i(\mathfrak{p}^\infty) \oplus \bigoplus_{j=1}^n f_{11} A_i(\overline{\mathfrak{p}}^\infty)$$

where e_{11} and f_{11} are idempotents of the first, respectively second, factor of $M_n^{\mathrm{opp}}(\mathbb{Z}_p) \times M_n^{\mathrm{opp}}(\mathbb{Z}_p)$. By functoriality this gives a chain

$$e_{11}A_0(\mathfrak{p}^\infty) \to e_{11}A_1(\mathfrak{p}^\infty) \to \cdots \to e_{11}A_{n-1}(\mathfrak{p}^\infty) \to e_{11}A_0(\mathfrak{p}^\infty)$$

of isogenies of degree p with the composition $e_{11}A_0(\mathfrak{p}^\infty) \to e_{11}A_0(\mathfrak{p}^\infty)$ being multiplication by p. We then set

$$G_i = \ker\left(e_{11}A_i(\mathfrak{p}^\infty) \to e_{11}A_{i+1}(\mathfrak{p}^\infty)\right)$$

noting that G_i is a finite flat group scheme of order p.

Now consider the symplectic case. Here the maps $A_i \to A_{i+1}$ are isogenies of degree p, and we set $G_i = \ker(A_i \to A_{i+1})$.

Definition 4.1.1. For a group scheme G/S, define $\omega_G = \omega_{G/S}$ to be the sheaf on S given by $\varepsilon^*(\Omega^1_{G/S})$ where $\varepsilon : S \to G$ is the identity section.

In the following proposition in the unitary case, recall that we are using φ_i for both the morphism $\mathscr{L}_{i,S} \oplus \mathscr{L}^*_{i,S} \to \mathscr{L}_{i+1,S} \oplus \mathscr{L}^*_{i+1,S}$ and its restriction to $\Lambda_{i,S} \to \Lambda_{i+1,S}$.

Proposition 4.1.2. *Let $S = \mathrm{Spec}(k)$, $x : S \to \mathcal{A}_0$, and $y : S \to M^{loc}$ corresponding to x as described above inducing the data $\{\mathcal{F}_i \hookrightarrow \Lambda_{i,S}\}$. Then*

(i) In the unitary case, $\dim_k \omega_{H_i^*} = \dim_k(\omega_{i+1}/\varphi_i(\omega_i))$*, where H_i^* the Cartier dual of H_i.*
(ii) In the unitary case, $\dim_k \omega_{H_i} = \dim_k(\mathscr{L}_{i+1,S} \oplus \mathscr{L}^*_{i+1,S})/\left(\varphi_i(\mathscr{L}_{i,S} \oplus \mathscr{L}^*_{i,S}) + \omega_{i+1}\right)$.
(iii) In both the unitary and symplectic cases, $\dim_k \omega_{G_i^*} = \dim_k(\mathcal{F}_{i+1}/\varphi_i(\mathcal{F}_i))$.
(iv) In both the unitary and symplectic cases, $\dim_k \omega_{G_i} = \dim_k \Lambda_{i+1,S}/\left(\varphi_i(\Lambda_{i,S}) + \mathcal{F}_{i+1}\right)$.

Proof. (i) H_i^* is given by the exact sequence

$$0 \to H_i^* \to \hat{A}_{i+1} \xrightarrow{\hat{\alpha}} \hat{A}_i.$$

The sequence of morphisms $\hat{A}_{i+1} \xrightarrow{\hat{\alpha}} \hat{A}_i \to \mathrm{Spec}(k)$ gives the standard exact sequence of Kähler differentials

$$\hat{\alpha}^*(\Omega^1_{\hat{A}_i}) \to \Omega^1_{\hat{A}_{i+1}} \to \Omega^1_{\hat{A}_{i+1}/\hat{A}_i} \to 0.$$

Now pull back this sequence by the identity section $\varepsilon_{\hat{A}_{i+1}}$

$$\omega_{\hat{A}_i} \to \omega_{\hat{A}_{i+1}} \to \varepsilon^*_{\hat{A}_{i+1}}(\Omega^1_{\hat{A}_{i+1}/\hat{A}_i}) \to 0.$$

As H_i^* can be described by the fibered product

$$\begin{tikzcd}
H_i^* \arrow[r, "i"] \arrow[d] & \hat{A}_{i+1} \arrow[d, "\alpha"] \\
\mathrm{Spec}(k) \arrow[r, "\varepsilon_{\hat{A}_i}"] & \hat{A}_i
\end{tikzcd}$$

and $i \circ \varepsilon_{H_i^*} = \varepsilon_{\hat{A}_{i+1}}$, we have the canonical isomorphisms

$$\Omega^1_{H_i^*} = i^*(\Omega^1_{\hat{A}_{i+1}/\hat{A}_i}) \quad \text{and} \quad \varepsilon^*_{\hat{A}_i}(\Omega^1_{\hat{A}_{i+1}/\hat{A}_i}) = (i \circ \varepsilon_{H_i^*})^*(\Omega^1_{\hat{A}_{i+1}/\hat{A}_i}) = \omega_{H_i^*}.$$

Therefore the exact sequence of invariant differentials becomes

$$\omega_{\hat{A}_i} \to \omega_{\hat{A}_{i+1}} \to \omega_{H_i^*} \to 0.$$

Now (i) follows since by definition $\omega_i = \omega_{\hat{A}_i}$.

(ii) Starting with the exact sequence

$$0 \to H_i \to A_i \xrightarrow{\alpha} A_{i+1}$$

as above we have

$$\omega_{A_{i+1}} \to \omega_{A_i} \to \omega_{H_i} \to 0.$$

From the Hodge filtration, $\omega_{A_i} \cong \big(M(A_i)/\omega_{\hat{A}_i}\big)^\vee$. As the map φ_i sends ω_i into ω_{i+1} we have

$$\dim_k \omega_{H_i} = \dim_k \Big(\mathscr{L}_{i+1} \oplus \mathscr{L}^*_{i+1})/\left(\varphi_i(\mathscr{L}_i \oplus \mathscr{L}^*_i) + \omega_{i+1}\right)\Big).$$

In the unitary case, parts (iii) and (iv) then follow from (i) and (ii) by the functoriality of our decompositions. In the symplectic case, (iii) and (iv) follow as in the proof of (i) and (ii) by replacing H_i with G_i, ω_{A_i} with $\mathcal{F}_i$, and $\mathscr{L}_i \oplus \mathscr{L}^*_i$ with Λ_i. □

For a general S-valued point $y : S \to M^{\mathrm{loc}}$, the maps $\varphi_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ and $\varphi^*_i : \Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ induce global sections q_i and q^*_i of the line bundles

$$\mathscr{Q}_i = \left(\bigwedge^{\mathrm{top}} \mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\mathrm{top}} \mathcal{F}_{i+1} \quad \text{and} \quad \mathscr{Q}^*_i = \left(\bigwedge^{\mathrm{top}} \Lambda_i/\mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\mathrm{top}} \Lambda_{i+1}/\mathcal{F}_{i+1}$$

respectively. In the case $S = \mathrm{Spec}(k)$, q_i and q^*_i are the determinants of the corresponding linear maps and it is immediate that $q_i \otimes q^*_i = \pi$.

Proposition 4.1.3. *Let $x : \mathrm{Spec}(k) \to \mathcal{A}_0$ and $y : \mathrm{Spec}(k) \to M^{loc}$ a corresponding geometric point inducing the sections q_i and q^*_i as described above.*

(i) $q_i = 0$ if and only if $\dim_k \omega_{G^*_i} = 1$.

(ii) $q^*_i = 0$ if and only if $\dim_k \omega_{G_i} = 1$.

Proof. This follows immediately as $q_i \neq 0$ if and only if $\mathcal{F}_i$ is carried isomorphically onto $\mathcal{F}_{i+1}$, and similarly with q^*_i. □

We now recall the classification of finite flat group schemes of order p over an algebraically closed field (see Section A.4.1 for details). If $\mathrm{char}(k) \neq p$, then any finite flat group scheme of order p over $\mathrm{Spec}(k)$ is isomorphic to the constant group scheme $\mathbb{Z}/p\mathbb{Z}$. If $\mathrm{char}(k) = p$, then there are, up to isomorphism, three finite flat group schemes G of order p over $\mathrm{Spec}(k)$: $\mathbb{Z}/p\mathbb{Z}$, μ_p, and α_p. The Cartier dual of $\mathbb{Z}/p\mathbb{Z}$ is μ_p, and the Cartier dual of α_p is α_p itself. The following table is computed in Section A.4.1.

G	μ_p	$\mathbb{Z}/p\mathbb{Z}$	α_p
$(\dim_k \omega_G, \dim_k \omega_{G^*})$	(1,0)	(0,1)	(1,1)

Table 4.1: Dimension of invariant differentials of group schemes of order p.

Thus knowing q_i and q_i^*, one can determine the isomorphism type of G_i.

Corollary 4.1.4. *Given a morphism $S \to \mathcal{A}_0$, consider the divisor on S defined by the vanishing of q_i^*. Then the support of this divisor is precisely the locus given by the collection of closed points*

$$\{x \in S \otimes \mathbb{F}_p : x : \mathrm{Spec}(k(x)) \to \mathcal{A}_0 \textit{ induces } G_i \cong \mu_p \textit{ or } G_i \cong \alpha_p\}.$$

4.2 Integral and local model of $\mathcal{A}_1$

Oort-Tate theory, as described in Section A.4.3, can be summarized as follows.

Theorem 4.2.1. *[HR, Theorem 3.3.1] Let OT be the $\mathbb{Z}_p$-stack representing finite flat group schemes of order p.*

(i). OT is an Artin stack isomorphic to

$$[(\mathrm{Spec}\, \mathbb{Z}_p[X, Y]/(XY - w_p))/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts via $\lambda \cdot (X, Y) = (\lambda^{p-1}X, \lambda^{1-p}Y)$. Here w_p denotes an explicit element of $p\mathbb{Z}_p^\times$ given in loc. cit.

(ii). The universal group scheme $\mathcal{G}_{OT}$ over OT is

$$\mathcal{G}_{OT} = [(\mathrm{Spec}_{OT}\,\mathcal{O}[Z]/(Z^p - XZ))/\mathbb{G}_m],$$

(where $\mathbb{G}_m$ acts via $Z \to \lambda Z$), with zero section $Z = 0$.

(iii). Cartier duality acts on OT by interchanging X and Y.

As in [HR], we denote $\mathcal{G}_{OT}^\times$ to be the closed subscheme of $\mathcal{G}_{OT}$ defined by the ideal $(Z^{p-1} - X)$. The morphism $\mathcal{G}_{OT}^\times \to OT$ is relatively representable, finite, and flat of degree $p-1$. The notation is justified by the following proposition.

Proposition 4.2.2. *[HR, 3.3.2] (cf. [Pap1, 5.1]) Let S and G be as in Definition A.4.11, so G corresponds to a morphism $\varphi : S \to OT$ determined by the condition $G = \varphi^*(\mathcal{G}_{OT})$. Set $G^\times = \varphi^*(\mathcal{G}_{OT}^\times)$. For $c \in G(S)$, $c \in G^\times(S)$ if and only if c is an Oort-Tate generator.*

Remark 4.2.3. Let $\varphi : S \to OT$ correspond to G/S with G finite flat of order p. Then $\varphi^*(X)$ cuts out the locus of closed point of S where G is infinitesimal.

Given an S-valued point of $\mathcal{A}_0$, we have associated with it the group schemes $\{G_i\}_{i=0}^{n-1}$ over S in the previous section. This defines a morphism

$$\varphi : \mathcal{A}_0 \to \overbrace{OT \times_{\mathbb{Z}_p} \cdots \times_{\mathbb{Z}_p} OT}^{n \text{ times}}.$$

Definition 4.2.4. $\mathcal{A}_1$ is the fibered product

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r] \arrow[d, "\pi"'] & \mathcal{G}_{OT}^{\times} \times \cdots \times \mathcal{G}_{OT}^{\times} \arrow[d] \\
\mathcal{A}_0 \arrow[r, "\varphi"] & \overbrace{OT \times \cdots \times OT}^{n \text{ times}} .
\end{tikzcd}$$

Remark 4.2.5. With $\mathcal{G}_{OT}^{\times} \to OT$ relatively representable, we have $\mathcal{A}_1$ is represented by a quasi-projective scheme. It is also immediate from the definition that $\mathcal{A}_1$ is finite and flat over $\mathcal{A}_0$. Since $\mathcal{A}_0$ is flat over $\mathrm{Spec}(\mathbb{Z}_p)$ [Gör1] [Gör2], we have $\mathcal{A}_1$ is flat over $\mathrm{Spec}(\mathbb{Z}_p)$ as well.

Given a geometric point $x : \mathrm{Spec}(k) \to \mathcal{A}_0$, as noted in the previous section there exists an étale neighborhood $V \to \mathcal{A}_0$ and an étale morphism $\psi : V \to M^{\mathrm{loc}}$. Assume that ψ factors through an open subscheme $U \subset M^{\mathrm{loc}}$ where each $\mathcal{Q}_i^*$ is trivial. Choosing a trivialization, by abuse of notation we will write $q_i^* \in \Gamma(U, \mathcal{O}_U)$, where q_i^* is the global section of $\mathcal{Q}_i^*$ defined in the previous section. Note that if $\mathcal{Q}_i^*$ is trivial then so is $\mathcal{Q}_i$. Thus we also have $q_i \in \Gamma(U, \mathcal{O}_U)$. Consider the following diagram

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\psi"'] & \mathcal{A}_0 \arrow[r, "\varphi_i"] & OT \\
U
\end{tikzcd}$$

where $\varphi_i : \mathcal{A}_0 \to OT$ is φ followed by the ith projection.

Proposition 4.2.6. *The morphism $\rho_i : V \to OT$ in the diagram above is given by*

$$\rho_i^*(X_i) = \varepsilon_i \psi^*(q_i^*) \qquad \rho_i^*(Y_i) = w_p \varepsilon_i^{-1} \psi^*(q_i)$$

where ε_i is a unit in V and w_p is from the description of the Oort-Tate stack in Theorem

4.2.1.

Proof. The special fiber of M^{loc}, and therefore of V, is reduced [Gör1], [Gör2]. From the equalities $q_i q_i^* = \pi$ and $X_i Y_i = \omega_p \pi$ we have that the divisors defined by the vanishing of the global sections $Z(\psi^*(q_i^*))$ and $Z(\rho_i^*(X_i))$ are reduced. By Corollary 4.1.4, Example A.4.14, and Remark 4.2.3, the locus where $\psi^*(q_i^*)$ vanishes agrees with the locus where $\rho_i^*(X_i)$ vanishes. Therefore $Z(\psi^*(q_i^*)) = Z(\rho_i^*(X_i))$.

M^{loc} is flat over $\mathcal{O}_E$ with reduced special fiber [Gör1],[Gör2]. Furthermore, the generic fiber is normal (smooth, even). It follows that M^{loc} is normal [PZ, Proposition 8.2]. Since $V \to M^{\mathrm{loc}}$ is finite étale, we have that V is normal as well. Thus the equality of divisors above implies $\psi^*(q_i^*)$ and $\rho_i^*(X_i)$ are equal up to a unit, say ε_i. The same proof applies for the statement regarding $\rho_i^*(Y_i)$, and so $\psi^*(q_i)$ and $\rho_i^*(Y_i)$ are equal up to a unit. This unit must be $w_p \varepsilon_i^{-1}$ because $X_i Y_i = w_p \pi$ and $q_i q_i^* = \pi$. $\square$

Proposition 4.2.7. *Let $\widetilde{x} : \mathrm{Spec}(k) \to \mathcal{A}_1$ with $\pi(\widetilde{x}) = x : \mathrm{Spec}(k) \to \mathcal{A}_0$ and $V \to \mathcal{A}_0$ be an étale neighborhood of x which carries an étale morphism $V \to M^{loc}$. Suppose $V \to M^{loc}$ factors through an open affine subscheme $U \subset M^{loc}$ on which $\mathcal{Q}_i^*$ is trivial for each i. Set*

$$U_1 = \mathrm{Spec}_U \left(\mathcal{O}_U[u_0, \ldots, u_{n-1}] / \left(u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^* \right) \right).$$

Then there exists an étale neighborhood $\widetilde{V}$ of $\widetilde{x}$ and an étale morphism $\psi : \widetilde{V} \to U_1$.

Proof. Define $\widetilde{V} = V \times_{\mathcal{A}_0} \mathcal{A}_1$. Consider the diagram

$$\begin{array}{ccccccc}
U_1 & \longleftarrow & \widetilde{V} & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{G}_{OT}^{\times} \times \cdots \times \mathcal{G}_{OT}^{\times} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \longleftarrow & V & \longrightarrow & \mathcal{A}_0 & \longrightarrow & OT \times \cdots \times OT
\end{array}$$

and denote by η the morphism $\widetilde{V} \to \mathcal{G}_{OT}^{\times} \times \cdots \times \mathcal{G}_{OT}^{\times}$ in the diagram. The two right squares of the diagram are Cartesian and the morphism on the top left is defined by sending u_i to $\eta^*(Z_i)$. The diagram commutes by the proposition above. The morphism $\mathcal{G}_{OT}^{\times} \to OT$ is relatively representable and thus

$$\widetilde{V} \cong \mathrm{Spec}_V \left(\mathcal{O}_V[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - \eta^*(Z_0), \ldots, u_{n-1}^{p-1} - \eta^*(Z_{n-1}))\right).$$

As the top left morphism of the diagram above is given by sending u_i to $\eta^*(Z_i)$, it follows that $\widetilde{V} \cong U_1 \times_U V$. Therefore by the above diagram, the morphism $\widetilde{V} \to U_1$ is étale. □

Remark 4.2.8. Given a covering of affine open subschemes $\{U_j\}$ of $\mathcal{A}_0$ such that $\mathcal{Q}_i^*$ is trivial on every U_j for each i, it is tempting to hope that one may glue together the corresponding affine schemes defined in the proposition above to get a scheme "M_1^{loc}" with a morphism $M_1^{\mathrm{loc}} \to M^{\mathrm{loc}}$ which is a local model for $\mathcal{A}_1$. However this is not possible. Indeed, let $\{U_j\}$ be any open cover of M^{loc} so that for each j, $\mathcal{Q}_i^*|_{U_j}$ is trivial. Suppose that M_1^{loc} is a connected scheme which is a local model for $\mathcal{A}_1$ with a morphism $M_1^{\mathrm{loc}} \to M^{\mathrm{loc}}$ such that for each j,

$$U_j \times_{M^{\mathrm{loc}}} M_1^{\mathrm{loc}} \cong \mathrm{Spec}_{U_j} \left(\mathcal{O}_{U_j}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^*)\right).$$

Recall that the sections $q_0^*, \ldots, q_{n-1}^*$ vanish only on the special fiber. Therefore, the restriction of $M_1^{\mathrm{loc}} \to M^{\mathrm{loc}}$ to the generic fiber is finite étale. However the generic fiber of M^{loc} is the Grassmannian $\mathrm{Gr}(n, r)$, and is therefore simply connected. It follows that $M_1^{\mathrm{loc}} \to M^{\mathrm{loc}}$ is an isomorphism on the generic fiber, which is a contradiction.

Let $U \subset M^{\mathrm{loc}}$ be any open subset containing S_τ and consider the special fiber $U \otimes \mathbb{F}_p$. From the above proposition, $U \otimes \mathbb{F}_p$ intersects every stratum. From Proposition 3.8, the action of I on $M^{\mathrm{loc}} \otimes \mathbb{F}_p$ gives a surjective map $I \times (U \otimes \mathbb{F}_p) \to M^{\mathrm{loc}} \otimes \mathbb{F}_p$. Thus every closed point of M^{loc} has a Zariski neighborhood isomorphic to a neighborhood of some closed point of U. Combining this with Proposition 4.2.7, assume in addition that U is small enough so that each Q_i^* is trivial.

Theorem 4.2.9. *The scheme*

$$U_1 = \mathrm{Spec}_U \left(\mathcal{O}_U[u_0, \ldots, u_{n-1}]/\left(u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^*\right)\right)$$

is étale locally isomorphic to $\mathcal{A}_1$. More precisely, for every closed point x of $\mathcal{A}_0$, there exists an étale neighborhood V of x and an étale morphism $V \to U_1$.

By choosing $U = U_0$ from Chapter 3, Lemma 3.12 states that up to a unit, $q_i^* = a_{n-r,r}^{i+1}$ for $0 \leq i \leq n-1$. With this chosen presentation, the above theorem becomes the following.

Theorem 4.2.10. *The scheme*

$$U_1 = \mathrm{Spec}\left(B[u_0, \ldots, u_{n-1}]/\left(u_0^{p-1} - a_{n-r,r}^1, \ldots, u_{n-2}^{p-1} - a_{n-r,r}^{n-1}, u_{n-1}^{p-1} - a_{n-r,r}^0\right)\right)$$

is étale locally isomorphic to $\mathcal{A}_1$. More precisely, for every closed point x of $\mathcal{A}_0$, there exists an étale neighborhood V of x and an étale morphism $V \to U_1$.

4.3 Modification of U_1

As in the previous section, fix an affine open subscheme $U \subset M^{\mathrm{loc}}$ where $U = \mathrm{Spec}(B)$ such that U contains the "worst point" and for each i the line bundle $\mathcal{Q}_i|_U$ is trivial. We therefore identify $q_i^* \in \Gamma(U, \mathcal{Q}_i|_U)$ with a regular function on U. Assume we have a modification (i.e. proper birational morphism) $U' \to U$ which is an isomorphism on the generic fiber and such that U' is Zariski locally of the form

$$\mathrm{Spec}(\mathbb{Z}_p[x_1, \dots, x_t]/(x_1 \cdots x_s - p)).$$

Such modifications are known to exist for the local models associated with GSp_4 and GL_4, given by blowing up irreducible components of the special fiber [dJ1, Gör3]. We then get a modification of U_1 given by

$$U_1' = U_1 \times_U U' \to U_1.$$

Proposition 4.3.1. *Let $x \in U_1'$ be a closed point. Then there exists a Zariski open neighborhood of x of the form*

$$\mathrm{Spec}(\mathbb{Z}_p[x_1, \dots, x_t, u_0, \dots, u_{n-1}]/(x_1 \cdots x_s - p, u_0^{p-1} - q_0^*, \dots, u_{n-1}^{p-1} - q_{n-1}^*))$$

where each q_i^* is, up to a unit, a monomial in $x_1, \dots, x_s$ with each x_j occurring with multiplicity at most 1.

Proof. Let $x \in U_1'$ map to $y \in U'$ and let $V \subset U'$ be an affine open neighborhood of y of the form

$$V = \mathrm{Spec}(C), \quad C = \mathbb{Z}_p[x_1, \dots, x_t]/(x_1 \cdots x_s - p).$$

By the preceding section, U_1 is given by

$$\operatorname{Spec}(B[u_0, \dots, u_{n-1}]/(u_0^{p-1} - q_0^*, \dots, u_{n-1}^{p-1} - q_{n-1}^*)).$$

For each j, by abuse of notation write q_j and q_j^* for their images under the ring homomorphism $B \to C$. We then have an affine open chart of U_1' given by

$$\operatorname{Spec}(C[u_0, \dots, u_{n-1}]/(u_0^{p-1} - q_0^*, \dots, u_{n-1}^{p-1} - q_{n-1}^*)).$$

It remains to show that the q_j^* are, up to a unit, monomials in the variables $x_1, \dots, x_s$ with multiplicity at most 1. With $q_j q_j^* = p$ we have that, as a divisor on $V = \operatorname{Spec}(C)$, $Z(q_j^*)$ has support in the special fiber and hence must be a sum of the irreducible components of $V \otimes \mathbb{F}_p$. These are given by $Z(x_i)$ for $1 \leq i \leq s$. Suppose

$$Z(q_j^*) = \sum_i n_i Z(x_i).$$

Then $Z(q_j^*) = Z(x_1^{n_1} \cdots x_s^{n_s})$ and since V is normal we get $q_j^* = \varepsilon_j x_1^{n_1} \cdots x_s^{n_s}$ where ε_j is a unit in C. As the special fiber of V is reduced and $q_i q_i^* = 0$, each n_i is either 0 or 1 for all i. $\square$

Remark 4.3.2. Such a modification $U_1' \to U_1$ is not in general normal. An example of this will be seen when we consider such a modification for the group GSp_4 in the next chapter.

Chapter 5

Resolution of $\mathcal{A}_1$ associated with $\mathbf{GSp}_4$

All schemes in this section are of finite type over $\mathrm{Spec}(\mathbb{Z}_p)$ and all subschemes are locally closed, where p is an odd rational prime. We denote by $\mathcal{A}_1$ the integral model of the Shimura variety associated with GSp_4 equipped $K(N)$ level structure, where $N \geq 3$ and $p \nmid N$. Will construct and describe the resolution of singularities of $\mathcal{A}_1$ mentioned in the introduction.

To reach a resolution of $\mathcal{A}_1$ we will start with the known semi-stable resolution $\widetilde{\mathcal{A}_0}$ of $\mathcal{A}_0$. This is obtained by blowing up an irreducible component in the special fiber of $\mathcal{A}_0$ (see below). By fibering $\widetilde{\mathcal{A}_0} \to \mathcal{A}_0$ with $\mathcal{A}_1 \to \mathcal{A}_0$, the resulting modification (i.e. proper birational morphism) for $\mathcal{A}_1$ will take the form mentioned in Proposition 4.3.1; however it will fail to be even normal. A sequence of $p-1$ further blowups will produce the regular resolution described in the introduction.

At each stage of the above process, we will first work on a local model and then carry this over to the integral model. When performing a blowup of a local model, we require that the subscheme being blown up corresponds to a subscheme of the integral model. In order to understand more of the global structure of the resolution, such as the number of irreducible components and how they intersect, it is also necessary to track how certain subschemes transform with each modification. We begin by stating definitions and lemmas that will be used throughout the process.

5.1 Preliminaries

Notation 5.1.1. Let $f : X \to Y$ be a morphism of schemes and $Z \subset Y$ a subscheme. Then

- Z^{red} denotes the subscheme of Y given as a set by Z with reduced scheme structure; and
- $f^{-1}(Z)$ denotes the scheme-theoretic inverse image under f.

In particular, $f^{-1}(Z)^{\mathrm{red}}$ denotes the reduced inverse image of Z in X.

Definition 5.1.2 ([EH])**.** Let X be a scheme, $\mathcal{Z} \subset X$ a subscheme. We say that $\mathcal{Z}$ is Cartier at a closed point p in X if in an affine open neighborhood of p it is the zero locus of a single regular function which is not a zero divisor. We say that $\mathcal{Z}$ is a Cartier subscheme of X if $\mathcal{Z}$ is Cartier at all closed points of X.

Definition 5.1.3. Let $\rho : X' \to X$ be a modification.

- If ρ is given by the blowup of a closed subscheme $\mathcal{Z}$ of X, then $\mathcal{Z}$ is called a center of ρ.
- The true center of ρ, denoted by C_ρ, is defined to be the closed reduced subscheme of X given set-theoretically by the complement of the maximal open subscheme where ρ

is an isomorphism.

- The fundamental center of ρ, denoted by C_ρ^{fund}, is defined to be the reduced subscheme of C_ρ whose support is given by the closed points with fiber of dimension at least one.
- The residual locus of ρ, denoted by C_ρ^{res}, is defined to be the reduced subscheme of C_ρ whose support is given by the closed points with fiber of dimension zero.
- The exceptional locus of ρ is $\rho^{-1}(C_\rho)^{\text{red}}$.

Remark 5.1.4. For a modification ρ with a center $\mathcal{Z}$, we will often say "the center" when there is a canonical choice of $\mathcal{Z}$. The subscript ρ will be dropped from C_ρ, C_ρ^{fund}, and C_ρ^{res} when the morphism ρ is understood. By upper semi-continuity of the fiber, the fundamental center C^{fund} is a closed subscheme of C. By definition $C^{\text{res}} = C \setminus C^{\text{fund}}$, an open subscheme of C. Since all schemes are assumed to be of finite type over $\operatorname{Spec}(\mathbb{Z}_p)$, the fiber over a closed point of C^{res} is a finite collection of closed points.

Definition 5.1.5. Let $\rho : X' \to X$ be a modification and $W \subset X$ be a closed subscheme. Set $U = X \setminus C_\rho$. Then the strict transform of W with respect to ρ, denoted by $\mathrm{ST}_\rho(W)$ or by $\mathrm{ST}(W)$ if ρ is understood, is defined to be either

(i). the Zariski closure of $\rho^{-1}(W \setminus C_\rho)$ inside of X' if $W \not\subset C_\rho$; or

(ii). $\rho^{-1}(W)$ if $W \subset C_\rho$.

Definition 5.1.6. Let M be an étale local model of X. We say that a subscheme Z of M étale locally corresponds to a subscheme $\mathcal{Z}$ of X if there exists an étale cover $V \to X$ with an étale morphism $V \to M$ such that the scheme-theoretic pullback of Z and $\mathcal{Z}$ to V are equal as subschemes of V.

Remark 5.1.7. In the situation of the definition, we have in particular that Z is an étale local model of $\mathcal{Z}$.

Lemma 5.1.8. *Let M be an étale local model of X. Fix an étale cover $V \to X$ with an étale morphism $V \to M$. Suppose that with these fixed morphisms, for $i = 1, 2$ the reduced subscheme $\mathcal{Z}_i \subset X$ étale locally corresponds to the reduced subscheme $Z_i \subset M$. Then the following pairs étale locally correspond, where each is given the reduced scheme structure.*

$$\begin{array}{ccc} \mathcal{Z}_1 \cup \mathcal{Z}_2 & \text{and} & Z_1 \cup Z_2 \\ \mathcal{Z}_1 \cap \mathcal{Z}_2 & \text{and} & Z_1 \cap Z_2 \\ \mathcal{Z}_1 \setminus \mathcal{Z}_2 & \text{and} & Z_1 \setminus Z_2 \\ \overline{\mathcal{Z}_1} & \text{and} & \overline{Z_1} \end{array}$$

Here, $\overline{\mathcal{Z}_1}$ denotes the Zariski closure inside of X, and similarly with Z_1 inside of M.

Proof. Let $V_{\mathcal{Z}_i}$ denote the pullback of $\mathcal{Z}_i$ along the morphism $V \to X$, V_{Z_i} the pullback of Z_i along $V \to M$, and similarly with $V_{\mathcal{Z}_1 \cup \mathcal{Z}_2}$ and $V_{Z_1 \cup Z_2}$. Then $V_{\mathcal{Z}_1 \cup \mathcal{Z}_2} = V_{\mathcal{Z}_1} \cup V_{\mathcal{Z}_2}$ and $V_{Z_1 \cup Z_2} = V_{Z_1} \cup V_{Z_2}$ as sets. With $V_{\mathcal{Z}_i} = V_{Z_i}$ for $i = 1, 2$ we have $V_{\mathcal{Z}_1 \cup \mathcal{Z}_2} = V_{Z_1 \cup Z_2}$ as sets. Now since $\mathcal{Z}_1 \cup \mathcal{Z}_2$ is given the reduced scheme structure and $V \to X$ is étale, $V_{\mathcal{Z}_1 \cup \mathcal{Z}_2}$ is reduced. Likewise $V_{Z_1 \cup Z_2}$ is reduced. Therefore they are equal as subschemes of V.

The statement for the next two pairs of subschemes follows in a similar manner. The statement for $\overline{\mathcal{Z}_1}$ and $\overline{Z_1}$ follows from the fact that étale morphisms are flat and hence open, giving that the pullback of $\overline{\mathcal{Z}_1}$ to V is the Zariski closure of the pullback of $\mathcal{Z}_1$ to V. $\square$

Lemma 5.1.9. *Let X be a scheme with étale local model M, $\mathcal{Z} \subset X$ a closed subscheme étale locally corresponding to the closed subscheme $Z \subset M$. Then $Bl_M(Z)$ is an étale local model of $Bl_X(\mathcal{Z})$.*

Proof. Let $\varphi : V \to X$ be an étale cover of X with an étale morphism $\psi : V \to M$ such that $\mathcal{Z}_V = \varphi^{-1}(Z)$ is equal to $Z_V = \psi^{-1}(Z)$ as closed subschemes of V. Then since blowing up commutes with flat (and hence, étale) base extension, we have that the diagrams

$$\begin{array}{ccc} \mathrm{Bl}_V(\mathcal{Z}_V) & \longrightarrow & \mathrm{Bl}_X(\mathcal{Z}) \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} \mathrm{Bl}_V(Z_V) & \longrightarrow & \mathrm{Bl}_M(Z) \\ \downarrow & & \downarrow \\ V & \longrightarrow & M \end{array}$$

are cartesian. Of course, $\mathrm{Bl}_V(\mathcal{Z}_V) = \mathrm{Bl}_V(Z_V)$ and so $\mathrm{Bl}_V(\mathcal{Z}_V) \to \mathrm{Bl}_X(\mathcal{Z})$ is an étale cover with étale morphism $\mathrm{Bl}_V(\mathcal{Z}_V) = \mathrm{Bl}_V(Z_V) \to \mathrm{Bl}_M(Z)$. Therefore $\mathrm{Bl}_M(Z)$ is an étale local model of $\mathrm{Bl}_X(\mathcal{Z})$. □

Remark 5.1.10. In the proof of the above lemma, given an étale cover $V \to X$ and étale morphism $V \to M$ there is constructed a canonical étale cover $\mathrm{Bl}_V(\mathcal{Z}_V) \to \mathrm{Bl}_X(\mathcal{Z})$ and étale morphism $\mathrm{Bl}_V(\mathcal{Z}_V) \to \mathrm{Bl}_M(Z)$. This will be vital for applying Lemma 5.1.8 throughout the construction.

Lemma 5.1.11. *Let $\mathcal{Z} \subset X$ be a closed subscheme and $\varphi : V \to X$ be an étale morphism. Then $\mathcal{Z}$ is Cartier at a closed point $x \in X$ if and only if $\varphi^{-1}(\mathcal{Z})$ is Cartier at some closed point of $\varphi^{-1}(x)$.*

Proof. Let $\mathcal{Z}$ be Cartier at a closed point x, so in fact $\mathcal{Z}$ is Cartier on some open affine neighborhood U of x. Since φ is étale, we have in particular that $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is flat. It follows that $\varphi^{-1}(\mathcal{Z})$ is Cartier on $\varphi^{-1}(U)$, and in particular, at every closed point of $\varphi^{-1}(x)$.

Conversely, suppose $\varphi^{-1}(\mathcal{Z})$ is Cartier at some closed point y of $\varphi^{-1}(x)$. Let $\mathrm{Spec}(A) \subset X$ be an affine open neighborhood of x, and let $\mathrm{Spec}(B) \subset \varphi^{-1}(S)$ be an affine open neighborhood of y where $\varphi^{-1}(\mathcal{Z})$ is Cartier. Let $\eta : A \to B$ be the ring homomorphism corresponding to $\varphi|_{\mathrm{Spec}(B)}$, $I \subset A$ the ideal corresponding to $\mathcal{Z}$, $m_x \subset A$ the maximal ideal corresponding to x, and $m_y \subset B$ the maximal ideal corresponding to y. With $\hat{A}$ and $\hat{B}$ denoting the completion with respect to m_x and m_y respectively, we have $\hat{\eta} : \hat{A} \to \hat{B}$ is an isomorphism after possibly some base extension. As $\eta(I)B$ is principal and generated by a nonzero divisor, the same is true for $I\hat{A}$. By Nakayama's lemma, it suffices to show that $IA_m \subset A_m$ is principal and generated by a nonzero divisor. This is the content of the following lemma. □

Lemma 5.1.12. *Let (R, m) be a Noetherian local ring and $I \subset R$ an ideal. Denote by $\hat{}$ the completion with respect to m. Suppose that $\hat{I} = I \otimes_R \hat{R}$ is principal and generated by a nonzero divisor. Then I is principal and generated by a nonzero divisor as well.*

Proof. Once it is shown that $I = (t)$ for some $t \in R$, it follows that t is a nonzero divisor from the faithfully flat map $R \to \hat{R}$.

By Nakayama's lemma, it suffices to show that I/mI is generated by a single element as an R/m-module. In the following, all isomorphisms are as R/m-modules.

$$\begin{aligned} I/mI &\cong I \otimes_R R/m \\ &\cong I \otimes_R \hat{R}/\hat{m} \\ &\cong (I \otimes_R \hat{R}) \otimes_{\hat{R}} \hat{R}/\hat{m} \\ &\cong \hat{I}/\hat{m}\hat{I} \end{aligned}$$

Now $\hat{I}$ is generated by a single element as an $\hat{R}$-module, and hence so is $\hat{I}/\hat{m}\hat{I}$ as an $\hat{R}/\hat{m}$-module. But as $\hat{R}/\hat{m} = R/m$, we have that $\hat{I}/\hat{m}\hat{I}$, and hence I/mI, is generated by a single element as an R/m-module. □

Lemma 5.1.13. *Let $\mathcal{Z} \subset X$ be a closed subscheme, $\varphi : V \to X$ étale, $\mathcal{Z}_V = \varphi^{-1}(\mathcal{Z})$, and $\rho_X : Bl_X(\mathcal{Z}) \to X$ (respectively $\rho_V : Bl_V(\mathcal{Z}_V) \to V$) be the blowing up of $\mathcal{Z}$ in X (respectively $\mathcal{Z}_V$ in V). Then*

$$C_{\rho_V} = \varphi^{-1}(C_{\rho_X}), \qquad C^{fund}_{\rho_V} = \varphi^{-1}(C^{fund}_{\rho_X}), \quad \text{and} \quad C^{res}_{\rho_V} = \varphi^{-1}(C^{res}_{\rho_X}).$$

Proof. Note that since φ is étale, the pullback of a reduced subscheme is reduced. As such, all subschemes of V in the above equalities have reduced scheme structure, and it suffices to verify the equalities as sets.

That $C_{\rho_V} = \varphi^{-1}(C_{\rho_X})$ follows immediately from Lemma 5.1.11. The second equality $C^{\text{fund}}_{\rho_V} = \varphi^{-1}(C^{\text{fund}}_{\rho_X})$ is clear from the cartesian diagram

$$\begin{tikzcd}
\mathrm{Bl}_V(\mathcal{Z}_V) \arrow[r] \arrow[d] & \mathrm{Bl}_X(\mathcal{Z}) \arrow[d] \\
V \arrow[r] & X
\end{tikzcd}$$

as mentioned in Lemma 5.1.9. Finally, $C^{\text{res}}_{\rho_V} = \varphi^{-1}(C^{\text{res}}_{\rho_X})$ follows immediately from the previous two statements. □

Corollary 5.1.14. *Let X be a scheme with étale local model M, $\mathcal{Z} \subset X$ a closed subscheme étale locally corresponding to the closed subscheme $Z \subset M$. Then the true center, fundamental center, and residual locus of $Bl_X(\mathcal{Z}) \to X$ étale locally corresponds respectively to the true center, fundamental center, and residual locus of $Bl_M(Z) \to M$.*

The next proposition says that strict transforms may be calculated étale locally.

Proposition 5.1.15. *Let $\mathcal{Z} \subset X$ be a closed subscheme, $\varphi : V \to X$ be étale, and $\mathcal{Z}_V = \varphi^{-1}(\mathcal{Z})$ giving the following cartesian diagram.*

$$\begin{array}{ccc} Bl_V(\mathcal{Z}_V) & \xrightarrow{\varphi'} & Bl_X(\mathcal{Z}) \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \rho_X} \\ V & \xrightarrow{\varphi} & X \end{array}$$

Then

$$ST_{\rho_V}(\varphi^{-1}(\mathcal{Z})) = (\varphi')^{-1}(ST_{\rho_X}(\mathcal{Z})).$$

Proof. In the case that $\mathcal{Z}$ is contained in the true center of ρ_X, the strict transform of $\mathcal{Z}$ under ρ_X is defined to be $\rho_X^{-1}(\mathcal{Z})$. Then the claim follows immediately from the definitions and Lemma 5.1.13.

So assume that $\mathcal{Z}$ is not contained in the true center $\mathcal{C}_{\rho_X}$. Then

$$\begin{aligned} \mathrm{ST}_{\rho_V}(\varphi^{-1}(\mathcal{Z})) &= \overline{\rho_V^{-1}(\varphi^{-1}(\mathcal{Z}) \setminus C_{\rho_V})} \\ &= \overline{\rho_V^{-1}(\varphi^{-1}(\mathcal{Z} \setminus C_{\rho_X}))} && \text{by Lemma 5.1.13} \\ &= \overline{(\varphi')^{-1}(\rho_X^{-1}(\mathcal{Z} \setminus C_{\rho_X}))} \\ &= (\varphi')^{-1}\left(\overline{\rho_X^{-1}(\mathcal{Z} \setminus C_{\rho_X})}\right) && \text{since } \varphi' \text{ is flat and hence open} \\ &= (\varphi')^{-1}(\mathrm{ST}_{\rho_X}(\mathcal{Z})). \end{aligned}$$

$\square$

Lemma 5.1.16. *Let M and M' be étale local models of X and X' respectively. Suppose there is an étale cover $\varphi : V \to X$ with an étale morphism $\psi : V \to M$ along with morphisms ρ_X and ρ_M giving the diagram*

$$
\begin{tikzcd}
X' \arrow[d, "\rho_X"'] & V' \arrow[l, "\varphi'"'] \arrow[r, "\psi'"] \arrow[d, "\rho_V"] & M' \arrow[d, "\rho_M"] \\
X & V \arrow[l, "\varphi"'] \arrow[r, "\psi"] & M
\end{tikzcd}
$$

where the left and right squares are cartesian.

(i). Let $\mathcal{Z} \subset X$ be a closed subscheme étale locally corresponding to a closed subscheme $Z \subset M$ with respect to φ and ψ. Then $\rho_X^{-1}(\mathcal{Z})$ étale locally corresponds to $\rho_M^{-1}(Z)$.

(ii). Let $v' \in V'$ be a closed point, and set

$$x' = \varphi'(v'), \quad y' = \psi'(v'), \quad v = \rho_V(v'), \quad x = \varphi(v), \quad y = \psi(v).$$

Suppose further that $k(x) = k(v)$. Then $k(x') = k(v')$ and

$$\rho_X^{-1}(x) \cong \rho_M^{-1}(y) \times_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(x)).$$

Proof. (i) Pulling back $\rho_X^{-1}(\mathcal{Z})$ (respectively $\rho_M^{-1}(Z)$) to V' is the same as pulling back $\varphi^{-1}(\mathcal{Z})$ (respectively $\psi^{-1}(Z)$) to V'. With $\varphi^{-1}(\mathcal{Z}) = \psi^{-1}(Z)$ as subschemes of V, the claim follows.

(ii) With $v' \in V'$ a closed point mapping to $x' \in X'$ and $v \in V$, $k(v')$ is a quotient of $k(x') \otimes_{k(x)} k(v) = k(x')$. Thus $k(x') = k(v')$.

For the second statement, note that there is an inclusion $k(y) \subset k(v) = k(x)$ giving the morphism $\mathrm{Spec}(k(x)) \to \mathrm{Spec}(k(y))$. Thus

$$\begin{aligned}
\rho_X^{-1}(x) &= X' \times_X \mathrm{Spec}(k(x)) \\
&= (X' \times_X V) \times_V \mathrm{Spec}(k(x)) \\
&= (M' \times_M V) \times_V \mathrm{Spec}(k(x)) \\
&= M' \times_M \mathrm{Spec}(k(x)) \\
&= (M' \times_M \mathrm{Spec}(k(y))) \times_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(x)) \\
&= \rho_M^{-1}(y) \times_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(x)).
\end{aligned}$$

□

Lemma 5.1.17. *Let $f : X \to Y$ be a morphism of schemes. Suppose there exists a closed point $y \in Y$ such that $f^{-1}(y)$ is connected and the set-theoretic image of every connected component of X under f contains y. Then X is connected.*

Proof. Let $X = X_1 \coprod X_2$ where X_i is open and closed in X for $i = 1, 2$. Then

$$f^{-1}(y) = (f^{-1}(y) \cap X_1) \cup (f^{-1}(y) \cap X_2)$$

with $f^{-1}(y) \cap X_i$ being open and closed in $f^{-1}(y)$ for $i = 1, 2$. Since $f^{-1}(y)$ is connected, without loss of generality suppose $f^{-1}(y) = f^{-1}(y) \cap X_1$. Then $f^{-1}(y) \cap X_2 = \emptyset$. As the image of each connected of X contains y, it must be that X_2 is not a union of connected components; i.e. $X_2 = \emptyset$. Therefore X is connected. □

5.1.1 Étale cover

In view of Proposition 5.1.8, it will be important that we work with a fixed étale cover of the integral model and a fixed étale morphism to the local model in each step. It is also important that we chose such a cover so that we are able to apply Proposition 5.1.16 part (ii) as well as Proposition 4.2.6. We now describe the covers and morphisms to be used throughout the construction.

Proposition 5.1.18. *There exists an étale cover $\varphi : V \to \mathcal{A}_0$ and an étale morphism $\psi : V \to U_0$ such that for each closed point $x \in \mathcal{A}_0$, there exists a closed point $v \in V$ with $x = \varphi(v)$ and $k(x) = k(v)$.*

Proof. This follows from modifying the argument given in [DP, Section 3]. Let us briefly sketch how. Let $x : \mathrm{Spec}(k) \to \mathcal{A}_0$ be a closed point with $k = k(x)$ and

$$\mathcal{A}_0^{\mathrm{univ}} \to \mathcal{A}_1^{\mathrm{univ}} \to \mathcal{A}_2^{\mathrm{univ}}$$

be the chain of universal abelian schemes over $\mathcal{A}_0$. By [RZ, Proposition A.56], there exists a Zariski open neighborhood $U \subset \mathcal{A}_0$ of x where $H^1_{dR}(A_i^{\mathrm{univ}})$ is free for all i. This gives a section $U \to \widetilde{\mathcal{A}_0}$ of $\widetilde{\mathcal{A}_0} \to \mathcal{A}_0$ and hence, restricting if necessary, a morphism

$$\varphi : U \to \widetilde{\mathcal{A}_0} \to U_0.$$

By taking V to be the Zariski open subset of U where φ is étale, the proposition will follow from showing φ is étale at x.

Let $y = \varphi(x)$, $\mathcal{A}_0^{\wedge x}$ the completion of $\mathcal{A}_0$ at x, $U_0^{\wedge y}$ the completion of U_0 at y, and $U_0^{\wedge y} \otimes k$

be the extension of residue field of $U_0^{\wedge y}$ to k. It suffices to show that the induced map $\mathcal{A}_0^{\wedge x} \to U_0^{\wedge y} \otimes k$ is an isomorphism.

Let $B = k[\varepsilon]/(\varepsilon^2)$ and

$$(A_0 \to A_1 \to A_2, \lambda_0, \lambda_n, \overline{\eta})$$

correspond to $x : \mathrm{Spec}(k) \to \mathcal{A}_0$. As in [dJ2, Section 2] the principal polarizations induce the nondegenerate alternating pairings

$$e_{\lambda_j} : \mathbb{D}(A_j) \times \mathbb{D}(A_j) \to \mathcal{O}_{k_{crys}} \quad \text{for } j = 0, n.$$

Let $A'_0 \to \cdots \to A'_n$ be a chain of abelian schemes over $\mathrm{Spec}(B)$ with A'_i a deformation of A_i for every i. Then as shown in [dJ2, Proposition 4.5], the polarization induced by x lifts giving a deformation $\mathrm{Spec}(B) \to \mathcal{A}_{0,x}^{\wedge}$ of x if and only if for $j = 0, n$ the corresponding filtration $\mathrm{Fil}_j^1 \subset \mathbb{D}(A_j)_{\mathrm{Spec}(B)}$ is totally isotropic with respect to the pairing $e_{\lambda_j, \mathrm{Spec}(B)}$. It follows that the $\mathrm{Spec}(B)$-valued points of $\mathcal{A}_0^{\wedge x}$ and $U_0^{\wedge y} \otimes k$ are the same and hence the map $\mathcal{A}_0^{\wedge x} \to U_0^{\wedge y} \otimes k$ induces an isomorphism on Zariski tangent spaces.

Now the proof given in [DP, Theorem 3.3] below Lemma 3.5 shows that the map $\mathcal{A}_0^{\wedge x} \to U_0^{\wedge y} \otimes k$ is indeed an isomorphism. $\square$

Remark 5.1.19. In the course of the proof of the above proposition, φ and ψ were chosen so that we have the diagram

$$\begin{tikzcd} & & \widetilde{\mathcal{A}_0} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ V \arrow[r, "\varphi"] \arrow[urr, bend left] & \mathcal{A}_0 & & U_0 \end{tikzcd}$$

where ψ is given by the composition $V \to \widetilde{\mathcal{A}_0} \to U_0$. Thus Proposition 4.2.6 may be applied. In fact, since Ψ is surjective [Gen, Proposition 1.3.2], we also choose $V \to U_0$ such that ψ is surjective.

We fix such an étale cover $\varphi : V \to \mathcal{A}_0$ and étale morphism $\psi : V \to U_0$ as in the proposition.

In Step I we construct

$$\mathcal{A}_0' = \mathrm{Bl}_{\mathcal{A}_0}(\mathcal{Z}_1) \quad \text{and} \quad U_0' = \mathrm{Bl}_{U_0}(Z_1)$$

taking the induced étale cover $\varphi' : V' \to \mathcal{A}_0'$ with étale morphism $\psi' : V' \to U_0'$ as described in the proof of Proposition 5.6.1. In Step II, the schemes constructed are

$$\mathcal{A}_1'' = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_0' \qquad U_1'' = U_1 \times_{U_0} U_0'.$$

The étale cover and morphism in Step II are given by the top horizontal arrows of the diagram

$$\begin{tikzcd}
\mathcal{A}_1'' \arrow[d, "\rho_{\mathcal{A}}''"'] & V'' \arrow[l, "\varphi''"'] \arrow[r, "\psi''"] \arrow[d, "\rho_V''"] & U_1'' \arrow[d, "\rho_U''"] \\
\mathcal{A}_0' & V' \arrow[l, "\varphi'"'] \arrow[r, "\psi'"] & U_0'
\end{tikzcd}$$

where both squares are cartesian. With each step after II being given by a blowup, we again take the étale cover and étale morphism described in the proof of Proposition 5.6.1.

Repeated applications of Proposition 5.1.16 part (ii) give the following.

Proposition 5.1.20. *Let $1 \leq i \leq p+1$ and $x^{[i-1]} \in \mathcal{A}_j^{[i-1]}$ be a closed point, where $j = 0$ if $i = 1, 2$ and $j = 1$ if $i > 2$. Then there exists a closed point $v^{[i-1]} \in V^{[i-1]}$ with $x^{[i-1]} =$*

$\varphi^{[i-1]}(v^{[i-1]})$ such that $k(x^{[i-1]}) = k(v^{[i-1]})$, where $k(\cdot)$ denotes the residue field. Let $v^{[i]} \in V^{[i]}$ be any closed point such that $\rho_V^{[i]}(v^{[i]}) = v^{[i-1]}$ and set $y^{[i-1]} = \psi(v^{[i-1]})$. Then

$$(\rho_{\mathcal{A}}^{[i]})^{-1}(x^{[i-1]}) \cong (\rho_U^{[i]})^{-1}(y^{[i-1]}) \times_{\mathrm{Spec}(k(y^{[i-1]}))} \mathrm{Spec}(k(x^{[i-1]})).$$

In particular if the fiber $(\rho_U^{[i]})^{-1}(y^{[i-1]})$ is geometrically connected, then $(\rho_{\mathcal{A}}^{[i]})^{-1}(x^{[i-1]})$ is connected.

5.1.2 Notation

Before beginning the construction of the resolution, we explain some notation that will be used throughout.

Roman letters such as Z, C, and E will be used to denote subschemes of the local models. Calligraphic letters such as $\mathcal{Z}$, $\mathcal{C}$, and $\mathcal{E}$ denote subschemes of the integral models that étale locally correspond to their Roman counterparts.

The construction will proceed in steps, where Step II is constructed by a fiber product and every other step is constructed by a blowup. The schemes constructed in each additional step will be decorated with an additional tick mark $'$, and the superscript $[i]$ denotes i tick marks (e.g. $\mathcal{A}_0^{[0]} = \mathcal{A}_0$, $\mathcal{A}_0^{[1]} = \mathcal{A}_0'$). The integral models that will be constructed are

$$\mathcal{A}_1^{[p+1]} \xrightarrow{\rho_{\mathcal{A}}^{[p+1]}} \mathcal{A}_1^{[p]} \xrightarrow{\rho_{\mathcal{A}}^{[p]}} \dots \xrightarrow{\rho_{\mathcal{A}}^{[5]}} \mathcal{A}_1^{[4]} \xrightarrow{\rho_{\mathcal{A}}^{[4]}} \mathcal{A}_1''' \xrightarrow{\rho_{\mathcal{A}}'''} \mathcal{A}_1'' \xrightarrow{\rho_{\mathcal{A}}''} \mathcal{A}_0' \xrightarrow{\rho_{\mathcal{A}}'} \mathcal{A}_0$$

with their corresponding étale local models

$$U_1^{[p+1]} \xrightarrow{\rho_U^{[p+1]}} U_1^{[p]} \xrightarrow{\rho_U^{[p]}} \dots \xrightarrow{\rho_U^{[5]}} U_1^{[4]} \xrightarrow{\rho_U^{[4]}} U_1''' \xrightarrow{\rho_U'''} U_1'' \xrightarrow{\rho_U''} U_0' \xrightarrow{\rho_U'} U_0.$$

Moreover, any subscheme will also be decorated by tick marks, e.g. $\mathcal{Z}''$ signifies that $\mathcal{Z}'' \subset \mathcal{A}_1''$.

The subschemes that will be blown up in each step will arise from subschemes of $\mathcal{A}_0$. As such, it will be necessary to observe how these subschemes of $\mathcal{A}_0$ transform (either their strict transform or scheme-theoretic inverse image) in each step. To keep track of this, we will use a subscript to denote which step the subscheme will be used in. So for example, $\mathcal{C}_4$ is a subscheme of $\mathcal{A}_0$. The $\mathcal{C}$ indicates it will transform to be the true center of some blowup, and the 4 indicates that it will become the true center of Step 4. So in this example, we start with the strict transform $\mathcal{C}_4' = \mathrm{ST}_{\rho'}(\mathcal{C}_4)$ and then $\mathcal{C}_4'' = (\rho'')^{-1}(\mathcal{C}_4')^{\mathrm{red}}$. Finally $\mathcal{C}_4''' = \mathrm{ST}_{\rho'''}(\mathcal{C}_4'')$ and, as we will show, this is the true center of the blowup $\mathcal{A}_1^{[4]} \to \mathcal{A}_1'''$ in Step 4.

In general, C and $\mathcal{C}$ will denote true centers, E and $\mathcal{E}$ will denote exceptional loci, and Z_{ij} and $\mathcal{Z}_{ij}$ will denote irreducible components. For $x \in \mathrm{Perm}(\mu)$, $\mathcal{A}_x$ will denote the KR-stratum corresponding to x and n_x the number of connected components of $\mathcal{A}_x$ (see Chapter 3 and the next subsection).

5.1.3 Connected components

Let $(A_0 \to A_1 \to A_2, \lambda_0, \lambda_2, \overline{\eta})$ correspond to a closed point $\mathrm{Spec}(k) \to \mathcal{A}_0$. The functor $\mathcal{A}_0 \to \mathcal{A}_{n,1,N}$ from Section 2.4 induces $A_0[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^4$. Combining this with the Weil pairing induced by λ_0, we get the homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^4 \times (\mathbb{Z}/N\mathbb{Z})^4 \xrightarrow{\sim} A[N] \times A[N] \to \mu_N(k)$$

where μ_N denotes the Nth roots of unity. Taking the highest exterior power gives $\mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N$ which is equivalent to choosing a primitive Nth root of unity ζ_N. Therefore the structure morphism of $\mathcal{A}_0$ over $\mathrm{Spec}(\mathbb{Z}_p)$ factors as

$$\mathcal{A}_0 \to \mathrm{Spec}(\mathbb{Z}_p[\zeta_N]) \to \mathrm{Spec}(\mathbb{Z}_p).$$

The fibers over the closed points of $\mathrm{Spec}(\mathbb{Z}_p[\zeta_N])$ are connected [Hai, Lemma 13.2].

Furthermore, in each connected component every KR-stratum is nonempty [Hai, Lemma 13.2] and every KR-stratum has the same number of connected components [GY1]. In view of the this, for our construction of the resolution it suffices to treat a single connected component of $\mathcal{A}_0 \to \mathrm{Spec}(\mathbb{Z}_p[\zeta_N])$. By abuse of notation we will write $\mathcal{A}_0$ for such a connected component over $\mathrm{Spec}(\mathbb{Z}_p[\zeta_N])$. Similarly, we write $\mathcal{A}_1$ for the union of connected components lying over $\mathcal{A}_0$ with respect to the map $\pi : \mathcal{A}_1 \to \mathcal{A}_0$.

5.1.4 KR Strata

The KR strata of $\mathcal{A}_0$ will play a fundamental role in what follows. Recall that the strata of $M^{\mathrm{loc}} \otimes \mathbb{F}_p$ correspond to the μ-permissible set of W^{aff}.

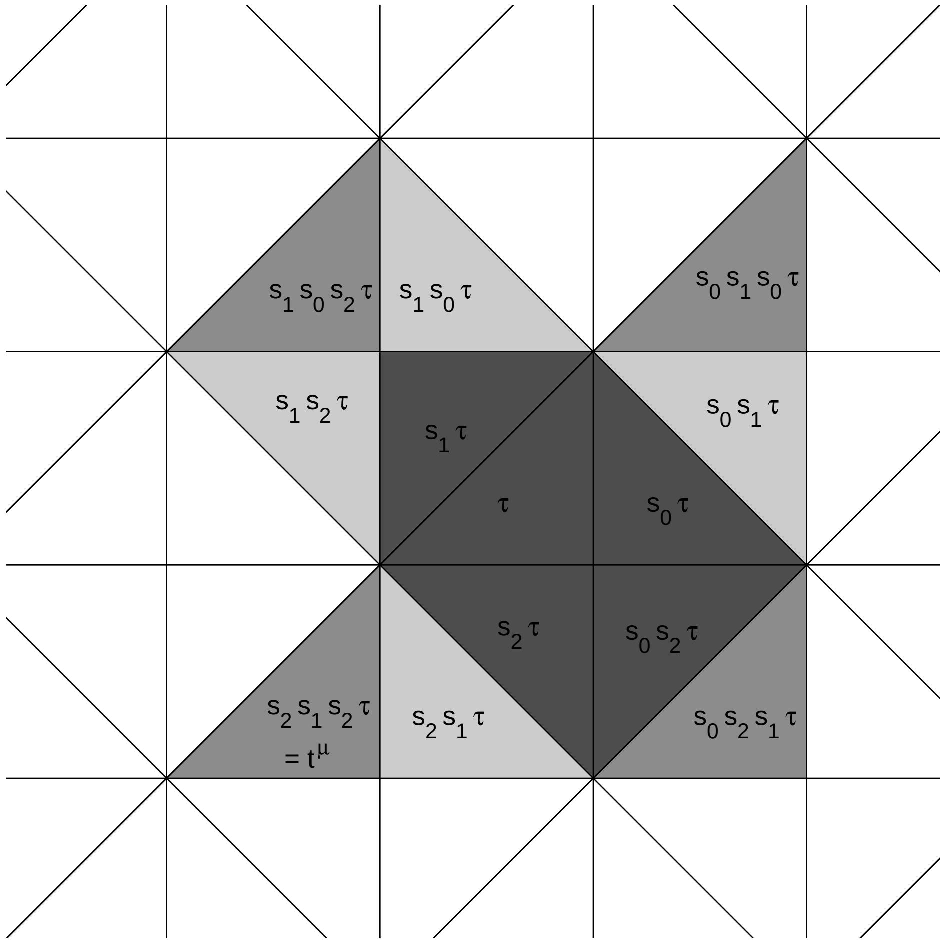

Figure 5.1: μ-permissible set for GSp_4

The alcoves that make up the μ-permissible set are shown above in various shades of gray. The collection $\{s_i\}$ are the standard generators of the affine Weyl group of GSp_4 (see Section A.2). The dimension of each stratum can be read off the corresponding $w \in W^{\mathrm{aff}}$ by $\ell(w)$, where $\ell(\cdot)$ is the length with respect to the Bruhat order (see Section A.2 and Proposition A.3.9). The stratum S_τ corresponding to the base alcove τ is the unique KR-stratum of dimension zero, which corresponds to the worst point of the local model. The irreducible components are the extreme alcoves which are shaded medium gray. By [GY2, Theorem 1.5] the extreme alcoves are connected, and as they are smooth, it follows that there are precisely four irreducible components of $\mathcal{A}_0 \otimes \mathbb{F}_p$. The supersingular locus is pictured in dark gray, given by $\overline{\mathcal{A}_{s_0 s_2 \tau} \cup \mathcal{A}_{s_1 \tau}}$. Note that the supersingular locus is precisely the locus where the zero section is an Oort-Tate generator of both of the corresponding group schemes.

5.2 Step 0: $\mathcal{A}_0$, U_0, and $\mathcal{A}_1$

5.2.1 Description of the local model U_0

A presentation of U_0 is given as the closed subscheme of $\mathrm{Spec}(\mathbb{Z}_p[a^i_{jk}; i = 0, \ldots, 3, j, k = 1, 2])$ cut out by the following two collections of equations. This first collection comes from the equations of the local model associated with GL_4.

$$a^i_{21} = a^{i+1}_{12} a^i_{11}, \quad a^i_{22} = a^{i+1}_{11} + a^{i+1}_{12} a^i_{12}, \quad i \in \mathbb{Z}/4\mathbb{Z}$$

$$(a^{i+2}_{11} + a^{i+2}_{12} a^{i+1}_{12}) a^i_{11} - p, \quad i \in \mathbb{Z}/4\mathbb{Z}$$

$$a^{i+2}_{12} a^{i+1}_{11} + (a^{i+2}_{11} + a^{i+2}_{12} a^{i+1}_{12}) a^i_{12}, \quad i \in \mathbb{Z}/4\mathbb{Z}$$

We also have to include those coming from the duality condition (see Proposition 3.10).

$$a^0_{22} = a^0_{11} \quad a^2_{22} = a^2_{11}$$

$$a_{11}^3 = a_{22}^1 \quad a_{12}^3 = -a_{12}^1 \quad a_{21}^3 = -a_{21}^1 \quad a_{22}^3 = a_{11}^1.$$

By setting

$$x = a_{22}^1 \quad y = a_{11}^0 \quad a = a_{12}^0 \quad b = a_{12}^2 \quad c = -a_{12}^1$$

we arrive at the equations derived in [dJ2]:

$$U_0 = \mathrm{Spec}(B) \quad \text{where} \quad B = \mathbb{Z}_p[x, y, a, b, c]/(xy - p, ax + by + abc).$$

With this presentation we have that, up to a unit:

$$q_0 = y, \quad q_0^* = x, \quad q_1 = y + ac, \quad q_1^* = x + bc.$$

Following [dJ2, Section 5], we label the four irreducible components of $U_0 \otimes \mathbb{F}_p$ as

$$Z_{00} = Z(y, a)$$

$$Z_{01} = Z(y, x + bc)$$

$$Z_{10} = Z(x, y + ac)$$

$$Z_{11} = Z(x, b).$$

5.2.2 Description of the integral model $\mathcal{A}_0$

Still following [dJ2, Section 5], using the local model diagram

$$\begin{tikzcd} & \widetilde{\mathcal{A}_0} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{A}_0 & & U_0 \end{tikzcd}$$

we define $\mathcal{Z}_{ij} = \Phi(\Psi^{-1}(Z_{ij}))$. Since there are precisely four irreducible components of $\mathcal{A}_0 \otimes \mathbb{F}_p$ [Yu], the $\mathcal{Z}_{ij}$ make up all of the irreducible components of $\mathcal{A}_0 \otimes \mathbb{F}_p$.

Proposition 5.2.1. *$\mathcal{Z}_{ij}$ étale locally corresponds to Z_{ij}. Moreover this holds for arbitrary unions, intersections, and complements of the $\mathcal{Z}_{ij}$, e.g. $\mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$ étale locally corresponds to $Z_{11} \cup (Z_{01} \cap Z_{10})$, where each is given the reduced scheme structure.*

Proof. Let $x \in \mathcal{A}_0$ be a closed point of $\mathcal{Z}_{ij}$. With $\mathcal{A}_0 \leftarrow V \rightarrow U_0$ our chosen étale cover and étale morphism, by Remark 5.1.19 we may apply Proposition 4.2.6. Thus we have that pullback of q_0 and y to V induce the same divisor on V, and likewise for q_0^*, q_1, and q_1^* with x, $y + ac$, $x + bc$ respectively. Now each $\mathcal{Z}_{ij}$ can be described as the Zariski closure of the locus given by certain conditions on the functions q_i and q_i^*.

Irreducible component	Conditions
$\mathcal{Z}_{00}$	$q_0 = 0,\ q_0^* \neq 0,\ q_1 = 0,\ q_1^* \neq 0$
$\mathcal{Z}_{01}$	$q_0 = 0,\ q_0^* \neq 0,\ q_1 \neq 0,\ q_1^* = 0$
$\mathcal{Z}_{10}$	$q_0 \neq 0,\ q_0^* = 0,\ q_1 = 0,\ q_1^* \neq 0$
$\mathcal{Z}_{11}$	$q_0 \neq 0,\ q_0^* = 0,\ q_1 \neq 0,\ q_1^* = 0$

Table 5.1: Irreducible components of $\mathcal{A}_0$

Thus we have that the pullback of $\mathcal{Z}_{ij}$ and Z_{ij} to V define the same closed subscheme and hence étale locally correspond. The claim for arbitrary unions, intersections, and complements follows by applying Lemma 5.1.8. □

We now define subschemes that will be used in the steps throughout the construction of the resolution.

Local model U_0			Integral model $\mathcal{A}_0$		
Z_1	$=$	$Z(x,b)$	$\mathcal{Z}_1$	$=$	$\mathcal{Z}_{11}$
C_1	$=$	$Z(x,y,a,b)$	$\mathcal{C}_1$	$=$	$\mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}$
Z_3	$=$	$Z(x,bc)$	$\mathcal{Z}_3$	$=$	$\mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$
C_3	$=$	$Z(x,bc)$	$\mathcal{C}_3$	$=$	$\mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$
Z_4	$=$	$Z(x,b)$	$\mathcal{Z}_4$	$=$	$\mathcal{Z}_{11}$
C_4	$=$	$Z(x,y,b,c)$	$\mathcal{C}_4$	$=$	$\overline{(\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}) \setminus \mathcal{Z}_{00}}$

Table 5.2: Subschemes of $\mathcal{A}_0$

Proposition 5.2.1 shows that the subschemes on the left étale locally correspond to those on the right.

Proposition 5.2.2. *Writing each subscheme below as a disjoint union of KR-strata, we have the following.*

$$\begin{array}{ccccc} \mathcal{Z}_{11} & = & \overline{\mathcal{A}_{s_2 s_1 s_2 \tau}} & = & \mathcal{A}_\tau \cup \mathcal{A}_{s_1\tau} \cup \mathcal{A}_{s_2\tau} \cup \mathcal{A}_{s_2 s_1 \tau} \cup \mathcal{A}_{s_2 s_1 s_2 \tau} \\ \mathcal{Z}_{01} \cap \mathcal{Z}_{10} & = & \overline{\mathcal{A}_{s_0 s_2 \tau} \cup \mathcal{A}_{s_1 \tau}} & = & \mathcal{A}_\tau \cup \mathcal{A}_{s_0\tau} \cup \mathcal{A}_{s_1\tau} \cup \mathcal{A}_{s_2\tau} \cup \mathcal{A}_{s_0 s_2 \tau} \\ \mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11} & = & \overline{\mathcal{A}_{s_1\tau}} & = & \mathcal{A}_\tau \cup \mathcal{A}_{s_1\tau} \\ \overline{(\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}) \setminus \mathcal{Z}_{00}} & = & \overline{\mathcal{A}_{s_2\tau}} & = & \mathcal{A}_\tau \cup \mathcal{A}_{s_2\tau} \end{array}$$

Proof. With our chosen presentation, the locus on U_0 corresponding to the supersingular locus of $\mathcal{A}_1$ is $Z_{01} \cap Z_{10}$. By Proposition 5.2.1 it follows that $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ is the supersingular locus. The supersingular locus is given by $\overline{\mathcal{A}_{s_0 s_2 \tau} \cup \mathcal{A}_{s_1\tau}}$ [GY1, Section 2] and the claim follows.

To show $\mathcal{Z}_{11} = \overline{\mathcal{A}_{s_2 s_1 s_2 \tau}}$ we choose the point $x = a = b = c = 0$ and $y = 1$ of U_0 which lies solely on the irreducible component Z_{11}. Using the equations in Section 5.2.1 and Chapter

3, this point corresponds to the following flag.

$$\mathcal{F}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the notation of Proposition A.3.10 this gives the alcove

$$\begin{pmatrix} 1 & & & \\ & \pi & & 1 \\ 1 & & \pi & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 1 & & \pi \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \pi^{-1} & & & \\ & 1 & & \\ \pi^{-1} & & 1 & \\ & & & 1 \end{pmatrix}$$

where we omit any entry which is zero. Our chosen point lies in $\mathcal{A}_{s_2 s_1 s_2 \tau}$ if and only if there is an element b in the Iwahori subgroup such that $b \cdot s_2 s_1 s_2 \tau$ gives the same alcove as above. With $s_2 s_1 s_2 \tau$ given by the alcove

$$\begin{pmatrix} \pi & & & \\ & \pi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \pi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \pi^{-1} & \\ & & & 1 \end{pmatrix}$$

it is easy to check that

$$b = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

suffices. Therefore $\mathcal{Z}_{11} = \overline{\mathcal{A}_{s_2 s_1 s_2 \tau}}$. Since $Z_{00} \cap Z_{11}$ is the union of a two dimensional and one dimensional scheme, we see from the μ-permissible set (Section 5.1.4) it must be that $\mathcal{Z}_{00} = \overline{\mathcal{A}_{s_0 s_1 s_0 \tau}}$. That $\mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11} = \mathcal{A}_\tau \cup \mathcal{A}_{s_1 \tau}$ again follows from the diagram in Section 5.1.4. Finally

$$\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11} = \mathcal{A}_\tau \cup \mathcal{A}_{s_1 \tau} \cup \mathcal{A}_{s_2 \tau}$$

and thus we have

$$(\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}) \setminus \mathcal{Z}_{00} = \mathcal{A}_{s_2 \tau}.$$

□

Proposition 5.2.3. *The number of connected and irreducible components of the subschemes of $\mathcal{A}_0$ are as follows.*

Subscheme of $\mathcal{A}_0$	# of connected components	# irreducible components
$\mathcal{Z}_1$	1	1
$\mathcal{C}_1$	$n_{s_1\tau}$	$n_{s_1\tau}$
$\mathcal{Z}_3$	1	$1 + n_{s_0 s_2 \tau}$
$\mathcal{C}_3$	1	$1 + n_{s_0 s_2 \tau}$
$\mathcal{Z}_4$	1	1
$\mathcal{C}_4$	$n_{s_2\tau}$	$n_{s_2\tau}$

Table 5.3: Number of connected and irreducible components of subschemes of $\mathcal{A}_0$

Proof. $\mathcal{Z}_1 = \mathcal{Z}_4 = \mathcal{Z}_{11}$: This is an irreducible component.

$\mathcal{Z}_3 = \mathcal{C}_3 = \mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$: To see this subscheme is connected, it suffices to show that each connected component of $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ meets $\mathcal{Z}_{11}$. Let $\mathcal{W}$ be such a connected component. With $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ being a union of KR-strata, by possibly shrinking $\mathcal{W}$ we may assume $\mathcal{W}$ is a connected component of some KR-stratum. By [GY2, Theorem 6.4] $\overline{\mathcal{W}} \cap \mathcal{A}_\tau \neq \emptyset$, where $\overline{\mathcal{W}}$ is the Zariski closure of $\mathcal{W}$ inside of $\mathcal{A}_0$. As $\mathcal{A}_\tau \subset \mathcal{Z}_{11}$, the claim follows.

To find the number of irreducible components, note that $\mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$ is a union of three and two dimensional irreducible components: the unique three dimensional component is $\mathcal{Z}_{11}$ and the two dimensional components are given by the irreducible components of

$$\overline{(\mathcal{Z}_{01} \cap \mathcal{Z}_{10}) \setminus \mathcal{Z}_{11}} = \overline{\mathcal{A}_{s_0 s_2 \tau}}.$$

As $\overline{(Z_{01} \cap Z_{10}) \setminus Z_{11}}$ corresponds to the ideal (x, y, a, b) of B, we have that $\overline{\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \setminus \mathcal{Z}_{11}}$ is smooth. Thus as $\overline{\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \setminus \mathcal{Z}_{11}}$ has $n_{s_0 s_2 \tau}$ connected components and each component is irreducible, we have a total of $1 + n_{s_0 s_2 \tau}$ irreducible components of $\mathcal{Z}_3 = \mathcal{C}_3$.

$\mathcal{C}_1 = \mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}$: From Proposition 5.2.2, this subscheme is given by $\overline{\mathcal{A}_{s_1 \tau}}$. Since C_1 and hence $\overline{\mathcal{A}_{s_1 \tau}}$ is smooth, each connected component of $\overline{\mathcal{A}_{s_1 \tau}}$ is irreducible and $\overline{\mathcal{A}_{s_1 \tau}}$ has the same number of connected components as $\mathcal{A}_{s_1 \tau}$.

$\mathcal{C}_4$: This follows by a similar argument to that given for the statement about $\mathcal{C}_1$. $\square$

5.2.3 Description of $\mathcal{A}_1$

As remarked in section 5.1.4, $\mathcal{A}_0 \otimes \mathbb{F}_p$ has four irreducible components. Using the notation of [dJ2, Section 5], we denote these irreducible components by $\mathcal{Z}_{00}$, $\mathcal{Z}_{01}$, $\mathcal{Z}_{10}$, and $\mathcal{Z}_{11}$. To determine the irreducible components of $\mathcal{A}_1 \otimes \mathbb{F}_p$, we will need the following.

Lemma 5.2.4. *The irreducible components of $U_1 \otimes \mathbb{F}_p$ are normal.*

Proof. The four irreducible components of $U_1 \otimes \mathbb{F}_p$ correspond to the following ideals in

$\mathbb{Z}_p[x, y, a, b, c, u, v]/(xy - p, ax + by + abc, u^{p-1} - x, v^{p-1} - x - bc)$.

$$(u, v, b), \quad (u, y + ac), \quad (y, v), \quad (y, a)$$

The first irreducible component is smooth, while the other three are the spectra of

$$\mathbb{F}_p[y, a, b, c, v]/(v^{p-1} - bc), \quad \mathbb{F}_p[a, b, c, u, v]/(u^{p-1} - bc), \quad \text{and} \quad \mathbb{F}_p[b, c, u, v]/(v^{p-1} - u^{p-1} - bc).$$

Note that each is a complete intersection and hence Cohen-Macaulay. The Jacobian Criterion shows that the singular-locus has codimension greater than one. By Serre's Criterion [Mat, Theorem 23.8], each is normal. □

Proposition 5.2.5. *$\mathcal{A}_1 \otimes \mathbb{F}_p$ is connected, equidimensional of dimension three, and the irreducible components are normal. Furthermore $\mathcal{A}_1 \otimes \mathbb{F}_p$ and has precisely four irreducible components and these irreducible components are given by $\pi^{-1}(\mathcal{Z}_{ij})^{red}$ where $\pi : \mathcal{A}_1 \to \mathcal{A}_0$.*

Proof. That $\mathcal{A}_1 \otimes \mathbb{F}_p$ is equidimensional of dimension three is immediate from inspection of the local model $U_1 \otimes \mathbb{F}_p$. As $\mathcal{A}_\tau$ is in the supersingular locus, the fiber above a closed point of $\mathcal{A}_\tau \subset \mathcal{A}_0$ with respect to π consists of a single closed point. Since π is finite and surjective, each irreducible component of $\mathcal{A}_1 \otimes \mathbb{F}_p$ maps surjectively onto an irreducible component of $\mathcal{A}_0 \otimes \mathbb{F}_p$. Finally as every irreducible component of $\mathcal{A}_0 \otimes \mathbb{F}_p$ contains $\mathcal{A}_\tau$, we conclude that $\mathcal{A}_1 \otimes \mathbb{F}_p$ is connected by Lemma 5.1.17.

Let $\mathcal{Z}'_{ij}$ denote a fixed irreducible component of $\mathcal{A}_1 \otimes \mathbb{F}_p$ which maps onto $\mathcal{Z}_{ij}$ and suppose there is a fifth irreducible component $\mathcal{Z}'$ of $\mathcal{A}_1 \otimes \mathbb{F}_p$. By [GY2, Theorem 6.4], each irreducible component of $\mathcal{A}_0 \otimes \mathbb{F}_p$ contains $\mathcal{A}_\tau$, and thus $\mathcal{Z}'_{00}, \mathcal{Z}'_{01}, \mathcal{Z}'_{10}, \mathcal{Z}'_{11}$, and $\mathcal{Z}'$ simultaneously intersect at some closed point x of $\mathcal{A}_1 \otimes \mathbb{F}_p$. We now show that this is not possible.

Let $\mathcal{A}_1 \otimes \mathbb{F}_p \xleftarrow{\varphi} V \xrightarrow{\psi} U_1 \otimes \mathbb{F}_p$ be an étale cover of $\mathcal{A}_1 \otimes \mathbb{F}_p$ with étale morphism to $U_1 \otimes \mathbb{F}_p$. Choose a closed point $p \in V$ such that $\varphi(p) = x$. Set $y = \psi(p)$. Since the irreducible components of $U_1 \otimes \mathbb{F}_p$ are integral, normal, and excellent, the completion of any irreducible component at y is also an integral domain and normal. Therefore, there are at most four components of V passing through p. Thus the number of irreducible components passing through $x \in \mathcal{A}_1 \otimes \mathbb{F}_p$ is at most four, giving the contradiction that we are seeking. □

5.3 Step I: Semi-stable resolution of $\mathcal{A}_0$

We define the integral and local models

$$\mathcal{A}_0' = \mathrm{Bl}_{\mathcal{A}_0}(\mathcal{Z}_1) \qquad \text{and} \qquad U_0' = \mathrm{Bl}_{U_0}(Z_1).$$

5.3.1 Description of the local model U_0'

With $\mathcal{I}$ the ideal sheaf on $\mathrm{Spec}(B)$ corresponding to the ideal $(x, b) \subset B$, the blowup is given by

$$U_0' = \mathrm{Proj}_{U_0}\left(\mathcal{O} \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots\right).$$

Note that we have the morphism

$$T = \mathrm{Proj}_{U_0}\left(B[\widetilde{x}, \widetilde{b}]/(a\widetilde{x} + \widetilde{b}y + a\widetilde{b}c, x\widetilde{b} - \widetilde{x}b)\right) \to U_0'$$

sending $\widetilde{x}$ and $\widetilde{b}$ respectively to x and b in grade one. The two standard affine open charts of T are given by

$$\widetilde{x} \neq 0: \quad T_1 = \operatorname{Spec}\left(\mathbb{Z}_p\left[x, y, a, c, \frac{\widetilde{b}}{\widetilde{x}}\right] / \left(xy - p, a + \frac{\widetilde{b}}{\widetilde{x}} y + a\frac{\widetilde{b}}{\widetilde{x}} c\right)\right)$$

and

$$\widetilde{b} \neq 0: \quad T_2 = \operatorname{Spec}\left(\mathbb{Z}_p\left[a, b, c, \frac{\widetilde{x}}{\widetilde{b}}\right] / \left(\frac{\widetilde{x}}{\widetilde{b}} b(-a)\left(\frac{\widetilde{x}}{\widetilde{b}} + c\right) - p\right)\right).$$

T_1 is covered by the two open subschemes defined by the conditions $\widetilde{b} \neq 0$ and $1 + \widetilde{b}c \neq 0$. Noting that the condition $\widetilde{b} \neq 0$ makes the first of these open subschemes a subscheme of T_2, we merely write down the presentation of the second, given by $1 + \widetilde{b}c \neq 0$:

$$T_1' = \operatorname{Spec}\left(\mathbb{Z}_p[x, y, c, \widetilde{b}, (1 + \widetilde{b}c)^{-1}]/(xy - p)\right).$$

From the affine open cover $T_1' \cup T_2$, we see that T is integral. As the blowup U_0' may be, a priori, cut out by more equations, we at least have a closed immersion $U_0' \to T$. This closed immersion is an isomorphism on the generic fiber and with T integral, it must be an isomorphism.

The morphism $U_0' \to U_0$ is given as homomorphisms of coordinate rings, sending the ordered set of global sections to the ordered set of global sections, as

$$T_1': \quad \{x, y, a, b, c\} \to \{\widetilde{x}b, -a(\widetilde{x} + c), a, b, c\}$$

$$T_2: \quad \{x, y, a, b, c\} \to \left\{x, y, \widetilde{b}y(1 + \widetilde{b}c)^{-1}, x\widetilde{b}, c\right\}.$$

The true center of $U_0' \to U_0$ is the closed subscheme where the ideal sheaf induced by the ideal

(x, b) is not Cartier. From the relation $ax + by + abc = 0$, we see that C_1 must be contained in the closed subscheme $Z(x, y, a, b)$. From the presentation of the morphism above, we see that the fiber over any closed point of $Z(x, y, a, b)$ is the affine line given by the coordinate $\widetilde{x}$. Therefore, $C_1 = Z(x, y, a, b)$ and is of dimension one, the true center of ρ'_U is equal to the fundamental center, and the exceptional locus of ρ'_U is two dimensional. The affine cover $T'_1 \cup T_2$ also shows that $U'_0 \otimes \mathbb{F}_p$ is equidimensional of dimension three. The strict transforms of the closed subschemes given in Step 0 are as follows.

$Z_3 = C_3 = Z(x, bc)$: The complement $Z_3 \setminus C_1$ a union of three subschemes defined by the following conditions.

$$x = 0, \quad b = 0, \quad a \neq 0$$

$$x = 0, \quad b = 0, \quad y \neq 0$$

$$x = 0, \quad b \neq 0, \quad a \neq 0, \quad c = 0$$

Now consider the inverse image of $Z_3 \setminus C_1$ under the morphisms $\rho'_U : U'_0 \to U_0$. The Zariski closure of the inverse image of the first two subschemes give the same subscheme, namely the subscheme corresponding to the ideals $(\widetilde{x}, b)$ and (x, b) in the coordinate rings of T'_1 and T_2 respectively. Likewise the third corresponds to $(x, y, c, \widetilde{x})$ and (1) in T'_1 and T_2 respectively. Thus $Z'_3 = C'_3$ is given by $Z(x, b) \cup Z(x, y, c, \widetilde{x})$.

$Z_4 = Z(x, b)$: This must necessarily be the irreducible component lying above $Z(x, b)$, and it is also given by $Z(x, b)$.

$C_4 = Z(x, y, b, c)$: The complement $Z_3 \setminus C_1$ given by $x = y = b = c = 0$ and $a \neq 0$. The Zariski closure of the inverse image is $Z(x, y, b, c)$.

In summary, we have:

$$Z_3' = Z(x,b) \cup Z(x,y,c,\widetilde{x})$$

$$C_3' = Z(x,b) \cup Z(x,y,c,\widetilde{x})$$

$$Z_4' = Z(x,b)$$

$$C_4' = Z(x,y,b,c).$$

5.3.2 Description of the integral model $\mathcal{A}_0'$

With U_0' an étale local model of $\mathcal{A}_0'$, Lemma 5.1.13 and Proposition 5.2.1 give that the true center of $\mathcal{A}_0' \to \mathcal{A}_0$ is $\mathcal{C}_1$. By the remarks in the previous section, $\mathcal{A}_0' \otimes \mathbb{F}_p$ is equidimensional of dimension three. From Lemma 5.1.16, the exceptional locus of $\mathcal{A}_0' \to \mathcal{A}_0$ is two dimensional. Thus no irreducible component of $\mathcal{A}_0' \otimes \mathbb{F}_p$ is contained in the exceptional locus. Therefore $\mathcal{A}_0' \otimes \mathbb{F}_p$ has four irreducible components, each being given by the strict transform of an irreducible component of $\mathcal{A}_0 \otimes \mathbb{F}_p$. We denote these strict transforms by $\mathcal{Z}_{00}'$, $\mathcal{Z}_{01}'$, $\mathcal{Z}_{10}'$, and $\mathcal{Z}_{11}'$.

Proposition 5.3.1. *The number of connected and irreducible components of the subschemes of $\mathcal{A}_0'$ are as follows.*

Closed subscheme of $\mathcal{A}_0'$	# connected components	# irreducible components
$\mathcal{Z}_3' = \mathrm{ST}(\mathcal{Z}_3)$	1	$n_{s_0 s_2 \tau} + 1$
$\mathcal{C}_3' = \mathrm{ST}(\mathcal{C}_3)$	1	$n_{s_0 s_2 \tau} + 1$
$\mathcal{Z}_4' = \mathrm{ST}(\mathcal{Z}_4)$	1	1
$\mathcal{C}_4' = \mathrm{ST}(\mathcal{C}_4)$	$n_{s_2 \tau}$	$n_{s_2 \tau}$

Table 5.4: Number of connected and irreducible components of subschemes of $\mathcal{A}_0'$

Proof. $\mathcal{Z}_3' = \mathcal{C}_3'$: We start by showing $\mathcal{Z}_3'$ is connected. From Proposition 5.2.3, $\mathcal{Z}_3 = \mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$ is a union of three and two dimensional components intersecting in a one dimensional closed subscheme. We claim that this one dimensional subscheme intersects with the true center $\mathcal{C}_1 = \mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}$ in a zero dimensional subscheme. This can easily be seen by writing each as a union of KR-strata. Indeed, set $\mathcal{W} = \overline{\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \setminus \mathcal{Z}_{11}}$ which is equidimensional of dimension two. Then $\mathcal{Z}_3 = \mathcal{Z}_{11} \cup \mathcal{W}$ and from Proposition 5.2.2, each is given as a union of KR-strata as follows.

$$
\begin{aligned}
\mathcal{Z}_{11} &= \mathcal{A}_\tau \cup \mathcal{A}_{s_1\tau} \cup \mathcal{A}_{s_2\tau} \cup \mathcal{A}_{s_2s_1\tau} \cup \mathcal{A}_{s_2s_1s_2\tau} \\
\mathcal{W} &= \mathcal{A}_\tau \cup \mathcal{A}_{s_0\tau} \cup \mathcal{A}_{s_2\tau} \cup \mathcal{A}_{s_0s_2\tau} \\
\mathcal{C}_1 &= \mathcal{A}_\tau \cup \mathcal{A}_{s_1\tau}
\end{aligned}
$$

Therefore the one dimensional subscheme $\mathcal{Z}_{11} \cap \mathcal{W}$ intersects with $\mathcal{C}_1$ in $\mathcal{A}_\tau$, a zero dimensional subscheme as claimed. With $\mathcal{Z}_{11}$ and $\mathcal{W}$ smooth it follows immediately that $\mathcal{Z}_3 \setminus \mathcal{C}_1$ is connected, and thus so is the strict transform of $\mathcal{Z}_3$. Therefore $\mathcal{Z}_3' = \mathcal{C}_3'$ is connected.

Now we show that $\mathcal{Z}_3' = \mathcal{C}_3'$ has $n_{s_0s_2} + 1$ irreducible components. From the proof of Proposition 5.2.3 we have $\mathcal{Z}_3$ is a union $n_{s_0s_2} + 1$ irreducible components, each of dimension two or three. Since the true center of $\mathcal{A}_0' \to \mathcal{A}_0$ is of dimension one, each irreducible component is not contained in the true center. Hence the strict transform of each irreducible component is irreducible as well and the claim follows.

$\mathcal{Z}_4'$: Note that $\mathcal{Z}_4$ is connected, smooth, and three dimensional giving that $\mathcal{Z}_4 \setminus \mathcal{C}_1$ is connected as well. It follows immediately that $\mathcal{Z}_4'$ is connected and irreducible.

$\mathcal{C}_4'$: As $\mathcal{C}_4$ is smooth, each irreducible component is a connected component. Also, $\mathcal{C}_4$ intersects the true center of $\mathcal{A}_0' \to \mathcal{A}_0$ in a zero-dimensional subscheme and therefore strict transform of each irreducible component of $\mathcal{C}_4$ is irreducible. $\square$

5.4 Step II: Fiber with $\mathcal{A}_1$

We define the integral and local models

$$\mathcal{A}_1'' = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_0' \qquad \text{and} \qquad U_1'' = U_1 \times_{U_0} U_0'.$$

5.4.1 Description of the local model U_1''

As in Theorem 4.2.10, U_1 is given in the chosen presentation by adjoining the variables u and v along with the relations $u^{p-1} - x$ and $v^{p-1} - (x + bc)$ to U_0. We thus have

$$U_1'' = \mathrm{Proj}_{U_0} \left(B[u,v][\widetilde{x}, \widetilde{b}]/(a\widetilde{x} + \widetilde{b}y + a\widetilde{b}c, x\widetilde{b} - \widetilde{x}b, u^{p-1} - x, v^{p-1} - (x + bc) \right)$$

where $B[u,v]$ is of grade 0 and $\widetilde{x}$ and $\widetilde{b}$ are of grade 1. We define Z_{ij}'' to be the reduced inverse image of Z_{ij}' under the morphism $\rho_U'' : U_1'' \to U_0'$. These are the irreducible components of

U_1''. The reduced inverse images under $U_1'' \to U_0'$ are given by

$$
\begin{aligned}
Z_{00}'' &= Z(y, a) \\
Z_{01}'' &= Z(y, v) \\
Z_{10}'' &= Z(u, y + ac) \\
Z_{11}'' &= Z(u, v, b) \\
Z_3'' &= Z(u, v, b) \cup Z(u, v, y, c, \widetilde{x}) \\
C_3'' &= Z(u, v, b) \cup Z(u, v, y, c, \widetilde{x}) \\
Z_4'' &= Z(u, v, b) \\
C_4'' &= Z(u, v, y, b, c, \widetilde{x}).
\end{aligned}
$$

Note that $Z(u, v, b) \cup Z(u, v, y, c, \widetilde{x}) = Z(u, v)$ as the relation $v^{p-1} - u^{p-1} - bc$ implies that if u and v are zero, then $bc = 0$ giving the two components.

5.4.2 Description of the integral model $\mathcal{A}_1''$

With $\mathcal{A}_1'' = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_0'$, the projection $\mathcal{A}_1'' \to \mathcal{A}_1$ is proper and birational and so it is a modification. Also note that the projection $\rho'' : \mathcal{A}_1'' \to \mathcal{A}_0'$ is finite and flat. As claimed in the introduction, we have the following.

Proposition 5.4.1. *$\mathcal{A}_1''$ is not normal.*

Proof. Since $\psi : V \to U_0$ and hence $\psi'' : V'' \to U_1''$ is surjective, it suffices to show that U_1'' is not normal. Consider the irreducible component Z_{11}'' of $U_1'' \otimes \mathbb{F}_p$. In the local ring of the generic point of this component, the maximal ideal is given by (u, v). This ideal is not principal since $u \notin (v)$ and $v \notin (u)$. Therefore U_1'' is not normal by Serre's Criterion [Mat, Theorem 23.8]. $\square$

Proposition 5.4.2. *Set $\mathcal{Z}_{ij}'' = (\rho'')^{-1}(\mathcal{Z}_{ij}')^{red}$. Each $\mathcal{Z}_{ij}''$ is an irreducible component of $\mathcal{A}_1'' \otimes \mathbb{F}_p$, and these give all the irreducible components of $\mathcal{A}_1'' \otimes \mathbb{F}_p$.*

Proof. From Proposition 5.2.5 we have that $\mathcal{W}_{ij} := \pi^{-1}(\mathcal{Z}_{ij})^{\mathrm{red}}$ is irreducible, where $\pi : \mathcal{A}_1 \to \mathcal{A}_0$. Note that the morphism $\mathcal{A}_1'' \to \mathcal{A}_1$ is a modification with true center of dimension at most one. As such, $\mathcal{W}_{ij}$ is not contained in the true center and therefore its strict transform $\mathcal{W}_{ij}''$ with respect to $\mathcal{A}_1'' \to \mathcal{A}_1$ is irreducible.

Set

$$\mathcal{U} = \mathcal{A}_0 \setminus \mathcal{C}_1, \qquad \mathcal{U}' = (\rho_{\mathcal{A}}')^{-1}(\mathcal{U}), \quad \text{and} \quad \mathcal{U}'' = (\rho_{\mathcal{A}}'')^{-1}(\mathcal{U}').$$

Then $\mathcal{Z}_{ij}'' \cap \mathcal{U}'' = \mathcal{W}_{ij}'' \cap \mathcal{U}''$ because both can be described as the reduced inverse image of $\mathcal{Z}_{ij} \cap (\mathcal{A}_0 \setminus \mathcal{C}_1)$ under the two paths in the following cartesian diagram.

$$\begin{array}{ccc} \mathcal{A}_1'' & \longrightarrow & \mathcal{A}_0' \\ \downarrow & & \downarrow \\ \mathcal{A}_1 & \longrightarrow & \mathcal{A}_0 \end{array}$$

As sets we have

$$
\begin{aligned}
\mathcal{Z}''_{ij} &= (\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_{ij}) \\
&= (\rho''_{\mathcal{A}})^{-1}(\overline{\mathcal{Z}'_{ij} \cap \mathcal{U}'}) && \text{since } \mathcal{Z}'_{ij} \text{ is irreducible} \\
&= \overline{(\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_{ij} \cap \mathcal{U}')} && \text{since } \rho''_{\mathcal{A}} \text{ is flat} \\
&= \overline{(\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_{ij}) \cap (\rho''_{\mathcal{A}})^{-1}(\mathcal{U}')} \\
&= \overline{\mathcal{Z}''_{ij} \cap \mathcal{U}''}.
\end{aligned}
$$

It thus suffices to show that $\mathcal{Z}''_{ij} \cap \mathcal{U}''$ is irreducible. But this is immediate since $\mathcal{Z}''_{ij} \cap \mathcal{U}'' = \mathcal{W}''_{ij} \cap \mathcal{U}''$ with $\mathcal{W}''_{ij}$ irreducible.

That the collection $\{\mathcal{Z}''_{ij}\}$ gives all the irreducible components is immediate. □

Note that by Lemma 5.1.16 part (i) we have that Z''_{ij} étale locally corresponds to $\mathcal{Z}''_{ij}$.

Proposition 5.4.3. *The number of connected and irreducible components of the subschemes of $\mathcal{A}''_1$ are as follows.*

Closed subscheme of $\mathcal{A}''_1$	# connected components	# irreducible components
$\mathcal{Z}''_3 = (\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_3)^{\mathrm{red}}$	1	$n_{s_0 s_2 \tau} + 1$
$\mathcal{C}''_3 = (\rho''_{\mathcal{A}})^{-1}(\mathcal{C}'_3)^{\mathrm{red}}$	1	$n_{s_0 s_2 \tau} + 1$
$\mathcal{Z}''_4 = (\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_4)^{\mathrm{red}}$	1	1
$\mathcal{C}''_4 = (\rho''_{\mathcal{A}})^{-1}(\mathcal{C}'_4)^{\mathrm{red}}$	$n_{s_2 \tau}$	$n_{s_2 \tau}$

Table 5.5: Number of connected and irreducible components of subschemes of $\mathcal{A}''_1$

Proof. $\mathcal{Z}''_3 = \mathcal{C}''_3$: Let $\mathcal{W}' \subset \mathcal{Z}'_3$ be an irreducible component. We claim that $(\rho''_{\mathcal{A}})^{-1}(\mathcal{W}')$ is irreducible. As shown in the proof of Proposition 5.3.1, $\mathcal{W}'$ arises as the strict transform of an irreducible component of $\mathcal{Z}_3$. First consider the case where $\mathcal{W}'$ is the strict transform of $\mathcal{Z}_{11}$. Then $\mathcal{W}' = \mathcal{Z}'_{11}$ and Proposition 5.4.2 says that $(\rho''_{\mathcal{A}})^{-1}(\mathcal{Z}'_{11})$ is irreducible.

So assume now that $\mathcal{W}'$ is the strict transform of some two dimensional irreducible component of $\mathcal{Z}_3$. From the proof of Proposition 5.2.3 it must be that this two dimensional component of $\mathcal{Z}_3$ is contained in $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ and hence $\rho'_{\mathcal{A}}(\mathcal{W}') \subset \mathcal{Z}_{01} \cap \mathcal{Z}_{10}$.

For convenience, we remind the reader of the following cartesian diagram for the next argument.

$$\begin{tikzcd}
\mathcal{A}''_1 \arrow[r, "\rho''_{\mathcal{A}}"] \arrow[d] & \mathcal{A}'_0 \arrow[d, "\rho'_{\mathcal{A}}"] \\
\mathcal{A}_1 \arrow[r, "\pi"] & \mathcal{A}_0
\end{tikzcd}$$

Let $x' \in \mathcal{W}'$ be a closed point and so $x = \rho'_{\mathcal{A}}(x') \in \mathcal{Z}_{01} \cap \mathcal{Z}_{10}$. Then as x is in the supersingular locus, $\pi^{-1}(x)$ consists of a single closed point. Therefore the fiber $(\rho''_{\mathcal{A}})^{-1}(x')$ also consists of a single closed point. With ρ'' finite and flat, it must be that $(\rho''_{\mathcal{A}})^{-1}(\mathcal{W}')$ is irreducible.

Thus we conclude that each irreducible component of $\mathcal{Z}''_3 = \mathcal{C}''_3$ arises as the reduced inverse image of an irreducible component of $\mathcal{Z}'_3$, and therefore the number of irreducible components of $\mathcal{Z}''_3$ is $n_{s_0 s_2 \tau} + 1$. That $\mathcal{Z}''_3 = \mathcal{C}''_3$ is connected follows from the fact that the fiber above any closed point of a two dimension component of $\mathcal{Z}'_{11}$ with respect to the morphism $\rho''_{\mathcal{A}}$ consists of a single closed point.

$\mathcal{C}''_4$: Let $\mathcal{W}'$ be a connected component of $\mathcal{C}'_4$. From the proof of Proposition 5.2.3, $\mathcal{W}'$ is irreducible and arises as the strict transform of some irreducible component of $\mathcal{C}_4 \subset \mathcal{A}_0$. Let $x' \in \mathcal{W}'$ be a closed point. Then $x = \rho'_{\mathcal{A}}(x') \in \mathcal{C}_4$ is contained in the supersingular locus of $\mathcal{A}_0$. As such, $\pi^{-1}(x)$ consists of a single closed point. Thus it follows that the fiber $(\rho''_{\mathcal{A}})^{-1}(\mathcal{W}')$ consists of a single closed point as well. Hence the reduced inverse image of $\mathcal{W}'$ under $\mathcal{A}''_1 \to \mathcal{A}'_0$ is connected and irreducible. Therefore, $\mathcal{C}''_4$ has the same number of connected and irreducible components as $\mathcal{C}'_4$, namely $n_{s_2 \tau}$. $\square$

5.5 Step III: Blowup of $\mathcal{Z}_3''$.

$$\mathcal{A}_1''' = \mathrm{Bl}_{\mathcal{A}_1''}(\mathcal{Z}_3'') \quad \text{and} \quad U_1''' = \mathrm{Bl}_{U_1''}(Z_3'')$$

5.5.1 Description of the local model U_1'''

In each affine chart of U_1'', the subscheme Z_3'' corresponds to the ideal (u, v). We start by describing a scheme X which is given by, a priori, a subset of the equations defining U_1'''. Once we show that X is integral, from an argument similar to that given in Step I it will follow that $X = U_1'''$. A presentation of X is given by the closed subscheme of $\mathrm{Proj}_{U_1''}(\mathcal{O}[\widetilde{u}, \widetilde{v}])$ where u and v are of grade 1 cut out by the following equations.

$$u\widetilde{v} - \widetilde{u}v, \quad (\widetilde{x} + \widetilde{b}c)\widetilde{u}^{p-1} - \widetilde{x}\widetilde{v}^{p-1}, \quad y\widetilde{u}^{p-1} - (y + ac)\widetilde{v}^{p-1}$$

Using these equation along with those in the presentation for U_1'', we have that X is covered by four standard affine charts.

$$\begin{array}{llll}
X_{00} & \widetilde{x} = 1 & \widetilde{u} = 1 & \mathbb{Z}_p[y, a, c, u, \widetilde{b}, \widetilde{v}]/(u^{p-1}y - p, \widetilde{v}^{p-1} - (1 + \widetilde{b}c), a(1 + \widetilde{b}c) + \widetilde{b}y) \\
X_{01} & \widetilde{x} = 1 & \widetilde{v} = 1 & \mathbb{Z}_p[y, c, v, \widetilde{b}, \widetilde{u}]/(v^{p-1}\widetilde{u}^{p-1}y - p, \widetilde{u}^{p-1}(1 + \widetilde{b}c) - 1) \\
X_{10} & \widetilde{b} = 1 & \widetilde{u} = 1 & \mathbb{Z}_p[a, b, u, \widetilde{x}, \widetilde{v}]/(b\widetilde{x}^2\widetilde{v}^{p-1}(-a) - p, u^{p-1} - b\widetilde{x}) \\
X_{11} & \widetilde{b} = 1 & \widetilde{v} = 1 & \mathbb{Z}_p[a, b, s, v, \widetilde{u}]/(bs^2\widetilde{u}^{p-1}(-a) - p, v^{p-1} - bs).
\end{array}$$

Note: The last chart uses a change of coordinates $s = \widetilde{x} + c$.

We can cover X_{00} with two open subschemes each respectively defined by the condition $\widetilde{b}$ and $1 + \widetilde{b}c$ is invertible. These open subschemes are

$$
\begin{array}{lll}
X'_{00} & \widetilde{b} \neq 0 & \mathbb{Z}_p[a, u, \widetilde{b}^{\pm 1}, \widetilde{v}]/(u^{p-1}(-a)\widetilde{v}^{p-1}\widetilde{b}^{-1} - p) \\
X''_{00} & 1 + \widetilde{b}c \neq 0 & \mathbb{Z}_p[y, c, u, \widetilde{b}, \widetilde{v}, (1 + \widetilde{b}c)^{-1}]/(u^{p-1}y - p, \widetilde{v}^{p-1} - (1 + \widetilde{b}c)).
\end{array}
$$

Since $X'_{00} \subset X_{10}$ as an open subscheme, X is covered by X''_{00}, X_{01}, X_{10}, and X_{11}. Thus we see that X is integral and so $X \cong U'''_1$.

Proposition 5.5.1. X *is normal.*

Proof. The statement is clear for the charts X''_{00} and X_{01}. Focusing now on the chart X_{10}, note first that it is a complete intersection and hence Cohen-Macaulay. With the generic fiber smooth, by Serre's Criterion it suffices to check that the generic points of the irreducible components of the special fiber are regular, i.e. their maximal ideals are generated by a single element. Their maximal ideals, written in the corresponding local ring, are given by

$$(b, u) = (u), \quad (\widetilde{x}, u) = (u), \quad (\widetilde{v}), \quad (a).$$

The normality of the chart X_{11} follows from the argument just given by a change of variables. Therefore X is normal. □

As remarked in the previous section, $Z(u, v) = Z(u, v, b) \cup Z(u, v, y, c, \widetilde{x})$ inside of U''_1. The true center is a closed subscheme of $Z(u, v)$, and we see from the above charts that the fiber above a closed point of $Z(u, v, y, c, \widetilde{x})$ consists of the projective line given by $[\widetilde{u} : \widetilde{v}]$ which lies inside $Z_{10} \cup Z_{11}$. Also, the fiber over a closed point of $Z(u, v, b)$ outside of $Z(u, v, y, c, \widetilde{x})$ consists of a $p - 1$ closed points. Therefore the true center is C''_3, the fundamental center is $Z(u, v, \widetilde{x}, y, c) = Z''_{01} \cap Z''_{10}$, and the residual locus is $Z''_{11} \setminus (Z''_{01} \cap Z''_{10})$. Moreover the fundamental center is two dimensional and smooth, the residual locus is three dimensional, and the exceptional locus is equidimensional of dimension three. Taking the strict transform under ρ'''_U we have

$$
\begin{aligned}
Z'''_{00} &= Z(y, a) \\
Z'''_{01} &= Z(y, v, \widetilde{x} + \widetilde{b}c, \widetilde{v}) \\
Z'''_{10} &= Z(u, y + ac, \widetilde{x}, \widetilde{u}) \\
Z'''_{11} &= Z(u, v, b) \\
Z'''_{4} &= Z(u, v, b) \\
C'''_{4} &= Z(u, v, y, b, c, \widetilde{x}).
\end{aligned}
$$

We note that C''_4 is smooth of dimension two and C''_4 intersects with the fundamental center of $U'''_1 \to U''_1$ in a smooth one dimensional subscheme.

5.5.2 Description of the integral model $\mathcal{A}'''_1$

$\mathcal{A}'''_1$ has U'''_1 as an étale local model and it is immediate that $\mathcal{A}'''_1$ is normal. As the true center of $U'''_1 \to U''_1$ is C''_3, we have that the true center of $\mathcal{A}'''_1 \to \mathcal{A}''_1$ is $\mathcal{C}''_3$.

Proposition 5.5.2. *$\mathcal{A}'''_1 \otimes \mathbb{F}_p$ has precisely $4 + n_{s_0 s_2 \tau}$ irreducible components. Three are given by the strict transforms of $\mathcal{Z}''_{00}$, $\mathcal{Z}''_{01}$, and $\mathcal{Z}''_{10}$. The other $1 + n_{s_0 s_2 \tau}$ are contained in the exceptional locus: one lying above $\mathcal{Z}''_{11}$ and one lying above each two dimensional irreducible component of $\mathcal{Z}''_3$.*

Proof. That the strict transforms of $\mathcal{Z}''_{00}$, $\mathcal{Z}''_{01}$, and $\mathcal{Z}''_{10}$ are irreducible follows immediately from the fact that they are not contained in the true center $\mathcal{C}''_3$.

From Section 5.5.1, the exceptional locus $\mathcal{E}'''$ of $\mathcal{A}'''_1 \to \mathcal{A}''_1$ is equidimensional of dimension three. As the exceptional locus maps surjectively onto the true center $\mathcal{C}_3''$ and $\mathcal{C}_3''$ has $1+n_{s_0s_2\tau}$ irreducible components by Proposition 5.4.3, we conclude that $\mathcal{E}'''$ must consist of at least $1+n_{s_0s_2\tau}$ irreducible components. Denote these irreducible components by $\{\mathcal{W}_i\}$. Without loss of generality assume that $\rho_{\mathcal{A}}'''(\mathcal{W}_1''') \subset \mathcal{Z}_{11}''$ and $\rho_{\mathcal{A}}'''(\mathcal{W}_2'''), \ldots, \rho_{\mathcal{A}}'''(\mathcal{W}_{1+n_{s_0s_2\tau}}''')$ are each contained in a unique two dimensional irreducible component of $\mathcal{Z}_3''$.

We claim that if $\rho_{\mathcal{A}}'''(\mathcal{W}_i''') \subset \mathcal{Z}_{11}''$, then $\rho_{\mathcal{A}}'''(\mathcal{W}_i''') = \mathcal{Z}_{11}''$. Indeed, since $\rho_{\mathcal{A}}'''$ is proper it suffices to show that $\rho_{\mathcal{A}}'''(\mathcal{W}_i''')$ is three dimensional. As the fiber above any closed point of U_1'' with respect to ρ_U''' is at most one dimensional, we conclude that the same is true for $\rho_{\mathcal{A}}'''$ and thus $\rho_{\mathcal{A}}'''(\mathcal{W}_i''')$ is at least two dimensional. So by way of contradiction, suppose $\rho_{\mathcal{A}}'''(\mathcal{W}_i''')$ has dimension exactly two. From Step II we have that $Z_{11} = Z(u, v, b)$ and from the previous section $C_{\text{fund},3}'' = Z(u, v, \widetilde{x}, y, c)$. Thus they intersect in a smooth one dimensional scheme. It follows that the intersection of $\mathcal{Z}_{11}''$ with any two dimensional component of $\mathcal{C}_{\text{fund},3}''$ is one dimensional. As such, there exists a closed point $x'' \in \rho_{\mathcal{A}}'''(\mathcal{W}_i''')$ lying solely on the component $\mathcal{Z}_{11}''$ such that the fiber above x'' is one dimensional. Let y'' be a closed point of U_1'' corresponding to x''. Then it must be that y'' lies solely on the irreducible component Z_{11}''. Since no closed point of $C_{\text{res},3}'' = Z_{11}'' \setminus (Z_{01}'' \cap Z_{10}'')$ has fiber of dimension one with respect to the morphism $U_1''' \to U_1''$, using Proposition 5.1.20 we arrive at a contradiction. Therefore, for any i with $\mathcal{W}_i''' \to \mathcal{Z}_{11}''$ it must be that the image is three dimensional as claimed.

Now consider a closed point $x'' \in \mathcal{C}_3'' \setminus \mathcal{Z}_{11}''$. From Proposition 5.1.20 and $U_1''' \to U_1''$ we see that the fiber above x'' is connected and smooth. It follows from Lemma 5.1.17 that for each irreducible component of $\overline{\mathcal{C}_3'' \setminus \mathcal{Z}_{11}''}$, there is a single irreducible component of the exceptional locus of ρ''' mapping surjectively onto it. Recall that we have labeled these components $\mathcal{W}_2''', \ldots, \mathcal{W}_{1+n_{s_0s_2\tau}}'''$.

We claim that $(\rho''')^{-1}(\mathcal{Z}_{11}'')^{\text{red}}$ is irreducible. Indeed, since the reduced inverse image of

Z''_{11} under the morphism $U'''_1 \to U''_1$ is smooth and equidimensional of dimension three, so is $(\rho''')^{-1}(\mathcal{Z}''_{11})^{\mathrm{red}}$. This implies that $(\rho''')^{-1}(\mathcal{Z}''_{11})^{\mathrm{red}}$ is a disjoint union of irreducible components of $\mathcal{A}'''_1 \otimes \mathbb{F}_p$. As each of these irreducible components maps into $\mathcal{Z}''_{11}$, they must indeed map surjectively onto $\mathcal{Z}''_{11}$. It follows that the image of each irreducible component contains a closed point $x'' \in \mathcal{Z}''_{11} \cap \overline{\mathcal{C}''_3 \setminus \mathcal{Z}''_{11}}$. From Proposition 5.1.20 we have the fiber above x'' is connected, and hence $(\rho''')^{-1}(\mathcal{Z}''_{11})^{\mathrm{red}}$ is connected. The claim follows immediately.

Now suppose there exists another irreducible component $\mathcal{W}_{2+n_{s_0s_2\tau}}$. By the above, it must be that $\mathcal{W}_{2+n_{s_0s_2\tau}} \to \mathcal{Z}''_{11}$. But then it follows that in fact

$$\mathcal{W}_{2+n_{s_0s_2\tau}} \subset (\rho''')^{-1}(\mathcal{Z}''_{11})^{\mathrm{red}} = \mathcal{W}_1$$

and therefore $\mathcal{W}_{2+n_{s_0s_2\tau}} = \mathcal{W}_1$. □

Proposition 5.5.3. *The number of connected and irreducible components of the subschemes of $\mathcal{A}'''_1$ are as follows.*

Closed subscheme of $\mathcal{A}'''_1$	# connected components	# irreducible components
$\mathcal{Z}'''_4 = \mathrm{ST}(\mathcal{Z}''_{11})$	1	1
$\mathcal{C}'''_4 = \mathrm{ST}(\mathcal{C}''_4)$	$n_{s_2\tau}$	$n_{s_2\tau}$

Table 5.6: Number of connected and irreducible components of subschemes of $\mathcal{A}'''_1$

Proof. That $\mathcal{Z}'''_4$ is irreducible was shown in the proof of the previous proposition.

As C''_4 and C'''_4 are both smooth, $\mathcal{C}''_4$ and $\mathcal{C}'''_4$ are as well giving that each connected component is irreducible. Let $\mathcal{W}'' \subset \mathcal{C}''_4$ be some connected component. Recalling that $\mathcal{C}''_4$ has $n_{s_2\tau}$ such connected components, the proposition will follow by showing that the inverse image of $\mathcal{W}''$ with respect to $\mathcal{A}'''_1 \to \mathcal{A}''_1$ is connected. But this follows immediately since the fiber above every closed point of C''_4 with respect to the morphism $U'''_1 \to U''_1$ is smooth and connected. $\square$

5.6 Step IV: $p-2$ blowups of $\mathcal{Z}_{11}'''$.

In this last step we define the integral models $\mathcal{A}_1^{[i]}$ for $4 \leq i \leq p+1$ by first blowing up $\mathcal{Z}_4'''$ in $\mathcal{A}_1'''$ and then blowing up the strict transform of $\mathcal{Z}_4'''$ in each successive step. Likewise, we define the local models $U_1^{[i]}$ for $4 \leq i \leq p+1$ by blowing up Z_4''' in U_1''' and then blowing up the strict transform of Z_4''' in each successive step.

5.6.1 Description of the local model $U_1^{[i]}$

Recall that Z_{11}''' is given by $Z(u,v,b)$. Z_{11}''' may be described on each affine chart of $X = U_1'''$ by giving its corresponding ideal.

Chart	X_{00}''	X_{01}	X_{10}	X_{11}
Ideal	(u)	(v)	(u,b)	(v,b)

Table 5.7: Ideal sheaf of Z_{11}'''

Write $U_1^{[3]} = U_1'''$ and $Z_{11}^{[3]} = Z_{11}'''$. As Z_{11}''' is Cartier on X_{00}'' and X_{01}, the blowups of Z_{11}''' and its strict transforms are isomorphisms over these open subschemes. Focusing now on X_{10} and X_{11}, each of these two charts are given by a scheme with the presentation

$$Y = \mathrm{Spec}(A), \quad A = \mathbb{Z}_p[x_1, x_2, x_3, x_4, u]/(x_1 x_2^2 x_3^{p-1} x_4 - p, u^{p-1} - x_1 x_2)$$

and Z_3''' is given by the subscheme W corresponding to the ideal (u, x_1) in this presentation. To describe the blowups, we write $x_1^{[0]} = x_1$, and $x_1^{[1]} = \widetilde{x}_1$, $x_1^{[2]} = \widetilde{\widetilde{x}}_1$, etc. for projective coordinates.

Proposition 5.6.1. *Set* $Y^{[0]} = Y$, $W_0 = W$, *and for* $1 \le i \le p-2$, *define* $Y^{[i]}$ *inside* $Y \times \overbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}^{i-times}$ *by*

$$u^{[i]}u^{p-i-1} - x_1^{[i]}x_2, \qquad ux_1^{[1]} - x_1u^{[1]}$$

$$uu^{[j-1]}x_1^{[j]} - x_1^{[j-1]}u^{[j]} \quad \text{for} \quad 2 \le j \le i.$$

Let W_i *be the strict transform of* W_{i-1} *in* $Y^{[i]}$ *for each* $i \ge 1$. *Then for* $1 \le i \le p-2$ *we have the following.*

(i). $Y^{[i]} \cong Bl_{W_{i-1}}(Y^{[i-1]})$.

(ii). The true center of $Y^{[i]} \to Y^{[i-1]}$ *is one dimensional and smooth.*

(iii). The fundamental center of $Y^{[i]} \to Y^{[i-1]}$ *is equal to the true center.*

(iv). The exceptional locus of $Y^{[i]} \to Y^{[i-1]}$ *is smooth and two dimensional.*

Furthermore, $Y^{[p-2]}$ *is regular with special fiber a divisor with normal crossings.*

Proof. (i) We proceed by induction. So assume W_{i-1} corresponds to the ideal $(u, x_1^{[i-1]})$, which is certainly true for $i = 1$. By explicit computation, the claimed equations are part of those defining $\mathrm{Bl}_{W_{i-1}}(Y^{[i-1]})$. The standard affine charts of $Y^{[i]}$, indexed by $1 \le k \le i+1$, are described by the conditions

$$u^{[j]} \neq 0 \quad \text{for} \quad 1 \le j < k \quad \text{and} \quad x_1^{[j]} \neq 0 \quad \text{for} \quad k \le j \le i.$$

In order to explicitly write them, we must consider three cases.

$k = 1$: The equations of $Y^{[i]}$ become

$$x_2 = x_1^{p-2} \left(\frac{u^{[1]}}{x_1^{[1]}} \right)^{p-1}, \quad u = x_1 \frac{u^{[1]}}{x_1^{[1]}}, \qquad \frac{u^{[j]}}{x_1^{[j]}} = x_1^{j-1} \left(\frac{u^{[1]}}{x_1^{[1]}} \right)^{j}, \quad 2 \le j \le i,$$

and the coordinate ring is

$$\mathbb{Z}_p \left[x_1, \frac{u^{[1]}}{x_1^{[1]}}, x_3, x_4 \right] / \left(x_1^{2p-3} \left(\frac{u^{[1]}}{x_1^{[1]}} \right)^{2p-2} x_3^{p-1} x_4 - p \right).$$

$1 < k \le i$: The equations of $Y^{[i]}$ become

$$x_1 = \left(\frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{k} \left(\frac{u^{[k]}}{x_1^{[k]}} \right)^{k-1}, \quad x_2 = \left(\frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{p-k-1} \left(\frac{u^{[k]}}{x_1^{[k]}} \right)^{p-k}, \quad u = \frac{x_1^{[k-1]}}{u^{[k-1]}} \frac{u^{[k]}}{x_1^{[k]}}$$

$$\frac{x_1^{[j-1]}}{u^{[j-1]}} = \left(\frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{k-j+1} \left(\frac{u^{[k]}}{x_1^{[k]}} \right)^{k-j}, \quad \text{for} \quad 2 \le j < k,$$

$$\frac{u^{[j]}}{x_1^{[j]}} = \left(\frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{j-k} \left(\frac{u^{[k]}}{x_1^{[k]}} \right)^{j-k+1} \quad \text{for} \quad k < j \le i,$$

and the coordinate ring is

$$\mathbb{Z}_p \left[\frac{x_1^{[k-1]}}{u^{[k-1]}}, \frac{u^{[k]}}{x_1^{[k]}}, x_3, x_4 \right] / \left(\left(\frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{2p-k-2} \left(\frac{u^{[k]}}{x_1^{[k]}} \right)^{2p-k-1} x_3^{p-1} x_4 - p \right).$$

$k = i + 1$: The equations of $Y^{[i]}$ become

$$x_1 = u \frac{x_1^{[1]}}{u^{[1]}} \quad u^{p-i-1} - \frac{x_1^{[i]}}{u^{[i]}} x_2 = 0$$

$$\frac{x_1^{[j]}}{u^{[j]}} = u^{i-j+1} \frac{x_1^{[i+1]}}{u^{[i+1]}} \quad \text{for} \quad 2 \le j \le i$$

and the coordinate ring is

$$\mathbb{Z}_p \left[\frac{x_1^{[i]}}{u^{[i]}}, x_2, , x_3, x_4, u \right] / \left(u^i \frac{x_1^{[i]}}{u^{[i]}} x_2^2 x_3^{p-1} x_4 - p, u^{p-i-1} - \frac{x_1^{[i]}}{u^{[i]}} x_2 \right).$$

Note that each chart is integral. Since the equations defining $Y^{[i]}$ are part of those defining $\mathrm{Bl}_{W_{i-1}}(Y^{[i-1]})$, there is a closed immersion $\iota : \mathrm{Bl}_{W_{i-1}}(Y^{[i-1]}) \to Y^{[i]}$ which is an isomorphism on the generic fiber. With $Y^{[i]}$ integral and of the same dimension as $\mathrm{Bl}_{W_{i-1}}(Y^{[i-1]})$, this implies ι is an isomorphism.

To complete the induction, we must show that the strict transform of the subscheme of $Y^{[i-1]}$ given by $Z(u, x_1^{[i-1]})$ corresponds to the subscheme of $Y^{[i]}$ given by $Z(u, x_1^{[i]})$. From the charts above, the true center of the blowup in $Y^{[i-1]}$ is given by $Z(u, x_1^{[i-1]}, x_2)$. Thus taking the inverse image away from the center we have that x_2 is invertible, and so from the relation $u^{[i]}u^{p-i-1} - x_1^{[i]}x_2$ of $Y^{[i]}$ we get that $x_1^{[i]}$ is in the ideal defining the strict transform. As subschemes of $Y^{[i]}$, $Z(u, x_1^{[i-1]}, x_1^{[i]}) = Z(u, x_1^{[i]})$. This subscheme is irreducible and of dimension three and therefore we conclude it must be the strict transform of W_{i-1}.

We now inspect these charts to deduce the remainder of the proposition. To calculate the true center of $U_1^{[i+1]} \to U_1^{[i]}$, one need only to consider the chart indexed by $k = i + 1$ since $Z(u, x_1^{[i]})$ is Cartier in all others. Here we see that the true center is contained in $Z(u, x_1^{[i]}, x_2)$. Now consider the fiber over any closed point of $Z(u, x_1^{[i]}, x_2)$ with respect to the morphism $U_1^{[i+1]} \to U_1^{[i]}$. We have that both of the relations $u^{[i+1]}u^{p-i-2} - x_1^{[i+1]}x_2$ and $uu^{[i]}x_1^{[i+1]} - x_1^{[i]}u^{[i+1]}$ vanish, since we are assuming that u, $x_1^{[i]}$, and x_2 are all zero. No other relation involving $u^{[i+1]}$ and $x_1^{[i+1]}$ exists and therefore the fiber over this closed point is isomorphic to $\mathbb{P}^1_{\mathbb{F}_p}$. This gives (ii), (iii), and (iv). $\square$

Using the explicit equations above, we record the global structure of the irreducible components of the special fiber.

Lemma 5.6.2. *The irreducible components of $U_1^{[p-2]} \otimes \mathbb{F}_p$ are described as follows.*

- *There are $p+3$ components.*
- *Three components are given by $Z(\widetilde{u})$, $Z(\widetilde{v})$ and $Z(a)$. We index the other components by $1 \leq i \leq p$. For $1 \leq i \leq p-1$, the ith irreducible component is given by the locus*

$$Z_i = Z(u, b, \widetilde{x}, b^{[1]}, b^{[2]}, \ldots, b^{[i-2]}, u^{[i]}, u^{[i+1]}, \ldots, u^{[p-2]})$$

  and the pth irreducible component is given by the locus

$$Z_p = Z(u, b, b^{[1]}, b^{[2]}, \ldots, b^{[p-2]}).$$

- *The components given by $Z(\widetilde{u})$, $Z(\widetilde{v})$, $Z(a)$, and Z_i have multiplicity $p-1$, $p-1$, 1, and $2p-i-1$ respectively. In particular, Z_{p-1} is the only component with multiplicity divisible by p.*
- *The components Z_1 and Z_p are isomorphic to $\mathbb{A}^3_{\mathbb{F}_p}$. The components Z_i with $2 \leq i \leq p-1$ are isomorphic to $\mathbb{P}^1_{\mathbb{F}_p} \times \mathbb{A}^2_{\mathbb{F}_p}$.*
- *The components intersect as indicated in the following "dual complex", drawn for $p=5$. Each vertex represents an irreducible component where the label indicates the multiplicity of the irreducible component. Each edge indicates that the two irreducible components intersect.*

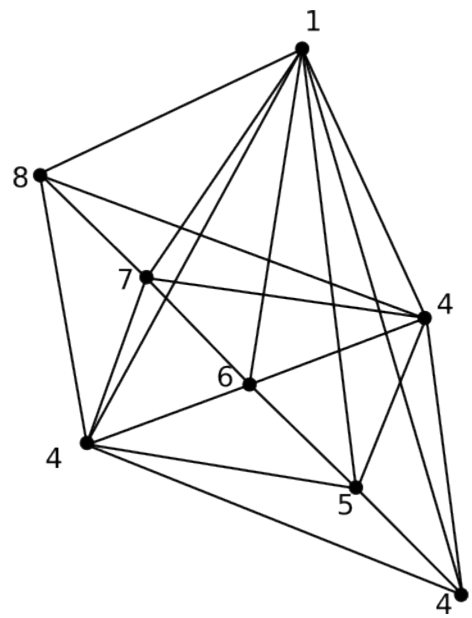

Figure 5.2: Dual complex of $U_1^{[p+2]}$ for $p = 5$

Moreover, consider a k-simplex appearing in the complex where every pair of vertices within the simplex is directly connected by an edge. Such a simplex indicates a $(k+1)$-fold intersection of the irreducible components.

- *A k-fold intersection of the components has dimension $3 - k$ over* $\mathrm{Spec}(\mathbb{F}_p)$.

5.6.2 Description of the integral model

Proposition 5.6.3. *For $3 \leq i \leq p+1$, the number of irreducible components of $\mathcal{A}_1^{[i]}$ is $4 + n_{s_0 s_2 \tau} + n_{s_2 \tau} \cdot (i-3)$.*

Proof. We recall the following facts:

(i). $\mathcal{C}_4^{[3]}$ has $n_{s_2\tau}$ connected components and each is smooth of dimension two;

(ii). For $3 \leq i \leq p+1$, the fiber over a closed point of the true center of $U_1^{[i]} \to U_1^{[i-1]}$ is one dimensional, smooth, and connected; and

(iii). For $3 \leq i \leq p+1$, $U_1^{[i]} \otimes \mathbb{F}_p$ is equidimensional of dimension three.

We proceed by induction, starting with the modification $\mathcal{A}_1^{[4]} \to \mathcal{A}_1^{[3]}$. Now (i) and (ii) imply that the exceptional locus of $\mathcal{A}_1^{[4]} \to \mathcal{A}_1^{[3]}$ has the same number of connected components as the true center $\mathcal{C}_4^{[3]}$, and furthermore that each such connected component is three dimensional and smooth. By (iii) each of these components is an irreducible component of $\mathcal{A}_1^{[4]} \otimes \mathbb{F}_p$, with all of the other irreducible components of $\mathcal{A}_1^{[4]} \otimes \mathbb{F}_p$ being given by the strict transform of the irreducible components of $\mathcal{A}_1^{[3]} \otimes \mathbb{F}_p$. Therefore there are $4 + n_{s_0 s_2 \tau} + n_{s_2 \tau}$ irreducible components of $\mathcal{A}_1^{[4]} \otimes \mathbb{F}_p$.

Now assume the result is true for $i-1$ with $4 < i \leq p+1$. We must show that, $\mathcal{C}_i^{[i-1]}$ has $n_{s_2\tau}$ connected components and each is smooth of dimension two. Indeed, then the induction will follow using the same argument as in the above paragraph. Note that from the local model we have each connected component of $\mathcal{C}_i^{[i-1]}$ is two dimensional and smooth, so it is left to show that there are $n_{s_2\tau}$ connected components of $\mathcal{C}_i^{[i-1]}$.

Now $\mathcal{Z}_i^{[i-1]}$, $\mathcal{E}^{[i-1]}$, and $\mathcal{C}_i^{[i-1]}$ étale locally correspond to $Z_i^{[i-1]}$, $E^{[i-1]}$, and $C_i^{[i-1]}$ respectively. As $C_i^{[i-1]} = Z_i^{[i-1]} \cap E^{[i-1]}$ we get that $\mathcal{C}_i^{[i-1]} = \mathcal{Z}_i^{[i-1]} \cap \mathcal{E}^{[i-1]}$.

Consider a connected component of $\mathcal{C}_{i-1}^{[i-2]}$, which is irreducible because $\mathcal{C}_{i-1}^{[i-2]}$ is smooth. The fiber above this component is connected since $\mathcal{A}_1^{[i-2]}$ is normal by Zariski's Main Theorem. Thus $\mathcal{E}^{[i-1]}$ has the same number of connected components as $\mathcal{C}_{i-1}^{[i-2]}$. Now $\mathcal{Z}_i^{[i-1]} = \mathrm{ST}(\mathcal{Z}_{i-1}^{[i-2]})$ maps surjectively onto $\mathcal{Z}_{i-1}^{[i-2]}$ via $\rho_{\mathcal{A}}^{[i-1]}$ and hence the image meets each connected component of $\mathcal{C}_{i-1}^{[i-2]}$. As such, $\mathcal{Z}_i^{[i-1]}$ meets each connected component of $\mathcal{E}^{[i-1]}$. Therefore $\mathcal{C}_i^{[i-1]} = \mathcal{Z}_i^{[i-1]} \cap \mathcal{E}^{[i-1]}$ has the same number of connected components as $\mathcal{C}_{i-1}^{[i-2]}$; namely $n_{s_2\tau}$. $\square$

We will use the following graph to describe how these irreducible components of the special fiber intersect.

Definition 5.6.4. Let p be an odd rational prime and $K^p \subset G(\mathbb{A}_f^p)$ so that K^p determines the numbers $n_{s_2\tau}$ and $n_{s_0 s_2\tau}$ of $\mathcal{A}_{0,K^p}$, i.e. the number of connected components of the KR strata $s_2\tau$ and $s_0 s_2\tau$. We then define the vertex-labeled graph Γ_{p,K^p} as follows.

(i). Begin with $n_{s_2\tau}$ batons, each having $p-2$ vertices. Label the vertices $2p-3, 2p-4, \ldots, p$ from head to tail.

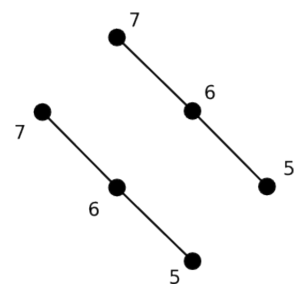

Figure 5.3: Batons of Γ_{p,K^p} where $p = 5$, $n_{s_2\tau} = 2$

(ii). Add one vertex labeled $2p-2$ (top left) and attach edges between this vertex and the heads of the batons. Add two more vertices labeled $p-1$ (bottom left and top right) and connect these two vertices to every vertex in the batons, as well as the (unique) vertex labeled $2p-2$. Add $n_{s_0 s_2\tau}$ vertices labeled $p-1$ (bottom right) and attach edges between these and the tails of the batons, as well as the two vertices labeled $p-1$ added in the previous sentence.

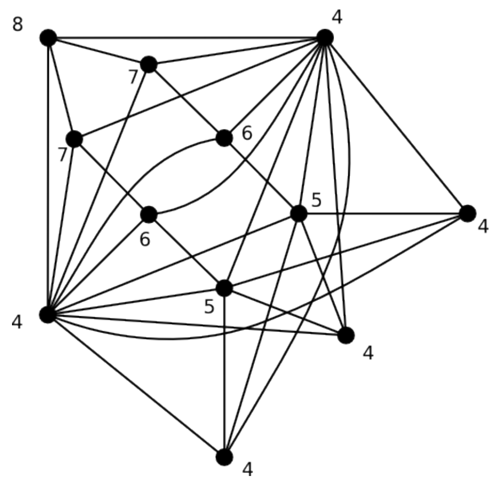

Figure 5.4: Base of Γ_{p,K^p} where $p = 5$, $n_{s_2\tau} = 2$, $n_{s_0 s_2 \tau} = 3$

(iii). Add one vertex labeled 1 and attach edges from this to every vertex constructed in the above two steps.

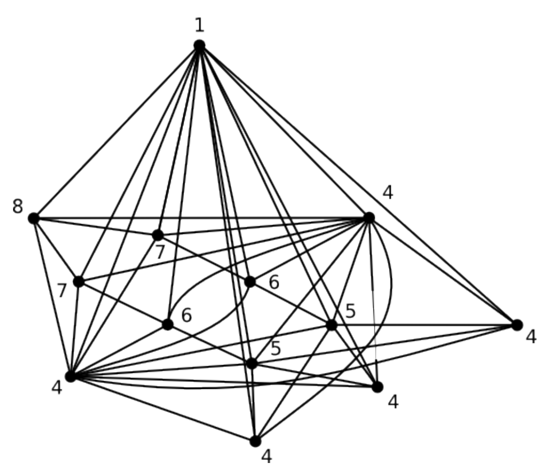

Figure 5.5: Γ_{p,K^p} where $p = 5$, $n_{s_2\tau} = 2$, $n_{s_0 s_2 \tau} = 3$

Definition 5.6.5. We define the following subsets of the vertices of Γ_{p,K^p}.

- The batons consist of the vertices given in step (i) above. They may be identified as the vertices with label in $[p, 2p-3]$.
- The front consists of the vertices labeled $p-1$ on the bottom right of the diagram

directly above. They may be identified as the vertices of label $p-1$ and (edge) degree $3+n_{s_2\tau}$ that share edges with precisely two vertices labeled 4.

- The sides consist of the vertices labeled $p-1$ which are not in the front.

Recall that we are writing $\mathcal{A}_0$ for a single connected component of $\mathcal{A}_0 \to \mathrm{Spec}(\mathbb{Z}_p[\zeta_N])$ and similarly with $\mathcal{A}_1$ (see Section 5.1.3 for details).

Theorem 5.6.6. *$\mathcal{A}_1^{[p+1]} \to \mathcal{A}_1$ is a resolution of singularities and the special fiber of $\mathcal{A}_1^{[p+1]}$ is a nonreduced divisor with normal crossings. $\mathcal{A}_1^{[p+1]} \otimes \mathbb{F}_p$ has $4+n_{s_0s_2\tau}+n_{s_2\tau}(p-2)$ irreducible components whose intersections are described by the vertex-labeled graph Γ_{p,K^p} as follows.*

(i). Each vertex represents an irreducible component. The label of the vertex is the multiplicity of the component.

(ii). A k-simplex of Γ_{p,K^p} indicates a $(k+1)$-fold intersection of irreducible components corresponding to the vertices of the k-simplex. Such an intersection has dimension $3-k$ over $\mathrm{Spec}(\mathbb{F}_p)$.

(iii). Let $x^{[p+1]} \in \mathcal{A}_1^{[p+1]}$ be a closed point and $\{e_1, \ldots, e_t\}$ be the multiset of the multiplicities of the irreducible components which $x^{[p+1]}$ lies on. Then there is an étale neighborhood of $x^{[p+1]}$ of the form

$$\mathrm{Spec}(\mathbb{Z}_p[x_1, x_2, x_3, x_4]/(x_1^{e_1} \ldots x_t^{e_t} - p)).$$

(iv). The following table gives the image of each irreducible component under the map $\mathcal{A}_1^{[p+1]} \to \mathcal{A}_0$.

Description	Image
Front	Each irreducible component surjects onto a connected component of $\overline{\mathcal{A}_{s_0s_2\tau}}$
Sides	These two irreducible components surject onto the irreducible components $\mathcal{Z}_{01}$ and $\mathcal{Z}_{10}$ respectively.
Vertex labeled $2p-2$	Surjects onto $\mathcal{Z}_{11}$.
Vertex labeled 1	Surjects onto $\mathcal{Z}_{00}$.
Batons	Fix a baton B. The irreducible components corresponding to a vertices in B all surject onto the same connected component of $\overline{\mathcal{A}_{s_2\tau}}$. This induces a bijection between the set of batons and the connected components of $\overline{\mathcal{A}_{s_2\tau}}$.

Table 5.8: Images of irreducible components of $\mathcal{A}_1^{[p+1]}$

APPENDIX

A.1 Determinants

Let R be a ring and M a finite locally free R-module. For $r \in R$, denote by $[r] : M \to M$ the homomorphism given by multiplication by r. We then define

$$\det : \mathrm{End}_R(M) \to \bigwedge^{\mathrm{rk} M} \mathrm{End}_R(M) \xrightarrow{\sim} R$$

as follows. The R-homomorphism $\mathrm{End}_R(M) \to \bigwedge^{\mathrm{rk} M} \mathrm{End}_R(M)$ is given by $f \to \bigwedge^{\mathrm{rk} M} f$. The second map in the above composition is the inverse of the isomorphism

$$R \to \bigwedge^{\mathrm{rk} M} \mathrm{End}_R(M) \quad \text{sending} \quad r \to \bigwedge^{\mathrm{rk} M} [r].$$

That this map is indeed an isomorphism can be verified locally, and hence we are reduced to the case where M is free.

Let A be an R-algebra and let M be a left A-module which is finite and locally free as an R-module. Define $\mathbb{V}_A$ to be the functor on the category of R-algebras given by $\mathbb{V}_A(S) = A \otimes_R S$. We define the morphism $\det_{M,A} : \mathbb{V}_A \to \mathbb{A}^1_R$ on S-valued points by

$$x \to \det_S(x | M \otimes_R S).$$

If A is finite and free as an R-algebra, let $\{a_1, \ldots, a_t\}$ be a R-basis of A. Then we have

$$S^t \xrightarrow{\sim} A \otimes_R S \quad \text{by} \quad (x_1, \ldots, x_t) \to a_1 \otimes x_1 + \cdots + a_t \otimes x_t.$$

Therefore $\mathbb{V}$ is representable by $\mathbb{A}^t_R$ and $\det_M$ corresponds to the polynomial

$$\det_{R[X_1,\ldots,X_t]}\big(a_1 \otimes X_1 + \cdots + a_t \otimes X_t | M \otimes_R R[X_1,\ldots,X_t]\big).$$

Proposition A.1.1. *Suppose $R = k$ is a field and A is a finite dimensional semisimple k-algebra. Let M and N be A-modules. Then $M \cong N$ as A-modules if and only if $\det_M = \det_N$.*

Proof. If $M \cong N$ then certainly $\det_M = \det_N$. So suppose $\det_M = \det_N$. Write $A = A_1 \times \cdots \times A_r$ where each A_i as simple. Let $\{a_1, \ldots a_r\}$ be a set of mutually orthogonal idempotents with $a_i \in A_i$, so $a_i^2 = a_i$ and $a_i a_j = 0$ for $i \neq j$. Then we have the decompositions

$$M = M_1 \times \cdots \times M_r \qquad \text{and} \qquad N = N_1 \times \cdots \times N_r$$

where $M_i = a_i M$ and $N_i = a_i N$. Set $S = R_i[T]$ and consider the element $1 \otimes T \in \mathbb{V}_A(S) = A \otimes_R R_i[T]$. Then

$$\mathrm{rk}_{R_i} M_i = \det_{R_i[T]}(T | M \otimes_R R_i[T]) = \det_{R_i[T]}(T | N \otimes_R R_i[T]) = \mathrm{rk}_{R_i} N_i.$$

Since each A_i is simple there exists a unique irreducible A_i-module up to isomorphism, and therefore $M_i \cong N_i$. The proposition immediately follows. $\square$

Proposition A.1.2. *Let $f \in R$ and R_f denote the localization of R with respect to the set $\{1, f, f^2, \ldots\}$. Then $\det_{M_f} = \det_M \otimes_R R_f$, where $\det_{M_f}$ is with respect to the R_f-module A_f.*

Proof. Note that $V \otimes R_f = V_{A \otimes_R R_f}$ is the functor on the category of R_f-algebras sending S to $A_f \otimes_{R_f} S$. Then $\det_M \otimes_R R_f : V \otimes R_f \to \mathbb{A}^1_{R_f}$ which on an R_f-algebra S sends $x \in S$ to $\det_S(x | (M \otimes_R R_f)_S)$. Since $M \otimes_R R_f = M_f$ by definition, the result is immediate. $\square$

A.2 Weyl groups

Let G be a split reductive linear algebraic group over a field k, B a Borel subgroup, and $T \subset B$ a maximal torus defined over k. Let Φ be the set of roots given by T, Φ^+ denote the set of positive roots distinguished by B, and Q denote the subgroup of the affine transformations of V^* generated by $\Phi^\vee$. Write $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, the cocharacter lattice of T.

Definition A.2.1. With respect to the above data, we define the following groups.

- The Weyl group $W = N_G(T)/T$;
- The affine Weyl group $W^{\mathrm{aff}} = Q \rtimes W$;
- The extended affine Weyl group $\widetilde{W} = W \ltimes X_*(T)$.

Example A.2.2.

- The Weyl group of $G = \mathrm{SL}_n$ is isomorphic to S_n. The affine Weyl group W_a is the semidirect product of S_n with the subgroup of $\mathbb{Z}^n$ consisting of all tuples $(a_1, \ldots, a_n)$ with $\sum_i a_i = 0$, where S_n acts on $\mathbb{Z}^n$ via permutation of the coordinates.
- The Weyl group of Sp_{2n} can be realized as a subgroup of S_{2n} consisting of the permutations that commute with the permutation $(1, 2n)(2, 2n-1)\ldots(n, n+1) \in S_{2n}$. The affine Weyl group of Sp_{2n} is the subgroup of $\mathbb{Z}^{2n} \rtimes S_{2n}$ generated by the permutations $(i, i+1)(2n+1-i, 2n-i)$ for $1 \leq i \leq n-1$, the permutation $(n, n+1)$, and the element $(-1, 0, \ldots, 0, 1)(1, 2n)$.

Definition A.2.3. Let (W, S) be a Coxeter system. Then for $u, v \in W$ we write $u \leq v$ if there is a reduced word $v = s_1 s_2 \ldots s_d$ and a sequence $1 \leq i_1 < i_2 < \cdots < i_r \leq d$ such that

$w = s_{i_1}s_{i_2}\dots s_{i_r}$ is a reduced word for w. This is a partial order on W.

As W and W^{aff} are both Coxeter groups, they can be equipped with the Bruhat order. The Bruhat order, denoted by $\leq$, may be extended to $\widetilde{W} = W \rtimes X_*$ as follows. With $x, x' \in \widetilde{W}$, they may be uniquely decomposed as $x = wc$ and $x' = w'c'$ where $w, w' \in W$ and $c, c' \in X_*$. Then $x \leq x'$ means $w \leq w'$ and $c = c'$.

A.3 Affine flag variety

Let G be a reductive linear algebraic group over a field k, B a Borel subgroup of G.

Definition A.3.1. Define the following functors from k-algebra to sets.

- The loop group LG
$$R \to G(R((T))).$$
- The positive loop group L^+G
$$R \to G(R[[T]])).$$
- Let I denote the Iwahori subgroup of LG induced by B, given by identifying the inverse image of B under $L^+G \to G$ with a subgroup of LG using $L^+G \hookrightarrow LG$. Then the affine flag variety $\mathcal{F}_G = LG/I$ where the quotient is as fpqc-sheaves on k-schemes.

Proposition A.3.2. *We have the following properties.*

(i). L^+G is represented by an affine scheme over $\mathrm{Spec}(k)$.

(ii). LG is represented by an ind-scheme over $\mathrm{Spec}(k)$.

Proof.

(i). Let us first show this for $G = \mathrm{GL}_n$. Here we identify $\mathrm{GL}_n(R[[t]])$ with the set of matrices

$$\{(A, B) \in M_n(R[[t]]) \times M_n(R[[t]]) : AB = 1\}.$$

We can consider $M_n(R[[t]])$ as $\prod_{i\geq 0} \mathbb{A}_R^{n^2}$, where each index i gives the coefficient of t^i. It then follows that GL_n is a closed subscheme of the affine scheme $\prod_{i\geq 0} \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$.

For an arbitrary linear group G, a closed embedding of G into GL_n for some n gives a closed embedding of L^+G into $L^+\mathrm{GL}_n$.

(ii). It is clear that

$$LG(R) = \varinjlim_{i\leq 0} G(t^i R[[t]])$$

with the directed system given by the inclusion homomorphisms, so that LG is an ind-scheme.

$\square$

The affine flag variety can be realized as a space of lattices subject to additional conditions. We now explain this in detail in the case $G = \mathrm{SL}_n$ or $G = \mathrm{GSp}_{2n}$.

Definition A.3.3. A lattice $\mathcal{L} \subset R((t))^n$ is a locally free $R[[t]]$-submodule such that $\mathcal{L} \otimes_{R[[t]]} R((t)) = R((t))^n$.

We say that a lattice $\mathcal{L}$ is r-special if $\bigwedge^n \mathcal{L} = t^r \Lambda_R$.

Definition A.3.4. A sequence $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{n-1} \subset t^{-1}\mathcal{L}_0$ of lattices in $R((t))^n$ is called a complete lattice chain if $\mathcal{L}_{i+1}/\mathcal{L}_i$ is a locally free R-module of rank one for all i.

Proposition A.3.5. *Fix $r \in \mathbb{Z}$. There is a functorial isomorphism*

$$\mathcal{F}(R) \xrightarrow{\sim} \{r\text{-special complete lattice chains in } R((t))^n\}.$$

Proof. The morphism is given by

$$\overline{g} \to g \cdot (\lambda_i)_i, \quad \text{where } \lambda_i = R[[t]]^{n-r+i} \oplus (tR[[t]])^{r-i}.$$

Since I is the stabilizer of the standard lattice chain $(\lambda_i)_i$, this map is well-defined and injective. It remains to show that if $(\mathcal{L}_i)_i$ is an r-special lattice chain, then Zariski-locally on $\operatorname{Spec}(R)$ there exists an element $g \in \mathrm{SL}_n(R[[t]])$ such that $(\mathcal{L}_i)_i = g \cdot (\lambda_i)_i$. Now each $\mathcal{L}_i$ is locally free and so there exists, Zariski-locally on R, $g' \in \mathrm{GL}_n(R[[t]])$ such that $(\mathcal{L}_i)_i = g' \cdot (\lambda_i)_i$. As $\mathcal{L}_0$ is r-special, we have

$$\begin{tikzcd}
\bigwedge^n \lambda_0 \arrow[r, "\wedge g'"] \arrow[d] & \bigwedge^n \mathcal{L}_0 \arrow[d] \\
t^r R[[t]] \arrow[r, "\sim"] & t^r R[[t]]
\end{tikzcd}$$

giving that $\det(g') \in R[[t]]^\times$. As such, we can find an element $g \in \mathrm{SL}_n(R[[t]])$ such that $(\mathcal{L}_i)_i = g \cdot (\lambda_i)_i$. □

Definition A.3.6. Let $\langle \cdot, \cdot \rangle$ denote the alternating pairing on $R((t))^n$ given by the matrix

$$J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}, \quad \text{where } J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

We say that a lattice chain $\mathcal{L}$ is self-dual if for all lattices $\Lambda \in \mathcal{L}$, the dual

$$\Lambda^{\perp} = \{x \in K^n : \langle x, y\rangle \in R[[t]] \text{ for all } y \in \Lambda\}$$

also occurs in the lattice chain.

Proposition A.3.7. *There is a functorial isomorphism*

$$\mathcal{F}(R) \xrightarrow{\sim} \{0\text{-}\textit{special self-dual complete lattice chains in } R((t))^n\}.$$

Proof. The morphism is given by

$$\overline{g} \to g \cdot (\lambda_i)_i, \quad \text{where } \lambda_i = R[[t]]^{n-r+i} \oplus tR[[t]]^{r-i}.$$

Since I is the stabilizer of the standard lattice chain $(\lambda_i)_i$, this map is well-defined and injective. It remains to show that if $(\mathcal{L}_i)_i$ is an r-special lattice chain, then Zariski-locally on $\operatorname{Spec}(R)$ there exists an element $g \in \mathrm{Sp}_n(R[[t]])$ such that $(\mathcal{L}_i)_i = g \cdot (\lambda_i)_i$. This follows from [RZ, Proposition A.21]. □

The affine flag varieties admit a stratification by Schubert cells. By a stratification of a space X we mean that there exists a collection $\{X_i \subset X\}_{i \in I}$ where I has a partial order $\leq$ such that

$$X = \coprod_i X_i \qquad \text{and} \qquad \overline{X_i} = \bigcup_{j \leq i} X_j \text{ for every } i \in I.$$

In the case of the affine flag variety associated with SL_n or Sp_{2n}, there is a canonical stratification where $I = \widetilde{W}$ and for $w \in \widetilde{W}$, X_w is the associated Schubert cell.

In the following, we regard all algebraic groups as the group given by their $k((t))$-valued points. Thus we will write G for $G(k((t)))$, etc. We define an embedding $\iota : X_*(T) \hookrightarrow G$ as follows. For $\lambda \in X_*(T)$, we have $\lambda : \mathbb{G}_m \to T$ and thus set $\iota(\lambda) = \lambda(t) \in T \subset G$.

When $G = \mathrm{SL}_n$ we can identify $W = N_G T/T$ with the group of permutation matrices. When $G = \mathrm{Sp}_{2n}$ we can identify W with the subgroup of generalized permutation matrices. Thus in either case we have an extension $\iota : \widetilde{W} \hookrightarrow G$. From now on, we will identify $\widetilde{W}$ as a subset of G via the embedding ι.

Definition A.3.8. Let $w \in \widetilde{W}$. The Schubert cell associated to w is given by $IwI/I \subset \mathcal{F}_G$. The Schubert variety associated with w, denoted by X_w, is the Zariski closure of IwI/I inside of $\mathcal{F}_G$.

Proposition A.3.9. *For $w \in \widetilde{W}$, we have*

(i). $X_w = \cup_{v \leq w} IvI/I$*;*

(ii). For $v \in W^{aff}$*,* $X_v \subset X_w$ *if and only if* $v \leq w$*;*

(iii). $\dim X_w = \ell(w)$*, where* $\ell(w)$ *is the length of a reduced expression for* w*;*

and $\mathcal{F}_G$ *admits a stratification*

$$\mathcal{F}_G = \coprod_{w \in \widetilde{W}} IwI/I.$$

The standard apartment of (the Bruhat-Tits building associated with) $\mathcal{F}$ is defined as follows. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $k((t))^n$. Then the vertices of the standard apartment are given by lattices generated by

$$\langle t^{-r_1} e_1, \ldots, t^{-r_n} e_n \rangle.$$

Identify such a lattice with the n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$. Two lattices $(r_1, \ldots, r_n)$ and $(s_1, \ldots s_n)$ are considered equivalent if there exists an integer m such that

$$(s_1, \ldots, s_n) = (r_1 + m, \ldots, r_n + m).$$

Thus the set of vertices of the standard apartment can be identified with $\mathbb{Z}^n$ modulo $\mathbb{Z}$, where $\mathbb{Z}$ acts diagonally by addition.

The alcoves of the standard apartment of $\mathcal{F}$ are by definition tuples $(x_0, \ldots, x_{n-1})$ where each x_i is a vertex (i.e. an element of $\mathbb{Z}^n/\mathbb{Z}$) such that for some choice of lifts $\widetilde{x}_i \in \mathbb{Z}^n$ we have

$$\widetilde{x}_0 \leq \widetilde{x}_1 \leq \cdots \leq \widetilde{x}_{n-1} \leq \widetilde{x}_n := \widetilde{x}_0 + (1, \ldots, 1)$$

and

$$\sum_j \widetilde{x}_{i+1}(j) = \sum_j \widetilde{x}_i(j) + 1 \text{ for all } i.$$

Here $\widetilde{x}_i \leq \widetilde{x}_j$ is defined coordinate-wise: $\widetilde{x}_i(j) \leq \widetilde{x}_{i+1}(j)$ for all i, j. In the case $G = \mathrm{Sp}_n$, we also impose the additional condition that for $1 \leq i \leq 2n$ we have

$$x_{n-i} = \theta(x_i), \quad \text{where} \quad \theta(r_1, r_2, \ldots, r_n) = (-r_n, -r_{n-1}, \ldots, -r_1).$$

Note that such an alcove $(x_1, \ldots, x_n)$ naturally corresponds to a complete periodic (and in the case GSp_{2n}, self-dual) lattice chain, i.e. an element of $\mathcal{F}$.

We now fix the alcoves

$$\omega = (\omega_0, \ldots, \omega_{n-1}), \quad \omega_i = (1^i, 0^{n-i})$$

$$\tau = \big((1^r, 0^{n-r}), (1^{r+1}, 0^{n-r-1}), \ldots, (2^{r-2}, 1^{n-r}), (2^{r-1}, 1^{n-r+1})\big)$$

noting that both are alcoves of $\mathcal{F}_{\mathrm{Sp}}$ and hence also alcoves of $\mathcal{F}_{\mathrm{SL}}$. We define the size of an alcove x to be

$$\sum_{j=0}^{n-1} x_i(j) - \omega_i(j)$$

which is constant with respect to the choice of i. The affine Weyl group in each case naturally acts on the set of alcoves of size r in the standard apartment, given by acting on each vertex. This action is simply transitive. With our fixed base alcove τ we can thus identify the affine Weyl group with the set of alcoves of size r in the standard apartment.

Proposition A.3.10. *Let $x = (x_0, \ldots, x_{n-1})$ be an and $(\mathcal{L}_i)_i \in S_x$. Then there exists $b \in I$ such that*

$$\mathcal{L}_i = b \cdot \begin{pmatrix} t^{-x_i(1)+1} & & & \\ & t^{-x_i(2)+1} & & \\ & & \ddots & \\ & & & t^{-x_i(n)+1} \end{pmatrix}.$$

Here we are identifying the matrix above with the lattice generated by its columns.

Proof. This is immediate as S_x is defined as the Iwahori-orbit of x in $\mathcal{F}$. □

A.4 Group schemes

Definition A.4.1. Let S be a scheme. A group scheme G over S is an S-scheme equipped with S-morphisms

$$\mu : G \times_S G \to G \qquad \varepsilon : S \to G \qquad \iota : G \to G$$

such that the following diagrams commute.

(i). Associativity

$$\begin{tikzcd}
G \times_S G \times_S G \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"'] & G \times_S G \arrow[d, "\mu"] \\
G \times_S G \arrow[r, "\mu"] & G
\end{tikzcd}$$

(ii). Identity

$$\begin{tikzcd}
S \times_S G \arrow[r, "\varepsilon \times \mathrm{id}"] \arrow[d, "\mathrm{pr}_2"'] & G \times_S G \arrow[dl, "\mu"] \\
G & G \times_S S \arrow[l, "\mathrm{pr}_1"] \arrow[u, "\mathrm{id} \times \varepsilon"']
\end{tikzcd}$$

(iii). Inverse

$$\begin{tikzcd}
 & G \times_S G \arrow[dr, "\mu"] & \\
G \arrow[ur, "(\mathrm{id}, \iota)"] \arrow[r] \arrow[dr, "(\iota, \mathrm{id})"'] & S \arrow[r, "\varepsilon"] & G \\
 & G \times_S G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Definition A.4.2. Let G/S be a group scheme and $\sigma : G \times_S G \to G \times_S G$ be the morphism which interchanges the factors. Then we say G is a commutative group scheme if the following diagram commutes.

$$\begin{tikzcd}
G \times_S G \arrow[r, "\mu"] \arrow[d, "\sigma"'] & G \\
G \times_S G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Definition A.4.3. Let S be a Noetherian scheme. A finite flat group scheme over S is a group scheme G/S such that the structure morphism $G \to S$ is finite and flat. Equivalently, the structure morphism makes $\mathcal{O}_G$ into a locally free $\mathcal{O}_S$-module of finite rank. The rank is then a locally constant function on S, and when the rank function is constant, we will refer to it as the order of G/S, or simply the order of G if S is understood.

Definition A.4.4. Let $S = \mathrm{Spec}(R)$ be a Noetherian scheme and $G = \mathrm{Spec}(A)$ a finite flat group scheme over S. The Cartier dual of G is given by $G^* = \mathrm{Spec}(\mathrm{Hom}_R(A, R))$ equipped with the following maps.

- $\varepsilon^* : S \to G^*$. This morphism is defined on the coordinate rings $\mathrm{Hom}_R(A, R) \to R$ by sending an R-module homomorphism $\varphi : A \to R$ to $\varphi \circ \pi^{\#}(1)$ where $\pi^{\#} : R \to A$ is the structure morphism of G/S.
- $\mu^* : G^* \times G^* \to G^*$. This morphism is defined on the coordinate rings $\mathrm{Hom}_R(A, R) \to \mathrm{Hom}_R(A, R) \otimes_R \mathrm{Hom}_R(A, R)$ by first identifying

  $$\mathrm{Hom}_R(A, R) \otimes_R \mathrm{Hom}_R(A, R) = \mathrm{Hom}_R(A \otimes_R A, R)$$

  and sending an R-module homomorphism $\varphi : A \to R$ to

  $$A \otimes_R A \xrightarrow{m} A \xrightarrow{\varphi} R$$

  where $m(a \otimes b) = ab$.
- $\iota^* : G^* \to G^*$. This morphism is defined on the coordinate rings $\mathrm{Hom}_R(A, R) \to \mathrm{Hom}_R(A, R)$ by sending an R-module homomorphism $\varphi : A \to R$ to $A \xrightarrow{\iota^{\#}} A \xrightarrow{\varphi} R$.

One can check that these maps give G^* the structure of a finite flat group scheme over S, and that $(G^*)^*$ is canonically isomorphic to G.

Definition A.4.5. Let G/S be a group scheme, $\varepsilon : S \to G$ the identity section. The sheaf of invariant differentials of G/S is defined to be

$$\omega_{G/S} = \varepsilon^*(\Omega_{G/S}).$$

A.4.1 Finite group schemes of order p

Throughout this section, k denotes an algebraically closed field and p a rational prime. Set $S = \mathrm{Spec}(k)$. If the characteristic of k is not p, there is (up to isomorphism) precisely one finite flat group scheme of order p over S. On the other hand, if the characteristic of k is p, then there are (up to isomorphism) three finite flat group schemes of order p over S. We will present these three group schemes, calculate their Cartier duals, and calculate the dimension of their invariant differentials.

Example A.4.6. The constant group scheme $G = (\mathbb{Z}/p\mathbb{Z})_S$.

Description of G: As a scheme, G is given by the disjoint union of p copies of $\mathrm{Spec}(k)$ and we fix an indexing by $0, 1, \ldots, p-1$ which we will write as $S_i \subset G$ for $0 \leq i \leq p-1$. Then this induces an indexing of $G \times_S G$ by pairs (i, j) with $0 \leq i, j \leq p-1$, which we write as $S_{ij} \subset G \times_S G$. Define the multiplication map $G \times_S G \to G$ by $S_{ij} \to S_{i+j}$ where the addition is in $\mathbb{Z}/p\mathbb{Z}$ and the morphism is the identity map. The identity and inverse are respectively defined as

$$S \to S_0 \qquad \text{and} \qquad S_i \to S_{-i}$$

again using the identity maps. It is straightforward to see that this makes G/S into a group scheme and that G is finite and flat over S.

The Cartier dual of G: The collection of (set) maps $\{e_i : \mathbb{Z}/p\mathbb{Z} \to k\}$ where e_i is defined by

$e_i(j) = \delta_{ij}$ gives a basis of the k-vector space $\Gamma(G, \mathcal{O}_G)$. The morphisms μ, ε, and ι are given on the coordinate rings as

$$\mu^{\#}(e_i \otimes e_j) = \delta_{ij} e_i \qquad \varepsilon^{\#}(e_i) = \delta_{0i} \qquad \iota^{\#}(e_i) = e_{-i}.$$

To calculate the Cartier dual, let $\{e_i^*\}$ be the dual basis defined by $e_i^*(e_j) = \delta_{ij}$. Then μ^*, ε^*, and ι^* of the Cartier dual G^* are given on the coordinate rings as

$$(\mu^*)^{\#}(e_i^*) = e_i^* \otimes e_i^* \qquad (\varepsilon^*)^{\#}(e_i^*) = 1 \qquad (\iota^*)^{\#}(e_i^*) = e_{-i}^*.$$

From this description, it is immediate that $(\mathbb{Z}/p\mathbb{Z})_S^* \cong (\mu_p)_S$ (see the example below).

The invariant differentials of G: Since $G \to S$ is étale, it is immediate that $\Omega_{G/S} = 0$ and hence $\omega_{G/S} = 0$.

Example A.4.7. The roots of unity $G = (\mu_p)_S$.

Description of G: Define $G = \operatorname{Spec}(k[T]/(T^p - 1))$ over $\operatorname{Spec}(k)$ with the morphisms μ, ε, and ι defined on the coordinate rings as

$$T \to T \otimes T \qquad T \to 1 \qquad T \to T^{p-1}.$$

It is easy to check that this morphisms satisfy the required commutative diagrams, and G is visibly flat and finite over $\operatorname{Spec}(k)$. Furthermore G is étale over S if and only if the characteristic of k is not p.

The Cartier dual of G: As the double (Cartier) dual of a group scheme is canonically isomorphic to the group scheme itself, we have that the Cartier dual of G is $(\mathbb{Z}/p\mathbb{Z})_S$.

The invariant differentials of G: If the characteristic of k is not p, then μ_S is étale over S and thus the dimension of its invariant differentials is automatically zero. We thus assume that the characteristic of k is p. Set $I = (x^p - 1)$ and $A = k[x]/I$. We have the standard exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{k[x]/k} \otimes_{k[x]} A \to \Omega_{R/k} \to 0$$

where the first map is given by $\delta(\overline{\alpha}) = d\alpha \otimes 1$. The image of the first map is thus generated by px^{p-1} which is zero since the characteristic of k is p. We therefore have that $\Omega_{G/S}$ is given by the module $k[x]dx/(x^p - 1)$. The pullback of this module to k is given by $k[x]\langle dx\rangle/(x^p - 1) \otimes_{k[x]/(x^p-1)} k$. It follows that the invariant differentials are given by $k\langle dx\rangle$ and therefore are of dimension one.

Example A.4.8. $G = (\alpha_p)_S$, where p is a rational prime and k is of characteristic p.

Description of G: Define $G = \operatorname{Spec}(k[T]/(T^p))$ with the morphisms μ, ε, and ι defined on the coordinate rings as

$$T \to T \otimes 1 + 1 \otimes T \qquad T \to 0 \qquad T \to -T.$$

Note that the first map above is a ring homomorphism precisely because the characteristic of k is p. Again, it is easy to check that this morphisms satisfy the required commutative diagrams, and α_p is visibly flat and finite over $\operatorname{Spec}(k)$, but certainly not étale since G is not reduced.

The Cartier dual of G: Set $R = k[T]/(T^p)$. We have a k-basis of R given by $\{T^i\}$. Let $\{u_i\}$ denote the dual basis, i.e. $u_i(T^j) = \delta_{ij}$. Then the k-linear map

$$\varphi : \operatorname{Hom}_k(R, k) \to R \quad \text{sending} \quad u_i \to T^i/i!$$

is k-isomorphism. Furthermore, it preserves the morphisms giving the group structure of G. To see this, first note that the multiplication map m is given on coordinate rings by

$$m^{\#}(T^i) = \sum_{j=0}^{i} \binom{i}{j} T^j \otimes T^{i-j}.$$

The multiplication map for the Cartier dual of G is given on coordinate rings by

$$(m^*)^{\#}(u_i) = \sum_{j=0}^{i} u^j \otimes u^{i-j}.$$

Thus we have that the diagram (given on generators)

$$\begin{tikzcd} u_i \arrow[r, "\varphi"] \arrow[d, "(m^*)^{\#}"'] & T^i/i! \arrow[d, "m^{\#}"] \\ \sum_{j=0}^{i} u^j \otimes u^{i-j} \arrow[r, "\varphi \otimes \varphi"] & \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} T^j \otimes T^{i-j} \end{tikzcd}$$

commutes for all i. The inverse and identity maps may be checked in a similar fashion and it therefore follows that $G \cong G^*$.

The invariant differentials of G: Set $I = (x^p)$ and $A = k[x]/I$. We have the standard exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{k[x]/k} \otimes_{k[x]} A \to \Omega_{R/k} \to 0$$

where the first map is given by $\delta(\overline{\alpha}) = d\alpha \otimes 1$. The image of the first map is thus generated by $px^{p-1} = 0$ since the characteristic of k is p. We therefore have that $\Omega_{G/S}$ is given by the module $k[x]\langle dx \rangle/(x^p)$. The pullback of this module to k is given by $k[x]\langle dx \rangle/(x^p) \otimes_{k[x]/(x^p)} k$. It follows that the invariant differentials are given by $k\langle dx \rangle$ and therefore are of dimension one.

Theorem A.4.9. *[OT, Lemma 1] Let k be an algebraically closed field. If the characteristic of k is not p, then $(\mathbb{Z}/p\mathbb{Z})_k$ is the only finite flat group scheme of order p up to isomorphism. If the characteristic of k is p, then there are three nonisomorphic finite flat group schemes of order p:*

$$(\mathbb{Z}/p\mathbb{Z})_k, \quad (\mu_p)_k, \quad \textit{and} \quad (\alpha_p)_k.$$

A.4.2 Oort-Tate generators

Using the notion of a "full-set of sections" [KM] for a finite flat group scheme G/S, we recall the definition of an Oort-Tate generator from [HR].

Definition A.4.10. [KM, 1.8.2] Let G be finite flat group scheme over S of rank $N \geq 1$. Then we say that a set of sections $P_1, \ldots, P_N$ in $G(S)$ is a full set of sections of G/S if for every affine S-scheme $T = \mathrm{Spec}(R)$, and for every function $f \in B = \Gamma(G \times_S T, \mathcal{O}_{Z \times_S T})$, we have:

$$\mathrm{Norm}_{B/R}(f) = \prod_{i=1}^{N} f(P_i).$$

Definition A.4.11. Let S be a $\mathbb{Z}_p$-scheme and $\pi : G \to S$ finite flat group scheme of order p with finite presentation over S. Suppose $\sigma : S \to G$ is a section of π. Then we have a collection of sections $\{\varepsilon, \sigma, [2]\sigma, \ldots, [p-1]\sigma\}$. We say that σ is an Oort-Tate generator if this collection is a full set of sections of G/S.

Proposition A.4.12. *[KM, Lemma 1.8.3] Let S be a connected scheme, G/S be a finite étale group scheme of order N, and $\sigma_1, \ldots, \sigma_N$ be a collection of sections $S \to G$ of G. Then the following conditions are equivalent.*

(i). The S-morphism

$$\coprod_{i=1}^{N} S \to G$$

defined by the sections $\{\sigma_i\}$ is an isomorphism of S schemes.

(ii). $\{\sigma_1, \ldots, \sigma_N\}$ form a full set of sections of G/S.

Proof. Suppose the morphism $\coprod S \to G$ is an isomorphism. Then for $T = \mathrm{Spec}(R)$ in the definition above, $B = \oplus_{i=1}^{N} R$. Choosing the standard R-basis $\{e_i\}$, we have that for $f = \sum_i r_i e_i$ the matrix representing "multiplication by f" in this basis is $\mathrm{diag}(r_1, \ldots, r_n)$ and hence $\mathrm{Norm}_{B/R}(f) = \prod_i r_i$. Since $f(P_i) = r_i$, the conclusion follows.

Conversely, suppose $\{\sigma_i\}$ form a full set of sections of G/S. Then on each connected component of the source, the morphism

$$\coprod_{i=1}^{N} S \to G$$

restricts to the identity map onto its image in G. Thus to show that this is an isomorphism, we only need to show that no two connected components of the source are sent to the same connected component of G.

It thus suffices to show that for every geometric point $\mathrm{Spec}(k) \to S$, the N points $\sigma_{i,k}$: $\mathrm{Spec}(k) \to G_k$ are all distinct. To see this, let $Q_1, \ldots, Q_N$ denote the reduced closed points whose disjoint union forms G_k. Let $f : G_k \to \mathbb{A}^1_k$ such that $f(Q_i)$ are all distinct values. Then the characteristic polynomial of f is

$$\det(T - f) = \prod_{i=1}^{N} (T - f(Q_i))$$

and since $\{\sigma_i\}$ form a full set of sections, we also have

$$\det(T - f) = \prod_{i=1}^{N} (T - f(\sigma_{i,k})).$$

It follows that $f(\sigma_{i,k})$ are all distinct, meaning of course that $\sigma_{i,k}$ are as well. $\square$

Proposition A.4.13. *Let $Z = \mathrm{Spec}(k[x]/(x-a)^n)$ and $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Then the collection of sections $\{P_1, \ldots, P_n\}$ where each is defined on coordinate rings as $x \to a$ gives a full set of sections. Furthermore, this collection is the only full set of sections.*

Proof. Let R be a k-algebra, $B = R[x]/(x-a)^n$, and $f \in B$. We first calculate $\mathrm{Norm}_{B/R}(f)$. Note that $\{1, x-a, \ldots, (x-a)^{n-1}\}$ forms an R-basis of B and write $f = \sum_{i=0}^{n} f_i (x-a)^i$. Then the matrix representing the map "multiplication by f" with respect to this basis is

$$\begin{pmatrix} f_0 & & & \\ f_1 & f_0 & & \\ \vdots & \vdots & \ddots & \\ f_{n-1} & f_{n-2} & \ldots & f_0 \end{pmatrix}.$$

Therefore $\mathrm{Norm}_{B/R}(f) = f_0^n$ and this is precisely $\prod_{i=1}^{n} f(a)$.

That this is the only collection forming a full set of sections is immediate, as a collection of sections in this case is determined by the collections cardinality. $\square$

We now describe the Oort-Tate generators of the three group schemes we are primarily interested in.

Example A.4.14.

(i). $G = (\mathbb{Z}/p\mathbb{Z})_k$, $k = \overline{k}$ a field. This group scheme is étale and hence the $p-1$ nonzero sections are all Oort-Tate generators.

(ii). $G = (\mu_p)_k$, $k = \overline{k}$ a field of characteristic p. This group scheme is the spectrum of a nonreduced point, and hence the zero section is the only generator.

(iii). $G = (\alpha_p)_k$, $k = \overline{k}$ a field of characteristic p. This group scheme is the spectrum of a nonreduced point, and hence the zero section is the only generator.

A.4.3 Oort-Tate Theory

In [OT], Oort and Tate classify finite flat group schemes over

$$\mathrm{Spec}(\Lambda_p), \quad \text{where } \Lambda_p = \mathbb{Z}\left[\zeta, \frac{1}{p(p-1)}\right] \cap \mathbb{Z}_p.$$

In particular, the classification applies to schemes S over $\mathrm{Spec}(\mathbb{Z}_p)$, the case of interest. We will state the classification and then recast it using stacks as in [HR].

Theorem A.4.15. *[OT, Theorem 2] Let S be a scheme over* $\mathrm{Spec}(\Lambda_p)$. *Then there is a natural 1-1 correspondence finite flat group schemes of order p and the collection of (L, a, b) where*

- *L is an invertible sheaf on S;*
- *$a \in \Gamma(S, L^{\otimes(p-1)})$, $b \in \Gamma(S, L^{\otimes(1-p)})$; and*
- *$a \otimes b = w_p \in \Gamma(S, \mathcal{O}_s)$ where $w_p = \varepsilon \cdot p$ with $\varepsilon \in \Lambda_p^\times$.*

Sketch. Let us merely indicate how to construct (L, a, b) from a group scheme G. We first define

$$e_i = \frac{1}{p-1} \sum_{j \in \mathbb{Z}/p\mathbb{Z}^\times} \chi^{-i}(j)[j] \in \mathcal{O}_S[\mathbb{Z}/p\mathbb{Z}^\times].$$

Here $\chi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p$ is the Teichmüller representative (whose image consists of the $(p-1)st$ roots of unity) and $[j] : G \to G$ is multiplication by j. Let $m_0 = \ker(\varepsilon^\#)$ where $\varepsilon : S \to G$ is the identity section. One can show that the e_i are orthogonal idempotents, and thus we have

$$m_0 = \bigoplus_{i=1}^{p-1} e_i m_0.$$

Set $I = e_1 m_0$. The above construction applies equally as well to the Cartier dual G^*, and we likewise define I^D in the same manner but with respect to G^*. Now it turns out the p-fold multiplication of G sends $I^{\otimes p} \to I$ which is to say that there is a homomorphism in

$$\mathrm{Hom}_{\mathcal{O}_S}(I^{\otimes p}, I) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, I^{\otimes 1-p}) = \Gamma(S, I^{\otimes 1-p})$$

giving the global section a. Applying the same to G^* we get b. Now one cay show $a \otimes b = w_p$ where w_p is defined as follows. In the ring $\Lambda_p[z]/(z^p - 1)$ set

$$y_j = \sum_{m \in \mathbb{Z}/p\mathbb{Z}^\times} \chi^{-j}(m)(1 - z^m)$$

and define w_p by the equation $y_1^p = w_p y_p$. Finally one shows that w_p is indeed $p \cdot \varepsilon$ where ε is a unit in Λ_p. □

A.5 de Rham cohomology

Let $F^\bullet$ be a complex of sheaves of abelian groups on a topological space X such that $F^i = 0$ for $i << 0$. Denote by $d^i : F^i \to F^{i+1}$ the differential of the complex, so that $d^{i+1}d^i = 0$. Then the i^{th} cohomology of the complex $F^\bullet$ is given by

$$h^i(F^\bullet) = \ker d^i / \operatorname{Img} d^{i-1}.$$

We say that a map of complexes $\varphi : F^\bullet \to G^\bullet$ of sheaves of abelian groups on a topological space X is called a quasi-isomorphism if it induces an isomorphism $h^i(\varphi) : h^i(F^\bullet) \to h^i(G^\bullet)$ for all i.

An injective resolution is a quasi-isomorphism $f^\bullet : F^\bullet \to I^\bullet$ where I^i is an injective sheaf for all i. Then the hypercohomology of $F^\bullet$ is

$$\mathbb{H}^i(X, F^\bullet) = h^i(\Gamma(X, I^\bullet))$$

where Γ is the global sections functor. The hypercohomology groups are independent of the injective resolution chosen.

Now let S be a scheme and A/S be an abelian scheme of relative dimension n. The de Rham complex of A/S is given by

$$\Omega^1_{A/S} \xrightarrow{d^1} \Omega^2_{A/S} \xrightarrow{d^2} \dots \xrightarrow{d^{n-1}} \Omega^n_{A/S}$$

with differential $d^i : \Omega^i_{A/S} \to \Omega^{i+1}_{A/S}$. We define the de Rham cohomology of A/S to be the hypercohomology of A/S with respect to the de Rham complex:

$$H^1_{\mathrm{dR}}(A/S) = \mathbb{H}^1(A, \Omega^\bullet_{A/S}).$$

Proposition A.5.1. *[BBM] Set $\omega_{A/S} = e^*(\Omega^1_{A/S})$ where $e : S \to A$ is the identity section; $\omega_{A/S}$ is called the sheaf of invariant differentials of A/S. Let $Lie(A^\vee/S)$ denote the Lie algebra of the dual abelian scheme $A^\vee/S$. There is an exact sequence of locally free modules over S*

$$0 \to \omega_{A/S} \to H^1_{dR}(A/S) \to Lie(A^\vee/S) \to 0$$

whose formation commutes with base change.

A.6 Abelian schemes

We collect here definitions and propositions relating to abelian schemes. We include only the essentials needed.

Definition A.6.1. Let S be a Noetherian scheme, and $\pi : A \to S$ be an abelian scheme of relative dimension n. That is, A/S is a group scheme where π is smooth, proper, and the geometric fibers of π are connected. Denote the multiplication by $\mu : A \times_S A \to A$, the inverse by $\iota : A \to A$, and the identity by $\varepsilon : S \to A$.

Definition A.6.2. The dual abelian scheme over S is

$$A^\vee = \mathrm{Pic}^0(A/S)$$

where Pic^0 is the connected component of the identity of the scheme representing the Picard functor.

Definition A.6.3. The Lie algebra of A/S is

$$\mathrm{Lie}(A/S) = \varepsilon^* T_{A/S}$$

where $T_{A/S} = \mathcal{H}om_{\mathcal{O}_A}(\Omega_{A/S}, \mathcal{O}_A)$ is the tangent bundle of A/S.

Definition A.6.4. The p-divisible group of A/S is

$$A[p^\infty] = \varinjlim_n A[p^n]$$

where the maps of the directed system are given by inclusion.

Definition A.6.5. A full level N structure on A/S consists of a collection of sections $\sigma_i : S \to A$ where $1 \leq i \leq 2N$ such that

(i). for all geometric points s of S, the images $\sigma_i(s)$ form a basis for $A_s[N]$; and

(ii). $[N] \circ \sigma_i = \varepsilon$ where $[N] : A \to A$ is multiplication by N.

We now seek to define a polarization of an abelian scheme. Let L be an invertible sheaf on A and consider the invertible sheaf on $A \times_S A$ given by

$$\mu^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1} \quad \text{where } A \times_S A \underset{p_1}{\overset{p_2}{\rightrightarrows}} A.$$

Regarding $A \times_S A$ as a scheme over A via p_1, this sheaf defines an A-valued point $\Lambda(L) : A \to$

Pic(A/S). Now $\Lambda(L) \circ \varepsilon : S \to \text{Pic}(A/S)$ is the identity and thus ψ is a homomorphism. Also, since the geometric fibers of A over S are connected, we have $\Lambda(L)$ factors through $A^\vee = \text{Pic}^0(A/S) \hookrightarrow \text{Pic}(A/S)$.

Definition A.6.6. A polarization of an abelian scheme A/S is an S-homomorphism $\lambda : A \to A^\vee$ such that for all geometric points s of S, the induced homomorphism $\lambda_s : A_s \to A_s^\vee$ is of the form $\Lambda(L_s)$ for some ample invertible sheaf L_s on A_s. Such a polarization is said to be principal if λ is an isomorphism.

Theorem A.6.7. *[MFK, Theorem 7.9] Let $\mathcal{A}_{n,1,N}$ denote the moduli functor of abelian schemes of relative dimension n, equipped with a principal polarization and level N structure. If $N \geq 3$, then a fine moduli scheme for $\mathcal{A}_{n,1,N}$ exists.*

We now describe the Rosati involution. Let A be an abelian variety over an algebraically closed field k equipped with a polarization $\lambda : A \to A^\vee$. Let $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$.

Definition A.6.8. The Rosati involution on $\text{End}^0(A)$ with respect to λ is defined by

$$\varphi \to \varphi' = \lambda^{-1} \circ \varphi^\vee \circ \lambda$$

for $\varphi \in \text{End}^0(A)$ where $\varphi^\vee \in \text{End}^0(A^\vee)$ is the dual isogeny of φ.

Proposition A.6.9. *[Mum, pg. 189-190] The Rosati involution satisfies the following properties.*

(i). The map $(\cdot)' : End^0(A) \to End^0(A)$ is a $\mathbb{Q}$-algebra homomorphism.

(ii). $e_\lambda(\varphi x, y) = e^\lambda(x, \varphi' y)$, where e_λ is the Weil pairing induced by λ. As e_λ is a non-

degenerate bilinear form, this immediately implies that $\varphi \to \varphi'$ is an involution.

(iii). The Rosati involution is positive.

Definition A.6.10. Suppose R is a subring of $\mathbb{Q}$. An R-isogeny $f : A \to A'$ between two abelian S-schemes is an isomorphism in the category of whose objects are abelian schemes over S and whose morphisms consist of $\mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} R$.

Let A/k be an abelian variety of dimension n with k algebraically closed and let $\ell \neq \mathrm{char}(k)$ be a rational prime. Then $A[\ell^i] \cong (\mathbb{Z}/l^i\mathbb{Z})^n$. We have surjective homomorphisms $[\ell] : A[\ell^{i+1}] \to A[\ell^i]$ given by multiplication by ℓ. These homomorphisms are compatible in the sense that they form an inverse system.

Definition A.6.11. Let A/k be an abelian variety with k algebraically closed. The ℓ-adic Tate module of A/k is defined to be the inverse limit

$$T_\ell(A) = \varprojlim_i A[\ell^i].$$

With A/k still an abelian variety over an algebraically closed field, suppose A is equipped with a principal polarization $\lambda : A \to \hat{A}$. Then the polarization induces the Weil pairing

$$A[\ell^i] \times A[\ell^i] \to \mu_{\ell^i}$$

where μ_{ℓ^i} is the group of ℓ^i roots of unity in k. Define the ℓ-adic Tate module $\mathbb{Z}_\ell(1) = \varprojlim_i \mu_{\ell^i}$, taking the inverse limit over the Weil pairing gives

$$T_\ell(A) \times T_\ell(A) \to \mathbb{Z}_\ell(1).$$

With $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$ noncanonically, we may choose an isomorphism $\mathbb{Z}_\ell(1) \xrightarrow{\sim} \mathbb{Z}_\ell$ and thus have the Weil pairing on the ℓ-adic Tate modules take values in $\mathbb{Z}_\ell$. As the choice of such an isomorphism is up to some $\mathbb{Z}_\ell^\times$-multiple, $T_\ell \times T_\ell \to \mathbb{Z}_\ell$ is well-defined up to a $\mathbb{Z}_\ell^\times$ multiple.

Proposition A.6.12. *If $f : A \to A'$ is an isogeny with kernel N, there is an exact sequence*

$$0 \to T_\ell(A) \xrightarrow{T_\ell(f)} T_\ell(A') \to N_\ell \to 0$$

where N_ℓ is the pro-ℓ part of N.

Proposition A.6.13. *Let $f : A \to A'$ be an isogeny that is also a $\mathbb{Z}_{(p)}$-isogeny. Then f is an isomorphism if and only if for all primes $\ell \neq p$, the induced homomorphism $T_\ell(f) : T_\ell(A) \to T_\ell(A')$ is an isomorphism.*

The paring on $T_\ell(A)$ induces a pairing on the rational Tate module $V_\ell := T_\ell(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ which we also call the Weil pairing.

Proposition A.6.14. *If $f : A \to A'$ is a $\mathbb{Z}_{(p)}$-isogeny, then the induced homomorphism $V_\ell(f) : V_\ell(A) \to V_\ell(A')$ is an isomorphism.*

Define

$$H_1(A, \hat{\mathbb{Z}}^{(p)}) = \prod_{\ell \neq p} T_\ell(A) \qquad \text{and} \qquad H_1(A, \mathbb{A}_f^p) = \prod_{\ell \neq p} T_\ell(A) \otimes \mathbb{Q}.$$

We again have the Weil pairing on $H_1(A, \mathbb{A}_f^p)$ taking values in $\mathbb{A}_f^p(1) := \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \otimes \mathbb{Q}$.

Proposition A.6.15. *If $f : A \to A'$ is a $\mathbb{Z}_{(p)}$-isogeny, then f induces an isomorphism*

$$H_1(f) : H_1(A, \mathbb{A}_f^p) \xrightarrow{\sim} H_1(A', \mathbb{A}_f^p).$$

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