MICHIGAN STATE UNIVERSITY LIBRARIES
3 1293 01021 9628

This is to certify that the

dissertation entitled

The spectral flow, the Maslov index, and decompositions of manifolds

presented by

Liviu I. Nicolaescu

has been accepted towards fulfillment of the requirements for

Ph.D. degree in Mathematics

Thomas Parker
Major professor

Date May 31 1994

MSU is an Affirmative Action/Equal Opportunity Institution

O-12771

LIBRARY
Michigan State
University

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.

DATE DUE	DATE DUE	DATE DUE

MSU Is An Affirmative Action/Equal Opportunity Institution
c:\circ\datedue.pm3-p.1

THE SPECTRAL FLOW,THE MASLOV INDEX AND DECOMPOSITIONS OF MANIFOLDS

By

Liviu I. Nicolaescu

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1994

Thomas Parker, Major Professor

ABSTRACT

THE SPECTRAL FLOW,THE MASLOV INDEX AND DECOMPOSITIONS OF MANIFOLDS

BY

LIVIU I. NICOLAESCU

Let $\mathcal{E} \to M$ be a Clifford bundle over a compact oriented Riemann manifold (M,g) and let $D(t) : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ be a path of selfadjoint Dirac operators. M is divided into two manifolds-with-boundary by a hypersurface $\Sigma \subset M$. Set $\mathcal{E}_0 = \mathcal{E}|_\Sigma$, $D_j(t) = D(t)|_{M_j}$ $\Lambda_j(t) = \ker D_j(t)|_\Sigma$ j=1,2. The Clifford multiplication defines a symplectic structure in $L^2(\mathcal{E}_0)$ such that $\Lambda_j(t)$ are (infinite dimensional) lagrangian subspaces. The main result of this thesis shows that the spectral flow of the family D(t) is equal to the Maslov index of the (continuous) path $t \mapsto (\Lambda_1(t), \Lambda_2(t))$. We then show that an adiabatic deformation of the neck reduces the computation of the Maslov index to a finite dimensional situation.

To my parents and my wife

ACKNOWLEDGEMENT

I am deeply indebted to my advisor, Professor Thomas Parker. The conversations with him have dramatically changed the way I view mathematics. His patience, insights and advises were and will always be an excellent guide and support.

It makes me a great pleasure to thank Professor Ronald Fintushel. The discussions with him taught me not only gauge theory but they always transmitted me part of his enthusiasm and passion for mathematics. I spent many exciting hours with Professor Jon Wolfson discussing about symplectic geometry and mathematics in general. I want to thank him for his advises and encouragements which meant a lot for an aspiring grad student. While working on the results in this thesis I benefitted from the correspondence with Kryzstof Wojciechovski who supplied me with valuable materials and whose interest in my research kept me going. I want to thank both him and Paul Kirk for the stimulating days I spent in Indiana.

Chairman Richard Phillips always has had an open door for graduate students in general and for me in particular. I want to thank him for his support during my studies at MSU. This great MSU experience would not have happend if it were not for Professor Charles Seebeck. He took a chance with me by bringing me here and I am very much indebted to him. His support and advise during my first year here meant very much to me.

I owe very much to my wife who stood by me at all times. Her warmth helped me go through many rainy days. At last but not the least I want to thank my parents for their love and support they invested in me which shaped the person I am now.

Contents

List of Figures

Chapter 1

Introduction

Consider a closed, compact oriented Riemann manifold (M,g) and a Clifford bundle $\mathcal{E} \to M$ over M. The spectral flow of a smooth path of selfadjoint Dirac operators $D^t : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is the integer obtained by counting, with sign, the number of eigenvalues of D^t that cross zero as t varies; it is a homotopy invariant of the path (cf. [AS]). The aim of this thesis is to describe the spectral flow in terms of a decomposition of the manifold.

More precisely, suppose that M is divided into two manifolds-with-boundary M_1 and M_2 by an oriented hypersurface $\Sigma \subset M$. Assume that in a tubular neighborhood N of Σ the metric is a product and the operators D^t have the "cylindrical" form

$$D^t = c(ds)(\partial/\partial s + D_0^t) \tag{1.1}$$

where s is the normal coordinate in N, c(ds) is the Clifford multiplication by ds, and D_0^t is independent of s. Set $\mathcal{E}_0 = \mathcal{E}|_\Sigma$ and denote by D_1^t and D_2^t the restriction of D^t to M_1 and M_2.

The kernels of D_j^t are infinite dimensional spaces of solutions of $D_j^t\psi = 0$ on

M_j. Restriction to Σ gives the "Cauchy data spaces"(CD spaces)

$$\Lambda_1(t) = \operatorname{Ker} D_1^t|_\Sigma, \qquad \Lambda_2(t) = \operatorname{Ker} D_2^t|_\Sigma$$

in $L^2(\mathcal{E}_0)$. Note that the intersection $\Lambda_1(t) \cap \Lambda_2(t)$ is the finite dimensional space of solutions of $D^t\psi = 0$ on M.

This setup has a rich symplectic structure. Multiplication by c(ds) introduces a complex structure in $L^2(\mathcal{E}_0)$ and hence a symplectic structure in this space. The CD spaces $\Lambda_j(t)$ are then infinite dimensional lagrangian subspaces of $L^2(\mathcal{E}_0)$ that vary smoothly with t, and the pair $(\Lambda_1(t), \Lambda_2(t))$ is a Fredholm pair (as defined in chapter 2). As in the finite dimensional case, one can associate to a path of Fredholm pairs of lagrangians an integer called the Maslov index. The main result of this thesis (Theorem 5.14) states that this Maslov index equals the spectral flow of the family D^t.

The lagrangians defined by the CD spaces are infinite dimensional, but the setup can be reduced to finite dimensional symplectic geometry by "stretching the neck". This is done by changing the metric on M to one in which the neck is isometric to a long cylinder $(-r, r) \times \Sigma$. We study the Cauchy data spaces in the adiabatic limits $r \to \infty$. These limits exist if we assume D is " neck-compatible", i.e. is cylindrical and the operator D_0 in (1.1) is selfadjoint. If, moreover, certain nondegeneracy conditions are satisfied these limits have a nice description and our Cauchy data spaces $\Lambda_j(t)$ stabilize to asymptotic Cauchy data spaces Λ_j^∞ . These limiting spaces arise naturally in the Atiyah-Patodi-Singer index problem ([APS1-3]). A related adiabatic analysis was considered in [CLM2]. After performing this adiabatic deformation we can reduce the Maslov index computation to a finite dimensional situation by passing to a symplectic quotient. This generalizes a recent result of Yoshida [Y] in the context of Floer's instanton homology.

The thesis consists of seven chapters. In chapter 2 we translate some basic facts of finite-dimensional symplectic topology into infinite dimensions. The main result here is the homotopical description of the space of Fredholm pairs of lagrangians: it has the homotopy type of the classifying space of KO^{-7}.

Chapter 3 deals with the Maslov index in infinite dimensions. Using Arnold's definition ([Ar]) as a model, we define it as an intersection number. We then derive some computational formulae which play a crucial part later.

Chapter 4 contains the main analytical technicalities of this paper. Many of these results are known but we have reformulated them in a symplectic context (see [BW4] for an extended presentation of this subject).

Chapter 5 contains contains our main result: The Maslov index equals the spectral flow. The idea of the proof is to reduce the general problem via successive homotopies to a simple situation. For this we rely on a genericity result first used by Floer ([F]) in the context of symplectic homology (we give a complete proof in an Appendix). After reducing to the case of piecewise affine homotopies, the theorem follows by an integration by parts formula. Again, this has an elegant symplectic interpretation.

In chapter 6 we take up the problem of stretching the neck. This entails studying the behavior of the Cauchy data spaces of a neck-compatible Dirac on a manifold M as the length of the neck tends to infinity. We begin by studying a related finite dimensional problem. Namely, suppose that A is a $2n \times 2n$ symmetric matrix that anticommutes with the canonical complex structure J on on $\mathbf{R}^{2n}$. We then get a 1-parameter group of symplectic transformations $r \rightarrow e^{-rA}$, and hence a flow on the lagrangian grassmanian $\Lambda(n)$ of $\mathbf{R}^{2n}$. In Corollary 6.1 we show that each trajectory in $\Lambda(n)$ has a unique limit point as $r \to \infty$; this limit is an A-invariant lagrangian in $\mathbf{R}^{2n}$. This follows from a simple trick we learned from Tom Parker. We then return to the infinite dimensional problem, where we can regard the CD spaces as infinite dimensional lagrangians evolving by the "flow" $r \to e^{-rD_0}$ as the neck length $r \to \infty$. By passing to a carefully defined symplectic quotient, we relate this to the above finite dimensional situation. This yields Theorem 6.9, which shows that as the neck length $r \to \infty$ the Cauchy data spaces stabilize to limiting infinite dimensional lagrangians. We can then obtain the Maslov index from a computation in the finite dimensional symplectic quotient (Corollary 6.14).

Finally in chapter 7 we present a simple application where the splitting formula of Theorem 5.14 is used to identify the Conley-Zehnder index of [CZ] with a spectral flow.

We would like to mention that Tom Mrowka informed the author that he recently proved these results using a similar approach. After this work was completed the author learned that Ulrich Bunke independently obtained a splitting formula for the spectral flow (see [Bu2]) as consequence of a gluing result for the eta function of a neck-compatible Dirac (see [Bu1]). The results of this thesis were announced in [N1].

Chapter 2

Infinite dimensional lagrangian subspaces

In this chapter we study lagrangian subspaces in an infinite dimensional symplectic space. In contrast to the finite dimensional situation, the grassmanian of lagrangian subspaces is contractible. A related, but more topologically interesting space is the space of Fredholm pairs of lagrangians. We will show this is a classifying space for KO^1.

Let H be a separable real Hilbert space with inner product (,). We will denote the *-algebra of bounded linear operators on H by B(H). Let GL(H) be the group of invertible elements in B(H) and O(H) be the subgroup of bounded orthogonal operators. For $A, B \in B(H)$ define the commutator and the anticommutator as usual

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA.$$

Fix once and for all a complex structure on H, that is, an operator $J \in O(H)$ with $J^2 = -I$. Thus H becomes a symplectic space with symplectic form

$$\omega(x, y) = (Jx, y) \quad \forall x, y \in H.$$

We can then introduce the basic notions of symplectic geometry. Let W be a subspace of H (the word subspace will always mean closed subspace). Its **annihilator** is the subspace

$$W^0 = \{y \in H \ ; \ \omega(w,y) = 0 \ \forall \, w \in W\}.$$

It is easily seen that $W^0 = JW^\perp$ where $W^\perp$ is the orthogonal complement of W in H.

Definition A subspace W of H is called **isotropic** if $W \subset W^0$, **coisotropic** if $W^0 \subset W$, and **lagrangian** if $W^0 = W$. Equivalently, W is lagrangian if $W^\perp = JW$.

Let $\mathcal{L} = \mathcal{L}_J$ be the set of lagrangian subspaces of H. To topologize $\mathcal{L}$ we identify it with a space of operators using the following construction. Associated to each lagrangian are three operators: the orthogonal projection P_L onto L, the complementary projection $Q_L = I - P_L$ onto the orthogonal complement of L and the conjugation operator (reflection through L)

$$C_L = P_L - Q_L = 2P_L - I \ .$$

Note that $C = C_L$ satisfies

$$C = C^*, \qquad C^2 = I, \qquad \{C, J\} = 0. \tag{2.1}$$

It is easy to see that if C satisfies (2.1) then Ker(I - C) is a lagrangian subspace with projection $P_L = 1/2(I + C)$. Thus we can identify $\mathcal{L}_J$ with

$$\mathcal{C}_J = \{C \ ; \text{C satisfies } \ (2.1)\} \tag{2.2}$$

and topologize it using the operator norm. We will use this identification $\mathcal{L}_J = \mathcal{C}_J$ frequently below.

Now thinking of (H,J) as a complex Hilbert space, the unitary group

$$\mathcal{U}_J(H) = \{U \in O(H) \;\; ; \;\; [U, J] = 0\}$$

is a topological group which is contractible by Kuiper's theorem ([Ku]) and which acts on on $\mathcal{L}$ by

$$C \mapsto UCU^{-1}.$$

This action is transitive (just as in the finite dimensional case, cf. [GS]). The stabilizer of C is

$$O_C = \{U \in \mathcal{U}_J \;\; ; [U, C] = 0\}.$$

By standard arguments ([BW2] or [AS]) we have a fibration

$$O(L) \to \mathcal{U}_J \to \mathcal{L}$$

where, again by Kuiper's theorem, O(L) is contractible. The long exact sequence in homotopy implies the following result.

Proposition 2.1 *$\mathcal{L}$ is contractible.*

Thus in infinite dimensions $\mathcal{L}$ has no interesting topology. To get something interesting we will consider

$$\mathcal{L}^{(2)} \; = \; \{(\Lambda_1, \Lambda_2) \in \mathcal{L}^2 \; ; \; (\Lambda_1, \Lambda_2) \text{ Fredholm pair}\}.$$

Recall that a pair of (V,W) of infinite dimensional subspaces of H is called Fredholm if both subspaces have infinite codimension, V+W is closed and both $\dim(V \cap W)$ and codim (V+W) are finite. The Fredholm index of this pair is defined as

$$i(V, W) \; = \; \dim(V \cap W) \; - \; \dim(V \; + \; W).$$

(For basic facts about Fredholm pairs we refer to [C] or [K]). Note that Fredholm pairs of lagrangians automatically have index O since

$$i(\Lambda_1,\Lambda_2) = \dim(\Lambda_1 \cap \Lambda_2) - \dim(\Lambda_1^{\perp},\Lambda_2^{\perp}) = \dim(\Lambda_1 \cap \Lambda_2) - \dim J(\Lambda_1 \cap \Lambda_2) = 0. \quad (2.3)$$

We can also describe $\mathcal{L}^{(2)}$ in terms of conjugation operators. By Lemma 2.6 of [BW2], $(\Lambda_1,\Lambda_2) \in \mathcal{L}^{(2)}$ iff the corresponding conjugations satisfy

$$C_1 + C_2 \in \mathcal{K}$$

where $\mathcal{K}$ is the space of compact operators on H. Thus

$$\mathcal{L}^{(2)} = \{(C_1,C_2) \in \mathcal{C}^2 \ ; C_1 + C_2 \in \mathcal{K}\}.$$

Now fix $C_0 \in \mathcal{C}$. We have a fibration

$$\mathcal{L}_C \hookrightarrow \mathcal{L}^{(2)} \xrightarrow{p} \mathcal{L}$$

where $p(C_1,C_2) = C_1$ and $\mathcal{L}_0 = p^{-1}(C_0) = (-C_0 + \mathcal{K}) \cap \mathcal{C}$. Since $\mathcal{L}$ is contractible we get a weak homotopy equivalence

$$\mathcal{L}^{(2)} \cong \mathcal{L}_C.$$

Set $\mathcal{U}_{\mathcal{K}} = \mathcal{U}_J \cap (I + \mathcal{K})$ and for $C \in \mathcal{C}$ set $O_{\mathcal{K},C} = (I + \mathcal{K}) \cap O_C$.

Theorem 2.2 *There exists a weak homotopy equivalence.*

$$\mathcal{L}_0 \cong U(\infty)/O(\infty)$$

where

$$U(\infty) = \varinjlim U(n) \quad O(\infty) = \varinjlim O(n).$$

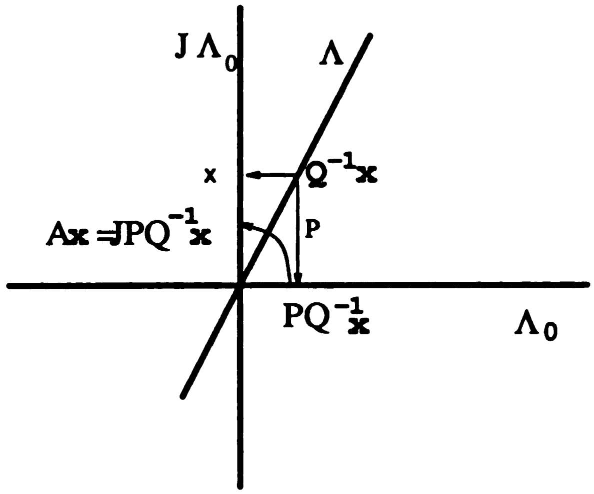


Figure 2.1: *Lagrangian subspaces can be viewed as graphs of symmetric operators*

Proof: The proof will be carried out in several steps, with some intervening Lemmas.

Step 1 *$\mathcal{L}_0$ is path connected.*

Associated to each finite dimensional subspace $V \subset \Lambda_0$ is a subspace

$$\mathcal{L}_0(V) = \{\Lambda \in \mathcal{L}_0 \ ; \Lambda \cap \Lambda_0 \subset V\}$$

of $\mathcal{L}_0$; these define a filtration of $\mathcal{L}_0$. To show that $\mathcal{L}_0$ is connected it suffices to show that each $\mathcal{L}_0(V)$ is connected. Now in finite dimensions the space of lagrangians in V+JV is connected (see [GS]). Hence any lagrangian in $\mathcal{L}_0(V)$ can be connected in $\mathcal{L}_0(V)$ to a lagrangian in $\mathcal{L}_0(0)$. Thus it suffices to show that $\mathcal{L}_0(0)$ is connected. This follows immediately from the next lemma, which gives an alternate description of $\mathcal{L}_0(0)$. The idea, which is standard in the finite dimensional case, is to regard lagrangian subspaces as the graphs of symmetric operators (cf. [Ar], [GS]).

Lemma 2.3 *There is an identification*

$$\mathcal{L}_0(0) \cong \{\text{selfadjoint operators } J\Lambda_0 \to J\Lambda_0\}$$

and hence $\mathcal{L}_0(0)$ is contractible.

Proof: Suppose that (Λ, Λ_0) is a Fredholm pair of transversal lagrangians. Let $P = P_{\Lambda_0}$ and Q = I-P. We deduce that $H = \Lambda + \Lambda_0$. In particular this implies that $Q(\Lambda) = J\Lambda_0$ (see Fig. 2.1). Using the fact that $\Lambda \cap \Lambda_0 = 0$ we see that $Q : \Lambda \rightarrow J\Lambda_0$ is also injective. The open mapping theorem implies that Q is an isomorphism. Construct the operator $A : J\Lambda_0 \rightarrow J\Lambda_0$ by

$$l^{\perp} \mapsto Q^{-1}l^{\perp} \mapsto PQ^{-1}l^{\perp} \mapsto JPQ^{-1}l^{\perp}.$$

Clearly A is a bounded operator (by the closed graph theorem). Note that:

(i) each $u \in \Lambda$ can be uniquely written as $u = l^{\perp} - JAl^{\perp}$ where $l^{\perp} = Qu \in J\Lambda_0$;

(ii) the condition that L is lagrangian is equivalent to A being selfadjoint.

Conversely, given a selfadjoint operator $A : J\Lambda_0 \rightarrow J\Lambda_0$ its "graph"

$$\Lambda_A = \{l^{\perp} - JAl^{\perp} \; ; \; l^{\perp} \in J\Lambda_0\}$$

is a lagrangian. Note that $\Lambda_A \cap \Lambda_0 = 0$. Now consider the operator

$$\overline{A} : H \rightarrow H \;\; , \;\; \overline{A}(l, l^{\perp}) = JAl^{\perp} \;\; , \;\; (l, l^{\perp}) \in \Lambda_0 \oplus J\Lambda_0.$$

One sees that $\text{Range}(I - \overline{A}) = \Lambda_0 + \Lambda_A$ so that the transversality of the pair (Λ_A, Λ_0) is equivalent to the surjectivity of $I - \overline{A}$. Since $\overline{A}^2 = 0$ for any selfadjoint $A : J\Lambda_0 \rightarrow J\Lambda_0$ we deduce that for any such A, $I - \overline{A}$ is invertible. Hence (Λ_A, Λ_0) is a transversal Fredholm pair. ◇

Step 2: If $C_1, C_2 \in \mathcal{L}_0$ satisfy $\|C_1 - C_2\| < 2$ then there is a T in

$$GL_{\mathcal{K}} = \{T \in GL(H) \cap (I + \mathcal{K}) \; ; \; [T, J] = 0\}$$

such that

$$C_2 = TC_1T^{-1}. \tag{2.4}$$

Following [W] we set $T = I + 1/2(C_1 - C_2)C_1$. Then T is invertible since $\|(C_1 - C_2)C_1\| < 2$, and T commutes with J because C_1 and C_2 anticommute with J. On the other hand, C_1 and C_2 lie in $-C + \mathcal{K}$, so $T \in GL_{\mathcal{K}}$. A simple computation shows that (2.4) holds.

Step 3: For each pair $C_1, C_2 \in \mathcal{L}_0$ there is a $T \in GL_{\mathcal{K}}$ such that

$$C_2 = TC_1T^{-1}. \tag{2.5}$$

This follows from Step 2 and the path-connectedness of $\mathcal{L}_0$; the details are left to the reader.

To proceed further we need the following technical result.

Lemma 2.4 *If $T \in GL_{\mathcal{K}}$ then $(T^*T)^{1/2} \in GL_{\mathcal{K}}$.*

Proof: Set $S = T^*T$. Clearly $S^{1/2} \in GL(H)$ and $S^{1/2}$ commutes with J. We have to show that $S^{1/2} \in I + \mathcal{K}$. Then $S \in I + \mathcal{K}$. To find $S^{1/2}$ we use Newton's iteration as in [Ku]

$$S_0 = I \ , \ S_{n+1} = 1/2(S_n + S_n^{-1}S).$$

Note that this iteration is well defined since all S_n's are invertible (they are positive selfadjoint operators with their spectra bounded away from 0). One sees inductively that the right hand side of the iteration is an affine combination of terms in $I + \mathcal{K}$. Thus $S^{1/2} = \lim S_n \in I + \mathcal{K}$. ◇

Step 4 For each pair $C_1, C_2 \in \mathcal{L}_0$ there is a $U \in \mathcal{U}_{\mathcal{K}}$ such that

$$C_2 = UC_1U^*. \tag{2.6}$$

We follow the idea in [B], Prop.4.6.5. Consider $T \in GL_{\mathcal{K}}$ as in (2.5). Then $T C_1 = C_2 T$ and $C_1 T^* = T^* C_2$. It follows that C_1 commutes with $S = S = T^*T$, and hence with $S^{1/2}$. Setting $U = T(TT^*)^{-1/2}$, we clearly have $U^*U = I$, and $U \in GL_{\mathcal{K}}$ by Lemma 2.4. Therefore U is in $\mathcal{U}_{\mathcal{K}}$ and satisfies (2.6).

Step 4 shows that $\mathcal{U}_{\mathcal{K}}$ acts transitively on $\mathcal{L}$. For $C \in \mathcal{L}_0$ the stabilizer of this action is $O_{\mathcal{K},C}$. Thus

$$\mathcal{L}_0 \cong \mathcal{U}_{\mathcal{K}}/O_{\mathcal{K},C} \quad , \quad C \in \mathcal{L}_0. \tag{2.7}$$

Step 5 There are homotopy equivalences

$$\mathcal{U}_{\mathcal{K}} \cong GL_{\mathcal{K}} \cong GL(\infty, \mathbf{C}) \ , \ O_{\mathcal{K},C} \cong GL(\infty, \mathbf{R}). \tag{2.8}$$

The proof of $\mathcal{U}_{\mathcal{K}} \cong GL_{\mathcal{K}}$ is identical to the proof of Lemma 2.9 of [BW2]. It essentially uses the polar decomposition which by Lemma 2.4 is an internal decomposition in $GL_{\mathcal{K}}$, followed by an affine deformation of the positive symmetric term of the polarization. $I + \mathcal{K}$ is an affine space so this deformation stays within $GL_{\mathcal{K}}$. Then by the results of Palais [P1] we have a homotopy equivalence

$$GL_{\mathcal{K}} = GL(\infty, \mathbf{C}).$$

The second part is completely analogous. Classically

$$U(\infty) \cong GL(\infty, \mathbf{C}) \ , \ O(\infty) \cong GL(\infty, \mathbf{R}) \ \text{ homotopically.} \tag{2.9}$$

Theorem 2.2 follows form (2.7), (2.8) and (2.9). ◇

Remark 2.5 *A related result was proved in [W], [BW3]. In that context $\mathcal{K}$ represents compact* **pseudodifferential** *operators in some* **complex** L^2 *space.*

The above arguments apply in finite dimensions to show that the grassmanian $\Lambda(n)$ of lagrangians in $\mathbf{C}^n$ is diffeomorphic to U(n)/O(n). Taking the direct limits over the embeddings $\Lambda(n) \hookrightarrow \Lambda(n+1)$ then gives

$$\Lambda(\infty) = \varinjlim \Lambda(n) \cong U(\infty)/O(\infty).$$

Hence we get

Corollary 2.6

$$\mathcal{L}^{(2)} \cong \mathcal{L}_0 \cong U(\infty)/O(\infty) \cong \Lambda(\infty).$$

It is known that U/O is a classifying space for KO^1 (cf. [Kar]). On the other hand Atiyah-Singer [AS] have shown that this classifying space can also be identified (up to homotopy) with the space of selfadjoint Fredholm operators on a real Hilbert space. Its fundamental group is isomorphic to **Z**. The isomorphism is given by the the spectral flow (of a loop of selfadjoint Fredholm operators). Obviously

$$\pi_1(U/O) \cong \mathbf{Z}$$

and the isomorphism is given by the Maslov index. Thus Corollary 2.6 displays the double nature of $\mathcal{L}^{(2)}$: the operator theoretic nature and the symplectic nature. In the sequel we will further analyze this duality.

Chapter 3

The Maslov index in infinite dimensions

The purpose of this chapter is to provide a more computational description of the Maslov index introduced in the previous chapter. In the finite dimensional situation there are many excellent presentations of the Maslov index (see e.g.[Ar], [CLM1], [D1-2], [GS], [RS]). However, all these assume the finite dimensionality, especially when dealing with orientability questions. For a Banach manifold orientability is a delicate question. To avoid this issue we will give a meaning to a local intersection number without any elaborate considerations of orientability. Our approach is inspired from Arnold's description of the finite dimensional index ([Ar]).

Let (H,J) be a Hilbert space with a complex structure and consider a lagrangian $\Lambda_0 = \mathrm{Ker}(I - C_0)$ specified by the conjugation C_0. The next several lemmas describe the geometry of the space

$$\mathcal{L}_0 = \{\Lambda \in \mathcal{L} \;/\; (\Lambda_0, \Lambda) \text{ is a Fredholm pair}\}.$$

Lemma 3.1 *$\mathcal{L}_0$ is a smooth Banach manifold modelled on the space $Sym(J\Lambda_0)$ of bounded symmetric operators on $J\Lambda_0$.*

Proof To each finite dimensional subspace V of Λ_0 we associate an orthogonal operator I_V commuting with J by

$$I_V(v) = \begin{cases} Jv & \text{for } v \in V \\ v & \text{for } v \in \Lambda \cap V^{\perp}. \end{cases}$$

and the open subset

$$\mathcal{D}_V = \{\Lambda \in \mathcal{L} \;\; \Lambda \cap I_V \Lambda_0 = 0\}.$$

Thus $\mathcal{D}_V = I_V \mathcal{L}_0^*$, where

$$\mathcal{L}_0^* = \{\Lambda \in \mathcal{L}_0 \;/\; \Lambda \cap \Lambda_0 = 0\}.$$

is the dense open set of transverse pairs (in particular, $\mathcal{D}_0 = \mathcal{L}_0^*$). Notice that $I_V \in \mathcal{U}_{\mathcal{K}}$, so $(I_V\Lambda, \Lambda_0)$ is a Fredholm pair and thus $\mathcal{D}_V \subset \mathcal{L}_0$. The sets $\mathcal{D}_V$ cover $\mathcal{L}_0$: if $\Lambda \in \mathcal{L}_0$ then $\Lambda \in \mathcal{D}_V$ for $V = \Lambda \cap \Lambda_0$.

The isomorphism of Lemma 2.3 is a map $\Psi_0 : \mathcal{D}_0 = \mathcal{L}_0^* \to Sym(J\Lambda_0)$. For other V, set

$$\Psi_V = \Psi_0 \circ I_V^{-1} : \mathcal{D}_V \to Sym(J\Lambda_0).$$

Then the collection

$$\{(\mathcal{D}_V, \Psi_V) \;\; ; \; V \in \mathcal{V}, \Psi_V : \mathcal{D}_V \to Sym(J\Lambda_0)\}$$

forms an atlas of $\mathcal{L}_0$. The verification that the transition functions are smooth is accomplished by writing the conjugation operator C associated to a lagrangian in terms of of these coordinates. Thus if $\Lambda \in \mathcal{L}_0^*$ and we identify Λ_0 with $J\Lambda_0$ via J so that J becomes

$$J = \begin{bmatrix} 0 & -1_{\Lambda_0} \\ 1_{\Lambda_0} & 0 \end{bmatrix}$$

then the conjugation associated to Λ, $C : \Lambda_0 \oplus \Lambda_0 \to \Lambda_0 \oplus \Lambda_0$ can be described as

$$C = \begin{bmatrix} (1+S^2)^{-1}(1-S^2) & 2(1+S^2)^{-1}S \\ 2(1+S^2)^{-1}S & -(1+S^2)^{-1}(1-S^2) \end{bmatrix}$$

where $S = \Psi_0(\Lambda)$. The details are left to the reader. $\diamond$

The manifold $\mathcal{L}_0$ is filtered by the subspaces

$$\mathcal{L}_0^m = \{\Lambda \in \mathcal{L}_0 \ ; \ \dim(\Lambda \cap \Lambda_0) = m\}.$$

In fact, these are subvarieties, as follows. Note that $\mathcal{L}_0^m$ is covered by charts of the form $\mathcal{D}_V$ with $\dim V = m$. Fix one such chart and write $S = \Psi_V$. By an elementary argument of Arnold ([Ar] Lemma 3.3.3) one sees that $\Lambda \in \mathcal{D}_V$ lies in $\mathcal{L}_0^m$ if and only if

$$(SJu, Jv) = 0 \qquad \text{for all } u, v \in V. \tag{3.1}$$

Since S is symmetric and $\dim V = m$ this describes $\mathcal{L}_0^m$ in this chart as the solution set of $m(m+1)/2$ algebraic equations.

In particular, if $\Lambda \in \mathcal{L}_0^1$, then $\Lambda \cap \Lambda_0$ is a 1-dimensional space $V_0 = \mathbf{R}e$, $\Lambda \in \mathcal{D}_{V_0}$, and $S = \Psi_{V_0}(\Lambda)$ then

$$(SJe, Je) = 0. \tag{3.2}$$

Corollary 3.2 *The closure $\overline{\mathcal{L}_0^1}$ is a codimension 1 subvariety of $\mathcal{L}_0$ called the* **resonance divisor**. *It is stratified by subvarieties $\mathcal{L}_0^m$ of codimension $m(m+1)/2$.*

We may think of $\overline{\mathcal{L}_0^1}$ as a divisor in $\mathcal{L}_0$ defining an element in $H^1(\mathcal{L}_0, \mathbf{Z}) \cong \mathbf{Z}$ dual to the generator of $H_1(\mathcal{L}_0, \mathbf{Z})$. Dually, given a loop γ in $\mathcal{L}_0$, we may think of its Maslov index $\mu(\gamma)$ as being the intersection number $\gamma \cap \mathcal{L}_0$. Most of the rest of this chapter is devoted to making this intuition rigorous. We will first show that if a path γ intersects $\mathcal{L}_0^1$ transversally, one can associated a sign to each intersection point. The sum of then intersection numbers is a homotopy invariant of the path. As a byproduct, we will get several formulas for the local intersection number; these will be extremely useful later.

Consider the vector field χ over $\mathcal{L}_0$ defined by

$$\chi(\Lambda) = \frac{d}{dt}\bigg|_{t=0} (e^{Jt}\Lambda).$$

Proposition 3.3 *χ defines a transversal orientation on $\mathcal{L}_0^1$.*

Proof This follows easily from a computation of Arnold ([Ar] Lemma 3.5.3). Consider $\Lambda \in \mathcal{L}_0$ and assume Λ lies in a coordinate chart $\mathcal{D}_V$, $\dim V = 1$. If $S = \Psi_V(\Lambda)$ are the coordinates of Λ then the coordinates of χ are given by the formula:

$$\chi(\Lambda) = -(I + S^2).$$

Putting this in equation (3.2) shows that $(\chi(\Lambda)Je, Je) < 0$, where V=span(e). Thus χ is transversal to $\mathcal{L}_0^1$ and defines a transversal orientation. ◇

Consider a path $\Lambda(t)$ which for $|t|$ small lies in a single chart $\mathcal{D}_v = \mathcal{D}_{\mathbf{R}v}$ and such that

$$\Lambda(0) \cap \Lambda_0 = \mathbf{R}e_0 \ , \ |e_0| = 1.$$

Let $S_t^v = \Psi_v(\Lambda(t))$. Assume $\Lambda(t)$ intersects $\mathcal{L}_0^1$ transversally at t=0. The transversality can be rewritten as

$$(\dot{S}_t^v Je_0, Je_0) \neq 0.$$

where — here and below — the dot denotes $\frac{d}{dt}$ at $t = 0$. Let $\mathcal{M} = \{v \in \Lambda_0 \ ; \ |v| = 1 \ , \ \Lambda(0) \in \mathcal{D}_v\}$ and define a map $\sigma = \sigma_{\Lambda(\cdot)} : \mathcal{M} \to \{\pm 1\}$ by

$$\sigma(v) = \text{sign}(\dot{S}_t^v Je_0, Je_0).$$

Lemma 3.4 *For a path $\Lambda(t)$ as above the map $\sigma_{\Lambda(\cdot)}$ is constant.*

Proof One can alternatively characterize $\mathcal{M}$ as $\{v \in \Lambda_0 \ ; \ |v| = 1 \ , \ (v, e_0) \neq 0\}$. Hence $\mathcal{M}$ has two components:

$$\mathcal{M}_\pm = \{v \in \mathcal{M} \ ; \ \pm(v, e_0) > 0\}.$$

Now $\dot{S}_t^v$ varies continuously with v, and obviously $\sigma(v) = \sigma(-v)$. Thus $\sigma : \mathcal{M} \to \{\pm 1\}$ a continuous even map, so is constant. ◇

Definition *For a path $\Lambda(t)$ as in Lemma 3.4 we define the local Maslov index by*

$$\mu(\Lambda_0, \Lambda(t)) = \sigma_{L(\cdot)}(v) \ , \ v \in \mathcal{M}. \tag{3.3}$$

By Lemma 3.4 this definition is independent of coordinates.

We will next give several more concrete versions of formula (3.3). To begin, note that in (3.3) $\sigma_\Lambda(v)$ is independent of v, so we are free to choose v as we please. Choose $v = e_0$. Set $f_0 = Je_0$ and $R_t = \Psi_0(I_0^{-1}\Lambda(t))$ where $I_0 = I_{\mathbf{R}e_0}$. (3.3) becomes:

$$\mu(\Lambda_0, \Lambda(t)) = \operatorname{sign}(\dot{R}_t f_0, f_0). \tag{3.4}$$

Now consider the path

$$x_t = f_0 - JR_t f_0 \in I_0^{-1}\Lambda(t).$$

Then $x_0 = f_0$ (since $e_0 \in \Lambda(0)$ so that $f_0 = -I_0^{-1}e_0 \in I_0^{-1}\Lambda(0)$) and hence

$$(x_t - x_0, e_0) = -(JR_t f_0, e_0) = (R_t f_0, f_0).$$

Differentiating at t=0 we get

$$(\dot{x}_t, e_0) = (\dot{R}_t f_0 \, , \, f_0). \tag{3.5}$$

Now introduce the conjugation $D = D(t)$ associated to $I_0^{-1}\Lambda(t)$. Since $x_t \in I_0^{-1}\Lambda(t)$ we have $x_t = D(t)x_t$. Differentiating this at t=0, taking the inner product with e_0, and noting that $D(0)e_0 = -e_0$ we get

$$(\dot{x}_0, e_0) = (\dot{D}f_0, e_0) - (\dot{x}\,,\, e_0)$$

and therefore

$$2(\dot{x}_0, e_0) = (\dot{D}f_0, e_0) = (J\dot{C}f_0\,,\, f_0). \tag{3.6}$$

The conjugation associated with $\Lambda(t)$ is $C(t) = I_0^{-1}D(t)I_0$. Using this in (3.6) we deduce

$$2(\dot{x}_0, e_0) = \left(JI_0\dot{C}I_0^{-1}f_0, f_0\right) = (J\dot{C}e_0, e_0). \tag{3.7}$$

Combining (3.4), (3.5) and (3.7) we get:

Corollary 3.5

$$\mu(\Lambda_0, \Lambda(t)) = \text{sign}\,(J\dot{C}e_0\,,\, e_0) = \text{sign}\;\omega(\dot{C}e_0, e_0)$$

where $\Lambda(0) \cap \Lambda_0 = \mathbf{R}e_0$ *and* $\omega(x,y) = (Jx,y)$ *is the symplectic form.*

Note that the above formula is independent of coordinates. For the application we have in mind we will need another variant of this formula. Consider a family $U(t) \in \mathcal{U}_J$ with

$$U(0) = I \;\;,\;\; C(t) = U(t)C(0)U(t)^*.$$

If we write $\dot{U} = JA$ where A commutes with J and A is selfadjoint then

$$\dot{C} = JAC(0) - C(0)JA = JAC(0) + JC(0)A$$

$$\dot{C}e_0 = JAC(0)e_0 + JC(0)Ae_0 = J(Ae_0 + C(0)Ae_0) = J(I + C(0))Ae_0.$$

But $P(0) = 1/2(I + C(0))$ is the orthogonal projection onto $\Lambda(0)$, so

$$(J\dot{C}e_0\,,\, e_0) = -2(P(0)Ae_0\,,\, e_0) = -2(Ae_0\,,\, e_0).$$

Hence we have the following result.

Corollary 3.6 *If* $\Lambda(t) = U(t)\Lambda(0)$ *with* $U(t) = I + tJA + O(t^2)$ *then*

$$\mu(\Lambda_0, \Lambda(t))|_{t=0} = -\operatorname{sign}(Ae_0\,,\, e_0) \;=\; \operatorname{sign}\omega(\dot{U}e_0, e_0). \tag{3.8}$$

Remark 3.7 *There is an ambiguity in the definition of the Maslov index and without a proper normalization the Maslov index is well defined up to a sign. This is easily seen in the "mirror symmetry" of the Maslov index (cf. [CLM1], Prop.XI):*

$$\mu(\Lambda_1(t), \Lambda_2(t)) \;=\; -\mu(\Lambda_2(t), \Lambda_1(t)).$$

We consider as standard normalization the one in Property VII of [CLM1] and we want to compare it with our definition of Maslov index. For this we consider $\mathbf{R}^2$ *with the standard symplectic structure*

$$\omega(x,y) \;=\; -(Jx,y) \quad , \quad J \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $L_0 \;=\; \operatorname{span}(e_0)$ *where* $e_0 \;=\; (1,0)$ *and consider the path* $L_t \;=\; e^{Jt}L_0$ *for* t *in a small neighborhood of 0. Corollary 3.6 gives*

$$\mu(L_0, L_t) \;=\; \operatorname{sign}\, \omega(Je_0, e_0) \;=\; 1$$

which agrees with the standard normalization.

Consider

$$\mathcal{P}_0 = \{\gamma : (I\,,\, \partial I) \to (\mathcal{L}_0, \mathcal{L}_0^*)\} \quad , \quad I = [a,b] - \text{compact interval}$$

$$\mathcal{P}_0{}^* = \{\gamma \in \mathcal{P}_0\,;\ \ \gamma(t)\,\text{intersects}\,\mathcal{L}_0^1\,\text{transversally}\,\}.$$

Since $\operatorname{codim}\mathcal{L}_0^k = k(k+1)/2 \geq 3$ if $k \geq 2$ we see that any path γ in $\mathcal{P}_0$ can be deformed (in $\mathcal{P}_0$) to a path in $\mathcal{P}_0{}^*$. For $\gamma^* \in \mathcal{P}_0{}^*$ define

$$\mu(\Lambda_0, \gamma^*) = \sum_{\gamma^*(t_i)\in\mathcal{L}_0^1} \mu(\Lambda_0, \gamma^*(t)\,|_{|t-t_i|<\varepsilon}).$$

This is the usual definition of an intersection number. In particular standard arguments show that the above μ can be extended to the whole $\mathcal{P}_0$ as a homotopy invariant function. Now define

$$\mathcal{L}^{(2)}{}_* = \{(\Lambda_1, \Lambda_2) \in \mathcal{L}^{(2)}\ \ ;\ \ \Lambda_1 \cap \Lambda_2 = 0\}$$

and

$$\mathcal{P}^{(2)} = \{\gamma : (I\,,\partial I) \to (\mathcal{L}^{(2)}\,,\,\mathcal{L}^{(2)}{}_*\}.$$

Any $\gamma \in \mathcal{P}^{(2)}$ looks like $\gamma(t) = (\Lambda_1(t)\,,\,\Lambda_2(t))$. Without any loss of generality we may assume that $\Lambda_1(0) = \Lambda_0$. We can find a smooth family of unitary operators $U(t) \in \mathcal{U}_K$ such that

$$\Lambda_1(t) = U(t)\Lambda_0\ \ ,\ \ U(0) = I.$$

Consider the family of paths $\gamma_s \in \mathcal{P}^{(2)}$ defined by

$$\gamma_s(t) = (U(s)^{-1}\Lambda_1(t)\,,\,U(s)^{-1}\Lambda_2(t)).$$

Define

$$\mu(\gamma) = \mu(\Lambda_1(t), \Lambda_2(t)) = \mu(\Lambda_0\,,\,U(t)^{-1}\Lambda_2(t)).$$

We will check two things.

A. $\mu(\gamma)$ is independent of the family U(t) .

Indeed if $U(t), V(t) \in \mathcal{U}_{\mathcal{K}}$ are such that $U(t)\Lambda_0 = V(t)\Lambda_0$ then $T(t) \in \mathcal{O}_{\mathcal{K}}(\Lambda_0)$. Set

$$\Lambda' = U(1)^{-1}\Lambda_1(t) \ \ , \ \ \Lambda''(t) = V(t)^{-1}\Lambda_2(t).$$

If $\Lambda^s(t) = T(s)\Lambda'(T)$ then $(\Lambda_0\,,\,\Lambda^s(t)) \in \mathcal{L}^{(2)})$ and $\Lambda^0(t) = \Lambda'(t)$, $\Lambda^1(t) = \Lambda''(t)$ i.e.

$$\mu(\Lambda_0\,,\,\Lambda'(t)) = \mu(\Lambda_0\,,\,\Lambda''(t))$$

and this proves **A**.

B. $\mu(\gamma)$ is a homotopy invariant. The proof is entirely similar to the proof of **A**. In fact both rely on the fact that the inclusion

$$\mathcal{L}_0 \hookrightarrow \mathcal{L}^{(2)}$$

is a homotopy equivalence. The details are left to the reader .

An immediate consequence of the above considerations is that μ defines a morphism

$$\mu : \pi_1(\mathcal{L}^{(2)}) \to \mathbf{Z}.$$

The finite dimensional Maslov index behaves nicely with respect to symplectic reductions. So does this infinite dimensional version of it. Recall first the process of reduction.

Lemma 3.8 *Consider $\Lambda \subset H$ a lagrangian of H, an isotropic subspace W and its annihilator W^0. If (Λ, W^0) is a Fredholm pair then:*

(i) $\mathcal{H}_0 = W^0/W$ has an induced symplectic structure;

(ii) $\Lambda^W = (\Lambda \cap W^0)/W$ is a lagrangian subspace in W/W^0.

Proof (i) is straightforward and is left to the reader. We now prove (ii) in a special case which is precisely the situation we will ever need. We will assume that

Λ is **clean mod W** i.e. $\Lambda \cap W = 0$. We will identify $\mathcal{H}_0$ with the orthogonal complement of W in W^0. Finally set $U = (W^0)^\perp$. Then

$$H = \mathcal{H}_0 \oplus W \oplus U.$$

Denote by P_0 (resp P_W, P_U) the orthogonal projections onto $\mathcal{H}_0$ (resp. W,U). Obviously Λ^W is an isotropic subspace of $\mathcal{H}_0$. We show it is maximal isotropic. Let $h_0 \in \mathcal{H}_0$ such that $(Jh_0, \Lambda^W) = 0$. Then

$$Jh_0 \perp (\Lambda^W + W) \Rightarrow Jh_0 \perp (\Lambda \cap W^0)$$

i.e.

$$h_0 \in J(\Lambda \cap W^0)^\perp = J(\Lambda^\perp + U) \quad \text{(since } (\Lambda, \mathrm{W}^0) \text{ is Fredholm)}$$

$$= J(J\Lambda + U) = \Lambda + W \quad \text{(since W is isotropic).}$$

Thus $h_0 \in \mathcal{H}_0 \cap (\Lambda + W)$ i.e. $h_0 \in \Lambda^W$. Lemma is proved. ◇

For any isotropic subspace W, JW is also isotropic and we define

$$\mathcal{L}^{(2)}{}_W(H) = \{(\Lambda_1, \Lambda_2) \in \mathcal{L}^{(2)} \;/\; (\Lambda_1, W) \text{ is Fredholm } \Lambda_1 \cap W = \Lambda_2 \cap JW = 0\}.$$

(The pairs of $\mathcal{L}^{(2)}{}_W$ are called **clean mod W**). Note that if $(\Lambda_1, \Lambda_2) \in \mathcal{L}^{(2)}{}_W$ then $(\Lambda_2, (JW)^0))$ is Fredholm and we have a natural identification $W^0/W \equiv (JW)^0/JW$ (given by J). The reduction process described in lemma 3.9 induces a map

$$\pi_W : \mathcal{L}^{(2)}{}_W(H) \to \mathcal{L}^{(2)}(\mathcal{H}_0) \;\; (\Lambda_1, \Lambda_2) \mapsto (\Lambda_1^W, \Lambda_2^{JW}).$$

Since the reduction is clean mod W we deduce as in the finite dimensional case that π_W is continuous (see [GS]). As in finite dimensions we have the following result.

Proposition 3.9 (Invariance under clean reductions) *If $\gamma(t) \in \mathcal{P}^{(2)}$ is clean mod W at any time then*

$$\mu(\gamma) = \mu(\pi_W(\gamma)).$$

Proof: As before it suffices to consider only the special case $\gamma(t) = (\Lambda_0, \Lambda_1(t))$ where t is very small. We can assume without any loss of generality that

$$\Lambda_1(t) = U(t)\Lambda_1(0) \ , \ U(0) = I \ , \ U(t)|_{JW} \equiv I.$$

Let $\dot{U}(0) = JA$. Clearly $A \equiv 0$ on JW. Using (3.10) to compute the local Maslov index we see that W has no contribution in the formula and thus nothing changes if we mod W out. $\diamond$

Remark 3.10 *One can show that the map π_W is actually a homotopy equivalence. A similar result holds if we allow W to vary with t. As long as the reductions stay clean we have the invariance of the Maslov class (see [DP], [V] for a related result). We leave the details to the reader.*

Using the homotopy long exact sequence for the pair $(\mathcal{L}^{(2)}, \mathcal{L}^{(2)}_*)$ and the results proved so far we deduce

Theorem 3.11 *Let $\gamma_0, \gamma_1 \in \mathcal{P}^{(2)}$. Then γ_0 is homotopic to γ_1 if and only if*

$$\mu(\gamma_0) = \mu(\gamma_1).$$

In particular $\mu : \pi_1(\mathcal{L}^{(2)}) \to \mathbf{Z}$ is an isomorphism.

The details are left to the reader.

We now have a flexible definition of the Maslov index. In the following chapters we will apply it in connection to spectral flow computations.

Chapter 4

Boundary value problems for selfadjoint Dirac operators

We gather in this chapter various analytical facts about boundary value problems for Dirac operators. Many of these results are known (see [BW4]) but we reformulate them in a form suitable to our purposes.

Consider an oriented Riemann manifold (M,g) and $\mathcal{E} \to M$ an euclidian vector bundle over M. Denote by C(M) the bundle of Clifford algebras over M whose fiber at $x \in M$ is the Clifford algebra $C(T_x^*M)$. We will assume that $\mathcal{E}$ is a *selfadjoint C(M)-module* that is for each 1-form $\eta \in \Omega^1(M)$ the Clifford multiplication $c(\eta) \in End(\mathcal{E})$ is skew-adjoint.

Definition 4.1 *([BGV], Chap.3) Let $D : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ be a 1st order differential operator. Then D is called a Dirac operator if* $\forall\, f \in C^\infty(M)$

$$c(df)u = [D\,,\, f]u \;=\; D(fu) - f(Du) \quad \forall u \in C^\infty(\mathcal{E}).$$

In the sequel all Dirac operators will be assumed formally selfadjoint.

It is easy to construct Dirac operators. Let ∇ be the extension of the Levi-Civita connection to C(M). Fix a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ compatible with the Clifford multiplication in the sense that $\nabla c = 0$, that is

$$\nabla(c(u)) = c((\nabla u)). \tag{4.1}$$

Then the composition

$$C^\infty(\mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(T^*M \otimes \mathcal{E}) \xrightarrow{c} C^\infty(\mathcal{E})$$

is a Dirac operator $D = D(\nabla^{\mathcal{E}})$. In dual local framings

$$D = \sum c(e^i)\nabla^{\mathcal{E}}_{e_i}.$$

The space of selfadjoint Dirac operators compatible with a given Clifford action is an affine space modelled on the space of symmetric endomorphisms of $\mathcal{E}$.

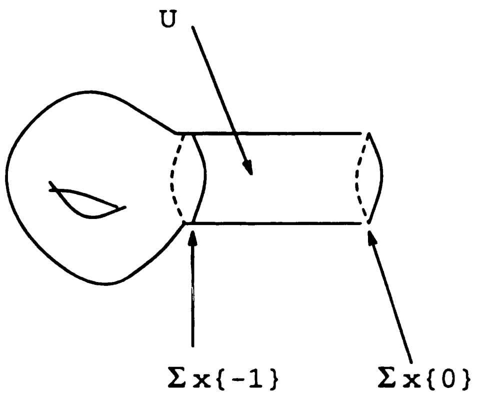


Figure 4.1: *The metric is cylindrical in a neighborhood of the boundary*

Let M be a compact oriented manifold with boundary $\Sigma = \partial M$ and suppose it is endowed with a cylindrical metric in a neighborhood of the boundary. More precisely if $U \subset M$ is a collar neighborhood of Σ in M with an identification $\psi : U \cong \Sigma \times (-1, 0]$ then in these coordinates the riemannian metric on M satisfies $g|_U = h + ds^2$ where h is a riemannian metric on Σ (Fig. 4.1). Denote by $\nabla = \nabla^g$ the corresponding Levi-Civita connection. Let $\mathcal{E}$ be a selfadjoint C(M)-module and

$\hat{\nabla}$ a Clifford connection on $\mathcal{E}$. Set $\mathcal{E}_0 = \mathcal{E}|_\Sigma$ and pick an isomorphism (cylindrical gauge)

$$\Psi : \mathcal{E}|_U \cong \mathcal{E}_0 \times (-1, 0]$$

covering ψ such that over the neck

$$\hat{\nabla} = \hat{\nabla}^0 + ds \otimes \partial/\partial s$$

where $\hat{\nabla}^0 = \hat{\nabla}|_\Sigma$. Fix once and for all the isomorphisms ψ, Ψ, the connection $\hat{\nabla}$ and the metric g.

Definition 4.2 *A Dirac operator D is called* **cylindrical** *if over U it has the form*

$$D \;=\; c(ds)(\partial/\partial s \;+\; D_0) \tag{4.2}$$

where $D_0 : C^\infty(\mathcal{E}_0) \to C^\infty(\mathcal{E}_0)$ is independent of s over U'. In addition if D_0 is selfadjoint then D is called **neck-compatible** *(*__n.c.__*).*

In the sequel all Dirac operators on manifolds with boundary will be assumed cylindrical.

For example $\hat{D} = D(\hat{\nabla})$ is a n.c. operator. To see this consider $e_1, e_2, \cdots, e_n, \partial/\partial s$ a local orthonormal frame in U (n $=$ $\dim \Sigma$). Then over the neck U

$$\hat{D} \;=\; c(ds)(\partial/\partial s \;+\; \hat{D}_0)$$

where

$$\hat{D}_0 = \sum c(e^i)c(ds)\hat{\nabla}^0_{e_i} \tag{4.3}$$

is independent of s and selfadjoint because $c(e^i)c(ds)$ is skewadjoint and commutes with $\hat{\nabla}^0_{e_i}$.

Remark 4.3 *D_0 is a Dirac operator over Σ. To see this fix $x \in \Sigma$ and set $V_0 = T^*_x\Sigma$, $V = V_0 \oplus \langle ds\rangle \cong T^*_x M$, $E_0 = \mathcal{E}_0|_x$. Let $e^1, e^2, \cdots, e^n$ be an orthonormal frame of V_0. E_0 is a $C(V)$- module. It inherits a structure of selfadjoint $C(V_0)$-module induced by the embedding of Clifford algebras*

$$C(V_0) \hookrightarrow C(V) \ , \quad e^I \mapsto e^I \cdot ds$$

(here I is an ordered multiindex: $1 \leq i_1 < \cdots i_k \leq n$). Denote by c_0 the action of $C(V_0)$ on E_0 so that $c_0(e^I) = c(e^I)c(ds)$. In particular we deduce that $\hat{\nabla}^0$ is a Clifford connection compatible with the above Clifford action and $\hat{D}_0 = D_0(\hat{\nabla}^0)$ is the Dirac operator associated to ∇^0.

If A is a **cylindrical** endomorphism of $\mathcal{E}$ i.e. a selfadjoint endomorphisms satisfying $\partial/\partial s A = 0$ over U then $\hat{D} + A$ is cylindrical. We deduce that the space of cylindrical Dirac operators is an affine space modelled on the space of cylindrical endomorphisms.

If A is a **neck compatible** endomorphism of $\mathcal{E}$ i.e. a cylindrical endomorphism anticommuting with $d(ds)$

$$\{A\, , c(ds)\} = 0 \tag{4.4}$$

over U then $D = \hat{D} + A$ is a n.c. Dirac operator. Indeed it is cylindrical and if we set $A_0 = A|_\Sigma$, $B_0 = A_0 c(ds)$ then we deduce that B_0 is selfadjoint since A_0 is selfadjoint, c(ds) is skewadjoint and A_0 and c(ds) anticommute. Over U

$$D = c(ds)(\partial/\partial s + D_0) \ , \quad D_0 = \hat{D}_0 + B_0.$$

We next turn to the analytic aspects of cylindrical Dirac operators. The adequate functional framework for all our future considerations is that of Sobolev spaces L^2_σ (functions "σ-times differentiable" with derivatives in L^2). We will denote the norm of L^2_σ by $|\ |_\sigma$ and the L^2 norm by $|\ |$.

Let D be a Dirac operator. Following Seeley [S] we consider the spaces

$$\mathcal{K}(D) = \{u \in C^\infty(\mathcal{E}) \ ; \ Du = 0; \text{in } M\}$$

$$\mathcal{K}_\sigma(D) \ = \ \mathcal{K}(D) \cap L^2_\sigma$$

Since D is elliptic $\mathcal{K}_\sigma(D) \subset C^\infty(\mathcal{E})$. We are interested in the subspace spanned by the restrictions over Σ of the sections in $\mathcal{K}_\sigma(D)$. For $\sigma > 1/2$ the existence of these restrictions is a consequence of classical trace results for Sobolev spaces (cf. [LiMa]). The case $\sigma = 1/2$ requires a more subtle treatment since the usual trace map is not defined. One uses the fact that $\mathcal{K}_{1/2}(D)$ is a distinguished subspace of sections satisfying an elliptic PDE and a growth condition near the boundary. For $s \in (0,1)$ consider the restriction map

$$R_s : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}_0) \quad u \mapsto u|_{\Sigma\times\{s\}}$$

For any $u \in \mathcal{K}_{1/2}(D)$ the limit (in $L^2(\mathcal{E}_0)$)

$$R_0 u \stackrel{def}{=} \lim_{s\to 0} R_s u$$

exists and is uniform in $\{|u|_{1/2} < 1\} \cap \mathcal{K}_{1/2}$ (see [BW4], [S]). This limit map has two important properties.

Proposition 4.4 $R_0 : \mathcal{K}_{1/2} \to L^2(\mathcal{E}_0)$ *is a continuous map satisfying*

*(a)***Unique continuation***: If* $u \in \mathcal{K}_{1/2}(D)$ *and* $R_0(u) = 0$ *then u=0.*

*(b)***Boundary estimates***: If* $u \in \mathcal{K}_{1/2}(D)$ *then*

$$|u|_{1/2} \leq const.|R_0 u|$$

For the proof we refer to [BW4] or [S].

Definition 4.5 *The* **Cauchy-data** *space of* D *(***CD space***) is the subspace* $\Lambda(D) \subset L^2(\mathcal{E}_0)$ *defined as*

$$\Lambda(D) = R_0(\mathcal{K}_{1/2}(D)) = \mathcal{K}_{1/2}(D)|_{\Sigma}.$$

One sees that $\Lambda(D)$ is a closed subspace of $L^2(\mathcal{E}_0)$. It is roughly the subspace consisting of those sections $u \in L^2(\mathcal{E}_0)$ which extend to a solution of $DU = 0$ over M. Proposition 4.4 shows that R_0 is a linear isomorphism between $\mathcal{K}_{1/2}(D)$ and $\Lambda(D)$. The orthogonal projection P(D) onto $\Lambda(D)$ is usually called the **Caldéron projector** of D. By the classical results of [S] this projection is induced by a 0th order pseudodifferential operator whose symbol can be explicitly computed ([BW4], [P2], [S]).

The dependence of the Caldéron projector on the operator is rather nice. The method of constructing the Caldéron projectors detailed in [BW4], Theorem 12.4(b) (see also [S]) can be used to prove the following result.

Proposition 4.6 *Let* $\{D^t\}$ *be a family of cylindrical Dirac operators on* M *compatible with a fixed Clifford action. Assume* D^t *is smooth in some Sobolev norm* L^2_k*, where* k *is sufficiently large so that* $L^2_k \hookrightarrow C^2$*, (e.g.* $k \geq N/2+2$*,* $N = \dim M$*). Then the path of orthogonal projections* Π_t *onto* $\Lambda(t) = \Lambda(D(t))$ *is* C^1 *as a path in the Banach space of bounded operators* $L^2(\mathcal{E}_0) \to L^2(\mathcal{E}_0)$*.*

Proof We begin by briefly recalling the construction of the Caldéron projection.

Let $\tilde{M}$ denote the double of M: $\tilde{M} = M \cup_{\Sigma} (-M)$. Continue to denote by s the longitudinal coordinate along a tubular neighborhood N of Σ in $\tilde{M}$ so that $N \cong \Sigma \times (-1, 1)$ and $s \leq 0$ on M.

For each Dirac operator D over M denote by $\mathcal{D} = \mathcal{D}(D)$ the *invertible double of D* constructed in Thm.9.1 of [BW4]. This is an invertible Dirac operator on a bundle $\tilde{\mathcal{E}}$ over $\tilde{M}$, extending D. Moreover $\mathcal{D}$ depends smoothly upon D.

For every $u \in C^\infty(\mathcal{E}_0)$ denote by $\delta \otimes u$ the vector-valued distribution over $\tilde{M}$ defined by

$$\langle \delta \otimes u, V \rangle = \int_\Sigma (u, V\,|_\Sigma) \quad V \in C^\infty(\tilde{\mathcal{E}}).$$

Note that supp $\delta \otimes u \subset \Sigma$ and $\delta \otimes u \in L^2_{-1/2-\varepsilon}$ for $0 < \varepsilon < 1/2$. This follows from an equivalent description of the map $u \mapsto \delta \otimes u$ as the adjoint of the trace map

$$\gamma : C^\infty(\tilde{\mathcal{E}}) \to C^\infty(\mathcal{E}_0) \quad V \mapsto V\,|_\Sigma\,.$$

This adjoint is a continuous operator $\gamma^* : L^2_{-\sigma} \to L^2_{-1/2-\sigma}$ for all $\sigma > 0$.

Given $u \in C^\infty(\mathcal{E}_0)$ denote by $U = U(u)$ the distribution over $\tilde{M}$ defined by $U = \mathcal{D}^{-1}(\delta \otimes u)$. By classical regularity results sing supp U $\subset \Sigma$ and $U \in L^2_{1/2-\varepsilon}$. In particular U is smooth over the interior of M and in [BW4; Thm12.4] or [S] it is shown that

$$R_0^- U = \lim_{s \to 0^-} U \mid \Sigma \times \{s\}$$

exists in any C^k norm. The basic result is that

$$\Pi(D)u = R_0^- U \quad \forall\, u \in C^\infty(\mathcal{E}_0).$$

Now let D^t be a smooth path of Dirac operators over M and set $\Pi_t = \Pi(D^t)$, $\mathcal{D}_t = \mathcal{D}(D^t)$ and let $\|\cdot\|$ denote the natural norm in the space of bounded linear operators $L^2\mathcal{E}_0) \to L^2(\mathcal{E}_0)$. One fact that will be frequently used in the sequel is the following inequality for distributions over a compact manifold.

$$|a\phi|_\sigma \le C(\sigma)\|a\|_{C^2}|\phi|_\sigma \quad 0 \le \sigma \le 2,\ a \in C^2,\ \phi \in L^2_\sigma. \tag{4.5}$$

The proof of (4.5) is immediate. For $\sigma = 0$ or $\sigma = 2$ this is simply the Holder inequality. For the other σ it follows by interpolation.

The proof that $t \mapsto \Pi_t$ is smooth is carried out in several steps. For every $u \in C^\infty(\mathcal{E}_0)$ and any t set $U_t = \mathcal{D}_t^{-1}(\delta \otimes u)$. Fix $\varepsilon \in (0, 1/2)$. Note that

$$|U_t|_{1/2-\varepsilon} \leq C|u| \tag{4.6}$$

where $C > 0$ is independent of t at least for all small t.

Step1 We will prove that $U_t - U_0 \in L^2_{3/2-\varepsilon}$ and

$$|U_t - U_0|_{3/2-\varepsilon} \leq Ct|u| \quad \forall u \in C^\infty(\mathcal{E}_0) \tag{4.7}$$

where $C > 0$ is independent of t (small).

To prove (4.7) write $\mathcal{D}_t$ as $\mathcal{D}_t = \mathcal{D}_0 + A(t)$ where $A(t) \in End(\tilde{\mathcal{E}})$ satisfies

$$\|A(t)\|_{C^2} = O(t) \text{ as } t \to 0. \tag{4.8}$$

U_t satisfies the equation $\mathcal{D}_0 U_t + A(t)U_t = \delta \otimes u$ so that

$$\mathcal{D}_0(U_t - U_0) = -A(t)U_t \in L^2_{1/2-\varepsilon}.$$

By standard elliptic regularity we deduce $U_t - U_0 \in L^2_{3/2-\varepsilon}$. Using elliptic estimates and the invertibility of $\mathcal{D}_0$ we deduce

$$|U_t - U_0|_{3/2-\varepsilon} \leq C|A(t)U_t|_{1/2-\varepsilon}.$$

The estimate (4.7) follows immediately using (4.5), (4.6) and (4.8).

Step 2

$$\|\Pi_t - \Pi_0\| = O(t) \quad \text{as } t \to 0. \tag{4.9}$$

For $u \in C^\infty(\mathcal{E}_0)$ we have

$$\Pi_t u = R_0^- U_t = R_0^- U_0 + R_0^-(U_t - U_0) = \Pi_0 u + R_0^-(U_t - U_0).$$

The existence of $R_0^-(U_t - U_0)$ follows from the regularity established at Step 1 and classical trace results. In particular

$$|\Pi_t u - \Pi_0 u| = |R_0^-(U_t - U_0)| \leq C|U_t - U_0|_{3/2-\varepsilon}.$$

The estimate (4.9) now follows from (4.7). In particular we proved that Π_t depends continuously upon t.

For any $u \in C^\infty(\mathcal{E}_0)$ let V_0 be defined as the unique solution of the equation

$$\mathcal{D}_0 V_0 + \dot{A}(0) U_0 \;=\; 0 \tag{4.10}$$

where as before $\mathcal{D}_t = \mathcal{D}_0 + A(t)$, $U_0 = \mathcal{D}_0^{-1}(\delta \otimes u)$ and the dot denotes the differentiation at t=0.

Step 3

$$|U_t - U_0 - tV_0|_{3/2-\varepsilon} \leq Ct^2|u| \quad \text{for all } u \in C^\infty(\mathcal{E}_0) \text{ and all t small.} \tag{4.11}$$

To prove (4.11) write $A(t) = t\dot{A}(0) + R(t)$ where $R(t) \in End(\tilde{\mathcal{E}})$ and

$$\|R(t)\|_{C^2} \;=\; O(t^2) \;\text{ as } t \to 0. \tag{4.12}$$

The equation $\mathcal{D}_t U_t = \delta \otimes u$ can be rewritten as

$$\mathcal{D}_0 U_t + t\dot{A}(0) U_t + R(t) U_t \;=\; \delta \otimes u.$$

Using (4.10) and $\mathcal{D}_0 U_0 = \delta \otimes u$ we deduce

$$\mathcal{D}_0(U_t - U_0 - tV_0) \;=\; -t\dot{A}(0)(U_t - U_0) - R(t)U_t.$$

Hence by elliptic estimates we have

$$|U_t - U_0 - tV_0|_{3/2-\varepsilon} \leq C\left(t|\dot{A}(0)(U_t - U_0)|_{1/2-\varepsilon} + |R(t)U_t|_{1/2-\varepsilon}\right).$$

The estimate (4.11) follows easily using (4.5), (4.7) and (4.12).

Coupling elliptic estimates in (4.10) with the relations (4.5) and (4.6) we get

$$|V_0|_{3/2-\epsilon} \leq C|u|. \tag{4.13}$$

The reader can now verify immediately that $t \to \Pi_t$ is C^1 and

$$\dot{\Pi}_0 u = R_0^- V_0$$

Proposition 4.6 is proved. ◇

We can now relate the Dirac operators and their CD spaces to the infinite dimensional symplectic topology of the previous chapters. All this setup lies over a natural symplectic background. Indeed c(ds) is a fiberwise isometry so it defines a unitary operator

$$J : L^2(\mathcal{E}_0) \to L^2(\mathcal{E}_0) \quad , \quad J^2 = -I$$

i.e. J is a complex structure on $L^2(\mathcal{E}_0)$ thus defining a symplectic structure

$$\omega(u,v) = \int_\Sigma (Ju, v)$$

for all $u, v \in L^2(\mathcal{E}_0)$.

The next result is the key fact which unifies all the topics discussed so far. It is another manifestation of the duality *Selfadjoint operators* $\sim$ *Lagrangian subspaces.*

Proposition 4.7 *([BW2], Prop.3.2) $\Lambda(D)$ is a lagrangian subspace of $L^2(\mathcal{E}_0)$ with respect to the natural symplectic structure induced by the Clifford multiplication with ds.*

Finally consider the following situation. (M,g) is a compact oriented Riemann manifold and $\mathcal{E}$ a selfadjoint C(M)-module over M. Let Σ be an oriented

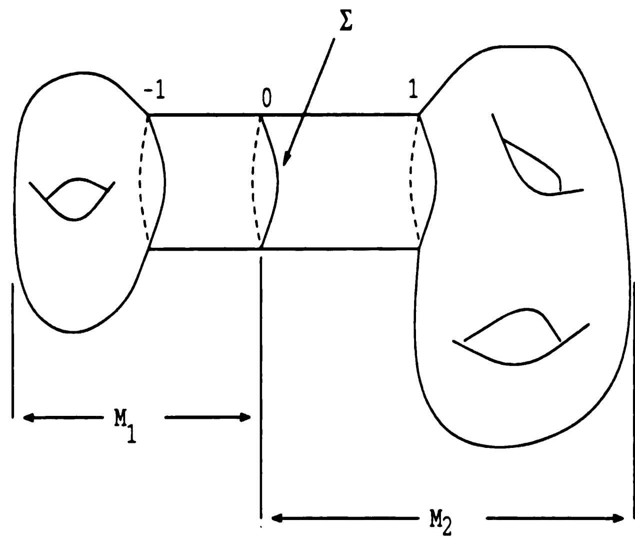


Figure 4.2: *A metrically nice splitting*

hypersurface in M which divides it into two manifolds-with-boundary M_1, M_2. Choose N_1, N_2 tubular neighborhoods of M_1, M_2 such that $N_1 \cong \Sigma \times (-1, 0]$, $N_2 \cong \Sigma \times [0, 1)$ (see Fig. 4.2). Set $N = N_1 \cup N_2$. We assume the metric g is a product metric on N i.e. $g|_N = ds^2 + h$, where h is a metric on Σ and $-1 < s < 1$ is the longitudinal coordinate on N. Let $D : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ be a Dirac operator on M. Denote by D_1 (resp. D_2) its restrictions to M_1 (resp. M_2). D will be called **cylindrical** if both D_1 and D_2 are cylindrical. As usual set $\mathcal{E}_0 = \mathcal{E}|_\Sigma$. $L^2(\mathcal{E}_0)$ has a symplectic structure induced by the Clifford action. The CD spaces $\Lambda_1(D)$, $\Lambda_2(D)$ of D_1 and D_2 are lagrangian subspaces in $L^2(\mathcal{E}_0)$. In fact more can be said.

Proposition 4.8 $(\Lambda_1(D), \Lambda_2(D))$ *is a Fredholm pair. Moreover* (Λ_1, Λ_2) *is a transversal pair if and only if D is invertible.*

Proof Let P_j be the orthogonal projection onto Λ_j, j=1,2. We have seen that these are 0-th order pseudodifferential operators in $L^2(\mathcal{E}_0)$. In [S] (see also [P2], Chap.XVII) it is proved that their symbols satisfy

$$\sigma(P_1)(\xi) + \sigma(P_2)(\xi) = Id.$$

Thus $P_1 - (I - P_2)$ is a pseudodifferential operator of order ≤ -1 in $L^2(\mathcal{E}_0)$. In particular $P_1 - (I - P_2)$ is compact so that (Λ_1, Λ_2) is a Fredholm pair. The second part is intuitively clear (see also [BW2] Corollary 3.4). $\diamond$

Chapter 5

The Maslov index and the spectral flow

The setting of this chapter is identical to the one at the end of Chapter 4. We endow the space of cylindrical Dirac operators $\mathcal{D}$ with a Sobolev topology, given by a L_k^2 norm with k sufficiently large so that $L_k^2 \hookrightarrow C^2$. Inside $\mathcal{D}$ sits

$$\mathcal{D}^* = \{D \in \mathcal{D} \ ; \ D \text{ is invertible}\}$$

To any continuous path $\gamma = D(t)$ in $\mathcal{D}$ with endpoints in $\mathcal{D}^*$ one can associate an integer, *the spectral flow* $SF(\gamma)$ (see [APS3], [BW1]) defined as the number of eigenvalues of $D(t)$ that change from negative to positive minus the number of eigenvalues that change from positive to negative. This is a homotopy invariant of γ (cf. [AS], [BW1]) with an obvious additivity property. If $\gamma_1, \gamma_2 : (I, \partial I) \to (\mathcal{D}, \mathcal{D}^*)$ with $\gamma_1(1) = \gamma_2(0)$ then

$$SF(\gamma_1 \cdot \gamma_2) = SF(\gamma_1) + SF(\gamma_2).$$

so the spectral flow can be viewed as a homomorphism

$$SF : \pi_1(\mathcal{D}, \mathcal{D}^*) \to \mathbf{Z}.$$

In the previous chapter have defined a continuous map

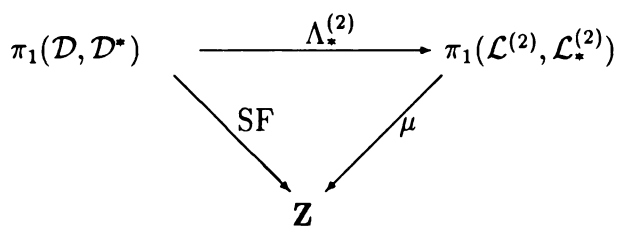
$$\Lambda^{(2)} : (\mathcal{D}, \mathcal{D}^*) \to (\mathcal{L}^{(2)}, \mathcal{L}^{(2)}_*) \ , \ \ D \mapsto (\Lambda_1(D), \Lambda_2(D)).$$

Denote by $\Lambda^{(2)}_*$ the homomorphism between π_1's induced by this map.

In Chapter 3 we constructed the Maslov index isomorphism

$$\mu : \pi_1(\mathcal{L}^{(2)}, \mathcal{L}^{(2)}_*) \to \mathbf{Z}.$$

We will prove that the following diagram is commutative.



To this end we will need a localization procedure for the spectral flow. Let $t \mapsto D(t) \in \mathcal{D}$ ($|t| \leq \varepsilon$) be a smooth family of cylindrical Dirac operators such that D(t) is invertible for $t \neq 0$. Let $K_0 = \ker D(0)$ and denote by P_0 the orthogonal projection onto K_0. We form the **resonance matrix**:

$$R = R(A) : K_0 \to K_0 \quad R = P_0 \dot{D}(0)$$

We can view R as a symmetric matrix. We have the following result ([DRS]).

Theorem 5.1 *Let D and A as above satisfying (1). If the resonance matrix R(A) is nondegenerate then its signature gives the spectral flow*

$$SF(D(t)\, ; |t| \leq \varepsilon) = \operatorname{sign} R(A)$$

The above formula follows from an abstract result of Kato (Thm.II.5.4 and II.6.8 of [K])which we recall now. H is a separable Hilbert space and A(t) $t \in \mathbf{R}$ a family of unbounded selfadjoint operators with a fixed dense domain W. W becomes a Hilbert space in its own right using the graph norms. We assume that the embedding $W \hookrightarrow H$ is compact and that the resolvent set of A(t) is nonempty for every t. Then A(t) has compact resolvent and its spectrum consists entirely of eigenvalues with finite multiplicities. A(t) can also be interpreted as bounded operators $W \rightarrow H$. As such we assume that A(t) depends smoothly upon t. The following result gives a precise information about how the eigenvalues of A vary.

Theorem 5.2 (Kato Selection Theorem) *Let $t_0 \in \mathbf{R}$ and $c_0 > 0$ such that $\pm c_0 \notin \sigma(A(t_0))$. Then there exists a constant $\varepsilon > 0$ and differentiable functions $\lambda_j : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow (-c_0, c_0)$, $j = 1, 2, \cdots, N$ (N is the dimension of the subspace spanned by the eigenvectors corresponding to eigenvalues in $(-c_0, c_0)$) such that $\lambda_j(t) \in \sigma(A(t))$ and*

$$\dot{\lambda}_j(t) \in \sigma(P_j(t))\dot{A}(t)P_j(t)$$

where $P_j(t) : H \rightarrow H$ denotes the orthogonal projection onto $ker\,(\lambda_j(t)I - A(t))$. Moreover if $\lambda \in \sigma(A(t)) \cap (c_0, c_0)$ with corresponding spectral projection $P : H \rightarrow ker\,(\lambda I - A(t))$ and $\theta \in \sigma(P\dot{A}(t)P$ is an eigenvalue of multiplicity m then there are precisely m indices $j_1, \cdots, j_m$ such that $\lambda_{j_\nu}(t) = \lambda$ and $\dot{\lambda}_{j_\nu}(t) = \theta$ for $\nu = 1, \cdots, m$.

Kato Selection Theorem has a corollary particularly important for our purposes. To formulate it introduce the set of **positive** cylindrical endomorphisms

$$Cyl(\mathcal{E})_+ = \{A \in Cyl(\mathcal{E}) \,/\, \exists\lambda > 0 : \inf\sigma(A(x)) \geq \lambda \;\forall x \in M\}$$

where $\sigma(A(x))$ is the spectrum of the selfadjoint endomorphism

$$A(x) : \mathcal{E}_x \rightarrow \mathcal{E}_x.$$

Set

$$\mathcal{P} = \{\gamma : (I, \partial I) \to (\mathcal{D}, \mathcal{D}^*) \ ; \ \gamma \in C^1\}.$$

A path $\gamma \in \mathcal{P}$ is called **positive** if $\dot\gamma \in Cyl(\mathcal{E})_+$ and **negative** if $-\dot\gamma \in Cyl(\mathcal{E})_+$. The set of positive (resp. negative) paths is denoted by $\mathcal{P}_+$ (resp. $\mathcal{P}_-$). The **resonance** set $Z = Z(\gamma)$ of a path $\gamma \in \mathcal{P}$ is defined as

$$Z = \{t \in I \ ; \ \ker D(t) \neq 0\}.$$

We can now formulate

Lemma 5.3 *The resonance set of a positive path is finite.*

Proof Let $\gamma = D(t) \in \mathcal{P}_+$ and $t_0 \in Z(\gamma)$. Since $\dot D(t_0) \in Cyl(\mathcal{E})_+$ the resonance matrix is positive definite and by Kato selection theorem we deduce that D(t) is invertible when t is in some ε-neighborhood of t_0. Therefore $Z(\gamma)$ is a discrete set. $\diamond$

Positive paths have other important properties.

Lemma 5.4 *Any path $\gamma \in \mathcal{P}$ is homotopic to a product of a positive path with a negative path. (In the sequel all the homotopies of paths $(I, \partial I) \to (\mathcal{D}, \mathcal{D}^*)$ will be understood as relative homotopies - the endpoints stay invertible during the deformation).*

Proof The difference $A = D(1) - D(0) \in Cyl(\mathcal{E})$ is a bounded selfadjoint endomorphism of $\mathcal{E}$. Choose $C > 0$ such that

$$C \geq 1 + |\sup \sigma(A(x))| \ \forall x \in M \tag{5.1}$$

$$D(0) + C \cdot Id_{\mathcal{E}} \in \mathcal{D}^*. \tag{5.2}$$

The choice (5.2) is possible by Lemma 5.3. Now consider

$$\alpha_+ = D(0) + t\,C \cdot Id_{\mathcal{E}} \quad t \in I$$

$$\alpha_- = D(0) + C \cdot Id_{\mathcal{E}} + t\,(A - C \cdot Id_{\mathcal{E}}) \quad t \in I.$$

By (5.1) and (5.2) $\alpha_\pm \in \mathcal{P}_\pm$. γ is homotopic to $\alpha_+ \cdot \alpha_-$ via an affine homotopy. ◇

Definition 5.5 *A C^1 path $\gamma : (I, \partial I) \to (\mathcal{D}, \mathcal{D}^*)$, $t \mapsto D(t)$ is called*

*(i)***locally affine** *if $\dot\gamma$ = const. in a neighborhood of any $t \in Z(\gamma)$;*

(ii) **standard** *if $Z(\gamma)$ is finite and $\forall\, t \in Z(\gamma)$ dim ker $D(t) = 1$.*

A key step in our deformation program is a genericity result which states that almost any path of Dirac operators is standard.

Proposition 5.6 *Let D be a cylindrical Dirac operator and assume* rank$\mathcal{E} \geq 2$. *Then there exists a Baire set $\mathcal{A}_{\mathrm{reg}} \subset \mathcal{A}$ such that for $A \in \mathcal{A}_{\mathrm{reg}}$ the path $D(t) = D + A(t)$ is standard.*

The proof of this proposition is carried out in the Appendix.

In particular since $\mathcal{P}_+$ is open in $\mathcal{P}$ we deduce

Corollary 5.7 *Any positive path is homotopic to a positive standard path.*

A simple application of Kato's selection theorem yields

Lemma 5.8 *A positive standard path $\gamma \in \mathcal{P}$ is homotopic to a locally affine positive standard path $\tilde\gamma$ such that:*

(i) $Z(\gamma) = Z(\tilde\gamma)$

(ii) $\forall\, t \in Z(\gamma) \;:\; \gamma(t) = \tilde\gamma(t)$.

Proof The underlying idea is natural: any path is locally homotopic to the tangent line at a point on the path. The only thing we have to prove is that we can find a **relative** homotopy achieving this. Assume $\gamma : [-1,1] \to \mathcal{D}$ an d $Z(\gamma) = \{0\}$. Set D(t) =D(0) + A(t) and $A_0 = \dot{A}(0)$. A_0 is a positive cylindrical endomorphism of $\mathcal{E}$. Consider

$$\tilde{D}_s(t) = (1-s)D(t) + stA_0 \quad s \in [0,1]$$

By Kato Selection Theorem there exists $\varepsilon > 0$ such that $\forall\ 0 < |t| \leq \varepsilon$ D(t) is invertible and its inverse $E(t)$ satisfies

$$\|E(t)\| = O\left(\frac{1}{t\|A_0\|}\right) \tag{5.3}$$

Now

$$\tilde{D}_s(t) = D(t) + R_s(t)$$

where $R_s(t) = s(tA_0 - A(t))$ satisfies

$$\|R_s(t)\| = o(t) \quad \text{uniformly in s} \tag{5.4}$$

Thus

$$E(t)\tilde{D}_s(t) = I + K_s(t) \quad K_s(t) = E(t)R_s(t) \tag{5.5}$$

where (by (5.3))

$$\|K_s(t)\| = o(1) \quad \text{uniformly in s} \tag{5.6}$$

Hence we can find $t_0 > 0$ such that

$$\|K_s(\pm t_0)\| < 1/2 \quad \forall s \in [0,1]$$

and from (5.5) we deduce that $\tilde{D}_s(\pm t_0)$ is invertible for all s. Therefore $\tilde{D}_s(t)$ is an admissible homotopy between γ and a locally affine path satisfying properties (i) and (ii) in the lemma. ◇

The homotopies constructed so far were between paths close to each other in the C^1 distance. Our next result describes one instance of homotopic paths which can be C^1-far apart (but still C^0-close).

Lemma 5.9 *Let $D \in \mathcal{D}$ and $A \in Cyl(\mathcal{E})$ such that dim ker $D = 1$,*

$$\ker D = \operatorname{span}(U)$$

and

$$(AU, U) \neq 0.$$

If $B \in Cyl(\mathcal{E})$ is such that

$$(BU, U) = (AU, U)$$

then $\exists\, \varepsilon > 0$ such that $\forall\, 0 < |t| \leq \varepsilon$ and $\forall\, s \in I$

$$\gamma_s(t) = D + (1-s)(tA) + s(tB) \in \mathcal{D}^*.$$

In particular $\gamma_s(\cdot) \in \mathcal{P}$ realizes an affine homotopy between $\gamma_0(t) = D + tA$ and $\gamma_1(t) = D + tB$ ($|t| \leq \varepsilon$).

Proof The paths $\gamma_s(t)$ are analytic in t (being affine). In such situations more powerful perturbation results are available. In particular by Thm.VII 3.9 of [K] there exists $\varepsilon_1 > 0$ and analytic functions

$$\lambda_{n,s} : [-\varepsilon_1, \varepsilon_1] \rightarrow \mathbf{R} \quad n \in \mathbf{Z} \quad s \in [0,1]$$

such that

$$\sigma(\gamma_s(t)) = \{\lambda_{n,s}(t) \ / \ n \in \mathbf{Z}\} \text{ (multiplicities included).}$$

We labelled the eigenvalues so that $\lambda_{0,s}(0) = 0 \in \ker D$. Note that $\lambda_{n,s}(0)$ is independent of s for all $n \in \mathbf{Z}$. We will denote it by λ_n. On the other hand we can find $a, b > 0$ such that $\forall v \in C^\infty(\mathcal{E})$

$$\|\dot{\gamma}_s(t)v\| \leq a\|v\| + b\|\gamma_s(t)v\| \;\; \forall |t| \leq \varepsilon \;\; \forall s \in [0,1].$$

Theorem VII 3.6 of [K] implies

$$|\lambda_{n,s}(t) - \lambda_n| \leq C(1+\lambda_n)t$$

where $C = C(a,b) > 0$ is independent of $n \in \mathbf{Z}$ and $s \in [0,1]$. In particular for

$$0 < |t| \leq \varepsilon_2 = \inf\left\{\frac{|\lambda_n(0)|}{2C(1+|\lambda_n(0)|)} \;/\; n \in \mathbf{Z} \setminus \{0\}\right\} \cup \{\varepsilon_1\}$$

$$|\lambda_{n,s}(t)| \geq 1/2|\lambda_n(0)| > 0. \tag{5.7}$$

On the other hand by Kato Selection Theorem

$$\dot{\lambda}_{0,s}(0) = (AU, U) \neq 0 \;\; \forall s \in [0,1].$$

Arguing by contradiction we can find $0 < \varepsilon < \varepsilon_2$ such that

$$\lambda_{0,s}(\pm\varepsilon) \neq 0. \tag{5.8}$$

In particular (5.7) and (5.8) show that the operators $\gamma_s(\pm\varepsilon)$ are invertible for any s and Lemma 5.9 is proved. ◇

Definition 5.10 *A standard path $\gamma \in \mathcal{P}$ is called* **elementary** *if $\forall t \in Z(\gamma)$*

$$\dot{\gamma}(t) = \alpha Id_{\mathcal{E}}$$

for some $\alpha \in C_0^\infty(M)$ a function supported in $M_2 \setminus N$ and not changing sign (see Fig. 5.1).

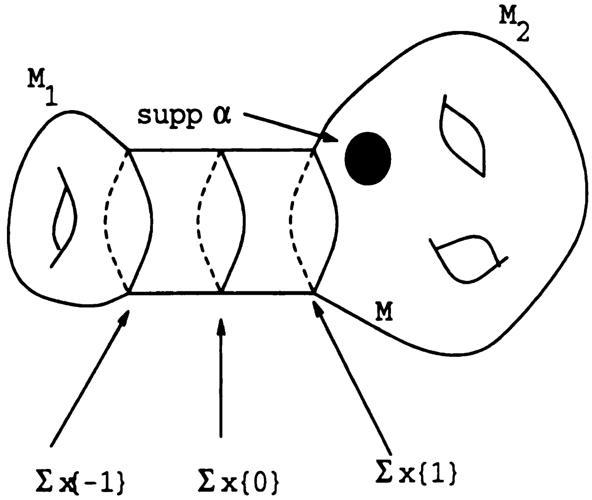


Figure 5.1: *The cutoff function* α

Remark 5.11 *If* $\alpha \in C_0^\infty(M)$ *is as in Definition 5.2 and* $D \in \mathcal{D}$ *then the unique continuation principle for Dirac operators ([BW4], Chap.8) implies that*

$$(\alpha U, U) \neq 0 \;\; \forall U \in \ker D \setminus \{0\}.$$

Lemmata 5.3, 5.4, 5.8, 5.9, Corollary 5.7 and Remark 5.11 have the following corollary.

Corollary 5.12 *Any path* $\gamma \in \mathcal{P}$ *is homotopic to an elementary path.*

In particular we have the following abstract result.

Proposition 5.13 *Let* $\phi : \mathcal{P} \to \mathbf{Z}$ *be a* **continuous, additive** *function such that for any elementary standard path* ω

$$\phi(\omega) = SF(\omega)$$

and $\Phi(\gamma) = 0$ *for every* $\gamma : I \to \mathcal{D}^*$. *Then* $\forall\, \gamma \in \mathcal{P} : \phi(\gamma) = SF(\gamma)$

Let $\gamma \in \mathcal{P}$, $\gamma(t) = D(t)$. Denote by $D_j(t)$ the restriction of D(t) to M_j j=1,2. Let $\Lambda_j(t)$ be the CD space of $D_j(t)$ j=1,2. Since D(0) and D(1) are invertible we deduce $\Lambda_1(t) \cap \Lambda_2(t) = 0$ for t=0,1. The results of section 4 show that the Fredholm pairs of lagrangians $(\Lambda_1(t), \Lambda_2(t))$ vary smoothly with t. In particular the Maslov index $\mu(\Lambda_1(t), \Lambda_2(t))$ is well defined. We can now state the main result of this paper

Theorem 5.14 *For any path $\gamma \in \mathcal{P}$ as above we have*

$$SF(\gamma) = \mu(\Lambda_1(t), \Lambda_2(t)) \tag{5.9}$$

Proof We have defined a map $\phi : \mathcal{P} \to \mathbf{Z}$

$$\phi : \gamma = D(t) \mapsto \mu\left(\Lambda_1(D(t)), \Lambda_2(D(t))\right).$$

By Propositions 4.6 and 4.8 we see that ϕ is continuous and $\phi = 0$ on the paths in $\mathcal{D}^*$. By Proposition 5.13 it suffices to check (5.9) on elementary paths. Thus fix a cylindrical Dirac operator such that

$$\ker D = \text{span}(F_0) \ , \ |F_0| = 1$$

and consider the family $D(t) = D + t\alpha I$ with $|t| \leq \varepsilon$ where α is a smooth not-changing-sign function ,compactly supported inside M_2, away from the neck N. The operator $D_1(t)$ is not changing since α is supported outside M_1. Thus

$$\Lambda_1(t) \stackrel{def}{\equiv} \Lambda_0$$

is constant and $\Lambda(t) \stackrel{def}{=} \Lambda_2(t)$ is varying.

Let U(t) be a smooth path of unitary operators on $L^2(\mathcal{E}_0)$ such that

$$U(0) = I \ , \ \Lambda(t) = U(t)\Lambda(0).$$

Set $f_0 = RF_0$,$f_t = U(t)f_0$ be the restriction of F_0 to Σ (we adopt the convention of using capital letters for sections of $\mathcal{E}$ defined over M, M_1 or M_2 and small letters for sections of $\mathcal{E}$ defined only over Σ). Then f_t lies in $\Lambda(t)$ so there exists an unique $F_t \in \ker D_2(t)$ such that

$$\begin{cases} D_2(t)F_t &= 0 \quad in\ M_2 \\ RF_t &= U_t f_0 \quad on\ \Sigma \end{cases} \tag{5.10}$$

$U_t f_0$ varies smoothly with t and the boundary estimates of Prop.4.4 imply that F_t depends smoothly upon t as well. Derivating (5.10) at t= 0 (the dot will denote the t-derivative at t=0) and noting that $\dot{D}_2 = \alpha I$ we get

$$\begin{cases} D_2(0)\dot{F}_0 \ + \ \alpha F_0 &= 0 \quad in\ M_2 \\ \qquad\qquad R\dot{F}_0 &= \dot{U} f_0 \quad on\ \Sigma \end{cases}$$

Multiplying by F_0 we get

$$-(\alpha F_0\,,\,F_0) \;=\; (D_2(0)\dot{F}_0\,,\,F_0).$$

Now if we integrate by parts in the above equality and use (5.10) we obtain

$$\int_{M_2}\langle D_2(0)\dot{F}_0\,,\,F_0\rangle \;=\; -\int_{\Sigma}\langle J\dot{F}_0\,,\,F_0\rangle \;+\; \int_{M_2}\langle \dot{F}_0\,,\,D_2(0)F_0\rangle = -(J\dot{U}f_0\,,\,f_0)$$

Thus

$$(\alpha F_0\,,\,F_0) \;=\; (J\dot{U}(0)f_0\,,\,f_0) \;=\; \omega(\dot{U}f_0\,,\,f_0). \tag{5.11}$$

By unique continuation $(\alpha F_0\,,\,F_0) \neq 0$. The sign of the left hand side of (5.11) is equal to SF(D(t); $|t| \leq \varepsilon$) by Theorem 5.1. The sign of the right hand side is equal to the Maslov index $\mu(\Lambda, \Lambda(t))$ by Corollary 3.6. This completes the proof. ◇

Chapter 6

Adiabatic limits of CD spaces

Consider a manifold with boundary M as in Chapter 4 and D a neck-compatible Dirac operator on M. Define $M(r) = M \cup \Sigma \times [0,r]$. M(r) is usually called an *adiabatic deformation* of M; (see Fig. 6.1). D has a natural extension D(r) as a neck-compatible Dirac on M(r) and denote by $\Lambda^r \subset L^2(\mathcal{E} \mid_{\partial M(r)})$ the CD space of D(r).

In this chapter we will study the behavior of Λ^r as $r \to \infty$. On the tube $\Sigma \times [0,\infty)$ the operator D has the cylindrical form D = c(ds)($\partial/\partial s + D_0$) so that at least formally we may write $\Lambda^r = e^{-D_0 r}\Lambda^0$ i.e. we are dealing with a dynamics problem on a lagrangian grassmanian. From this representation we see that the part of Λ^0 "interacting" with the negative spectrum of D_0 will have a dominant effect as $r \to \infty$ while we expect that the "interactions" with the positive spectrum will "soften" as r increases. We may continue our formal discussion by observing that since D_0 anticommutes with J it lies in the "Lie algebra" of the infinite dimensional symplectic group so that the "flow" $e^{-D_0 r}$ is a 1-parameter group of symplectic transformations of H and the family Λ^r is a trajectory in an infinite dimensional lagrangian grassmanian. Unfortunately these observations are purely formal since D_0 cannot generate a semigroup (the spectrum is unbounded both

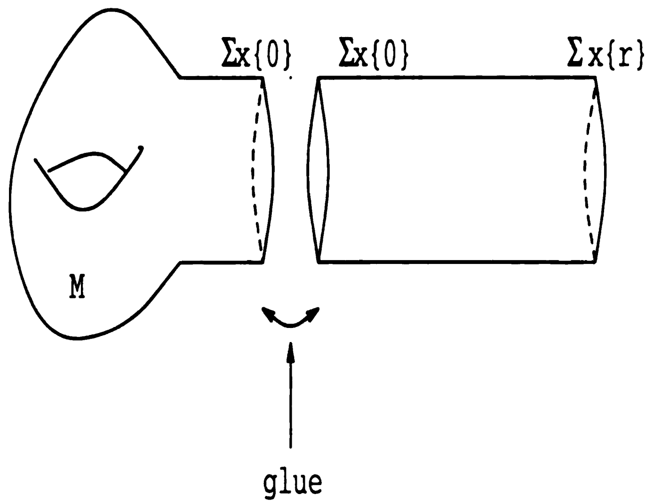


Figure 6.1: *Adiabatic deformation of the neck*

from bellow and from above). However, in finite dimensions this discussion makes sense and the first result of this section, Proposition 6.3, describes the asymptotics of this flow. The study of the infinite dimensional situation will ultimately rely on this result via a careful symplectic reduction.

Since we will be dealing with asymptotics of families of subspaces it is appropriate to begin our presentation by discussing ways to measure the distance between two closed subspaces in a Hilbert space. The right notion is provided by the **gap distance** between two subspaces introduced in [K].

Let X,Y be two closed subspaces of a Hilbert space H. Define

$$\delta(X,Y) \;=\; \sup\{\operatorname{dist}(x,Y)\;\;;\; x \in X \;\; |x| = 1\}.$$

δ is in general not symmetric in X and Y. We symmetrize it by defining the **gap** between X and Y as

$$\hat{\delta}(X,Y) \;=\; \max\{\delta(X,Y), \delta(Y,X)\}.$$

Note that $\delta(X,Y)$ can also be characterized as the smallest number δ such that

$$\operatorname{dist}(x,Y) \leq \delta|x| \;\;\; \forall \;\; x \in X.$$

We say $X_n \to X$ if $\hat{\delta}(X_n, X) \to 0$. In particular if P_n are the orthogonal projections onto X_n then

$$X_n \to X \iff P_n \to P \text{ in norm.}$$

Thus if H is symplectic the gap topology on the space $\mathcal{L}$ of lagrangians is equivalent with the natural topology (defined by the identification (2.2)). Although the function $\delta(.,.)$ is in general not symmetric in its arguments it becomes symmetric when restricted to $\mathcal{L}$. Indeed by Thm. IV.2.9 of [K] we have

$$\delta(L_1, L_2) = \delta(L_2^\perp, L_1^\perp).$$

Since L_1 , L_2 are lagragians

$$\delta(L_2^\perp, L_1^\perp) = \delta(JL_2, JL_1) = \delta(L_2, L_1).$$

At the last step we have used the fact that J is an isometry. Thus

$$L_n \to L_* \text{ in } \mathcal{L} \iff \delta(L_n, L_*) \to 0.$$

In studying convergence of sequences of subspaces it is very convenient to have a method to "renormalize" them (much like the homogeneous coordinates in the projective spaces). We can achieve this if we can represent these subspaces as graphs of linear operators. This representation is possible once some obvious transversality conditions are assumed (compare with Arnold's charts on lagrangian grassmanians). When these renormalizations are possible there are ways to relate the gap topology with the norm topology of linear operators. In particular we will frequently use the following results. Their proofs can be found in [K].

Lemma 6.1 *Let H_1 and H_2 be two separable Hilbert spaces and consider a sequence (T_n) of bounded linear operators $T_n : H_1 \to H_2$ with graphs $G(T_n) \subset H_1 \oplus H_2$. Then the following are equivalent* *(i)* $T_n \to T$ *as* $n \to \infty$ *in norm;*

(ii)$G(T_n) \to G(T)$ *as* $n \to \infty$ *in gap.*

Now consider H $= \mathbf{R}^{2n}$ with the complex structure J induced by the identification $\mathbf{R}^{2n} \cong \mathbf{C}^n$. J defines a symplectic structure ω by $\omega(x,y) = (Jx,y)$ for all $x, y \in H$. The symplectic group is then

$$Sp(n,\mathbf{R}) = \{T \in GL(2n,\mathbf{R}) \;/\; T^*JT = I\}.$$

$Sp(n,\mathbf{R})$ is a Lie group with Lie algebra

$$\underline{sp}(n,\mathbf{R}) = \{A \in \underline{gl}(2n,\mathbf{R}) \;/\; A^*J + JA = 0\}.$$

Inside $\underline{sp}(n,\mathbf{R})$ sits the subspace

$$\sigma(n) = \{A \in \underline{sp}(n,\mathbf{R}) \;/\; A = A^*\}$$

consisting of selfadjoint matrices anticommuting with J. Denote by $\Lambda(n)$ the lagrangian grassmanian of $(\mathbf{R}^{2n}, J)$. $Sp(n,\mathbf{R})$ acts (transitively) on $\Lambda(n)$. In particular any $A \in \sigma(n)$ defines a 1-parameter group of diffeomorphisms of $\Lambda(n)$: $r \mapsto e^{-rA}$. The problem we intend to discuss is that of the asymptotic behavior of the above flow on $\Lambda(n)$. Fix $A \in \sigma(n)$ and consider

$$I_A = \{L \in \Lambda(n) \;/\; AL \subset L\}$$

the family of invariant lagrangians of A. The lagrangians in I_A are stationary points of the flow $r \mapsto e^{-rA}$.

Let us now describe the dynamics of e^{-rA} in a simple but instructive case.

Example 6.2 Take n=1 and fix $A \in \sigma(1)\backslash\{0\}$. We can then choose $e \in \mathbf{R}^2$, $|e| = 1$ such that in the basis (e,Je) the operator A has the form A $=$ diag $(\lambda\,, -\lambda\,)$.

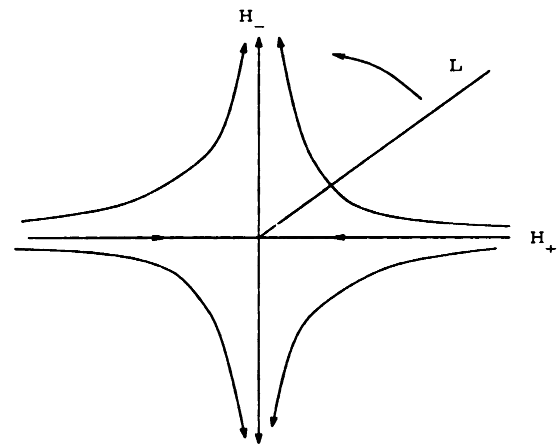


Figure 6.2: *Hyperbolic flow*

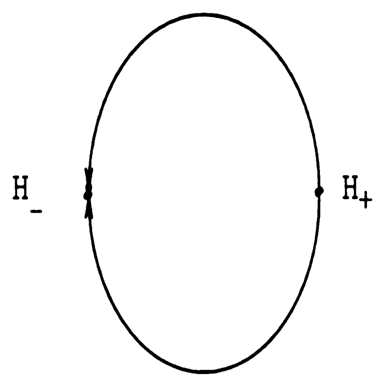


Figure 6.3: *Dynamics on* $\Lambda(1)$

Viewed as a (linear) flow on $\mathbf{R}^2$ e^{-rA} has the hyperbolic phase portrait depicted in Fig. 6.2. The lagrangians of $\mathbf{R}^2$ are the lines through the origin so that $\Lambda(1) \cong \mathbf{RP}^1 \cong S^1$. e^{-rA} becomes diag$(e^{-\lambda r}, e^{\lambda r})$. $H_- = \text{span}(f)$ and $H_+ = \text{span}(e)$ are the only stationary points of the flow. If $L \neq H_+$ then one sees from Fig. 6.2 that

$$e^{-rA}L \longrightarrow H_- \quad \text{exponentially as } r \longrightarrow \infty.$$

The phase portrait of e^{-rA} on $\Lambda(1)$ is then the one described in Fig. 6.3. In particular we have shown that $\forall L \in \Lambda(1)$ $e^{-rA}L$ has a limit in I_A as $r \longrightarrow \infty$. $\diamond$

The situation presented in the example above is a manifestation of a more general phenomenon.

Proposition 6.3 *Let a be a real $n \times n$ symmetric matrix. Then for any subspace $U \subset \mathbf{R}^n$ there exists U_∞ - an invariant subspace of A such that*

$$\lim_{r \to \infty} e^{-rA} U = U_\infty$$

Proof Assume $\sigma(A) = \{\lambda_1 \leq \cdots \leq \lambda_n\}$ with the corresponding orthonormal spectral basis $e_1, \cdots, e_n$.Pick $u_1, \cdots, u_m$ (m = dim U) a basis of U. Then

$$u_i = \sum_{j=1}^{n} c_{ij} e_j \quad i = 1, \cdots, m$$

and we can form the matrix

$$C = (c_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$$

We may assume C is upper triangular. Otherwise we can reduce it to this form by performing row operations (which is equivalent to choosing a different basis for U). For each $1 \leq i \leq m$ let j(i) be the smallest j such that $c_{ij} \neq 0$. Since C is upper triangular

$$j(1) < \cdots < j(m). \tag{6.1}$$

For $r \geq 0$ let

$$v_i(r) = \frac{1}{c_{ij(i)}} e^{r\lambda_{j(i)}} e^{-rA} u_i.$$

$v_1(r), \cdots, v_m(r)$ form a basis of $e^{-rA}U$ and moreover

$$v_i(\infty) = \lim_{r \to \infty} v_i(r)$$

exists and for all i and

$$v_i(\infty) = e_{j(i)} + \sum_{k > j(i)} \nu_{ik} e_k.$$

From (6.1) we deduce that $v_i(\infty)$ are linearly independent and therefore

$$e^{-rA}U \to U_\infty = \text{span}(v_1(\infty), \cdots, v_m(\infty)).$$

Proposition 6.3 is proved. $\diamond$

As a consequence we have

Corollary 6.4 *Let $L \in \Lambda(n)$ and $A \in \sigma(n) \setminus \{0\}$. Then there exists $L_\infty \in I_A$ such that*

$$\lim_{r \to \infty} e^{-rA} L = L_\infty$$

Remark 6.5 *In [N2] we showed that the flow $L \mapsto e^{At} L$ on $\Lambda(n)$ is the gradient flow of a $\mathbf{Z}_2$-perfect Morse function. Moreover the unstable manifolds corresponding to the critical submanifolds of this define a Schubert-type decomposition of the lagrangian grassmanian.*

We now return to our original problem. Thus M is a manifold with boundary and D is a neck-compatible Dirac operator (throughout this section all Dirac operators on manifolds with boundary will be assummed neck-compatible) and set $M(r) = M \cup \Sigma \times [0, r]$. D has a natural extension D(r) as a neck-compatible Dirac on M(r) and denote by Λ^r the CD space of D(r). We are interested in the adiabatic limit $\lim_{r \to \infty} \Lambda^r$. As usual set $D_0 = D|_\Sigma$. For any real number E we denote by $\mathcal{H}^E_>$ (resp. $\mathcal{H}^E_\geq$, $\mathcal{H}^E_<$, $\mathcal{H}^E_\leq$, $\mathcal{H}^E_0$) the subspace of $L^2(\mathcal{E}_0)$ spanned by eigenvectors corresponding to eigenvalues $> E$ ($\geq E$, $< E$, $\leq E$ and resp. in $[-|E|, |E|\,]$). In the sequel we will frequently use the following technical result.

Lemma 6.6 *For any $U \subset L^2(\mathcal{E}_0)$ finite dimensional subspace and any real E the pair $(\Lambda^r(D), \mathcal{H}^E_> \oplus U)$ is Fredholm.*

For a proof of this lemma we refer to [BW4]. For nonnegative E the space $\mathcal{H}^E_>$ is an isotropic subspace of $L^2(\mathcal{E}_0)$. By the above lemma the pair $(\Lambda^r(D), \mathcal{H}^E_>)$ is Fredholm so according to Lemma 3.9 we can construct the symplectic reduction of Λ^r mod $\mathcal{H}^E_>$:

$$L^r_E = \frac{\left(\Lambda^r \cap \mathcal{H}^{-E}_{\geq}\right)}{\mathcal{H}^E_>}. \tag{6.2}$$

(The symplectic reduction of $\Lambda = \Lambda^0$ mod $\mathcal{H}^E_>$ will be denoted by L_E). These are lagrangian subspaces in the symplectic vector space $\mathcal{H}^E_0$. Set $A_E = D_0|_{\mathcal{H}^E_0}$.

Lemma 6.7 *The set*

$$\mathcal{N}(D) = \{E \geq 0 \;/\; \Lambda(D) \cap \mathcal{H}^E_> = 0\}$$

is a nonempty, closed, unbounded interval.

Proof Consider an increasing sequence $E_n \to \infty$. Using Lemma 6.6 we obtain a decreasing sequence of finite dimensional vector spaces

$$U_n = \Lambda \cap \mathcal{H}^{E_n}_>.$$

In particular there exists an $m > 0$ such that

$$U_m = U_{m+1} = \cdots$$

On the other hand $\bigcap U_n = 0$. Thus $U_m = 0$ and therefore $E_m \in \mathcal{N}(D)$. Since the spectrum of D_0 is discrete we deduce that $\mathcal{N}(D)$ is closed. It is an unbounded interval because $\left(\mathcal{H}^E_>\right)_{E\geq 0}$ is a decreasing family of (isotropic) subspaces of $L^2(\mathcal{E}_0)$. $\diamond$

Definition 6.8 *The set* $\mathcal{N}(D)$ *is called the* **nonresonance range** *of* D.

$\nu(D) = \min \mathcal{N}(D)$ *is called the* **nonresonance level** *of* D. *When* $\nu(D) = 0$ *i.e.* $\mathcal{N}(D) = [0, \infty)$ *the operator* D *is called* **nonresonant**.

We can now formulate the main result of this chapter which shows that the family Λ^r has a limit as $r \to \infty$.

Theorem 6.9 *Let M and D as above and $E \geq \nu(D)$. As $r \to \infty$*

$$\Lambda^r \to L_E^\infty \oplus \mathcal{H}_<^{-E}$$

where

$$L_E^\infty = \lim_{r \to \infty} L_E^r = \lim_{r \to \infty} e^{-rA_E} L_E.$$

Proof Fix $E \in \mathcal{N}(D)$. The proof is carried out in several steps.

Step1: A dynamical description of Λ^r

Let $\mathcal{E}_r$ be the extension of $\mathcal{E}$ to M(r) and $\mathcal{K}(r) = \mathcal{K}_{1/2}(D(r))$. For each $0 \leq s \leq r$ let

$$T_s : \mathcal{K}(r) \to L^2(\mathcal{E}_0)$$

be the restriction map $U \mapsto U|_{\Sigma \times \{s\}}$ whose image lies in Λ^s. The CD space Λ^r can be equivalently described as $\Lambda^r = T_0(\mathcal{K}(r))$. By Proposition 4.4, $T_0 : \mathcal{K}(r) \to \Lambda^r$ is bijective with continuous inverse. These traces define a *backward translation operator* $G_r : \Lambda^r \to \Lambda^0$ defined as the composition

$$G_r : \Lambda^r \stackrel{T_r^{-1}}{\to} \mathcal{K}(r) \stackrel{T_0}{\to} \Lambda^0. \tag{6.3}$$

On the cylindrical portion $C_r = \Sigma \times [0, r)$ of M(r), D(r) has the form

$$D(r) = c(ds)(\frac{\partial}{\partial s} + D_0).$$

Thus any $U \in \mathcal{K}(r)$ satisfies on C_r an evolution like equation

$$DU = \frac{\partial}{\partial s} U + D_0 U = 0.$$

For any $u \in L^2(\mathcal{E}_0)$ we write $u = u_+ + u_0 + u_-$ according to the spectral decomposition

$$L^2\mathcal{E}_0 = \mathcal{H}^E_> \oplus \mathcal{H}^E_0 \oplus \mathcal{H}^{-E}_<$$

which is independent of s. Thus we can decompose $U(s) = T_s U$ as

$$U(s) = U(s)_+ + U(s)_0 + U(s)_-.$$

Each of these three pieces satisfies the same evolution-like equation as U (formally $U(s) = e^{-sD_0}U(0)$). Since the spectrum of D_0 is discrete we can find $\mu > 0$ such that the set $[-\mu, -E) \cup (E, \mu]$ contains no eigenvalues of D_0. Then we deduce (by standard Fourier techniques)

$$|(T_s U)_+| \leq const.\ \exp(-\mu s)|(T_0 U)_+| \tag{6.4}$$

$$|(T_s U)_-| \geq const.\ \exp(\mu s)|(T_0 U)_-|. \tag{6.5}$$

Using (6.4)-(6.5) we deduce that $\forall u \in \Lambda^r$

$$|u_+|^2 \leq const.\ \exp(-\mu r)|(G_r u)_+|^2 \tag{6.6}$$

$$|u_-|^2 \geq const.\ \exp(\mu r)|(G_r u)_-|^2. \tag{6.7}$$

Intersecting Λ^r with the coisotropic subspace $\mathcal{H}^{-E}_{\geq}$ we get (by Lemma 6.6) the finite dimensional space

$$\overline{\Lambda}^r = \Lambda^r \cap \mathcal{H}^{-E}_{\geq}$$

which leads to the symplectic reduction L^r_E defined in (6.2). Using the Fourier decomposition for D_0 we deduce easily that for any $L \in \mathbf{R}$ D_0 restricted to $\mathcal{H}^L_{\geq}$ defines a C_0- semigroup which we denote by e^{-rD_0} $r \geq 0$. In particular

$$\overline{\Lambda}^r = e^{-rD_0}\overline{\Lambda}$$

$$L_E^r = e^{-rD_0} L_E = e^{-rA_E} L_E.$$

Let $L_E^\infty = \lim_{r\to\infty} e^{-rA_E} L_E$ (which exists by Corollary 6.4) and is a lagrangian in $\mathcal{H}_0^E$.

Step2: Asymptotic transversality If $E \geq \nu(D)$ then for r large Λ^r is transverse to the lagrangian subspace

$$W = JL_E^\infty + \mathcal{H}_>^E.$$

First suppose $u_r \in \Lambda^r \cap W$. Since $JL_E^\infty \subset \mathcal{H}_0^E$, u_r lies in $\overline{\Lambda}^r$ so its orthogonal projection $\overline{u}_r$ on $\mathcal{H}_0^E$ lies in $L_E^r \cap JL_E^\infty$. But L_E^r converges to L_E^∞ which is transverse to J_E^∞ so $\overline{u}_r = 0$ for large r. Our nonresonant choice $E \geq \nu(D)$ then implies $u_r = 0$, so for large r

$$\Lambda^r \cap W = 0. \tag{6.8}$$

Now, according to Lemma 6.6 (Λ^r , W) is a Fredholm pair of lagrangians and so has index 0 by (2.3). Then (6.8) and the definition of the index imply that Λ^r and W span so

$$\Lambda^r + W = L^2(\mathcal{E}_0). \tag{6.9}$$

Step 3

$$\lim_{r\to\infty} \overline{\Lambda}^r = L_E^\infty. \tag{6.10}$$

By Step 2 $\overline{\Lambda}^r \cap W = 0$. Since $\overline{\Lambda}^r$, L_E^r, L_E^∞ have the same dimension we can represent $\overline{\Lambda}^r$ as the the graph of a bounded linear map

$$B_r : L_E^\infty \to W = JL_E^\infty + \mathcal{H}_>^E.$$

To describe B_r we first represent L_E^r as the graph of a symmetric operator S_r : $L_E^\infty \to L_E^\infty$ (ee Fig. 6.4)

$$L_E^r = \{u + JS_r u \;/\; u \in L_E^\infty\}$$

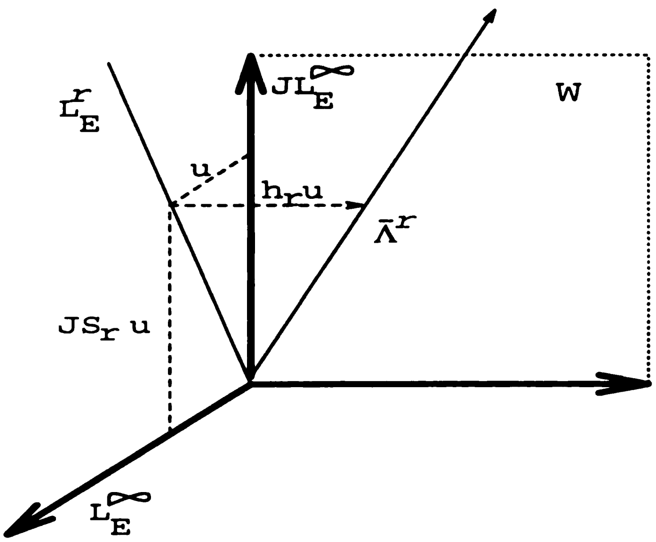


Figure 6.4: $\overline{\Lambda}^r$ *is the graph of* $B_r = JS_r + h_r$

(where $S_r \to 0$ since $L_E^r \to L_E^\infty$). Next (since Λ^r is clean mod $\mathcal{H}_>^E$) there exists a bounded linear map $h_r : L_E^\infty \to \mathcal{H}_>^E$ such that

$$\overline{\Lambda}^r = \{u + JS_r u + h_r(u) \;/\; u \in L_E^\infty\} \;,\quad B_r(u) = (JS_r u, h_r(u)).$$

But recall that

$$\overline{\Lambda}^r = e^{-rD_0}\overline{\Lambda} = \{e^{-rA_E}u + e^{-rA_E}JS_0 u + e^{-rD_0}h_0(u) \;/\; u \in L_E^\infty\}$$

$$= \{v + Je^{rA_E}S_0 e^{rA_E}v + e^{-rD_0}h_0(e^{rA_E}v) \;/\; v \in L_E^\infty\}.$$

Therefore we have

$$S_r = e^{rA_E}S_0 e^{rA_E} \;\text{ and }\; h_r(v) = e^{-rD_0}h_0(e^{rA_E}v).$$

Then the estimate

$$\|h_r\| \le \|e^{-rD_0}\|_{\mathcal{H}_>^E}\,\|h_0\|\,\|e^{rA_E}\|_{\mathcal{H}_0^E} = \le e^{-r\mu}\,e^{rE}\,\|h_0\|$$

shows that $h_r \to 0$ exponentially (we chose $\mu > E$) ; we then deduce (6.10) using Lemma 6.1.

Step 4 Convergence. The conditions (6.8) and (6.9) can be used as in the proof of Theorem 2.2 to represent Λ^r as the graph of a bounded selfadjoint operator $M_r : L_E^\infty \oplus \mathcal{H}_<^{-E} \to L_E^\infty \oplus \mathcal{H}_<^{-E}$ i.e.

$$\Lambda^r = \{u + JM_r u \;/\; u \in L_E^\infty \oplus \mathcal{H}_<^{-E}\}.$$

M_r has a block decomposition

$$M_r = \begin{pmatrix} S_r & (-Jh_r)^* \\ -Jh_r & C_r \end{pmatrix} \quad , \quad C_r : \mathcal{H}_<^{-E} \to \mathcal{H}_<^{-E}.$$

We already know that $S_r \to 0$, $h_r \to 0$ and we will now show that $\|C_r\| \to 0$. The theorem will follow from Lemma 6.1.

Remark 6.10 *Let P_∞ denote the orthogonal projection onto L_E^∞ which is a closed D_0-invariant subspace of $L^2(\mathcal{E}_0)$. If $U \in \mathcal{K}(r)$ then $w(s) = P_\infty T_s U$ satisfies the o.d.e.*

$$\dot{w}(s) + A_E w(s) = 0 \qquad s \in [0, r).$$

In particular if $w(r) = P_\infty T_r U = 0$ then the backward translation $w(s) = 0$ for all $s \in [0, r]$.

For any $f \in \mathcal{H}_<^{-E}$ consider

$$u = u(f) = f + JM_r f = f + J(-Jh_r)^* f + JC_r f \in \Lambda^r.$$

In particular $P_\infty u = 0$ and any $u \in \Lambda^r$ with this property can be written in the above form. Since $J(-Jh_r)^* : \mathcal{H}_<^{-E} \to JL_E^\infty$ we deduce

$$u(f)_- = f \quad \text{and} \quad u(f)_+ = JC_r f = JC_r u_-. \tag{6.11}$$

By Remark 6.10 the backward translation of u – defined in (6.3)– $v = G_r u \in \Lambda$ satisfies $P_\infty v = 0$ and as in (6.11) we deduce $v_+ = JC_0 v_-$. C_0 is continuous and we get

$$\frac{|v_+|}{|v_-|} \leq const. \tag{6.12}$$

On the other hand (6.6), (6.7) and (6.11) imply

$$\frac{|v_+|}{|v_-|} = \frac{|(G_r u)_+|}{|(G_r u)_-|} \geq \frac{e^{r\mu}|u_+|}{e^{-r\mu}|u_-|} = e^{2r\mu}\frac{|JC_r f|}{|f|}. \tag{6.13}$$

The relations (6.12) and (6.13) imply that $\|C_r\| = O(e^{-2r\mu})$. Theorem 6.9 is proved.◇

Theorem 6.9 has many interesting corollaries. We will consider only a special situation motivated by problems in topology (see [Y]). Assume D is nonresonant i.e.

$$\nu(D) = 0$$

In this case we will use the simplified notation

$$\mathcal{H}_-(D) = \mathcal{H}^0_< , \ \mathcal{H}_0(D) = \mathcal{H}^0_0 , \ \mathcal{H}_+(D) = \mathcal{H}^0_>.$$

Here $\mathcal{H}_0$=ker D is finite dimensional and the spaces $\mathcal{H}_\pm$ are spanned by the positive/negative eigenmodes of D_0. We call $\mathcal{H}_0$ the *harmonic* space of D. Both $\mathcal{H}_\pm$ are isotropic subspaces of $L^2(\mathcal{E}_0)$. The annihilator of $\mathcal{H}_\pm$ is $\mathcal{H}_0 \oplus \mathcal{H}_\pm$. The corresponding symplectic reduction

$$L(D) = (\Lambda \cap (\mathcal{H}_0 \oplus \mathcal{H}_+))/\mathcal{H}_+ \tag{6.14}$$

will be called the **reduced Cauchy data (RCD)** space of D. It can be identified with a lagrangian in the harmonic space. To see this consider the Atiyah-Patodi-Singer (APS) boundary value problem i.e.

$$(D, APS): \quad Du = 0 \ \ in\ M \quad R_0 u \in \mathcal{H}_+ \oplus \mathcal{H}_0$$

with adjoint

$$(D, APS)^* : \qquad Du = 0 \;\; \in \; M \quad R_0 u \in \mathcal{H}_+.$$

One sees that

$$\dim L(D) \;=\; \operatorname{ind}(D, APS) \;=\; 1/2 \dim \mathcal{H}_0(D).$$

This agrees with the APS formula since D is selfadjoint so its index density is 0 and D_0 has a symmetric spectrum (it anticommutes with J) so its eta invariant vanishes. In [APS1], $\Lambda(D) \cap (\mathcal{H}_0 \oplus \mathcal{H}_+)$ was called **the space of extended L^2 solutions** and L(D) was identified with the subspace in $\mathcal{H}_0$ of asymptotic values of extended L^2 solutions. Using the reduced CD space L(D) we can form the **asymptotic** CD space

$$\Lambda^\infty(D) \;=\; L(D) \,\oplus\, \mathcal{H}_-(D).$$

The definition of the asymptotic CD space is orientation sensitive. Changing the orientation of M without changing that of Σ will have the effect of replacing $\mathcal{H}_-$ with $\mathcal{H}_+$ in the above definition. We see that D is nonresonant iff $(D, APS)^*$ has only the trivial solution. The pleasant thing in the nonresonance case is that the finite dimensional dynamics is not present since A_E is identically 0 when E = 0 so that $L(D) \;\equiv\; L^r(D)$, $\forall\, r \geq 0$. We deduce immediately the following

Corollary 6.11 *Assume that D is* **nonresonant**. *Then*

$$\lim_{r \to \infty} \Lambda^r \;=\; \Lambda^\infty.$$

Corollary 6.12 *Let $\{D(t) \;\; ; \; 0 \leq t \leq 1\}$ be a continuous family of neck-compatible Dirac operators on M such that each D(t) is nonresonant. Let $D^r(t)$ denote their* *extensions to M(r) and $\Lambda^r(t)$ denote their CD spaces. If* $\dim \mathrm{Ker}(D_0(t))$ *is* **independent of t** *then*

$$\lim_{r\to\infty} \Lambda^r(t) = \Lambda^\infty(t) \quad \text{uniformly in t.}$$

In particular $(\Lambda^\infty(t))$ *is a continuous family of lagrangians in* $L^2(\mathcal{E}_0)$

One can use the existence of an adiabatic limit when computing the spectral flow. We analyze what happens to the terms in Theorem 5.14 as we "stretch the neck". Assume we have a path $\gamma = D(t) \in \mathcal{P}$ such that for every t the operators $D_1(t)$ and $D_2(t)$ are nonresonant. We can form the adiabatic deformation M(r) of (M,g) by replacing the neck $N \cong \Sigma \times (-1,1)$ by a longer one $N_r \cong \Sigma \times (-r,r)$. Let $D^r(t)$ be the obvious extension of D(t) to M(r). Denote by $\Lambda_j^\infty(t)$ the asymptotic CD space of $D_j(t)$. We have the following result

Corollary 6.13 *Let D(t) be a nonresonant path of neck-compatible Dirac operators such that* $dim\mathcal{H}_0(t)$ *is* **independent of t**. *Assume*

$$\Lambda_1^\infty(j) \cap \Lambda_2^\infty(j) = 0 \quad j = 0,1 \tag{6.15}$$

Then for r large enough $D^r(0)$ *and* $D^r(1)$ *are invertible and*

$$SF(D^r(t)) = \mu(\Lambda_1^\infty(t), \Lambda_2^\infty(t)) \tag{6.16}$$

Proof The fact that $D^r(0)$ and $D^r(1)$ are invertible for large r follows easily from (6.14) using "adiabatic analysis" as in Thm.6.9. Alternatively we can quote the results of [CLM2] from which the above conclusion follows trivially. (6.15) follows from Theorem 5.14 combined with Corollary 6.12. ◇

The nonresonance of the operators D(t) can be translated symplectically by saying that $\Lambda_1(t)$ is clean mod $\mathcal{H}_+(D_1(t))$ and $\Lambda_2(t)$ is clean mod $\mathcal{H}_-(D_2(t))$. Using the invariance of the Maslov index under clean reductions we deduce

Corollary 6.14 *Let D(t) as in Corollary 6.13. Then*

$$SF(D^r(t)) = \mu(L_1(t), L_2(t))$$

for r large enough, where $L_i(t) = L(D_i(t))$ is the RCD space of $D_i(t)$.

This last result generalizes a result of [Y]. In that case the Dirac operators arise as the deformation complexes of the flat connection equation on a homology 3-sphere.

Finally we want to address a natural question. Assume that $D(t)$ is a path of neck compatible Dirac operators on M_1 and suppose that some of them have positive nonresonace levels. For simplicity suppose $\nu(D(t)) = \nu_0 > 0$ for all t and that the boundary operators $D_0(t) = D(t)\mid_\Sigma$ are independent of. Then by Theorem 6.1 we can find lagrangians $L^\infty(t)$ in $\mathcal{H}_0^{\nu_0}$ such that

$$\lim_{r\to\infty} \Lambda^r(t) = L^\infty(t) \oplus \mathcal{H}_<^{\nu_0} \quad \forall t \tag{6.17}$$

Is the convergence in (6.17) uniform in t ?

We sketch a simple heuristic argument which suggests that the answer one should expect is in general negative. Let us specialize and assume that the restriction of D_0 to $V_0 = \mathcal{H}_{\nu_0}$ (henceforth denoted by A) has only **simple eigenvalues.** In particular A is invertible because it anticommutes with J. Denote by $L^r(t)$ the symplectic reduction of $\Lambda^r(t)$ mod $\mathcal{H}_>^{\nu_0}$. We have seen that

$$\gamma_r(t) = L^r(t) = e^{-Ar}L^0 t = e^{-Ar}\gamma_0(t) \quad \forall t.$$

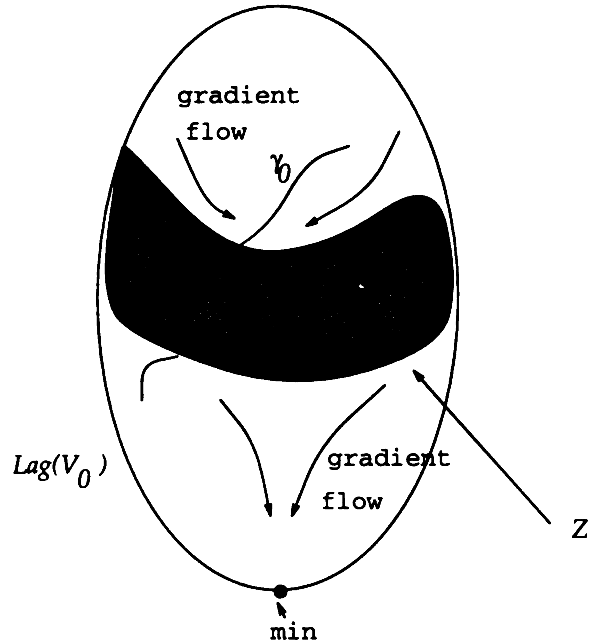


Figure 6.5: *A Morse flow on the lagrangian grassmanian*

Denote by $Lag(V_0)$ the lagrangian grassmanian associated to the symplectic space V_0. The results of [N2] (see Remark 6.5) show that e^{-Ar} is the negative gradient flow of some function $Lag(V_0)$. Since A has only simple eigenvalues all the critical points are nondegenerate. The function has a unique critical point P of index 1. The stable manifold of this point is a codimension 1 submanifold $\mathcal{Z}$ of $Lag(V_0)$ whose closure $\overline{\mathcal{Z}}$ is the Poincaré dual of the Maslov index (see Fig. 6.5). Now if we let the path γ_0 flow along the gradient lines it will "desintegrate"as $r \to \infty$ into a finite set of critical points. Hence the only time $\gamma_r(t)$ can converge uniformly in t is when γ_0 lies entirely in the stable manifold of some critical point. Generically this has to be the region of attraction of the minimum which is the complement of $\overline{\mathcal{Z}}$. Via a small perturbation we may assume $\gamma_0(0)$, $\gamma_1(1)$ lie in this attraction region and thus we obtain a Maslov index

$$\mu(\gamma_0) = \#\gamma_0 \cap \overline{\mathcal{Z}}.$$

This number is stable under small perturbations. In particular if $\mu(\gamma_0) \neq 0$ then the endpoints of γ_0 will flow towards the minimum and some point on this curve will flow inside $\mathcal{Z}$ towards the critical point P and hence we do not have uniform convergence.

Chapter 7

The Conley-Zehnder index

We will illustrate the general splitting formula on a simple example arising in the study of periodic trajectories of hamiltonian equations.

We begin by reviewing the Conley-Zehnder index. For details we refer the reader to [CZ], [RS] or [SZ].

Let $E = E_n$ be the standard symplectic space $(\mathbf{R}^{2n}, \omega_0)$ where

$$\omega_0(x, y) = -(J, x, y) = (x, Jy)$$

and

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Sym(E) will denote the space of symmetric matrices $A : E \to E$. Set

$$\Sigma_n = C^1(S^1, Sym(E)).$$

Associated to any loop $A(\theta) \in \Sigma_n$ is a selfadjoint 1-dimensional Dirac operator

$$D_A : C^\infty(S^1, E) \to C^\infty(S^1, E) \quad u(\theta) \mapsto J\frac{du}{d\theta} + A(\theta)u.$$

Define

$$\Sigma_n^* = \{A \in \Sigma_n \;/\; \operatorname{Ker} D_A = 0\}.$$

Σ_n^* is an open subset of Σ_n. It has countably many paths components. Let

$$C_\varepsilon = \begin{bmatrix} \varepsilon I_n & 0 \\ 0 & -\varepsilon I_n \end{bmatrix} \in Sym(E).$$

We think of C_ε as a constant map $S^1 \to Sym(E)$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ we have $C_\varepsilon \in \Sigma_n^*$.

Following [CZ] we define an injection

$$\nu_n : \pi_0(\Sigma_n^*, C_\varepsilon) \to \mathbf{Z}$$

called **the Conley-Zehnder index**.

Its construction is carried out in several steps. First, one shows that any connected component of Σ_n^* contains a constant loop. Next, if $S(\theta) \equiv S \in \Sigma_n^*$ is such a loop then 1 is not an eigenvalue of $\exp(2\pi JS)$ so that

$$\sigma(JS) \cap i\mathbf{Z} = \emptyset$$

The eigenvalues of JS occur in pairs $(\lambda, \overline{\lambda})$. We consider only those pairs of purely imaginary eigenvalues $(\lambda, \overline{\lambda})$. If $(e, \overline{e})$ are the corresponding eigenvectors then $\tilde{\omega}_0(\overline{e}, e)$ is purely imaginary. Here $\tilde{\omega}_0$ denotes the complex bilinear extension of ω_0 to $\mathbf{C}^{2n}$. Set

$$\sigma(\lambda) = \operatorname{sign} Im\, \tilde{\omega}_0(\overline{e}, e)$$

and

$$\alpha(\lambda) = \sigma(\lambda) Im\, \lambda.$$

The Conley-Zehnder index of S is

$$\nu_n(S) = \sum_{\lambda \in \sigma(JS) \cap i\mathbf{R}} ([\alpha(\lambda)] + \frac{1}{2})$$

where [,] denotes the integer part and in the above sum each eigenvalue is repeated as many times as its multiplicity. If JS has no purely imaginary eigenvalues than set $\nu_n(S) = 0$.

Finally one shows that $S_1, S_2 \in Sym(E) \cap \Sigma_n^*$ belong to the same component of Σ_n^* iff $\nu_n(S_1) = \nu_n(S_2)$ so that ν_n correctly defines an injection $\nu_n : \pi_0(\Sigma_n^*, C_\varepsilon) \to \mathbf{Z}$. If moreover $n \geq 2$ then ν_n is actually a bijection.

The Conley-Zehnder index has an obvious additivity property

$$\nu_{n_1+n_2}(A_1 \oplus A_2) = \nu_{n_1}(A_1) + \nu_{n_2}(A_2) \quad \forall A_i \in \Sigma_{n_i}^*\ ,\ i = 1,2. \tag{7.1}$$

Example 7.1 *Let $E = E_1 \cong (\mathbf{R}^2, \omega_0)$ and*

$$S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$S \in \Sigma_1^*$ iff $\Delta = det\ S$ is not a perfect square. If $\Delta \leq 0$ then JS has only real eigenvalues so that $\nu_1(S) = 0$.

If $\Delta > 0$ then the eigenvalues of JS are $\lambda_\pm = \pm i\sqrt{\Delta}$ with eigenvectors $e_\pm = (1\ ,\ \mp i\sqrt{\Delta})$. We compute

$$\tilde{\omega}_0(e_-\ ,\ e_+) = 2i\frac{\sqrt{\Delta}}{\lambda_2}$$

so that

$$\sigma(\lambda_\pm) = \pm \mathrm{sign}\, tr(S)\ \ ,\ \ \alpha(\lambda_\pm) = \mathrm{sign}\, tr(S)\sqrt{\Delta}.$$

Hence

$$\nu_1(S) = 2[\mathrm{sign}\, tr(S)\sqrt{\Delta}] + 1.$$

Using the additivity property of the index we deduce

$$\nu_n(C_\varepsilon) = 0 \quad \forall n \geq 1.$$

The Conley-Zehnder index is a Maslov index in disguise. We briefly describe this point of view following [D2] or [RS].

Let $A(\theta) \in \Sigma_n^*$. Denote by $\Phi(\theta, \theta_0)$ the path of symplectic matrices satisfying the initial value problem

$$\begin{cases} \frac{d\Phi}{d\theta}(\theta,\theta_0) & = & JA(\theta)\Phi(\theta,\theta_0) \quad \theta,\theta_0 \in [0,2\pi] \\ \Phi(\theta_0,\theta_0) & = & I_{E_n} \end{cases} \tag{7.2}$$

Note that Φ_t satisfies the *cocycle condition*:

$$\Phi_t(\theta_1,\theta_3) = \Phi_t(\theta_1,\theta_2)\cdot\Phi_t(\theta_2,\theta_3) \quad \forall\theta_1,\theta_2,\theta_3 \in [0,2\pi].$$

Denote by $\Gamma(\theta)$ the graph of $\Phi(\theta, 0)$

$$\Gamma(\theta) = \{(x, \Phi(\theta,0)x) \mathbin{/} x \in E_n\}.$$

$\Gamma(\theta)$ is a lagrangian subspace in $E_n \oplus E_n$ endowed with the symplectic structure $-(\omega_0) \oplus \omega_0$. In [D2], [RS] it is proved

$$\nu_n(A(\theta)) = \mu(\Delta\,,\,\Gamma(\theta)) \tag{7.3}$$

where Δ is the diagonal $\Delta = \{(x,x) \mathbin{/} x \in E_n\} \subset E_n \oplus E_n$. Note that $\Delta = \Gamma(0)$ so the endpoints of this pair of paths are not transversal. The Maslov index can still be defined in this situation and the Maslov index has all the wished for properties: path additivity and homotopy (rel endpoints) invariance.

Now consider a path $t \mapsto A_t(\theta) \in \Sigma_n$ such that $A_j(\theta) \in \Sigma_n^*$ for j = 0,1. We get a path of selfadjoint Dirac operators $D(t) = D_{A_t}$. We want to apply the splitting formula to this path.

To describe the various CD spaces, consider for each $t \in [0,1]$ the path of symplectic matrices $\Phi_t(\theta, \theta_0)$ defined as in (7.2) with $A = A_t$. The graphs of $\Phi_t(\theta, 0)$ will be denoted by $\Gamma_t(\theta)$. Represent S^1 as in Fig.7.1.

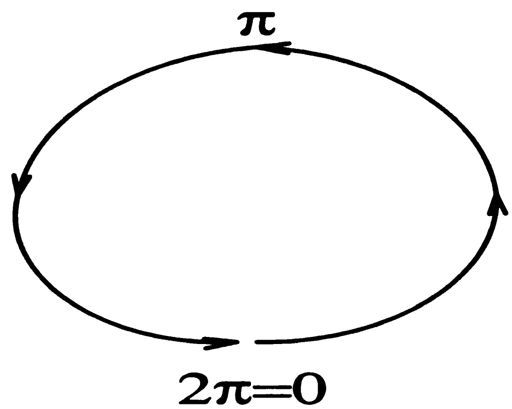


Figure 7.1: *Splitting* S^1

The objects defined over the LHS of Fig.7.1 will have " - "subscripts and those on the RHS will have "+" subscripts. Then $\mathcal{E} = S^1 \times E$,$\mathcal{E}_0 = E_0 \oplus E_\pi$. The symplectic form on $E_0 \oplus E_\pi$ will be $\omega = (-\omega_0) \oplus \omega_0$. The CD spaces are

$$\Lambda_-(t) = \{(\Phi_t(2\pi, \pi)v, v) \;/\; v \in E_\pi\}$$

$$\Lambda_+(t) = \{(u, \Phi_t(\pi, 0)u) \;/\; u \in E_0\}.$$

Thus $\Lambda_-(t)$ is the graph of $\Phi_t^{-1}(2\pi, \pi) : E_{2\pi} = E_0 \to E_\pi$ and $\Lambda_+(t)$ is the graph of $\Phi(\pi, 0) : E_0 \to E_\pi$. Note that

$$\mathrm{Ker}D(t) \neq 0 \iff \Lambda_-(t) \cap \Lambda_+(t) \neq 0$$

so that the pair $(\Lambda_-(t), \Lambda_+(t))$ has transversal endpoints. For $s \in [\pi, 2\pi]$ set

$$\Lambda_-^s(t) = \{(\Phi_t(2\pi, s)v, v) \;/\; v \in E_\pi\} \;\; , \;\; \Lambda_+^s(t) = \{(u, \Phi_t(s, 0)u) \;/\; u \in E_0\}.$$

These paths define a homotopy

$$(\Lambda_-(t), \Lambda_+(t)) \sim (\Lambda_-^s(t), \Lambda_+^s(t)) \sim (\Delta, \Xi_t)$$

where $\Xi_t \subset E_0 \oplus E_\pi$ is the graph of $\Phi_t(2\pi, 0)$. Using the cocycle condition satisfied by Φ_t we see that the endpoints of $(\Lambda_-^s(\cdot), \Lambda_+^s(\cdot))$ stay transversal during the

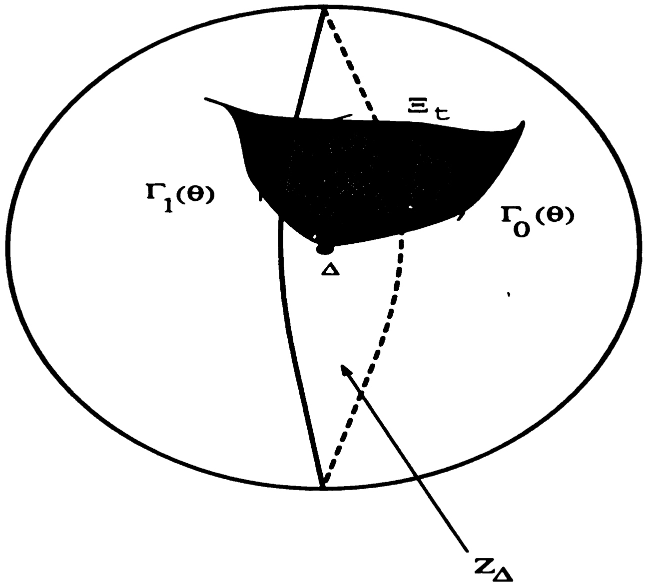


Figure 7.2: *Intersecting the resonance divisor*

deformation. Hence by the splitting formula we have

$$SF(D(t)) = \mu(\Lambda_-(t)), \Lambda_+(t)) = \mu(\Delta, \Xi_t). \tag{7.4}$$

We get a loop

$$\gamma = \{\Gamma_0(\theta)\ ;\ \theta \in [0, 2\pi]\} + \{\Xi_t\ ;\ t \in [0,1]\} - \{\Gamma_1(\theta)\ ;\ \theta \in [0, 2\pi]\}$$

in the lagrangian grassmanian of $E_0 \oplus E_\pi$, as in Fig.7.2, which is clearly a contractible loop. In Fig.7.2 Z_Δ is the resonance divisor determined by Δ consisting of all lagrangian subspaces of $-E \oplus E$ which intersect Δ nontrivially. Hence

$$0 = \mu(\Delta, \gamma) = \mu(\Delta, \Gamma_0(\theta) + \mu(\Delta, \Xi_t) - \mu(\Delta, \Gamma_1(\theta)). \tag{7.5}$$

Using (7.3) and (7.4) in the above equality we deduce

$$SF(D(t)) = \nu_n(A_1(\theta)) - \nu_n(A_0(\theta)) \tag{7.6}$$

which is precisely the content of Theorem 4.1 of [SZ].

We conclude this section with a numerical verification of (7.6). Consider a smooth path of diagonal matrices

$$t \mapsto A_t = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix} \quad t \in [-\varepsilon, \varepsilon].$$

Now form the path of 1-dimensional Dirac operators:

$$D(t) : C^\infty(S^1; E) \to C^\infty(S^1; E) \quad u \mapsto J\frac{du}{d\theta} + A_t u(\theta).$$

We will discuss two cases. Set $\Delta(t) = \det A_t$.

A: The Even Case. We will assume that

$$\Delta(t) > 0 \quad \forall\, |t| \le \varepsilon \tag{7.7}$$

$$\dot{\Delta}(0) \neq 0 \tag{7.8}$$

$$\delta_t = \sqrt{\Delta(t)} \in \mathbf{Z}_+ \iff t = 0. \tag{7.9}$$

We have the following result

Proposition 7.2 *D(t) is invertible for all $t \neq 0$ and*

$$SF(D(t); |t| \le \varepsilon) = 2\mathrm{sign}\,(\dot{\Delta}(0)\mathrm{tr} A_0)$$

Proof If $u \in \mathrm{Ker} D(t)$ then

$$\begin{cases} u'(\theta) &= JA_t u(\theta) \\ u(2\pi) &= u(0) \end{cases} \tag{7.10}$$

Set

$$B_t = JA_t = \begin{bmatrix} 0 & -\lambda_2(t) \\ \lambda_1(t) & 0 \end{bmatrix}.$$

Then, if u satisfies (7.10) we have

$$u(\theta) = e^{B_t\theta}u(0) \;,\; u(0) = e^{2\pi B_t}.$$

Thus Ker$D(t) \neq 0$ iff 1 is an eigenvalue of of $e^{2\pi B_t}$. Note that $B^2 = -\Delta I$ (we omit the subscript t for simplicity). A simple computation shows

$$e^{B\theta} = I\cos\delta\theta + \frac{1}{\delta}\sin\delta\theta B.$$

We now compute ($\theta = 2\pi$)

$$\det(I - e^{2\pi B}) = \det((1-\cos 2\pi\delta)I - \frac{1}{\delta}\sin 2\pi\delta) = 4\sin^2\pi\delta.$$

Thus 1 is an eigenvalue of $e^{2\pi B_t}$ iff $\delta_t \in \mathbf{Z}_+$ i.e. by (7.9) t = 0. In this case $e^{2\pi B_0} = I$ and

$$\text{Ker}D(0) = \text{span}\left\{u_1 = \begin{bmatrix} \cos\delta\theta \\ \frac{\lambda_1}{\delta}\sin\delta\theta \end{bmatrix}, \quad u_2 = \begin{bmatrix} -\frac{\lambda_2}{\delta}\sin\delta\theta \\ \cos\delta\theta \end{bmatrix}\right\}.$$

Note that $u_1 \perp u_2$ in $L^2(S^1)$. Clearly $\dot{D}(0) = \dot{A}_0$ and a simple computation shows

$$(\dot{A}_0 u_1\,,\, u_2)_{L^2(S^1)} = (\dot{A}_0 u_2\,,\, u_1)_{L^2(S^1)} = 0$$

$$(\dot{A}_0 u_1\,,\, u_1)_{L^2(S^1)} = \pi\frac{\dot{\Delta}(0)}{\lambda_2}$$

$$(\dot{A}_0 u_2\,,\, u_2)_{L^2(S^1)} = \pi\frac{\dot{\Delta}(0)}{\lambda_1}$$

from which we deduce that

$$SF(D(t)\,;\,|t| \leq \varepsilon) = \text{sign}\begin{bmatrix} \pi\frac{\dot{\Delta}(0)}{\lambda_2} & 0 \\ 0 & \pi\frac{\dot{\Delta}(0)}{\lambda_1} \end{bmatrix} = 2\text{sign}\,(\dot{\Delta}(0) tr A_0).$$

Proposition 7.2 is proved. □

B: The Odd Case. We assume

$$|\Delta(t)| \leq 1/2 \quad \forall\, |t| \leq \varepsilon \tag{7.11}$$

$$\Delta(t) = 0 \iff t = 0 \tag{7.12}$$

$$\dot{\Delta}(0)\,,\; tr\, A_0 \neq 0. \tag{7.13}$$

The spectral flow of the family D(t) is computed as in Proposition 7.2.

Proposition 7.3 *D(t) is invertible for all t $\neq 0$ and*

$$SF(D(t)) = \operatorname{sign}(\dot{\Delta}(0)\operatorname{tr} A_0)$$

Proof The first part is established as before and we deduce Ker$D(t) \neq 0$ iff t= 0. From (7.11) and (7.12) we deduce that only one of the eigenvalues of A_0 is zero and say $\lambda_2(0) \neq 0$. We deduce

$$e^{2\pi B_0} = 1 + B_0$$

so that

$$\operatorname{Ker} D(0) = \operatorname{span}\{f = (0,1)\}.$$

We compute easily

$$(\dot{A}_0 f, f)_{L^2(S^1)} = 2\pi\dot{\lambda}_2(0) = 2\pi\frac{\dot{\Delta}_0}{\lambda_1}$$

and Proposition 7.3 is now obvious. □.

Note that when $\Delta(t) < 0$ the operator D(t) is invertible an thus there is no change in the spectral flow.

The computations in Propositions 7.2-3 have a nice geometric interpretation. Any diagonal matrix as above can be viewed as a point in the plane (λ_1, λ_2) (see Fig.3). A path of such matrices is a path in this plane. For each integer n we have a curve

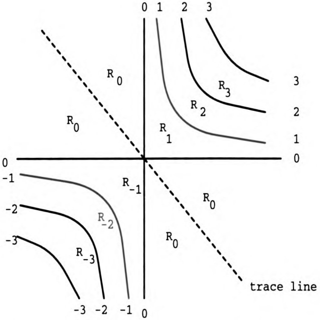


Figure 7.3: *An easier way to compute the Conley-Zehnder index*

$H_n = \{\lambda_1\lambda_2 = n^2 \ , \ n(\lambda_1 + \lambda_2) \geq 0\}$ labelled by n in Fig.12. These curves form the **resonance locus** $\mathcal{R}$: if a path crosses one of these curves we get a change in the spectral flow. The labels in Fig.7.3 also define a transversal orientation of the resonace locus. If H_n , $n \neq 0$ we get a ± 2 change in the spectral flow. If a path crosses H_0 away from the origin we get a ± 1 change. We can partition the complement of the resonance locus as

$$\mathbf{R}^2 \setminus \mathcal{R} = \bigcup_{n \in \mathbf{Z}} R_n$$

where $R_0 = \{\lambda_1\lambda_2 < 0\}$ and for $n \neq 0$ R_n is the region in between the hyperbolae H_n and $H_{n-\sigma_n}$ (here $\sigma_n = \operatorname{sign} n$. Define

$$m : \mathbf{R}^2 \setminus \mathcal{R} \to \mathbf{Z}$$

by

$$m(A) = \begin{cases} 0 & , \quad A \in R_0 \\ \sigma_n(2|n| - 1) & , \quad A \in R_n \quad n \neq 0 \end{cases} \tag{7.14}$$

Then the above discussion shows

$$SF(D(t)) = m(A(1)) - m(A(0)) \tag{7.15}$$

Comparing (7.14) with the computations in Example 7.1 we see that $m(A) = \nu_1(A)$ and this is in perfect agreement with (7.6).

Appendix A

The proof of Proposition 5.6

Proposition 5.6 is a consequence of the Sard-Smale theorem. We roughly follow an outline given by Floer (Prop.3.1 in [F]) making several necessary modifications (Floer overlooked the hypothesis in Lemma A.2; fixing this requires applying Sard-Smale to a modified map). To define it, choose k large enough so that $L^2_k([0,1] \times M) \hookrightarrow C^2([0,1] \times M)$ (N=dim M) and set

$$\mathcal{A} \;=\; \{A \in L^2_k\,(End([0,1] \times \mathcal{E}))\ /\ A(t) \in Cyl(\mathcal{E}) \quad \forall\, t \in [0,1]\,\}$$

We will parametrize the 2-dimensional planes in $L^2_1(\mathcal{E})$ by

$$W \;=\; \{(\xi,\eta) \in L^2_1(\mathcal{E}) \times L^2_1(\mathcal{E})\ /\ \langle \xi,\eta\rangle_{L^2} = 0\ ,\ |\xi|_{L^2} = |\eta|_{L_2} = 1\ \}$$

This is a Banach manifold. Its tangent space at (ξ,η) consists of all pairs ϕ, $\psi \in L^2_1(\mathcal{E})$ that satisfy

$$\langle \xi,\psi\rangle + \langle \phi,\eta\rangle = \langle \xi,\phi\rangle = \langle \eta,\psi\rangle = 0 \tag{A.1}$$

In the proof of our genericity results we will need the following lemmata.

Lemma A.1 *Let D be a cylindrical Dirac and $(\xi,\eta) \in W$ such that $D\xi = D\eta = 0$. Then there exists an open subset $U \subset M_2$ away from the neck such that ξ and η are pointwise linearly independent over U.*

Proof By unique continuation the set

$$S = \{x \in M \ / \ \xi(x) \neq 0 \text{ and } \eta(x) \neq 0 \}$$

is open and dense as an intersection of two open and dense sets. Set $S_2 = S \cap (M_2 \setminus \text{neck})$. The set

$$\mathcal{I} = \{x \in S_2 \ / \ \xi(x) \,\&\, \eta(x) \ \text{ are linearly independent } \}$$

is open if nonempty. The Lemma is proved if we show that $\mathcal{I} \neq \emptyset$. Assume the contrary. This means there exists $\alpha \in C^\infty(S_2)$ such that

$$\xi(x) = \alpha(x)\eta(x) \quad \forall\, x \in S_2$$

$\xi, \eta \neq 0$ on S_2 so that $\alpha \neq 0$ on S_2. Since $\xi \perp \eta$ we deduce from the unique continuation that α is not constant on S_2 i.e. $d\alpha \not\equiv 0$ on S_2. On the other hand from Definition 4.1

$$0 = D\xi = D\alpha\eta = D\eta + [D, \alpha]\eta = c(d\alpha)\eta.$$

This is a contradiction since the Clifford multiplication $c(d\alpha)$ is an isomorphism when $d\alpha \neq 0$. Lemma A.1 is proved. $\diamond$

For $k \geq 0$ let S_k denote the linear space of real , symmetric $k \times k$ matrices ($S_0 \equiv 0$)

Lemma A.2 *Let $\xi, \eta \in \mathbf{R}^k$ ($k \geq 2$) two linearly independent vectors. Then for any vectors $u, v \in \mathbf{R}^k$ satisfying*

$$\langle \xi, v \rangle = \langle \eta, u \rangle$$

there exists $A \in S_k$ such that $(A\xi, A\eta) = (u, v)$.

Proof Define

$$H_{\xi,\eta} : S_k \to \mathbf{R}^{2k} \quad A \mapsto (A\xi, A\eta)$$

We have to prove that

$$\text{Range} H_{\xi,\eta} = V_{\xi,\eta} = \{(u,v) \in \mathbf{R}^k \times \mathbf{R}^k \ / \ \langle \xi, v\rangle = \langle \eta, u\rangle\}.$$

Note that $V_{\xi,\eta} = (\text{span}(-\eta,\xi))^{\perp}$ and for any $A \in S_k$

$$\langle (A\xi, A\eta), (-\eta, \xi)\rangle = -\langle A\xi, \eta\rangle + \langle A\eta, \xi\rangle = 0$$

so that

$$\text{Range} H_{\xi,\eta} \subset V_{\xi,\eta} \tag{A.2}$$

On the other hand

$$\dim \text{Range} H_{\xi,\eta} = \dim S_k - \dim \text{Ker} H_{\xi,\eta}$$

Since ξ and η are linearly independent we can identify

$$\text{Ker} H_{\xi,\eta} \cong S_{k-2}$$

Thus

$$\dim \text{Range} H_{\xi,\eta} = k(k+1)/2 - (k-2)(k-1)/2 = 2k-1 = \dim V_{\xi,\eta} \tag{A.3}$$

Lemma A.2 follows from(A.2)and (A.3). $\diamond$

Proof of proposition 5.6 We will apply the Sard-Smale theorem to the smooth function

$$F : X = \mathcal{A} \times (0,1) \times W \times \mathbf{R} \to Y = L^2(\mathcal{E}) \times L^2(\mathcal{E})$$

defined by

$$(A(\cdot), t, \xi, \eta, \lambda) \mapsto (D(t)\xi - \lambda\eta, D(t)\eta + \lambda\xi)$$

Let Z = $F^{-1}(0)$. The proof of Proposition 5.6 is done in two steps.

Step 1 Z is a smooth Banach manifold.

To prove this we will use the implicit function theorem. Given $z \in Z$ we will show that $DF(z) : T_zX \to Y$ is onto. More precisely we will show that DF(z) has closed range and its cokernel is zero. Let $z = (A, t, \xi, \eta, \lambda) \in Z$. Note that this implies $\lambda = 0$. Indeed we have $D(t)\xi = \lambda\eta$ and $D(t)\eta = -\lambda\xi$ so that

$$D(t)^2\eta \;=\; -\lambda D(t)\xi \;=\; -\lambda^2\eta.$$

Since D(t) is selfadjoint we deduce $|D(t)\eta|^2 \;=\; -\lambda^2|\eta|^2$ which is possible iff $\lambda = 0$.

Now consider the variation on the direction $(a, \tau, \phi, \psi, \mu) \in T_zX$.The partial derivatives of F are

$$D_A F(z)(a) \;=\; (a(t)\xi, a(t)\eta) \tag{A.4}$$

$$D_t F(z)(\tau) \;=\; \tau(\dot{A}(t)\xi, \dot{A}(t)\eta) \tag{A.5}$$

$$D_{(\xi,\eta)} F(z)(\phi, \psi) \;=\; (D(t)\phi, D(t)\psi) \tag{A.6}$$

$$D_\lambda F(z)(\mu) \;=\; \mu(-\eta, \xi) \tag{A.7}$$

where ϕ and ψ satisfy(A.1).

Since the operator D(t) is elliptic we deduce the range of $DF(z)$ is closed. Let$(u, v) \in$ Coker$DF(z_0)$. From (A.4) and (A.6) we deduce

$$\langle a(t)\xi, u\rangle + \langle a(t)\eta, v\rangle = 0 \quad \forall\, a \in \mathcal{A} \tag{A.8}$$

$$\langle D(t)\phi, u\rangle + \langle D(t)\psi, v\rangle = 0 \tag{A.9}$$

$\forall\, \phi, \psi$ satisfying (A.1) . Let $(e_n)_{n\in\mathbf{Z}}$ be the eigenvectors of $D(t)$ corresponding to the **nonzero eigenvalues**. If we let $\phi = e_n$ and $\psi = 0$ in (A.9) we deduce

$$\langle e_n, u\rangle \;=\; 0 \quad \forall\, n \in \mathbf{Z}$$

so that $u \in \mathrm{Ker}D(t)$. We deduce similarly that $v \in \mathrm{Ker}D(t)$.

From Lemmata A.1 A.2 and (A.8) we deduce that on an open set $U \subset M_2$ away from the neck

$$(u(x), v(x)) = c(-\eta(x), \xi(x)) \ , \ \ \forall\, x \in U$$

for some $c \in \mathbf{R}$. By unique continuation the above equality holds for all $x \in M$. Pairing (u,v) with (A.7) we get that c = 0 i.e. $\mathrm{Coker}F(z) = 0$. Step 1 is completed.

Step 2 The natural projection $\pi : Z \to \mathcal{A}$ is Fredholm with index -1.

It is a standard fact that π is Fredholm if and only if

$$G = (\pi, F) : X \to \mathcal{A} \times Y$$

is Fredholm. Moreover $\pi|_Z$ and G have the same index. It suffices to study DG at a point $z \in Z$ of our choice. Thus let $z_0 = (A_0, t_0, \xi_0, \eta_0, 0) \in Z$ such that $\dot{A}_0(t_0)$ is a positive cylindrical endomorphism. Hence $(\xi_0, \eta_0) \in W$ and $D(t_0)\xi_0 = D(t_0)\eta_0 = 0$. The derivatives of G are given by (A.4)-(A.7) and

$$D_A\pi(z_0)(a) = a \ , \ \ a \in \mathcal{A} \tag{A.10}$$

Again the ellipticity of $D(t_0)$ implies that $DG(z_0)$ has closed range.

Let $(a, \tau, \phi, \eta, \mu) \in \mathrm{Ker}DG(z_0)$. This means $a = 0$,$\mu = 0$, $\tau \dot{A}_0(t_0)\xi_0 = \tau \dot{A}_0(t_0)\eta_0 = D(t_0)\xi_0 = D(t_0)\eta_0 = 0$. In particular since ϕ and ψ satisfy (A.1) they lie in a codimension 3 subspace of $\ker D(t_0) \times \ker D(t_0)$. Because $\dot{A}_0(t_0)$ is positive we deduce $\dot{A}_0(t_0)\xi_0$, $\dot{A}_0(t_0)\eta_0 \neq 0$ and therefore

$$\dim \mathrm{Ker}DG(z_0) = 2 \dim \mathrm{Ker}D(t_0) - 3. \tag{A.11}$$

Let $(\alpha, u, v) \in \text{Coker}DG(z_0)$. We deduce from (A.4) ,(A.6),(A.10) that

$$\langle a(t_0)\xi_0, u\rangle + \langle a(t_0)\eta_0, v\rangle + \langle a, \alpha\rangle_{\mathcal{A}} \;=\; 0 \quad \forall\, a \in \mathcal{A} \tag{A.12}$$

$$\tau\left(\langle \dot{A}_0(t_0)\xi_0, u\rangle + \langle \dot{A}_0(t_0)\eta_0, v\rangle\right) = 0 \quad \forall\, \tau \in \mathbf{R} \tag{A.13}$$

$$\langle D(t_0)\phi, u\rangle + \langle D(t_0)\psi, v\rangle = 0 \tag{A.14}$$

$$\mu\left(\langle \xi, v\rangle - \langle \eta, u\rangle\right) \;=\; 0 \quad \forall\, \mu \in \mathbf{R} \tag{A.15}$$

for all ϕ and ψ satisfying (A.1). In particular we deduce from (A.14) and (A.15) that $u, v \in \text{Ker}D(t_0)$ and $(u, v) \perp (-\eta_0, \xi_0)$. For any $u, v \in L^2(\mathcal{E})$ define $\alpha = \alpha(u, v) \in \mathcal{A}$ by

$$\langle a, \alpha\rangle_{\mathcal{A}} \;=\; -\langle a(t_0)\xi_0, u\rangle_{L^2(\mathcal{E})} + \langle a(t_0)\eta_0, v\rangle_{L^2(\mathcal{E})} \quad \forall\, a \in \mathcal{A}. \tag{A.16}$$

The existence and uniqueness of a such an α is a consequence of Riesz-Frechet representation theorem. Set

$$E \;=\; \{(\alpha(u, v), u, v) \;/\; u, v \in \ker D(t) \;\;\&\;\; (u, v) \perp (-\eta_0, \xi_0)\;\}.$$

We can now describe the cokernel of $DG(z_0)$ as

$$\text{Coker}DG(z_0) \;=\; \{(\alpha(u, v), u, v) \;/\; (u, v) \in E \;\;,\;\; \langle \dot{A}_0(t_0)\xi_0, u\rangle + \langle \dot{A}_0(t_0)\eta_0, v\rangle = 0\}.$$

Since $(\dot{A}_0(t_0)\xi_0, \dot{A}_0(t_0)\eta_0) \perp (-\eta_0, \xi_0)$ and $\dot{A}_0(t_0)$ is positive we get

$$\dim \text{Coker}DG(z_0) \;=\; \dim E - 1 \;=\; 2 \dim \text{Ker}D(t_0) - 2. \tag{A.17}$$

Step 2 follows from (A.11) and (A.17). Proposition 5.6 is now a consequence of Sard-Smale theorem. $\diamond$

Remark A.3 *The rank condition on $\mathcal{E}$ is always satisfied since from the representation theory for Clifford algebras the rank of a real C(M) - module is even.*

Remark A.4 *Let D be a cylindrical Dirac operator such that* $\dim \mathrm{Ker} D \geq 2$. *Set* $E = \mathrm{Ker} D \times \mathrm{Ker} D$. *Let $A \in Cyl(\mathcal{E})$ and $(\xi, \eta) \in E \cap W$. We get a point $\zeta \in Z$ by* $\zeta = ((D + (\cdot - 1/2)A), t_0 = 1/2, \xi, \eta, 0)$. *The equality*

$$\mathrm{index}\, DG(\zeta) = -1$$

has the following interesting consequence :

$$(A\xi, A\eta) = (0,0) \Longleftrightarrow \mathrm{Proj}_E(A\xi, A\eta) = (0,0).$$

Bibliography

[Ar] V.I. Arnold., *Une classe charactéristique intervenant dans les conditions de quantification*, Appendix to Maslov's book Théorie de perturbations et methodes asympotiques, Dunod, 1972.

[APS1-3] M.F. Atiyah, V.K. Patodi, I.M. Singer, *Spectral asymmetry and riemannian geometry I, II, III,* Math. Proc. Camb. Phil. Soc., 77 (1975), 43-69, 78 (1975), 405-432, 79 (1976), 77-99.

[AS] M.F. Atiyah, I.M. Singer, *Index theory for skew adjoint Fredholm operators*, Inst. Hautes Etudes Sci. Publ. Math., 37 (1969).

[B] B. Blackadar, *K-theory for operator algebras*, MSRI Publications, No. 5, Springer Verlag, 1986.

[BGV] N. Berligne, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*, Springer Verlag, 1992.

[BW1-2] B. Boos, K. Wojciechowski, *Desuspension of splitting elliptic symbols I,II*, Ann. Glob. Anal. and Geom., 3 (1985), 337-387, 4 (1986), 349-400.

[BW3] B.Boos, B K. Wojciechowski, *Pseudo-differential projections and the topology of certain spaces of elliptic boundary value problems*, Comm. Math. Phys., 121 (1989), 1-9.

[BW4] B. Boos, K. Wojciechowski, *Elliptic boundary values for Dirac operators*,Birckhauser 1993.

[Bu1] U.Bunke *A glueing formula for the η-invariant* , preprint Humboldt-Universitat Berlin, 1993.

[Bu2] U.Bunke *Splitting the spectral flow*, preprint Humboldt-Universitat Berlin, 1993.

[CLM1] S. Capell, R. Lee, E.Y. Miller, *On the Maslov index*, (preprint).

[CLM2] S. Capell, R. Lee, E.Y. Miller, *Selfadjoint elliptic operators and manifold decomposition Part I: Low eigenvalues and stretching*, (preprint).

[CZ] C.Conley, E Zehnder ,*Morse-type index theory for flows and periodic solutions of Hamiltonian equations* Comm. Pure Appl. Math 37 (1984), p 207-253.

[C] H.O. Cordes, *Elliptic Pseudodifferential operators*, Lectures Notes in Math., 756, Springer Verlag, 1979.

[D1] J.J. Duistermaat, *Fourier integral operators*, Courant Inst., New York, 1973.

[D2] J.J. Duistermaat, *On the Morse index in variational calculus* , Adv. in Math, 21 (1976), p. 173-195.

[DP] P. Dazord, G. Pattissier, *La premiere classe de Chern comme obstruction a la quantification asymptotique, Symplectic Geometry, Grupoids and Integrable Systems*, P. Dazord, A. Weinstein (Eds), MSRI Publ., 20, Springer Verlag, 1991.

[DRS] S. Dostoglou, J.W. Robin, D. Salamon, *The spectral flow and the Maslov index*, (preprint).

[F] A. Floer , *A relative Morse index for the symplectic action*, Comm. Pure and Appl. Math., 51 (1988), 393-407.

[GS] V. Guillemin, S. Sternberg, *Geometric asymptotics*, Math. Surveys, No. 14, AMS Providence, RI, 1977.

[Kar] M.Karoubi, *K-theory. An introduction*, Springer Verlag, 1977.

[K] T.Kato, *Perturbation theory for linear operators*, Springer Verlag, 1984.

[Ku] N.Kuiper, *The homotopy type of the unitary group of a Hilbert space*, Topology, 3 (1964-65), 19-30.

[LiMa] J.L.Lions, E. Magenes, *Problèmes aux limites nonhomogènes* vol. **1**, Travaux et recherches math., Paris Dunod, 1968.

[N1] L.I.Nicolaescu, *The spectral flow, the Maslov index and decompositions of manifolds*, C.R. Acad.Sci. Paris, t.**317** (1993), p. 515-519.

[N2] L.I.Nicolaescu, *Morse theory on grassmanians*, preprint, 1993 (to appear Anal.Sti.Univ.Iasi)

[P1] R.S.Palais, *On the homotopy type of certain groups of operators*, Topology, 3 (1964-65), 271-280.

[P2] R.S.Palais, *Seminar on the Atiyah - Singer index theorem*, Ann. of Math. Studies, No. 57, Princeton University Press, 1965.

[RS] J.Robbin, D.Salamon, *The Maslov index for paths* , Topology, 32 (1993), p.827-844.

[SZ] D.Salamon, E.Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. 45 (1992), p.1303-1360.

[S] R. Seeley, *Singular integrals and boundary value problems*, Amer. J. Math, 88 (1966), 781-809.

[V] C.Viterbo, *Intersections de sous varietes lagrangiennes, fonctionelles d'action et indice des systemes hamiltoniens*, Bull. Soc. Math France, 115 (1987), 361-390.

[W] K.Wojciechowski, *A note on the space of pseudodifferential projections with the same principal symbols*, J. Op. Theory, 15 (1986), 207-216.

[Y] T.Yoshida, *Floer homology and splittings of manifolds*, Ann. of Math, 134 (1991), 277-324.

MICHIGAN STATE UNIV. LIBRARIES
31293010219628