This is to certify that the

dissertation entitled

ON THE DOMINATION NUMBER OF A DIGRAPH

presented by

Changwoo Lee

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Edgar M. Palmer
Major professor

Date 18 OCT 1994

ON THE DOMINATION NUMBER OF A DIGRAPH

By

Changwoo Lee

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1994

ABSTRACT

ON THE DOMINATION NUMBER OF A DIGRAPH

BY

Changwoo Lee

A subset S of vertices of a digraph D is a *dominating set* of D if every vertex not in S is adjacent from a vertex in S, and the *domination number* of D is the number of vertices in any smallest dominating set of D. A subset I of vertices of D is an *independent set* of D if no two vertices of I are joined by an arc in D. The *independence number* of D is the number of vertices in any largest independent subset of vertices of D. If D has an independent and dominating set, the *independent domination number* of D is the number of vertices in any smallest independent and dominating subset of vertices of D.

We first establish bounds for the domination numbers of various types of digraphs and determine the domination number of a random digraph.

Next we study the relations among the domination number, the independent domination number, and the independence number of an oriented tree and a binary tree, respectively, and we estimate their bounds. We then derive a formula for the expected independent domination number of random binary trees and determine the asymptotic behavior of the expectation.

Finally we establish bounds for the domination number of tournaments and the Paley tournament, and we determine the domination number of a random tournament.

DEDICATION

To my academic father, Edgar M. Palmer,
for his sixtieth birthday

ACKNOWLEDGEMENTS

It is a great pleasure to thank my thesis advisor, Professor Edgar M. Palmer. His guidance, patience, understanding, and encouragement have been major factors contributing to this work. I would like to thank Professor Bruce E. Sagan for his introducing me to combinatorics as well as for his instruction and help. I also thank Professor David E. Blair, Professor Wei-Eihn Kuan, and Professor Jacob Plotkin for their time and advice.

I owe special thanks to Stephen T. Hedetniemi, who sent me about fifty of his articles on domination theory, and to Robert W. Robinson, who made numerous helpful comments.

Contents

List of Tables

Introduction

The earliest ideas of dominating sets, it would seem, date back to the origin of the game of chess, in which one studies sets of chess pieces that cover or dominate various opposing pieces or various squares of the chess board [HeL90(a)]. In more recent time, dominating concepts were raised in the form of the Five Queens Problem by König in 1936 [Ko36]. Finally the topic of domination was given formal mathematical definition in the books by Berge [Be58] in 1958 and Ore [Or62] in 1962. But relatively little had been done on this topic until Cockayne and Hedetniemi published a survey article [CoH77] in 1977. Since then over 500 papers have been published on the subject (see, for example, [HeL90(b)]). Among them there are many about the domination of undirected graphs but almost nothing for the domination number of directed graphs. In this thesis we will develop theory for the domination number of directed graphs.

In his book [Or62], which is the first graph theory book written in English, O. Ore says that a graph $G = (V, E)$ with no isolated vertices has domination number at most $\frac{1}{2}|V|$. W. McCuaig and B. Shepherd lowered this upper bound of the domination number to $\frac{2}{5}|V|$ for connected graphs with minimum degree at least 2 except for seven specific graphs [McS89]. B. Reed lowered it to $\frac{3}{8}|V|$ for graphs with minimum degree at least 3 [Re9x]. Moreover, using an elegant application of the probabilistic method, N. Alon and J. Spencer [AlS92] proved that any graph with minimum degree δ has domination number at most $\frac{1+\ln(\delta+1)}{\delta+1}|V|$. However, there has been no corresponding study of the domination number for digraphs.

The main goal of this thesis is to study the domination number for various types of digraphs and random digraphs.

We first establish an upper bound for the domination number of digraphs with minimum indegree δ^- at least one by applying the probabilistic method. This bound is good for large δ^- but quite loose for small δ^-. Finding a vertex disjoint star cover of D, we determine a sharp upper bound for the case $\delta^- = 1$. We then determine the domination number of a random digraph using the first and the second moment methods. The domination number of a random digraph turns out to be one of two consecutive numbers.

Next, we study the relations among the domination number, the independent domination number, and the independence number of an oriented tree and a binary tree, respectively, and we determine bounds. We then derive a formula for the expected value of the independent domination numbers of random binary trees and find the asymptotic behavior of the expectation.

Finally, using an algorithmic method, we establish an upper bound for the domination number of tournaments, which is a function of the number of vertices. To investigate the sharpness of this bound, we first find the domination number of a random tournament which is also one of the two consecutive numbers and next find bounds for the domination number of the Paley tournament, which is a typical quasi-random tournament.

Here are some of the basic definitions we need from graph theory. Those not included may be found in the books [Bo85], [HaNoC65], and [Pa85]. A *directed graph* (or *digraph*) D consists of a finite set of *vertices*, $V(D)$, together with a set of *arcs*, $E(D)$, which are ordered pairs of vertices. Usually an arc (u, v) is denoted by uv. The cardinality of $V(D)$ is the *order* of D and the cardinality of $E(D)$ is the *size* of

D. We will use the convention that $n = |V(D)|$. If $a = uv$ is an arc of a digraph D, then u is said to be the *initial vertex* of a and v the *terminal vertex* of a. We also say that a is an *outgoing arc* from u and that a is an *incoming arc* to v. We further say that a is *incident from* u and that a is *incident to* v, while u is *incident to* a and v is *incident from* a. Moreover, u is said to be *adjacent to* v and v is *adjacent from* u. The *outdegree* $od_D(v)$ of a vertex v in a digraph D is the number of vertices of D that are adjacent from v, and the *indegree* $id_D(v)$ of v is the number of vertices of D adjacent to v. The *minimum indegree* (or *outdegree*) of a digraph D, denoted $\delta^-(D)$ (or $\delta^+(D)$), is the minimum indegree (or outdegree) of a vertex in D, respectively. The *open in-neighborhood* of a set $S \subseteq V(D)$ is defined by $N_D^-(S) = \{v \in V(D) - S \mid v \text{ is adjacent to some } u \in S\}$ and the *open out-neighborhood* of a set $S \subseteq V(D)$ is defined by $N_D^+(S) = \{v \in V(D) - S \mid v \text{ is adjacent from some } u \in S\}$, while the *closed in-neighborhood* of a set $S \subseteq V(D)$ is defined by $N_D^-[S] = N_D^-(S) \cup S$ and the *closed out-neighborhood* of a set $S \subseteq V(D)$ is defined by $N_D^+[S] = N_D^+(S) \cup S$.

A *walk* in a digraph D is a sequence $v_1, v_2, \ldots, v_m$ of vertices such that v_i is adjacent to v_{i+1} for $i = 1$ to $m-1$. If $v_1 = v_m$, the walk is called *closed*. A *path* in D is a walk in which no vertex is repeated. If there is a path from u to v, then v is said to be *reachable from* u. The *length* of a path is the number of arcs in it. The *distance* between two vertices is the length of any shortest path between them. A *cycle* is a walk with at least two vertices in which the first and the last vertices are the only ones repeated. We denote a cycle of order m by C_m and a path of order m by P_m.

Each walk is directed from the first vertex to the last vertex. We also need a concept which does not have this property of direction and is analogous to a walk in a graph. A *semiwalk* is again a sequence $v_1, v_2, \ldots, v_m$ of vertices, but either $v_{i-1}v_i$ or v_iv_{i-1} is an arc for $i = 2, \ldots, m$. A *semipath, semicycle*, and so forth, are defined as expected.

A digraph D is *strong* if every two vertices are mutually reachable and D is *unilateral* if for any two vertices at least one is reachable from the other. We say that D is *weak* if every two vertices are joined by a semipath. A digraph is *disconnected* if it is not even weak.

A *subdigraph* H of a digraph D is a digraph such that $V(H) \subseteq V(D)$ and $E(H)$ is a subset of those arcs in $E(D)$ that are incident with only the vertices in $V(H)$. The subdigraph H of D *induced* by a set $S \subseteq V(D)$ is a subdigraph such that if $u, v \in V(H)$ and uv is an arc of D then uv is also an arc of H. For a set $S \subseteq V(D)$, $D[S]$ will represent the subdigraph of D induced by S. A subdigraph H of D *spans* D if $V(H) = V(D)$. A maximal, weak subdigraph of D is called a *weak component* of D.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then $a_n \to L$ means $\lim_{n\to\infty} a_n = L$. The big-$\mathcal{O}$ and little-o notation is defined as usual: $a_n = \mathcal{O}(b_n)$ means that there are constants K and N such that $|a_n| \leq K|b_n|$ for all $n > N$, and $a_n = o(b_n)$ means $\lim_{n\to\infty} |a_n/b_n| = 0$. If $a_n = (1 + o(1))b_n$, we say that a_n and b_n are *asymptotically equivalent* and we write $a_n \sim b_n$.

We use $\lfloor x \rfloor$ to denote the greatest integer that is at most x, while $\lceil x \rceil$ denotes the least integer that is at least x. For any positive integer n, $[n]$ denotes the set $\{1, 2, \cdots, n\}$. For any number n and positive integer k, $< n >_k$ denotes the falling factorial $< n >_k = n(n-1)\cdots(n-k+1)$ and $< n >_0 = 1$ for any n.

Chapter 1

Digraphs

1.1 Definitions and Preliminary Results

Let D be a digraph of order n. A subset S of $V(D)$ is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u, v) is an arc of D. Note that $V(D)$ itself is a dominating set of D. A *minimal dominating set* is a dominating set such that no proper subset dominates. A dominating set having smallest cardinality among all dominating sets of a given digraph D is called a *minimum dominating set* of D. The cardinality of a minimum dominating set of D is the *domination number* of D. We will reserve $\alpha(D)$ or just α for the domination number of D. For example, it is easily seen that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$ and $\alpha(C_n) = \lceil \frac{n}{2} \rceil$ by choosing every other vertex for a minimum dominating set. Note that if we add a new arc to a digraph D, then the domination number of the resulting digraph is at most that of D and that if we remove an arc of a given digraph D, then the domination number of the resulting digraph is at least that of D. For subsets S and T of $V(D)$, we say that S *dominates* T if S is a dominating set of $D[S \cup T]$.

For an undirected graph G, a subset S of $V(G)$ is a *dominating set* of G if for every vertex v not in S there exists a vertex u in S such that $\{u, v\}$ is an edge of G. The *domination number* of G is the minimum cardinality of all dominating sets of G.

A *minimal dominating set* of G, a *minimum dominating set* of G, and so forth, are defined as expected.

We are now ready to state some results in [Or62].

Theorem 1.1.1 ([Or62]) *Let G be a directed or undirected graph. A dominating set S is a minimal dominating set if and only if for each vertex v in S one of the two following conditions holds:*

(1) v is not adjacent from any vertex in S.

(2) There exists a vertex u not in S such that v is the only vertex in S adjacent to u. ■

Theorem 1.1.2 ([Or62]) *Any undirected graph G without isolated vertices has a dominating set S such that its complement $\overline{S}$ is also a dominating set.*

Proof: Let S be a minimal dominating set of G. Every vertex in S must be adjacent to some vertex in $\overline{S}$, or S would not be minimal. Thus $\overline{S}$ is also a dominating set. ■

This theorem implies that any undirected graph of order n without isolated vertices has domination number at most $n/2$. However, the corresponding theorem for digraphs does not hold as we can see, for example, in the case of a directed 3-cycle.

A set S of vertices of an undirected graph G is called an *independent set* of G if there are no edges between any of its vertices. The *independence number* of G, denoted $\beta(G)$, is the maximum cardinality taken over all independent sets of G. An *independent set* and the *independence number* of a directed graph are defined analogously.

Now we state a useful theorem relating the domination number of a graph G to the independence number of G.

Theorem 1.1.3 ([Or62]) *An independent set of an undirected graph G is maximal if and only if it is a dominating set.* ∎

This theorem implies that $\alpha(G) \leq \beta(G)$ for undirected graphs G. For directed graphs, however, it does not hold. A directed 3-cycle is an example.

1.2 The Domination Number of a Digraph

In this section we will establish bounds for the domination number of digraphs with minimum indegree at least one. In a graph, every isolated vertex must belong to any dominating set. Similarly, in a digraph, every vertex with indegree zero must belong to any dominating set. Therefore, it is quite natural to concentrate on digraphs with minimum indegree at least one.

Let X be a random variable on a probability space Ω, and let $E[X]$ be the expectation of X. Then we know that if $E[X] \leq c$ for some constant c, there is an $s \in \Omega$ such that $X(s) \leq c$. Let $X_1, X_2, \ldots, X_n$ be random variables, and let $X = c_1X_1 + \cdots + c_nX_n$, where c_i's are constants. Linearity of expectation states that $E[X] = c_1E[X_1] + \cdots + c_nE[X_n]$. The power of this property comes from the fact that there are no restrictions on the dependence or independence of the X_i's.

Using these simple observations, we prove the following theorem.

Theorem 1.2.1 *Let D be a digraph with order n and minimum indegree $\delta^- \geq 1$. Then D has a dominating set of size at most*

$$\{1 - (\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}} + (\frac{1}{1+\delta^-})^{\frac{1+\delta^-}{\delta^-}}\}n.$$

Proof: Fix p with $0 < p < 1$. Let us select, randomly and independently, each vertex in $V = V(D)$ with probability p. Let S be the random set of all vertices selected, and let T be the random set of all vertices not in S that do not have any in-neighbors in S. Then the expectation $E[|S|]$ of the random variable $|S|$ is $E[|S|] = np$ since $|S|$ has a binomial distribution with parameters n and p. To find $E[|T|]$, we let $|T| = \sum_{v \in V} \chi_v$, where $\chi_v = 1$ if $v \in T$ and $\chi_v = 0$ otherwise. Note that

$$\begin{aligned} P(v \in T) &= P(v \text{ and its in-neighbors are not in } S) \\ &= (1-p)^{1+id(v)} \\ &\leq (1-p)^{1+\delta^-} \end{aligned}$$

for each $v \in V$. Thus, we have

$$\begin{aligned} E[|T|] &= E[\sum_{v \in V} \chi_v] = \sum_{v \in V} E[\chi_v] \\ &= \sum_{v \in V} P(v \in T) \leq n(1-p)^{1+\delta^-}. \end{aligned}$$

Therefore, we have

$$E[|S| + |T|] \leq np + n(1-p)^{1+\delta^-}. \quad (1.1)$$

Using elementary calculus, we minimize the right side of (1.1) with respect to p. Then the minimum value of it is

$$\{1 - (\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}} + (\frac{1}{1+\delta^-})^{\frac{1+\delta^-}{\delta^-}}\}n,$$

which is attained when

$$p = 1 - (\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}}.$$

This means that there is at least one choice of S such that

$$|S| + |T| \leq \{1 - (\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}} + (\frac{1}{1+\delta^-})^{\frac{1+\delta^-}{\delta^-}}\}n.$$

The set $S \cup T$ is clearly a dominating set of D whose cardinality is at most

$$\{1 - (\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}} + (\frac{1}{1+\delta^-})^{\frac{1+\delta^-}{\delta^-}}\}n. \blacksquare$$

This theorem gives us a good upper bound for the domination number of a digraph with large minimum indegree. The coefficient of this upper bound goes to zero when the minimum indegree δ^- goes to infinity. See Table (1.1).

Table 1.1: Constants in upper bounds for domination number

δ^- or δ	$1-(\frac{1}{1+\delta^-})^{\frac{1}{\delta^-}}+(\frac{1}{1+\delta^-})^{\frac{1+\delta^-}{\delta^-}}$	$\frac{1+\ln\delta}{1+\delta}$	$\frac{\delta^-+1}{2\delta^-+1}$
1	.7500	.5000	.6666
2	.6150	.5643	.6000
3	.5275	.5246	.5714
4	.4650	.4772	.5555
5	.4176	.4349	.5454
6	.3802	.3988	.5384
7	.3498	.3682	.5333
8	.3245	.3421	.5294
9	.3031	.3197	.5263
10	.2847	.3002	.5238
10^2	.0545	.0554	.5024
10^3	.0078	.0078	.5002
10^4	.0010	.0010	.5000

Remark: Let G be an undirected graph with order n and minimum degree δ. Then, using the same argument as in Theorem 1.2.1, we can show that the domination number of G is at most

$$\{1-(\frac{1}{1+\delta})^{\frac{1}{\delta}}+(\frac{1}{1+\delta})^{\frac{1+\delta}{\delta}}\}n. \tag{1.2}$$

L. Lovász showed in [Lo75] that the domination number of G is at most

$$\frac{1+\ln\delta}{1+\delta}n, \tag{1.3}$$

and N. Alon and J. Spencer found a similar upper bound

$$\frac{1+\ln(\delta+1)}{1+\delta}n \tag{1.4}$$

(see [AlS92]). It is easily checked that these three upper bounds for the domination number of an undirected graph are asymptotically the same but our result (1.2) is smaller than (1.3) and (1.4) for $\delta \geq 4$. See Table (1.1).

It is easy to see that the domination number of a digraph D is the sum of the domination numbers of all weak components of D. But note that this is not true for unilateral or strong components. Since every vertex of indegree zero must belong to any dominating set, we consider weak digraphs with minimum indegree at least one. Then, what is the domination number of a digraph in which every vertex has indegree one? Such a digraph is called a *contrafunctional digraph.*

A vertex v of a digraph D is called a *source* of D if every vertex is reachable from v, and a *tree from a vertex* (or *arborescence*) is a digraph with a source but with no semicycles. A (*directed*) *star* S_n is a digraph on n vertices consisting of a center v and a set of arcs from v to $V(S_n) - \{v\}$.

Lemma 1.2.2 ([HaNoC65]) *A weak digraph is a tree from a vertex if and only if exactly one vertex has indegree zero and every other vertex has indegree one.* ∎

We need the above lemma to prove the following.

Theorem 1.2.3 *Every tree T from a vertex v has domination number*

$$1 \leq \alpha(T) \leq \lceil \frac{1}{2} |V(T)| \rceil .$$

Moreover, the bounds are sharp.

Proof: We shall state an algorithm which finds a dominating set for a tree T from a vertex v. This algorithm begins by selecting a largest star that is the farthest from the source v. Then we put the center of the star into a dominating set. Next we remove the vertices in the star from T to get a new tree from a vertex and repeat this process.

Algorithm: Let $T_1 = T$ be the given tree from the vertex v, and let $S_0 = \phi$. Put $i = 1$ and go to (1).

(1) Take a vertex v_i with maximum distance from v in T_i.

(2) If $v_i = v$, then let $S = S_{i-1} \cup \{v\}$ and stop. If $v_i \neq v$ (i.e., $id_{T_i}(v_i) = 1$), let u_i be the vertex of T_i that is adjacent to v_i and go to (3).

(3) If $od_{T_i}(u_i) = 1$ and $u_i = v$, then let $S = S_{i-1} \cup \{u_i\}$ and stop. If $od_{T_i}(u_i) = 1$ and $u_i \neq v$, then let $S_i = S_{i-1} \cup \{u_i\}$ and $T_{i+1} = T_i - \{u_i, v_i\}$ and next return to (1) putting $i = i + 1$. If $od_{T_i}(u_i) \geq 2$, go to (4).

(4) If $u_i = v$, then let $S = S_{i-1} \cup \{v\}$ and stop. If $u_i \neq v$, then let $S_i = S_{i-1} \cup \{u_i\}$ and $T_{i+1} = T_i - N^+[u_i]$, and next return to (1) putting $i = i + 1$.

From this algorithm, it is easily seen that S is a dominating set for T and that $|S| \leq \lceil \frac{1}{2}|V(T)| \rceil$ since in each step except (possibly) the last, we take at least two vertices and put only one vertex into S that dominates the rest of them.

Extremal digraphs are a star S_n on n vertices and a path P_n on n vertices. ∎

Here we note that the complexity of this algorithm is $\mathcal{O}(n^2)$, where $n = |V(T)|$.

Lemma 1.2.4 ([HaNoC65]) *The following statements are equivalent for a weak digraph D.*

(1) *D is contrafunctional.*

(2) *D has exactly one cycle C and the removal of any one arc of C results in a tree from a vertex.* ∎

The removal of any arc in a given digraph never decreases its domination number. Therefore, combining Theorem 1.2.3 and Lemma 1.2.4, we have the following corollary.

Corollary 1.2.5 *Every weak contrafunctional digraph D has domination number*

$$1 \leq \alpha(D) \leq \lceil \frac{1}{2}|V(D)| \rceil.$$

Moreover, the bounds are sharp.

To see the latter, we construct a digraph D as follows. We add one new vertex u to a star S_{n-1} and add two new arcs between u and the center of S_{n-1}. Then D is an extremal digraph, and a cycle C_n will do for the other extreme.

If a digraph D has a spanning subdigraph H of D such that H is a disjoint union of stars, then H is called a *vertex disjoint star cover* (*vds-cover*) of D.

Theorem 1.2.6 *Let D be a digraph with order n and minimum indegree $\delta^- \geq 1$. Then we have*

$$1 \leq \alpha(D) \leq \frac{\delta^- + 1}{2\delta^- + 1} n.$$

Proof: It is easy to see that D has a vds-cover H, namely, take H as the empty digraph on $V(D)$. Among all such vds-covers of D, let H^* be one with minimum number of copies of S_1. For each $k = 1, 2, \ldots$, let H_k^* be the subdigraph of H^* consisting of weak components that are isomorphic to S_k and let h_k denote the number of weak components in H_k^*.

The subdigraph of D induced by $V(H_1^*)$ has no arcs from vertices in $\bigcup_{k \neq 3} H_k^*$ to vertices in H_1^* because otherwise, H^* violates the minimality. However, each vertex in H_1^* is the terminal vertex of at least δ^- arcs. Hence these arcs must be incident from vertices in H_2^*. Let uv be a star in H_2^* with center u. Then, because of the minimality of H^*, u is not adjacent to any vertex in H_1^* and v is adjacent to at most one vertex in H_1^*. Since each vertex in H_1^* has indegree at least δ^-, we have $h_2 \geq \delta^- h_1$.

Now let S be the set of all centers of the stars in H^*. Then S is a dominating set of D and $|S| = \sum_{i\geq 1} h_i$. Note that

$$\frac{\delta^- + 1}{2\delta^- + 1} \geq \frac{1}{2} \geq \frac{1}{i}$$

for $i = 3, 4, \ldots$ and that

$$\frac{\delta^- + 1}{2\delta^- + 1}(h_1 + 2h_2) - (h_1 + h_2) = \frac{h_2 - \delta^- h_1}{2\delta^- + 1} \geq 0.$$

Since

$$|V(D)| = n = \sum_{i\geq 1} ih_i,$$

we have

$$\begin{aligned} \frac{\delta^- + 1}{2\delta^- + 1}n &= \frac{\delta^- + 1}{2\delta^- + 1}(h_1 + 2h_2) + \sum_{i\geq 3} \frac{\delta^- + 1}{2\delta^- + 1} ih_i \\ &\geq (h_1 + h_2) + \sum_{i\geq 3} h_i = |S|. \ \blacksquare \end{aligned}$$

This theorem gives a better upper bound for the domination number of a digraph with $\delta^- = 1$ or 2 than that of Theorem 1.2.1. See Table (1.1).

Corollary 1.2.7 *Let D be a weak contrafunctional digraph. Then we have the following:*

(1) $\alpha(D) = \frac{2}{3}|V|$ *if and only if* $D = C_3$.

(2) $\alpha(D) < \frac{2}{3}|V|$ *if and only if* $D \neq C_3$.

Here, C_3 denotes a directed 3*-cycle.*

Proof:(1) The sufficiency is trivial. For the necessity, first note that for integer $n \geq 2$, $\frac{2}{3}n \leq \lceil \frac{n}{2} \rceil$ iff $n = 3$. Suppose that $\alpha(D) = \frac{2}{3}|V|$. Then $\frac{2}{3}|V| = \alpha(D) \leq \lceil \frac{1}{2}|V| \rceil$ by Corollary 1.2.5 and so $|V| = 3$ by the note. Moreover, C_3 is the only digraph on 3 vertices whose domination number is 2. This completes the proof of the first part.

(2) Since a weak contrafunctional digraph D has $\delta^- = 1$, we have $\alpha(D) \leq \frac{2}{3}|V|$ by Theorem 1.2.6, and so the second part follows. ■

Theorem 1.2.8 *Let D be a contrafunctional digraph. Then we have the following:*

(1) $\alpha(D) = \frac{2}{3}|V|$ *if and only if D is a disjoint union of* 3*-cycles.*

(2) $\alpha(D) < \frac{2}{3}|V|$ *if and only if D is not a disjoint union of* 3*-cycles.*

Proof: (1) The sufficiency is trivial. To prove the necessity, let $\alpha(D) = \frac{2}{3}|V|$ and let $\{H_1, H_2, \cdots, H_l\}$ be the set of weak components of D. Suppose that there exists a component that is not a 3-cycle. Then by Corollary 1.2.7, we have

$$\frac{2}{3}|V| = \alpha(D) = \sum_{i=1}^{l} \alpha(H_i) < \sum_{i=1}^{l} \frac{2}{3}|V(H_i)| = \frac{2}{3}|V|,$$

which is a contradiction. Thus every weak component of D is a 3-cycle and hence D is a disjoint union of 3-cycles.

(2) Suppose that D is not a disjoint union of 3-cycles and let $\{H_1, H_2, \cdots, H_l\}$ be the set of weak components of D. Then all H_i's are weak contrafunctional digraphs, and $H_i \neq C_3$ for some i. Hence

$$\alpha(D) = \sum_{j=1}^{l} \alpha(H_j) < \sum_{j=1}^{l} \frac{2}{3}|V(H_j)| = \frac{2}{3}|V|$$

and so the sufficiency has been established.

To prove the necessity, we let $\alpha(D) < \frac{2}{3}|V|$ and assume D is a disjoint union of 3-cycle Z_i's. Then we have

$$\alpha(D) = \sum_{i \geq 1} \alpha(Z_i) = \sum_{i \geq 1} \frac{2}{3}|V(Z_i)| = \frac{2}{3}|V|,$$

which contradicts $\alpha(D) < \frac{2}{3}|V|$. Therefore D is not a disjoint union of 3-cycles. ■

The bound in Theorem 1.2.6 can be sharpened for weak digraphs with $3k$ vertices as follows.

Theorem 1.2.9 *Let D be a weak digraph with minimum indegree $\delta^- = 1$ and let $|V(D)| = n$. Then we have the following:*

(1) *If* $n \equiv 0 \pmod 3$ *and* $n \geq 6$, *then* $1 \leq \alpha(D) \leq \frac{2}{3}n - 1$.

(2) *If* $n \equiv 1 \pmod 3$ *and* $n \geq 4$, *then* $1 \leq \alpha(D) \leq \lfloor \frac{2}{3}n \rfloor$.

(3) *If* $n \equiv 2 \pmod 3$ *and* $n \geq 2$, *then* $1 \leq \alpha(D) \leq \lfloor \frac{2}{3}n \rfloor$.

Moreover, all bounds are sharp.

Proof: Since (2) and (3) are the same as Theorem 1.2.6, it suffices to prove (1). For each vertex in D, color one incoming arc green and the others red and next choose only green arcs. Then we have a spanning contrafunctional subdigraph H of D. First, consider the case that H is not a disjoint union of 3-cycles. Clearly, $\alpha(D) \leq \alpha(H) < \frac{2}{3}n$ by Theorem 1.2.8 and hence $\alpha(D) \leq \frac{2}{3}n - 1$. Next, consider the case that H is a disjoint union of 3-cycles. Since D is weak but H is not, the arc set $E(D)$ of D consists of $E(H)$ and some arcs not in H. In addition, if we add some arcs in $E(D) - E(H)$ to H, then the resulting digraph has a strictly smaller domination number than that of H. Therefore, $\alpha(D) < \alpha(H) = \frac{2}{3}n$ and hence $\alpha(D) \leq \frac{2}{3}n - 1$. This completes the proof of (1).

For the sharpness of the lower bound in all cases, we take a digraph D as follows:

$$
\begin{aligned}
V(D) &= \{v_1, v_2, \ldots, v_n\}, \\
E(D) &= \{v_2v_1, v_1v_2, v_1v_3, \ldots, v_1v_n\}.
\end{aligned}
$$

For an extremal digraph of the case (1), we define a digraph D as follows: Take a disjoint union of k 3-cycles $Z_1, Z_2, \ldots, Z_k$, and let v_i be a vertex in Z_i for each i. Add $k-1$ new arcs v_iv_1 for $i = 2, 3, \ldots, k$, and let D be the resulting digraph. Next, for an extremal digraph of the case (2), we define a digraph as follows: Take a disjoint union of k 3-cycles $Z_1, Z_2, \ldots, Z_k$ and a new vertex u. Let v_i be a vertex in Z_i for each

i. Add k new arcs v_iu and let D be the resulting digraph. Finally, for an extremal digraph of the case (3), we define a digraph D as follows: Take a disjoint union of k 3-cycles $Z_i's$ and a 2-cycle C_2. Let u be a vertex in C_2 and v_i in Z_i. Add k new arcs v_iu and let D be the resulting digraph. ∎

Every unilateral digraph has at most one vertex of indegree zero and at most one vertex of outdegree zero, while every strong digraph has the minimum indegree at least one and the minimun outdegree at least one. Therefore we do not need any more degree restrictions for unilateral or strong digraphs.

Theorem 1.2.10 *Every unilateral digraph D has*

$$1 \leq \alpha(D) \leq \lceil \frac{1}{2}|V(D)| \rceil.$$

Moreover, the bounds are sharp.

Proof: Let D be a unilateral digraph. Then D has at least one source (p.99, [HaNoC65]). We consider a spanning tree T from the source. Then

$$\alpha(D) \leq \alpha(T) \leq \lceil \frac{1}{2}|V(T)| \rceil = \lceil \frac{1}{2}|V(D)| \rceil.$$

Let S_n be a star with center u, and let v be another vertex in S_n. We construct a unilateral digraph D as follows:

$$\begin{aligned} V(D) &= V(S_n), \\ E(D) &= E(S_n) \cup \{wu \mid w \in V(S_n) - \{u, v\}\}. \end{aligned}$$

Then D is an extremal unilateral digraph, and P_n will do for the other extreme. ∎

Corollary 1.2.11 *Every strong digraph D has*

$$1 \leq \alpha(D) \leq \lceil \frac{1}{2}|V(D)| \rceil.$$

Moreover, the bounds are sharp.

Proof: Since every strong digraph is weak, it suffices to prove the sharpness of the bounds. Extremal digraphs are a symmetric star and a cycle. ∎

1.3 The Domination Number of a Random Digraph

In this section we will determine the domination number of a random digraph. To do this, we describe probability models commonly used in the study of random digraphs or random graphs and explain the first and the second moment methods.

For each positive integer n and each number p with $0 < p < 1$, the probability space $\mathcal{D}_{n,p}$ of digraphs is defined as follows: Each point in the space is a digraph with vertex set $V = \{1, 2, \ldots, n\}$ having no loops or multiple arcs, and the probability of a given digraph D with l arcs is given by $P(D) = p^l(1-p)^{n(n-1)-l}$. In other words, each arc is present with probability p, independently of the presence or absence of other arcs. In particular, if $p = 1/2$, then each digraph is assigned the same probability, namely $1/D_n$, where D_n is the total number of digraphs on V. On the other hand, the probability space $\mathcal{G}_{n,p}$ of graphs is defined analogously and so the probability of a given graph G with l edges is given by $P(G) = p^l(1-p)^{\binom{n}{2}-l}$.

In the study of random digraphs (or graphs), we cannot conclude anything about individual digraphs but what we do study are properties of sets of digraphs. Let $\mathcal{Q}$ be a property of digraphs. If $\mathcal{A}$ is the set of digraphs of order n with property $\mathcal{Q}$ and the probability $P(\mathcal{A})$ of $\mathcal{A}$ has limit 1 as $n \to \infty$, then we say *almost all digraphs have property* $\mathcal{Q}$ or *a random digraph has property* $\mathcal{Q}$ *almost surely.* We are studying a sequence of probability spaces and the limit of a sequence of probabilities.

The first and the second moment methods are important tools from probability theory which are used frequently in the study of random digraphs (or graphs).

Suppose X is a nonnegative integer-valued random variable. Let $E[X]$ denote the expected value of X and let $P(\mathcal{A})$ denote the probability of the event $\mathcal{A}$. Then we have $P(X \geq 1) \leq E[X]$ from Markov's inequality. Thus if $E[X] \to 0$, then $P(X \geq 1) \to 0$ and therefore $P(X = 0) \to 1$. On the other hand, if $E[X] \neq 0$, then we have $P(X = 0) \leq E[X^2]/E[X]^2 - 1$ from Chebyshev's inequality. Thus $E[X^2] \sim E[X]^2$ implies $P(X = 0) \to 0$ and therefore $P(X \geq 1) \to 1$.

In what follows log denotes the logarithm with base $1/(1-p)$ and ln denotes the logarithm with base e.

K. Weber determined the domination number for almost all graphs as follows.

Theorem 1.3.1 ([We81]) *For p fixed, $o < p < 1$, a random graph $G_n \in \mathcal{G}_{n,p}$ has domination number either*

$$\lfloor k^* \rfloor + 1 \text{ or } \lfloor k^* \rfloor + 2$$

almost surely, where

$$k^* = \log_b n - 2 \log_b \log_b n + \log_b \log_b e$$

and $\log_b$ *denotes the logarithm with base* $b = 1/p$.

Using the same techniques as in [We81] for analyzing the first and the second moments, we establish a similar result for digraphs.

Theorem 1.3.2 *For p fixed, $o < p < 1$, a random digraph $D_n \in \mathcal{D}_{n,p}$ has domination number either*

$$\lfloor k^* \rfloor + 1 \text{ or } \lfloor k^* \rfloor + 2$$

almost surely, where

$$k^* = \log n - 2 \log \log n + \log \log e$$

and log *denotes the logarithm with base* $1/(1-p)$.

Proof: Let X be a nonnegative random variable such that $X(D_n)$ is the number of dominating k-sets in D_n for each $D_n \in \mathcal{D}_{n,p}$. Since

$$P(\text{a fixed vertex } v \text{ does not dominate another fixed vertex } u) = 1 - p := q,$$

we have

$$P(\text{a fixed } k\text{-set } K \subseteq V \text{ does not dominate a fixed vertex in } V - K) = q^k$$

and hence

$$P(\text{a fixed } k\text{-set of vertices is a dominating set}) = (1 - q^k)^{n-k}.$$

Therefore, we have

$$\mu = \mu(k) = E[X] = \binom{n}{k}(1 - q^k)^{n-k}. \tag{1.5}$$

It is convenient to change the notation by setting $q = 1/r$ in (1.5), and we thus have

$$\mu = \binom{n}{k}(1 - r^{-k})^{n-k}. \tag{1.6}$$

Note that

$$\binom{n}{k} = \frac{n^k}{k!}\frac{(n)_k}{n^k} = (1 + o(1))\frac{n^k}{k!}, \tag{1.7}$$

when $k \to \infty$ and $k^2 = o(n)$. Substituting (1.7) in (1.6) and applying Stirling's formula for $k!$, we have

$$\mu = (1 + o(1))\frac{(en/k)^k}{\sqrt{2\pi k}}(1 - r^{-k})^{n-k}, \tag{1.8}$$

when $k \to \infty$ with $k^2 = o(n)$. By taking the ln of both sides of (1.8), we get

$$\begin{aligned} \ln \mu &= k + k \ln n - k \ln k - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln k \\ &\quad + (n - k)\ln(1 - r^{-k}) + \ln(1 + o(1)) \end{aligned} \tag{1.9}$$

when $k \to \infty$ with $k^2 = o(n)$. The term $\ln(1 - r^{-k})$ in (1.9) becomes

$$\ln(1 - r^{-k}) = -\frac{1}{r^k} - \frac{1}{2r^{2k}} - \frac{1}{3r^{3k}} - \cdots. \tag{1.10}$$

Substituting (1.10) in (1.9) and rearranging, we have

$$\ln \mu = k \ln n - nr^{-k} - k \ln k + k - \frac{1}{2} \ln k + \mathcal{O}(1) \tag{1.11}$$

when $k \to \infty$ with $k^2 = o(n)$ and $n = o(r^{2k})$. Converting ln in (1.11) to log, we have

$$\log \mu = k \log n - nr^{-k} \log e - k \log k + k \log e - \frac{1}{2} \log k + \mathcal{O}(1) \tag{1.12}$$

when $k \to \infty$ with $k^2 = o(n)$ and $n = o(r^{2k})$. Note that the function (1.8) is defined for integer k such that $k \to \infty$ as $n \to \infty$ and $k^2 = o(n)$. But we may regard (1.8) as a function defined for any real number k such that $k \to \infty$ as $n \to \infty$ and $k^2 = o(n)$. Let

$$k = k^* + \epsilon \text{ and } \epsilon = \mathcal{O}(\frac{\log \log n}{\log n}). \tag{1.13}$$

Then k satisfies $k \to \infty$, $k^2 = o(n)$, and $n = o(r^{2k})$ and thus it follows from the definition of k^*, (1.12), and (1.13) that

$$\log \mu = (\log n)^2(1 - r^{-\epsilon}) - 2(\log n)(\log \log n) - k^* \log k^* + \mathcal{O}(\log n). \tag{1.14}$$

Note that

$$\begin{aligned} 1 - r^{-\epsilon} &= 1 - e^{-\epsilon \ln r} \\ &= 1 - (1 - \epsilon \ln r + \mathcal{O}(\epsilon^2)) \\ &= \epsilon \ln r - \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0. \end{aligned} \tag{1.15}$$

Substituting (1.15) and the definition of k^* in (1.14) and rearranging, we have

$$\begin{aligned} \log \mu &= (\ln r)(\log n)^2 \epsilon - 3(\log n)(\log \log n) + \mathcal{O}(\log n) \\ &= (\ln r)(\log n)(\log \log n)\{\frac{\epsilon \log n}{\log \log n} - 3 \log e + \mathcal{O}(\frac{1}{\log \log n})\}. \end{aligned} \tag{1.16}$$

Therefore, we have

$$\log \mu \longrightarrow \begin{cases} -\infty & \text{if } \limsup_{n \to \infty} \frac{\epsilon \log n}{\log \log n} < 3 \log e \text{ and } \epsilon = \mathcal{O}(\frac{\log \log n}{\log n}) \\ \\ \infty & \text{if } \liminf_{n \to \infty} \frac{\epsilon \log n}{\log \log n} > 3 \log e \text{ and } \epsilon = \mathcal{O}(\frac{\log \log n}{\log n}) \end{cases}$$

and thus

$$\mu \longrightarrow \begin{cases} 0 & \text{if } \limsup_{n\to\infty} \frac{\epsilon \log n}{\log\log n} < 3\log e \text{ and } \epsilon = \mathcal{O}(\frac{\log\log n}{\log n}) \\ \\ \infty & \text{if } \liminf_{n\to\infty} \frac{\epsilon \log n}{\log\log n} > 3\log e \text{ and } \epsilon = \mathcal{O}(\frac{\log\log n}{\log n}). \end{cases}$$

It follows easily by observing the logarithmic derivative of the term

$$\frac{(en/k)^k}{\sqrt{2\pi k}}(1 - r^{-k})^{n-k}$$

in (1.8) that if $k \to \infty$ as $n \to \infty$ and $k^2 = o(n)$, then μ is asymptotic to an increasing function of k when $n \to \infty$. Hence, for any such real sequence k at all, we have

$$\mu \longrightarrow \begin{cases} 0 & \text{if } \limsup_{n\to\infty} \frac{(k-k^*)\log n}{\log\log n} < 3\log e \\ \\ \infty & \text{if } \liminf_{n\to\infty} \frac{(k-k^*)\log n}{\log\log n} > 3\log e. \end{cases}$$

Now, it is easy to see that

$$\mu \longrightarrow \begin{cases} 0 & \text{if } \quad k = \lfloor k^* \rfloor \\ \infty & \text{if } \quad k = \lfloor k^* \rfloor + 2. \end{cases}$$

This means that for any $k \leq \lfloor k^* \rfloor$, a random digraph has no dominating k-sets almost surely.

We have shown that $\mu = E[X] \to \infty$ for $k = \lfloor k^* \rfloor + 2$. Thus, using the second moment method, we want to show that $P(X \geq 1) \to 1$ for $k = \lfloor k^* \rfloor + 2$. To do this, it suffices to show that $E[X^2] \sim \mu^2$ for $k = \lfloor k^* \rfloor + 2$.

Let us estimate $E[X^2] - \mu^2$ for $k = \lfloor k^* \rfloor + 2$. Let a_s be the number of ordered pairs (K, K') of k-sets of vertices with $|K \cap K'| = s$, and let P_s be the probability that two fixed k-sets K and K' with $|K \cap K'| = s$ are dominating sets. Then

$$E[X^2] = \sum_{s=0}^{k} a_s P_s.$$

We want to estimate $a_s P_s$ up to the values of s. Now we have three cases to consider.

Case 1: Since a_k denotes the number of k-subsets of vertices and P_k the probability that a fixed k-subset of vertices is a dominating set, it follows that

$$a_k P_k = \mu = o(\mu^2) \quad \text{as} \quad \mu \to \infty.$$

Case 2: Since

$$\begin{aligned} P_0 &\leq P(\text{all vertices not in } K \cup K' \text{ are dominated by } K \text{ and } K') \\ &= \{(1 - r^{-k})^2\}^{n-2k} \end{aligned}$$

and

$$a_0 = \binom{n}{k}\binom{n-k}{k} \leq \binom{n}{k}^2,$$

we have

$$\begin{aligned} a_0 P_0 &\leq \binom{n}{k}^2 (1 - r^{-k})^{2(n-2k)} \\ &= \binom{n}{k}^2 (1 - r^{-k})^{2(n-k)} (1 - r^{-k})^{-2k} \\ &= \mu^2 (1 - r^{-k})^{-2k} \\ &= \mu^2 (1 + o(1)) e^{2k/r^k} \\ &= \mu^2 (1 + o(1))(1 + 2kr^{-k}) \\ &= \mu^2 (1 + \mathcal{O}(kr^{-k})). \end{aligned}$$

Therefore, we have

$$a_0 P_0 - \mu^2 = \mu^2 \mathcal{O}(kr^{-k}) = o(\mu^2).$$

Case 3: Let K and K' be two fixed k-sets of vertices with $|K \cap K'| = s$, $1 \leq s \leq k - 1$, and let $P(v)$ be the probability for a fixed vertex $v \in V - (K \cup K') := R$ to be dominated by both K and K'. Then $P(v) = P((K \cap K'$ dominates $v) \bigvee$ (both sets $K-S$ and $K'-S$ dominate v and $K\cap K'$ does not dominate v)).

Thus

$$\begin{aligned} P(v) &= (1-r^{-s})+(1-r^{-k+s})^2 r^{-s} \\ &= 1-r^{-2k}(2r^k-r^s). \end{aligned}$$

Therefore, we have

$$\begin{aligned} P_s &\leq P(\text{both sets } K \text{ and } K' \text{ dominate } R) \\ &= (1-r^{-2k}(2r^k-r^s))^{n-2k+s} := b_s. \end{aligned} \tag{1.17}$$

It is easily checked that

$$a_s = \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s} \leq (1+o(1))\binom{n}{k}^2 \frac{k^{2s}}{n^s}. \tag{1.18}$$

Let

$$c_s = n^{-s}(1-r^{-k})^{-2(n-k)} k^{2s+1} b_s. \tag{1.19}$$

Then, using (1.17) and (1.18), we have

$$\begin{aligned} \sum_{s=1}^{k-1} a_s P_s &\leq (1+o(1))\sum_{s=1}^{k-1}\binom{n}{k}^2 \frac{k^{2s}}{n^s} b_s \\ &= (1+o(1))\sum_{s=1}^{k-1}\frac{k^{2s}b_s}{n^s(1-r^{-k})^{2(n-k)}}\binom{n}{k}^2 (1-r^{-k})^{2(n-k)} \\ &\leq (1+o(1))\sum_{s=1}^{k-1}\frac{k^{2s+1}b_s}{(k-1)n^s(1-r^{-k})^{2(n-k)}}\binom{n}{k}^2 (1-r^{-k})^{2(n-k)} \\ &= (1+o(1))\sum_{s=1}^{k-1}\frac{c_s\mu^2}{k-1} \\ &\leq (1+o(1))\max\{c_s \mid 1\leq s\leq k-1\}\mu^2. \end{aligned} \tag{1.20}$$

Next, we will show $c_s \to 0$ for $1\leq s\leq k-1$. To do this, we estimate $\ln c_s$. Substitute (1.17) for b_s in (1.19) and next take the ln of the both sides of (1.19).

Then we have

$$\ln c_s = \quad -s \ln n + (n - 2k + s) \ln(1 - r^{-2k}(2r^k - r^s)) \\ -2(n-k)\ln(1 - r^{-k}) + (2s+1)\ln k. \qquad (1.21)$$

Expand two ln terms containing r in (1.21) and rearrange it. Here we need to recall $k = \lfloor k^* \rfloor + 2$. Then (1.21) becomes

$$\begin{aligned} \ln c_s &= -s \ln n + nr^{-2k+s} + \mathcal{O}(s \ln k) \\ &= -s \ln n + nr^{-2k+s} + \mathcal{O}(s \log \log n). \qquad (1.22) \end{aligned}$$

Subcase 1: Let $s = o(\log n)$. Using $k = \lfloor k^* \rfloor + 2 \geq k^*$, we have

$$\begin{aligned} nr^{-2k+s} &\leq r^s nr^{-2k^*} \\ &= r^s \frac{(\log n)^4}{n(\log e)^2} \\ &\leq r^{c \log n} \frac{(\log n)^4}{n(\log e)^2}, \qquad \text{where } 0 < c < 1 \\ &= \frac{(\log n)^4}{n^{1-c}(\log e)^2} = o(1). \qquad (1.23) \end{aligned}$$

Hence, from (1.22) and (1.23), we have

$$\begin{aligned} \ln c_s &= -s \ln n + nr^{-2k+s} + \mathcal{O}(s \log \log n) \\ &= -s \ln n + \mathcal{O}(s \log \log n) \longrightarrow -\infty \end{aligned}$$

and therefore

$$c_s \longrightarrow 0.$$

Subcase 2: Let

$$s = \log n - t \text{ and } t = o(\log n). \qquad (1.24)$$

Recall that $k = \lfloor k^* \rfloor + 2$ and $1 \leq s \leq k-1$. Using (1.24) and the definition of k^*, we have

$$\begin{aligned} k &\geq s+1 = \log n - (t-1) \qquad \text{and} \\ k &\geq k^* = \log n - (2\log\log n - \log\log e) \end{aligned}$$

and hence

$$k \geq \log n - \min\{t-1, 2\log\log n - \log\log e\}. \tag{1.25}$$

Now we evaluate a new term which will be used later:

$$n^2 r^{-2k-t} = r^{\log n^2} r^{-2k-t} = r^{2(\log n - k) - t}. \tag{1.26}$$

From (1.25), we have

$$2(\log n - k) \leq 2\min\{t-1, 2\log\log n - \log\log e\}.$$

Hence the utmost right side of (1.26) has an upper bound

$$r^{2\min\{t-1, 2\log\log n - \log\log e\} - t}. \tag{1.27}$$

If $t \leq 2\log\log n - \log\log e + 1$, (1.27) has an upper bound

$$r^{t-2}.$$

If $t > 2\log\log n - \log\log e + 1$, (1.27) has an upper bound

$$r^{2(2\log\log n - \log\log e) - (2\log\log n - \log\log e + 1)}.$$

In both cases, (1.27) has an upper bound

$$r^{2\log\log n - \log\log e - 1} = \frac{(\ln r)(\log n)^2}{r}. \tag{1.28}$$

Combining (1.26) through (1.28), we have

$$n^2 r^{-2k-t} \le \frac{(\ln r)(\log n)^2}{r}. \quad (1.29)$$

Substituting (1.24) for s in (1.22), we have

$$\ln c_s = -(\log n - t)\ln n + nr^{-2k+(\log n - t)} + (\log n - o(\log n))(\log\log n)\mathcal{O}(1). \quad (1.30)$$

Simplifying (1.30), we have

$$\ln c_s = -(\ln r)(\log n)^2 + n^2 r^{-2k-t} + t\ln n + \mathcal{O}((\log n)(\log\log n)). \quad (1.31)$$

Substituting (1.29) in (1.31), we have

$$\begin{aligned} \ln c_s \;\le\; & -(\ln r)(\log n)^2 + \frac{1}{r}(\ln r)(\log n)^2 \\ & + o(\log n)\ln n + \mathcal{O}((\log n)(\log\log n)). \end{aligned} \quad (1.32)$$

Simplifying (1.32), we finally have

$$\ln c_s \le -\frac{r-1}{r}(\ln r)(\log n)^2 + o((\log n)^2) \longrightarrow -\infty.$$

Therefore,

$$c_s \longrightarrow 0 \quad \text{for} \quad 1 \le s \le k-1$$

and hence

$$\sum_{s=1}^{k-1} a_s P_s \le (1+o(1))\max\{c_s \mid 1 \le s \le k-1\}\mu^2 = o(\mu^2).$$

So far, we showed the following:

Case 1: $a_k P_k = o(\mu^2)$.

Case 2: $a_0 P_0 - \mu^2 = o(\mu^2)$.

Case 3: $\sum_{s=1}^{k-1} a_s P_s = o(\mu^2)$.

Therefore, for $k = \lfloor k^* \rfloor + 2$, we have

$$\begin{aligned} P(X=0) &\leq \frac{E[X^2]-\mu^2}{\mu^2} \\ &= \frac{\sum_{s=0}^{k} a_s P_s - \mu^2}{\mu^2} \\ &= \frac{(a_0 P_0 - \mu^2) + a_k P_k + \sum_{s=1}^{k-1} a_s P_s}{\mu^2} \\ &= o(1). \end{aligned}$$

This implies that for any $k \geq \lfloor k^* \rfloor + 2$, a random digraph has a dominating k-set almost surely. Therefore a random digraph should have domination number either $\lfloor k^* \rfloor + 1$ or $\lfloor k^* \rfloor + 2$. This completes the proof. ∎

Remark: (1) We have shown in Theorem 1.3.2 that $\mu \to 0$ if $k = \lfloor k^* \rfloor$ and that $\mu \to \infty$ if $k = \lfloor k^* \rfloor + 2$. What if $k = \lfloor k^* \rfloor + 1$? We let $k = \lfloor k^* \rfloor + 1$ and for computational convenience take the probability $p = \frac{1}{2}$. When $n \to \infty$ in such a way that $n = 2^{2^i}$ for $i = 1, 2, \ldots$, we have $k - k^* = 1 - \log\log e > 0.4$ and so $\mu \to \infty$ as $i \to \infty$. When $n \to \infty$ in such a way that $n = \lfloor 2^{2^i+1} \ln 2 \rfloor$ for $i = 1, 2, \ldots$, we have $k - k^* = \mathcal{O}(\frac{1}{\log n})$ as $i \to \infty$ and so $\mu \to 0$ as $i \to \infty$. In Theorem 1.3.2, we analyzed the second moment under the condition that $k = \lfloor k^* \rfloor + 2$ but it is easily checked that the same result holds for $k = \lfloor k^* \rfloor + 1$ if $\mu \to \infty$. Therefore almost every D_n has domination number $\lfloor k^* \rfloor + 1$ (or $\lfloor k^* \rfloor + 2$) when $p = \frac{1}{2}$ and $n \to \infty$ in such a way that $n = 2^{2^i}$ (or $n = \lfloor 2^{2^i+1} \ln 2 \rfloor$), respectively. This means that the result of Theorem 1.3.2 is best possible.

(2) The *independence domination number* $\alpha'(D)$ of a digraph D is the minimum cardinality of all independent and dominating sets of D. Tomescu showed in [To90] that the independence domination number α' of every digraph in the model $\mathcal{D}_{n,1/2}$

satisfies

$$\log_2 n - \log_2 \log_2 n - 1.43 \leq \alpha' \leq \log_2 n - \log_2 \log_2 n + 2.11$$

almost surely and hence α' takes at most four distinct consecutive values. We have shown that the domination number α of every digraph in $\mathcal{D}_{n,1/2}$ is either

$$\lfloor \log_2 n - 2\log_2 \log_2 n + \log_2 \log_2 e + 1 \rfloor \text{ or } \lfloor \log_2 n - 2\log_2 \log_2 n + \log_2 \log_2 e + 2 \rfloor .$$

Note that it is easy to see $\alpha \leq \alpha'$ whenever an independent dominating set exists. The two results are consistent with the fact that $\alpha \leq \alpha'$.

Chapter 2

Oriented Trees

2.1 The Domination Number of an Oriented Tree

In this section we study the relations among the domination number, the independent domination number, and the independence number of an oriented tree and establish their bounds.

An *oriented tree* is a tree in which each edge is assigned a unique direction and an *oriented forest* is defined analogously. A *kernel* of a digraph D is an independent and dominating set of vertices of D and the *independent domination number* of D, denoted by $\alpha'(D)$, is the minimum cardinality of all kernels of D. A 3-cycle has no kernel and a 4-cycle has two kernels. But J. von Neumann and O. Morgenstern showed [NeM44] that every digraph without cycles has a unique kernel, and M. Richardson showed [Ri53] that every digraph without odd cycles has a kernel. The proofs were long and involved. However, for oriented forests (and hence oriented trees), we have the following short algorithmic proof.

Theorem 2.1.1 *Every oriented tree T has a kernel.*

Proof: It is sufficient to prove this theorem for oriented forests and so we shall state an algorithm which finds a kernel for an oriented forest T. The algorithm begins by putting vertices with indegree zero into a kernel. Next we remove the vertices that are already in the kernel together with their out-neighbors to get a new oriented forest and repeat this process for the new oriented forest.

Algorithm: Let $T_1 = T$ be the given oriented forest and let $K_0 = \phi$. Put $i = 1$ and go to (1).

(1) Choose the set S_i of all vertices with indegree zero in the oriented forest T_i and let $K_i = K_{i-1} \cup S_i$.

(2) Let T_{i+1} be the induced oriented forest $T_i[V - N^+[K_i]]$. If T_{i+1} is an empty digraph, let $K = K_i$ and stop. Otherwise, return to (1) putting $i = i + 1$. ∎

Let T' be an oriented tree with n vertices. Then the average indegree of T' is

$$\left(\sum_{v \in T'} indeg(v)\right)/n = \frac{n-1}{n} < 1.$$

Thus there is a vertex v of T' with indegree zero. This implies that the above algorithm terminates after finitely many steps.

It is obvious that K is a dominating set of T. To show that K is an independent set, we let u and v be in K. Assume there is an arc between u and v, say, uv in T. Then, by (1), u and v cannot be chosen for K in the same step. If u were chosen for K in an earlier step than the step in which v was chosen, then v would not be in K. Therefore v must be chosen for K in an earlier step i than the step in which u is chosen for K. For this, u should have been deleted in an earlier step than the step i. Thus u is not in K, which contradicts the fact that u is in K. ∎

We note that the complexity of this algorithm is $\mathcal{O}(n^2)$.

Theorem 2.1.2 *Every oriented tree T has a unique kernel.*

Proof: Suppose that T has two distinct kernels K and L. Then any one of K and L cannot be a proper subset of the other. Otherwise, one of them contains an arc and cannot be independent. Let v_1 be a vertex in $K - L$. Then there is a vertex v_2 in $L - K$ that dominates v_1 and next there is a vertex $v_3 \neq v_1$ in $K - L$ that dominates v_2. Repeat this argument in turn. Then we have a sequence $\{v_i\}$ of vertices such that $v_i \neq v_{i+2}$. Let j be the smallest integer such that $v_j = v_k$ for some $k < j$. Then $v_k = v_j, v_{j-1}, \ldots, v_k$ is a semicycle of length at least 3. This contradicts that T is an oriented tree. ■

Theorem 1.1.3 implies $\alpha(G) \leq \beta(G)$ for undirected graphs G. But it does not hold for directed graphs as we have already seen in a directed 3-cycle. However, for oriented trees, it still is true.

Corollary 2.1.3 *Let T be an oriented tree. Then we have*

$$1 \leq \alpha(T) \leq \alpha'(T) \leq \beta(T) \leq n - 1$$

and

$$\beta(T) \geq n/2.$$

Proof: The first part is immediate from the definitions. For the second part, observe that the independence number of an oriented tree is the same as that of the underlying unoriented tree. ■

Here is an example that shows that the three invariants need not be equal. Let $n \geq 4$ be an integer and let T be an oriented tree with $V = \{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ and $E = \{(u_1, u_j) \mid j = 2, \cdots, n\} \cup \{(v_1, v_j) \mid j = 2, \cdots, n\} \cup \{(u_1, v_1)\}$. Then it is easy to see that $\alpha(T) = 2$, $\alpha'(T) = n + 1$, and $\beta(T) = 2n - 2$. Therefore we have $\alpha(T) < \alpha'(T) < \beta(T)$.

Let α, α', β, and n be positive integers satisfying $1 \leq \alpha \leq \alpha' \leq \beta \leq n - 1$ and $\beta \geq n/2$. Then can we construct an oriented tree T of order n having $\alpha(T) = \alpha$, $\alpha'(T) = \alpha'$, and $\beta(T) = \beta$? By checking all oriented trees with four vertices, we know that all possible outcomes of $(\alpha,\ \alpha',\ \beta)$ are $(1,\ 1,\ 3)$, $(2,\ 2,\ 2)$, $(2,\ 3,\ 3)$, and $(3,\ 3,\ 3)$. Thus there are no oriented trees of order 4 having, for example, the outcome $(1, 2, 3)$.

Theorem 2.1.4 *Let $n \geq 2$ be an integer. Then for any α such that $1 \leq \alpha \leq n-1$, there is an oriented tree T of order n whose domination number is α.*

Proof: We construct T as follows. The vertex set of T is $V = [n]$ and the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha - 1$ and (n, j) for $j = \alpha, \alpha + 1, \ldots, n - 1$. Then T is an oriented tree and $\{1, 2, \ldots, \alpha - 1, n\}$ is a minimum dominating set of T. Therefore T has domination number α. ■

Theorem 2.1.5 *Let $n \geq 2$ be an integer. Then for any α' such that $1 \leq \alpha' \leq n-1$, there is an oriented tree T of order n whose independent domination number is α'.*

Proof: We construct T as follows. The vertex set of T is $V = [n]$. If $\alpha' \geq (n-1)/2$, then the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha'$ and $(j, j+\alpha')$ for $j = 1, 2, \ldots, n-\alpha'-1$. If $\alpha' < (n-1)/2$, then the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha'$, $(j, j + \alpha')$ for $j = 1, 2, \ldots, \alpha'$, and (α, k) for $k = 2\alpha + 1, \ldots, n - 1$. Then T is an oriented tree and $\{1, 2, \ldots, \alpha'\}$ is the kernel of T. Therefore T has independence domination number α'. ■

2.2 The Domination Number of a Binary Tree

In this section we study relations among the domination number, the independent domination number, and the independence number of a binary tree and establish their bounds.

A *binary (search) tree* is an oriented tree which enjoys the following properties (see [KoN73]):

(1) There is a unique vertex v_0 (called the root) such that for any vertex v distinct from v_0 there is one and only one path starting at v_0 and ending at v.

(2) For each vertex v the number of arcs beginning with v is zero or two. In the former case v is called a *leaf* while in the latter case it is called an *interior vertex.*

(3) The set of arcs is partitioned into two sets L and R (the left and right arcs, respectively). For each interior vertex there is precisely one left arc and one right arc starting with this vertex.

Equivalently (see [MeM77]), a binary (search) tree may be defined as an oriented rooted tree that consists either of a single vertex or is constructed from an ordered pair of smaller binary trees by joining their roots from a new vertex that serves as the root in the tree thus formed. The vertices are not labeled, although the root is distinguished from the remaining vertices, and two such trees are regarded as the same if and only if they have the same ordered pair of branches with respect to their roots. Notice that every vertex is incident with either zero or two arcs that lead away from the root; this fact implies that such trees must have an odd number of vertices.

Let T be a binary tree on $2n+1$ vertices. Then T has n interior vertices and $n+1$ leaves. Let I_0, I_1, I_2 be the sets of interior vertices with zero leaves, only one leaf, two leaves, respectively. It is of interest to observe that $|I_2| = |I_0| + 1$ since $|I_0| + |I_1| + |I_2| = n$ and $|I_1| + 2|I_2| = n + 1$.

Let T be a binary tree. The *level number* of a vertex v in T is the length of the unique path from the root to v in T and the *height* of T is the maximum of the level numbers of the vertices of T. A binary tree of height h is *balanced* if every leaf has distance h or $h-1$ from the root, while it is fully balanced if every leaf has distance h from the root.

Now we can state the main theorem of this section.

Theorem 2.2.1 *Let T be a binary tree on $2n+1$ vertices. Then we have*

$$(1)\quad \alpha(T) \leq \alpha'(T) \leq \beta(T),$$

$$(2)\quad \lceil \frac{2n+1}{3} \rceil \leq \alpha(T) \leq n,$$

$$(3)\quad n+1 \leq \beta(T) \leq \lfloor \frac{2(2n+1)+1}{3} \rfloor.$$

Proof: Corollary 2.1.3 implies (1). To prove (2), observe that every vertex in T dominates at most three vertices and that the set of all interior vertices of T is a dominating set for T. This establishes (2). The set of all leaves of T forms an independent set of cardinality $n+1$ and hence $n+1 \leq \beta(T)$.

Now we want to prove the last inequality. Let $|T|$ be the underlying tree of the binary tree T. Suppose $S = \{u_1, u_2, \cdots, u_k\}$ is any independent set in $|T|$. For each $i = 2, \cdots, k$, there is a unique $u_1 - u_i$ path in $|T|$. Let R be the set of all predecessors of u_i in the paths for $i = 2, \cdots, k$. Since the set R is disjoint from the set S, we have $|R| \leq (2n+1) - k$. In addition, since every vertex in $|T|$ has degree at most 3, we have $(k-1)/2 \leq |R|$. Therefore we have $(k-1)/2 \leq (2n+1) - k$ and hence $k \leq [2(2n+1)+1]/3$. ■

Here is an example that shows the three invariants in (1) need not be equal. Let n be an odd integer. Consider any binary tree of order $2n+1$ and height n. Such a tree always has a leaf adjacent from the root. Now attach two new vertices to this leaf. The resulting oriented tree T is a binary tree of order $2n+3$. It is easily seen that $\alpha(T) = n+1$, $\alpha'(T) = n+2$, and $\beta(T) = n+3$.

Now let us consider the sharpness of the bounds of (2) and (3) in Theorem 2.2.1 and let T_3 denote the binary tree of order 3.

The bounds in (2) are sharp. Let T be any binary tree of height n. Then the set of all interior vertices of T is a minimum dominating set for T and so $\alpha(T) = n$.

Hence the upper bound in (2) is sharp.

To see the sharpness of the lower bound of (2), there are three cases to consider.

Case 1: $2n + 1 = 3k$. Consider k copies of T_3. Put one of these copies with the root at the bottom and stack the remaining $k - 1$ copies one by one from left to right by joining the leaf of the bottom copy to the roots of two stacking copies. Observe that $k - 1$ is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k = (2n + 1)/3$.

Case 2: $2n + 1 = 3k + 1$. Consider k copies of T_3 and a single vertex. Put the single vertex at the bottom, which will serve as a root, and stack two copies by joining the root at the bottom to the roots of two stacking copies. Next stack the remaining $k - 2$ copies one by one from left to right by joining the leaf of the bottom to the roots of two stacking copies. Observe that k is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k + 1 = \lceil (2n + 1)/3 \rceil$.

Case 3: $2n+1 = 3k+2$. Consider k copies of T_3 and two vertices. Put one of these copies with the root at the bottom and stack the remaining $k - 1$ copies one by one from left to right by joining the leaf of the bottom to the roots of two stacking copies. Now join the remaining two vertices from any one of the leaves of the binary tree already constructed. Observe that $k - 1$ is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k + 1 = \lceil (2n + 1)/3 \rceil$.

The lower bound in (3) is sharp. A binary tree of order $2n + 1$ and height n has independence number $n + 1$.

There is a binary tree whose independence number attains the upper bound in (3) for infinitely many n. For example, a fully balanced binary tree of even height will do.

2.3 The Expected Independent Domination Number of Random Binary Trees

In this section we shall derive a formula for the expected value $\mu(2n+1)$ of the independent domination number of a random binary tree with $2n+1$ vertices and we shall determine the asymptotic behavior of $\mu(2n+1)$ as n goes to infinity.

Let T be a binary tree. If we remove the root r of T, along with all arcs incident from r, we obtain a (possibly empty) ordered pair of disjoint binary trees, or *1-branches*, whose roots were originally joined from r. Let y_{2n+1} denote the number of binary trees with $2n+1$ vertices. Clearly, $y_1 = 1$ and we know that

$$y_{2n+1} = \sum y_i y_j \tag{2.1}$$

for $n \geq 1$, where the sum is over all i and j such that i and j are odd and $i+j = 2n$. If we let

$$y = y(x) = \sum_{n=0}^{\infty} y_{2n+1} x^{2n+1}$$

be the ordinary generating function for binary trees, then it follows from equation (2.1) that

$$\begin{aligned} y &= \sum_{n=0}^{\infty} y_{2n+1} x^{2n+1} \\ &= y_1 x + \sum_{n=1}^{\infty} y_{2n+1} x^{2n+1} \\ &= x + \sum_{n=1}^{\infty} (\sum y_i y_j) x^{2n+1} \\ &= x + x \sum_{n=1}^{\infty} (\sum (y_i x^i)(y_j x^j)) \end{aligned}$$

$$= x(1+y^2) \tag{2.2}$$

$$= \frac{1}{2x}[1-(1-4x^2)^{1/2}] \tag{2.3}$$

$$= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^{2n+1}, \tag{2.4}$$

where the inner sums are over all i and j such that i and j are odd and $i+j=2n$. This, of course, is a well-known argument (see [Ca58] or [Mo83]).

On the other hand, we may find the generating function y for binary trees using slightly different approach. Let T be a binary tree with order at least 3 and root r and let T_3 denote the binary subtree of T with 3 vertices and the same root r. If we remove T_3 of T, along with all arcs incident with vertices in T_3, we obtain an ordered 4-tuple (B_1, B_2, B_3, B_4) of disjoint binary trees, or 2-*branches*, satisfying the following three conditions:

(i) Both B_1 and B_2 are either empty binary trees or both non-empty binary trees.

(ii) Both B_3 and B_4 are either empty binary trees or both non-empty binary trees.

(iii) The roots of B_1 and B_2 were originally joined from the left leaf of T_3 and the roots of B_3 and B_4 from the right leaf of T_3.

Now, using the same technique used to derive equation (2.2), we have

$$y = x + x^3(1+2y^2+y^4) \tag{2.5}$$

which is equivalent to $y = x(1+y^2)$.

Lemma 2.3.1 *Let T be a binary tree. Then the independent domination number of T is one more than the sum of the independent domination numbers of all* 2*-branches of T.*

Proof: This follows immediately from the algorithm in Theorem 2.1.1. ∎

For $1 \le k \le 2n+1$, let $y_{2n+1,k}$ denote the number of binary trees of order $2n+1$ whose independent domination number is exactly k. Let

$$Y = Y(x,z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{2n+1} y_{2n+1,k} z^k \right) x^{2n+1}.$$

It follows by a slight extension of the argument used to establish equation (2.5) that

$$Y = zx + zx^3(1 + 2Y^2 + Y^4). \tag{2.6}$$

The factor z is present in equation (2.6) because of Lemma 2.3.1. Here we note that $y = Y(x,1)$.

Theorem 2.3.2 *Let $\mu(2n+1)$ denote the expected value of the independent domination numbers of the y_{2n+1} binary trees with $2n+1$ vertices and define*

$$M = M(x) = \sum_{n=0}^{\infty} \mu(2n+1) y_{2n+1} x^{2n+1}.$$

Then we have

$$M = \frac{y}{1 - 4x^2y^2}. \tag{2.7}$$

Proof: It is easy to see that

$$M = M(x) = \sum_{n=0}^{\infty} \mu(2n+1) y_{2n+1} x^{2n+1} = Y_z(x,1).$$

If we differentiate both sides of equation (2.6) with respect to z, set $z = 1$, appeal to the fact that equations (2.2) and (2.5) are equivalent, and solve for $Y_z(x,1)$, we find the required result. ∎

Of course $M(x)$ is the ordinary generating function for the total sum of the independent domination numbers of binary trees. Therefore, using Maclaurin expansion of $M(x)$, we could find directly the expected value $\mu(2n+1)$ of the independent domination numbers of binary trees for small n. Actually, using (2.3), we have

$$M(x) = \frac{2x}{\sqrt{1-4x^2}(1+\sqrt{1-4x^2})(2-\sqrt{1-4x^2})}, \tag{2.8}$$

and routine use of *Mathematica* produces

$$M(x) = x + x^3 + 6x^5 + 17x^7 + 66x^9 + 234x^{11} + 876x^{13}$$
$$+3265x^{15} + 12330x^{17} + 46766x^{19} + \cdots.$$

Here is a table for $\mu(2n+1)$ and $\mu(2n+1)/(2n+1)$. The entries for $2n+1 \leq 9$ were verified by drawing all of the diagrams for binary trees with up to 9 vertices.

Table 2.1: Values of $\mu(2n+1)$ and $\mu(2n+1)/(2n+1)$

$2n+1$	y_{2n+1}	$\mu(2n+1)y_{2n+1}$	$\mu(2n+1)$	$\frac{\mu(2n+1)}{2n+1}$
1	1	1	1/1=1.00	1
3	1	1	1/1=1.00	.3333
5	2	6	6/2=3.00	.6000
7	5	17	17/5=3.40	.4857
9	14	66	66/14=4.71	.5238
11	42	234	234/42=5.57	.5064
13	132	876	876/132=6.63	.5104
15	429	3265	3265/429=7.61	.5073
17	1430	12330	12330/1430=8.62	.5071
19	4862	46766	46766/4862=9.61	.5062

Furthermore, we can derive a reasonably explicit formula for $\mu(2n+1)$ as follows.

Corollary 2.3.3 *The expected value of the independent domination numbers of binary trees of order* $2n+1$ *is*

$$\mu(2n+1) = \sum (k+1)2^k \frac{<n>_k}{<2n>_k}, \tag{2.9}$$

where the sum is over all even integers k *such that* $0 \leq k \leq n$.

Proof: The following identity appears in [Wi90]:

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{k=0}^{\infty} \frac{n(2k+n-1)!}{k!(k+n)!} x^k \tag{2.10}$$

for integer $n \geq 1$. Using (2.3) and (2.10), we have

$$\begin{aligned} y(2xy)^n &= 2^n y^{n+1} x^n \\ &= 2^n (\frac{1-\sqrt{1-4x^2}}{2x})^{n+1} x^n \\ &= 2^n (\frac{1-\sqrt{1-4x^2}}{2x^2})^{n+1} x^{2n+1} \\ &= 2^n (\sum_{k=0}^{\infty} \frac{(n+1)(2k+n)!}{k!(k+n+1)!} x^{2k}) x^{2n+1} \\ &= (n+1)2^n \sum_{k=n}^{\infty} \binom{2k+1-n}{k+1} \frac{x^{2k+1}}{2k+1-n}. \end{aligned} \tag{2.11}$$

Hence we have

$$\begin{aligned} M(x) &= \frac{y}{1-4x^2y^2} \\ &= \sum_{m=0}^{\infty} y(2xy)^{2m} \\ &= \sum_{m=0}^{\infty} (2m+1)2^{2m} \sum_{k=2m}^{\infty} \binom{2k+1-2m}{k+1} \frac{x^{2k+1}}{2k+1-2m}. \end{aligned} \tag{2.12}$$

Therefore, by equating the coefficients of x^{2n+1} in both sides of (2.12), we have

$$\mu(2n+1)\frac{\binom{2n}{n}}{n+1} = \sum (k+1)2^k \frac{\binom{2n+1-k}{n+1}}{2n+1-k}$$

and hence

$$\mu(2n+1) = \sum (k+1)2^k \frac{< n >_k}{< 2n >_k},$$

where the sums are over all even integers k such that $0 \leq k \leq n$. ∎

We have seen that $M(x)$ is the ordinary generating function for the total sum of the independent domination numbers of binary trees. On the other hand, it is easily seen from the algorithm in Theorem 2.1.1 that $M(x)$ counts the number of vertices at even levels of binary trees. We now want to find the ordinary generating function for the numbers of vertices at odd levels of binary trees. For $0 \leq k \leq 2n+1$, let $w_{2n+1,k}$ denote the number of binary trees of order $2n+1$ in which the number of vertices at odd levels is exactly k. Let

$$W = W(x,z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2n+1} w_{2n+1,k} z^k\right)x^{2n+1}.$$

By the same argument used to establish equation (2.6), we have

$$W = x + z^2x^3(1 + 2W^2 + W^4). \tag{2.13}$$

Here we note that $y = W(x,1)$.

Theorem 2.3.4 *Let $\lambda(2n+1)$ denote the expected number of vertices at odd levels of the y_{2n+1} binary trees with $2n+1$ vertices and define*

$$N = N(x) = \sum_{n=0}^{\infty} \lambda(2n+1) y_{2n+1} x^{2n+1}.$$

Then we have

$$N(x) = \frac{2xy^2}{1 - 4x^2y^2}. \tag{2.14}$$

Proof: It is easy to see that

$$N = N(x) = \sum_{n=0}^{\infty} \lambda(2n+1) y_{2n+1} x^{2n+1} = W_z(x,1).$$

If we differentiate both sides of equation (2.13) with respect to z, set $z = 1$, appeal to the equations (2.2) and (2.5), and solve for $W_z(x,1)$, we find the required result. ∎

From (2.9) and the fact that $\mu(2n+1) + \lambda(2n+1) = 2n+1$, we have a formula for $\lambda(2n+1)$:

$$\lambda(2n+1) = (2n+1) - \sum (k+1)2^k \frac{< n >_k}{< 2n >_k}, \tag{2.15}$$

where the sum is over all even integers k such that $0 \leq k \leq n$.

We also have a useful alternate formula for $\lambda(2n+1)$.

Corollary 2.3.5 *The expected value of the number of vertices at odd levels of binary trees of order* $2n+1$ *is*

$$\lambda(2n+1) = \sum (k+1)2^k \frac{< n >_k}{< 2n >_k} \tag{2.16}$$

for all $n \geq 1$*, where the sum is over all odd integers* k *such that* $0 \leq k \leq n$*. We, of course, have* $\lambda(1) = 0$*.*

Proof: We apply to (2.14) the same procedure as in Corollary 2.3.3. Then we have

$$\lambda(2n+1) = \sum (k+1)2^k \frac{< 2n-k >_n}{< 2n >_n}, \tag{2.17}$$

where the sum is over all odd integers k such that $0 \leq k \leq n$. It is easy to check that

$$\frac{< 2n-k >_n}{< 2n >_n} = \frac{< n >_k}{< 2n >_k}.$$

Therefore we have (2.16) from (2.17). ■

To determine the asymptotic behavior of $\mu(2n+1)/(2n+1)$, we need the following technical lemma, which is a slight modification of Theorem 2 in [Be74].

Lemma 2.3.6 *Let* $A(u) = \sum_{n=0}^{\infty} a_n u^n$ *and* $B(u) = \sum_{n=0}^{\infty} b_n u^n$ *be power series with radii of convergence* $\rho_1 \geq \rho_2$*, respectively. Suppose that* $A(u)$ *converges absolutely at* $u = \rho_1$*. Suppose that* $b_n > 0$ *for all* n *and that* b_{n-1}/b_n *approaches a limit* b *as* $n \to \infty$*. If* $\sum_{n=0}^{\infty} c_n u^n = A(u)B(u)$*, then* $c_n \sim A(b)b_n$*.*

Proof: It suffices to show that $c_n/b_n \sim A(b)$. Notice that $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ and $b = \rho_2$. By repeated application of the triangle inequality, we have

$$\begin{aligned} |A(b) - \frac{c_n}{b_n}| &= |(A(b) - \sum_{k=0}^{n} a_k b^k) + \sum_{k=K+1}^{n} a_k (b^k - \frac{b_{n-k}}{b_n}) \\ &\quad + \sum_{k=0}^{K} a_k (b^k - \frac{b_{n-k}}{b_n})| \end{aligned}$$

$$\leq \quad |A(b) - \sum_{k=0}^{n} a_k b^k| + \sum_{k=K+1}^{n} |a_k|(|b^k| + |\frac{b_{n-k}}{b_n}|)$$

$$+ \sum_{k=0}^{K} |a_k(b^k - \frac{b_{n-k}}{b_n})|, \tag{2.18}$$

for any K such that $0 \leq K < n$. As n goes to infinity, the first term of (2.18) goes to zero because $A(u)$ converges at $u = \rho_2 = b$. As n goes to infinity, the second term becomes the tail of a convergent series because $A(u)$ converges absolutely at $u = \rho_2 = b$ and $b_{n-k}/b_n \sim b^k$. As n goes to infinity, the third term goes to zero because $b_{n-k}/b_n \sim b^k$. Letting K become large, we obtain the lemma. ■

Recall that our generating function $M(x)$ has alternate zero coefficients. To eliminate these, we substitute u for x^2 and define

$$M_*(u) = \sum_{n=0}^{\infty} \mu(2n+1) y_{2n+1} u^n.$$

Now we can state the main result of this section.

Corollary 2.3.7 *The expected value of the independent domination numbers of binary trees of order* $2n+1$ *is*

$$\mu(2n+1) \sim \frac{1}{2}(2n+1)$$

and the expected value of the number of vertices at odd levels of binary trees of order $2n+1$ *is*

$$\lambda(2n+1) \sim \frac{1}{2}(2n+1).$$

Proof: It quickly follows from (2.8) that $M_*(u)$ becomes

$$M_*(u) = \frac{2}{\sqrt{1-4u}(1+\sqrt{1-4u})(2-\sqrt{1-4u})}.$$

Now we let

$$A(u) = \frac{2}{(1+\sqrt{1-4u})(2-\sqrt{1-4u})},$$

and

$$B(u) = \frac{1}{\sqrt{1-4u}}.$$

Note that $A(u)$ can be rewritten as:

$$A(u) = \frac{2}{3}(\frac{1-\sqrt{1-4u}}{4u} + \frac{2}{3+4u} + \frac{1}{3+4u}\sqrt{1-4u}),$$

which has a power series expansion in u with radius of convergence $1/4$. Moreover, it is not too hard to see this power series converges absolutely at $u = 1/4$ using the fact that $\sqrt{1-4u}$ has a power series expansion in u with radius of convergence $1/4$ which converges absolutely at $u = 1/4$ (see, for example, p.426, [Kn90]). On the other hand, we have

$$B(u) = \frac{1}{\sqrt{1-4u}} = \sum_{n=0}^{\infty}(-4)^n \binom{-\frac{1}{2}}{n} u^n$$

for $|u| < 1/4$. If we let

$$b_n = (-4)^n \binom{-\frac{1}{2}}{n},$$

it is easy to check that

$$\frac{b_{n-1}}{b_n} \to \frac{1}{4}$$

as $n \to \infty$ and that $b_n > 0$ for all n. Note that $M_*(u) = A(u)B(u)$. Therefore from Lemma 2.3.6 we have

$$\mu(2n+1)y_{2n+1} \sim A(\frac{1}{4})b_n = b_n$$

and hence

$$\begin{aligned}\mu(2n+1) &\sim \frac{b_n}{y_{2n+1}} = (-4)^n \binom{-\frac{1}{2}}{n} \frac{n+1}{\binom{2n}{n}} = n+1 \\ &\sim \frac{1}{2}(2n+1).\end{aligned}$$

This completes the proof of the first part of the lemma. The second part of the lemma comes immediately from the fact that $\lambda(2n+1) = (2n+1) - \mu(2n+1)$. ■

A. Meir and J. Moon showed (see [MeM73] or [MeM75]) that the expected independence number $\nu(2n+1)$ of binary trees of order $2n+1$ is

$$\nu(2n+1) \sim (.585786\cdots)(2n+1).$$

We observed in Theorem 2.2.1 that $\alpha'(T) \leq \beta(T)$ for any binary tree T. Our result

$$\mu(2n+1) \sim (.5)(2n+1)$$

is consistent with these two facts.

Chapter 3

Tournaments

3.1 The Domination Number of a Tournament

In this section we will investigate domination numbers of specific digraphs, known as tournaments. A *tournament* is a diraph in which every pair of distinct vertices has exactly one arc. A *transitive* tournament is a tournament such that if uv and vw are arcs then uw is also an arc.

First we introduce an algorithm which finds a dominating set of a given tournament. This algorithm is greedy in the sense that it selects a vertex that covers a maximum number of yet uncovered vertices in each step.

Algorithm 3.1.1 *Let $T_1 = T$ be the given tournament of order n and let $S_0 = \phi$. Put $i = 1$ and go to* (1).

(1) *Choose a vertex v_i with largest outdegree in T_i and let $S_i = S_{i-1} \cup \{v_i\}$.*

(2) *Let T_{i+1} be the subtournament of T_i induced by $V(T_i) - N^+[v_i]$.*

(3) *If T_{i+1} is an empty tournament, then let $S = S_i$ and stop. Otherwise, put $i = i + 1$ and return to* (1). ∎

We note that the complexity of this algorithm is $\mathcal{O}(n^2)$. But we will see shortly that this estimate can be improved.

Let T be a tournament of order n. Then we know that there exists a vertex v in T with $od(v) \geq (n-1)/2$ since $\sum_{v \in V} od(v) = n(n-1)/2$ and hence the average outdegree over all vertices is $(n-1)/2$. In addition, every subdigraph of a tournament induced by a subset of $V(T)$ is also a tournament.

Using these simple observations, we prove the following theorem.

Theorem 3.1.2 *Let T be a tournament of order n. Then Algorithm 3.1.1 terminates after at most $\lfloor \lg(n+1) \rfloor$ steps and S is a dominating set for T. Therefore we have*

$$1 \leq \alpha(T) \leq \lfloor \lg(n+1) \rfloor .$$

Here, lg *denotes the logarithm with base 2.*

Proof: *Step* 1: Let $T_1 = T$ and choose a vertex v_1 of T_1 having maximum outdegree.

Step 2: Let T_2 be the subtournament of T_1 induced by $V(T_1) - N^+_{T_1}[v_1]$. Since

$$|N^+_{T_1}[v_1]| \geq \frac{n-1}{2} + 1 = \frac{n+1}{2},$$

we have

$$n_2 := |V(T_2)| = n - |N^+_{T_1}[v_1]| \leq \frac{n-1}{2}.$$

Choose a vertex v_2 of T_2 having maximum outdegree.

Step 3: Let T_3 be the subtournament of T_2 induced by $V(T_2) - N^+_{T_2}[v_2]$. Since

$$|N^+_{T_2}[v_2]| \geq \frac{n_2+1}{2},$$

we have

$$n_3 := |V(T_3)| = n_2 - |N^+_{T_2}[v_2]| \leq \frac{n_2-1}{2} \leq \frac{n-(1+2)}{2^2}.$$

Choose a vertex v_3 of T_3 having maximum outdegree. We continue this process up to step k.

Step k: Let T_k be the subtournament of T_{k-1} induced by $V(T_{k-1}) - N^+_{T_{k-1}}[v_{k-1}]$. Then

$$\begin{aligned} n_k := |V(T_k)| &= n_{k-1} - |N^+_{T_{k-1}}[v_{k-1}]| \\ &\leq \frac{n_{k-1} - 1}{2} \\ &\leq \frac{n - (2^0 + 2^1 + \cdots + 2^{k-2})}{2^{k-1}}. \end{aligned}$$

Choose a vertex v_k of T_k having maximum outdegree.

After step k, the number of vertices in T that are not yet covered by $\{v_1, v_2, \ldots, v_k\}$ is

$$\begin{aligned} n_k - |N^+_{T_k}[v_k]| &\leq \frac{n_k - 1}{2} \\ &\leq \frac{n - (2^0 + 2^1 + \cdots + 2^{k-1})}{2^k}. \end{aligned} \qquad (3.1)$$

We want to find the minimum value k' of k that makes (3.1) zero. It is easy to see that $k' \leq \lg(n+1)$. Clearly, $\{v_1, v_2, \ldots, v_{k'}\}$ is a dominating set of T. ■

Now we can see from Theorem 3.1.2 that the complexity of Algorithm 3.1.1 is $\mathcal{O}(n \log n)$.

We will discuss the sharpness of the upper bound in the above theorem later. The lower bound is sharp. Any transitive tournament will do.

J. Moon stated in [Mo68] that $\lfloor \lg n - 2 \lg \lg n \rfloor \leq \alpha(T) \leq \lfloor \lg(n+1) \rfloor$ if $n \geq 2$, a result which was attributed to L. Moser. But this lower bound is incorrect as we have already shown.

It is easily seen that every tournament is unilateral and that every strong tournament has at least three vertices.

Corollary 3.1.3 *Let T be a strong tournament of order n. Then we have*

$$2 \leq \alpha(T) \leq \lfloor \lg(n+1) \rfloor.$$

Moreover, the lower bound is sharp.

Proof: We know that a tournament is strong if and only if there exists a spanning cycle of the tournament. Therefore any strong tournament has no vertices of outdegree $n-1$ and so $\alpha(T) \geq 2$. For the sharpness, we construct T as follows. Take an n-cycle C_n and let v be a fixed vertex of C_n. Join v to all possible vertices of C_n and choose the other arcs arbitrarily. Then the resulting tournament T is strong since it has a spanning cycle and $\alpha(T) = 2$. ■

A tournament T is called *reducible* if it is possible to partition its vertex set $V(T)$ into two nonempty sets V_1 and V_2 in such a way that every vertex in V_1 dominates all the vertices in V_2. Of course, a tournament is *irreducible* if it is not reducible. It is well-known that a tournament T is irreducible if and only if it is strong and that a tornament of order n is reducible if and only if $\sum_{i=1}^{k} od(v_i) = \binom{k}{2}$ for some $k < n$.

Now we need the following definition.

Definition 3.1.4 A *minimum subtournament*, denoted $m(T)$, of a reducible tournament T is the subtournament $T[V_1]$ induced by V_1 satisfying the following properties:

(1) $V(T)$ is partitioned into two nonempty sets V_1 and V_2 in such a way that every vertex in V_1 dominates all the vertices in V_2.

(2) V_1 has the minimum cardinality for which property (1) holds.

This definition says that only the arcs in $T[V_1]$ play an important role in the sense of domination. Therefore we have the following theorem.

Theorem 3.1.5 *Let T be a reducible tournament with a minimum subtournament $m(T)$. Then we have*

$$\alpha(T) = \alpha(m(T)).$$

Proof: Let $V_1 = V(m(T))$ and $V_2 = V(T) - V_1$. Then V_1 and V_2 are nonempty sets of vertices such that each vertex of V_1 dominates all vertices in V_2. Let S be a minimum dominating set of $m(T)$. Then S is clearly a dominating set of T and hence $\alpha(T) \leq \alpha(m(T))$.

Let R be a minimum dominating set of T. Then R cannot be a subset of V_2 and hence R intersects V_1. Therefore $R \cap V_1$ is a dominating set of T and so R is a subset of V_1. Moreover, R is a dominating set of $m(T)$. Thus $\alpha(m(T)) \leq \alpha(T)$. ■

3.2 The Domination Number of a Random Tournament and the Paley Tournament

Let us consider the probability space $\mathcal{T}_n$ consisting of random tournaments on the vertex set $V = \{1, 2, \ldots, n\}$. By a *random tournament* we mean here a tournament on V obtained by choosing, for each $1 \leq i < j \leq n$, independently, either the arc ij or the arc ji, where each of these two choices is equally likely. Observe that all the $2^{\binom{n}{2}}$ possible tournaments on V are equally likely.

Theorem 3.2.1 *A random tournament $T \in \mathcal{T}_n$ has domination number either*

$$\lfloor k_* \rfloor + 1 \text{ or } \lfloor k_* \rfloor + 2,$$

where

$$k_* = \lg n - 2 \lg \lg n + \lg \lg e$$

and lg *denotes the logarithm with base* 2.

Proof: For each $T \in \mathcal{T}_n$, let $X(T)$ be the number of dominating k-sets of T. If K is a fixed k-set of vertices, then

$$P(K \text{ dominates a fixed vertex in } V - K) = 1 - 2^{-k}$$

and

$$P(K \text{ dominates all vertices in } V - K) = (1 - 2^{-k})^{n-k}.$$

Therefore, we have

$$E[X] = \binom{n}{k}(1 - 2^{-k})^{n-k}.$$

The rest of this proof is exactly the same as the proof of Theorem 1.3.2 once we take $r = 2$. ∎

Now let us consider the sharpness of the upper bound of theorem 3.1.2. Theorem 3.2.1 says that not only do tournaments of order n with $\alpha = (1 + o(1)) \lg n$ exist, but when n is large, the overwhelming majority of tournaments will have a domination number near $\lg n$. Can we construct such a tournament?

The proof of Theorem 3.1.2 strongly suggests that a quasi-random tournament has a large domination number (see [ChG91]). Then do quasi-random tournaments really have domination number very close to the upper bound $\lfloor \lg(n+1) \rfloor$ for n sufficiently large? A well-known example of a quasi-random tournament is the so-called Paley tournament $Q_p(Z_p, E)$. For a prime $p \equiv 3 \pmod 4$, the vertices of Q_p consist of integers modulo p. A pair $(i, j) \in E$ iff $i - j$ is a non-zero quadratic residue modulo p, i.e., iff $(\frac{i-j}{p}) = 1$, where we use the familiar Legendre symbol. Then Q_p is a well-defined $(p-1)/2$-regular quasi-random tournament (see [ChG91]). It is easily checked that $\alpha(Q_p) = \lfloor \lg(p+1) \rfloor$ for $p = 3, 7, 11,$ and 19. But $\alpha(Q_{31}) \leq 4 < \lfloor \lg(31+1) \rfloor$ since $\{1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28\}$ is the set of all non-zero quadratic residues modulo 31 and hence $\{0, 27, 29, 31\}$ is a dominating set for Q_{31}. This shows that $\alpha(Q_p) = \lfloor \lg(p+1) \rfloor$ does not hold for some p. What if p is large enough? Now we consider Schütte property. We say that a tournament has *(Schütte) property* S_k if for every set of k vertices there is one vertex that dominates them all. For example, a directed 3-cycle has property S_1.

The following lemma is in [GrS71], but it was used to find a lower bound of p for Q_p to have property S_k.

Lemma 3.2.2 ([GrS71]) *If k satisfies the inequality*

$$p - \{(k-2)2^{k-1} + 1\}\sqrt{p} - 2^{k-1} > 0,$$

then the Paley tournament Q_p has property S_k.

Proof: It is easily seen that Q_p has property S_k if and only if for all $a_1, \ldots, a_n \in V$ there exists an $x \in V$ such that

$$\left(\frac{x - a_i}{p}\right) = 1 \quad \text{for} \quad 1 \leq i \leq k.$$

Set $\chi(a) = (\frac{a}{p})$ and let $A = \{a_1, \ldots, a_k\}$ denote a set of k arbitrary fixed vertices of Q_p. Define

$$f(A) = \sum_{x \in V - A} \prod_{j=1}^{k} \{1 + \chi(x - a_j)\}.$$

Then $f(A)2^{-k}$ counts the number of vertices that dominate all the vertices in A. Now define

$$g(A) = \sum_{x=0}^{p-1} \prod_{j=1}^{k} \{1 + \chi(x - a_j)\},$$

$$h(A) = \sum_{i=0}^{k} \prod_{j=1}^{k} \{1 + \chi(a_i - a_j)\}.$$

Then we have $f(A) = g(A) - h(A)$. Expanding the inner terms of $g(A)$, we have

$$g(A) = \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \sum_{j=1}^{k} \chi(x - a_j) + \sum_{x=0}^{p-1} \sum_{s=2}^{k} \sum_{j_1 < \cdots < j_s} \chi(x - a_{j_1}) \cdots \chi(x - a_{j_s}).$$

The first two terms of this are p and 0, respectively. To estimate the remaining terms we use the result of Burgess [Bu62]:

$$|\sum_{x=0}^{p-1} \chi(x-a_{j_1})\cdots\chi(x-a_{j_s})| \leq (s-1)\sqrt{p}$$

for $a_{j_1},\ldots,a_{j_s}$ distinct. Thus we have

$$\begin{aligned} |g(A)-p| &\leq \sum_{s=2}^{k} |\sum_{x=0}^{p-1} \sum_{j_1<\cdots<j_s} \chi(x-a_{j_1})\cdots\chi(x-a_{j_s})| \\ &\leq \sum_{s=2}^{k} \binom{k}{s}(s-1)\sqrt{p} \\ &= \{(k-2)2^{k-1}+1\}\sqrt{p}. \end{aligned}$$

Therefore, we have

$$g(A) \geq p - \{(k-2)2^{k-1}+1\}\sqrt{p}.$$

Now consider $h(A)$. If $h(A) \neq 0$, then for some i_0, $\prod_{j=1}^{k}\{1+\chi(a_{i_0}-a_j)\}$ is nonzero. Thus, for all j, $\chi(a_{i_0}-a_j) \neq 1$ so that for all $j \neq i_0$, $\chi(a_{i_0}-a_j) = 1$. But this implies $\chi(a_j - a_{i_0}) = -1$ for all $j \neq i_0$ and consequently

$$\prod_{j=1}^{k}\{1+\chi(a_i-a_j)\} = \begin{cases} 0 & \text{for } i \neq i_0 \\ 2^{k-1} & \text{for } i = i_0. \end{cases}$$

Therefore, in any case, we have

$$h(A) \leq 2^{k-1}.$$

Thus, we have

$$\begin{aligned} f(A) &= g(A) - h(A) \\ &\geq p - \{(k-2)2^{k-1}+1\}\sqrt{p} - 2^{k-1} \end{aligned}$$

and hence Q_p has property S_k if $p - \{(k-2)2^{k-1}+1\}\sqrt{p} - 2^{k-1} > 0$. ■

Now we are ready to state the following theorem.

Theorem 3.2.3 *The domination number of the Paley tournament Q_p satisfies*

$$\alpha(Q_p) > (1 + o(1))\frac{1}{2}\lg p.$$

Proof: Suppose Q_p satisfies property S_k. Then for every set S of k vertices there exists a vertex not in S that is dominated by S and hence every dominating set must have more than k vertices. Consequently, $\alpha(Q_p) > k$ if Q_p satisfies property S_k.

Now we know that Q_p satisfies S_k if

$$\{(k-2)2^{k-1} + 1\}\sqrt{p} + 2^{k-1} < p \tag{3.2}$$

and hence we want to find the maximum value k' of k satisfying (3.2) when p is large. But it is easy to check $k' < \lg(p+1)$ and so we let

$$k = c \lg p - d \lg\lg p + 1, \quad c > 0 \text{ and } d \geq 0.$$

Then the left side of (3.2) becomes

$$p\{\frac{p^{c-1/2}}{(\lg p)^d}\lg(\frac{p^c}{2(\lg p)^d}) + \frac{1}{\sqrt{p}} + \frac{p^{c-1}}{(\lg p)^d}\}. \tag{3.3}$$

To make the second factor of (3.3) smaller than 1 when $p \to \infty$, we must have $c \leq 1/2$. But the maximum value k' of k can be obtained when $c = 1/2$ and $d > 0$. Therefore

$$k' = \frac{1}{2}\lg p - d\lg\lg p + 1, \quad d > 0$$

and so

$$\alpha(Q_p) > k' = (1 + o(1))\frac{1}{2}\lg p. \quad \blacksquare$$

Open Problems

This thesis covers only a portion of the topic of domination for graphs and digraphs. But we believe that many more results will be forthcoming in the near future. In the course of our researches, we struggled with many difficult questions. Among them, we would like to state the following unsolved problems.

(1) We have shown that a digraph D with order n and minimum indegree $\delta^- \geq 1$ has domination number

$$\alpha(D) \leq \lfloor \frac{\delta^- + 1}{2\delta^- + 1} n \rfloor$$

in Theorem 1.2.6 and that this upper bound is sharp for infinitely many n when $\delta^- = 1$ in Theorem 1.2.9. For $\delta^- = 2$, can we either sharpen this upper bound or construct a digraph with order n and $\delta^- = 2$ whose domination number is $\lfloor \frac{\delta^- + 1}{2\delta^- + 1} n \rfloor$?

(2) Regarding binary trees as undirected graphs, A. Meir and J. Moon showed in [MeM77] that the expected domination number of a random binary tree with $2n + 1$ vertices is asymptotic to $(.3782\cdots)(2n + 1)$. What about the asymptotics of the expected domination number of a random binary tree in our sense, that is, if we regard binary trees as directed away from the root? The details will be published elsewhere.

(3) What is the asymptotic behabior of the expected domination number and the expected independent domination number of a random oriented tree? This seems to be a difficult problem even for the simplest families of oriented trees, such as

orientations of the paths of order n.

(4) We have shown that a tournament T of order n has domination number

$$\alpha(T) \leq \lfloor \lg(n+1) \rfloor$$

in Theorem 3.1.2. Here lg denotes the logarithm with base 2. Can we either sharpen this upper bound or construct a tournament with n vertices whose domination number is this upper bound?

(5) Can we find the domination number of the Paley tournament Q_p as a function of p? What about the asymptotics for the domination number of Q_p?

(6) Finally, we want to state a conjecture, which is not unrelated to the main topic of this thesis.

Conjecture: Let G be a connected cubic (or 3-regular) graph with n vertices. Then

$$\alpha(G) \leq \lceil \frac{n}{3} \rceil.$$

The author encountered the same conjecture in [Re9x] but our conjecture was established independently due to the following clues. First, it is true for connected cubic graphs with order up to 14. We checked 621 diagrams in [ReW9x] for unlabeled connected cubic graphs with up to 14 vertices. Second, Robinson and Wormald showed in [RoW92] that almost all cubic graphs are hamiltonian. Since we know that a cycle of order n has domination number $\lceil n/3 \rceil$, it follows that almost all cubic graphs satisfy the conjecture.

In addition, this conjecture is best possible. To see this, consider a cubic graph G with $6n$ vertices consisting of the cycle $v_1 v_2 \cdots v_{6n} v_1$ and the edges $v_{6i+1}v_{6i+4}$, $v_{6i+2}v_{6i+5}$, and $v_{6i+3}v_{6i+6}$ for $i = 0, \ldots, n-1$. Since G contains a Hamiltonian cycle, $\alpha(G) \leq \lceil 6n/3 \rceil = 2n$. On the other hand, any dominating set of G must contain at least two of the six vertices $v_{6i+1}, \cdots, v_{6i+6}$ for each i and hence $\alpha(G) \geq 2n$. Therefore $\alpha(G) = 2n = \lceil 6n/3 \rceil$.

We note that the conjecture requires that all vertices of G should have degree exactly three rather than at least three. To see this, we consider a cubic graph H with 8 vertices consisting of the cycle $v_0v_1 \cdots v_7v_0$ and the edges v_0v_4, v_1v_7, v_2v_5, and v_3v_6. Next construct a graph G from $3n$ disjoint copies of H by adding an edge between all pairs of vertices both of which are labeled v_0. Then it is easily checked that $\alpha(G) = 9n > 8n = \lceil |V|/3 \rceil$.

Bibliography

[AlS92] N. Alon and J. H. Spencer, *The Probabibistic Method*, Wiley, New York 1992.

[Be74] E. A. Bender, Asymptotic Method in Enumeration, *SIAM Review* **16** (1974), 485-515.

[Be58] C. Berge, *Theory of Graphs and its Applications*, Dunrod, Paris 1958.

[Bo85] B. Bollobás, *Random Graphs*, Academic, London 1985.

[Bu62] D. A. Burgess, On character sum and primitive roots, *Proc. London Math. Soc.* **12** (1962), 179-192.

[Ca58] A. Caley, On the analytical forms called trees, *Philos. Mag.* **28** (1858), 374-378. [*Collected Mathematical Papers*, Cambridge **4** (1891), 112-115.]

[ChG91] F. R. K. Chung and R. L. Graham, Quasi-random tournaments, *J. of Graph Theory* **15** (1991), 173-198.

[CoH77] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977), 247-261.

[GrS71] R. L. Graham and J. H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* **14** (1971), 45-48.

[HaNoC65] F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models*, Wiley, New York 1965.

[HeL90(a)] S. T. Hedetniemi and R. C. Laskar, Introduction, *Discrete Math.* **86** (1990), 3-9.

[HeL90(b)] S. T. Hedetniemi and R. C. Laskar, Bibliography on domination in graphs and some basic definitions of domination parameters, *Discrete Math.* **86** (1990), 257-277.

[Kn90] K. Knopp, *Theory and Application of Infinite Series*, Dover, Mineola, 1990.

[KoN73] A. G. Konheim and D. J. Newman, A note on growing binary trees, *J. of Graph Theory* **4** (1973), 57-63.

[Ko36] D. König, *Theory of Finite and Infinite Graphs*, Birkhäuser, Boston 1990.

[Lo75] L. Lovász, On the ratio of optimal integral and fractional covers, *Discrete Math.* **13** (1975), 383-390.

[MeM73] A. Meir and J. W. Moon, The Expected Node-Independence Number of Random Trees, *Proc. Kon. Ned. Akad. v. Wetensch* **76** (1973), 335-341.

[MeM75] A. Meir and J. W. Moon, The Expected Node-Independence Number of Various Types of Trees, *Recent Advances in Graph Theory*, Praha (1975), 351-363.

[MeM77] A. Meir and J. W. Moon, Packing and covering constants for certain families of trees. I, *J. of Graph Theory* **1** (1977), 157-174.

[McS89] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two, *J. of Graph Theory* **13** (1989), 749-762.

[Mo68] J. W. Moon, *Topics on Tournament*, Holt, Rinehart & Winston, New York 1968.

[Mo83] J. W. Moon, On level numbers of t-ary trees, *SIAM J. Alg. Disc. Math.* **4** (1983), 8-13.

[NeM44] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton Univ. Press, Princeton 1944.

[Or62] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Collq. Publ. **38**, Amer. Math. Soc., Providence 1962.

[Pa85] E.M. Palmer, *Graphical Evolution*, Wiley, New York 1985.

[ReW9x] R. C. Read and R. J. Wilson, *Atlas of Graph Theory*, Oxford Univ. Press (to appear).

[Re9x] B. Reed, Paths, stars, and the number three, preprint.

[Ri53] M. Richardson, Solutions of irreflexive relations, *Ann. Math.* **58** (1953), 573-590.

[RoW92] R. W. Robinson and N. C. Wormald, Almost all cubic graphs are hamiltonian, *Random Struct. Alg.* **3** (1992), 117-125.

[To90] I. Tomescu, Almost all digraphs have a kernel, *Random Graphs*, vol 2 (M. Karoński, J. Jaworski, and A. Ruciński, ed.), Wiley, Chichester, 1990, 325-340.

[We81] K. Weber, Domination number for almost every graph, *Rostock. Math. Kolloq.* **16** (1981), 31-43.

[Wi90] H. S. Wilf, *generatingfuctionology*, Academic, San Diego 1990.