This is to certify that the

dissertation entitled

INDEPENDENT INCREASES IN RISK

AND THEIR COMPARATIVE STATICS

presented by

HELEI QU

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Economics

Major professor

Date March 25, 1994

INDEPENDENT INCREASES IN RISK AND THEIR COMPARATIVE STATICS

By

Helei Qu

AN ABSTRACT OF A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1994

ABSTRACT

INDEPENDENT INCREASES IN RISK AND
THEIR COMPARATIVE STATICS

By

Helei Qu

This paper introduces a special type of Rothschild and Stiglitz increases in risk. This is prompted by the fact that Rothschild and Stiglitz increases in risk are too broadly defined to generate determinate comparative statics in most decision models. Typically very severe restrictions on the utility functions are needed in order for the comparative static effect to be determinate. An alternative is to further restrict the increases in risk. Many cases of special Rothschild and Stiglitz increases in risk have thus been proposed and examined. We introduce our own increase in risk in this paper.

An independent increase in risk further restricts a Rothschild and Stiglitz increase in risk by requiring $\tilde{\epsilon}$ to be independent of $\tilde{x}$. Random variable $\tilde{y}$ is an independent increase in risk from random variable $\tilde{x}$ if

$$\tilde{y} =^d \tilde{x} + \tilde{\epsilon},$$

where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$. An independent increase in risk is a special Rothschild and Stiglitz increase in risk as $E(\tilde{\epsilon}) = E(\tilde{\epsilon}|x) = 0$.

Under an independent increase in risk, the distribution of random variable $\tilde{\epsilon}$ is the same no matter what realized value of random variable $\tilde{x}$ is. The distribution of $\tilde{\epsilon}$ is independent of x. This uniform property makes the comparative static effect of an independent increase in risk determinate in many instances.

Like a strong increase in risk, an independent increase in risk is also a generalization of an introduction of risk. An independent increase in risk can, however, generate any Rothschild and Stiglitz increases in risk, when the initial random variable $\tilde{x}$ is degenerate at a point.

An independent increase in risk imposes no restrictions on the two distribution functions in the center of the supports. The two CDF's may cross many times. The support of $F(x)$ is, however, contained inside the support of $G(y)$.

The conditions for generating comparative statics are acceptable for an independent increase in risk. The independent increases in risk have a wide range of applications. Background risk, savings and uncertainty, asset proportion, portfolio problem and the output level of a competitive firm are few examples.

Dedicated to My Parents, Yuee Li and Jinlian Qu

ACKNOWLEDGEMENT

A dissertation cannot be accomplished without the support of many people. This is especially true for this dissertation. There would not be this dissertation without the supports from my committee members and many friends.

During this long process, my adviser, Professor Jack Meyer suffered as much as I did if not more than I did. Dr. Meyer read many drafts of this dissertation and made lots of comments in addition to his insightful guidance on this subject. His efforts have always been appreciated.

I am fortunate to have Dr. Ching-Fan Chung on my committee. His kindness, sincere efforts made this dissertation possible. His courage helped me through some of the most difficult times. His supports are beyond the expression of English language. Without him this dissertation can not be finished.

The same can be said to Professor Carl Davidson, another member of my committee. Numerous faculties and students have helped me through this ordeal, Professor Lawrence W. Martin, Professor Robert H. Rashe, Dr. Gyemyung Choi and John "Bud" Schulz, to name a few. It would be too tedious to list every one here.

Mr. Hailong Qian is worth mentioning specifically. It was Mr. Qian who came to the author's rescue and read part of this dissertation when the author was at difficulty. He was always there extending a helpful hand when the author needed it. And he shared the excitements and frustrations with the author as this dissertation progressed. Without him this dissertation would have been still in working. His friendship is always appreciated.

During my prolonged education, I lost the most precious person in my life, My Dear Mother, Yuee Li. I was not at her side when she passed away 22 December 1991, which is and will be the most regrettable moment in my life. Dear Mom, you were, are and will be with me every step along the long life journey.

My dear father, Jinlian Qu, has been giving me his best support through my education. My beautiful sister, Lanlan Qu, has been helping the family through so many turbulent years, while I have been far away. My brother-in-law, Rongjun Liu, is a great help to the family, and this is always appreciated. Without the support of my family, I would not have finished this.

This dissertation is dedicated to my beloved parents.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

In economics there are many unknowns. For instance, the determinants of supply and demand and therefore the market price have significant stochastic components. Like any other science, economics takes these into consideration. Including randomness enriches the content of economics. Explaining the impact of uncertainty is an important aspect of economic theory. Risk and uncertainty theory is the branch of economics that deals with these issues.

Risk, uncertainty or lack of information is represented by including random variables as parameters in a decision model. A random variable has a probability distribution function (PDF) and a cumulative distribution function (CDF). These describe all the possible outcomes and the likelihood that each of the outcomes occurs. We will use the PDF and CDF to describe the risk and uncertainty.

Von Neumann and Morgenstern (1944) place conditions on preferences over the random outcomes. These conditions, known as expected utility theory, imply that the ranking of random alternatives is given by the expectation of the utility of the possible outcomes.

An economic agent's preferences can be classified into risk averse, risk loving or risk neutral according to their attitude towards risk. Some economic agents buy lottery tickets as well as insure their properties. That is, there are agents who do not belong to any of the above groups, they are both risk averse and risk loving.

Different economic agents have different attitudes toward risk, some are more risk averse than others. Risk neutral economic agents are those who are indifferent between taking a random outcome and taking the certain expected value of the random outcome. Risk averters are those who prefer the expected value of a random outcome to the random outcome itself. Risk lovers are those who prefer a random outcome to the certain expected value of the random outcome. These attitudes are reflected in the shape of the utility function. Risk averters have a concave utility function, risk lovers have a convex utility function, and risk neutral economic agents have a linear utility function. Concave utility functions are usually best suited to the maximization of expected utility.

An economic agent's attitude toward risk can be measured. Arrow (1965) and Pratt (1964) use absolute and relative risk aversion to measure the curvature of the utility functions. Absolute risk aversion is defined as $A(z) = -u''(z)/u'(z)$, and relative risk aversion is defined as $R(z) = -z \cdot u''(z)/u'(z) = z \cdot A(z)$, where u is the utility function, z is the outcome parameter. $A(z) \geq 0$ is for the risk averters, $A(z) \leq 0$ for the risk lovers and $A(z) = 0$ for the risk neutral economic agents.

Risk aversion depends on the level of the outcome parameter. When *A(z)* decreases as z increases, this is referred to as decreasing absolute risk aversion (DARA). Increasing absolute risk aversion (IARA) occurs when *A(z)* increases as *z* increases. Finally, constant absolute risk aversion (CARA) prevails when *A(z)* does not change as *z* changes. Arrow (1965) argues that absolute risk aversion *A(z)* is a decreasing function (DARA) of *z* where *z* is wealth.

Similar definitions apply to relative risk aversion. Arrow (1965) argues that the relative risk aversion *R(z)* is an increasing function (IRRA) of wealth. IRRA means that if both the wealth and the size of the random variable are increased in the same proportion, the willingness to accept the risk should decrease.

When one group of economic agents prefer one random variable to another, this generates the definition of dominance among the random variables, or stochastic dominance. Hanoch and Levy (1969) define dominance for all agents with non-decreasing utility functions and also for those with non-decreasing and concave utility functions. Hadar and Russell (1969) also give these definitions and call them first order stochastic dominance (FSD) and second order stochastic dominance (SSD), respectively. Hanoch and Levy, Hadar and Russell prove that a distribution *F(x)* FSD a distribution *G(x)* if and only if all economic agents whose utility function is non-decreasing in *x* prefer *F(x)* to *G(x)*. Similarly *F(x)* SSD *G(x)* if and only if all economic agents whose utility function is non-decreasing and concave in *x* prefer *F(x)* to *G(x)*.

Stochastic dominance is a unanimous preference concept. FSD is for the group of non-decreasing utility functions, including risk averters, risk lovers and risk neutral economic agents. SSD applies to non-decreasing and concave utility functions, that is only risk averters are in this group.

A related concept defined by Rothschild and Stiglitz (R-S) (1970) is a Mean Preserving Spread (MPS) increase in risk. Rothschild and Stiglitz give three definitions of a risk change. They consider unanimous preference for all economic agents with concave utility functions. Distribution *G(x)* is a Rothschild and Stiglitz increase in risk from *F(x)* if the risk averters prefer *F(x)* to *G(x)* and *F(x)* and *G(x)* have the same means. Note that *u(x)* is not required to be increasing or decreasing.

The most commonly studied decision model contains only one random variable. Often this is presented in a one random variable, one choice parameter and one outcome parameter (1-1-1) format, Choi (1992). In this model, the final outcome is a function of the random variable, the choice parameter and possibly other exogenous parameters. This is the model used in much of the research in risk and uncertainty theory. The model is employed in chapters 2 and 3 and the notation will be introduced at that time.

Both the stochastic dominance and the Rothschild and Stiglitz increase in risk definitions allow two particular comparative static questions to be posed in this decision mode. When the random variable undergoes an FSD, SSD or Rothschild and Stiglitz risk change, how does expected utility change, and how does the decision made by the agent change? These comparative static questions are an important part of risk and uncertainty analysis over the past twenty years.

This dissertation is organized as follows. In chapter 2, we will review the literature on different approaches to changes in randomness and their comparative statics and the findings which have been published so far. Changes in randomness are usually represented as CDF changes. The initial and the final CDF are specified and their difference is restricted. The comparative static question this dissertation focuses on concerns the change in the optimal choice parameter as the CDF changes. This change will always have an already known effect on expected utility. The general MPS, FSD or SSD changes are often too broad to generate determinate comparative static results. Much research therefore has focused on how to generate special types of changes in randomness to obtain determinate comparative static results. These special types of changes in randomness and their comparative static results are also reviewed in chapter 2.

Chapter 3 introduces an independent increase in risk, which is the main focus of this dissertation. An independent increase in risk changes a random variable by adding an independent random variable to it. This generates a special type of Rothschild and Stiglitz increase in risk for which determinate comparative static results can be demonstrated.

Associated with independent increases in risk are Independent Mean Preserving Spreads (IMPS). An IMPS is a set of MPS that differ from one another by linear shifts. In chapter 3 we connect independent increases in risk and IMPS.

The last chapter is devoted to comparative statics. To get determinate comparative statics after all is the purpose of introducing the independent increases in risk. An independent increase in risk indeed generates determinate comparative statics under quite general conditions. We will see this and examples of applications in chapter 4.

CHAPTER 2
LITERATURE REVIEW

In this chapter we will review the literature on changes in randomness and their comparative statics. Stochastic dominance and Mean Preserving Spread (MPS) increases in risk are perhaps the two most important concepts in the risk and uncertainty literature, we therefore start with these concepts.

In this paper the notations and assumptions are as follows, $\tilde{x}$ and $\tilde{y}$ represent the random variables with CDF's $F(x)$ and $G(y)$ respectively, $F(x)$ has bounded support $[b,B]$ and $G(y)$ $[a,A]$, $a \le b \le c \le C \le B \le A$, c and C are two points inside the supports. λ and ξ represent the exogenous non-random parameters, α represents the choice parameter, u is the utility function. Thus a 1-1-1 model is $Eu[z(x,\alpha)]$, where z is the outcome. Economic agents choose α to maximize expected utility. To guarantee an interior solution, $z_\alpha(x,\alpha) = 0$ for a finite α $\forall$ $x \in [a,A]$ is assumed. $z_{\alpha\alpha}(x,\alpha) < 0$ is also assumed to guarantee the second order condition for maximization is satisfied.

Hanoch and Levy (1969) and Hadar and Russell (1969) define first order stochastic dominance (FSD) and second order stochastic dominance (SSD) as follows.

Definition 2.1 (Hanoch and Levy, Hadar and Russell):

(1). CDF $F(x)$ is said to be at least as large as CDF $G(x)$ in the sense of FSD if and only if $G(x) \geq F(x)\ \forall\ x \in [a,A]$;

(2). CDF $F(x)$ is said to be at least as large as CDF $G(x)$ in the sense of SSD if and only if $\int_a^y [G(x) - F(x)]dx \geq 0\ \forall\ y \in [a,A]$.

These definitions of stochastic dominance are for CDF's not for random variables. Hanoch and Levy and Hadar and Russell also show that the dominating distribution is unanimously preferred to the other distribution by a certain group of economic agents. $F(x)$ FSD $G(x)$ if and only if all economic agents with an increasing utility function prefer $F(x)$ to $G(x)$. $F(x)$ SSD $G(x)$ if and only if all economic agents with an increasing and concave utility function prefer $F(x)$ to $G(x)$. Thus the impact of a CDF change on expected utility is known, the impact on the choice variable is of concern here.

Rothschild and Stiglitz (1970) give three definitions of an increase in risk and prove that these definitions are equivalent. We state this result as a definition:

Definition 2.2 (Rothschild and Stiglitz): The following three definitions of an increase in risk are equivalent.

(1). Every risk averter prefers random variable $\tilde{x}$ to random variable $\tilde{y}$, that is, $\int_a^A u(x)dF(x) \geq \int_a^A u(x)dG(x)$, where $u'' \le 0$;

(2). Random variable $\tilde{y}$ is equal in distribution to random variable $\tilde{x}$ plus some noise $\tilde{\epsilon}$. That is, $\tilde{y} =^{d} \tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ satisfies $E(\tilde{\epsilon}|x) = 0$ and " $=^{d}$ " represents "is equal in distribution to";

(3). (a) $\int_a^y [G(x) - F(x)]dx \ge 0, \forall\ y \in [a,A]$;

(b) $\int_a^A [G(x) - F(x)]dx = 0$, where $[a,A]$ contains the supports of $\tilde{x}$ and $\tilde{y}$.

These definitions define an MPS increase in risk and definition (3) is often referred to as the integral conditions of an MPS increase in risk. An MPS increase in risk moves probability mass from the center of a distribution to its two tails while preserving the mean. This change produces a new distribution which has the same mean as the original distribution and is defined to be a risk increase. Definition (1) shows the impact on expected utility. Rothschild and Stiglitz also provide the framework for determining the impact of a risk change on the decision made by an expected utility maximizing agent. The method of defining a risk change and determining its comparative static effect focuses on the integral conditions in definition (3). Rothschild and Stiglitz (1971) use this CDF approach and most have followed ever since.

An alternative method of representing a change in the riskiness of a random variable involves transforming the random variable and has been used by Sandmo (1969, 1970, 1971) and others, and it is formalized under the name deterministic transformation by Meyer and Ormiston (1989). A deterministic transformation changes the random variable and hence the CDF by deterministic function $t(x)$, which maps every realization value of random variable $\tilde{x}$ into a new point. Appropriate restrictions on $t(x)$ will generate MPS, FSD or SSD changes in randomness. This method has proven to be a simple and effective way to represent a random variable change, we will review this method in section 2.

Over the years economists have found that a general Rothschild and Stiglitz increase in risk or an FSD or SSD change in randomness is too broad to yield determinate comparative static results. Typically very severe restrictions on utility functions are needed in order for the comparative static results to be determinate. An alternative is to further restrict the change in randomness. Many subsets of these FSD, SSD and Rothschild and Stiglitz changes in randomness have thus been proposed and examined, these changes and their comparative static results are reviewed next.

2.1 CDF Approach to Risk Change and its Comparative Statics

CDF approach to comparative static analysis specifies the initial and the final CDF's and then compares the optimal values for the choice parameter under the two CDF's.

The change in the optimal values is the comparative static effect of the CDF change. In this section, we shall investigate the comparative static effect of CDF changes referred to as strong increases in risk, relatively strong increases in risk, relatively weak increases in risk and monotonic likelihood ratio risk changes and stochastic dominance.

§ 2.1.1 An Impossibility Theorem

For an expected utility maximizer $Eu[z(x,\alpha)]$, his optimal choice parameter α satisfies first order condition (FOC) $Eu'[z(x,\alpha)]\cdot z_\alpha(x,\alpha) = 0$ in the 1-1-1 model. By the first definition of an MPS increase in risk, if $u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha)$ is concave in $\tilde{x}$, Rothschild and Stiglitz then conclude that an MPS increase in risk will decrease α. Rothschild and Stiglitz continue to explore what this condition implies in specific economic models. Concavity of $u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha)$ is a general requirement, it is very restrictive when translated into conditions on utility function $u(z)$ and payoff function $z(x,\alpha)$.

Meyer and Ormiston (1983) ask when an arbitrary Rothschild and Stiglitz increase in risk causes all risk averse economic agents to adjust their optimal choice parameters in the same direction in a 1-1-1 model. Unfortunately the answer to this question is negative, they have the following theorem regarding this problem.

Theorem 2.1 (Meyer and Ormiston): Rothschild and Stiglitz increases in risk cause all risk averse economic agents to decrease optimal choice parameter α if and only if there exists a α_0 such that $z_\alpha(x,\alpha_0) = 0 \; \forall \; x \in [a,A]$.

Result concerning the same problem with regarding to FSD is in the following corollary.

Corollary 2.2 (Meyer and Ormiston): All changes in a CDF such that the final CDF is dominated in the sense of FSD by the initial CDF cause all risk averse economic agents to decrease the choice parameter if and only if there exists a α_0 such that $z_\alpha(x,\alpha_0) = 0 \; \forall \; x \in [a,A]$.

An SSD shift is a combination of an FSD and an MPS shifts, Hadar and Seo (1990). A similar result can thus be derived for an SSD change in randomness.

The implication of condition $z_\alpha(x,\alpha_0) = 0$ for all x is that optimal choice parameter α_0 does not depend on random variable $\tilde{x}$. Obviously this restriction eliminates all interesting models. A general Rothschild and Stiglitz increase in risk does not yield determinate comparative static results for all risk averse economic agents in any meaningful 1-1-1 model. This leaves us two choices. One is to further restrict the utility function such as requiring it to be DARA. The other choice is to restrict the change in randomness. Most literature on this subject defines special types of changes in randomness. These special types of changes in randomness and their comparative statics are reviewed next.

§ 2.1.2 Strong Increases in Risk

Meyer and Ormiston (1985) introduce a strong increase in risk. A strong increase in risk transfers probability mass from the original interval $[b,B]$, which contains the support of initial CDF $F(x)$, to intervals $[a,b]$ and $[B,A]$.

Definition 2.3 (Meyer and Ormiston): CDF $G(x)$ is a strong increase in risk from CDF $F(x)$ if their difference $G(x) - F(x)$ satisfies the following conditions.

(a). $\int_a^y [G(x) - F(x)]dx \geq 0, \ \forall \ y \in [a,A];$

(b). $\int_a^A [G(x) - F(x)]dx = 0 \ ;$

(c). $G(x) - F(x)$ is non-increasing in interval (b,B).

A strong increase in risk is a generalization of an introduction of risk, which is an increase in risk from an initial non-random situation. Note that the two CDF's only cross once in the case of a strong increase in risk. The opposite is however not true. Not all CDF pairs which cross once are strong increases in risk.

Meyer and Ormiston (1985) have the following theorem regarding a strong increase in risk.

Theorem 2.2 (Meyer and Ormiston): Assume that decision makers choose α to maximize $Eu[z(x,\alpha)]$, where $u' \geq 0$, $u'' \leq 0$, then all risk averse economic agents, when faced with a strong increase in risk, will decrease the optimal value of α if

(a). $z_x \geq 0$ and $z_{\alpha\alpha} \leq 0 \ \forall \ x \in [a,A]$;

(b). $z_{\alpha x} \geq 0$, $z_{\alpha x x} \leq 0 \ \forall \ x \in [a,A]$.

Condition $z_x \geq 0$ together with $u' \geq 0$ guarantees that the higher values of random variable are preferred to the lower values. The case where $z_x \leq 0$ can be treated the same way with minor modifications. The conditions $z_{\alpha x} \geq 0$ and $z_{\alpha x x} \leq 0$ are restrictions needed to generate the comparative static results, they are conditions on the payoff function.

§ 2.1.3 Relatively Strong Increases in Risk

A relatively strong increase in risk is proposed by Black and Bulkley (1989) to be less restrictive than a strong increase in risk. A relatively strong increase in risk allows some of the probability mass that is transferred to intervals $[a,b]$ and $[B,A]$ in the case of a strong increase in risk to stay inside interval $[b,B]$ and yet preserves the comparative static result associated with a strong increase in risk.

Definition 2.4 (Black and Bulkley): CDF $G(x)$ is a relatively strong increase in risk from CDF $F(x)$ if:

(a). $\int_a^A [G(x) - F(x)]dx = 0$;

(b). For all points in interval $[c,C]$, $G(x) - F(x)$ is non-increasing; For all points outside this interval, $G(x) - F(x)$ is non-decreasing, where c and C are two points inside $[b,B]$ and $c \leq C$;

(c). $f(x)/g(x)$ is non-decreasing in interval $[b,c)$;

(d). $f(x)/g(x)$ is non-increasing in interval $(C,B]$.

Conditions (c) and (d) relax a strong increase in risk. A relatively strong increase in risk allows the density function of the riskier random variable to be bigger than that of the less riskier random variable in intervals $[b,c)$ and $(C,B]$.

Assume α_F and α_G are the optimal choice parameters under distributions $F(x)$ and $G(x)$ respectively. Black and Bulkley (1989) have the following theorem concerning a relatively strong increase in risk.

Theorem 2.3 (Black and Bulkley): The sufficient conditions for $\alpha_G < \alpha_F$ for all risk averse economic agents are:

(a). $G(x)$ represents a relatively strong increase in risk from $F(x)$;

(b). $z_x \geq 0$, $z_{\alpha x} \geq 0$, $z_{\alpha xx} \leq 0$ and $z_{\alpha\alpha} < 0$ for all x and α.

This theorem requires no more conditions on utility function than those required by a strong increase in risk.

The difference between a strong increase in risk and a relatively strong increase in risk is that density function $f(x)$ is bigger than density function $g(x)$ on entire interval $[b,B]$ under a strong increase in risk, but under a relatively strong increase in risk density function $f(x)$ may be smaller than $g(x)$ in intervals $[b,c]$ and $[C,B]$.

§ 2.1.4 Relatively Weak Increases in Risk

Dionne, Eeckhoudt and Gollier (1993) propose a relatively weak increase in risk when studying a special type of payoff function. They study models with a payoff function that is linear in both random variable and choice parameter. This group of payoff functions satisfies the restrictions on payoff functions required by a relatively strong increase in risk.

The model Dionne et al. use is a general linear model $z(x,\alpha) = \alpha(x-\lambda) + \xi$, where λ and ξ are the exogenous parameters. There are two cases $\alpha \geq 0$ and $\alpha \leq 0$. In an application, the sign of α is usually known. We only consider $\alpha \geq 0$, for $\alpha \leq 0$ can be handled with minor modifications. The conditions on the payoff function are $z_{\alpha x} > 0$, $z_{\alpha\alpha} = z_{xx} = z_{\alpha x x} = 0$. These stronger conditions on $z(x,\alpha)$ allow one to relax some restrictions on the type of changes in randomness. We start with the definition of a relatively weak increase in risk, assume two density functions cross at two points c and C.

Definition 2.5 (Dionne, Eeckhoudt and Gollier): CDF $G(x)$ represents a relatively weak increase in risk from CDF $F(x)$ if for parameter γ

(i). When $\gamma \in [c,C]$,

(a) $\int_a^A [G(x) - F(x)]dx = 0$;

(b) For all points in interval $[c,C]$, $G(x) - F(x)$ is non-increasing; For all points outside the interval, $G(x) - F(x)$ is non-decreasing;

(ii). When $\gamma \in [b,c)$, then besides conditions (a) and (b) one needs the following conditions;

(c) $f(x)/g(x) \leq f(\gamma)/g(\gamma)$, $b \leq x \leq \gamma$; $f(x)/g(x) \geq f(\gamma)/g(\gamma)$, $\gamma \leq x < c$;

(iii). When $\gamma \in (C,B]$, then besides conditions (a) and (b) one needs the following conditions;

(d) $f(x)/g(x) \geq f(\gamma)/g(\gamma)$, $C < x \leq \gamma$; $f(x)/g(x) \leq f(\gamma)/g(\gamma)$, $\gamma \leq x \leq B$.

γ plays a critical role in this definition, different conditions are needed when γ belongs to different intervals. On intervals $[b,c)$ and $(C,B]$ a relatively strong increase in risk imposes condition on $f(x)/g(x)$ on the entire intervals, while a relatively weak increase in risk imposes condition on $f(x)/g(x)$ relative to point γ, or $f(\gamma)/g(\gamma)$.

The comparative static result for a relatively weak increase in risk is in the following theorem.

Theorem 2.4 (Dionne, Eeckhoudt and Gollier): Suppose that α_F and α_G are the optimal choice parameters under CDF's *F(x)* and *G(x)* respectively and α_F is an interior solution. Then the sufficient conditions for $\alpha_G \leq \alpha_F$ for all strictly risk averse economic agents are:

(a). *G(x)* represents a relatively weak increase in risk from *F(x)*;

(b). Payoff function $z(x,\alpha)$ is a linear function of both random variable $\tilde{x}$ and choice parameter α, $z_{xx} = z_{\alpha\alpha} = 0$.

This theorem depends on parameter γ, if γ is in different intervals, the required conditions are different.

§ 2.1.5 Monotone Likelihood Ratio Changes in Randomness

Ormiston and Schlee (1991) use Monotone Likelihood Ratio (MLR) stochastic dominance which is a subset of FSD to study the tradeoff between restricting the utility function and restricting the types of changes in randomness. Ormiston and Schlee specify a set of economic agents whose behavior is known under certainty, then they investigate the behavior of these economic agents under uncertainty. For a class of economic agents whose preferences under certainty are such that the optimal choice parameter increases with the increase of non-random variable x, what type of CDF change for $\tilde{x}$ when $\tilde{x}$ is random causes all economic agents in this class to increase optimal choice parameter α under uncertainty?

CDF $F(x)$ has a support of $[b,A]$ and CDF $G(x)$ has a support of $[a,B]$. $m(x)$ is a non-negative and non-decreasing function. $H(x)$ is the difference between the two CDF's $H(x) = G(x) - F(x)$. Ormiston and Schlee have the following MLR definition.

Definition 2.6 (Ormiston and Schlee): CDF $G(x)$ is MLR dominated by CDF $F(x)$ if there exists a non-negative and non-decreasing function $m(x)$ defined in interval $[b,B]$ and the following conditions are satisfied.

(a). $H(x) = G(x)$ in $[a,b)$;

(b). $dH(x) = [1-m(x)]dG(x)$ in $[b,B]$, where $dH(x)$ is the derivative of $H(x)$, $dG(x)$ is the derivative of $G(x)$;

(c). $H(x) = -F(x)$ in $(B,A]$.

Ormiston and Schlee consider model $Eu(x,\alpha)$, where $\tilde{x}$ is the random variable, α is the choice parameter. They deal with random variable and choice parameter directly without payoff function z. If α_F and α_G maximize the expected utility function $Eu(x,\alpha)$ under CDF's $F(x)$ and $G(x)$ respectively, then

Theorem 2.5 (Ormiston and Schlee): The following conditions are equivalent.

(a). $\alpha_F \geq \alpha_G$ whenever CDF G(x) is MLR dominated by CDF *F(x)*;

(b). $u_{\alpha\alpha}(x,\alpha) \geq 0$ whenever $u_\alpha(x,\alpha) = 0$.

Condition $u_{\alpha\alpha}(x,\alpha) \geq 0$ whenever $u_\alpha(x,\alpha) = 0$ is a condition under certainty. This theorem says that whenever economic agents increase their optimal choice parameters under certainty, they will increase their optimal choice parameters under uncertainty if facing an MLR risk shift.

§ 2.1.6 Summary: Relationship Among Different Type of Changes in Randomness

It is important to see the relationship among these special types of changes in randomness. In terms of density function, a strong increase in risk requires that the density function g(x) of the riskier random variable, Figure 2.1, to be smaller than that of the less risky random variable in interval *[b,B]*, the riskier random variable also distributes in intervals *[a,b]* and *[B,A]*. A relatively strong increase in risk allows the density function of the riskier random variable be bigger than that of the less riskier random variable in two intervals *[b,c)* and *(C,B]*. But in interval *[b,c)*, $f(x)/g(x)$ is non-decreasing; And in interval *(C,B]*, $f(x)/g(x)$ is non-increasing. The MLR

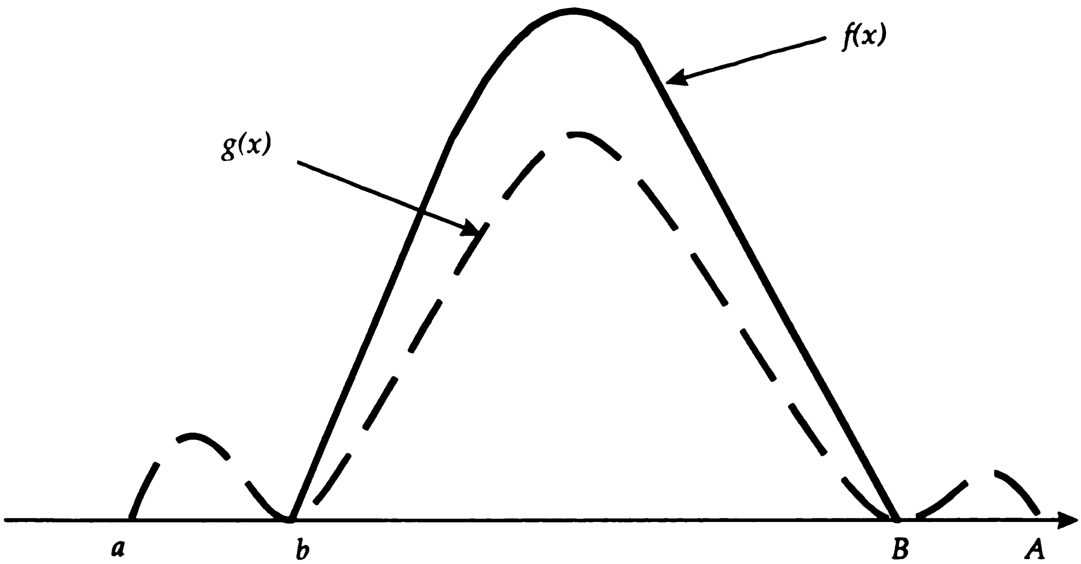

Figure 2.1. $f(x)$ is the density function of the initial random variable, $g(x)$ is the density function of the final random variable.

requires that the support of the dominated random variable is somewhere lower than that of the dominating random variable, and $d[G(x)-F(x)] = [1-m(x)]dG(x)$ is satisfied in interval $[b,B]$, where $m(x)$ is a non-negative and non-decreasing function.

We can see that a strong increase in risk is the most restrictive type of increase in risk. It implies a relatively strong increase in risk which in turn implies a relatively weak increase in risk. A relatively strong increase in risk and a relatively weak increase in risk have different restrictions in intervals $[b,c)$ and $(C,B]$.

2.2 Deterministic Transformation Approach

The deterministic transformation approach changes the initial random variable and hence the CDF by mapping each possible outcome of the random variable into a new value. Meyer and Ormiston (1989) define deterministic function $t(x)$ as deterministic transformation function which transforms random variable $\tilde{x}$ into a new random variable. Deterministic function $t(x)$ is a non-decreasing, continuous and piecewise differentiable function. The non-decreasing assumption combined with the monotonic preferences for outcomes ensures that the transformation does not reverse the preference ordering over the various possible outcomes of the original random variable. A special group of deterministic transformations are simple transformations. We will review the comparative static result for simple transformations.

§ 2.2.1 Deterministic Transformations

To study the marginal risk change effect of price, Sandmo (1971) transforms random variable $\tilde{x}$ into a new random variable $(\gamma\tilde{x} + \theta)$, where γ is the multiplicative shift parameter and θ is the additive shift parameter. A change in θ will only change the mean of the random variable, Sandmo gives the following result regarding the changes in parameter θ. The model is $Eu[\alpha x+\lambda(\alpha)+\xi]$.

Theorem 2.6 (Sandmo): The decreasing absolute risk aversion (DARA) is a sufficient condition for optimal choice parameter α to increase with an increase in parameter θ.

This result is a local one, it has to be evaluated at $\gamma = 1$ and $\theta = 0$.

A change in γ (from point $\gamma = 1$ and $\theta = 0$) will change the mean of random variable $\tilde{x}$. We however only need a mean preserving increase in risk, Sandmo therefore reduces θ simultaneously. To restore the mean we need $d\theta/d\gamma = -\mu$, where μ is the mean of random variable $\tilde{x}$. Ishii (1977) proves a comparative static result for a change in parameter γ.

Theorem 2.7 (Ishii): Decreasing absolute risk aversion (DARA) is a sufficient condition for optimal choice variable α to decrease with an increase in parameter γ.

The transformation Sandmo uses is a prototype of a deterministic transformation. Meyer and Ormiston (1989) prove that a general deterministic transformation is a fourth characterization of a Rothschild and Stiglitz MPS increase in risk. They have the following theorem concerning transformation function $t(x)$.

Theorem 2.8 (Meyer and Ormiston): Deterministic transformation $t(x)$ represents a Rothschild and Stiglitz increase in risk for the random variable given by CDF $F(x)$ if function $k(x) = t(x) - x$ satisfies the following conditions:

(a). $\int_a^A k(x)dF(x) = 0$, where $[a,A]$ is the support of $\tilde{x}$;

(b). $\int_a^y k(x)dF(x) \leq 0,\ \forall\ y \in [a,A]$.

The advantage of deterministic transformation is that the changes in randomness can be restricted in different ways from that under the CDF approach. Restricting function $t(x)$ beyond those in theorem 2.8 will generate special changes in randomness. Meyer (1989) uses deterministic transformation to define an FSD change.

Theorem 2.9 (Meyer): Transformed random variable $t(x)$ dominates initial random variable $\tilde{x}$ in the sense of an FSD if and only if $[t(x)-x]\cdot f(x) \geq 0 \; \forall \; x \in [a,A]$.

The FSD dominating CDF lies below the initial CDF, it generates a greater expected utility for all economic agents with non-decreasing utility functions.

For SSD risk changes, Meyer has:

Theorem 2.10 (Meyer): Transformed random variable $t(x)$ dominates initial random variable $\tilde{x}$ in the sense of an SSD if and only if $\int_a^y [t(x)-x]f(x)dx \geq 0, \; \forall \; y \in [a,A]$.

Recall that SSD changes in randomness yield a higher expected utility for all economic agents with a non-decreasing and concave utility function.

§ 2.2.2 Simple Increases in Risk and Comparative Statics

One way to further restrict $t(x)$ so that determinate comparative static results can be obtained is to restrict the difference between the two random variables, $k(x) = t(x) - x$. Simple transformations require $k(x)$ to be a monotonic function. A simple transformation produces a special Rothschild and Stiglitz increase in risk. A simple increase in risk is the case where $k'(x) \geq 0$ if $k(x)$ differentiable, it implies that there exists a value $x^{*} \in [a,A]$ such that the values of $\tilde{x}$ to the right or to the left of x^* are moved away from x^* as the risk increases. The original random variable is stretched out around particular value x^* to get the new random variable.

The economic agents maximize expected utility function $Eu[z(x,\alpha)]$. The comparative static result from the simple transformation is in the following theorem.

Theorem 2.11 (Meyer and Ormiston): The economic agents choosing α to maximize $Eu[z(x,\alpha)]$ will decrease optimal choice parameter α when the random variable undergoes a simple increase in risk, if

(a). Utility function u(z) displays decreasing absolute risk aversion (DARA);

(b). $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$ and $z_{\alpha xx} \leq 0$.

This is a generalization of Sandmo and Ishii's results for the competitive firm. Condition DARA is widely used and accepted.

The simple transformation, like the general deterministic transformation, can generate FSD and SSD stochastic dominate changes in randomness. Comparative statics for these FSD and SSD changes are also possible. Ormiston (1990) defines a simple FSD transformation as following.

Definition 2.7 (Ormiston): The random variable given by simple transformation $t(x)$ first degree stochastically dominates (FSD) random variable $\tilde{x}$ if and only if function $k(x) = t(x) - x$ satisfies $k(x) \geq 0 \; \forall \; x \in [a,A]$.

The comparative static result for this FSD simple transformation is in the following theorem.

Theorem 2.12 (Ormiston): The optimal value of the choice parameter increases for any simple FSD transformation if

(a). $u' > 0$, $u'' \leq 0$ and $A' \leq 0$;

(b). $z_x > 0$, $z_{xx} \leq 0$ and $z_{\alpha x} \geq 0$;

(c). $k(x) \geq 0$ and $k'(x) \leq 0$.

A simple SSD change in randomness is an SSD shift generated from a simple transformation which is defined as the following.

Definition 2.8 (Ormiston): The random variable given by simple transformation $t(x)$ second degree stochastically dominates (SSD) random variable $\tilde{x}$ if function $k(x) = t(x) - x$ satisfies $\int_a^y k(x)dF(x) \geq 0 \; \forall \; y \in [a,A]$.

The comparative static result for a simple SSD transformation is in the following theorem.

Theorem 2.13 (Ormiston): The optimal value of the choice parameter increases for any simple SSD transformation if

(a). $u' > 0$, $u'' \leq 0$ and $A' \leq 0$;

(b). $z_x > 0$, $z_{xx} \leq 0$, $z_{\alpha} \geq 0$ and $z_{\alpha x x} \leq 0$;

(c). $k'(x) \leq 0$ and $\int_a^y k(x)dF(x) \geq 0 \;\forall\; y \in [a,A]$.

A simple SSD transformation requires not only that z_α be monotonically increasing in random variable $\tilde{x}$, but also it requires that z_α be concave in the random variable. A relaxation of the restriction on the transformation requires an extra condition on the payoff function, that is $z_{\alpha x x} \leq 0$. This highlights the tradeoff between the restrictions on the payoff function and the restrictions on the types of changes in randomness.

We have reviewed strong increases in risk, relatively strong increases in risk and relatively weak increases in risk. All these are subset of MPS. We also reviewed the alternative approach to comparative statics, the transformation approach. In chapter three we will introduce a new increase in risk, an independent increase in risk.

CHAPTER 3
INDEPENDENT INCREASES IN RISK

In this chapter we will introduce a special type of increase in risk, an independent increase in risk. A random variable $\tilde{y}$ is said to be an independent increase in risk from the random variable $\tilde{x}$ if $\tilde{y} =^d \tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$. Here, " $=^d$ " represents "is equal in distribution to". Independent increases in risk can produce determinate comparative statics in the 1-1-1 models which will be discussed in chapter 4. In this chapter, section 1 discusses stochastic transformations and defines independent increases in risk. In section 2, we will provide a method to recover the independent random variable given the distribution of $\tilde{x}$ and $\tilde{y}$.

3.1 Independent Increases In Risk

The second of Rothschild and Stiglitz's three definitions of an increase in risk is that random variable $\tilde{y}$ is riskier than random variable $\tilde{x}$ if

$$\tilde{y} =^d \tilde{x} + \tilde{\epsilon},$$

where

$$E(\tilde{\epsilon}|x) = 0.$$

An independent increase in risk further restricts the Rothschild and Stiglitz's definition by requiring $\tilde{\epsilon}$ to be independent of $\tilde{x}$.

Definition 3.1: A random variable $\tilde{y}$ is an independent increase in risk from the random variable $\tilde{x}$ if

$$\tilde{y} =^d \tilde{x} + \tilde{\epsilon},$$

where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$.

To better understand what an independent increase in risk is, we shall examine the conditions placed on the CDF's for $\tilde{x}$ and $\tilde{y}$ for the special case of discrete random variables with a finite number of mass points. The supports of all the random variables are assumed to be contained in compact intervals on the real line. Assume that random variable $\tilde{x}$ has probability function $f(x)$ and CDF $F(x)$ with support $[b,B]$. The probability function $f(x)$ has mass $p_i = f(x_i)$ at $x_1 < x_2 < \cdots < x_n$. Let $\tilde{\epsilon}$ be a random variable with the probability function $h(\epsilon)$ and CDF $H(\epsilon)$ on $[\epsilon_1, \epsilon_m]$. We assume $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$. The probability function $h(\epsilon)$ has mass $q_j = h(\epsilon_j)$ at $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_m$ so that $E(\tilde{\epsilon}) = \sum_{j=1}^{m} q_j \cdot \epsilon_j = 0$. Note that $\epsilon_j = 0$ for some j is possible. Suppose $g(x)$ and $G(x)$, respectively, are the probability function and CDF of the random variable $\tilde{y}$ which is defined by

$$\tilde{y} =^d \tilde{x} + \tilde{\epsilon}.$$

Then the probability function $g(x)$ has mass $p_{ij} = p_i \cdot q_j = g(x_{ij})$ at x_{ij}, where

$$x_{ij} \equiv x_i + \epsilon_j \text{ for } i = 1, \cdots n,\ j = 1, \cdot\cdot, m.$$

We may denote the support of the random variable $\tilde{y}$ by $[a,A]$.

Let $s(x)$ be the difference between the two probability functions g and f:

$$s(x) = g(x) - f(x).$$

Also, let $S(x)$ be the difference between the two corresponding CDF's:

$$S(x) = G(x) - F(x).$$

According to the Rothschild and Stiglitz, since $\tilde{y}$ is an R-S increase in risk from $\tilde{x}$, $s(x)$ can be decomposed into the sum of a number of MPS's.

In the following analysis we will examine the properties of $s(x)$ in details. Specifically, we first concentrate on the behavior of $s(x)$ over a subinterval of its support and then extend the analysis to the entire support. Initially to simplify the analysis, we make an important assumption about the "noise" random variable $\tilde{\epsilon}$: we assume the support $[\epsilon_1, \epsilon_m]$ of $\tilde{\epsilon}$ is relatively narrower in comparison with the support $[b,B]$ of $\tilde{x}$. In particular, if we define

$$x_i' \equiv x_i + \epsilon_1 \text{ and } x_i'' \equiv x_i + \epsilon_m, \quad \text{for } i = 1, \cdots, n,$$

then the subintervals $[x_i', x_i'']$, $i = 1, \cdots, n$, do not overlap. That is, the length of the support $[\epsilon_1, \epsilon_m]$ of $\tilde{\epsilon}$ is shorter than the distance between any two adjacent points x_i and x_{i+1}. (The analysis without such a restriction is in Appendix A.) Since $E(\tilde{\epsilon}) = 0$, we have $\epsilon_1 < 0$, $\epsilon_m > 0$, and $x_i' < x_i < x_i''$, for all i. Moreover, we have $x_{ij} \equiv x_i + \epsilon_j \in [x_i', x_i'']$, for all i and j, which implies the support of $\tilde{y}$, as well as that of $s(x)$, are included in $\bigcup_{i=1}^{n} [x_i', x_i'']$.

Given the non-overlapping subintervals $[x_i', x_i'']$, $i = 1, \cdots, n$, let us define $s_i(x)$ to be the restriction of $s(x)$ on $[x_i', x_i'']$. That is,

$$s_i(x) = \begin{cases} s(x), & \text{for } x \in [x_i', x_i'']; \\ 0, & \text{otherwise}. \end{cases}$$

A careful inspection of the density function $g(x)$ and the definition of $s(x)$ reveals that the mass of discrete function $s_i(x)$ are all positive except at point x_i and the sum of all the mass is always zero. So each $s_i(x)$ is an MPS function. Moreover the shapes of all $s_i(x)$, $i = 1, \cdots, n$, are all proportional to each other with p_i being the proportionality factors. Finally, we note $s(x) = \sum_{i=1}^{n} s_i(x)$.

To further explore the property that the shapes of $s_i(x)$ are proportional, we introduce the following definition of linear shift.

Definition 3.2: The discrete MPS $s_i(x)$ is a linear shift from $s_1(x)$ if

$$s_i(x) = \lambda_i \cdot s_1(x+\xi_i), \text{ for } x \in [x_i', x_i''] \text{ and } x + \xi_i \in [x_1', x_1''],$$

where

$$\lambda_i = p_i/p_1,$$
$$\xi_i = (x_1 - x_i),$$

are referred to as the linear shift parameters.

Since $s_1(y) = s_i(y-\xi_i)/\lambda_i$ with $y = x + \xi_i$, $s_1(x)$ is also a linear shift of $s_i(x)$ if $s_i(x)$ is a linear shift of $s_1(x)$.

Let us examine the relationships among a set of MPS which differ by linear shifts.

Definition 3.3: A discrete Independent Mean Preserving Spread (IMPS) for the discrete random variable $\tilde{x}$ is a set of discrete MPS's $\{s_i(x), i = 1, \cdots, n\}$ in which $s_i(x)$ are linear shifts of each other.

The IMPS is the key concept in determining the integral conditions for an independent increase in risk.

Define cumulative function $S_i(x)$ from $s_i(x)$ as follows.

$$S_i(x) = \int_a^x s_i(t)\,dt = \sum_{x_{ij}<x} s_i(x_{ij}),$$

where $x_{ij} \equiv x_i + \epsilon_j$. Note that $S_i(x)$ are step functions. Since

$$S_i(x) = \sum_{x_{ij}<x} s_i(x_{ij}) = \sum_{x_{ij}<x} \lambda_i s_1(x_{ij} + \xi_i) = \sum_{x_{1j}<x+\xi_i} \lambda_i s_1(x_{1j}) = \lambda_i S_1(x + \xi_i),$$

$S_i(x)$ is also a linear shift of $S_1(x)$. In fact all $S_i(x)$, $i = 1, \cdots, n$, are linear shifts of each other. Since $s(x) = \sum_{i=1}^{n} s_i(x)$, $S(x) = \int_a^x s(t)dt$ and $S_i(x) = \int_a^x s_i(t)dt$, we have $S(x) = \sum_{i=1}^{n} S_i(x)$. $S(x)$ is also a step function.

Now, let us define $T(x)$ as

$$T(x) = \int_a^x S(t)\,dt = \int_a^x [G(t) - F(t)]dt.$$

Note the Rothschild and Stiglitz's conditions for $G(x)$ to be an MPS increase in risk from $F(x)$ are

(1) $T(x) = \int_a^x [G(t)-F(t)]dt \geq 0,$ for $x \in [a,A]$;

(2) $T(A) = \int_a^A [G(t)-F(t)]dt = 0,$

where $[a,A]$ is the support of $G(x) - F(x)$. Define $T_i(x)$ by

$$T_i(x) = \int_a^x S_i(t)\,dt, \quad \text{for } x \in [x_i', x_i''],$$

so $T(x) = \sum_{i=1}^{n} T_i(x)$. Note that $T(x)$ and $T_i(x)$ are all continuous functions. Since $s_i(x)$ is an MPS, we have $T_i(x) \geq 0$, for $x \in [x_i', x_i'']$, and $T_i(x_i'') = 0$ by the Rothschild and

Stiglitz's conditions. Moreover, since

$$
\begin{aligned}
T_i(x) &= \int_a^x S_i(t)\,dt \\
&= \int_a^x \lambda_i S_1(t+\xi_i)\,d(t+\xi_i) \\
&= \int_{a+\xi_i}^{x+\xi_i} \lambda_i S_1(z)\,dz, \qquad z = t + \xi_i \\
&= \lambda_i T_1(x + \xi_i),
\end{aligned}
$$

$T_i(x)$, $i = 1, \cdots, n$, are also linear shifts of each other if $S_i(x)$, $i = 1, \cdots, n$, are.

For an IMPS increase in risk, $T(x)$ can be decomposed into $T(x) = \sum_{i=1}^{n} T_i(x)$, where $T_i(x)$ are linear shifts of each other with the shift parameters determined by the probability distribution of $\tilde{x}$. Note that if $T_i(x)$ are linear shifts of each other, then $S_i(x)$ are also linear shifts of each other, and so are $s_i(x)$.

The relationship between the independent increases in risk and the independent mean preserving spreads is presented in the following theorem:

Theorem 3.1: Given two discrete random variables $\tilde{x}$ and $\tilde{y}$, the following two statements are equivalent.

Statement I:

$T(x) = \int_a^x [G(t)-F(t)]dt$ satisfies the following three conditions:

I1. $T(x) \geq 0$, for $x \in [a,A]$;

I2. $T(A) = 0$;

I3. $T(x)$ can be decomposed into

$$T(x) = \sum_{i=1}^{n} T_i(x),$$

where

$$T_i(x) = \lambda_i \cdot T_1(x+\xi_i),$$

for $x \in [x_i', x_i'']$, for $x + \xi_i \in [x_1', x_1'']$, $\lambda_i = p_i/p_1$, and $\xi_i = (x_1 - x_i)$. Also $T_i(x) \geq 0$ and $T_i(x_i'') = 0$.

Statement II:

$\tilde{y} =^d \tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is discrete and independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$.

Proof: I implies II. Assume that $T(x) = \int_a^x [G(t)-F(t)]dt$ satisfies the three integral conditions. Then $S(x) = \sum_{i=1}^{n} S_i(x)$ and $S_i(x) = \lambda_i \cdot S_1(x+\xi_i)$; that is, $S_i(x)$ differ from each other by linear shifts. Note that $G(x) - F(x) = S(x) = \sum_{i=1}^{n} S_i(x)$.

Define $F_1(x) = F(x) + S_1(x)$, then $F_1(x)$ is a discrete MPS increase in risk from $F(x)$. Therefore, there exits a random variable $\tilde{\epsilon}_1$ such that $E(\tilde{\epsilon}_1|x) = 0$, $\tilde{x}$ has a CDF $F(x)$, and $\tilde{x} + \tilde{\epsilon}_1$ has a CDF $F_1(x)$ according to the Rothschild and Stiglitz's conditions. $S_1(x)$ contains all the information needed to determine the random variable $\tilde{\epsilon}_1$. Since $S_1(x)$ is non-zero only in the interval $[x_1', x_1'']$, $F_1(x)$ and $F(x)$ differ on $[x_1', x_1'']$, as shown in Figure 3.1.

$$F_1(x) = \begin{cases} S_1(x), & x_1' \le x < x_1; \\ f(x_1) + S_1(x), & x_1 \le x < x_1''; \\ F(x), & x_1'' \le x \le A. \end{cases}$$

The joint CDF of $\tilde{x}$ and $\tilde{\epsilon}_1$ is $F_1(x) = F(x) \cdot H_1(x)$, where $H_1(x)$ is the conditional CDF of $\tilde{\epsilon}_1$ given $\tilde{x} = x_1$.

$$F_1(x) = \begin{cases} f(x_1) H_1(x), & x_1' \le x < x_1''; \\ F(x), & x_1'' \le x \le A. \end{cases}$$

Hence,

$$H_1(x) = \begin{cases} \dfrac{S_1(x)}{f(x_1)}, & x_1' \le x < x_1; \\ 1 + \dfrac{S_1(x)}{f(x_1)}, & x_1 \le x \le x_1''. \end{cases}$$

Define $F_2(x) = F(x) + S_2(x)$, then $F_2(x)$ is an MPS increase in risk from $F(x)$. Again, there exists a random variable $\tilde{\epsilon}_2$ such that $E(\tilde{\epsilon}_2 | x_2) = 0$, $\tilde{x}$ has a CDF $F(x)$, and $\tilde{x} + \tilde{\epsilon}_2$ has a CDF $F_2(x)$ according to the Rothschild and Stiglitz's conditions. $S_2(x) = S_1(x+\xi_2) \cdot f(x_2)/f(x_1)$ contains all the information needed to determine the random variable $\tilde{\epsilon}_2$. Since $S_2(x) \neq 0$ only when $x \in [x_2', x_2'']$, $F_2(x)$ and $F(x)$ differ on $[x_2', x_2'']$, as shown in Figure 3.2.

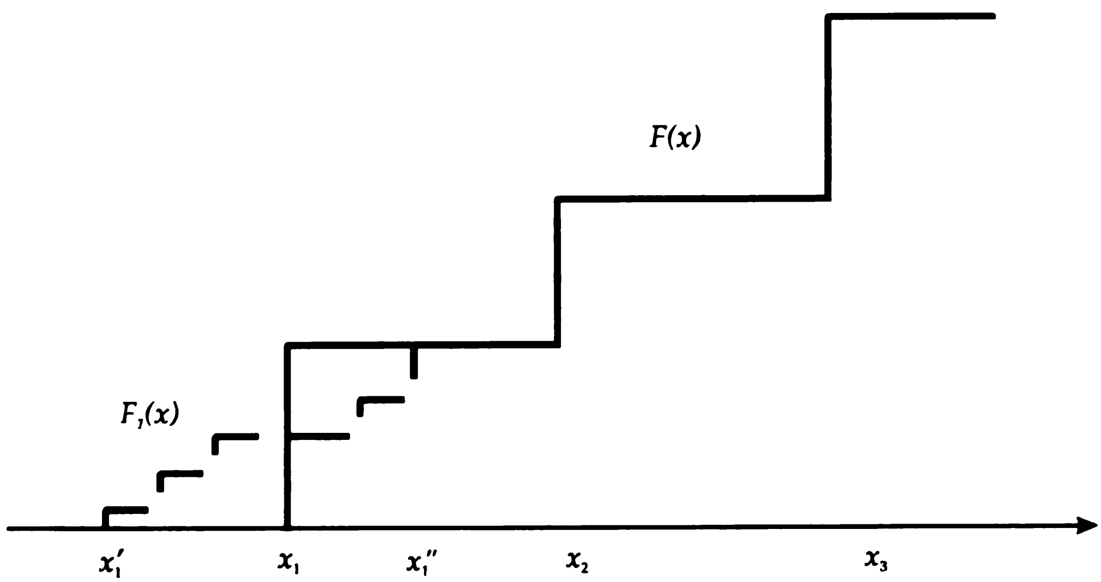

Figure 3.1. $F_1(x)$ and $F(x)$ differ on $[x_1', x_1'']$.

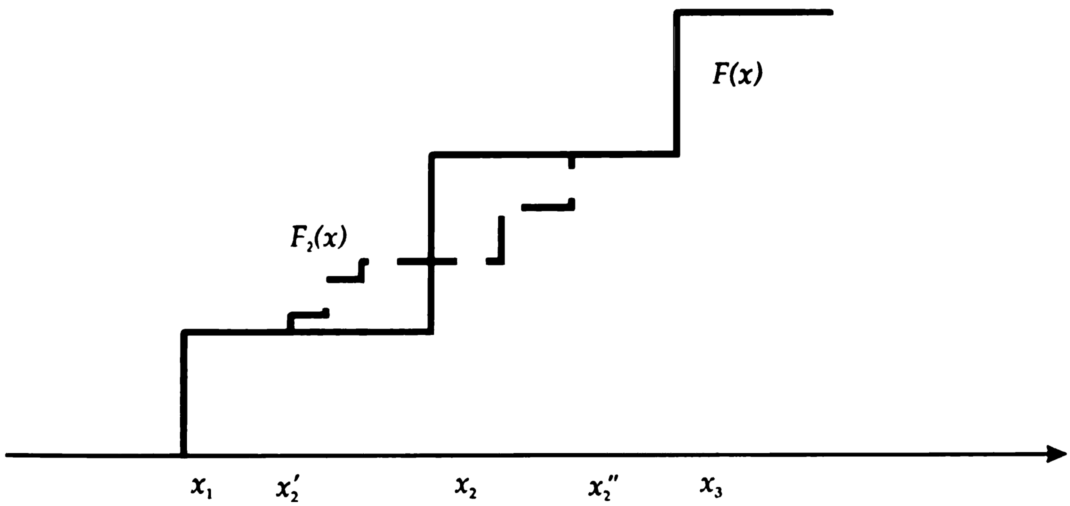

Figure 3.2. $F_2(x)$ and $F(x)$ differ on $[x_2', x_2'']$.

$$F_2(x) = \begin{cases} f(x_1), & x_1 \le x < x_2'; \\ f(x_1) + S_2(x), & x_2' \le x < x_2; \\ f(x_1) + f(x_2) + S_2(x), & x_2 \le x < x_2''; \\ F(x), & x_2'' \le x \le A. \end{cases}$$

The joint CDF of $\tilde{x}$ and $\tilde{\epsilon}_2$ is $F_2(x) = F(x) \cdot H_2(x)$, where $H_2(x)$ is the conditional CDF of $\tilde{\epsilon}_2$ given x_2.

$$F_2(x) = \begin{cases} f(x_1), & x_1 \le x < x_2'; \\ f(x_1) + f(x_2) H_2(x), & x_2' \le x < x_2''; \\ F(x), & x_2'' \le x \le A. \end{cases}$$

Hence,

$$H_2(x) = \begin{cases} \dfrac{S_2(x)}{f(x_2)} = \dfrac{S_1(x+\xi_2)}{f(x_1)}, & x_2' \le x < x_2; \\ 1 + \dfrac{S_2(x)}{f(x_2)} = 1 + \dfrac{S_1(x+\xi_2)}{f(x_1)}, & x_2 \le x \le x_2''. \end{cases}$$

Thus we have $H_2(x) = H_1(x+\xi_2)$, for $x \in [x_2', x_2'']$ and for $x + \xi_2 \in [x_1', x_1'']$; that is, the two conditional CDF's H_2 and H_1 are the same. These random variables therefore have the same conditional distributions $\tilde{\epsilon}_1 | x_1 =^d \tilde{\epsilon}_2 | x_2$.

By repeating the same process, we then have $H_i(x) = H_1(x+\xi_i)$, for all i, which implies that $\tilde{\epsilon}_i$ and $\tilde{\epsilon}_1$ have the same conditional distributions $\tilde{\epsilon}_1 | x_1 =^d \tilde{\epsilon}_i | x_i$, for all i. Let $\tilde{\epsilon}$ be

$$\tilde{\epsilon} = \tilde{\epsilon}_1 | x_1 =^d \tilde{\epsilon}_2 | x_2 =^d \cdots =^d \tilde{\epsilon}_i | x_i,$$

then $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$.

Define $G(x) = F(x) + \sum_{i=1}^{n} S_i(x)$, then $G(x)$ differs from $F(x)$ by a sequence of MPS's which differ by linear shifts. From $S_i(x)$, $i = 1, \cdots, n$, we could determine the random variable $\tilde{\epsilon}$ such that $\tilde{y} =^d \tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$.

II implies I. As to the proof of the inverse part, we note the previous argument that leads to the IMPS conditions in the first part of this section is actually a proof of this. Q.E.D.

Example 3.1: Consider a random variable $\tilde{x}$

$$\left\{-4, \frac{1}{4};\ -2, \frac{1}{4};\ 2, \frac{1}{4};\ 4, \frac{1}{4}\right\},$$

and let $\tilde{\epsilon}$ be an independent random variable

$$\left\{-2, \frac{1}{2};\ 2, \frac{1}{2}\right\}.$$

Then $\tilde{x} + \tilde{\epsilon}$ is

$$\left\{-6, \frac{1}{8};\ -4, \frac{1}{8};\ -2, \frac{1}{8};\ 0, \frac{1}{4};\ 2, \frac{1}{8};\ 4, \frac{1}{8};\ 6, \frac{1}{8}\right\},$$

which is an IMPS increase in risk from $\tilde{x}$. Note that the assumption $x_i^{*} < x_{i+1}^{'}$ is not imposed in this example. In this example, $G(x) - F(x)$ changes signs from positive to negative twice and there are four MPS's as shown in Figure 3.3. The

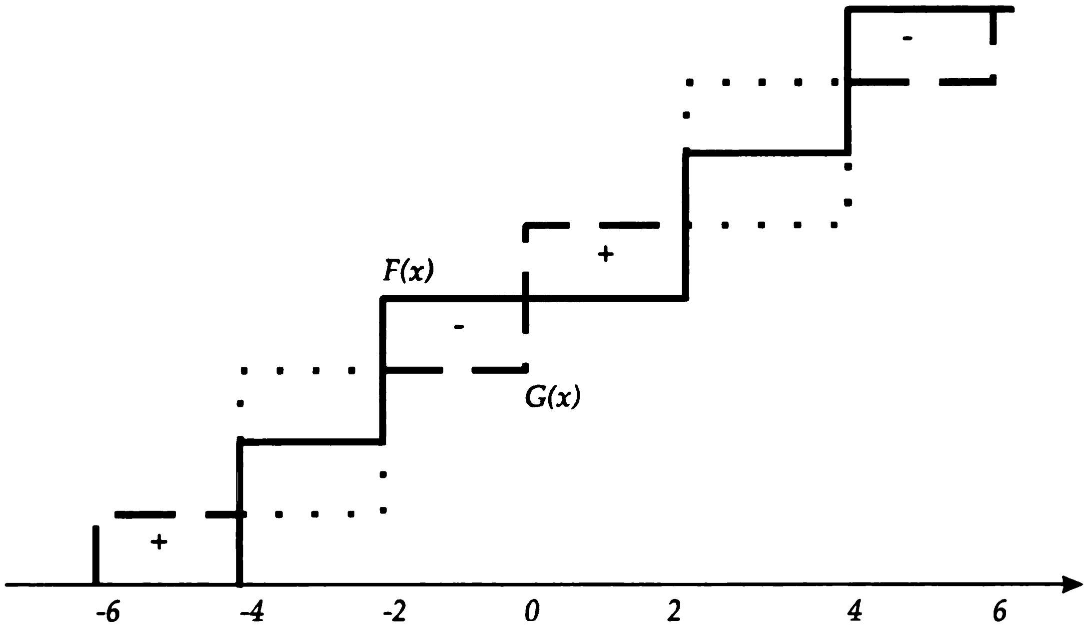

Figure 3.3. *G(x)* differs from *F(x)* by four MPS's and *G(x)* is an IMPS increase in risk from *F(x)*.

four MPS's are over the subintervals $[-6,-2]$, $[-4,0]$, $[0,4]$ and $[2,6]$, respectively. All four MPS's are linear shifts of each other. Actually the four MPS's are identical in this example since $\lambda_i = 1$ for all i.

3.2 Recovering the Independent Random Variable

Theorem 3.1 establishes the equivalence between an IMPS and an independent increase in risk for a discrete random variable. It does not, however, show how to find the independent random variable $\tilde{\epsilon}$ given the two random variables $\tilde{x}$ and $\tilde{y}$. Constructing such an independent random variable without imposing the condition $x_i^{\cdot} < x_{i+1}^{'}$ is the subject of this section.

Suppose that we have two random variables $\tilde{y}$ and $\tilde{x}$ with the CDF's $G(y)$ and $F(x)$, respectively, as shown in Figure 3.4. At each x_i, $F(x)$ increases by $p_{xi} = f(x_i)$, for $i = 1, \cdots, n$, where x_i are ordered as $x_i < x_{i+1}$. $G(y)$ has steps at y_j with height $p_{yj} = g(y_j)$, for $j = 1, \cdots, m$, where y_j are ordered as $y_j < y_{j+1}$.

Suppose $\tilde{y}$ is an IMPS increase in risk from $\tilde{x}$. We now demonstrate how to recover the independent random variable $\tilde{\epsilon}$ from the two density functions $f(x)$ and $g(y)$. It is easy to see that the support of $\tilde{\epsilon}$ is $[y_1-x_1, y_m-x_n]$ and the mass of the random variable $\tilde{\epsilon}$ at $y_1 - x_1$ is necessarily $g(y_1)/f(x_1)$ by the independence assumption. Therefore, the first mass point of the random variable $\tilde{\epsilon}$ is $\{y_1-x_1, g(y_1)/f(x_1)\}$.

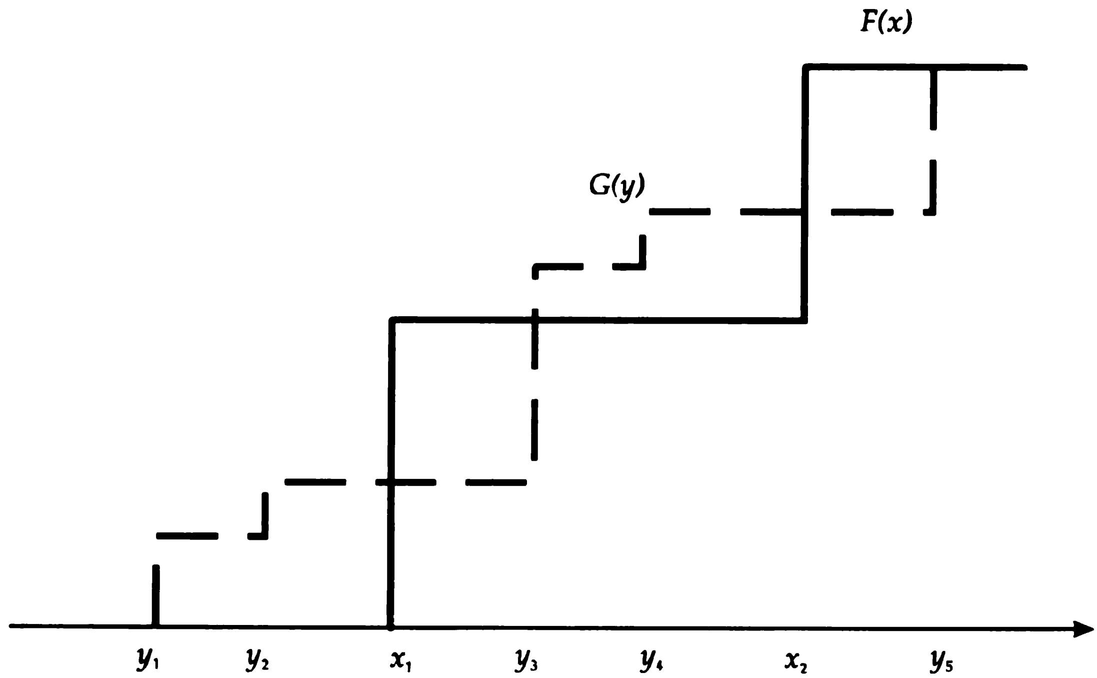

Figure 3.4. How to find the CDF of independent random variable $\tilde{\epsilon}$ from two CDF's.

By subtracting $f(x_i)g(y_1)/f(x_1)$ from $g(y)$ at $(x_i + y_1 - x_1)$ for every i to get rid of y_1, we are left with a new function which can be denoted as g^*. This new function g^* has at most $(m - 1)$ mass points at, say, $y_2^* < \cdots < y_m^*$.

We will repeat the same procedure to $g^*(y^*)$ as we did to $g(y)$ to get the second mass point of $\tilde{\epsilon}$. Since $g^*(y^*)$ at y_2^* is derived from $f(x)$ at x_1, the second mass point of $\tilde{\epsilon}$ is therefore $\{y_2^*-x_1,\ g^*(y_2^*)/f(x_1)\}$.

By subtracting $f(x_i)g^*(y_2^*)/f(x_1)$ from $g^*(y^*)$ at $(x_i + y_2^* - x_1)$, we then have another new function which will be denoted as g^{**}. It has at least one point fewer than g^* does. Again we can get the third point of $\tilde{\epsilon}$ which is $\{y_3^{**}-x_1,\ g^{**}(y_3^{**})/f(x_1)\}$.

Continuing this process until all the mass points of $g(y)$ are deleted, at which we then recover all the mass points of the random variable $\tilde{\epsilon}$. Note that if any of the above steps fails, we can infer that $\tilde{y}$ is not an IMPS increase in risk from $\tilde{x}$.

Example 3.2: Assume a random variable $\tilde{x}$ has the CDF $F(x)$ and the density function $f(x)$ as in Figure 3.5. So $\tilde{x}$ is

$$\left\{0, \frac{1}{3};\ 1, \frac{2}{3}\right\}.$$

If a random variable $\tilde{y}$ has a CDF $G(y)$ and a density function

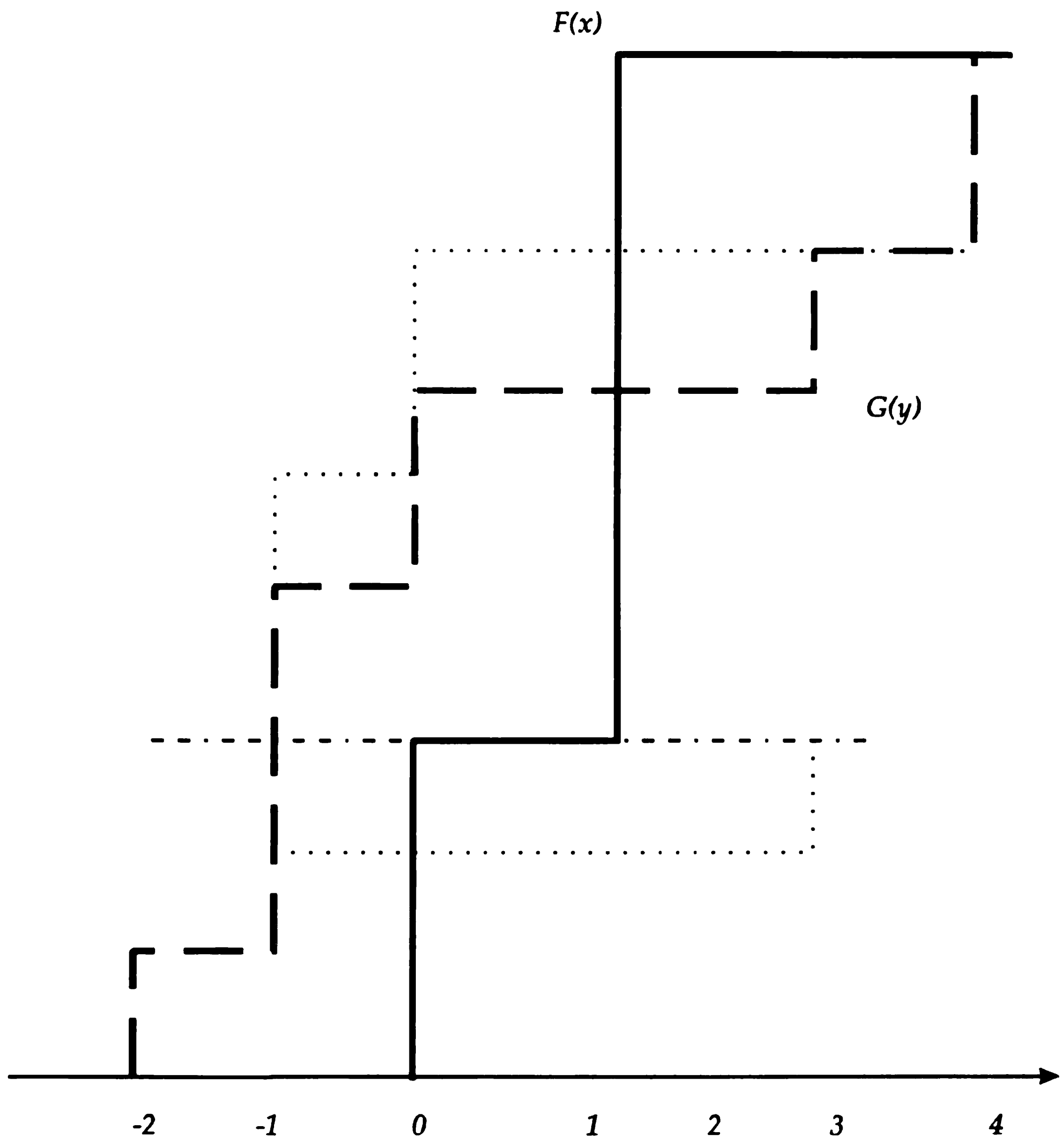

Figure 3.5. *G(y)* is an IMPS increase in risk from *F(x)*, *G(y)* can be broken into two parts in this example, the dotted line.

$g(y)$, and is defined by

$$\left\{-2, \frac{1}{9};\ -1, \frac{3}{9};\ 0, \frac{2}{9};\ 3, \frac{1}{9};\ 4, \frac{2}{9}\right\}.$$

Then it can be verified that $G(y)$ is an MPS increase in risk from $F(x)$. By assuming $G(y)$ is an IMPS increase in risk from $F(x)$, we should be able to find a random variable $\tilde{\epsilon}$ such that $\tilde{\epsilon}$ is independent of $\tilde{x}$, $E(\tilde{\epsilon}) = 0$, and $\tilde{y} =^d \tilde{x} + \tilde{\epsilon}$.

Since the mass of $\tilde{y}$ at -2 is derived from $\tilde{x}$ at 0, the probability mass of the independent random variable $\tilde{\epsilon}$ is $(1/9)/(1/3) = 1/3$ and the first point of $\tilde{\epsilon}$ is $\{-2, 1/3\}$. By subtracting $f(x_i)/3$ at $x_i - 2$ from $g(y)$, that is, subtracting $1/9$ at -2 and $2/9$ at -1 from $g(y)$, we get $g^*(y^*)$

$$\left\{-1, \frac{1}{9};\ 0, \frac{2}{9};\ 3, \frac{1}{9};\ 4, \frac{2}{9}\right\}.$$

$g^*(y^*)$ at -1 is derived from $f(x)$ at 0, So the second point of $\tilde{\epsilon}$ is $\{-1, 1/3\}$. By subtracting $f(x_i)/3$ at $x_i - 1$ from $g^*(y^*)$, that is, subtracting $1/9$ at -1 and $2/9$ at 0 from $g^*(y^*)$, we have $g^{**}(y^{**})$

$$\left\{3, \frac{1}{9};\ 4, \frac{2}{9}\right\}.$$

$g^{**}(y^{**})$ at 3 is derived from $f(x)$ at 0 and the third point of $\tilde{\epsilon}$ is $\{3, 1/3\}$. By subtracting $f(x_i)/3$ at $x_i + 3$ from $g^{**}(y^{**})$, that is, subtracting $1/9$ at 3 and $2/9$ at 4 from $g^{**}(y^{**})$, we then have $g^{***} = \{0\}$. Consequently we have recovered the random variable $\tilde{\epsilon}$

$$\left\{ -2, \frac{1}{3}; \quad -1, \frac{1}{3}; \quad 3, \frac{1}{3} \right\},$$

which is mean preserving, and $E(\tilde{\epsilon}) = 0$. We have shown that $G(y)$ is indeed an independent increase in risk from $F(x)$.

APPENDIX

Appendix

In this appendix we relax the non-overlapping condition $x_i^{'} < x_{i+1}^{'}$ and show that the proof of Theorem 3.1 still holds with some minor modifications.

Suppose $s_i(x)$ is a discrete function at points $\{x_i+\epsilon_1, \cdots, x_i+\epsilon_m\}$ and $s_{i+1}(x)$ a discrete function at points $\{x_{i+1}+\epsilon_1, \cdots, x_{i+1}+\epsilon_m\}$. Let us assume $x_i + \epsilon_j = x_{i+1} + \epsilon_k$, or $x_{ij} = x_{(i+1)k}$, for some positive integers j and k, then $s(x)$ at this point is equal to $s(x_{ij}) + s(x_{(i+1)k})$. Now, let us assume $s(x_{(i+1)k}) = 0$ when we define $s_i(x)$, and $s(x_{ij}) = 0$ when we define $s_{i+1}(x)$. With such redefinitions, both $s_i(x)$ and $s_{i+1}(x)$ still have the same properties as before. Therefore, $S_i(x)$ and $T_i(x)$ also have the same properties as before.

With the non-overlapping restriction, if $x \in [x_k^{'}, x_k^{''}]$ for some positive integer k, then

$$s(x) = \sum_{i=1}^{n} s_i(x) = s_k(x).$$

That is, if x is in subinterval $[x_k^{'}, x_k^{''}]$ of $[a,A]$, then $s_i(x) = 0$ for $i \neq k$.

Without the non-overlapping restriction, for some positive integers k and j, x may be in $j > 1$ subintervals, when $x \in [x_k^{'}, x_k^{''}]$, $\cdots$, $x \in [x_{k+j}^{'}, x_{k+j}^{''}]$, then

$$s(x) = \sum_{i=1}^{n} s_i(x) = s_k(x) + \cdots + s_{k+j}(x).$$

That is, these j subintervals overlap.

$S(x)$ and $T(x)$ also have the same interpretations as $s(x)$ does without the non-overlapping condition. That is, for some positive integers k and j, when $x \in [x_k', x_k'']$, $\cdots$, $x \in [x_{k+j}', x_{k+j}'']$, because $S_i(x_{im}) = 0$, we have

$$S(x) = \sum_{i=1}^{n} S_i(x) = S_k(x) + \cdots + S_{k+j}(x).$$

For $T(x)$, we have, for some positive integers k and j, when $x \in [x_k', x_k'']$, $\cdots$, $x \in [x_{k+j}', x_{k+j}'']$, because $T_i(x_{im}) = 0$.

$$T(x) = \sum_{i=1}^{n} T_i(x) = T_k(x) + \cdots + T_{k+j}(x).$$

Theorem 3.1 remains true except that $T(x) = \sum_{i=1}^{n} T_i(x)$ has a different interpretation.

CHAPTER 4
COMPARATIVE STATIC RESULTS

In chapter 3, we introduced independent increases in risk and IMPS, now we will study the comparative statics of the independent increases in risk. An independent stochastic transformation can also be used to generate first order stochastic dominance (FSD) and second order stochastic dominance (SSD) changes in randomness, we will consider the comparative statics for these changes in randomness as well.

We first provide a comparative static result for an independent increase in risk in section 1, and then proceed to consider its applications in section 2. The comparative statics is presented in a one random variable, one choice parameter and one argument (1-1-1) model. The 1-1-1 model can be written as $Eu[z(x,\alpha)]$, where $\tilde{x}$ is the random variable, α is the choice parameter and z is the argument. An economic agent chooses α to maximize the expected utility $Eu[z(x,\alpha)]$.

4.1 Comparative Statics

This section concentrates on the comparative statics of 1-1-1 models. The increases in risk (changes in randomness) are accomplished by the independent transformation. A random variable $\tilde{x}$ is transformed into $(\tilde{x} + \tilde{\epsilon})$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$. When $\tilde{\epsilon}$ has a non-positive support, $\tilde{x}$ FSD $\tilde{x} + \tilde{\epsilon}$, and $\tilde{x}$ SSD $\tilde{x} + \tilde{\epsilon}$ when $E(\tilde{\epsilon}) \leq 0$.

Before our main theorem, we prove the following corollary.

Corollary 4.1: Assume:

(a). Random variable $\tilde{\epsilon}$ has a support $[a,A]$ and a density function $h(\epsilon)$ and CDF $H(\epsilon)$;

(b). $w(\epsilon)$ and $v(\epsilon)$ are two continuous and differentiable functions of ϵ, $w'(\epsilon) \leq 0$ and $v'(\epsilon) \geq 0$;

Then $Ew(\epsilon)\cdot v(\epsilon) \leq Ew(\epsilon)\cdot Ev(\epsilon)$.

Proof: Let $W(\epsilon) = w(\epsilon) - Ew(\epsilon)$ and $V(\epsilon) = v(\epsilon) - Ev(\epsilon)$, we then need to prove $E\, W(\epsilon)\cdot V(\epsilon) \leq 0$. Let $dK(\epsilon) = V(\epsilon)\cdot dH(\epsilon)$, we have $K(\epsilon) = \int_a^\epsilon V(t)\cdot dH(t) + K(a)$ and

$$\int_a^A W(\epsilon)\cdot V(\epsilon)\cdot dH(\epsilon)$$
$$= \int_a^A W(\epsilon)\cdot dK(\epsilon)$$
$$= W(\epsilon)\cdot K(\epsilon)\,\Big|_a^A - \int_a^A K(\epsilon)\cdot W'(\epsilon)\cdot d\epsilon$$
$$= W(\epsilon)\cdot\Big[\int_a^\epsilon V(t)\cdot dH(t)+K(a)\Big]\ \Big|_a^A$$
$$- \int_a^A W'(\epsilon)\cdot\Big[\int_a^\epsilon V(t)\cdot dH(t)+K(a)\Big]\cdot d\epsilon$$
$$= - \int_a^A W'(\epsilon)\cdot\Big[\int_a^\epsilon V(t)\cdot dH(t)\Big]\cdot d\epsilon \leq 0.$$

The last inequality is because $W'(\epsilon) = w'(\epsilon) \leq 0$, and since $\int_a^A V(t)\cdot dH(t) = 0$ and $V'(t) = v'(t) \geq 0$ then $\int_a^\epsilon V(t)\cdot dH(t) \leq 0$. This completes the proof. Q.E.D.

We now prove our main theorem of the chapter concerning the comparative static effect of an independent increase in risk.

Theorem 4.1: Assume:

(a). Utility function u satisfies $u' \geq 0$ and $u'' \leq 0$, $A'(u) \leq 0$ and $P'(u) \leq 0$, where $A(u) = -u''/u'$, $P(u) = -u'''/u''$;

(b). $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, $z_{\alpha xx} \leq 0$ and $A'(z) \leq 0$, where $A(z) = -z_{xx}/z_x$;

(c). $\tilde{x}$ is replaced by $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$;

An economic agent maximizing $Eu[z(\tilde{x},\alpha)]$ will decrease optimal choice parameter α.

Proof: The FOC after the independent increase in risk is

$$E_x E_\epsilon\, u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha).$$

Since $u'[z(x+\epsilon,\alpha)]$ is a decreasing function of $\tilde{\epsilon}$ and $z_\alpha(x+\epsilon,\alpha)$ is an increasing function of $\tilde{\epsilon}$, the following inequality follows

$$E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$$
$$\leq E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot E_\epsilon z_\alpha(x+\epsilon,\alpha)$$
$$\leq E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x,\alpha).$$

The last inequality is because $z_\alpha(x+\epsilon,\alpha)$ is concave in $\tilde{\epsilon}$.

Let $v(x+\epsilon) = -u'[z(x+\epsilon,\alpha)]$, where α is the constant that maximizes the expected utility before the independent i

$\tilde{x}$

increase in risk. Since

$$v'(x+\epsilon) = -u''[z(x+\epsilon,\alpha)]\cdot z_x(x+\epsilon,\alpha) \geq 0,$$

$$v''(x+\epsilon) = -u'''[z(x+\epsilon,\alpha)]\cdot z_x^2(x+\epsilon,\alpha)$$

$$- u''[z(x+\epsilon,\alpha)]\cdot z_{xx}(x+\epsilon,\alpha) \leq 0,$$

therefore $v(x+\epsilon)$ is an increasing and concave function of $\tilde{\epsilon}$. Define risk premium $\psi(x,\epsilon)$ for risk $\tilde{\epsilon}$ under function $v(x+\epsilon)$ as $E_\epsilon v(x+\epsilon) = v[x-\psi(x,\epsilon)]$, $\psi(x,\epsilon)$ is then positive.

Let $A(v) = -v''(x+\epsilon)/v'(x+\epsilon)$, then

$$A(v) = -u'''[z(x+\epsilon,\alpha)]\cdot z_x(x+\epsilon,\alpha)/u''[z(x+\epsilon,\alpha)]$$

$$- z_{xx}(x+\epsilon,\alpha)/z_x(x+\epsilon,\alpha)$$

$$= P(u)\cdot z_x(x+\epsilon,\alpha) + A(z) \geq 0,$$

$$A'(v)\cdot v'(x+\epsilon) = P'(u)\cdot u'[z(x+\epsilon,\alpha)]\cdot z_x^2(x+\epsilon,\alpha)$$

$$+ P(u)\cdot z_{xx}(x+\epsilon,\alpha) + A'(z)\cdot z_x(x+\epsilon,\alpha) \leq 0.$$

Hence we conclude $A'(v) \leq 0$ and therefore $\psi_x(x,\epsilon) \leq 0$ by Pratt (1964).

Now the FOC after the independent increase in risk can be written as

$$E_xE_\epsilon\ u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$$

$$\leq E_xE_\epsilon\ u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x,\alpha)$$

$$= -E_xE_\epsilon\ v(x+\epsilon)\cdot z_\alpha(x,\alpha)$$

$$= -E_x\ v(x-\psi)\cdot z_\alpha(x,\alpha)$$

$$= E_x\ u'[z(x-\psi,\alpha)]\cdot z_\alpha(x,\alpha)$$

$$= E_x\ \{u'[z(x-\psi,\alpha)]/u'[z(x,\alpha)]\}\cdot u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha)$$

$$\leq 0,$$

if $D = u'[z(x-\psi,\alpha)]/u'[z(x,\alpha)]$ is positive and decreasing in $\tilde{x}$. $E_x u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha) = 0$ is the FOC before the

independent increase in risk.

Next we prove that D is positive and decreasing in $\tilde{x}$. D is positive obviously. The derivative of D with respect to $\tilde{x}$ has the same sign as the numerator of the derivative, which is

$$u''[z(x-\psi,\alpha)]\cdot z_x(x-\psi,\alpha)\cdot(1-\psi_x)\cdot u'[z(x,\alpha)]$$
$$- u'[z(x-\psi,\alpha)]\cdot u''[z(x,\alpha)]\cdot z_x(x,\alpha)$$
$$\leq u''[z(x-\psi,\alpha)]\cdot z_x(x,\alpha)\cdot u'[z(x,\alpha)]$$
$$- u'[z(x-\psi,\alpha)]\cdot u''[z(x,\alpha)]\cdot z_x(x,\alpha)$$
$$= u'[z(x-\psi,\alpha)]\cdot z_x(x,\alpha)\cdot u'[z(x,\alpha)]\cdot$$
$$\{u''[z(x-\psi,\alpha)]/u'[z(x-\psi,\alpha)] - u''[z(x,\alpha)]/u'[z(x,\alpha)]\}$$
$$= u'[z(x-\psi,\alpha)]\cdot z_x(x,\alpha)\cdot u'[z(x,\alpha)]\cdot\{A(u[z(x,\alpha)])$$
$$- A(u[z(x-\psi,\alpha)])\}$$
$$\leq 0, \qquad \text{if } A'(u) \leq 0.$$

$\psi_x(x,\epsilon) \leq 0$ if $A'(v) \leq 0$. $P'(u) \leq 0$ and $A'(z) \leq 0$ are sufficient conditions for $A'(v) \leq 0$.

Thus the FOC after the independent increase in risk is negative. In order to maximize the expected utility, α has to be decreased. This completes the proof. Q.E.D.

For a Rothschild and Stiglitz increase in risk, the distribution of random variable $\tilde{\epsilon}$ may be different for different values of random variable $\tilde{x}$. The distribution of $\tilde{\epsilon}$ depends on x. As the random variable $\tilde{x}$ moves across its support, it may be a different $\tilde{\epsilon}$ that is added to $\tilde{x}$. Under an independent increase in risk, the distribution of random

variable $\tilde{\epsilon}$ is the same no matter what realized value random variable $\tilde{x}$ takes. The distribution of $\tilde{\epsilon}$ is independent of x. The increase in risk is the same for different value of random variable $\tilde{x}$. This uniform property makes the comparative statics for an independent increase in risk attractive.

Like a strong increase in risk, an independent increase in risk is also a generalization of an introduction of risk. An independent increase in risk, however, generates a general Rothschild and Stiglitz increase in risk, when the initial random variable is degenerate at a point.

An independent increase in risk imposes no restrictions on the two distribution functions in the center of the supports. The two CDF's may cross many times. The support of $F(x)$ is however contained in the support of $G(y)$ for an independent increase in risk. A strong increase in risk, a relatively strong increase in risk and a relatively weak increase in risk all require that $F(x) - G(x)$ is non-decreasing in the center of the supports, and is non-increasing at the two ends of the supports.

$P(u) = -u'''/u''$ in our proof is termed absolute prudence by Kimball (1990) who first introduces it in a precautionary saving problem. Kimball defines precautionary premium $\phi(w,x)$ as the quantity satisfying

$$u'[w-\phi(w,x)] = E_{\epsilon}u'(w+x),$$

where w is the final wealth and $\tilde{x}$ is the random variable. It can be showed that $\phi(w,x)$ is approximately equal to $-(1/2)\cdot\sigma^2\cdot u'''/u''$, where σ^2 is the variance of random variable $\tilde{x}$ and $P(u) = -u'''/u''$ is the absolute prudence.

There are some basic assumptions about the absolute prudence. Kimball argues that the absolute prudence is the propensity to prepare oneself in the face of uncertainty. The absolute prudence is assumed to be a decreasing function of the initial wealth and it is greater than the absolute risk aversion measure if the utility function is decreasing absolute risk averse (DARA). A positive absolute prudence is a necessary condition for DARA, $A'(u) = A(u)\cdot[A(u) - P(u)]$. Note that in our proof $\psi(x,\epsilon)$ is the risk premium under function $v(x+\epsilon)$ and is the precautionary premium under utility function $u[z(x+\epsilon,\alpha)]$.

Eeckhoudt, Gollier and Schlesinger (1992) use absolute prudence in their background risk study. An economic agent has a utility function $u(w)$, where $w = \tilde{y} + \alpha\cdot\tilde{x}$ is the final wealth. Random variable $\tilde{y}$ is an exogenous and unavoidable background risk whose CDF is initially $G_1(y)$. There is a second source of uncertainty due to the existence of an independent and endogenous risk $\tilde{x}$ with CDF $F(x)$. α is the choice parameter. Eeckhoudt, Gollier and Schlesinger consider the impact on optimal choice parameter α when an agent faces a change in the distribution of unavoidable risk

$\tilde{y}$ from CDF $G_1(y)$ to $G_2(y)$.

The method Eeckhoudt, Gollier and Schlesinger use is independent transformation. When initial random variable $\tilde{y}$ first order stochastically dominates (FSD) $\tilde{y} + \tilde{\epsilon}$, where $\tilde{y}$ has CDF $G_1(y)$ and $\tilde{y} + \tilde{\epsilon}$ has CDF $G_2(y)$ and $\tilde{\epsilon}$ is the independent noise, the optimal choice parameter decreases if the utility function is DARA. When random variable $\tilde{y}$ second order stochastically dominates (SSD) $\tilde{y} + \tilde{\epsilon}$, the optimal choice parameter decreases if the utility function is standard risk aversion, Kimball (1993). A utility function is standard risk aversion if it is DARA $(A'(u) \leq 0)$ and decreasing absolute prudence (DAP) $(P'(u) \leq 0)$.

If independent random variable $\tilde{\epsilon}$ has a non-positive support, random variable $\tilde{x}$ FSD random variable $(\tilde{x} + \tilde{\epsilon})$. We have the following theorem regarding this FSD change in randomness.

Theorem 4.2: Assume:

(a). Utility function u satisfies $u' \geq 0$, $u'' \leq 0$ and $A'(u) \leq 0$;

(b). $z_x \geq 0$, $z_{xx} \leq 0$ and $z_{\alpha x} \geq 0$;

(c). $\tilde{x}$ is replaced by $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $\tilde{\epsilon}$ has a non-positive support;

Then an economic agent maximizing $Eu[z(x,\alpha)]$ will decrease optimal choice parameter α.

Proof: The FOC after an FSD change in randomness is

$$E_x E_\epsilon \; u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha).$$

Since $\tilde{\epsilon}$ has a non-positive support and $z_{\alpha x}(x,\alpha) \geq 0$, we have $z_\alpha(x+\epsilon,\alpha) \leq z_\alpha(x,\alpha)$, therefore

$$E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$$
$$\leq E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x,\alpha).$$

The FOC can be written as

$$E_x E_\epsilon \; u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$$
$$\leq E_x E_\epsilon \; u'[z(x+\epsilon,\alpha)]\cdot z_\alpha(x,\alpha)$$
$$= E_x\{E_\epsilon u'[z(x+\epsilon,\alpha)]/u'[z(x,\alpha)]\}\cdot u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha).$$

The expression $E_\epsilon u'[z(x+\epsilon,\alpha)]/u'[z(x,\alpha)]$ is positive and decreasing in $\tilde{x}$. It is obviously positive. Its first order derivative with respect to $\tilde{x}$ has the same sign as its numerator which is

$$u'[z(x,\alpha)]\cdot E_\epsilon u''[z(x+\epsilon,\alpha)]\cdot z_x(x+\epsilon,\alpha)$$
$$\quad - u''[z(x,\alpha)]\cdot z_x(x,\alpha)\cdot E_\epsilon u'[z(x+\epsilon,\alpha)]$$
$$\leq u'[z(x,\alpha)]\cdot E_\epsilon u''[z(x+\epsilon,\alpha)]\cdot z_x(x,\alpha)$$
$$\quad - u''[z(x,\alpha)]\cdot z_x(x,\alpha)\cdot E_\epsilon u'[z(x+\epsilon,\alpha)]$$
$$= u'[z(x,\alpha)]\cdot z_x(x,\alpha)\cdot E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot$$
$$\quad \{u''[z(x+\epsilon,\alpha)]/u'[z(x+\epsilon,\alpha)] - u''[z(x,\alpha)]/u'[z(x,\alpha)]\}$$
$$= u'[z(x,\alpha)]\cdot z_x(x,\alpha)\cdot E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot\{A(u[z(x,\alpha)])$$
$$\quad - A(u[z(x+\epsilon,\alpha)])\}$$
$$\leq u'[z(x,\alpha)]\cdot z_x(x,\alpha)\cdot E_\epsilon u'[z(x+\epsilon,\alpha)]\cdot\{A(u[z(x,\alpha)])$$
$$\quad - E_\epsilon A(u[z(x+\epsilon,\alpha)])\}$$
$$\leq 0, \qquad \text{if } A'(u) \leq 0.$$

The last inequality is because $u'[z(x+\epsilon,\alpha)]$ is decreasing in $\tilde{\epsilon}$ and $A(u[z(x,\alpha)]) - A(u[z(x+\epsilon,\alpha)])$ is increasing in $\tilde{\epsilon}$. Now we have

$$E_x\{E_\epsilon u'[z(x+\epsilon,\alpha)]/u'[z(x,\alpha)]\}\cdot u'[z(x,\alpha)]\cdot z_\alpha(x,\alpha) \leq 0.$$

That is, the FOC is negative. In order to maximize the expected utility, α has to be decreased. Q.E.D.

An FSD change in randomness is stronger than an MPS increase in risk, it therefore needs less restrictive conditions on the utility functions than an MPS increase in risk does. A similar comparative static result is also available for an SSD change in randomness. An SSD shift is a combination of an FSD and an MPS shifts, Hadar and Seo (1990). The comparative static result is then easily obtained. The comparative static result is in the following theorem.

Theorem 4.3: Assume:

(a). Utility function u satisfies $u' \geq 0$ and $u'' \leq 0$, $A'(u) \leq 0$ and $P'(u) \leq 0$, where $A(u) = -u''/u'$, $P(u) = -u'''/u''$;

(b). $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, $z_{\alpha xx} \leq 0$ and $A'(z) \leq 0$, where $A(z) = -z_{xx}/z_x$;

(c). $\tilde{x}$ is replaced by $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) \leq 0$;

An economic agent maximizing $Eu[z(x,\alpha)]$ will decrease optimal choice parameter α.

Proof: Combining the proofs of Theorems 4.1 and 4.2, we can prove this theorem. Q.E.D.

An SSD change in randomness is the least restricted change in randomness of the three and it thus needs the most restrictive conditions on the utility functions. This shows the tradeoff between the restrictions on the changes in randomness and the restrictions on the utility functions.

4.2 Examples of Applications

In this section, we will examine some applications of the independent increases in risk theorem. Rothschild and Stiglitz (1971) have discussed the savings and uncertainty, portfolio, a firm's production, the output level of a competitive firm and multi-stage planning models, we will consider these models here.

§ 4.2.1 Savings and Uncertainty

Rothschild and Stiglitz (1971) consider an economic agent who has a given wealth W_0 which he wishes to allocate between consumption today and consumption tomorrow. The savings today yields a random return e tomorrow, the expected utility is $Eu(C_1,C_2)$, where C_i is the consumption in period i, $i = 1,2$. This is a two period consumption model.

Rothschild and Stiglitz assume that the utility function is strictly increasing, strictly concave and separable,

$$Eu(C_1,C_2) = u(C_1) + (1-\delta)\cdot Eu(C_2),$$

where $C_1 = (1-s)\cdot W_0$ and $C_2 = s\cdot W_0\cdot e$, s is the savings rate, δ is the time discount rate. The first order condition (FOC) is

$$u'[(1-s)\cdot W_0] = Eu'(s\cdot W_0\cdot e)\cdot(1-\delta)\cdot e.$$

Rothschild and Stiglitz then conclude that an increase in risk in the sense of MPS will increase or decrease the optimal savings s depends on the concavity of $u'(s\cdot W_0\cdot e)\cdot(1-\delta)\cdot e$.

Sandmo (1970) studies a two period consumption model without assuming the separable utility function. The first period budget constraint facing the economic agent is $I_1 = C_1 + S_1$, where I_1 is the income in the first period, S_1 is the savings and C_1 is the consumption. The second period consumption is $C_2 = I_2 + S_1\cdot\xi$, where ξ is the rate of return, I_2 is the income in the second period. The economic agent maximizes the utility function $u[C_1,I_2+(I_1-C_1)\xi]$, where C_1 is the choice parameter.

Sandmo assumes that the second period income is a random variable. Replacing I_2 with $\tilde{x}$, he rewrites the model as $Eu[\alpha,x+(\lambda-\alpha)\xi]$, where $\alpha = C_1$ is the choice parameter, λ

and ξ are the exogenous parameters and $\tilde{x} = I_2$ is the random variable. To study the comparative static effect of the random variable, Sandmo replaces the random variable with $\gamma + \theta \cdot \tilde{x}$, where γ is the additive shift parameter, $\theta > 0$ is the multiplicative shift parameter. The comparative static result concerning an increase in parameter γ is in the following theorem.

Theorem 4.4 (Sandmo): Assume:

(a). Utility function u satisfies $u' \geq 0$, $u'' \leq 0$ and $u_{12} - \xi \cdot u_{22} > 0$;

(b). γ_0 is replaced by γ_1, and $\gamma_1 \geq \gamma_0$;

(c). $Eu[\alpha_i, (\gamma_i + \theta x) + (\lambda - \alpha_i)\xi]$ is maximized at α_i, $i = 0,1$;

Then $\alpha_1 \geq \alpha_0$ if the utility function is decreasing absolute risk aversion (DARA).

This result is a local one, because it has to be evaluated at point $\theta = 1$, $\gamma = 0$.

Dardanoni (1988) discusses the same problem. Instead of taking the consumption in the first period as the choice parameter, Dardanoni takes the savings in the first period as the choice parameter. The model becomes $Eu[\lambda - \alpha, x + \alpha\xi]$. Dardanoni considers a Rothschild and Stiglitz increase in risk for the additive shift random variable. The comparative static result is stated in the following theorem.

Theorem 4.5 (Dardanoni): Assume:

(a). Utility function u satisfies $u' \geq 0$, $u'' \leq 0$ and $u_{12} - \xi \cdot u_{22} > 0$;

(b). $\tilde{x}^0$ is replaced by $\tilde{x}^1$, where $\tilde{x}^1$ is a Rothschild and Stiglitz increase in risk from $\tilde{x}^0$;

(c). $Eu[\lambda-\alpha_i, x^i+\alpha_i\xi]$ is maximized at α_i, $i = 0,1$;

Then $\alpha_1 \geq \alpha_0$ if the absolute risk aversion, $A_2 = -u_{22}/u_2$, is non-increasing in α.

The assumption that A_2 is non-increasing in α means that: (1). A_2 decreases when the random component of the utility function increases; (2). A_2 increases when the certain component of the utility function increases. The random component is the second argument and the certain component is the first argument. This assumption was used by Sandmo (1969).

Dardanoni also considers a multiplicative shift problem. The model is $Eu[\lambda-\alpha, \xi+\alpha x]$. He limits the choice variable $\alpha \geq 0$. The comparative static result is in the following theorem.

Theorem 4.6 (Dardanoni): Assume:

(a). Utility function u satisfies $u' \geq 0$ and $u'' \leq 0$ and $u_{12} - \xi \cdot u_{22} > 0$;

(b). α is an increasing function of x under certainty;

(c). $\tilde{x}^0$ is replaced by $\tilde{x}^1$, where $\tilde{x}^1$ is a Rothschild and

Stiglitz increase in risk from $\tilde{x}^0$;

(d). $Eu[\lambda-\alpha_i, x^i\alpha_i+\xi]$ is maximized at α_i, $i = 0,1$;

Then $\alpha_1 \geq \alpha_0$, if $\xi = 0$ and the relative risk aversion is non-decreasing in α.

The relative risk aversion is defined as

$$R(u_2) = -\alpha x \cdot u_{22}(\lambda-\alpha, \alpha x)/u_2(\lambda-\alpha, \alpha x).$$

For $\xi \neq 0$, the comparative static result will still hold, after replacing the relative risk aversion $R(u_2)$ with the proportional risk aversion $P(u_2)$. The proportional risk aversion is defined as

$$P(u_2) = -\alpha x \cdot u_{22}(\lambda-\alpha, \xi+\alpha x)/u_2(\lambda-\alpha, \xi+\alpha x).$$

The savings and uncertainty model is a two argument model, and it is actually presented in a one random variable, one choice parameter and two argument format (1-1-2), Choi (1992).

Now we will study a general savings and uncertainty model, $Eu[\lambda-\alpha, z(x,\alpha)]$, where z is the second argument, α is the choice parameter. When random variable $\tilde{x}$ undergoes an independent increase in risk, the comparative static result is presented in the following theorem.

Theorem 4.7: Assume:

(a). Utility function u is increasing and concave in both arguments and $u_{12} \geq 0$, $u_{122} \leq 0$, $u_{222} \geq 0$;

(b). $z_x \geq 0$, $z_{xx} \leq 0$, $z_\alpha \geq 0$, $z_{\alpha x} \leq 0$, $z_{\alpha xx} \geq 0$;

(c). $\tilde{x}$ is replaced by $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$ and $E(\tilde{\epsilon}) = 0$;

Then an economic agent maximizing $Eu[\lambda-\alpha, z(x,\alpha)]$ will increase the optimal choice parameter α.

Proof: The first order condition (FOC) before the independent increase in risk is

$$-Eu_1[\lambda-\alpha, z(x,\alpha)] + Eu_2[\lambda-\alpha, z(x,\alpha)]\cdot z_\alpha(x,\alpha) = 0.$$

The second order condition (SOC) is

$$Eu_{11} - 2\cdot Eu_{12}\cdot z_\alpha + Eu_{22}\cdot z_\alpha^2 + Eu_2\cdot z_{\alpha\alpha} \leq 0.$$

The FOC after the independent transformation is

$$-E_xE_\epsilon u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] + E_xE_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha).$$

The first term in the above expression, $C(\epsilon) = -u_1[\lambda-\alpha, z(x+\epsilon,\alpha)]$, is convex in ϵ, because

$C' = -u_{12}\cdot z_x \leq 0$,

$C'' = -u_{122}\cdot z_x^2 - u_{12}\cdot z_{xx} \geq 0.$

Therefore

$$E_\epsilon -u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] \geq -u_1[\lambda-\alpha, z(x,\alpha)].$$

The second term, $D(\epsilon) = u_2[\lambda-\alpha, z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$, is also convex in ϵ, because

$$D' = u_{22}\cdot z_x\cdot z_\alpha + u_2\cdot z_{\alpha x} \le 0,$$

$$D'' = u_{222}\cdot z_x^2\cdot z_\alpha + u_{22}\cdot z_{xx}\cdot z_\alpha + 2\cdot u_{22}\cdot z_x\cdot z_{\alpha x} + u_2\cdot z_{\alpha xx} \ge 0.$$

So that

$$E_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha) \ge u_2[\lambda-\alpha, z(x,\alpha)]\cdot z_\alpha(x,\alpha).$$

Thus we have

$$-E_xE_\epsilon u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] + E_xE_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)]\cdot z_\alpha(x+\epsilon,\alpha)$$
$$\ge -E_x u_1[\lambda-\alpha, z(x,\alpha)] + E_x u_2[\lambda-\alpha, z(x,\alpha)]\cdot z_\alpha(x,\alpha) = 0.$$

The FOC after the independent transformation is positive under the old optimal choice parameter, to maximize the expected utility, the choice parameter thus has to be increased. Q.E.D.

The comparative static result for an FSD independent transformation is also possible. We present it in the following theorem.

Theorem 4.8: Assume:

(a). Utility function u is increasing and concave in both arguments and $u_{12} \ge 0$, $u_{122} \le 0$;

(b). $z_x \ge 0$, $z_\alpha \ge 0$, $z_{\alpha x} \le 0$;

(c). $\tilde{x}$ is replaced by $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is independent of $\tilde{x}$

and $\tilde{\epsilon}$ has a non-positive support;

Then an economic agent maximizing $Eu[\lambda-\alpha, z(x,\alpha)]$ will increase the optimal choice parameter α.

Proof: The first order condition (FOC) and the second order condition (SOC) are the same as those in theorem 4.7. The FOC after the transformation is

$$-E_x E_\epsilon u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] + E_x E_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)] \cdot z_\alpha(x+\epsilon,\alpha).$$

The first term, $C(\epsilon) = -u_1[\lambda-\alpha, z(x+\epsilon,\alpha)]$, is decreasing in $\tilde{\epsilon}$, because $C' = -u_{12} \cdot z_x \leq 0$. Thus

$$E_\epsilon -u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] \geq -u_1[\lambda-\alpha, z(x,\alpha)].$$

The second term, $D(\epsilon) = u_2[\lambda-\alpha, z(x+\epsilon,\alpha)] \cdot z_\alpha(x+\epsilon,\alpha)$, is decreasing in $\tilde{\epsilon}$, because $D' = u_{22} \cdot z_x \cdot z_\alpha + u_2 \cdot z_{\alpha x} \leq 0$. So that

$$E_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)] \cdot z_\alpha(x+\epsilon,\alpha) \geq u_2[\lambda-\alpha, z(x,\alpha)] \cdot z_\alpha(x,\alpha).$$

Therefore we have

$$-E_x E_\epsilon u_1[\lambda-\alpha, z(x+\epsilon,\alpha)] + E_x E_\epsilon u_2[\lambda-\alpha, z(x+\epsilon,\alpha)] \cdot z_\alpha(x+\epsilon,\alpha)$$
$$\geq -E_x u_1[\lambda-\alpha, z(x,\alpha)] + E_x u_2[\lambda-\alpha, z(x,\alpha)] \cdot z_\alpha(x,\alpha) = 0.$$

This proves that the FOC is positive, to maximize the expected utility, the choice parameter thus has to be increased. Q.E.D.

§ 4.2.2 Asset Proportion

Rothschild and Stiglitz (1971) consider a one safe and one risky asset portfolio model. An economic agent allocates his asset in money, which yields zero return, and a risky asset, which yields a random return of e. The final wealth is $W(\alpha) = W_0 \cdot (\alpha \cdot e + 1)$, where α is the proportion held in the risky asset. The expected utility is $Eu[W(\alpha)]$ and the FOC for maximization is

$$W_0 \cdot Eu'[W(\alpha)] \cdot e = 0.$$

And the comparative statics of an increase in risk once again depends on the concavity of $u'[W(\alpha)] \cdot e$.

Hadar and Seo (1988) discuss an asset proportion problem in a two risky asset portfolio model. There is a correlation term in two random variable models. When one random variable changes, the correlation term often changes, too, which introduces a new dimension into the models. This makes the two random variable models very difficult to study. To avoid this problem, many models assume independence between the two random variables.

When a risk averse agent diversifies between two risky assets, the proportion held in the FSD, SSD or MPS dominating asset is not necessarily greater than half. For the case of FSD, Hadar and Seo (1988) offer the following intuitive explanation. When random variable $\tilde{x}$ FSD random variable $\tilde{y}$, not only risk averse agents prefer $\tilde{x}$ to $\tilde{y}$, but also the risk loving agents prefer $\tilde{x}$ to $\tilde{y}$. That is, $\tilde{x}$ has some characteristics which make it attractive to both risk loving agents and risk averse agents. Thus risk averse agents may invest more in $\tilde{y}$.

The economic agent is assumed to maximize the expected utility function $Eu(z)$, where $z = \alpha\tilde{x} + (1-\alpha)\tilde{y}$ is the final wealth, $\tilde{x}$ and $\tilde{y}$ are the two random variables representing the returns of the two risky assets, α is the proportion held in asset $\tilde{x}$. The two random variables are assumed to be independent and have CDF's $F(x)$ and $G(y)$ respectively, the joint CDF then is $H(x,y) = F(x)\cdot G(y)$.

Hadar and Seo has the following regarding FSD and Mean Preserving Contraction (MPC).

Theorem 4.9 (Hadar and Seo): Assume:

(a). Utility function u satisfies $u' > 0$, $u'' \leq 0$ and $u''' \geq 0$ (for MPC only);

(b). $\tilde{x}$ and $\tilde{y}$ are stochastically independent;

(c). $Eu[\alpha x+(1-\alpha)y]$ is maximized at α_0;

Then $\alpha_0 \geq 1/2$ for $\tilde{x}$ FSD $\tilde{y}$ if and only if $u'(z)\cdot z$ is non-decreasing in $z \geq 0$.

Then $\alpha_0 \geq 1/2$ for $\tilde{x}$ MPC $\tilde{y}$ if and only if $u'(z)\cdot z$ is concave in $z \geq 0$.

It is worth to mention that the payoff function z is linear in both random variables.

Assume that in a two random variable model, $Eu[z(x,y,\alpha)]$, the independent random variable is added to random variable $\tilde{x}$, and it is independent of both $\tilde{x}$ and $\tilde{y}$. First we have the following property regarding the independent transformation.

Theorem 4.10: An independent random variable $\tilde{\epsilon}$ transforms random variable $\tilde{x}$ in a two random variable model $Eu[z(x,y,\alpha)]$ and it is independent of both $\tilde{x}$ and $\tilde{y}$, then the independent transformation does not alter the covariance between random variables $\tilde{x}$ and $\tilde{y}$.

Proof: The covariance after the independent transformation is $cov(\tilde{x}+\tilde{\epsilon},\tilde{y})$, the covariance before the transformation is $cov(\tilde{x},\tilde{y})$. $cov(\tilde{x}+\tilde{\epsilon},\tilde{y}) = cov(\tilde{x},\tilde{y}) + cov(\tilde{\epsilon},\tilde{y}) = cov(\tilde{x},\tilde{y})$.
Q.E.D.

Now we are ready to consider the asset proportion problem using the independent transformation. The expected utility function is $Eu[\alpha x+(1-\alpha)y]$, where α is the choice parameter. Assume short term buying and selling is not allowed, $0 \leq \alpha \leq 1$, and $\tilde{x}$ and $\tilde{y}$ are stochastically dependent, the joint cumulative density function is $H(x,y)$. If one random variable is an independent transformation of the other one, then the proportion held in the less risky asset is greater than 1/2. We state this result in the following theorem.

Theorem 4.11: Assume:

(a). Utility function u satisfies $u' > 0$ and $u'' \leq 0$;

(b). Random variable $\tilde{y}$ is obtained from $\tilde{x}$ by adding an independent random variable $\tilde{\epsilon}$ where $E(\tilde{\epsilon}) = 0$;

(c). $Eu[\alpha x+(1-\alpha)y]$ is maximized at α_0;

Then $\alpha_0 \geq 1/2$.

Proof: It is sufficient to show that the first order condition for $\alpha \geq 1/2$ is greater than zero. The FOC is

$$E_{xy}u'[\alpha x+(1-\alpha)y]\cdot(x-y)$$

$$= E_{x\epsilon}u'[\alpha x+(1-\alpha)(x+\epsilon)]\cdot(-\epsilon)$$

$$= E_x E_\epsilon u'[(x+(1-\alpha)\epsilon)]\cdot(-\epsilon).$$

Let $\alpha \geq 1/2$, then

$$E_\epsilon u'[x+(1-\alpha)\epsilon]\cdot(-\epsilon)$$

$$\geq E_\epsilon u'[x+(1-\alpha)\epsilon]\cdot E_\epsilon(-\epsilon) = 0,$$

since $u'[x+(1-\alpha)\epsilon]$ is decreasing in ϵ and $(-\epsilon)$ is decreasing in ϵ. So that $E_{x\epsilon}u'[x+(1-\alpha)\epsilon]\cdot(-\epsilon) \geq 0$ for $\alpha \geq 1/2$.

Q.E.D.

Note that the above proof is independent of α, $FOC \geq 0$ if $\alpha = 1$. Thus we can say that the maximization occurs at $\alpha = 1$.

Similarly, we have the results for FSD changes in randomness, which is presented the following theorem.

Theorem 4.12: Assume:

(a). Utility function u satisfies $u' > 0$ and $u'' \leq 0$;

(b). Random variable $\tilde{y}$ is obtained from $\tilde{x}$ by adding an independent random variable $\tilde{\epsilon}$, where $\tilde{\epsilon}$ has a non-positive support;

(c). $Eu[\alpha x+(1-\alpha)y]$ is maximized at α_0;

Then $\alpha_0 = 1$.

Proof: It is sufficient to show the FOC is greater than zero for $\alpha = 1$. The FOC is

$$E_{xy}u'[\alpha x+(1-\alpha)y]\cdot(x-y)$$

$$= E_{x\epsilon}u'[\alpha x+(1-\alpha)(x+\epsilon)]\cdot(-\epsilon)$$

$$= E_x E_\epsilon u'[(x+(1-\alpha)\epsilon)]\cdot(-\epsilon) \geq 0.$$

The FOC is greater than 0 for $\alpha = 1$. To maximize the expected utility, α has to be set to one. Q.E.D.

§ 4.2.3 More Applications

In this part, we will consider more applications: a firm's production problem, the choice of output level for a competitive firm and a multi-stage planning problem, all of which have been studied by Rothschild and Stiglitz (1971).

A firm's production problem is this: the output level Q for next period is uncertain and the firm wishes to minimize the expected cost of producing Q. The cost consists of labor (L) and capital (K), the cost function, $C(L,K)$, is

$$C = r \cdot K + w \cdot L(K,Q),$$

where r is the cost of capital and w that of labor. Capital can not vary in the short run. $L(K,Q)$ is the labor required to produce Q given capital K, which is convex in Q if the production function is concave. Hence, an increase in risk always leads to an increase in the expected cost.

What happens to the optimum level of K when the output level Q changes to random? This is a standard expected utility function model, except that we are to minimize the expected cost instead of maximization. The FOC is

$$\frac{r}{w} = E\frac{\partial L(Q,K)}{\partial K}.$$

So that, a sufficient condition for decreasing K when faced with an increase in risk is the concavity of $\partial L/\partial k$.

Writing this model in our notations, we have

$$z(x,\alpha) = \alpha \cdot r + w \cdot L(x,\alpha),$$

where $\tilde{x}$ is the output level, α is the capital. Given the convexity of $L(x,\alpha)$, we have the convexity of $z(x,\alpha)$. By the comparative statics theorem in section 1, an independent increase in risk in the output will increase the capital needed.

A related problem is the competitive firm's output problem. Rothschild and Stiglitz (1971) assume that a firm chooses the output level for tomorrow, although the price p

of output Q is uncertain. The firm is assumed to maximize the expected utility of profit, $Eu(\pi)$, where the profit is

$$\pi = p \cdot Q - C(Q),$$

where $C(Q)$, as before, is the cost function and is assumed to be convex, p is the random price. The FOC is

$$Eu'(\pi) \cdot [p - C'(Q)] = 0.$$

Since the concavity of utility function, there is always less output under uncertainty than under certainty.

Sandmo (1971) and Ishii (1977) study the comparative statics of the same problem using linear transformation. They replace the price p with $(\gamma \cdot p + \theta)$, where $\gamma > 0$ is the multiplicative shift parameter and θ is the additive one. Changes in parameter γ generate special Rothschild and Stiglitz increases in risk and changes in parameter θ generate special FSD changes in randomness. Decreasing absolute risk aversion is a necessary and sufficient condition for increasing of output when there is an increase in parameter θ. DARA is a sufficient condition for decreasing of output when there is an increase in parameter γ.

This model, in our notations, is $z(x,\alpha) = \alpha \cdot x - C(\alpha)$. Given the convexity of $C(\alpha)$, $z(x,\alpha)$ is concave. Therefore by Theorem 4.1, an independent increase in risk will decrease the choice parameter α, that is, the output level will decrease when the firm is faced with an increase in risk.

We consider our last application next, the multi-stage planning problem of Rothschild and Stiglitz's (1971). In a simple economy, the final consumption good is produced by labor and an intermediate commodity y:

$$Q = P(L_2, y),$$

while y is produced by labor alone:

$$y = M(L_1) + e,$$

where e is the random variable associated with the production of y. The constraint on labor is $L = L_1 + L_2$. The social planner's problem is to allocate the labor between the two sectors efficiently. The FOC is

$$E(P_1 - P_2 \cdot M') = 0.$$

If e becomes riskier in the sense of Rothschild and Stiglitz increases in risk, what happens to the allocation of labor between the two sectors depends on the sign of

$$P_{122} - M' \cdot P_{222}.$$

Rewriting this model in our notations, we have

$$Eu[\lambda-\alpha, z(x,\alpha)] = EP[L-L_1, M(L_1)+e].$$

This is a two argument model as in the savings and uncertainty problem. Thus the results (theorem 4.7) we got from the savings and uncertainty problem are also applicable here.

We have considered the savings and uncertainty, asset proportion, a firm's production, output level of a competitive firm and multi-stage planing problems. It can be seen that the independent transformation is a useful tool in a wide range of applications. It is nice in generating determinate comparative statics.

REFERENCES

REFERENCES

Black, J. M. and G. Bulkley (1989): "A Ratio Criterion for Signing the Effects of an Increase in Uncertainty" *International Economic Review* 30, 119-130.

Block, M. K. and J. M. Heineke (1973): "The Allocation of Effort under Uncertainty: The Case of Risk Averse Behavior" *Journal of Political Economy* 81, 376-385.

Choi, G. Y. (1992): "Comparative Statics under Uncertainty for Decision Models with More Than One Choice Variable" *Dissertation* Michigan State University.

Dardanoni, V. (1988): "Optimal Choices under Uncertainty: The Case of Two-Argument Utility Functions" *The Economic Journal* 98, 429-450.

Diamond, P. A. and J. E. Stiglitz (1974): "Increases in Risk and in Risk Aversion" *Journal of Economic Theory* 8, 337-360

Dionne, G., L. Eeckhoudt and C. Gollier (1993): "Increases in Risk and Linear Payoffs" *International Economic Review* 34, 309-319.

Eeckhoudt, L, C. Gollier and H. Schlesinger (1992): "Changes in Background risk and Risk Taking behavior" *Working Paper*.

Feder, G. (1977): "The Impact of Uncertainty in a class of Objective Functions" *Journal of Economic Theory* 16, 504-512.

Hadar, J. and W. R. Russell (1969): "Rules for Ordering Uncertain Prospects" *American Economic Review*. 59, 25-34

Hadar, J. and T. K. Seo (1988): "Asset Proportions in Optimal Portfolios" *Review of Economic Studies* 459-468.

Hadar, J. and T. K. Seo (1990): "The Effects of Shifts in a Return Distribution on Optimal Portfolios" *International Economic Review* 31, 721-736.

Hadar, J. and T. K. Seo (1993): "Sensible Risk Aversion" *Working Paper*.

Hanoch, G, and H. Levy (1969): "The Efficiency Analysis of Choices Involving Risk" *Review of Economic Studies* 335-346.

Ishii, Y. (1977): "On the Theory of the Competitive Firm under Price Uncertainty: A Note" *American Economic Review* 67, 768-769.

Katz, E. (1983): "Relative Risk Aversion in Comparative Statics" *American Economic Review* 73, 452-453.

Kimball, M. (1990): "Precautionary Saving in the Small and in the Large" *Econometrica* 58, 53-73.

Kira, D. and W. T. Ziemba (1980): "The Demand for a Risky Asset" *Management Science* 26, 1158-1165.

Kraus, M. (1979): "A Comparative Statics Theorem for Choice under Risk" *Journal of Economic Theory* 21, 510-517.

Landsberger, M. and I. Meilijson (1990): "Demand for Risky Financial Assets: A Portfolio Analysis" *Journal of Economic Theory* 50, 204-213.

Menezes, C. F. and D. L. Hanson (1970): " On the Theory of Risk Aversion" *International Economic Review* 3, 481-487.

Meyer, J. (1987): "Two-Moment Decision Models and Expected Utility Maximization" *American Economic Review* 77, 421-430.

Meyer, J. and M. Ormiston (1983): "The Comparative Statics of Cumulative Distribution Function Changes for the Class of Risk Averse Agents" *Journal of Economic Theory* 31, 153-169.

Meyer, J. and M. Ormiston (1985): "Strong Increase in Risk and Their Comparative statics" *International Economic Review* 26, 425-437.

Meyer, J. and M. Ormiston (1989): "Deterministic Transformation of Random Variable and the Comparative Statics of Risk" *Journal of Risk and Uncertainty* 2, 179-188.

Meyer, J. and M. Ormiston (1993): "The Effect on Optimal Portfolios of Changing the Return to a Risky Asset: The Case of Dependent Risky Returns" *Working Paper*.

Meyer, J. and R. H. Rasche (1992): "Sufficient Conditions for Expected Utility to Imply Mean-Standard Deviation Rankings: Empirical Evidence Concerning the Location and Scale Condition" *The Economic Journal* 102,91-106.

Ormiston, M. B. (1992): "First and Second Degree Transformations and Comparative Statics under Uncertainty" *International Economic Review* 33, 33-44.

Pratt, John W. (1964): "Risk Aversion in the Small and in the

Large" *Econometrics* 32, 122-136.

Rothschild, M. and J. Stiglitz (1970): "Increasing Risk I: A Definition" *Journal of Economic Theory* 2, 225-243.

Rothschild, M. and J. Stiglitz (1971): "Increasing Risk II: Its Economic Consequences" *Journal of Economic Theory* 3, 66-84.

Sandmo, A. (1971): "On the Theory of the Competitive Firm Under Price Uncertainty" *American Economic Review* 61, 65-73.

Sandmo, A. (1970): "The Effect of Uncertainty on Savings Decisions" *Review of Economic Studies* 37, 353-360.