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# HANKEL OPERATORS ON HARMONIC BERGMAN SPACES

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ABSTRACT

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BY

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We study Toeplitz and Hankel operators on the harmonic Bergman space $b^2(B)$, where $B$ is the open unit ball in $R^n$, $n \geq 2$. Gregory Adams (see [1]) considered the case $n = 2$ and mostly studied the properties of the operator $T_z$. We show that if $f$ is in $C(\bar{B})$ then the Hankel operator with symbol $f$ is compact. For the proof we have to extend the definition of Hankel operators to the spaces $b^p(B)$, $1 < p < \infty$, and use an interpolation theorem. We also use the explicit formula for the orthogonal projection of $L^2(B, dV)$ onto $b^2(B)$. This result implies that the commutator and semi-commutator of Toeplitz operators with symbols in $C(\bar{B})$ are compact.

We also show that the space $b^2(B)$ decomposes as $b^2(B) = \oplus_{m=0}^{\infty} \mathcal{H}_m(B)$, where $\mathcal{H}_m(B)$ denotes the space of all homogeneous harmonic polynomials on $B$ of degree $m$. We prove that $H_{x_j}^* H_{x_j}(\mathcal{H}_m(B)) \subset \mathcal{H}_m(B)$ for all $j = 1, ..., n$ and all $m$. We have partial results that give some of the eigenvalues and eigenvectors of $H_{x_j}^* H_{x_j}$. Further, motivated by some calculations obtained with the aid of *Mathematica*, we have a conjecture concerning the eigenspace decomposition of the restriction of $H_{x_j}^* H_{x_j}$ to $\mathcal{H}_m(B)$.

DEDICATION

To my parents

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# Contents

# Chapter 1

# Preliminary Results

## 1.1 Spherical Harmonics

Let $\mathcal{H}_m(\mathbf{R}^n)$, $m \geq 0$, denote the space of all homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree $m$. A *spherical harmonic of degree* $m$ is the restriction to the unit sphere $S$ of an element of $\mathcal{H}_m(\mathbf{R}^n)$. The collection of all spherical harmonics of degree $m$ is denoted by $\mathcal{H}_m(S)$. Note that $\mathcal{H}_m(\mathbf{R}^n)$, and hence $\mathcal{H}_m(S)$, is a complex vector space.

Denote by $\mathcal{P}_m(\mathbf{R}^n)$ the complex vector space of all homogeneous polynomials on $\mathbf{R}^n$ of degree $m$. Define an inner product $\langle \ , \ \rangle_m$ on $\mathcal{P}_m(\mathbf{R}^n)$ in the following way: For $p(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$ and $q(x) = \sum_{|\beta|=m} b_\beta x^\beta$ set $\langle p, q \rangle_m = p(D)[\bar{q}]$, where $p(D)$ is the partial differential operator $\sum_{|\alpha|=m} a_\alpha D^\alpha$.

Note: The proofs of almost all the statements in this chapter can be found in [3].

The next theorem shows that any polynomial on $\mathbf{R}^n$, when restricted to $S$, is a sum of spherical harmonics. (See [3], Theorem 5.5.)

**Theorem 1.1** *Every $p \in \mathcal{P}_m(\mathbf{R}^n)$ can be uniquely written in the form*

$$p(x) = p_m(x) + |x|^2 p_{m-2}(x) + \ldots + |x|^{2k} p_{m-2k}(x),$$

*where $k = [\frac{m}{2}]$ and $p_{m-2j} \in \mathcal{H}_{m-2j}(\mathbf{R}^n)$ for $j = 0, 1, \ldots, k$.*

**Corollary 1.2** *If $p$ is a polynomial on $\mathbf{R}^n$ of degree $m$, then the restriction of $p$ to $S$ is a sum of spherical harmonics of degrees at most $m$.*

Let $L^2(S)$ be the usual Hilbert space of square-integrable functions on $S$ with inner product defined by $\langle f, g\rangle = \int_S f\bar{g}\, d\sigma$, where $\sigma$ is normalized surface-area measure on the unit sphere $S$.

Let us now view $\mathcal{H}_m(S)$ as an inner product space with the $L^2(S)$ inner product. Then we have the following two theorems. (See [3], Theorem 5.3 and Theorem 5.8.)

**Theorem 1.3** *If $m \neq k$, then $\mathcal{H}_m(S)$ is orthogonal to $\mathcal{H}_k(S)$ in $L^2(S)$.*

**Theorem 1.4** $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$

Fix a point $\eta \in S$, and consider the map $\Lambda : \mathcal{H}_m(S) \to \mathbf{C}$ defined by $\Lambda(p) = p(\eta)$. The map $\Lambda$ is clearly linear. By the self-duality of the finite dimensional Hilbert space $\mathcal{H}_m(S)$ there exists a unique $Z_\eta \in \mathcal{H}_m(S)$ such that

$$p(\eta) = \langle\, p, Z_\eta \,\rangle = \int_S p\overline{Z_\eta}\, d\sigma$$

for all $p \in \mathcal{H}_m(S)$. The spherical harmonic $Z_\eta$ is called the *zonal harmonic of degree* $m$ with pole $\eta$. At times it is convenient to write $Z_\eta(\zeta) = Z(\eta, \zeta) = Z_m(\eta, \zeta)$.

It is easy to show that each $Z_\eta$ is real valued, $Z_\eta(\zeta) = Z_\zeta(\eta)$ for all $\eta, \zeta \in S$ and $Z_\eta(\eta) = \|Z_\eta\|_2^2$ for all $\eta \in S$, where $\|\cdot\|_2$ denotes the norm in $L^2(S, d\sigma)$.

The dimension of $\mathcal{H}_m(\mathbf{R}^n)$ is given in the following lemma. (See [3], page 82.)

**Lemma 1.5** *Let $h_m$ denote the dimension (over $\mathbf{C}$) of the vector space $\mathcal{H}_m(S)$. Then*

$$h_m = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$$

*for $m \geq 2$ and $h_1 = n$, $h_0 = 1$.*

It is not hard to show that $Z_\eta(\eta) = h_m$ for all $\eta \in S$.

Our previous decomposition $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$ has the following restatement in terms of zonal harmonics. (See [3], Theorem 5.14.)

**Theorem 1.6** *If $f \in L^2(S)$, then $f(\eta) = \sum_{m=0}^{\infty} \langle f, Z_m(\eta, \cdot) \rangle$ in $L^2(S)$.*

Every element of $\mathcal{H}_m(S)$ has a unique extension to an element of $\mathcal{H}_m(\mathbf{R}^n)$; given $p \in \mathcal{H}_m(S)$ we will let $p$ denote this extension as well. (Note that this implies that the dimension of $\mathcal{H}_m(\mathbf{R}^n)$ is $h_m$.) In particular, the notation $Z_m(\,\cdot\,, \eta)$ will often refer to the extension of this zonal harmonic to an element of $\mathcal{H}_m(\mathbf{R}^n)$.

It is easy to show that for $x \in \mathbf{R}^n$ and $u \in \mathcal{H}_m(\mathbf{R}^n)$

$$u(x) = \int_S u(\zeta) Z_m(x, \zeta)\, d\sigma(\zeta). \tag{1.1}$$

An explicit formula for zonal harmonics is given in the following theorem. (See [3], Theorem 5.24.)

**Theorem 1.7** *Let $x \in \mathbf{R}^n$, $\zeta \in S$. Then*

$$Z_m(x, \zeta) = (n + 2m - 2) \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \frac{n(n+2)\ldots(n+2m-2k-4)}{2^k k!(m-2k)!} (x \cdot \zeta)^{m-2k} |x|^{2k}$$

*for each $m > 0$.*

Let us now extend the zonal harmonic $Z_m$ to a function on $\mathbf{R}^n \times \mathbf{R}^n$. We do this by making $Z_m$ homogeneous in the second variable as well as in the first; in other words, we set

$$Z_m(x, y) = |x|^m |y|^m Z_m(x/|x|, y/|y|).$$

If either $x$ or $y$ is 0, we define $Z_m(x, y)$ to be 0 when $m > 0$; when $m = 0$, we define $Z_0$ to be identically 1. With this definition, $Z_m(x,\,\cdot\,) \in \mathcal{H}_m(\mathbf{R}^n)$ for each $x \in \mathbf{R}^n$; also, $Z_m(x, y) = Z_m(y, x)$ for all $x, y \in \mathbf{R}^n$.

By using polar coordinates we can obtain the analogue of (1.1) for integration over $B$. For $u \in \mathcal{H}_m(\mathbf{R}^n)$, we have

$$u(x) = \frac{n+2m}{nV(B)} \int_B u(y) Z_m(x,y) dV(y) \tag{1.2}$$

for each $x \in \mathbf{R}^n$. In other words, for every $u \in \mathcal{H}_m(\mathbf{R}^n)$, $u(x)$ equals the inner product of $u$ with $\frac{n+2m}{nV(B)} Z_m(x, \cdot\,)$.

## 1.2 Harmonic Bergman Spaces

Let $B$ be the open unit ball in $\mathbf{R}^n$ for $n \geq 2$. Let $V$ be Lebesgue volume measure on $\mathbf{R}^n$ and let $1 \leq p < \infty$. The harmonic Bergman space $b^p(B)$ is the set of harmonic functions $u$ on $B$ such that

$$\|u\|_p = (\int_B |u|^p \, dV)^{1/p} < \infty.$$

We often view $b^p(B)$ as a subspace of $L^p(B, dV)$.

For fixed $x \in B$, the map $u \mapsto u(x)$ is a linear functional on $b^p(B)$. We refer to this map as *point evaluation at* $x$. The following proposition shows that point evaluation is continuous on $b^p(B)$. (See [3], Proposition 8.1.)

**Proposition 1.8** *Suppose* $x \in B$. *Then*

$$|u(x)| \leq \frac{1}{V(B)^{1/p}(1-|x|)^{n/p}} \|u\|_p$$

*for every* $u \in b^p(B)$.

The next result shows that $b^p(B)$ is a Banach space. (See [3], Proposition 8.3.)

**Proposition 1.9** *The space* $b^p(B)$ *is a closed subspace of* $L^p(B, dV)$.

Taking $p = 2$, we see that Proposition 1.9 shows that $b^2(B)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_B u\bar{v}\, dV. \tag{1.3}$$

Because the map $u \mapsto u(x)$ is a bounded linear functional on $b^2(B)$ for each $x \in B$, there exists a unique function $R_B(x, \cdot\,) \in b^2(B)$ such that

$$u(x) = \int_B u(y)\overline{R_B(x,y)}\, dV(y)$$

for every $u \in b^2(B)$. The function $R_B$, which can be viewed as a function on $B \times B$, is called the *reproducing kernel* of $B$. We list some basic properties of $R_B$.

(i) Each $R_B$ is real valued.

(ii) $R_B(x,y) = R_B(y,x)$ for all $x, y \in B$.

(iii) $\|R_B(x, \cdot\,)\|^2 = \langle R_B(x, \cdot\,), R_B(x, \cdot\,)\rangle = R_B(x,x)$ for every $x \in B$.

The following lemma is useful in finding an explicit formula for the reproducing kernel of the ball. (See [3], Lemma 8.8.)

**Lemma 1.10** *The set of harmonic polynomials is dense in $b^p(B)$, for $1 \le p < \infty$.*

The reproducing kernel of the ball is given by the following theorem. (See [3], Theorem 8.9.)

**Theorem 1.11** *If $x, y \in B$ then*

$$R_B(x,y) = \frac{1}{nV(B)} \sum_{m=0}^{\infty} (n+2m) Z_m(x,y).$$

*The series converges absolutely and uniformly on $K \times B$ for every compact $K \subset B$.*

Let $Q$ denote the orthogonal projection of $L^2(B, dV)$ onto $b^2(B)$. Then for every $f \in L^2(B, dV)$ and $x \in B$ we have

$$(Qf)(x) = \langle Qf,\ R_B(x,\ \cdot\ )\rangle \ = \ \langle f,\ R_B(x,\ \cdot\ )\rangle \ = \int_B f(y) R_B(x,y)\, dV(y) \qquad (1.4)$$

Hence we can show the following theorem. (See [3], Theorem 8.14.)

**Theorem 1.12** *Let $p$ be a polynomial on $\mathbf{R}^n$ of degree $m$. Then $Qp$ is a polynomial of degree at most $m$. Moreover*

$$(Qp)(x) = \frac{1}{nV(B)} \sum_{k=0}^{m} (n+2k) \int_B p(y) Z_k(x,y)\, dV(y)$$

*for every $x \in B$.*

Now we can get a formula in closed form for $R_B(x,y)$. (See [3], Theorem 8.13.)

**Theorem 1.13** *Let $x, y \in B$. Then*

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x\cdot y + |x|^2|y|^2)^{1+n/2}}\ .$$

An easy calculation shows that

$$R_B(x,y) = \frac{n(1-|x|^2|y|^2)^2 - 4|x|^2|y|^2((1-|x|^2)(1-|y|^2) + |x-y|^2)}{nV(B)((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+n/2}}.$$

Remark: It is interesting to notice that $R_B$ is not positive everywhere on $B \times B$; in fact it is not even bounded from below on $B \times B$. To see this, we can choose $x$ and $y$ from $B$ such that $|x-y|^2 = 1/j$, $1 - |x|^2 = 1/j^2$, $1 - |y|^2 = 1/j^2$, $j \geq 2$. Then

$$\begin{aligned} R_B(x,y) \ &= \ \frac{n(1-(1-\frac{1}{j^2})^2)^2 - 4(1-\frac{1}{j^2})^2(\frac{1}{j^4}+\frac{1}{j})}{nV(B)(\frac{1}{j^4}+\frac{1}{j})^{1+n/2}} \\ &= \ \frac{-4 + \frac{8}{j^2} + \frac{4(n-1)}{j^3} + \frac{-4}{j^4} + \frac{-4(n-2)}{j^5} + \frac{n-4}{j^7}}{nV(B)(\frac{1}{j})^{\frac{n}{2}}(1+\frac{1}{j^3})^{1+n/2}}\ . \end{aligned}$$

If we let $j \to \infty$, then $R_B(x,y) \to -\infty$.

Since $R_B$ is a real valued function it follows that for every $u \in b^2(B)$

$$u(x) = \int_B u(y) R_B(x,y)\, dV(y).$$

Note that for fixed $x \in B$, the function $R_B(x, \cdot\,)$ is bounded on $B$. Thus it makes sense to ask whether the above equation holds not only for $u \in b^2(B)$, but also for $u \in b^p(B)$, where $1 \leq p < \infty$. The following lemma answers that question.

**Lemma 1.14** *For all $u \in b^p(B)$ and for all $p \in [1,\infty)$*

$$u(x) = \int_B u(y) R_B(x,y)\, dV(y). \tag{1.5}$$

Proof: Note that the left-hand side and the right-hand side of (1.5) are bounded linear functionals on $b^p(B)$ that agree on harmonic polynomials. Since, by Lemma 1.10, the set of harmonic polynomials is dense in $b^p(B)$ it follows that

$$u(x) = \int_B u(y) R_B(x,y)\, dV(y)$$

for all $u \in b^p(B)$. □

Let $\mathcal{H}_m(B)$ denote the space of all homogeneous harmonic polynomials on $B$ of degree $m$. Next we prove that the Hilbert space $b^2(B)$, with inner product defined by (1.3) is the direct sum of the spaces $\mathcal{H}_m(B)$.

**Theorem 1.15** $b^2(B) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(B)$

Proof: The finite dimensionality of $\mathcal{H}_m(B)$ implies that $\mathcal{H}_m(B)$ is a closed subspace of $b^2(B)$. Note that dimension of $\mathcal{H}_m(B)$ is $h_m$.

Let $m \neq k$, and $p \in \mathcal{H}_m(B)$, $q \in \mathcal{H}_k(B)$. Since $p$ and $q$ can be extended uniquely to $\mathbf{R}^n$, using polar coordinates and the homogeneity of $p$ and $q$ we have

$$\int_B p(x)q(x)\, dV(x) = nV(B) \int_0^1 r^{n-1+m+k}\, dr \int_S p(\zeta)\overline{q(\zeta)}\, d\sigma(\zeta) = 0$$

by Theorem 1.3. Hence $\mathcal{H}_m(B)$ is orthogonal to $\mathcal{H}_k(B)$ in $b^2(B)$.

To finish the proof it is enough to show that the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(B)$ is dense in $b^2(B)$. Since, by Lemma 1.10, the set of harmonic polynomials is dense in $b^2(B)$, it is enough to show that every harmonic polynomial belongs to the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(B)$. Let $p$ be a harmonic polynomial of degree $k$. Then $p = \sum_{j=0}^{k} p_j$, where $p_j \in \mathcal{P}_j(B)$, $j = 0, ..., k$. Also, $0 = \Delta p = \sum_{j=0}^{k} \Delta p_j$, where $\Delta p_j \in \mathcal{P}_{j-2}(B)$, $j = 2, ..., k$, and $\Delta p_0 = \Delta p_1 = 0$. This implies that $\Delta p_j = 0$ for $j = 0, ..., k$; in other words, $p_j \in \mathcal{H}_j(B)$, and the lemma is proved. □

## 1.3 The Kelvin Transform

The map $x \mapsto x^*$, where

$$x^* = \begin{cases} x/|x|^2 & \text{if } x \neq 0, \infty \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = 0 \end{cases}$$

is called the *inversion* of $\mathbf{R}^n \cup \{\infty\}$ relative to the unit sphere. For any subset $E$ of $\mathbf{R}^n \cup \{\infty\}$, we define $E^* = \{x^* : x \in E\}$.

Given a function $u$ defined on a set $E \subset \mathbf{R}^n \backslash \{0\}$, we define the function $K[u]$ on $E^*$ by

$$K[u](x) = |x|^{2-n} u(x^*).$$

The function $K[u]$ is called the *Kelvin transform* of $u$.

We easily see that $K[K[u]] = u$ for all functions $u$ as above. The transform $K$ is also linear: If $u$, $v$ are functions on $E$ and $b$, $c$ are constants, then $K[bu + cv] = bK[u] + cK[v]$ on $E^*$.

The crucial property of the Kelvin transform is given in the following theorem. (See [3], Theorem 4.4.)

**Theorem 1.16** *If $\Omega \subset \mathbf{R}^n \backslash \{0\}$, then $u$ is harmonic on $\Omega$ if and only if $K[u]$ is harmonic on $\Omega^*$.*

For $p$ a homogeneous harmonic polynomial we have the following identity. (See [3], Theorem 5.32.)

**Theorem 1.17** *Let $n > 2$ and let $p \in \mathcal{H}_m(\mathbf{R}^n)$. Then*

$$p = c_m K[p(D)|x|^{2-n}],$$

*where $c_m = \prod_{j=1}^{m}(4 - n - 2j)^{-1}$ for $m > 0$ and $c_0 = 1$.*

Now we have the following theorem. (See [3], Theorem 5.33.)

**Theorem 1.18** *Let $n > 2$. The space $\mathcal{H}_m(\mathbf{R}^n)$ is the linear span of*

$$\{K[D^\alpha |x|^{2-n}] : |\alpha| = m\}$$

*and $\mathcal{H}_m(S)$ is the linear span of*

$$\{(D^\alpha |x|^{2-n}) |_S : |\alpha| = m\}.$$

The next theorem gives a basis of $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$. (See [3], Theorem 5.34.)

**Theorem 1.19** *Let $n > 2$. The set*

$$\{K[D^\alpha |x|^{2-n}] : |\alpha| = m, \ \alpha_n = 0 \ \text{ or } \ 1\}$$

*is a vector space basis for $\mathcal{H}_m(\mathbf{R}^n)$ and the set*

$$\{D^\alpha |x|^{2-n} : |\alpha| = m, \ \alpha_n = 0 \ \text{ or } \ 1\}$$

*is a vector space basis for $\mathcal{H}_m(S)$.*

Note: We have concetrated here on the case $n > 2$. Analogous results hold when $n = 2$ if $|x|^{2-n}$ is replaced by $\log |x|$.

# Chapter 2

# Toeplitz and Hankel Operators

Since $b^2(B)$ is a closed subspace of the Hilbert space $L^2(B, dV)$, there exists a unique orthogonal projection $Q$ of $L^2(B, dV)$ onto $b^2(B)$. Then for $f \in L^2(B, dV)$ and $x \in B$ (1.4) gives

$$(Qf)(x) = \int_B f(y) R_B(x,y)\, dV(y).$$

Since the above integral makes sense whenever $f$ belongs to $L^1(B, dV)$ we can extend the definition of $Q$ to $L^1(B, dV)$. For $f \in L^1(B, dV)$ and $x \in B$ define

$$(Qf)(x) = \int_B f(y) R_B(x,y)\, dV(y). \tag{2.1}$$

The symmetry of $R_B$ and the harmonicity of $R_B(x, \cdot\,)$ imply that $R_B(\,\cdot\,, y)$ is a harmonic function on $B$ for every fixed $y \in B$. Differentiation with respect to $x$ under the integral sign shows that $Qf$ is a harmonic function on $B$ for every $f \in L^1(B, dV)$. By Lemma 1.14, $Qf$ equals $f$ for all $f \in b^1(B)$. We already know that $Q$ is a bounded projection of $L^2(B, dV)$ onto $b^2(B)$. In [7] Ligocka shows that $Q$, defined by (2.1), is a bounded projection from $L^p(B, dV)$ onto $b^p(B)$, and that the dual of $b^p(B)$ is $b^{p'}(B)$, where $1/p + 1/p' = 1$ and $1 < p < \infty$.

Remark: In [8] Ligocka shows that $Q$ maps $L^\infty(B, dV)$ continuously onto the space of Bloch harmonic functions on $B$, denoted by $BlHarm(B)$. $BlHarm(B)$ is defined to be the set of harmonic functions $u$ on $B$ such that

$$\sup_{x\in B}(1-|x|)|\nabla u(x)| < \infty.$$

Ligocka also shows that the space of Bloch harmonic functions on $B$ is the dual of the space $b^1(B)$.

For $f \in L^\infty(B, dV)$, define the Toeplitz operator $T_f : b^2(B) \to b^2(B)$ with symbol $f$ by

$$T_f u = Q(fu)$$

for $u \in b^2(B)$. It is clear that $T_f$ is a bounded operator and that $\|T_f\| \leq \|f\|_\infty$.

**Lemma 2.1** *Let $f, f_1, f_2 \in L^\infty(B, dV)$ and $a, b$ scalars. Then*

*(i)* $T_{af_1+bf_2} = aT_{f_1} + bT_{f_2}$

*(ii)* $T_f^* = T_{\bar{f}}$.

Proof: $(i)$ is obvious. To prove $(ii)$, let $u, v \in b^2(B)$. Then

$$\begin{aligned}
\langle T_f^* u, v\rangle &= \langle u, T_f v\rangle = \langle u, Q(fv)\rangle = \langle u, fv\rangle \\
&= \int_B u(z)\overline{f(z)v(z)}\, dV(z) = \langle u\bar{f}, v\rangle = \langle Q(\bar{f}u), v\rangle \\
&= \langle T_{\bar{f}} u, v\rangle.
\end{aligned}$$

We used that $Q$ is the orthogonal projection from $L^2(B, dV)$ onto $b^2(B)$. □

For $f \in L^\infty(B, dV)$, define the Hankel operator $H_f : b^2(B) \to b^2(B)^\perp$ with symbol $f$ by

$$H_f u = (I - Q)(fu)$$

for $u \in b^2(B)$.

**Lemma 2.2** *Let $f \in L^\infty(B, dV)$. Then $H_f$ is a bounded operator and $\|H_f\| \leq \|f\|_\infty$.*

Proof: Let $u \in b^2(B)$. Then

$$\|H_f u\|_2^2 = \|fu\|_2^2 - \|Q(fu)\|_2^2 \le \|fu\|_2^2 \le \|f\|_\infty^2 \|u\|_2^2.$$

Hence $\|H_f\| \le \|f\|_\infty$. □

The next lemma gives a formula for the adjoint of a Hankel operator.

**Lemma 2.3** *Let $f \in L^\infty(B, dV)$. Then $H_f^* : b^2(B)^\perp \to b^2(B)$ is given by*

$$H_f^* h = Q(\bar{f}h)$$

*for $h \in b^2(B)^\perp$.*

Proof: Let $u \in b^2(B)$, $h \in b^2(B)^\perp$. Then

$$\begin{aligned} \langle H_f^* h, u\rangle &= \langle h, H_f u\rangle = \langle h, (I-Q)(fu)\rangle \\ &= \langle h, fu\rangle - \langle h, Q(fu)\rangle = \langle h, fu\rangle \\ &= \langle \bar{f}h, u\rangle = \langle Q(\bar{f}h), u\rangle. \end{aligned}$$

Thus $H_f^* h = Q(\bar{f}h)$. □

The connection between Hankel and Toeplitz operators is provided by the formula given in the following lemma.

**Lemma 2.4** *Let $f, g \in L^\infty(B, dV)$. Then*

$$T_{fg} - T_f T_g = H_{\bar{f}}^* H_g.$$

Proof: Let $u \in b^2(B)$. Then

$$\begin{aligned} (H_{\bar{f}}^* H_g)u &= H_{\bar{f}}^*(I-Q)(gu) = H_{\bar{f}}^*(gu - Q(gu)) = H_{\bar{f}}^*(gu) - H_{\bar{f}}^*(Q(gu)) \\ &= Q(fgu) - Q(fQ(gu)) = T_{fg}u - T_f(Q(gu)) = T_{fg}u - (T_f T_g)u \\ &= (T_{fg} - T_f T_g)u. \end{aligned}$$

Therefore $T_{fg} - T_f T_g = H_{\bar{f}}^* H_g$. □

Let $f \in L^\infty(B, dV)$. Define $S_f : b^2(B)^\perp \to b^2(B)^\perp$ by

$$S_f h = (I - Q)(fh)$$

for $h \in b^2(B)^\perp$.

**Lemma 2.5** *Let $f, g \in L^\infty(B, dV)$. Then*

$$H_{fg} = S_f H_g + H_f T_g.$$

Proof: Let $u \in b^2(B)$. Then

$$\begin{aligned}
(S_f H_g + H_f T_g)u &= (S_f H_g)u + (H_f T_g)u = S_f(I - Q)(gu) + H_f(Q(gu)) \\
&= S_f(gu - Q(gu)) + (I - Q)(fQ(gu)) \\
&= (I - Q)(fgu - fQ(gu)) + fQ(gu) - Q(fQ(gu)) \\
&= fgu - fQ(gu) - Q(fgu) + Q(fQ(gu)) + fQ(gu) \\
&\quad - Q(fQ(gu)) = (I - Q)(fgu) = H_{fg}u.
\end{aligned}$$

Thus $H_{fg} = S_f H_g + H_f T_g$. □

Now, we have the following corollary.

**Corollary 2.6** *The set of all functions $f \in C(\bar{B})$ such that $H_f$ is a compact operator is a closed subalgebra of $C(\bar{B})$.*

Proof: The only nontrivial part of the corollary is the assertion that the set in question is closed under multiplication. This follows from Lemma 2.5. □

In order to prove our main result we need to extend the definition of Toeplitz and Hankel operators to the spaces $b^p(B)$ for $p \in (1, \infty)$.

Let $f \in L^\infty(B, dV)$ and let $1 < p < \infty$. The Toeplitz operator with symbol $f$ is the operator $T_f : b^p(B) \to b^p(B)$ defined by

$$T_f u = Q(fu)$$

for $u \in b^p(B)$. The Hankel operator $H_f : b^p(B) \to L^p(B, dV)$ with symbol $f$ is defined by

$$H_f u = (I - Q)(fu)$$

for $u \in b^p(B)$.

Note that $T_f$ and $H_f$ depend upon $p$, although $p$ does not appear in the notation. The domain of $T_f$ and $H_f$ will always be clear from the context.

Remark: We know that $Q : L^p(B, dV) \to b^p(B)$, defined by

$$(Qg)(x) = \int_B g(y) R_B(x, y)\, dV(y)$$

for $g \in L^p(B, dV)$, is a bounded operator for every $p \in (1, \infty)$. Also,

$$T_f = QM_f$$

and

$$H_f = (I - Q)M_f,$$

where $M_f$ is a bounded multiplication operator since $f \in L^\infty(B, dV)$. Thus it follows that $T_f$ and $H_f$ are bounded operators for every $p \in (1, \infty)$ and every $f \in L^\infty(B, dV)$.

We can extend the domain of the inner product given by (1.3) to include all pairs of functions $f, g$ measurable on $B$ such that $fg \in L^1(B, dV)$. For such a pair of functions define $\langle f, g\rangle$ by

$$\langle f, g\rangle = \int_B f\bar{g}\, dV.$$

From now on, let $p'$ denote the number such that $1/p + 1/p' = 1$, where $p \in (1, \infty)$.

Since $b^{p'}(B)$ is the dual of $b^p(B)$ with respect to the pairing $\langle\ ,\ \rangle$, define the Banach space adjoint of $T_f : b^p(B) \to b^p(B)$ to be the operator $T_f^* : b^{p'}(B) \to b^{p'}(B)$ such that

$$\langle T_f^* u, v\rangle \;=\; \langle u, T_f v\rangle$$

for $u \in b^{p'}(B)$ and $v \in b^p(B)$.

**Proposition 2.7** *Let $p \in (1,\infty)$, $a, b$ scalars, and $f, f_1, f_2 \in L^\infty(B, dV)$. Let $T_f$, $T_{f_1}$, $T_{f_2}$ and $T_{af_1+bf_2}$ be operators on $b^p(B)$. Then*

*(i) $T_{af_1+bf_2} = aT_{f_1} + bT_{f_2}$*

*(ii) $T_f^* = T_{\bar f}$, where $T_{\bar f}$ is the Toeplitz operator with symbol $\bar f$ defined on $b^{p'}(B)$.*

Proof: $(i)$ is obvious. To prove $(ii)$, first we show that

$$\langle (I-Q)g, Qh\rangle \;= 0 \tag{2.2}$$

for every $g \in L^p(B, dV)$ and every $h \in L^{p'}(B, dV)$. We know that $Q$, defined by (2.1), is the bounded projection from $L^q(B, dV)$ onto $b^q(B)$ for every $q \in (1,\infty)$. For $q = 2$, $Q$ is the orthogonal projection from $L^2(B, dV)$ onto $b^2(B)$, and hence (2.2) holds. Since $L^2(B, dV) \cap L^p(B, dV)$ is dense in $L^p(B, dV)$ and $L^2(B, dV) \cap L^{p'}(B, dV)$ is dense in $L^{p'}(B, dV)$, (2.2) holds for every $g \in L^p(B, dV)$ and every $h \in L^{p'}(B, dV)$.

Now, let $u \in b^{p'}(B)$ and $v \in b^p(B)$. Then

$$\begin{aligned}
\langle T_f^* u,\ v\rangle &= \langle u,\ T_f v\rangle \\
&= \langle u,\ Q(fv)\rangle \\
&= \langle u,\ (I-(I-Q))(fv)\rangle \\
&= \langle \bar f u,\ v\rangle \\
&= \langle ((I-Q)+Q)(\bar f u),\ v\rangle \\
&= \langle T_{\bar f} u,\ v\rangle .
\end{aligned}$$

We used that $Qw = w$ for $w \in b^q(B)$, $1 < q < \infty$, and (2.2). □

Let $p \in (1, \infty)$ and $H_f : b^p(B) \to L^p(B, dV)$. The Banach space adjoint of $H_f$ is the operator $H_f^* : L^{p'}(B, dV) \to b^{p'}(B)$ such that

$$\langle H_f^* u, v\rangle \;=\; \langle u, H_f v\rangle,$$

where $u \in L^{p'}(B, dV)$ and $v \in b^p(B)$.

As in the case $p = 2$, we have the following connection between Toeplitz and Hankel operators.

**Lemma 2.8** *Let $p \in (1, \infty)$ and let $f, g \in L^\infty(B, dV)$. Then*

$$T_{fg} - T_f T_g = H_{\bar{f}}^* H_g,$$

*where*

$T_f, T_g, T_{fg} : b^p(B) \to b^p(B)$, $H_g : b^p(B) \to L^p(B, dV)$, $H_{\bar{f}}^* : L^p(B, dV) \to b^p(B)$.

Proof: Let $u \in b^p(B)$, $v \in b^{p'}(B)$. Then

$$\begin{aligned} \langle H_{\bar{f}}^* H_g u,\ v\rangle \;&=\; \langle H_g u,\ H_{\bar{f}} v\rangle \;=\; \langle (I-Q)(gu),\ (I-Q)(\bar{f}v)\rangle \\ &= \langle gu,\ \bar{f}v\rangle - \langle gu,\ Q(\bar{f}v)\rangle - \langle Q(gu),\ \bar{f}v\rangle + \langle Q(gu),\ Q(\bar{f}v)\rangle \\ &= \langle (Q + (I-Q))(fgu),\ v\rangle - \langle (Q + (I-Q))(gu),\ Q(\bar{f}v)\rangle \\ &\quad - \langle Q(gu),\ (Q + (I-Q))(\bar{f}v)\rangle + \langle Q(gu),\ Q(\bar{f}v)\rangle \\ &= \langle T_{fg}u,\ v\rangle - \langle T_f T_g u,\ v\rangle \;=\; \langle (T_{fg} - T_f T_g)u,\ v\rangle. \end{aligned}$$

We used that $Qv = v$ for $v \in b^{p'}(B)$, (2.2), and Proposition 2.7. □

Let $f \in L^\infty(B, dV)$ and $p \in (1, \infty)$. The operator $S_f : L^p(B, dV) \to L^p(B, dV)$ is defined by

$$S_f h = (I - Q)(fh).$$

It is clear that $S_f$ is a bounded linear operator for every $f \in L^\infty(B, dV)$ and every $p \in (1, \infty)$. Straightforward calculations, as in Lemma 2.5, give the following lemma.

**Lemma 2.9** *Let $p \in (1, \infty)$ and let $f, g \in L^\infty(B, dV)$. Then*

$$H_{fg} = S_f H_g + H_f T_g,$$

*where*

$H_f, H_g, H_{fg} : b^p(B) \to L^p(B, dV)$, $T_g : b^p(B) \to b^p(B)$, $S_f : L^p(B, dV) \to L^p(B, dV)$.

As in the case $p = 2$ we have the next corollary.

**Corollary 2.10** *Let $p \in (1, \infty)$. The set of functions $f \in C(\bar{B})$ such that the Hankel operator $H_f : b^p(B) \to L^p(B, dV)$ is compact is a closed subalgebra of $C(\bar{B})$.*

Proof: This follows from Lemma 2.9 and the relation $H_f = (I - Q)M_f$. □

An operator $T$ that maps a subspace $X$ of $L^p(B, dV)$ into $L^p(B, dV)$ is called an *integral operator* if there exists a complex-valued, measurable function $k$ defined on $B \times B$ such that

$$(Tf)(x) = \int_B k(x, y) f(y)\, dV(y)$$

for all $f \in X$ and almost all $x \in B$. The function $k$ is called the *kernel* of $T$.

**Lemma 2.11** *Let $f \in L^\infty(B, dV)$ and $p \in (1, \infty)$. Then $H_f : b^p(B) \to L^p(B, dV)$ is an integral operator.*

Proof: Let $u \in b^p(B)$. Then

$$\begin{aligned} (H_f u)(x) &= (I - Q)(fu)(x) = f(x)u(x) - Q(fu)(x) \\ &= f(x) \int_B u(y) R_B(x, y)\, dV(y) - \int_B f(y)u(y) R_B(x, y)\, dV(y) \\ &= \int_B (f(x) - f(y)) R_B(x, y) u(y)\, dV(y) = \int_B k_f(x, y) u(y)\, dV(y), \end{aligned}$$

where $k_f(x,y) = (f(x) - f(y))R_B(x,y)$ is the kernel of $H_f$. □

For $1 \le p, q < \infty$ the mixed norm space $L^{p,q}(B \times B)$ is defined to be the space of functions $k$ on $B \times B$ such that

$$\int_B \left( \int_B |k(x,y)|^p \, dV(y) \right)^{q/p} dV(x) < \infty.$$

In [4] Benedek and Panzone prove that $L^{p,q}(B \times B)$ is a Banach space with the norm

$$\|k\|_{p,q} = \left( \int_B \left( \int_B |k(x,y)|^p \, dV(y) \right)^{q/p} dV(x) \right)^{1/q}.$$

They also show that the dual of $L^{p,q}(B \times B)$ can be identified with $L^{p',q'}(B \times B)$. More precisely, every bounded linear functional $\varphi$ on $L^{p,q}(B \times B)$ is of the form

$$\varphi(k) = \int_B \int_B k(x,y) v(x,y) \, dV(y) dV(x)$$

for some unique $v \in L^{p',q'}(B \times B)$. Furthermore, $\|\varphi\| = \|v\|_{p',q'}$.

The next lemma shows that each $k \in L^{p',p}(B \times B)$ defines a bounded integral operator on $L^p(B, dV)$.

**Lemma 2.12** *Let $p \in (1, \infty)$ and let $k \in L^{p',p}(B \times B)$. Then the integral operator $T : L^p(B, dV) \to L^p(B, dV)$, defined by*

$$(Tf)(x) = \int_B k(x,y) f(y) \, dV(y),$$

*is bounded.*

Proof: Let $f \in L^p(B, dV)$. Then we have

$$\begin{aligned} \|Tf\|_p^p &= \int_B |Tf|^p \, dV(x) = \int_B \left| \int_B k(x,y) f(y) \, dV(y) \right|^p dV(x) \\ &\le \int_B \left( \int_B |k(x,y)|^{p'} \, dV(y) \right)^{p/p'} \left( \int_B |f(y)|^p \, dV(y) \right) dV(x) = \|f\|_p^p \|k\|_{p',p}^p. \end{aligned}$$

Hence $\|T\| \le \|k\|_{p',p}$. □

For $g \in L^p(B, dV)$ and $h \in L^{p'}(B, dV)$, the *tensor product* of $g$ and $h$ is a function on $B \times B$ defined by

$$(g \otimes h)(x, y) = g(x)h(y).$$

**Lemma 2.13** *Let $p \in (1, \infty)$. Then the linear span of the set*

$$\{g \otimes h : g \in L^p(B, dV), h \in L^{p'}(B, dV)\}$$

*is dense in $L^{p',p}(B \times B)$.*

Proof: First we show that for $g \in L^p(B, dV)$ and $h \in L^{p'}(B, dV)$, $g \otimes h$ belongs to $L^{p',p}(B \times B)$. We have

$$\begin{aligned}
\int_B \Big(\int_B |(g \otimes h)(x, y)|^{p'} \, dV(y)\Big)^{p/p'} \, dV(x) &= \int_B \Big(\int_B |g(x)h(y)|^{p'} \, dV(y)\Big)^{p/p'} \, dV(x) \\
&= \int_B |g(x)|^p \|h\|_{p'}^p \, dV(x) \\
&= \|h\|_{p'}^p \|g\|_p^p < \infty.
\end{aligned}$$

Therefore $g \otimes h \in L^{p',p}(B \times B)$.

Suppose $\varphi$ is a bounded linear functional on $L^{p',p}(B \times B)$ such that $\varphi(g \otimes h) = 0$ for every $g \in L^p(B, dV)$ and every $h \in L^{p'}(B, dV)$. Since $\varphi \in (L^{p',p}(B \times B))^* \cong L^{p,p'}(B \times B)$, there exists $v \in L^{p,p'}(B \times B)$ such that

$$\varphi(k) = \int_B \int_B k(x, y) v(x, y) \, dV(y) dV(x)$$

for all $k \in L^{p',p}(B \times B)$.

Therefore for every $g \in L^p(B, dV)$ and every $h \in L^{p'}(B, dV)$

$$\begin{aligned}
0 = \varphi(g \otimes h) &= \int_B \int_B g(x)h(y)v(x, y) \, dV(y)dV(x) \\
&= \int_B g(x) \int_B h(y)v(x, y) \, dV(y) \, dV(x). \qquad (2.3)
\end{aligned}$$

We know that

$$\int_B h(y)v(x,y)\,dV(y) \in L^{p'}(B,dV)$$

because

$$\left(\int_B \left|\int_B h(y)v(x,y)\,dV(y)\right|^{p'} dV(x)\right)^{1/p'} \le \|h\|_{p'}\|v\|_{p,p'} < \infty.$$

Then (2.8) implies that

$$\int_B h(y)v(x,y)\,dV(y) = 0$$

for almost every $x$, where the set of measure zero can depend on $h \in L^{p'}(B,dV)$. Because $L^{p'}(B,dV)$ is separable, there exists a countable dense set $\{h_j\}_{j=1}^{\infty}$ in $L^{p'}(B,dV)$ and for every $j$, there exists a set of measure zero $E_j$ such that

$$\int_B h_j(y)v(x,y)\,dV(y) = 0$$

for $x \notin E_j$.

Let $E' = \bigcup_{j=1}^{\infty} E_j$. Then $E'$ is a set of measure zero, and for every $x \notin E'$ and every $j$ we have $\int_B h_j(y)v(x,y)\,dV(y) = 0$.

Since $v \in L^{p,p'}(B \times B)$, which means that $\int_B(\int_B |v(x,y)|^p\,dV(y))^{p'/p}\,dV(x) < \infty$, it follows that $(\int_B |v(x,y)|^p dV(y))^{1/p} < \infty$ for almost every $x$. Hence there exists a set of measure zero $E'' \subset B$ such that $(\int_B |v(x,y)|^p\,dV(y))^{1/p} < \infty$ for $x \notin E''$, i.e., $v_x(\cdot) = v(x,\,\cdot\,) \in L^p(B,dV)$ for all $x \notin E''$.

Let $E = E' \bigcup E''$. The measure of the set $E$ is zero, and for every $x \notin E$, $v(x,\,\cdot\,)$ defines a bounded linear functional $\varphi_x$ on $L^{p'}(B,dV)$ by

$$\varphi_x(h) = \int_B h(y)v(x,y)\,dV(y).$$

For every $j$, $\varphi_x(h_j) = 0$, and the density of $\{h_j\}_{j=1}^{\infty}$ in $L^{p'}(B,dV)$ implies that $\varphi_x \equiv 0$ as a linear functional on $L^{p'}(B,dV)$. Hence $v_x(\cdot) = 0$ in $L^p(B,dV)$, which means $(\int_B |v(x,y)|^p dV(y))^{1/p} = 0$ for all $x \notin E$. Then it follows that $v \equiv 0$ in $L^{p,p'}(B \times B)$

since $(\int_B (\int_B |v(x,y)|^p \, dV(y))^{p'/p} \, dV(x))^{1/p'} = 0$. Therefore $\varphi \equiv 0$ on $L^{p',p}(B \times B)$, which proves the lemma. $\square$

The following proposition shows that the integral operator $T$ with kernel $k$ in $L^{p',p}(B \times B)$ is compact.

**Proposition 2.14** *Let $p \in (1, \infty)$ and let $k \in L^{p',p}(B \times B)$. Then the integral operator $T : L^p(B, dV) \to L^p(B, dV)$, defined by*

$$(Tf)(x) = \int_B k(x,y) f(y) \, dV(y),$$

*is compact.*

Proof: Let $k \in L^{p',p}(B \times B)$. By Lemma 2.13, for every $\epsilon > 0$ there exist functions $g_j \in L^p(B, dV)$, $h_j \in L^{p'}(B, dV)$ and complex numbers $\alpha_j$, $j = 1, ..., m(\epsilon)$, such that $\|k - \sum_{j=1}^{m(\epsilon)} \alpha_j g_j \otimes h_j\|_{p',p} < \epsilon$.

Define an operator $T_\epsilon : L^p(B, dV) \to L^p(B, dV)$ by

$$(T_\epsilon f)(x) = \int_B (\sum_{j=1}^{m(\epsilon)} \alpha_j g_j(x) h_j(y)) f(y) \, dV(y)$$

for $f \in L^p(B, dV)$. $T_\epsilon$ is a finite rank operator since

$$T_\epsilon f = \sum_{j=1}^{m(\epsilon)} \alpha_j (\int_B h_j(y) f(y) \, dV(y)) g_j.$$

We also have

$$\|(T - T_\epsilon) f\|_p^p = \int_B |(T - T_\epsilon)(f)|^p \, dV(x)$$

$$= \int_B | \int_B (k(x,y) - \sum_{j=1}^{m(\epsilon)} \alpha_j g_j(x) h_j(y)) f(y) \, dV(y)|^p \, dV(x)$$

$$\leq (\int_B |f(y)|^p \, dV(y)) (\int_B (\int_B |k(x,y) - \sum_{j=1}^{m(\epsilon)} \alpha_j g_j(x) h_j(y)|^{p'} \, dV(y))^{p/p'} \, dV(x))$$

$$= \|f\|_p^p \|k - \sum_{j=1}^{m(\epsilon)} \alpha_j g_j \otimes h_j\|_{p',p}^p \leq \epsilon^p \|f\|_p^p.$$

Hence $\|T - T_\epsilon\| \leq \epsilon$, which implies that $T$ is a norm limit of finite rank operators, and therefore compact. □

In order to prove our main result we will need the following interpolation theorem given in [5]. (See Theorem 2.9, Chapter IV)

**Theorem 2.15** *Let $(X, \mu)$ and $(Y, \nu)$ be finite measure spaces. Suppose that $1 \leq q_j, r_j \leq \infty$, $(j = 0, 1)$, and let $T$ be a linear operator that satisfies*

$$T : L^{q_0}(\mu) \to L^{r_0}(\nu) \quad \textit{boundedly}$$

*and*

$$T : L^{q_1}(\mu) \to L^{r_1}(\nu) \quad \textit{compactly.}$$

*If $0 < \theta < 1$ and $q, r$ are defined by*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$$

*then*

$$T : L^{q}(\mu) \to L^{r}(\nu) \quad \textit{compactly.}$$

Now we can prove our main result.

**Theorem 2.16** *Let $p \in (1, \infty)$ and let $f$ be a continuous function on the closure of $B$. Then the Hankel operator $H_f : b^p(B) \to L^p(B, dV)$ is compact.*

Proof : By Corollary 2.10 $\mathcal{A} = \{ f \in C(\bar{B}) : H_f$ is compact $\}$ is a closed subalgebra of $C(\bar{B})$ . We want to show that $\mathcal{A} = C(\bar{B})$.

By the Stone-Weierstrass Theorem it will be enough to show that $H_{x_j}$ is compact for every $j = 1, ..., n$. Note that because of the symmetry with respect to the coordinates, it is enough to consider only $H_{x_1}$. Hence it is enough to show that $H_{x_1} : b^p(B) \to L^p(B, dV)$, defined by $H_{x_1}(u) = (I - Q)(x_1 u)$, is compact.

By Lemma 2.11

$$\begin{aligned}(H_{x_1}u)(x) &= (I-Q)(x_1u)(x)\\ &= \int_B (x_1-y_1)R_B(x,y)u(y)\,dV(y)\\ &= \int_B k_1(x,y)u(y)\,dV(y),\end{aligned}$$

where $k_1(x,y) = (x_1-y_1)R_B(x,y)$.

Let

$$I = \int_B \left(\int_B |k_1(x,y)|^{q'}\,dV(y)\right)^{q/q'} dV(x),$$

where

$$k_1(x,y) = \frac{(x_1-y_1)(n(1-|x|^2|y|^2)^2 - 4|x|^2|y|^2((1-|x|^2)(1-|y|^2)+|x-y|^2)}{nV(B)((1-|x|^2)(1-|y|^2)+|x-y|^2)^{1+n/2}}.$$

If we can show that $I$ is finite for some $q$, that would imply that $k_1 \in L^{q',q}(B\times B)$ and hence, by Proposition 2.14, it would follow that $H_{x_1} : b^q(B) \to L^q(B,dV)$ is compact. Then, using Theorem 2.15, we could prove that $H_{x_1} : b^p(B) \to L^p(B,dV)$ is compact.

In order to show that $I$ is finite, we divide the region of integration into three parts.

$$\begin{aligned} I \le\ & C(q)\{\underbrace{\int_B \left(\int_{|x-y|\le 1-|x|} |k_1(x,y)|^{q'}\,dV(y)\right)^{q/q'} dV(x)}_{I_1} \\ & + \underbrace{\int_B \left(\int_{|x-y|>1-|x|,|x-y|>1-|y|} |k_1(x,y)|^{q'}\,dV(y)\right)^{q/q'} dV(x)}_{I_2} \\ & + \underbrace{\int_B \left(\int_{1-|x|<|x-y|\le 1-|y|} |k_1(x,y)|^{q'}\,dV(y)\right)^{q/q'} dV(x)}_{I_3}\}. \end{aligned}$$

We used the following inequality : If $\alpha > 1$, then for arbitrary complex numbers $a$

and $b$

$$|a+b|^\alpha \le 2^{\alpha-1}(|a|^\alpha + |b|^\alpha). \tag{2.4}$$

To show the finiteness of $I_1, I_2$ and $I_3$ we will need the following relationships as well:

$$1-|x|^2|y|^2 = (1-|x|^2) + |x|^2(1-|y|^2) \tag{2.5}$$

and

$$(1-|x|^2)(1-|y|^2) + |x-y|^2 \ge \frac{1}{4}((1-|x|^2)^2 + (1-|y|^2)^2) \tag{2.6}$$

for $x, y \in B$.

In $I_1$, $|x-y| \le 1-|x|$, which implies $1-|y| \le 2(1-|x|)$, and using (2.5) and (2.6)

$$|k_1(x,y)| \le C_1(n)|x_1-y_1|\frac{(1-|x|)^2}{(1-|x|)^{2+n}} = C_1(n)\frac{|x_1-y_1|}{(1-|x|)^n}.$$

Hence

$$I_1 \le C_1(n,q)\int_B \left(\int_{|x-y|\le 1-|x|} \frac{|x_1-y_1|^{q'}}{(1-|x|)^{nq'}}\,dV(y)\right)^{q/q'} dV(x).$$

By the change of variables formula with $z = x-y$ we get

$$I_1 \le C_1(n,q)\int_B \frac{1}{(1-|x|)^{nq}}\left(\int_{|z|\le 1-|x|} |z_1|^{q'}\,dV(z)\right)^{q/q'} dV(x).$$

Using the polar coordinates formula for integration on $\mathbf{R}^n$ we have

$$\begin{aligned} I_1 &\le C_1'(n,q)\int_B \frac{1}{(1-|x|)^{nq}}\left(\int_0^{1-|x|} r^{n-1+q'}\,dr\right)^{q/q'} dV(x) \\ &\le C_1''(n,q)\int_B \frac{(1-|x|)^{(n+q')q/q'}}{(1-|x|)^{nq}}\,dV(x) \\ &\le C_1''(n,q)\int_B (1-|x|)^{q-n}\,dV(x). \end{aligned}$$

The last integral is finite if and only if $q > n-1$, and hence $I_1$ is finite for $q > n-1$.

Similarly for $I_2$, since $|x-y|>1-|x|$ and $|x-y|>1-|y|$, using (2.5)

$$|k_1(x,y)| \leq C_2(n)|x_1-y_1|\frac{|x-y|^2}{|x-y|^{2+n}} = C_2(n)\frac{|x_1-y_1|}{|x-y|^n}.$$

Hence

$$I_2 \leq C_2(n,q)\int_B\Big(\int_{|x-y|>1-|x|,|x-y|>1-|y|}\frac{|x_1-y_1|^{q'}}{|x-y|^{nq'}}\,dV(y)\Big)^{q/q'}\,dV(x).$$

The change of variables $z=x-y$ gives

$$I_2 \leq C_2(n,q)\int_B\Big(\int_{1-|x|<|z|\leq 2}\frac{|z_1|^{q'}}{|z|^{nq'}}\,dV(z)\Big)^{q/q'}\,dV(x).$$

Using the polar coordinates formula we have

$$I_2 \leq C_2'(n,q)\int_B\Big(\int_{1-|x|}^{2} r^{n-1+q'-nq'}\,dr\Big)^{q/q'}\,dV(x).$$

We consider two cases:

(i) $n-1+q'-nq' \neq -1$, i.e., $q \neq n$

Then

$$I_2 \leq C_2'(n,q)\int_B\Big(\frac{2^{n+q'-nq'}}{n+q'-nq'} - \frac{(1-|x|)^{n+q'-nq'}}{n+q'-nq'}\Big)^{q/q'}\,dV(x).$$

Using (2.4) we get

$$\begin{aligned} I_2 &\leq C_2''(n,q) + C_2'''(n,q)\int_B(1-|x|)^{(n+q'-nq')q/q'}\,dV(x) \\ &\leq C_2''(n,q) + C_2'''(n,q)\int_B(1-|x|)^{q-n}\,dV(x). \end{aligned}$$

The last integral is finite if and only if $q>n-1$, and hence $I_2$ is finite for $q>n-1$ and $q\neq n$.

(ii) $q=n$

Then $q/q'=n-1$ and we have

$$I_2 \leq C_2'(n)\int_B(\ln 2 - \ln(1-|x|))^{n-1}\,dV(x).$$

By (2.4) it follows

$$I_2 \leq C_2''(n) + C_2'''(n) \int_B |\ln(1-|x|)|^{n-1}\, dV(x),$$

and the last integral is finite.

Hence for every $q > n-1$, $I_2$ is finite.

To show the finiteness of $I_3$, we proceed in the similar way. Using $1-|x| < |x-y| \leq 1-|y|$, (2.5), and (2.6) we have

$$|k_1(x,y)| \leq C_3(n)|x_1 - y_1| \frac{(1-|y|)^2}{(1-|y|)^{2+n}} \leq C_3(n)\frac{|x_1-y_1|}{(1-|y|)^n} \leq C_3(n)\frac{|x_1-y_1|}{|x-y|^n}.$$

Now, as in the case of $I_2$,

$$I_3 \leq C_3(n,q) \int_B (\int_{1-|x|<|x-y|\leq 1-|y|} \frac{|x_1-y_1|^{q'}}{|x-y|^{nq'}}\, dV(y))^{q/q'}\, dV(x).$$

Using the change of variables formula with $z = x-y$ we have

$$I_3 \leq C_3(n,q) \int_B (\int_{1-|x|<|z|\leq 1} \frac{|z_1|^{q'}}{|z|^{nq'}}\, dV(z))^{q/q'}\, dV(x).$$

Using the polar coordinates formula we have

$$I_3 \leq C_3'(n,q) \int_B (\int_{1-|x|}^1 r^{n-1+q'-nq'}\, dr)^{q/q'}\, dV(x).$$

We consider the following two cases:

(i) $n-1+q'-nq' \neq -1$, i.e., $q \neq n$

Then

$$I_3 \leq C_3'(n,q) \int_B (\frac{1}{n+q'-nq'} - \frac{(1-|x|)^{n+q'-nq'}}{n+q'-nq'})^{q/q'}\, dV(x).$$

By (2.4) it follows that

$$I_3 \leq C_3''(n,q) + C_3'''(n,q) \int_B (1-|x|)^{(n+q'-nq')q/q'}\, dV(x).$$

The last integral is finite if and only if $q > n-1$, and hence $I_3$ is finite for $q > n-1$ and $q \neq n$.

(ii) $q = n$

Then $q/q' = n - 1$ and we have

$$I_3 \le C_3'(n) \int_B (\ln 1 - \ln(1 - |x|))^{n-1} \, dV(x),$$

i.e.,

$$I_3 \le C_3'(n) \int_B (-\ln(1 - |x|))^{n-1} \, dV(x),$$

and the above integral is finite.

Hence for every $q > n - 1$, $I_3$ is finite.

Therefore $k_1 \in L^{q',q}(B \times B)$ for $q > n - 1$. By Proposition 2.14 it follows that the operator $T_1 : L^q(B, dV) \to L^q(B, dV)$, defined by $(T_1 f)(x) = \int_B k_1(x, y) f(y) \, dV(y)$, is compact whenever $q > n - 1$.

In the Interpolation Theorem 2.15 let $X = Y = B$, $\mu = \nu = V$, $q_j = r_j$, $j = 0, 1$, and $T = H_{x_1} Q : L^q(B, dV) \to L^q(B, dV)$ for $q \in (1, \infty)$. Then $T$ is bounded on $L^q(B, dV)$, for $q \in (1, \infty)$ since $H_{x_1}$ and $Q$ are bounded for $1 < q < \infty$, and $T$ is compact for $q > n - 1$ since $T = T_1 Q$. Fix $q_0$ such that $1 < q_0 < p$ and $q_1$ such that $q_1 > \max\{n - 1, p\}$. By Theorem 2.15 $T$ maps $L^q(B, dV)$ compactly into $L^q(B, dV)$ for every $q$ such that $q_0 < q < q_1$, and hence for $q = p$.

Since $H_{x_1} : b^p(B) \to L^p(B, dV)$ is the restriction of $T : L^p(B, dV) \to L^p(B, dV)$ onto $b^p(B)$, it follows that $H_{x_1}$ is compact, completing the proof. □

We can now prove the following corollary.

**Corollary 2.17** *Let $f, g \in L^\infty(B, dV)$. If either $f$ or $g$ is in $C(\bar{B})$, then the operators $T_{fg} - T_f T_g$ and $T_f T_g - T_g T_f$ on $b^p(B)$, where $1 < p < \infty$, are compact.*

Proof: This follows from Theorem 2.16 and Lemma 2.8. □

Let $\mathcal{H}$ be a separable Hilbert space, $\{e_m\}_{m=1}^{\infty}$ an orthonormal basis for $\mathcal{H}$. A bounded operator $A$ on $\mathcal{H}$ is called *trace class* if and only if $\sum_{m=1}^{\infty}\langle |A|e_m, e_m\rangle < \infty$. We define the trace of $A$ to be

$$\text{tr}A = \sum_{m=1}^{\infty}\langle Ae_m, e_m\rangle.$$

This sum is finite and independent of the orthonormal basis $\{e_m\}_{m=1}^{\infty}$. (See [9], Theorem 6.24.)

For any bounded linear operator $T$ on $b^2(B)$ define

$$\|T\|_r = \{\text{tr}(T^*T)^{r/2}\}^{1/r}, \quad 1 \leq r < \infty.$$

The Schatten $r$-class $\mathcal{S}_r$ is the set of operators $T$ with $\|T\|_r < \infty$.

**Proposition 2.18** *If* $r > \max\{n-1, 2\}$, *the Hankel operators* $H_{x_j}$ *mapping* $b^2(B)$ *to* $b^2(B)^{\perp}$, $j = 1, ..., n$, *belong to Schatten* $r$*-class* $\mathcal{S}_r$.

Proof: By the Hausdorff-Young Theorem for integral operators [11], (see also [6], Theorem A), and Proposition 2.14 it follows that for $r > \max\{n-1, 2\}$

$$\|H_{x_j}\|_r \leq (\|k_j\|_{r',r}\|k_j^*\|_{r',r})^{1/2} < \infty,$$

where $k_j^*(x,y) = \overline{k_j(y,x)}$ and $k_j(x,y) = (x_j - y_j)R_B(x,y)$, $j = 1, ..., n$. Thus $H_{x_j} \in \mathcal{S}_r$ for $r > \max\{n-1, 2\}$ and $j = 1, ..., n$. □

Remark: If we want to show Theorem 2.16 only for $p = 2$ we could use Proposition 2.18 to show that $H_{x_j} : b^2(B) \to b^2(B)^{\perp}$ is compact for $j = 1, ..., n$.

# Chapter 3

# Hankel Operators on $\mathcal{H}_m(B)$

The Hankel operator $H_f : b^2(B) \to b^2(B)^\perp$ is defined by $H_f(u) = (I - Q)(fu)$. We consider the operators $H_{x_j}^* H_{x_j} : b^2(B) \to b^2(B)$, $j = 1, ..., n$ and we show that $H_{x_j}^* H_{x_j}$ maps $\mathcal{H}_m(B)$ into $\mathcal{H}_m(B)$ for every $m \geq 0$ and $j = 1, ..., n$.

We will need the following lemmas. (See [3], Exercise 1.20 and Corollary 5.23.)

**Lemma 3.1** *A polynomial $p$ is homogeneous of degree $m$ if and only if $\nabla p \cdot x = mp$.*

**Lemma 3.2** *If $u$ is a harmonic function on $B$, then there exist $h_m \in \mathcal{H}_m(\mathbf{R}^n)$ such that*

$$u(x) = \sum_{m=0}^{\infty} h_m(x)$$

*for all $x \in B$, the series converging absolutely and uniformly on compact subsets of $B$.*

Now we can show the following proposition.

**Proposition 3.3** *Let $u$ be harmonic. Then $\Delta^{k+1}(|x|^{2k}u) = 0$ for every $k \geq 0$.*

Proof: Lemma 3.2 implies that $u(x) = \sum_{m=0}^{\infty} h_m(x)$ for $h_m \in \mathcal{H}_m(\mathbf{R}^n)$. Hence it is enough to show that

$$\Delta^{k+1}(|x|^{2k}h_m) = 0$$

for $h_m \in \mathcal{H}_m(\mathbf{R}^n)$, $m \geq 0$ and $k \geq 0$.

We will use the following product rule. (See [3], page 13.)

$$\Delta(uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u. \tag{3.1}$$

Let $h_m \in \mathcal{H}_m(\mathbf{R}^n)$. Then

$$\Delta(|x|^{2k}h_m) = \Delta(|x|^{2k})h_m + 2\nabla|x|^{2k} \cdot \nabla h_m.$$

We have

$$\Delta|x|^{2k} = 2k(n + 2(k-1))|x|^{2(k-1)}$$

and

$$\nabla|x|^{2k} = 2k|x|^{2(k-1)}x.$$

By Lemma 3.1 it follows that

$$\Delta(|x|^{2k}h_m) = 2k(n + 2(k-1) + 2m)|x|^{2(k-1)}h_m.$$

After applying the Laplacian $k$-times to $|x|^{2k}h_m$ we have

$$\Delta^k(|x|^{2k}h_m) = 2^k k! \prod_{j=1}^{k}(n + 2(k-j) + 2m)h_m.$$

Therefore $\Delta^{k+1}(|x|^{2k}h_m) = 0$. □

We also have the next proposition.

**Proposition 3.4** *Let $u \in \mathcal{H}_m(\mathbf{R}^n)$. Then*

$$x_j u(x) = h_{m+1}(x) + |x|^2 h_{m-1}(x) \tag{3.2}$$

*and*

$$x_j^2 u(x) = h_{m+2}(x) + |x|^2 h_m(x) + |x|^4 h_{m-2}(x) \tag{3.3}$$

*where $h_i \in \mathcal{H}_i(\mathbf{R}^n)$, $i = m-2,\ m-1,\ m,\ m+1,\ m+2$, $j = 1, \ldots, n$.*

Proof: Theorem 1.1 implies that

$$x_j u(x) = h_{m+1}(x) + |x|^2 h_{m-1}(x) + \ldots + |x|^{2k} h_{m+1-2k}(x),$$

where $k = [\frac{m+1}{2}]$ and $h_{m+1-2j} \in \mathcal{H}_{m+1-2j}(\mathbf{R}^n)$, $j = 0, \ldots, k$.

For $m \leq 2$ (3.2) is obvious. Suppose now that $m \geq 3$. We first show that $x_j u$ is orthogonal to $|x|^{2k}\mathcal{P}_{m+1-2k}(\mathbf{R}^n)$ with respect to the inner product $\langle\ ,\ \rangle_{m+1}$ for $k \geq 2$ and $j = 1, \ldots, n$.

Let $q \in \mathcal{P}_{m+1-2k}(\mathbf{R}^n)$ and let $k \geq 2$. Then

$$\langle\ |x|^{2k} q, x_j u\ \rangle_{m+1} = q(D)[\Delta^k(x_j \bar{u})].$$

Using (3.1) we have

$$\Delta(x_j \bar{u}) = \Delta x_j \bar{u} + 2\nabla x_j \cdot \nabla \bar{u} + x_j \Delta \bar{u} = 2\frac{\partial \bar{u}}{\partial x_j}$$

and hence

$$\Delta^2(x_j \bar{u}) = \Delta(2\frac{\partial \bar{u}}{\partial x_j}) = 2\frac{\partial}{\partial x_j}(\Delta \bar{u}) = 0.$$

Thus $q(D)[\Delta^k(x_j \bar{u})] = 0$ for $k \geq 2$; i.e., $x_j u$ is orthogonal to $|x|^{2k}\mathcal{P}_{m+1-2k}(\mathbf{R}^n)$ with respect to $\langle\ ,\ \rangle_{m+1}$.

Also $h_{m+1}$ and $|x|^2 h_{m-1}$ are orthogonal to $|x|^{2k}\mathcal{P}_{m+1-2k}(\mathbf{R}^n)$ with respect to the inner product $\langle\ ,\ \rangle_{m+1}$ for $k \geq 2$. To see this let $q \in \mathcal{P}_{m+1-2k}(\mathbf{R}^n)$. Then

$$\langle |x|^{2k} q, h_{m+1} \rangle_{m+1} \ = \ \langle |x|^{2k-2} q, \Delta h_{m+1} \rangle_{m-1} \ = 0$$

since $h_{m+1}$ is harmonic. Proposition 3.3 implies that $\Delta^2(|x|^2 h_{m-1}) = 0$ and hence

$$\langle |x|^{2k} q, |x|^2 h_{m-1} \rangle_{m+1} \ = \ \langle |x|^{2k-4} q, \Delta^2(|x|^2 h_{m-1}) \rangle_{m-3} \ = 0.$$

Therefore $x_j u - h_{m+1} - |x|^2 h_{m-1}$ is orthogonal to $|x|^4\mathcal{P}_{m-3}(\mathbf{R}^n)$ with respect to $\langle\ ,\ \rangle_{m+1}$ and $x_j u - h_{m+1} - |x|^2 h_{m-1} = |x|^4 h_{m-3} + \ldots + |x|^{2k} h_{m+1-2k} \in |x|^4\mathcal{P}_{m-3}(\mathbf{R}^n)$, and (3.2) is proved.

To show (3.3) we use (3.2) in the following way:

$$\begin{aligned} x_j^2 u(x) &= x_j(h_{m+1}(x) + |x|^2 h_{m-1}(x)) \\ &= x_j h_{m+1}(x) + |x|^2 x_j h_{m-1}(x) \\ &= a_{m+2}(x) + |x|^2 a_m(x) + |x|^2(b_m(x) + |x|^2 b_{m-2}(x)) \\ &= h_{m+2}(x) + |x|^2 h_m(x) + |x|^4 h_{m-2}(x). \end{aligned}$$

Hence (3.3) holds. □

Now we can prove the following theorem.

**Theorem 3.5** *For $m \geq 0$ and $j = 1, \ldots, n$ the operator $H^*_{x_j} H_{x_j}$ maps $\mathcal{H}_m(B)$ into $\mathcal{H}_m(B)$.*

Proof: Let $u \in \mathcal{H}_m(B)$. Then

$$H^*_{x_j} H_{x_j} u = H^*_{x_j}((I - Q)(x_j u)) = Q(x_j(x_j u - Q(x_j u))).$$

By Proposition 3.4

$$x_j u(x) = h_{m+1}(x) + |x|^2 h_{m-1}(x),$$

where $h_i \in \mathcal{H}_i(B)$ for $i = m-1,\ m+1$. Then by Theorem 1.12

$$\begin{aligned} Q(x_j u)(x) &= \frac{1}{nV(B)} \sum_{k=0}^{m+1} (n+2k) \int_B Z_k(x,y) y_j u(y)\, dV(y) \\ &= \frac{1}{nV(B)} \sum_{k=0}^{m+1} (n+2k) \int_B Z_k(x,y)(h_{m+1}(y) + |y|^2 h_{m-1}(y))\, dV(y) \\ &= h_{m+1}(x) + \frac{n+2(m-1)}{n+2m} h_{m-1}(x). \end{aligned}$$

We used the polar coordinates formula, Theorem 1.3 and (1.2).

Hence

$$(x_j u - Q(x_j u))(x) = (|x|^2 - \frac{n+2(m-1)}{n+2m}) h_{m-1}(x)$$

and

$$x_j(x_j u - Q(x_j u))(x) = x_j(|x|^2 - \frac{n+2(m-1)}{n+2m})h_{m-1}(x)$$

$$= (|x|^2 - \frac{n+2(m-1)}{n+2m})(h_m(x) + |x|^2 h_{m-2}(x)),$$

where $h_i \in \mathcal{H}_i(B)$ for $i = m,\ m-2$. In the same way as above

$$H_{x_j}^* H_{x_j} u(x) = Q(x_j(x_j u - Q(x_j u)))(x)$$

$$= \frac{1}{nV(B)} \sum_{k=0}^{m+2} (n+2k) \int_B Z_k(x,y) y_j (y_j u(y) - Q(y_j u)(y))\, dV(y)$$

$$= \frac{1}{nV(B)} \sum_{k=0}^{m+2} (n+2k) \int_B Z_k(x,y)(|y|^2 - \frac{n+2(m-1)}{n+2m})h_m(y)\, dV(y)$$

$$+ \frac{1}{nV(B)} \sum_{k=0}^{m+2} (n+2k) \int_B Z_k(x,y)(|y|^2 - \frac{n+2(m-1)}{n+2m})|y|^2 h_{m-2}(y)\, dV(y)$$

$$= (n+2m)(\int_0^1 r^{n-1+2+2m}\, dr - \frac{n+2(m-1)}{n+2m} \int_0^1 r^{n-1+2m}\, dr)h_m(x)$$

$$+ (n+2(m-2))(\int_0^1 r^{n-1+2m}\, dr - \frac{n+2(m-1)}{n+2m} \int_0^1 r^{n-3+2m}\, dr)h_{m-2}(x)$$

$$= (n+2m)h_m(x)(\frac{1}{n+2+2m} - \frac{n+2(m-1)}{n+2m}\frac{1}{n+2m})$$

$$+ (n+2(m-2))h_{m-2}(x)(\frac{1}{n+2m} - \frac{n+2(m-1)}{n+2m}\frac{1}{n+2m-2})$$

$$= \frac{4}{(n+2m)(n+2m+2)} h_m(x).$$

Therefore $H_{x_j}^* H_{x_j}(\mathcal{H}_m(B)) \subset \mathcal{H}_m(B)$. □

Since $b^2(B) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(B)$ and $H_{x_j}^* H_{x_j}(\mathcal{H}_m(B)) \subset \mathcal{H}_m(B)$ it is enough to study the restriction of the operators $H_{x_j}^* H_{x_j}$ to $\mathcal{H}_m(B)$. Note that $H_{x_j}^* H_{x_j}$ is self-adjoint so its restriction to $\mathcal{H}_m(B)$ has an orthonormal basis of eigenvectors.

Motivated by some calculations obtained with the aid of *Mathematica*, we have a conjecture concerning this orthonormal basis of eigenvectors. It will be stated for

$H^*_{x_1}H_{x_1}$, but it holds for all $H^*_{x_j}H_{x_j}$, $j = 1, ..., n$, with an appropriate choice of basis for $\mathcal{H}_m(B)$.

**Conjecture 3.6** *Let $n > 2$. Then the eigenvalues of the restriction of $H^*_{x_1}H_{x_1}$ to $\mathcal{H}_m(B)$ are:*

$$\frac{4(m-k)(m+n+k-3)}{(2m+n-4)(2m+n-2)(2m+n)(2m+n+2)},$$

*for $k = 0, 1, ..., m$, with multiplicities*

$$\binom{n+k-2}{n-2} - \binom{n+k-4}{n-2},$$

*where we define $\binom{l}{r} = 0$ for $l < r$. The corresponding eigenvectors are obtained from the basis $\{K[D^\alpha|x|^{2-n}] : |\alpha| = m,\ \alpha_1 = m-k,\ \alpha_n = 0 \text{ or } 1,\ k = 0, ..., m\}$, using the Gram-Schmidt Orthogonalization Process.*

We can prove this conjecture for $k = 0,\ 1$.

**Lemma 3.7** *Let $n > 2$ and $m \geq 1$. Then for $x \neq 0$*

$$|x|^2 D_1^{m+1}|x|^{2-n} = -(2m+n-2)x_1 D_1^m|x|^{2-n} - m(m+n-3)D_1^{m-1}|x|^{2-n}. \quad (3.4)$$

Proof: Induction on $m$. It is easy to see that (3.4) holds for $m = 1$. Suppose it is true for $m = k$. Then we have

$$|x|^2 D_1^{k+1}|x|^{2-n} = -(2k+n-2)x_1 D_1^k|x|^{2-n} - k(k+n-3)D_1^{k-1}|x|^{2-n}.$$

Differentiating with respect to $x_1$ we get

$$2x_1 D_1^{k+1}|x|^{2-n} + |x|^2 D_1^{k+2}|x|^{2-n} = -(2k+n-2)D_1^k|x|^{2-n}$$
$$-(2k+n-2)x_1 D_1^{k+1}|x|^{2-n} - k(k+n-3)D_1^k|x|^{2-n}.$$

Hence

$$|x|^2 D_1^{k+1}|x|^{2-n} = -(2k+n)x_1 D_1^k |x|^{2-n} - (k+1)(k+n-2)D_1^k|x|^{2-n},$$

and the lemma is proved. □

**Corollary 3.8** *Let $n > 2$ and $m \geq 1$. Then*

$$K[D_1^{m+1}|x|^{2-n}] = -(2m+n-2)x_1 K[D_1^m|x|^{2-n}] - m(m+n-3)|x|^2 K[D_1^{m-1}|x|^{2-n}].$$

Proof: Using the definition of Kelvin transform and Lemma 3.7 we get

$$\begin{aligned} K[D_1^{m+1}|x|^{2-n}] &= |x|^{2-n} D_1^{m+1}|x^*|^{2-n} \\ &= |x|^{2-n}\frac{1}{|x^*|^2}(|x^*|^2 D_1^{m+1}|x^*|^{2-n}) \\ &= |x|^{2-n+2}(-(2m+n-2)x_1^* D_1^m|x^*|^{2-n} - m(m+n-3)D_1^{m-1}|x^*|^{2-n}) \\ &= -(2m+n-2)x_1 K[D_1^m|x|^{2-n}] - m(m+n-3)|x|^2 K[D_1^{m-1}|x|^{2-n}], \end{aligned}$$

and the corollary is proved. □

Remark: Corollary 3.8 implies that

$$x_1 K[D_1^m|x|^{2-n}] = -\frac{1}{2m+n-2}(K[D_1^{m+1}|x|^{2-n}] + m(m+n-3)|x|^2 K[D_1^{m-1}|x|^{2-n}]).$$

Now we can prove the following theorem.

**Theorem 3.9** *Let $n > 2$. Then $K[D_1^m|x|^{2-n}]$ is an eigenvector of $H_{x_1}^* H_{x_1}$ with eigenvalue*

$$\frac{4m(m+n-3)}{(2m+n-4)(2m+n-2)(2m+n)(2m+n+2)}.$$

Proof: From Proposition 3.4 using (3.2) it follows that for $u \in \mathcal{H}_m(B)$ we have

$$x_1 u(x) = h_{m+1}(x) + |x|^2 h_{m-1}(x)$$

and

$$x_1 h_{m-1}(x) = h_m(x) + |x|^2 h_{m-2}(x), \tag{3.5}$$

where $h_i \in \mathcal{H}_m(B)$, $i = m-2,\ m-1,\ m,\ m+1$.

The proof of Theorem 3.5 shows that

$$H^*_{x_1} H_{x_1} u(x) = \frac{4}{(2m+n)(2m+n+2)} h_m(x).$$

Hence it is enough to show that in the case $u(x) = K[D_1^m |x|^{2-n}]$ the corresponding $h_m$ from (3.5) is given by

$$\frac{m(m+n-3)}{(2m+n-4)(2m+n-2)} u.$$

From the remark after Corollary 3.8 it follows that

$$h_{m-1}(x) = -\frac{m(m+n-3)}{2m+n-2} K[D_1^{m-1}|x|^{2-n}].$$

Similarly, the remark after Corollary 3.8 implies that

$$x_1 h_{m-1}(x) = -\frac{m(m+n-3)}{2m+n-2}\left(-\frac{1}{2(m-1)+n-2}\right) K[D_1^m |x|^{2-n}] + |x|^2 h_{m-2}(x).$$

Therefore

$$h_m(x) = \frac{m(m+n-3)}{2m+n-2}\frac{1}{2m+n-4} u(x),$$

which proves the theorem. □

We can also show that $K[D_1^{m-1} D_j |x|^{2-n}]$, $j = 2, ..., n$ are eigenvectors of $H^*_{x_1} H_{x_1}$ with the eigenvalue

$$\frac{4(m-1)(m+n-2)}{(2m+n-4)(2m+n-2)(2m+n)(2m+n+2)}.$$

To show this we will need the following lemmas.

**Lemma 3.10** *Let $n > 2$ and $m \geq 2$. Then for $j = 2, ..., n$ and $x \neq 0$*

$$\begin{aligned} x_1 D_1^{m-1} D_j |x|^{2-n} &= \frac{-1}{2m+n-4}\left(|x|^2 D_1^m D_j |x|^{2-n} + 2x_j D_1^m |x|^{2-n}\right. \\ &\quad \left. + (m-1)(m+n-4) D_1^{m-2} D_j |x|^{2-n}\right). \end{aligned}$$

Proof: Let $n > 2$. From Lemma 3.7 we have

$$|x|^2 D_1^m |x|^{2-n} = -(2m+n-4)x_1 D_1^{m-1}|x|^{2-n} - (m-1)(m+n-4)D_1^{m-2}|x|^{2-n}.$$

Differentiating with respect to $x_j$ gives

$$\begin{aligned} 2x_j D_1^m |x|^{2-n} + |x|^2 D_1^m D_j |x|^{2-n} &= -(2m+n-4)x_1 D_1^{m-1} D_j |x|^{2-n} \\ &\quad -(m-1)(m+n-4)D_1^{m-2} D_j |x|^{2-n}. \end{aligned}$$

Therefore

$$\begin{aligned} x_1 D_1^{m-1} D_j |x|^{2-n} &= \frac{-1}{2m+n-4}(|x|^2 D_1^m D_j |x|^{2-n} + 2x_j D_1^m |x|^{2-n} \\ &\quad +(m-1)(m+n-4)D_1^{m-2} D_j |x|^{2-n}), \end{aligned}$$

and the lemma is proved. □

**Corollary 3.11** *Let $n > 2$. Then for $j = 2, ..., n$*

$$\begin{aligned} x_1 K[D_1^{m-1} D_j |x|^{2-n}] &= \frac{-1}{2m+n-4}(K[D_1^m D_j |x|^{2-n}] + 2x_j K[D_1^m |x|^{2-n}] \\ &\quad +(m-1)(m+n-4)|x|^2 K[D_1^{m-2} D_j |x|^{2-n}]). \end{aligned}$$

Proof: From Lemma 3.10 we have

$$\begin{aligned} x_1^* D_1^{m-1} D_j |x^*|^{2-n} &= \frac{-1}{2m+n-4}(|x^*|^2 D_1^m D_j |x^*|^{2-n} + 2x_j^* D_1^m |x^*|^{2-n} \\ &\quad +(m-1)(m+n-4)D_1^{m-2} D_j |x^*|^{2-n}). \end{aligned}$$

The Kelvin transform gives

$$\begin{aligned} \frac{x_1}{|x|^2} K[D_1^{m-1} D_j |x|^{2-n}] &= \frac{-1}{2m+n-4}(\frac{1}{|x|^2} K[D_1^m D_j |x|^{2-n}] + 2\frac{x_j}{|x|^2} K[D_1^m |x|^{2-n}] \\ &\quad +(m-1)(m+n-4)K[D_1^{m-2} D_j |x|^{2-n}]). \end{aligned}$$

Therefore

$$\begin{aligned} x_1 K[D_1^{m-1} D_j |x|^{2-n}] &= \frac{-1}{2m+n-4}(K[D_1^m D_j |x|^{2-n}] + 2x_j K[D_1^m |x|^{2-n}] \\ &\quad +(m-1)(m+n-4)|x|^2 K[D_1^{m-2} D_j |x|^{2-n}]), \end{aligned}$$

which finishes the proof. □

**Lemma 3.12** *Let $n > 2$, $m \geq 2$ and $j = 2, ..., n$. Then for $x \neq 0$*

$$x_j D_1^m |x|^{2-n} = \frac{-1}{2m+n-2}(|x|^2 D_1^m D_j |x|^{2-n} - m(m-1) D_1^{m-2} D_j |x|^{2-n}). \qquad (3.6)$$

Proof: Induction on $m$. It is easy to see that the statement holds for $m = 2$. Suppose it holds for $m = k$. Hence we have

$$x_j D_1^k |x|^{2-n} = \frac{-1}{2k+n-2}(|x|^2 D_1^k D_j |x|^{2-n} - k(k-1) D_1^{k-2} D_j |x|^{2-n}).$$

Differentiating with respect to $x_1$ gives

$$\begin{aligned} x_j D_1^{k+1} |x|^{2-n} &= \frac{-1}{2k+n-2}(2x_1 D_1^k D_j |x|^{2-n} + |x|^2 D_1^{k+1} D_j |x|^{2-n} \\ &\quad -k(k-1) D_1^{k-1} D_j |x|^{2-n}). \end{aligned}$$

Lemma 3.10 implies

$$\begin{aligned} x_j D_1^{k+1} |x|^{2-n} &= \frac{-1}{2k+n-2}(\frac{-2}{2k+n-2}(|x|^2 D_1^{k+1} D_j |x|^{2-n} + 2x_j D_1^{k+1} |x|^{2-n} \\ &\quad +k(k+n-3) D_1^{k-1} D_j |x|^{2-n}) + |x|^2 D_1^{k+1} D_j |x|^{2-n} \\ &\quad -k(k-1) D_1^{k-1} D_j |x|^{2-n}). \end{aligned}$$

Thus we have

$$(1 - \frac{4}{(2k+n-2)^2}) x_j D_1^{k+1} |x|^{2-n} = \frac{-1}{2k+n-2}(\frac{2k+n-4}{2k+n-2}|x|^2 D_1^{k+1} D_j |x|^{2-n}$$
$$-\frac{2k(k+n-3) + k(k-1)(2k+n-2)}{2k+n-2} D_1^{k-1} D_j |x|^{2-n}).$$

In other words,

$$x_j D_1^{k+1} |x|^{2-n} = \frac{-1}{2k+n}(|x|^2 D_1^{k+1} D_j |x|^{2-n} - k(k+1) D_1^{k-1} D_j |x|^{2-n}),$$

and the lemma is proved. □

**Corollary 3.13** *Let $n > 2$ and $j = 2, ..., n$. Then*

$$x_j K[D_1^m |x|^{2-n}] = -\frac{1}{2m+n-2} K[D_1^m D_j |x|^{2-n}] + \frac{m(m-1)}{2m+n-2} |x|^2 K[D_1^{m-2} D_j |x|^{2-n}].$$

Proof: By Lemma 3.12

$$x_j^* D_1^m |x^*|^{2-n} = \frac{-1}{2m+n-2} (|x^*|^2 D_1^m D_j |x^*|^{2-n} - m(m-1) D_1^{m-2} D_j |x^*|^{2-n}).$$

The Kelvin transform gives

$$\frac{x_j}{|x|^2} K[D_1^m |x|^{2-n}] = \frac{-1}{2m+n-2} (\frac{1}{|x|^2} K[D_1^m D_j |x|^{2-n}] - m(m-1) K[D_1^{m-2} D_j |x|^{2-n}]).$$

Therefore

$$x_j K[D_1^m |x|^{2-n}] = \frac{-1}{2m+n-2} (K[D_1^m D_j |x|^{2-n}] - m(m-1)|x|^2 K[D_1^{m-2} D_j |x|^{2-n}]),$$

which proves the corollary. □

Now we can prove the following theorem.

**Theorem 3.14** *Let $n > 2$ and $j = 2, ..., n$. Then $K[D_1^{m-1} D_j |x|^{2-n}]$ is an eigenvector of $H_{x_1}^* H_{x_1}$ with eigenvalue*

$$\frac{4(m-1)(m+n-2)}{(2m+n-4)(2m+n-2)(2m+n)(2m+n+2)}.$$

Proof: As in the proof of Theorem 3.9, we know that for $u \in \mathcal{H}_m(B)$

$$x_1 u(x) = h_{m+1}(x) + |x|^2 h_{m-1}(x)$$

and

$$x_1 h_{m-1}(x) = h_m(x) + |x|^2 h_{m-2}(x), \tag{3.7}$$

where $h_i \in \mathcal{H}_i(B)$, $i = m-2,\ m-1,\ m,\ m+1$. Also, the proof of Theorem 3.5 implies that

$$H_{x_1}^* H_{x_1} u(x) = \frac{4}{(2m+n)(2m+n+2)} h_m(x).$$

Hence it is enough to show that in the case $u = K[D_1^{m-1} D_j |x|^{2-n}]$ the corresponding $h_m$ from (3.7) is given by

$$\frac{(m-1)(m+n-2)}{(2m+n-4)(2m+n-2)} u.$$

Using Corollary 3.11 and Corollary 3.13 we have

$$\begin{aligned} x_1 K[D_1^{m-1} D_j |x|^{2-n}] &= -\frac{1}{2m+n-4} K[D_1^m D_j |x|^{2-n}] \\ &\quad -\frac{2}{2m+n-4}\Big(-\frac{1}{2m+n-2} K[D_1^m D_j |x|^{2-n}] \\ &\quad +\frac{m(m-1)}{2m+n-2} |x|^2 K[D_1^{m-2} D_j |x|^{2-n}]\Big) \\ &\quad -\frac{(m-1)(m+n-4)}{2m+n-4} |x|^2 K[D_1^{m-2} D_j |x|^{2-n}]. \end{aligned}$$

Thus the corresponding $h_{m-1}$ is

$$\begin{aligned} h_{m-1}(x) &= \Big(-\frac{2m(m-1)}{(2m+n-4)(2m+n-2)} - \frac{(m-1)(m+n-4)}{2m+n-4}\Big) K[D_1^{m-2} D_j |x|^{2-n}] \\ &= -\frac{2m(m-1) + (m-1)(m+n-4)(2m+n-2)}{(2m+n-4)(2m+n-2)} K[D_1^{m-2} D_j |x|^{2-n}] \\ &= -\frac{(m-1)(m+n-2)}{(2m+n-2)} K[D_1^{m-2} D_j |x|^{2-n}]. \end{aligned}$$

Using Corollary 3.11 and Corollary 3.13 we get

$$\begin{aligned} x_1 h_{m-1}(x) &= -\frac{(m-1)(m+n-2)}{2m+n-2} x_1 K[D_1^{m-2} D_j |x|^{2-n}] \\ &= -\frac{(m-1)(m+n-2)}{2m+n-2}\Big(-\frac{1}{2m+n-6} K[D_1^{m-1} D_j |x|^{2-n}] \end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{2m+n-6}\Big(-\frac{1}{2m+n-4}K[D_1^{m-1}D_j|x|^{2-n}] \\
&+ \frac{(m-1)(m-2)}{2m+n-4}|x|^2K[D_1^{m-3}D_j|x|^{2-n}]\Big) \\
&- \frac{(m-2)(m+n-5)}{2m+n-6}|x|^2K[D_1^{m-3}D_j|x|^{2-n}]\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
h_m(x) &= -\frac{(m-1)(m+n-2)}{2m+n-2}\Big(-\frac{1}{2m+n-6} \\
&+ \frac{2}{(2m+n-6)(2m+n-4)}\Big)K[D_1^{m-1}D_j|x|^{2-n}] \\
&= \frac{(m-1)(m+n-2)}{(2m+n-4)(2m+n-2)}K[D_1^{m-1}D_j|x|^{2-n}],
\end{aligned}
$$

which finishes the proof. □

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