ROBUST TRACKING CONTROL FOR NONLINEAR SYSTEMS USING OUTPUT FEEDBACK

By

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ABSTRACT

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In this work we use output feedback to study the regional as well as the semi-global tracking and disturbance rejection problems. The class of systems that we are dealing with is that of uncertain minimum phase single-input, single-output nonlinear systems that are transformable into the normal form, uniformly in a set of disturbances and uncertain parameters that belong to a known compact set. The uncertainties encompass both parameter uncertainty as well as modeling errors.

We first address the regulation problem where parameters and/or disturbances are constant. We show that with the addition of an integrator driven by the tracking error we create an equilibrium point at which the tracking error vanishes for all admissible uncertainties. We then design a partial state feedback controller that relies on feedback of some of the states to stabilize this unknown equilibrium point.

Next, we tackle the more general tracking problem where disturbances and references are, in general, time-varying and generated by a linear exosystem. In this approach

we surpass the issue of partial state feedback by extending the system with the introduction of m integrators at the input. With this we change an n-th order system to an $(n+m)$-th order system with m states available for measurement. Instead of giving a specific controller we present conditions that will characterize a class of state feedback controllers.

To recover the asymptotic properties achieved under state feedback in both cases, we saturate the state feedback control over a compact set of interest then implement it as an observer-based control using a linear high gain observer. We provide estimates of the region of attraction that are not shrinking. On the contrary, they are limited only by the region of validity of our model. If this region encompasses the whole state space, then the estimates can be chosen arbitrarily large and our semi-global result follows directly. Finally we test the two design methodologies through simulations on some examples, both physical and contrived. The results obtained from those simulations are in good agreement with the predicted behavior of the system.

To The Memory of My Father: **Abdelfattah H. M. Sabi**

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CHAPTER 1

Introduction

Most engineering systems encountered in practice exhibit significant nonlinear behavior. For systems exhibiting nonlinearities, the normal design procedure in the past has employed a linearized approximation of the model followed by the application of linear control methodology. However, this procedure can yield unsatisfactory performance, especially when the system is highly nonlinear and operates over wide nonlinear regimes as it is the case in aircraft control and many chemical processes. During the past fifteen years, motivated by progress in the nonlinear differential geometry, techniques have been developed to solve the feedback linearization problem. Feedback linearization utilizes state feedback, after a possible change of coordinates, to transform a given nonlinear system into a linear and controllable one. Then already developed linear control tools are available for design. This application is characterized in the nonlinear literature as exact linearization.

This idea is probably an old one. An illustration is the so called *computed torque* method in robotics. Consider a single-link manipulator model [1]

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\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a\sin(x_1) + u
\end{aligned}
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The control u can be chosen to cancel the nonlinear function, i.e., $u = a\sin(x_1) + v$. Then we can use the linear control $v = k_1x_1 + k_2x_2$ where k_1, k_2 are chosen to stabilize the resulting matrix.

Exact linearization techniques can be classified as follows: input-output linearization where the objective is to get a linear response between the input and the output; input-state linearization where the objective is to get a linear response between the input and the state of the system; and finally full linearization, where the desire is to achieve an input-state linearization as well as a linear mapping between the states and the output. The class of *input-output* feedback linearizable systems is more general and our emphasis here is on this class of nonlinear systems.

Exact linearization techniques have been applied to solve many practical engineering problems. Without quoting numerous references, we only list a representative sample of the work that has been done using these techniques. In [2] the tools were used to design an automatic flight controller for an aircraft; in [3] they controlled a brushless DC motor for direct-drive robotic application; in [4] they controlled the position of PM stepper motor, and in power systems, [5] employed feedback linearization to enhance transient stability and achieve voltage regulation. Adaptive control schemes were also developed for this class of systems; see for example [6, 7, 8, 9].

An attractive structural property of linearizable systems is the fact that it can be transformed into a form (normal form) where the the nonlinearity satisfies the matching condition, and this permits cancellation of the nonlinearity using state feedback. As in any cancellation scheme, exact mathematical cancellation can not be achieved due to parameter variations as well as hardware limitations. Hence the issue of robustness, i.e., insensitivity to parameter perturbations, must be addressed; also modeling errors will have to be taken into account.

In this work we focus on the robust tracking problem for minimum phase input-output linearizable uncertain systems. We allow parameter uncertainty as well as modeling errors, hence the robustness issue is one of our primary concerns. Since measurement of the states is not always feasible, we use dynamic output feedback in our design approach through building an observer. The choice of dynamic instead of static output feedback is motivated by the fact that most systems can not be stabilized through the use of static output feedback (e.g., $\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1$).

The details of the design procedure is carried out in two steps. first, a robust state feedback control is designed to achieve the control task, and in the second step we implement it as a globally bounded control through designing an observer to recover the robustness and asymptotic properties achieved under state feedback.

In Chapter 2 we study in detail the regulation problem. More precisely, we present a solution to the problem of asymptotically regulating the output of a nonlinear system to a constant reference with zero error, in the presence of uncertain constant parameters/disturbances. We do this through the introduction of an integrator driven by the tracking error. This will create an equilibrium point at which the tracking error is zero for all admissible perturbations. We then stabilize the system via the use of a min-max state feedback controller although, other methods such as variable structure control or high-gain control could have been used as well.

In Chapter 3 we address the more general tracking problem for systems that are represented by input-output models where disturbances and/or references are time-varying and generated by a linear exosystem. In this chapter we utilize the idea of [10] in extending the system with the advantage of being able to implement less restrictive control schemes. The nonlinearities of the system are restricted to introduce only a finite number of harmonics of the original modes. This will enable us to identify the internal model as a linear servo-compensator. In this chapter we present conditions that will characterize a class of state feedback stabilizing controllers as opposed to presenting only one specific design of controllers.

In Chapter 4 we illustrate the controllers of Chapters 2 and 3 by solving a speed regulation problem for a field-controlled DC motor in the presence of uncertain parameters. The results of the simulations are in good agreement of what we expected and give a good example of how the theory presented can be utilized in solving engineering problems.

There is a fundamental difference between the two design methodologies implemented in this work. In Chapter 2 no feedback from the zero dynamic states was utilized, while in Chapter 3 we were able to use feedback from the extended states. This is done with a price, since the strategy adopted in Chapter 3 will, in general, lead to a more complex controller with higher dimension. This is evident in Chapter 4 where both strategies are tested. Using the approach of Chapter 2 we did not need an observer while it was needed using the other approach. Common to both methodologies, however, is the choice of the linear high gain observer, which is basically an approximate differentiator of the output [11], and the idea of globally bounded control introduced first in [12]. Finally, in Chapter 5 we give some concluding thoughts and prospects for future work.

CHAPTER 2

Asymptotic Regulation of Minimum Phase Nonlinear Systems

2.1 Introduction

One of the important problems in control systems is the servomechanism problem; that is, to get the plant output to asymptotically track a reference while asymptotically rejecting disturbances, when both the reference and disturbance signals satisfy a given differential equation model. For linear systems this problem was extensively studied by many researchers; see for example Davison [13] and Davison and Ferguson [14]; a self-contained exposition is found in Desoer and Wang [15]; a different approach to the solution of this problem was presented by Francis [16]. Since our focus in this work is on nonlinear systems, we highlight some of the previous contributions to the solution of this problem. In [17] Desoer and Wang studied this problem for a class of nonlinear distributed systems where the nonlinearity appears as causal operators at the input and output channels of a linear system. Desoer and Lin [18] studied this problem using PI controllers for exponentially stable plants having a strictly increasing dc steady-state input-output map with references and disturbances tending to constant vectors. Isidori and Byrnes [19] provided necessary and sufficient conditions for the local solution of the problem for a general case where the disturbance and reference signals can be time-varying but small, and initial states were required to be small. Huang and Rugh [20], [21], using the method of extended linearization, designed dynamic output feedback controllers for the solution of this problem. In [20] they considered the case of sufficiently small constant or slowly-varying external signals. In [21], they allowed large slowly-varying external signals but still required proximity of the initial states to the zero-error manifold. In [22] Priscolli provided a local solution, with robustness, to the general problem addressed in [19]. Except for [17], [22] and the integral control of [20], the other papers [18, 19, 21] did not explicitly address robustness of asymptotic tracking to modeling errors. Aside from these servomechanism papers, tracking and disturbance rejection problems for feedback linearizable systems have been tackled by many authors. Related to this work are the results of [9, 23]. In [9] Marino and Tomei used adaptive control techniques to solve this problem globally for a class of systems that is characterized by geometric conditions where the zero dynamics are restricted, in suitable coordinates, to be linear. In [23] Khalil studied a special case of this problem for a class of single-input, single-output (SISO) nonlinear systems with relative degree $r = n$ that admit a *disturbance-strict-feedback* form, but he allowed time-varying disturbances and reference signals. For feedback linearizable systems, robust continuous feedback control laws can be designed to ensure convergence of the tracking error to a small ball while rejecting bounded disturbances. However, making the error arbitrarily small requires the use of high gain feedback near the origin; see for example [24, 25, 26, 27].

In this work we use integral control to ensure asymptotic regulation in the case of constant references, for a SISO minimum phase nonlinear system that is transformable into the normal form, uniformly in a set of constant disturbances and uncertain parameters. The introduction of the integrator creates an equilibrium point at which the tracking error is zero for all possible parameters and/or disturbances that belong to a known compact set. We provide estimates of the region of attraction that are limited only by the region of validity of our model. If the domain becomes global, those estimates can be made arbitrarily large. As a consequence of this, we have the semi-global result which follows directly. In this case we do not impose global linear growth conditions on the nonlinearities nor do we require global exponential stability of the zero dynamics.

In order to recover the asymptotic properties achieved under state feedback we saturate the state feedback control over a compact set of interest then implement this globally bounded control as an observer-based control using a linear high gain observer.

This chapter is organized as follows: The class of systems that will be considered is presented in Section 2.2 Asymptotic regulation is achieved under state feedback; this is shown in Section 2.3. In Section 2.4, output feedback is used to recover the robustness and asymptotic regulation properties of the state feedback controller. The main result of this section is Theorem 2.1. In Section 2.5 we present the semi-global result. A stabilization result is given in Section 2.6 for a special case of the class of systems. In Section 2.7 we investigate the performance of the integral control in the presence of certain time-varying signals. Finally, some concluding remarks are given in Section 2.8.

2.2 System Description

Consider a SISO nonlinear system, modeled by

$$\left.\begin{aligned} \dot{\xi} &= f(\xi,\theta) + g(\xi,\theta)u \\ y &= h(\xi,\theta) \end{aligned}\right\} \tag{2.1}$$

where $\xi \in R^n$ is the state, $u \in R$ is the control input, $y \in R$ is the measured output, θ is a vector of unknown but constant parameters and disturbance inputs which belongs to a compact set $\Theta \subset R^l$. We consider a case where the output $y(t)$ is to track a constant reference $\nu \in \Gamma$ where $\Gamma \subset R$ is compact. For all $\theta \in \Theta$, we assume the following: f, g are smooth vector fields on U_θ, an open subset of R^n that might depend on θ, h is a smooth function in ξ from $U_\theta \rightarrow R$.

In this work we are interested in input-output linearizable minimum phase nonlinear systems where f, h do not necessarily vanish at the origin, i.e., $f(0,\theta) \neq 0$, $h(0,\theta) \neq 0$. We consider the case where the system has a well defined normal form and possibly nontrivial zero dynamics. With this in mind, we assume the following about the system (2.1).

Assumption 2.1 *$\forall\ \nu \in \Gamma$ and $\forall\ \theta \in \Theta$, there exist an equilibrium point $\xi_0(\nu,\theta)$ and a control $u(\nu,\theta)$ such that*

$$\left.\begin{aligned} 0 &= f(\xi_0(\nu,\theta),\theta) + g(\xi_0(\nu,\theta),\theta)u(\nu,\theta) \\ \nu &= h(\xi_0(\nu,\theta),\theta) \end{aligned}\right\} \tag{2.2}$$

Moreover, $\xi_0(\nu,\theta)$ is the only such equilibrium point in U_θ.

The existence of $\xi_0(\nu,\theta)$ and $u(\nu,\theta)$ satisfying (2.2) is a necessary requirement for the system to maintain equilibrium at $y = \nu$.

Assumption 2.2 *$\forall\ \theta \in \Theta$ there exists a mapping*

$$\begin{bmatrix} x \\ z \end{bmatrix} = T(\xi, \theta) \tag{2.3}$$

which is a diffeomorphism of U_θ onto its image, that transforms (2.1) into the normal form

$$\left.\begin{array}{rcl} \dot{x}_i &=& x_{i+1},\ 1 \leq i \leq r-1 \\ \dot{x}_r &=& \bar{f}(x,z,\theta) + \bar{g}(x,z,\theta)u \\ \dot{z} &=& \phi(x,z,\theta) \\ y &=& x_1 \end{array}\right\} \tag{2.4}$$

or, more compactly,

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + B[\bar{f}(x,z,\theta) + \bar{g}(x,z,\theta)u] \\ \dot{z} &=& \phi(x,z,\theta) \\ y &=& Cx \end{array}\right\} \tag{2.5}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{r\times r}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r\times 1},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{1\times r}$$

Conditions under which Assumption 2.2 holds locally or globally, when $\theta = \theta_0$ (known), are given in [28, Proposition 3.2b, Corollary 5.6]. Global conditions are also given in [28, Corollary 5.7] for the case when $\phi = \phi(x_1, z)$. We point out here

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that, in contrast to a local or a global diffeomorphism, the mapping (2.3) is required to be a diffeomorphism of a domain U_θ onto its image. A result of [29] gives necessary and sufficient conditions for a smooth mapping that maps U into V to be a diffeomorphism of U onto V. A smooth mapping $F : U \rightarrow F(U)$ is a diffeomorphism of U onto its image if and only if (a) $detJ_\xi \neq 0$ through out U and (b) F is a proper map of U into $F(U)$; where J_ξ denotes the Jacobian matrix of F at a general point $\xi \in U$. At this juncture, we recognize that there are no results available in the literature that will guarantee the existence of the mapping T for a given domain U_θ.

Usually one starts from local conditions and in the process of transforming the system into the normal form (2.5) a region over which the normal form holds is identified. Requiring the normal form (2.5) to hold uniformly in θ is clearly more restrictive than requiring it to hold for a given value of θ. For many examples of physical systems which are transformable into the normal form, it is indeed true that the normal form holds uniformly in the system parameters, at least over a compact set of these parameters; see for example the field-controlled DC motor and the robot arm examples of [30, section 4.10]. There is also the result of [6] which characterizes a class of systems for which such representation is valid. That paper considers a special case of (2.1) where θ appears linearly in the model, i.e.,

$$\left.\begin{array}{rcl} \dot{\xi} & = & f(\xi) + g(\xi)u + \sum_{i=1}^{l} p_i(\xi)\theta_i \\ y & = & h(\xi) \end{array}\right\} \tag{2.6}$$

For this class of systems it was shown in [6] that if there exists a parameter-independent global diffeomorphism that, when $\theta_i = 0$, transforms (2.6) into the normal form (2.5) with $\phi = \phi(x_1, \cdots, x_{q+1}, z)$ for some integer $1 \leq q \leq r-1$ then, under certain geometric conditions on the vector fields $p_i(\xi)$, the same diffeomorphism will t

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transform (2.6) when $\theta_i \neq 0$ into the parametric-strict-feedback form:

$$\left.\begin{array}{rcl} \dot{x}_i & = & x_{i+1} + \theta^T \gamma_i(x_1, \cdots, x_i), \ 1 \leq i \leq q-1 \\ \dot{x}_i & = & x_{i+1} + \theta^T \gamma_i(x_1, \cdots, x_i, z), \ q \leq i \leq r-1 \\ \dot{x}_r & = & \bar{f}(x,z) + \bar{g}(x,z)u + \theta^T \gamma_r(x,z) \\ \dot{z} & = & \phi(x_1, \cdots, x_{q+1}, z) + \sum_{i=1}^{l} \theta_i \gamma_i^z(x_1, \cdots, x_{q+1}, z) \\ y & = & x_1 \end{array}\right\} \tag{2.7}$$

where $\gamma_i(.) \in R^l, \forall\ 1 \leq i \leq r$, $\gamma_i^z(.) \in R^{n-r}, \forall\ 1 \leq i \leq l$. It can be easily shown that, using a parameter-dependent transformation, system (2.7) is transformable into a global normal form uniformly in θ.

Now, let us introduce a new set of coordinates in terms of the tracking error and its derivatives. Let

$$\left.\begin{array}{rcl} e_1 & = & x_1 - \nu \\ e_{i+1} & = & \dot{e}_i = x_{i+1}, \ 1 \leq i \leq r-1 \\ z & = & z \end{array}\right\} \tag{2.8}$$

Let us denote the transformation (2.8) by

$$\begin{bmatrix} e \\ z \end{bmatrix} = \begin{bmatrix} \Psi_0(x,\nu) \\ z \end{bmatrix} = \Psi(x,z,\nu)$$

and rewrite (2.4), in the newly defined error coordinates, as

$$\left.\begin{aligned}
\dot{e}_i &= e_{i+1},\ 1 \le i \le r-1 \\
\dot{e}_r &= \bar{f}(e_1+\nu, e_2, \cdots, e_r, z, \theta) + \bar{g}(e_1+\nu, e_2, \cdots, e_r, z, \theta)u \\
&\stackrel{def}{=} f_1(e, z, \nu, \theta) + g_1(e, z, \nu, \theta)u \\
\dot{z} &= \phi(e_1+\nu, e_2, \cdots, e_r, z, \theta) \\
&\stackrel{def}{=} \phi_0(e, z, \nu, \theta) \\
y_m &= e_1
\end{aligned}\right\} \tag{2.9}$$

where y_m denotes the measured output, and $e = [e_1, \cdots, e_r]^T$. We remark that the transformation (2.8) is a diffeomorphism for all $\nu \in \Gamma$. To simplify the notation, we set $d = (\nu, \theta)$ and $D = \Gamma \times \Theta$.

2.3 Integral Control

We augment system (2.9) with an integrator driven by the tracking error, i.e.,

$$\sigma = \int_0^t (y(\tau) - \nu)\, d\tau$$

The augmented system is given by

$$\left.\begin{aligned}
\dot{\sigma} &= e_1 \\
\dot{e}_i &= e_{i+1},\ 1 \le i \le r-1 \\
\dot{e}_r &= f_1(e, z, d) + g_1(e, z, d)u \\
\dot{z} &= \phi_0(e, z, d) \\
y_m &= e_1
\end{aligned}\right\} \tag{2.10}$$

Rewrite (2.10) in the compact form

$$\dot{\zeta} = \mathcal{A}\zeta + \mathcal{B}[f_1(e,z,d) + g_1(e,z,d)u] \tag{2.11}$$

$$\dot{z} = \phi_0(e,z,d) \tag{2.12}$$

$$y_m = Ce$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \ \zeta = \begin{bmatrix} \sigma \\ e \end{bmatrix}$$

We remark that in (2.11) f_1 and g_1 satisfy the matching condition, and the pair $(\mathcal{A}, \mathcal{B})$ is controllable. We assume the following:

Assumption 2.3 *There exists a domain $N_d \subset R^n$, that contains the origin, such that $(\Psi \circ T)^{-1}(N_d) \subset U_\theta$ and contains the point ξ_0, for all $d \in D$.*

Let $M_d = R \times N_d$. We restrict our analysis to the domain of interest M_d.

We note that as a consequence of Assumptions 2.1 and 2.3 together with the fact that the composite transformation $\Psi \circ T$ is a diffeomorphism, equation (2.12), with $e = 0$, has a unique equilibrium point in the domain of interest, M_d, that will be denoted by $z^0 = \lambda(d)$, i.e., $\phi_0(0, \lambda(d), d) \equiv 0$, for all $d \in D$.

Before stating the next assumption, let $\mathcal{S} \subset R^{r+1}$ and $\mathcal{U} \subset R^{n-r}$ be open sets such that $\mathcal{S} \times \mathcal{U} \subset M_d$. Also define the balls, $\mathcal{S}_0 = \{\zeta \in \mathcal{S} : \|\zeta\| < r_1\}$ and $\mathcal{U}_0 = \{z \in \mathcal{U} : \|z - z^0\| < r_2\}$ where $\|.\|$ denotes the Euclidean norm and $r_i > 0$, $(i = 1,2)$ are chosen to give the maximum balls in $\mathcal{S}$ and $\mathcal{U}$, uniformly in d.

Assumption 2.4 *With e as a driving input to (2.12), there exist a C^1 proper function $W : \mathcal{U} \to \mathcal{R}_+$ and class $\mathcal{K}$ functions, $\alpha_i : [0, r_2) \to R_+$, $(i = 1,2,3)$ and $\gamma_1 : [0, r_1) \to R_+$ such that*

$$\alpha_1(\|z - z^0\|) \leq W(z) \leq \alpha_2(\|z - z^0\|) \tag{2.13}$$

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$$\frac{\partial W}{\partial z}\phi_0(e,z,d) \le -\alpha_3(\|z - z^0\|),\ \forall\ \|z - z^0\| \ge \gamma_1(\|e\|) \tag{2.14}$$

$\forall\ (\zeta, z, d) \in \mathcal{S}_0 \times \mathcal{U}_0 \times D.$

This assumption implies that when $e = 0$, the equilibrium z^0 is asymptotically stable. Moreover, when $e \neq 0$ but bounded it can be shown [31, Theorem 4.10] that the solution $z(t)$ satisfies the estimate

$$\|z(t) - z^0\| \le \beta_1(\|z(0) - z^0\|, t) + \gamma(|e|),\ \forall\ t \ge 0 \tag{2.15}$$

where $|e| = \sup\{\|e(t)\| :\ t \ge 0\}$, β_1 is a class $\mathcal{KL}$ function and γ is a class $\mathcal{K}$ function; see [31] for the definition of these classes of functions.

We point out that for semi-global results, to be considered in section 2.5, Assumption 2.4 must hold globally. In this case the estimate (2.15) implies that the system (2.12) is input to state stable, (ISS) for short, as it is defined in [32]. In light of a recent result of [33], this assumption is also necessary for the system to be ISS. However, for regional results it becomes less restrictive. To see this, we consider the following example where the z-dynamics, with x as input, are given by

$$\dot{z} = -z + (z^2 + 1)x$$

This system, as it was indicated in [32], is not bounded-input bounded-state (BIBS) stable, hence it is not ISS. However, if we restrict our domain of interest to the region defined by $\{(z, x) : |z| \le 1, |x| \le .25\}$ we can see that $W(z) = (1/2)z^2$ satisfies (2.13)-(2.14) since

$$\dot{W} \le -(1/2)z^2,\ \forall\ |z| \ge 4\,|x|$$

There are also some global results available in the literature that guarantee the existence of $W(z)$ satisfying (2.13)-(2.14) when the zero dynamics $\dot{z} = \phi_0(0, z)$ are globally exponentially stable and ϕ_0 is globally Lipschitz in x; see for example [34], [35] and [7]. However, Assumption 2.4 is less restrictive. To illustrate, consider the example

$$\dot{z} = -z - 2z^3 + (z^2+1)x^2 = \phi_0(x,z) \tag{2.16}$$

When $x = 0$, the system has a globally exponentially stable equilibrium point at $z = 0$, but ϕ_0 is not globally Lipschitz in x. It can be shown that $W(z) = \frac{1}{2}z^2$ satisfies Assumption 2.4 with

$$\dot{W} \leq -(1/2)z^2, \quad \forall\ |z| \geq 2\,|x|^2$$

We proceed now to design a robust state feedback controller assuming that the state e is available for measurement. This is not a reasonable assumption because of the dependence of the transformation (2.3) on the unknown parameter θ. However, the final controller will be an observer-based controller and no measurement of the state will be used. For feedback linearizable systems where the uncertain terms f_1, g_1 satisfy the matching condition, several methods are available to design robust state feedback controls such as min-max, high gain and variable structure control. In this work we chose to use the min-max controller of [26], but other methods could have been used as well.

We start the design procedure by choosing K such that $(\mathcal{A}+\mathcal{B}K)$ is Hurwitz. Let $P{=}P^T > 0$ be the solution of the Lyapunov equation

$$P(\mathcal{A}+\mathcal{B}K) + (\mathcal{A}+\mathcal{B}K)^T P = -I \tag{2.17}$$

Take $V(\zeta){=}\zeta^T P\zeta$, and for $c_i > 0$, $(i = 1,2)$ define

$$\Omega_{c_1} \stackrel{\text{def}}{=} \{\zeta \in R^{r+1} : V(\zeta) \leq c_1\} \tag{2.18}$$

$$\Omega_{c_2} \overset{\text{def}}{=} \{z \in R^{n-r} : W(z) \leq c_2\} \tag{2.19}$$

Since our assumptions are required to hold in a given region, we require both Ω_{c_1} and Ω_{c_2} to belong to the domain of validity of our assumptions, i.e., $\Omega_{c_1} \times \Omega_{c_2} \subset \mathcal{S}_0 \times \mathcal{U}_0$. Moreover, since e acts as a driving input to (2.12), we need to choose c_1 and c_2 such that if ζ is contained in Ω_{c_1} for all t, then z will be contained in Ω_{c_2} for all t. Inside Ω_{c_1} we have

$$\|\zeta\|^2 \leq \frac{c_1}{\lambda_{min}(P)}$$

For Ω_{c_1} to be in the interior of $\mathcal{S}_0$, we require

$$c_1 < \lambda_{min}(P) r_1^2 \tag{2.20}$$

when ζ is contained in Ω_{c_1} for all t, inequality (2.14) holds outside the ball $\{\|z - z^0\| \leq \gamma_1\left(\sqrt{c_1/\lambda_{min}(P)}\right)\}$. To contain this ball inside Ω_{c_2}, we require

$$\alpha_4\left(\sqrt{c_1/\lambda_{min}(P)}\right) \leq c_2$$

where $\alpha_4 = \alpha_2 \circ \gamma_1$. Again for Ω_{c_2} to be in the interior of $\mathcal{U}_0$ we must have

$$c_2 < \alpha_1(r_2)$$

Therefore, c_2 must satisfy

$$\alpha_4\left(\sqrt{c_1/\lambda_{min}(P)}\right) \leq c_2 < \alpha_1(r_2) \tag{2.21}$$

Hence, to satisfy both (2.20) and (2.21) we choose c_1 small enough to satisfy

$$c_1 < \min\left\{ \lambda_{min}(P) r_1^2, \lambda_{min}(P)(\alpha_4^{-1} \circ \alpha_1(r_2))^2 \right\} \tag{2.22}$$

From now on, we fix c_1, c_2 as chosen above. Thus we are guaranteed, under Assumption 2.4, that as long as $\zeta(t) \in \Omega_{c_1}$, Ω_{c_2} will be a positively invariant set.
Let $f_0(e,\nu)$ and $g_0(e,\nu)$ be known nominal models of $f_1(e,z,d)$ and $g_1(e,z,d)$, respectively, that are not allowed to depend on z. If no such nominal functions are known, we can take f_0=0 and g_0=sgn(g_1).

Assumption 2.5 *There exist a scalar nonnegative locally Lipschitz function $\rho(\zeta)$ and a positive constant k, both known, such that*

$$\left| f_1(e,z,d) - g_1(e,z,d) g_0^{-1}(e,\nu) f_0(e,\nu) - K\zeta \right| \leq \rho(\zeta) \tag{2.23}$$

$$g_1(e,z,d) g_0^{-1}(e,\nu) \geq k \tag{2.24}$$

$\forall\ (\zeta, z, d) \in \Omega_{c_1} \times \Omega_{c_2} \times D.$

Since Inequality (2.23) is required to hold on a compact set, it is always possible to find $\rho(\zeta)$ that satisfies (2.23). In fact we can always take ρ to be constant. However, allowing $\rho(\zeta)$ to depend on ζ will lead, in general, to less conservative bounds. We note also that ρ is not allowed to depend on z even though the left-hand side of (2.23) is a function of z. This is a built-in conservative measure of our design which is adopted to allow the use of a partial state feedback independent of z. Inequality (2.24) is a sign definiteness requirement on g_1 that is sometimes referred to in the literature as the high frequency gain assumption [36]. Continuing the design procedure, we consider the following partial state feedback control [26]

$$u = \varphi(\zeta,\nu) = -g_0^{-1}(e,\nu) f_0(e,\nu) - \frac{1}{k} g_0^{-1}(e,\nu) \eta(\zeta) \mathcal{N}_\mu(s, \eta(\zeta)) \tag{2.25}$$

where $\eta(\zeta) \geq \rho(\zeta)$ and

$$\mathcal{N}_\mu(s,\eta) = \begin{cases} \frac{s}{|s|}, & \text{if } \eta\,|s| > \mu > 0 \\ \eta\frac{s}{\mu}, & \text{if } \eta\,|s| \leq \mu \end{cases} \tag{2.26}$$

with $s{=}2\mathcal{B}^T P\zeta$ and μ is a design parameter whose role will be discussed soon.

As it was emphasized in [36], handling local convergence to the equilibrium point separate from showing boundedness of solutions has some advantages, as certain assumptions on the nonlinearities of the system are required to hold only locally on a small set around that equilibrium point. Then, if we force the trajectories, using nonlinear or high gain control, to enter in finite time a positively invariant set that is in the interior of the set of validity of our local assumptions, one can utilize these assumptions in the local analysis to show convergence to the equilibrium point. Our control (2.25) achieves this task by using the nonlinear term $-(1/k)g_0^{-1}\eta\mathcal{N}_\mu$, which is preferable to using a high gain term that results in a larger control effort far from the origin. In light of this discussion, our immediate task is to perform regional analysis to show boundedness of solutions. To do this, we first show using Lyapunov techniques that the state (ζ, z) will enter and thereafter remain inside a positively invariant set around $(0, z^0)$ which can be made arbitrarily small. Thus, if the system has an equilibrium point it has to be inside this residual set. To that end, use $V(\zeta)$ as a Lyapunov function candidate for the system

$$\dot{\zeta} = (\mathcal{A} + \mathcal{B}K)\zeta + \mathcal{B}(f_1 - g_1 g_0^{-1} f_0 - K\zeta) - \frac{1}{k}\mathcal{B}g_1 g_0^{-1}\eta(\zeta)\mathcal{N}_\mu \tag{2.27}$$

then for all $(\zeta, z, d) \in \{\zeta : \{\eta(\zeta)\,|s| > \mu\} \bigcap \Omega_{c_1}\} \times \Omega_{c_2} \times D$, we have

$$\dot{V} \leq -\|\zeta\|^2$$

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while for all $(\zeta, z, d) \in \{\zeta : \{\eta(\zeta)\,|s| \le \mu\} \bigcap \Omega_{c_1}\} \times \Omega_{c_2} \times D$, we have

$$\begin{aligned}\dot{V} &\le -\|\zeta\|^2 + \eta(\zeta)\,|s| - \frac{\eta^2(\zeta)s^2}{\mu} \\ &\le -\|\zeta\|^2 + \frac{\mu}{4}\end{aligned}$$

Hence, we conclude that

$$\dot{V} \le -\|\zeta\|^2 + \frac{\mu}{4},\ \forall\,(\zeta, z, d) \in \Omega_{c_1} \times \Omega_{c_2} \times D \tag{2.28}$$

For $\mu \le 4c_1/\alpha\lambda_{max}(P)$, with $\alpha > 1$,

$$\dot{V} < 0,\ \forall\,(\zeta, z, d) \in \Omega_{c_1} \times \Omega_{c_2} \times D \text{ and } V(\zeta) = c_1 \tag{2.29}$$

From (2.14)

$$\dot{W} < 0,\ \forall\,(\zeta, z, d) \in \Omega_{c_1} \times \Omega_{c_2} \times D \text{ and } W(z) = c_2 \tag{2.30}$$

From (2.29) and (2.30) we conclude that $\Omega_{c_1} \times \Omega_{c_2}$ is a positively invariant set; i.e., all trajectories $(\zeta(t), z(t))$ starting at $(\zeta(0), z(0)) \in \Omega_{c_1} \times \Omega_{c_2}$ will remain in $\Omega_{c_1} \times \Omega_{c_2}$ for all $t \ge 0$. Note also that for any positive constants a_1 and a_2 such that $a_1 < c_1$ and

$$\alpha_4\left(\sqrt{\frac{a_1}{\lambda_{min}(P)}}\right) \le a_2 < c_2$$

we can repeat the foregoing argument to show that for $\mu \le 4a_1/\alpha\lambda_{max}(P)$, the set $\Omega_{a_1} \times \Omega_{a_2}$ is a positively invariant set.

Now, choose $a_1 = \beta_1 = (\lambda_{max}(P)\alpha\mu)/4$ for some $\alpha > 1$ and $a_2 = \beta_2 = \alpha_4\left(\sqrt{\frac{\beta_1}{\lambda_{min}(P)}}\right)$, and take $\mu \le 4b_1/\alpha\lambda_{max}(P)$, where $0 < b_1 < c_1$. This particular choice of a_1, a_2 and μ implies that $\beta_1 \le b_1 < c_1$ and $\beta_2 < c_2$. Then the previous argument shows that

$$\mathcal{R}_\mu \stackrel{def}{=} \Omega_{\beta_1} \times \Omega_{\beta_2} \subset \Omega_{b_1} \times \Omega_{c_2} \subset \Omega_{c_1} \times \Omega_{c_2} \tag{2.31}$$

is a positively invariant set.

Lemma 2.1 *All trajectories* $(\zeta(t), z(t))$ *that start in* $(\zeta(0), z(0)) \in \Omega_{c_1} \times \Omega_{c_2}$ *will enter the set* $\mathcal{R}_\mu$ *in finite time.*

Proof: From (2.28) we have

$$\dot{V} \leq -\frac{V}{\lambda_{max}(P)} + \frac{\mu}{4}, \; \forall\, (\zeta, z, d) \in \Omega_{c_1} \times \Omega_{c_2} \times D$$

When $V \geq \beta_1$, we have

$$\dot{V} \leq -\frac{\mu}{4}(\alpha - 1) \tag{2.32}$$

which implies that V will be bounded by a bound that tends to zero in finite time. Hence, it will clearly tend to β_1 in finite time. This proves that the trajectories $(\zeta(t), z(t))$ will enter the set $\Omega_{\beta_1} \times \Omega_{c_2}$ in finite time and remain thereafter. Note that (2.32) explains why we included the factor $\alpha > 1$ in the choice of β_1. Now, for all trajectories inside $\Omega_{\beta_1} \times \Omega_{c_2}$ but outside $\mathcal{R}_\mu = \Omega_{\beta_1} \times \Omega_{\beta_2}$, we have from Assumption 2.4

$$\dot{W} \leq -\alpha_3 \circ \gamma_1 \left(\sqrt{\frac{\beta_1}{\lambda_{min}(P)}} \right)$$

which shows that W will reach β_2 in finite time. Thus we conclude that $(\zeta(t), z(t))$ will enter the set $\mathcal{R}_\mu$ in finite time. •

As a consequence of this lemma, we see that $(\zeta, z - z^0)$ are ultimately bounded with ultimate bounds which are class $\mathcal{K}$ functions of μ; hence the trajectory can be made arbitrarily close to $(0, z^0)$ by choosing μ small enough.

We turn now to the behavior of the trajectory inside $\mathcal{R}_\mu$. We want to establish that there is an equilibrium point inside $\mathcal{R}_\mu$ at which the tracking error is zero, and that every trajectory in $\mathcal{R}_\mu$ converges to this point as $t \to \infty$. Our approach to that end is to force the state ζ to enter and thereafter remain inside the boundary layer $\{\eta(\zeta)\, |s| \leq \mu\}$ and to have the nonlinear term $\eta(\zeta)\mathcal{N}_\mu(s, \eta(\zeta))$ reduce to a linear

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function of s inside this boundary layer. We consider the function $\bar{V} = \frac{1}{2}s^2$ whose derivative along the trajectories of (2.27) satisfies

$$\begin{aligned}\dot{\bar{V}} &= s[2\mathcal{B}^T P(\mathcal{A}+\mathcal{B}K)\zeta + 2\mathcal{B}^T P\mathcal{B}(f_1 - g_1 g_0^{-1} f_0 - K\zeta) \\ &\quad -(2/k)\mathcal{B}^T P\mathcal{B} g_1 g_0^{-1}\eta(\zeta)\mathcal{N}_\mu]\end{aligned}$$

Using (2.23), (2.24) and $\mathcal{B}^T P\mathcal{B} = p$, where p is the $(r+1)$-th diagonal element of P, we get

$$\dot{\bar{V}} \leq 2\,|s|\,[\|\mathcal{B}^T P(\mathcal{A}+\mathcal{B}K)\zeta\| + p\rho(\zeta)] - 2sp\eta(\zeta)\mathcal{N}_\mu \tag{2.33}$$

$$\begin{aligned}&\leq 2p\,|s|\,\max_{\zeta\in\Omega_{\beta_1}}\{\rho(\zeta) + \frac{\|\mathcal{B}^T P(\mathcal{A}+\mathcal{B}K)\zeta\|}{p}\} \\ &\quad -2sp\eta(\zeta)\mathcal{N}_\mu\end{aligned} \tag{2.34}$$

Let

$$\eta(\zeta) = \max\{\eta_0, \rho(\zeta)\} \tag{2.35}$$

where

$$\eta_0 \geq \max_{\zeta\in\Omega_{\beta_1}}\{\rho(\zeta) + \frac{\|\mathcal{B}^T P(\mathcal{A}+\mathcal{B}K)\zeta\|}{p}\} + \mu_1 \tag{2.36}$$

for some $\mu_1 > 0$, so that inside Ω_{β_1}, $\eta(\zeta) = \eta_0$. This step simplifies the boundary layer inside Ω_{β_1} to $\{\eta_0\,|s| \leq \mu\}$. Also, it follows from (2.36) that

$$\dot{\bar{V}} \leq 2p\,|s|\,(\eta_0 - \mu_1) - 2sp\eta_0\mathcal{N}_\mu \tag{2.37}$$

Hence

$$\dot{\bar{V}} \leq -2\,|s|\,p\mu_1 \tag{2.38}$$

for all $\eta_0\,|s| \geq \mu$. This implies that the set $S_\mu \stackrel{\text{def}}{=} \{\eta_0\,|s| \leq \mu\}$ is positively invariant and all trajectories inside Ω_{β_1} will enter it in finite time. Thus, ζ will approach the

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set $\Lambda_\mu \stackrel{\text{def}}{=} \Omega_{\beta_1} \cap S_\mu$ in finite time. Inside Λ_μ, $\eta(\zeta)\mathcal{N}_\mu(s,\eta(\zeta))$ will be linear in s, i.e., $\eta\mathcal{N}_\mu = \frac{\eta_0^2}{\mu}s$. Therefore, the trajectory $(\zeta(t), z(t))$ enters the positively invariant set $\Lambda_\mu \times \Omega_{\beta_2}$ in finite time. From that time on the closed-loop system is given by

$$\left.\begin{aligned} \dot{\zeta} &= (\mathcal{A}+\mathcal{B}K)\zeta + \mathcal{B}[f_1 - g_1 g_0^{-1} f_0 - K\zeta] \\ &- \tfrac{2\eta_0^2}{k\mu}\mathcal{B}g_1 g_0^{-1}\mathcal{B}^T P\zeta \\ \dot{z} &= \phi_0(e,z,d) \end{aligned}\right\} \tag{2.39}$$

The next step is to establish the convergence of the trajectories of (2.39) to an equilibrium point at which $e = 0$. Before doing that, we will impose the following local growth assumption that is required to hold in the neighborhood of z^0.

Assumption 2.6 *There exists a C^1 function $\tilde{V}$ and a continuous function ψ, both positive definite in $\tilde{z} = z - z^0$ and vanish at the origin, such that $\forall\ d \in D$*

$$\frac{\partial \tilde{V}}{\partial \tilde{z}}\phi_0(0,z,d) \le -\alpha_0\psi(\tilde{z}),\ \alpha_0 > 0$$

$$\begin{aligned} \left|f_1(0,z,d) - g_1(0,z,d)g_1^{-1}(0,z^0,d)f_1(0,z^0,d)\right| &\le k_1\psi^a(\tilde{z}) \\ 0 < a \le 1/2,\ k_1 &> 0 \end{aligned} \tag{2.40}$$

$$\frac{\partial \tilde{V}}{\partial \tilde{z}}[\phi_0(e,z,d) - \phi_0(0,z,d)] \le k_2\psi^b(\tilde{z})\|e\|^c,\ 0 < b < 1,\ c = \frac{1-b}{a},\ k_2 > 0$$

This local growth assumption is adopted from [37]. It is less restrictive than local exponential stability of the zero dynamics. In particular, if the zero dynamics are locally exponentially stable, then by the converse Lyapunov Theorem [31, Theorem 4.5] there is a Lyapunov function $\tilde{V}$ which satisfies Assumption 2.6 with $a = 1/2, b = 1/2, c = 1$ and $\psi(\tilde{z}) = \|\tilde{z}\|^2$. Assumption 2.6 is also less restrictive than the quadratic-type Lyapunov function assumption used in [38]. In [38] the emphasis is on completing s

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squares, while here we follow [37] in repeatedly using the following fact which is a special case of Young's Inequality [39, Theorem 156].

Fact 2.1 *$\forall\, x, y \in R_+$, $\forall\, p > 1$ and $\forall\, \epsilon_0 > 0$*

$$xy \leq \frac{1}{\epsilon_0}|x|^p + (\epsilon_0)^{p_0}|y|^{\frac{p}{p-1}} \tag{2.41}$$

where $p_0 = \frac{1}{p-1}$.

Lemma 2.2 *There exists $\mu^* > 0$ such that $\forall\, \mu \leq \mu^*$ and $\forall\, d \in D$, the closed-loop system (2.11), (2.12) and (2.25) has a unique equilibrium point ($\sigma = \bar{\sigma}, e = 0, z = z^0$) and all closed-loop trajectories starting in $\Omega_{c_1} \times \Omega_{c_2}$ at $t = 0$ will remain in $\Omega_{c_1} \times \Omega_{c_2}, \forall\, t \geq 0$ and will converge to this equilibrium point as $t \to \infty$.*

Proof: We have already established that trajectories starting in $\Omega_{c_1} \times \Omega_{c_2}$ enter $\Lambda_\mu \times \Omega_{\beta_2}$ in finite time. We choose μ^* small enough to ensure that for $\mu \leq \mu^*$ the set $\Lambda_\mu \times \Omega_{\beta_2}$ is in the interior of the set of validity of Assumption 2.6. We start by showing the existence of an equilibrium point in $\Lambda_\mu \times \Omega_{\beta_2}$. At equilibrium, we have from (2.39) and Assumptions 2.1 and 2.3 that $e = 0, z = z^0$ and

$$\begin{aligned} B^T P_{12}^T \bar{\sigma} &= \frac{k\mu}{2\eta_0^2}[g_0(0,\nu)g_1^{-1}(0,z^0,d)f_1(0,z^0,d) - f_0(0,\nu)] \\ &\stackrel{def}{=} \frac{k\mu}{2\eta_0^2}a(d) \end{aligned} \tag{2.42}$$

where P is partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

with $dim(P_{11}) = 1$. Using (2.23), (2.24) and (2.36) it can be verified that $k\,|a(d)| < \eta_0$ for all $d \in D$, and this ensures that $\bar{\sigma}$ is confined to the boundary layer S_μ.

Equation (2.42) has a unique solution if the scalar constant $(B^T P_{12}^T)$ is nonzero. To show this, observe that from equation (2.17) we have $2K_1 B^T P_{12}^T = -1$ where K is partitioned as $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. The fact that $(\mathcal{A} + \mathcal{B}K)$ is Hurwitz implies that $K_1 \neq 0$. Hence, $B^T P_{12}^T \neq 0$. Combining this with Assumptions 2.1 and 2.3, we conclude that $(\sigma = \bar{\sigma}, e = 0, z = z^0)$ is the unique equilibrium point of the closed-loop system (2.39). To study convergence to this equilibrium point, we shift it to the origin by the change of variables

$$\tilde{\zeta} = \begin{bmatrix} \sigma - \bar{\sigma} \\ e \end{bmatrix} = \zeta - \begin{bmatrix} \bar{\sigma} \\ 0 \end{bmatrix}, \quad \tilde{z} = z - z^0$$

Then the closed-loop system becomes

$$\dot{\tilde{\zeta}} = (\mathcal{A} + \mathcal{B}K)\tilde{\zeta} + \mathcal{B}[\delta(e, z, d) - K\tilde{\zeta}] - \frac{\eta_0^2}{k\mu}\mathcal{B} g_1 g_0^{-1} \tilde{s} \tag{2.43}$$

$$\dot{\tilde{z}} = \phi_0(e, \tilde{z} + z^0, d) \tag{2.44}$$

where $\tilde{s} = 2\mathcal{B}^T P \tilde{\zeta}$ and

$$\begin{aligned} \delta(e, z, d) &= f_1(e, z, d) - g_1(e, z, d) g_0^{-1}(e, \nu) f_0(e, \nu) \\ &\quad - g_1(e, z, d) g_0^{-1}(e, \nu) g_0(0, \nu) g_1^{-1}(0, z^0, d) f_1(0, z^0, d) \\ &\quad + g_1(e, z, d) g_0^{-1}(e, \nu) f_0(0, \nu) \end{aligned}$$

Observe that $\delta(0, z, d) = f_1(0, z, d) - g_1(0, z, d) g_1^{-1}(0, z^0, d) f_1(0, z^0, d)$ which is required to satisfy inequality (2.40). After adding and subtracting the term $\delta(0, z, d)$ to the bracketed term in (2.43), it follows from Assumption 2.6 and smoothness of the nonlinearities that

$$\begin{aligned} \left|\delta(e, z, d) - K\tilde{\zeta}\right| &\leq k_0\|e\| + k_1 \psi^a(\tilde{z}) + \|K\| \|\tilde{\zeta}\| \\ &\leq k_1 \psi^a(\tilde{z}) + a_1 \|\tilde{\zeta}\| \end{aligned}$$

$$\leq \; \bar{a}_1(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)$$

where $\bar{a}_1 \geq \max\{k_1, a_1\}$. Let $\mathcal{V}(\tilde{\zeta}, \tilde{z}) = \tilde{V}(\tilde{z}) + \lambda(\tilde{\zeta}^T P \tilde{\zeta})^\gamma, \gamma = 1/2a \geq 1$, be a Lyapunov function candidate for the system (2.43)-(2.44). It can be shown that

$$\begin{aligned} \dot{\mathcal{V}} \;&\leq\; -\alpha_0\psi(\tilde{z}) + k_2\psi^b(\tilde{z})\|\tilde{\zeta}\|^c - \lambda a_2\|\tilde{\zeta}\|^{2\gamma} - \frac{\eta_0^2\lambda a_2}{\mu}\|\tilde{\zeta}\|^{2\gamma-2}\tilde{s}^2 \\ &\quad +\lambda\gamma\|P\|^{\gamma-1}\|\tilde{\zeta}\|^{2\gamma-2}\left|\delta - K\tilde{\zeta}\right|\left|\tilde{s}\right| \end{aligned} \tag{2.45}$$

where $a_2 = \gamma(\lambda_{min}(P))^{\gamma-1}$. Using the inequalities $-\|\tilde{\zeta}\|^{2\gamma-2} \leq -a_3\left|\tilde{s}\right|^{2\gamma-2}$ (for some $a_3 > 0$) and $\|\tilde{\zeta}\| \leq (\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)$, we get

$$\begin{aligned} \dot{\mathcal{V}} \;&\leq\; -\alpha_0\psi(\tilde{z}) + k_2\psi^b(\tilde{z})\|\tilde{\zeta}\|^c - \lambda a_2\|\tilde{\zeta}\|^{2\gamma} - \frac{\lambda\rho_0}{\mu}\left|\tilde{s}\right|^{2\gamma} \\ &\quad +\lambda\gamma\|P\|^{\gamma-1}\bar{a}_1\left|\tilde{s}\right|(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma-1} \end{aligned}$$

where $\rho_0 = \eta_0^2 a_2 a_3$. We apply (2.41) to $\psi^b(\tilde{z})\|\tilde{\zeta}\|^c$ and $\left|\tilde{s}\right|(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma-1}$, with $p = 1/b$ and $p = 2\gamma$, respectively, to obtain

$$\begin{aligned} \dot{\mathcal{V}} \;&\leq\; -\alpha_0\psi(\tilde{z}) + \frac{k_2}{\mu_0}\psi(\tilde{z}) + k_2(\mu_0)^{p_0}\|\tilde{\zeta}\|^{2\gamma} - \lambda a_2\|\tilde{\zeta}\|^{2\gamma} - \frac{\lambda\rho_0}{\mu}\left|\tilde{s}\right|^{2\gamma} \\ &\quad +(\epsilon_0)^{p_1}\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} + \frac{\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}}{\epsilon_0}\left|\tilde{s}\right|^{2\gamma} \end{aligned} \tag{2.46}$$

where $p_0 = b/(1-b)$, $p_1 = 1/(2\gamma-1)$ and μ_0, ϵ_0 are arbitrary positive constants. Choose μ_0 large enough so that $\alpha_0 - \frac{k_2}{\mu_0} \geq \frac{\alpha_0}{2}$. Then choose λ large enough so that $\lambda a_2 - k_2(\mu_0)^{p_0} \geq \frac{\lambda a_2}{2}$; we get

$$\begin{aligned} \dot{\mathcal{V}} \;&\leq\; -\bar{\alpha}(\psi(\tilde{z}) + \|\tilde{\zeta}\|^{2\gamma}) + (\epsilon_0)^{p_1}\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} \\ &\quad +\frac{\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}}{\epsilon_0}\left|\tilde{s}\right|^{2\gamma} - \frac{\lambda\rho_0}{\mu}\left|\tilde{s}\right|^{2\gamma} \end{aligned}$$

where $\bar{\alpha} = \min(\alpha_0/2, \lambda a_2/2)$. Using the inequality *

$$(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} \leq c_0(\psi(\tilde{z}) + \|\tilde{\zeta}\|^{2\gamma}), \; c_0 > 0 \tag{2.47}$$

we obtain

$$\begin{aligned} \dot{\mathcal{V}} \leq \; & -\frac{\bar{\alpha}}{c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} + (\epsilon_0)^{p_1}\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} \\ & + \frac{\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}}{\epsilon_0}|\tilde{s}|^{2\gamma} - \frac{\lambda\rho_0}{\mu}|\tilde{s}|^{2\gamma} \end{aligned}$$

Choose ϵ_0 small enough such that

$$\frac{\bar{\alpha}}{c_0} - (\epsilon_0)^{p_1}\bar{a}_1\gamma\lambda\|P\|^{\gamma-1} \geq \frac{\bar{\alpha}}{2c_0}$$

Finally, choose μ^* small enough so that

$$\frac{\lambda\rho_0}{\mu^*} - \frac{\bar{a}_1\lambda\gamma\|P\|^{\gamma-1}}{\epsilon_0} \geq \frac{\lambda\rho_0}{2\mu^*}$$

Then, for all $\mu \leq \mu^*$, we have

$$\dot{\mathcal{V}} \leq -\frac{\bar{\alpha}}{2c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} - \frac{\lambda\rho_0}{2\mu}|\tilde{s}|^{2\gamma} \bullet$$

2.4 Output Feedback Controller

To implement the controller of the previous section as an observer-based controller, we need to estimate the state ζ. However, since σ is available as the output of the

*see Appendix A for the proof of (2.47).

integrator, we will only estimate the state e using the following observer [40]

$$\left.\begin{aligned} \dot{\hat{e}}_i &= \hat{e}_{i+1} + \tfrac{\alpha_i}{\epsilon^i}(e_1 - \hat{e}_1), \quad i = 1, \cdots, r-1 \\ \dot{\hat{e}}_r &= \tfrac{\alpha_r}{\epsilon^r}(e_1 - \hat{e}_1) \end{aligned}\right\} \tag{2.48}$$

where $\epsilon > 0$ is a design parameter to be specified, and the positive constants α_i are to be chosen such that the roots of

$$s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r = 0 \tag{2.49}$$

are in the open left half plane. This is a high-gain observer motivated by our desire to recover the robustness properties of the state feedback control. To eliminate the peaking phenomenon associated with this high-gain observer, we implement the idea of [12] in saturating the control $u = \varphi(\zeta, \nu)$ defined by (2.25) outside the region of interest $\Omega_{c_1} \times \Gamma$. Let $S = \max_{(\zeta \in \Omega_{c_1}, \nu \in \Gamma)} |\varphi(\zeta, \nu)|$ and define

$$\varphi^s(\zeta, \nu) = S \text{ sat}\left(\frac{\varphi(\zeta, \nu)}{S}\right)$$

where sat(.) is the saturation function defined by

$$\text{sat}(x) = \begin{cases} \text{sign}(x), & \text{if } |x| > 1 \\ x, & \text{otherwise} \end{cases} \tag{2.50}$$

then $\varphi^s(\zeta, \nu) = \varphi(\zeta, \nu)$, $\forall\ \zeta \in \Omega_{c_1}$ and the conclusions obtained earlier for the state feedback control $u = \varphi(\zeta, \nu)$ hold for the state feedback control $u = \varphi^s(\zeta, \nu)$. The output feedback controller will be taken as $u = \varphi^s(\sigma, \hat{e}, \nu)$. Define the scaled estimation error

$$\chi_i = \frac{1}{\epsilon^{r-i}}(e_i - \hat{e}_i), \ 1 \leq i \leq r \tag{2.51}$$

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From (2.51) it follows that $\hat{e} = e - D(\epsilon)\chi$, where $D(\epsilon)$ is a diagonal matrix with ϵ^{r-i} as the ith diagonal element. Then, the closed-loop system will have the following singularly perturbed form

$$\dot{\zeta} = \mathcal{A}\zeta + \mathcal{B}[f_1(e,z,d) + g_1(e,z,d)\varphi^s(\sigma,\hat{e},\nu)] \tag{2.52}$$

$$\dot{z} = \phi_0(e,z,d) \tag{2.53}$$

$$\epsilon\dot{\chi} = (A - HC)\chi + \epsilon B[f_1(e,z,d) + g_1(e,z,d)\varphi^s(\sigma,\hat{e},\nu)] \tag{2.54}$$

where

$$\chi = \begin{bmatrix} \chi_1 \\ \vdots \\ \chi_r \end{bmatrix}_{r\times 1}, H = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}_{r\times 1}$$

It can be shown that the characteristic polynomial of the matrix $(A - HC)$ is given by the left hand side of (2.49). Hence, $(A - HC)$ is Hurwitz. The boundary-layer model of (2.52)-(2.54) is exponentially stable, and the reduced model is the closed-loop system under state feedback. Moreover, in view of the scaling (2.51), the initial conditions of the fast variables are of order $O(\epsilon^{-r+1})$. This causes an impulsive-like behavior in χ [12]. Since χ enters the slow equation (2.52) through the bounded function $\varphi^s(\sigma, e - D(\epsilon,\nu)\chi)$, the slow variable ζ does not exhibit a similar impulsive-like behavior. However, we will show that after some arbitrarily small time T_1, the scaled estimation error will decay to a level where $\|\chi\|$ is of the order $O(\epsilon)$ (Lemma 2.3) and then approaches zero asymptotically (Theorem 2.1). During the time period $[0, T_1]$ the estimate of ζ could be very far from the actual state; this is not of any concern since the control saturates and the state (ζ, z) will remain in the region of attraction. In the following lemma and the subsequent analysis we will establish that under output feedback and by choosing ϵ small enough, the state ζ will, after some finite time, enter a positively invariant set that is in the interior of the set Λ_μ. Again, Ass

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Assumption 2.4 will imply the boundedness of z.

Lemma 2.3 *Let $\bar{b}_1$ be a positive constant such that $0 < \bar{b}_1 < b_1 < c_1$. Suppose that $(\zeta(0), z(0)) \in \Omega_{\bar{b}_1} \times \Omega_{c_2}$ and let T_2 be the first time $(\zeta(t), z(t))$ exits from the set $\Omega_{b_1} \times \Omega_{c_2}$. Then $T_2 > 0$ and for sufficiently small ϵ, there exists a finite time T_1 $(0 < T_1 < T_2)$ such that $\forall\ t \in [T_1, T_2)$, $\|\chi\|$ is of order $O(\epsilon)$.*

Proof: We know that $(\zeta(0), z(0)) \in \Omega_{\bar{b}_1} \times \Omega_{c_2} \subset \Omega_{b_1} \times \Omega_{c_2}$. Since

$$\alpha_4\left(\sqrt{\frac{b_1}{\lambda_{min}(P)}}\right) < \alpha_4\left(\sqrt{\frac{c_1}{\lambda_{min}(P)}}\right) \leq c_2$$

we know from Assumption 2.4 that

$$\dot{W} < 0,\ \forall\ (\zeta, z, d) \in \Omega_{b_1} \times \Omega_{c_2} \times D \text{ and } W(z) = c_2$$

Hence, the trajectory $(\zeta(t), z(t))$ can leave the set $\Omega_{b_1} \times \Omega_{c_2}$ only through the boundary $V(\zeta) = b_1$. Since $\varphi^s(\sigma, \hat{e}, \nu)$ is globally bounded, we have

$$\|f_1 + g_1\varphi^s - K\zeta\| \leq k_3 \tag{2.55}$$

$\forall\ (\zeta, z, d) \in \Omega_{b_1} \times \Omega_{c_2} \times D$, for some $k_3 > 0$. Taking $V(\zeta) = \zeta^T P\zeta$ and evaluating its derivative along the trajectories of (2.52), we obtain

$$\dot{V} \leq -\|\zeta\|^2 + 2k_3\|P\mathcal{B}\|\|\zeta\| \tag{2.56}$$

$$\leq -2\gamma_2 V + 2\beta_3\sqrt{V},\ \ \forall\ (\zeta, z, d) \in \Omega_{b_1} \times \Omega_{c_2} \times D \tag{2.57}$$

where $\gamma_2 = 1/2\lambda_{max}(P)$ and $\beta_3 = k_3\|P\mathcal{B}\|/\sqrt{\lambda_{min}(P)}$. From (2.57) we have

$$\sqrt{V(t)} \leq \sqrt{V(0)}e^{-\gamma_2 t} + \frac{\beta_3}{\gamma_2}(1 - e^{-\gamma_2 t})$$

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Since $V(0) \leq \bar{b}_1 < b_1$, there exists a finite time $\bar{T}_2$, independent of ϵ, such that $T_2 \geq \bar{T}_2$. Now, let us turn to the fast equation (2.54) and study its solution over the interval $[0, T_2)$. Let $\mathcal{W} = \chi^T \bar{P} \chi$, where $\bar{P} = \bar{P}^T > 0$ is the solution of the Lyapunov equation

$$\bar{P}(A - HC) + (A - HC)^T \bar{P} = -I$$

From (2.55) we have

$$\|f_1 + g_1 \varphi^s\| \leq k_4$$

$\forall\ (\zeta, z, d) \in \Omega_{b_1} \times \Omega_{c_2} \times D$, for some $k_4 > 0$. Then, similar to (2.56)-(2.57), it can be shown that $\forall\ t \in [0, T_2)$,

$$\dot{\mathcal{W}} \leq -\frac{1}{\epsilon \lambda_{max}(\bar{P})} \mathcal{W} + \frac{2\|\bar{P}B\| k_4}{\sqrt{\lambda_{min}(\bar{P})}} \sqrt{\mathcal{W}}$$

$$\leq -\frac{\gamma_3}{\epsilon} \mathcal{W}, \quad \text{for } \mathcal{W} \geq \epsilon^2 \beta_4$$

where $\beta_4 = 16\|\bar{P}B\|^2 k_4^2 \lambda_{max}^2(\bar{P}) / \lambda_{min}(\bar{P})$ and $\gamma_3 = 1/2\lambda_{max}(\bar{P})$. Thus, as long as $(\zeta, z, d) \in \Omega_{b_1} \times \Omega_{c_2} \times D$ and $\mathcal{W} \geq \epsilon^2 \beta_4$, we have

$$\mathcal{W}(t) \leq \mathcal{W}(0) e^{-\gamma_3 t/\epsilon} \leq \frac{k_5}{\epsilon^{2r-2}} e^{-\gamma_3 t/\epsilon}$$

for some $k_5 > 0$. Let ϵ_1 be small enough such that

$$T_1(\epsilon) = \frac{\epsilon}{\gamma_3} \ln \left(\frac{k_5}{\beta_4 \epsilon^{2r}} \right) \leq \frac{1}{2} \bar{T}_2$$

$\forall\ \epsilon \in (0, \epsilon_1]$. This is possible since the left-hand side of the foregoing inequality tends to zero as $\epsilon \to 0$. Hence, for all $\epsilon \in (0, \epsilon_1]$, there exist $T_1 \leq \frac{1}{2}\bar{T}_2$ such that $\forall\ t \in [T_1, T_2)$, $\mathcal{W}(t) \leq \epsilon^2 \beta_4$ and this implies that $\|\chi\|$ is of order $O(\epsilon)$. In case $T_2 = \infty$, the foregoing analysis implies that $\|\chi\|$ is of order $O(\epsilon)$ for all $t \geq T_1$. •

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In the forthcoming analysis, our purpose is to establish that under output feedback ζ will enter the set $\Lambda_1 \stackrel{\text{def}}{=} \Omega_{\beta_1} \cap \{\zeta : \eta_0 |s| \leq \mu_2\}$ in finite time, where $\mu_2 < \mu$ is a positive constant independent of ϵ. We note that the set Λ_1 is chosen with $\mu_2 < \mu$ to ensure that when $|s| < \frac{\mu_2}{\eta_0}$, we will have $|\hat{s}| < \frac{\mu}{\eta_0}$ for sufficiently small ϵ; consequently, the nonlinearity $\eta(\hat{\zeta})\mathcal{N}_\mu(\hat{s}, \eta(\hat{\zeta}))$ will reduce to $\eta_0^2 \frac{\hat{s}}{\mu}$. To that end, let us study the slow equation (2.52) over the time interval $[T_1, T_2)$. Rewrite (2.52) as

$$\begin{aligned} \dot{\zeta} &= \mathcal{A}\zeta + \mathcal{B}[f_1(e,z,d) + g_1(e,z,d)\varphi^s(\sigma,e,\nu)] \\ &\quad + \mathcal{B}g_1(e,z,d)[\varphi^s(\sigma,\hat{e},\nu) - \varphi^s(\sigma,e,\nu)] \end{aligned} \tag{2.58}$$

and observe that $\|\hat{e} - e\| = \|D(\epsilon)\chi\| \leq k_6\epsilon$, $(\|D(\epsilon)\| = 1, \forall\ \epsilon \leq 1)$. Since Ω_{b_1} is in the interior of Ω_{c_1}, choosing ϵ small enough ensures that whenever $\zeta \in \Omega_{b_1}$, $\hat{\zeta} \in \Omega_{c_1}$. Equation (2.58) can be viewed as a perturbation of the closed-loop equation under state feedback over the time period $[T_1, T_2)$, with the perturbation term of the order of $\|\chi\|$. Define $\Omega_\epsilon = \{\mathcal{W} \leq \epsilon^2\beta_4\}$, and recall that $\forall\ t \in [T_1, T_2)$, $\|\chi\|$ is of order $O(\epsilon)$. Hence we can revise the estimate of $\dot{V}$ given in (2.28) to

$$\dot{V} \leq -\|\zeta\|^2 + \frac{\mu}{4} + k_7\epsilon, \quad \forall\ (\zeta, z, \chi, d) \in \Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon \times D$$

for some $k_7 > 0$. For

$$\epsilon \leq (\alpha - 1)\frac{\mu}{8k_7} \tag{2.59}$$

we obtain

$$\dot{V} \leq -\frac{\mu}{8}(\alpha - 1), \ \forall\ V \geq \beta_1$$

This shows that Ω_{β_1} is a positively invariant set. Since $\beta_1 \leq b_1$, the trajectory $(\zeta(t), z(t), \chi(t))$ can not leave the set $\Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon$ through the boundary $V(\zeta) = b_1$. Repeating previous arguments, it can be shown that the set $\Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon$ is a

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positively invariant set, and trajectories inside $\Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon$ reach the positively invariant set $\mathcal{R}_\mu \times \Omega_\epsilon \subset \Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon$ in finite time. Therefore, the set $\Omega_{b_1} \times \Omega_{c_2} \times \Omega_\epsilon$ has no finite exit time, i.e., $T_2 = \infty$; hence, $\|\chi\|$ is $O(\epsilon)$ for all $t \geq T_1$.

The next step is to establish the attractivity of $S_1 \stackrel{\text{def}}{=} \{\zeta : \eta_0 |s| \leq \mu_2\}$. By repeating the analysis carried out for the state feedback case, taking into consideration the effect of $O(\epsilon)$ perturbation due to output feedback, one can show that, inside $\mathcal{R}_\mu \times \Omega_\epsilon$,

$$
\begin{aligned}
\dot{V} &= 2s[\mathcal{B}^T P(\mathcal{A} + \mathcal{B}K)\zeta + \mathcal{B}^T P\mathcal{B}(f_1 + g_1\varphi^s - K\zeta)] \\
&\quad + 2s\mathcal{B}^T P\mathcal{B} g_1[\varphi^s(\sigma, \hat{e}, \nu) - \varphi^s(\sigma, e, \nu)]
\end{aligned}
\tag{2.60}
$$

The first bracketed term of (2.60) is the expression we had under state feedback, while the second term represents the perturbation due to output feedback. Therefore, using (2.37) we obtain

$$
\dot{V} \leq 2|s|\, p(\eta_0 - \mu_1) - 2sp\eta_0\mathcal{N}_\mu + 2p|s|\, k_6 k_8 L_1 \epsilon \tag{2.61}
$$

where L_1 is a Lipschitz constant for $\varphi^s(\zeta, \nu)$ and $k_8 = \max\{|g_1(e, z, d)| : (\zeta, z, d) \in \mathcal{R}_\mu \times D\}$. Notice that L_1 depends on $1/\mu$. It can be shown that for all $\epsilon \leq \mu_1/2k_6k_8L_1$ we have

$$
\dot{V} \leq -\frac{1}{2}|s|\, p\mu_1, \text{ when } \eta_0 |s| \geq \mu_2
$$

where $\mu_2 = \mu(1 - \frac{\mu_1}{4\eta_0}) < \mu$. Thus, we obtain the desired result, namely the trajectory will enter $\Lambda_1 \times \Omega_{\beta_2} \times \Omega_\epsilon$ in finite time and remain inside thereafter. By choosing ϵ small enough we can ensure that for all $(\zeta, z, \chi) \in \Lambda_1 \times \Omega_{\beta_2} \times \Omega_\epsilon$, we have $\eta(\hat{\zeta}) = \eta_0$ and $\mathcal{N}_\mu(\hat{s}, \eta_0) = \eta_0\frac{\hat{s}}{\mu}$. Therefore inside $\Lambda_1 \times \Omega_{\beta_2} \times \Omega_\epsilon$ the closed-loop system is given by

$$
\dot{\tilde{\zeta}} = (\mathcal{A} + \mathcal{B}K)\tilde{\zeta} + \mathcal{B}[f_1 + g_1\varphi^s(\zeta, \nu) - K\tilde{\zeta}] + \mathcal{B}g_1[\varphi^s(\hat{\zeta}, \nu) - \varphi^s(\zeta, \nu)] \tag{2.62}
$$

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$$\dot{z} = \phi_0(e,z,d) \tag{2.63}$$

$$\epsilon\dot{\chi} = (A-HC)\chi + \epsilon B[f_1 + g_1\varphi^s(\zeta,\nu)] + \epsilon B g_1[\varphi^s(\hat{\zeta},\nu) - \varphi^s(\zeta,\nu)] \tag{2.64}$$

where in $\varphi^s(\zeta,\nu)$ and $\varphi^s(\hat{\zeta},\nu)$ above, $\mathcal{N}_\mu(\eta,s) = \eta_0\frac{s}{\mu}$ and $\mathcal{N}_\mu(\eta,\hat{s}) = \eta_0\frac{\hat{s}}{\mu}$. The system (2.62)-(2.64) is a singular perturbation of (2.39) and has an equilibrium point at $(\tilde{\zeta}=0, z=z^0, \chi=0)$.

The last step in our analysis is to show asymptotic regulation, i.e., all trajectories inside $\Lambda_1 \times \Omega_{\beta_2} \times \Omega_\epsilon$ approach the equilibrium point as $t \to \infty$. Let

$$\tilde{W}(\tilde{\zeta},\tilde{z},\chi) = \tilde{V}(\tilde{z}) + \lambda(\tilde{\zeta}^T P\tilde{\zeta})^\gamma + (\chi^T\bar{P}\chi)^\gamma$$

be a Lyapunov function candidate for the system (2.62)-(2.64) where λ and γ are chosen as in the state feedback case. It can be shown that

$$\begin{aligned}\dot{\tilde{W}} \leq\ & -\frac{\bar{\alpha}}{2c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} - \frac{\lambda\rho_0}{2\mu}|\tilde{s}|^{2\gamma} \\ & +\lambda a_6(\gamma_0)^{p_1}\|\tilde{\zeta}\|^{2\gamma} + \frac{\lambda a_6}{\gamma_0}\|\chi\|^{2\gamma} - \frac{a_8}{\epsilon}\|\chi\|^{2\gamma} \\ & +a_9\|\chi\|^{2\gamma} + a_{10}\|\chi\|^{2\gamma-1}|f_1 + g_1\varphi^s|\end{aligned} \tag{2.65}$$

for all $\mu \leq \mu^*$, where

$$a_6 = 2\gamma\|P\|^\gamma k_8 L_1, a_8 = \gamma(\lambda_{min}(\bar{P}))^{\gamma-1},$$

$$a_9 = 2\gamma k_8 L_1\|\bar{P}\|^\gamma, a_{10} = 2\gamma\|\bar{P}\|^\gamma,$$

and γ_0 is some arbitrary positive constant. Define $F(\sigma,e,z,d) = f_1(e,z,d) + g_1(e,z,d)\varphi^s(\sigma,e,\nu)$. Then, it can be shown that inside $\Lambda_1 \times \Omega_{\beta_2} \times \Omega_\epsilon$, $F(\bar{\sigma},0,z,d) = \delta(0,z,d)$. Hence by adding and subtracting the term $F(\bar{\sigma},0,z,d)$ to $F(\sigma,e,z,d)$ we ob

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obtain from (2.40) and smoothness of the nonlinearities

$$\left.\begin{array}{rcl} |F(\sigma,e,z,d)| & \leq & k_1\psi^a(\tilde{z}) + a_7\|\tilde{\zeta}\| \\ & \leq & \bar{a}_7(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|) \end{array}\right\} \qquad (2.66)$$

where $\bar{a}_7 > 0$. Using (2.66),(2.41) in (2.65) we get

$$\begin{aligned} \dot{\mathcal{W}} \leq & -\frac{\bar{\alpha}}{2c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} - \frac{\lambda\rho_0}{2\mu}|\tilde{s}|^{2\gamma} \\ & +\lambda a_6(\gamma_0)^{p_1}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} + \frac{a_{10}\bar{a}_7}{\mu_3}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} \\ & -[\frac{a_8}{\epsilon} - a_9 - a_{10}\bar{a}_7(\mu_3)^{p_1} - \frac{\lambda a_6}{\gamma_0}]\|\chi\|^{2\gamma} \end{aligned}$$

where μ_3 is some arbitrary positive constant. Choose γ_0 small enough and μ_3 large enough so that

$$\frac{\bar{\alpha}}{2c_0} - \lambda a_6(\gamma_0)^{p_1} - \frac{a_{10}\bar{a}_7}{\mu_3} \geq \frac{\bar{\alpha}}{4c_0}$$

Finally, choose $\epsilon^* \leq 1$ small enough so that

$$\frac{a_8}{\epsilon^*} - a_9 - a_{10}\bar{a}_7(\mu_3)^{p_1} - \frac{\lambda a_6}{\gamma_0} \geq \frac{a_8}{2\epsilon^*}$$

Then for all $\epsilon \leq \epsilon^*$ and for all $\mu \leq \mu^*$, we have

$$\dot{\mathcal{W}} \leq -\frac{\bar{\alpha}}{4c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} - \frac{\lambda\rho_0}{2\mu}|\tilde{s}|^{2\gamma} - \frac{a_8}{2\epsilon}\|\chi\|^{2\gamma}.$$

Our conclusions are summarized in the following theorem.

Theorem 2.1 *Suppose that Assumptions 2.1 through 2.6 are satisfied and consider the closed-loop system formed of the system (2.10), the observer (2.48) and the output feedback control* $u = \varphi^s(\hat{\zeta},\nu)$. *Suppose* $(\zeta(0), z(0)) \in \Omega_{\bar{b}_1} \times \Omega_{c_2}$ *and* $\hat{\zeta}(0) = \begin{bmatrix} \sigma(0) \\ \hat{e}(0) \end{bmatrix}$ *is bounded. Then, there exist* μ^* *such that* $\forall\ \mu \in (0,\mu^*]$ *there is* $\epsilon^* = \epsilon^*(\mu) \leq 1$ *such that*

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$\forall\ \epsilon \in (0, \epsilon^*]$ *all the state variables of the closed-loop system are bounded and* $e(t) \rightarrow 0$ *as* $t \rightarrow \infty$.

We remark that although the region of attraction in the transformed coordinates is independent of the parameters, it will be parameter-dependent in the original coordinates due to the dependence of the transformation $(\Psi \circ T)^{-1}$ on the parameters. To require this to hold uniformly in the parameters in the original coordinates as well will further restrict the allowable size of uncertainty.

2.5 Semi-global Regulation

In this section we will consider the semi-global case. Precisely, given any compact set of initial conditions of ξ, we can design an output feedback controller that ensures output regulation for all initial states in that set.

Corollary 2.1 *Suppose that* $U_\theta = R^n$ *for all* $\theta \in \Theta$, *the functions* $\alpha_i(.), (i = 1, 2, 3)$ *and* $\gamma_1(.)$ *are class* $\mathcal{K}_\infty$ *functions, Assumptions 2.1, 2.2, 2.3, 2.4 and 2.6 are satisfied, and that Assumption 2.5 is satisfied for all* $c_1, c_2 > 0$ *such that*

$$\alpha_4\left(\sqrt{c_1/\lambda_{min}(P)}\right) \leq c_2,$$

then for any given compact set N *and for all initial states* $\xi(0) \in N$ *there exists* $\mu^* > 0$ *such that* $\forall\ \mu \in (0, \mu^*]$ *there is* $\epsilon^* = \epsilon^*(\mu) \leq 1$ *such that* $\forall\ \epsilon \in (0, \epsilon^*]$, *the states of the closed-loop system, consisting of the system (2.10), the observer (2.48), and the output feedback control* $u = \varphi^s(\hat{\zeta}, \nu)$, *are bounded and* $e(t) \rightarrow 0$ *as* $t \rightarrow \infty$.

Proof: In the previous section we have already proved this result for initial states in $\Omega_{\bar{b}_1} \times \Omega_{c_2}$, $0 < \bar{b}_1 < b_1 < c_1$. Thus, It is enough to show that $\bar{b}_1, c_2$ can be chosen arbitrarily large to include any compact set N in $\Omega_{\bar{b}_1} \times \Omega_{c_2}$. We have $U_\theta = R^n$ and $T(U_\theta, \theta) = R^n$, for all $\theta \in \Theta$; clearly, Assumption 2.3 will be automatically satisfied.

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Hence, $M_d = R^{n+1}$ for all $d \in D$. Moreover, $\mathcal{S} = R^{r+1}$ and $\mathcal{U} = R^{n-r}$. Therefore, $r_i, (i = 1, 2)$ can be chosen arbitrarily large. It follows from (2.22),(2.21) that c_1, c_2 can be chosen arbitrarily large, respectively. Consequently, $\bar{b}_1$ can be chosen arbitrarily large. •

2.6 Stabilization Result

In solving the regulation problem we exercised special care in the choice of the control strategy and the analysis tools so as to arrive at a semi-global regulation result that avoids four restrictive features that appear in some earlier global or semi-global output feedback control of input-output linearizable systems; namely, global growth (Lipschitz) conditions, global exponential stability of the zero dynamics, linearity of the zero dynamics and restrictions on the way y and its derivatives appear in the zero dynamic equation (2.12). To put our contribution in a better perspective, we specialize our design to the thoroughly investigated output feedback stabilization problem. We consider the system (2.1) with $\theta = \theta_0$ (known) and $f(0, \theta_0) = h(0, \theta_0) = 0$, i.e., the origin is an open-loop equilibrium point. The goal is to design an output feedback control that achieves global or semi-global stabilization of the origin. Earlier results are available in [34, 35, 41, 42, 12, 40, 37, 36]. In [34], ϕ_0, f_1 were required to satisfy global Lipschitz conditions and the zero dynamics were globally exponentially stable. Also, only the output y was allowed to appear in ϕ_0. In [35] they used the same controller of [34] and allowed the output and its time derivatives to appear in ϕ_0, but required global exponential stability of the zero dynamics and imposed global Lipschitz conditions on ϕ_0, f_1, as in [34]. In [41] and [42] a solution was presented for a certain class of nonlinear systems that is characterized by geometric conditions but the zero dynamics, in suitable coordinates, were linear. In [12], a dynamic output feedback controller was designed to stabilize a fully linearizable nonlinear system

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where the zero dynamics, in suitable coordinates, were also linear. The results of [34], [35], [41] and [42] were global in nature. In a recent work, Teel and Praly [37], [36] presented a semi-global solution without imposing global growth conditions, but they allowed only the output y in ϕ_0. Also in [43] Praly and Jiang considered systems that admit a global normal form and, as [37], [36], only the output and not its time derivatives were allowed to appear in the nonlinearities. The controllers of [34], [35], and [37] were linear.

To develop our control strategy for this stabilization problem, we start from the normal form (2.5). The main difference from our regulation control is that we do not need to use integral control; recall that in the regulation problem we try to stabilize an uncertain equilibrium point where the tracking error is zero; a task that requires integral action. Without integral control, and with the reference $\nu = 0$, equations (2.11)-(2.12) coincide with (2.5). From this point on we proceed to design the output feedback controller as in the regulation problem case, leading to a semi-global result that does not suffer from any of the four restrictions stated at the beginning of this section. A concise description of this stabilization result can be found in [44].

2.7 Time-Varying External Signals

Although the integral control is used to ensure asymptotic tracking when the disturbances and references are constant, one would intuitively expect it to be effective for some cases of time-varying signals. Two such cases are time-varying signals which tend to constant limits as $t \rightarrow \infty$, and slowly-varying signals. In this section we study the tracking problem for these two cases. We start with time-varying signals which tend to constant limits. We assume that $d(t) \in D$, $\forall\, t \geq 0$ and $d(t) \rightarrow \bar{d}$ as $t \rightarrow \infty$. Since D is compact, $\bar{d} \in D$. Let $\tilde{x} \stackrel{def}{=} \left[\ \sigma - \bar{\sigma}(\bar{d}), (x - \bar{x}(\bar{d}))^T, (z - \lambda(\bar{d}))^T, \chi^T \ \right]^T$ and $\tilde{d} \stackrel{def}{=} d - \bar{d}$, where $\bar{\sigma}(.), \bar{x}(.)$ and $\lambda(.)$ denote the equilibrium point that corresponds to

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$d = \bar{d}$. Define $D_d = R \times \Psi^{-1}(N_d) \times R^r$; then, in the $\tilde{x}$ coordinates, the closed-loop system can be represented by

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \bar{d} + \tilde{d}) \tag{2.67}$$

$\forall\ \tilde{x} \in D_{r_1} \stackrel{def}{=} \{\tilde{x} \in R^{n+r+1} : \|\tilde{x}\| \leq r_1\}$ where r_1 is chosen small enough to ensure that $\forall\ \tilde{x} \in D_{r_1}$, $(x^T, z^T)^T \in \Psi^{-1}(N_d)$. The expression for $\tilde{f}$ can be easily determined from the previous section. The right hand side of (2.67) can be rewritten as

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \bar{d}) + \tilde{g}(\tilde{x}, d, \bar{d}) \tag{2.68}$$

where $\tilde{g}(\tilde{x}, d, \bar{d}) = \tilde{f}(\tilde{x}, d) - \tilde{f}(\tilde{x}, \bar{d})$. Due to the smoothness of $\tilde{f}$ we have $\|\tilde{g}(.)\| \leq m_1\|\tilde{d}\|$, $m_1 > 0$. Since $\bar{d} \in D$, we know that $\tilde{x} = 0$ is an asymptotically stable equilibrium point of the system

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \bar{d}) \tag{2.69}$$

Hence, using the Converse Lyapunov Theorems (see [31, Theorem 4.7]) there exists a Lyapunov function $V_0 : D_{r_0} \stackrel{def}{=} \{\tilde{x} \in R^{n+r+1} : \|\tilde{x}\| \leq r_0\} \rightarrow R^+$ for the system (2.69) that satisfies the inequalities:

$$\alpha_4(\|\tilde{x}\|) \leq V_0(\tilde{x}) \leq \alpha_5(\|\tilde{x}\|)$$

$$\frac{\partial V_0}{\partial \tilde{x}} \tilde{f}(\tilde{x}, \bar{d}) \leq -\alpha_6(\|\tilde{x}\|)$$

$$\|\frac{\partial V_0}{\partial \tilde{x}}\| \leq \alpha_7(\|\tilde{x}\|)$$

where r_0 is chosen small enough to ensure that the solution is always in D_{r_1} and α_i, $(i = 4, 5, 6, 7)$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. Now, let $\{T_n\}$ be a sequence such that $T_n \rightarrow \infty$ as $n \rightarrow \infty$ and $T_1 = t_0 = 0$. With V_0 as a Lyapunov function canc

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candidate for the system (2.68), it can be shown that

$$\dot{V}_0 < -(1-\theta_0)\alpha_6(\|\tilde{x}\|), \ \forall \ \|\tilde{x}\| \geq \alpha_6^{-1}(\frac{m_1\alpha_7(r_1)}{\theta_0}\|\tilde{d}\|), \ 0 < \theta_0 < 1 \tag{2.70}$$

From (2.70) we conclude that

$$\dot{V}_0 < -(1-\theta_0)\alpha_6(\|\tilde{x}\|), \ \forall \ \|\tilde{x}\| \geq \alpha_6^{-1}(\frac{m_1\alpha_7(r_1)}{\theta_0}\left|\tilde{d}\right|_{T_n}), \ \forall \ t \geq T_n \tag{2.71}$$

where $\left|\tilde{d}\right|_{T_n} = \sup\{\|\tilde{d}(t)\| \ : \ t \geq T_n\}$. Applying [31, Theorem 4.10] it can be shown that the solution $\tilde{x}(t)$ satisfies

$$\|\tilde{x}(t)\| \leq \bar{\beta}(\|\tilde{x}(T_n)\|, t - T_n) + \bar{\kappa}(\left|\tilde{d}\right|_{T_n}), \ \forall \ t \geq T_n$$

where $\bar{\beta}$, $\bar{\kappa}$ are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively, and

$$\|\tilde{x}(T_n)\| \leq \bar{\beta}(\|\tilde{x}(0)\|, T_n) + \bar{\kappa}(\left|\tilde{d}\right|_{T_1})$$

$\forall \ \tilde{x}(0) \in D_{r_0}$. Since $\bar{\beta}(\|\tilde{x}(T_n)\|, t) \to 0$ as $t \to \infty$, given any $\epsilon > 0$, there exist n_0 such that $\forall \ t \geq T_{n_0}$

$$\bar{\beta}(\|\tilde{x}(T_n)\|, t - T_n) < \epsilon/2$$

Also, since $\tilde{d} \to 0$ as $t \to \infty$, there exist n_1 such that $\forall \ t \geq T_{n_1}$

$$\|\tilde{d}(t)\| < \bar{\kappa}^{-1}(\epsilon/2) \Rightarrow \left|\tilde{d}\right|_{T_{n_1}} < \bar{\kappa}^{-1}(\epsilon/2)$$

Therefore, there exists $T > 0$ such that $\|\tilde{x}(t)\| < \epsilon$, $\forall \ t \geq T$. This implies that $\tilde{x}(t) \to 0$ as $t \to \infty$. Observe that $\tilde{x}_2(t) = e_1(t)$, therefore $e_1(t) \to 0$ as $t \to \infty$ or $y(t) \to \bar{\nu}$ as $t \to \infty$.

Next, we consider slowly-varying external signals where we assume that $\dot{d}$ is bounded, i.e.,

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i.e., $\|\dot{d}\| \leq c_3$. Recall from (2.67) that our system, in the $\tilde{x}$ coordinates, can be represented as

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, d) \tag{2.72}$$

where $\tilde{x}, \tilde{f}$ are as defined earlier and $d \in D$ is a parameter. We have shown previously that $\forall\ d \in D$, the equilibrium point $\tilde{x} = 0$ of (2.72) is asymptotically stable. Since D is closed and bounded, it follows from [45, Auxiliary Lemma] that $\forall\ \epsilon > 0$ there exists $\delta(\epsilon)$(independent of d) such that $\forall\ d \in D$

$$\|\tilde{x}(t)\| < \epsilon,\ \forall\ t \geq t_0,\ \text{if}\ \|\tilde{x}(t_0)\| < \delta(\epsilon)$$

and

$$\lim_{t\to\infty} \tilde{x}(t) = 0,\ \text{if}\ \|\tilde{x}(t_0)\| < \delta(\epsilon)$$

Here, we need to establish that the passage to the limit is uniform in d which was not established in the auxiliary lemma of [45]. Given $\eta > 0$ we have $\forall\ d \in D$

$$\|\tilde{x}(t,d)\| < \eta,\ \forall\ t \geq t_0,\ \text{if}\ \|\tilde{x}(t_0,d)\| < \delta(\eta) \tag{2.73}$$

Let $d_0 \in D$. Since $\tilde{x}(t,d_0) \to 0$ as $t \to \infty$, there exists $T(\eta, d_0) \geq t_0$ such that

$$\|\tilde{x}(t,d_0)\| < \delta/2,\ \forall\ t \geq T$$

Since the solution $\tilde{x}(t,d)$ depends continuously on the parameter d, given $\alpha > 0$ there exists $\gamma > 0$ such that $\forall\ d \in \{d : \|d - d_0\| < \gamma\}$,

$$\|\tilde{x}(t,d) - \tilde{x}(t,d_0)\| < \alpha,\ \forall\ t \in [t_0, T]$$

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This implies that, at $t = T$, $\|\tilde{x}(T, d)\| < \delta$ (by choosing $\alpha \leq \delta/2$).

Hence, from (2.73) and since the system (2.72) is autonomous, we conclude that $\|\tilde{x}(t, d)\| < \eta$, $\forall\ t \geq T$, for all $d \in \{d : \|d - d_0\| < \gamma\}$. Since D is compact, it is covered by a finite number of these neighborhoods. Hence, there exists $\bar{T}(\eta)$ such that

$$\|\tilde{x}(t, d)\| < \eta,\ \forall\ t \geq \bar{T},\ \forall\ d \in D$$

This shows that $\tilde{x} = 0$ is asymptotically stable uniformly in d. Following the proof of [31, Lemma 4.2], it can be shown that there exist a class $\mathcal{K}$ function, κ_1, and a class $\mathcal{L}$ function, σ_1 (see [46, pp 7] for the definition of class $\mathcal{L}$ functions) , independent of d such that

$$\|\tilde{x}(t, d)\| \leq \kappa_1(\|\tilde{x}(t_0)\|)\sigma_1(t - t_0),\ \forall\ d \in D$$

$\forall\ \tilde{x}(t_0) \in D_{r_2} \stackrel{def}{=} \{\tilde{x} \in R^{n+r+1} : \|\tilde{x}\| \leq r_2\} \subset D_{r_1}$. Thus, assumptions 1,2 and 3 of Hoppenstead's lemma [47, Lemma 1] are satisfied. By an application of this lemma, together with an argument similar to the one used in [47], we conclude that

$$\|\tilde{x}(t, d)\| \leq \bar{\beta}_1(\|\tilde{x}(t_0)\|, t) + \bar{\gamma}_1(|c_3|),\ \forall\ d \in D,\ \forall\ t \geq t_0$$

where $\bar{\beta}_1$, $\bar{\gamma}_1$ are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively. Hence, the tracking error is ultimately bounded by a class $\mathcal{K}$ function of c_3. Therefore, for slowly-varying signals where $|\dot{d}|$ is small, the tracking error will be small. If, in addition, $\dot{d} \to 0$ as $t \to \infty$, then an argument similar to the one used earlier in this section will show that $\tilde{x}(t) \to 0$ as $t \to \infty$ and this implies that the tracking error $\to 0$ as $t \to \infty$.

2.8 Conclusions

The basic idea of this work is the use of integral control to achieve asymptotic regulation and disturbance rejection for constant references and disturbances. By introd

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troducing the integrator we create an equilibrium point at which the tracking error is zero, since the integrator is driven by the tracking error. The main task of the controller after that is to stabilize the equilibrium point, which is a challenging task because the equilibrium point depends on the unknown parameter θ. This is not an issue in linear servomechanism where stabilizing the equilibrium point does not require knowledge of that point. There it is merely stabilization of a matrix which represents the homogeneous part of the closed-loop system. If we were interested in local results that would hold only for sufficiently small initial states and external signals, the stabilization problem for the nonlinear servomechanism would reduce to stabilization of a matrix. It is the desire to obtain regional and semi-global results that makes this stabilization problem a challenging one. Our approach to stabilize this unknown equilibrium point is to use a nonlinear robust controller to drive the trajectories toward the point $(\zeta, z) = (0, z^0)$. As far as that nonlinear controller is concerned, it is not trying to stabilize the closed-loop equilibrium point. It is only pushing trajectories to a small residual set around the point $(0, z^0)$, and by doing so it automatically pushes the closed-loop equilibrium point inside that set. Since the size of the residual set can be made arbitrarily small by the choice of a certain design parameter, we choose it small enough to ensure that inside the residual set the controller will act as a high-gain controller that stabilizes the closed-loop equilibrium point. At no point in our design were we required to know the exact location of the unknown closed-loop equilibrium point. This whole idea would not have been possible if we were to use state feedback, since the states in the error coordinates are again dependent on θ, and here comes the role of the high-gain observer and the globally bounded control idea of [12] to implement the controller using only measurement of the tracking error.

An important reason for using integral control is its robustness to model uncertainties. For linear servomechanism, it was shown that for any plant perturbation that does

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not destroy the asymptotic stability of the closed-loop system, asymptotic tracking and disturbance rejection will be achieved. In this work, as long as the uncertain parameter θ does not change the basic structure of the plant; namely, it preserves the relative degree and the minimum-phase property, asymptotic regulation will be achieved.

The control strategy used to solve the regulation problem was shown to solve the corresponding stabilization problem when the origin is an open-loop equilibrium point. Also, the semi-global result comes as a natural extension of the regional one. Our global assumption (for the semi-global case) relaxes some of the global growth conditions available in the literature, as we did not require exponential stability of the zero dynamics nor did we require the nonlinearities to satisfy global Lipschitz conditions. Asymptotic tracking was also achieved, using integral control, for a certain type of time-varying signals. these signals may represent references and disturbances as well as some time-varying system parameters.

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CHAPTER 3

Robust Tracking for Nonlinear Systems Represented By Input-Output Models

3.1 Introduction

In linear servomechanism theory, conditions under which asymptotic tracking is achieved using state or output feedback control have been completely established; see, e.g., [13, 14, 15, 16]. The nonlinear servomechanism problem, on the other hand, is not completely understood and continues to be an active area of research. It has been studied before under various assumptions on the class of systems being studied and/or the exogenous signals being tracked and/or rejected; see, e.g., [17, 18, 19, 20, 21, 23]. In this work we extend our previous work of Chapter 2. where we studied the regulation problem, to the more general tracking problem where the exogenous signals are time-varying. We consider a class of SISO nonlinear systems that are represented by input-output models. Using the idea of [10], we extend the dynamics of the system by augmenting a series of integrators at the input side. This makes the derivatives of the input available for feedback. With this approach we overcome one of the lim-

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itations of Chapter 2, namely being able to use only partial state feedback. In our design methodology we start by identify the internal model; a task that hinges on two basic assumptions. The first is the solvability of a partial differential equation, and the second is an equality that will be automatically satisfied if the nonlinearity is of polynomial type. After augmenting the original system with the servo-compensator, we design a state feedback control that stabilizes the augmented system. However, instead of using a specific stabilizing control, which is another limitation of Chapter 2, our objective here is to allow flexibility in the design of state feedback control. Toward that end, we present a set of conditions that a state feedback control should satisfy in order to be stabilizing. Then we proceed to design an observer to recover the asymptotic tracking properties achieved under state feedback. For that we used the technique introduced in [12] that comprises a high-gain observer with a globally bounded implementation of the control.

This chapter is organized as follows: In Section 3.2, we present some preliminaries and some basic assumptions. In Section 3.3, we give general characteristics of a state feedback stabilizing control. In Section 3.4, we present the observer structure and give our main result. In Section 3.5 we give examples of controllers. Since state-space models are prevalent in control theory, in Section 3.6, we give conditions under which a given state space model will have the input-output model assumed in our work. In Section 3.7, we illustrate our results by giving a special input-output model for which most of our assumptions are automatically satisfied. In Section 3.8 we present a design example with simulation results. Finally, we make some concluding remarks in Section 3.9.

3.2 Preliminaries

We consider a SISO nonlinear system which has an input-output model given by the n-th order differential equation:

$$\begin{aligned} y^{(n)} &= \bar{f}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(m-1)}, w(t)) \\ &+ \bar{g}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(m-1)}, w(t))u^{(m)} \end{aligned} \tag{3.1}$$

where $y^{(i)}$ and $u^{(i)}$ are the i-th derivatives, with respect to time, of y and u, respectively, $w(.)$ is a continuous time-varying disturbance signal, assumed to be contained in a compact set $D \subset R^p$, and $\bar{g}(.) \neq 0$, for all possible values of its arguments in a domain $U_1 \times U_2 \times D$ where $U_1 \subset R^n$ and $U_2 \subset R^m$; let $U = U_1 \times U_2$.

Let $r(t)$ be a time-varying reference signal that together with $w(t)$ are generated by a p_1-dimensional exosystem

$$\dot{\nu}(t) = S_0 \nu(t) \tag{3.2}$$

where all the eigenvalues of S_0 are on the imaginary axis and distinct. Clearly, $\nu(t)$ belongs to a compact set $D_1 \subset R^{p_1}$.

We note that for feedback linearizable systems with relative degree less than n the state feedback control, in some of the existing literature, uses only a part of the state vector, i.e., the states that describe the observable subsystem; see for example [37, 36, 44, 48] and the work presented earlier in Chapter 2. To avoid this in the present work, we utilize the idea advanced in [10] of augmenting a series of m integrators at the input side of the system. Define:

$$\begin{aligned} x_{i+1} &= y^{(i)}, \ i = 0, \cdots, n-1 \\ \zeta_{j+1} &= u^{(j)}, \ j = 0, \cdots, m-1 \\ v &= u^{(m)} \end{aligned}$$

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Augmenting a series of m integrators at the input side, and using (x, ζ) as an (n+m)-dimensional state vector, a state model of the system is given by

$$\left.\begin{array}{rcl} \dot{x}_i &=& x_{i+1},\ i = 1, \cdots, n-1 \\ \dot{x}_n &=& \bar{f}(x, \zeta, \nu(t)) + \bar{g}(x, \zeta, \nu(t))v \\ \dot{\zeta}_j &=& \zeta_{j+1},\ j = 1, \cdots, m-1 \\ \dot{\zeta}_m &=& v \\ y &=& x_1 \end{array}\right\} \tag{3.3}$$

where $x = (x_1, \cdots, x_n)^T$, $\zeta = (\zeta_1, \cdots, \zeta_m)^T$. We note that the state ζ is available for feedback. Hence in output feedback control, as it is the case here, we need only to estimate the state x. We rewrite the last m-equations of (3.3) as

$$\dot{\zeta} = A_2\zeta + B_2 v \tag{3.4}$$

where

$$A_2 = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{m\times m}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{m\times 1}$$

Observe that system (3.3) is input-output linearizable, but it is not in the normal form [30] due to the appearance of the control v in the last state equation. With this in mind, we state the following assumption

Assumption 3.1 *There exists a diffeomorphism*

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x \\ T_1(x, \zeta, \nu) \end{bmatrix} \stackrel{def}{=} T(x, \zeta, \nu) \tag{3.5}$$

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that maps (x,ζ) into (x,z) for all $\nu \in D_1$, and transforms the last m state equations of (3.3) into

$$\dot{z} = \psi_0(x, z, \nu(t)) \tag{3.6}$$

which, together with the first n state equations of (3.3), define a normal form which we assume to hold in the domain $\mathcal{D} \stackrel{def}{=} T(U \times D_1) \subset R^{n+m}$.

Remark 3.1 *The above assumption requires the normal form to be valid on a given region rather than being valid only locally. A similar assumption was used in Chapter 2; see Assumption 2.2 and the subsequent discussion for more details. If $\bar{g}(.)$ is independent of x_n and ζ_m, i.e., $\bar{g}(.) = \bar{g}(x_1, \cdots, x_{n-1}, \zeta_1, \cdots, \zeta_{m-1}, w(t))$, then the transformation*

$$\begin{aligned} z_i &= \zeta_i, \ i = 1, \cdots, m-1 \\ z_m &= \zeta_m - \frac{x_n}{\bar{g}} \end{aligned}$$

will satisfy Assumption 3.1.

Define the tracking error $e_1 = x_1 - r(t)$. Let

$$e_{i+1} = \dot{e}_i = x_{i+1} - r^{(i)}, \ i = 1, \cdots, n-1 \tag{3.7}$$

Then, in the new coordinates, the system (3.3) becomes

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \ i = 1, \cdots, n-1 && (3.8) \\ \dot{e}_n &= \bar{f}(e_1 + r(t), \cdots, e_n + r^{(n-1)}(t), \zeta, \nu(t)) \\ &+ \bar{g}(e_1 + r(t), \cdots, e_n + r^{(n-1)}(t), \zeta, \nu(t))v - r^{(n)}(t) \\ &\stackrel{def}{=} f(e, \zeta, \nu(t)) + g(e, \zeta, \nu(t))v && (3.9) \\ \dot{\zeta} &= A_2\zeta + B_2 v && (3.10) \end{aligned}$$

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$$y_m = e_1 \tag{3.11}$$

where $e = (e_1, \cdots, e_n)^T$ and y_m is the measured output.

Our objective is to identify the internal model. Of particular interest to us are the equations that govern the system (3.8)-(3.11) on the zero-error manifold. These are the equations that will result when we restrict $e_1 \equiv 0$, $\Rightarrow e \equiv 0$. Hence, we get

$$f(0,\zeta,\nu(t)) + g(0,\zeta,\nu(t))v = 0 \tag{3.12}$$

$$\dot{\zeta} = A_2\zeta + B_2\left[\frac{-f(0,\zeta,\nu(t))}{g(0,\zeta,\nu(t))}\right] \tag{3.13}$$

Equation (3.13) is referred to in the literature as the *tracking dynamics* [8]. It gives the zero dynamics of the system (3.8)-(3.11) when the output is the tracking error. In general, the state feedback controller will take the form $v = g_0^{-1}(e,\zeta,\nu(t))[-f_0(e,\zeta,\nu(t)) + \tilde{v}]$, where $g_0(.)$ and $f_0(.)$ are nominal functions of $g(.)$ and $f(.)$, respectively, that are allowed to depend on e, ζ and $\nu(t)$ (possibly the reference signal and its derivatives). We observe that although the state ζ is measurable, the state z is not so, due to the dependence of the transformation T on the disturbance. Substituting the expression for v in (3.12), we obtain

$$\begin{aligned} 0 &= g_0(0,\zeta,\nu(t))g^{-1}(0,\zeta,\nu(t))f(0,\zeta,\nu(t)) \\ &\quad -f_0(0,\zeta,\nu(t)) + \tilde{v} \end{aligned} \tag{3.14}$$

Assumption 3.2 *(a) There exists a unique mapping $\zeta = \lambda_0(\nu)$ which solves the partial differential equation*

$$\frac{\partial\lambda_0}{\partial\nu}S_0\nu = A_2\lambda_0(\nu) + B_2\left[\frac{-f(0,\lambda_0(\nu),\nu)}{g(0,\lambda_0(\nu),\nu)}\right] \tag{3.15}$$

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(b) There exist a $q \times q$ matrix S, $q \geq p_1$, with distinct eigenvalues on the imaginary axis, and a $1 \times q$ constant matrix Γ such that

$$g_0(0, \lambda_0(\nu(t)), \nu(t)) g^{-1}(0, \lambda_0(\nu(t)), \nu(t)) f(0, \lambda_0(\nu(t)), \nu(t))$$
$$-f_0(0, \lambda_0(\nu(t)), \nu(t)) = \Gamma \mathcal{V}(t) \tag{3.16}$$

$\forall\, t \in R$, where $\mathcal{V}(t)$ is a solution of the linear differential equation

$$\dot{\mathcal{V}}(t) = S\mathcal{V}(t) \tag{3.17}$$

Remark 3.2 *Note that if $\zeta = \lambda_0(\nu)$, on the zero-error manifold, then it follows that z is a well defined function of $\nu(t)$ through $T_1(.)$, e.g, $z = T_1(e_1 + r(t), \cdots, e_n + r^{(n-1)}(t), \lambda_0(\nu), \nu)|_{e=0} \stackrel{def}{=} \lambda(\nu)$. In the special case of constant exogenous signals that we studied in Chapter 2, Assumption 3.2(b) is automatically satisfied while 3.2(a) reduces to an assumption on the existence of an equilibrium point, see Assumptions 2.1 and 2.3.*

Assumption 3.2(b) was first used by Khalil [23] and it is fundamental to this work. Essentially this assumption requires the nonlinearities in the system to generate only a finite number of harmonics of the original modes as modeled by equation (3.2). This being the case, then the q-dimensional exosystem can be modeled by a linear differential equation. Clearly, $\mathcal{V}(t)$ belongs to a compact set; $D_2 \subset R^q$. To simplify the notation we will drop, from now on, the time variable t from $\nu(t)$ and $\mathcal{V}(t)$, although the fact remains that they are time dependent.

Remark 3.3 *Related to this assumption is the one used by [22]. The paper [22] considers a nonlinear system*

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where the signal ν *satisfies a linear equation* $\dot{\nu} = S_0\nu$ *and the matrix* S_0 *has all its eigenvalues on the imaginary axis. The paper assumes that for all admissible perturbations of the system, there exist functions* $\pi(\nu)$ *and* $\rho(\nu)$ *which satisfy the equations*

$$\frac{\partial \pi}{\partial \nu} S_0\nu = f(\pi(\nu)) + g(\pi(\nu))\rho(\nu) + p(\pi(\nu))\nu \qquad (3.18)$$

$$0 = h(\pi(\nu)) + q(\nu) \qquad (3.19)$$

and ρ *is a polynomial in* ν*. Applying these equations to our system in the error coordinates with* $\pi(\nu) = \begin{bmatrix} 0 \\ \lambda_0(\nu) \end{bmatrix}$ *and* $v = g_0^{-1}(-f_0 + \rho)$*, it can be verified that we obtain (3.15) and the function* ρ *is the left-hand side of (3.16). The fact that* ρ *is a polynomial in* ν *guarantees that (3.16) is satisfied.*

Rewrite the transformation (3.7) as

$$\begin{bmatrix} e \\ z \end{bmatrix} = \begin{bmatrix} x - L_1\nu \\ z \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \Psi_1(x,\nu) \\ z \end{bmatrix}$$

where L_1 is an $n \times p_1$ matrix. We assume the following

Assumption 3.3 *There exists a domain* $N = N_1 \times N_2$*, that contains the origin, such that* $\Psi_1^{-1}(N_1) \subset U_1$ *and* $T_1^{-1}(N_2) \subset U_2$ *for all* $\nu \in D_1$*.*

Basically, this assumption will restrict the size of the signal ν. To see this, consider the transformations $e = x - r(t)$, $z = \zeta - xr(t)$, and suppose the sets U_1, U_2 and D_1 are given by $U_1 = \{x : |x| \leq 1\}$, $U_2 = \{\zeta : |\zeta| \leq 1\}$ and $D_1 = \{r(t) : |r(t)| \leq r_0\}$. Clearly $\forall\, r_0 > 1$ both N_1 and N_2 will be empty.

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Let $X_1 = R^q \times N_1$. Also, define a new variable $\tilde{z} = z - \lambda(\nu)$. For the purpose of stating the next assumption, we need to express the system in the $(e, \tilde{z})$-coordinates. First, observe that from (3.6) we have

$$\dot{z} = \phi(e, z, \nu) \tag{3.20}$$

where $\phi(.) = \psi_0(e + L_1\nu, z, \nu)$. Then the system in the $(e, \tilde{z})$-coordinates becomes

$$\left.\begin{array}{rcl} \dot{e}_i &=& e_{i+1},\ i = 1, \cdots, n-1 \\ \dot{e}_n &=& f(e, \zeta, \nu) + g(e, \zeta, \nu)v|_{\zeta = T_1^{-1}(e + L_1\nu, \tilde{z} + \lambda(\nu), \nu)} \\ \dot{\tilde{z}} &=& \phi_0(e, \tilde{z}, \nu) \end{array}\right\} \tag{3.21}$$

where $\phi_0(.) = \phi(e, \tilde{z} + \lambda(\nu), \nu) - \frac{\partial\lambda}{\partial\nu} S_0\nu$. We state the following assumption.

Assumption 3.4 *Assume for the system* $\dot{\tilde{z}} = \phi_0(e, \tilde{z}, \nu)$ *there exists a* C^1 *function* $W : R^m \to R_+$ *which satisfies*

$$\alpha_1(\|\tilde{z}\|) \le W(\tilde{z}) \le \alpha_2(\|\tilde{z}\|)$$

$$\frac{\partial W}{\partial \tilde{z}} \phi_0(e, \tilde{z}, \nu) \le -\phi_1(\|\tilde{z}\|),\ \forall\ \|\tilde{z}\| \ge \gamma(\|e\|)$$

for all $(e, z, \nu) \in N_1 \times N_2 \times D_1$, *where* α_i, $(i = 1, 2)$, ϕ_1 *and* γ *are class* $\mathcal{K}$ *functions.*

Let $\Omega_{a_2} = \{z : W(\tilde{z}) \le a_2\}$ where a_2 is chosen such that $\Omega_{a_2} \subset N_2$. This assumption, when $e = 0$, implies that the origin of the system $\dot{\tilde{z}} = \phi_0(0, \tilde{z}, \nu)$ is asymptotically stable, i.e., the zero dynamics have the minimum phase property. It also implies input-to-state stability from the input e to the state $\tilde{z}$ [32].

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3.3 State Feedback Design

We augment the linear servo-compensator

$$\dot{\sigma} = S\sigma + Je_1 \tag{3.22}$$

where (S, J) is a controllable pair, with (3.8)-(3.11) to obtain the *augmented system:*

$$\begin{aligned} \dot{\sigma} &= S\sigma + Je_1 && (3.23)\\ \dot{e} &= Ae + B[f(e,\zeta,\nu) + g(e,\zeta,\nu)v] && (3.24)\\ \dot{\zeta} &= A_2\zeta + B_2 v && (3.25)\\ y_m &= Ce && (3.26) \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{1\times n}$$

We consider a state feedback control of the form

$$v = g_0^{-1}(e,\zeta,\nu)[-f_0(e,\zeta,\nu) + \tilde{\varphi}(\xi,\zeta,\nu)] \stackrel{def}{=} \varphi(\xi,\zeta,\nu)$$

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where $\tilde{\varphi}(.)$ is locally Lipschitz in ξ uniformly in ζ, ν. For compactness of the presentation we rewrite the closed-loop system as

$$\begin{aligned} \dot{\xi} &= \mathcal{A}\xi + \mathcal{B}[f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi(\xi,\zeta,\nu)] \\ &\stackrel{def}{=} h_1(\xi,\zeta,\nu) \qquad (3.27) \\ \dot{\zeta} &= A_2\zeta + B_2\varphi(\xi,\zeta,\nu) \qquad (3.28) \\ y_m &= Ce \end{aligned}$$

where

$$\mathcal{A} = \begin{bmatrix} S & JC \\ 0 & A \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \ \xi = \begin{bmatrix} \sigma \\ e \end{bmatrix}$$

and h_1 is defined in an obvious way.

To motivate the upcoming assumption, we recall the design strategy as documented in some of the literature; see for example [23, 44, 37] and our earlier work in Chapter 2. The essence of this strategy is to achieve the control task in two steps. First the controller will ensure that the trajectories of the system are ultimately bounded, then if the system satisfies certain local properties, the controller will achieve asymptotic convergence to an equilibrium point.

Assumption 3.5 *Assume for the system $\dot{\xi} = h_1(\xi,\zeta,\nu)$ there exists a C^1 function $V : R^{n+q} \rightarrow R_+$ which for all $\xi \in X_1$ satisfies*

$$\beta_1(\|\xi\|) \leq V(\xi) \leq \beta_2(\|\xi\|)$$

$$\frac{\partial V}{\partial \xi} h_1(\xi,\zeta,\nu) \leq -\phi_2(\|\xi\|), \ \forall \ V(\xi) \geq \beta(\mu) \qquad (3.29)$$

uniformly for all $(\zeta,\nu) \in U_2 \times D_1$, where β, β_i, $(i = 1,2)$ and ϕ_2 are class $\mathcal{K}$ functions and $\mu > 0$ is a design parameter.

Define $\Omega_{a_1} = \{\xi : V(\xi) \le a_1\}$ where $a_1 > 0$ is chosen such that $\Omega_{a_1} \subset X_1$.
Assumption 3.5 implies that $\xi(t)$ will be eventually confined to the set $\Lambda_\mu \stackrel{def}{=} \{\xi \in X_1 : V(\xi) \le \beta(\mu)\}$, a neighborhood of $\xi = 0$, where the size of this set can be made arbitrarily small by choosing μ small enough. In the upcoming analysis we will use Assumptions 3.4 and 3.5 to establish that $z(t)$ will also be confined to a small set in the neighborhood of $\lambda(\nu)$. To do this we first express the closed-loop system in the $(\xi, \tilde{z})$ coordinates, i.e.,

$$\begin{aligned} \dot{\xi} &= \mathcal{A}\xi + \mathcal{B}[f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi(\xi,\zeta,\nu)]|_{\zeta = T_1^{-1}(e+L_1\nu, \tilde{z}+\lambda(\nu),\nu)} \\ &\stackrel{def}{=} \mathcal{A}\xi + \mathcal{B}[f_1(e,\tilde{z},\nu) + g_1(e,\tilde{z},\nu)\varphi_1(\xi,\tilde{z},\nu)] \qquad (3.30) \\ \dot{\tilde{z}} &= \phi_0(e,\tilde{z},\nu) \qquad (3.31) \end{aligned}$$

then show that the product set $\Omega_{a_1} \times \Omega_{a_2}$ is positively invariant. Recall that we have a similar situation in Chapter 2 except their it is done for a specific controller. In the forthcoming analysis we will show that the set $\Omega_{a_1} \times \Omega_{a_2}$ is positively invariant for a more general setup. However in our argument we will highlight the main points of the proof and refer the reader to Chapter 2 for more details.
To that end, we note that Choosing $a_2 \ge \alpha_2 \circ \gamma \circ \beta_1^{-1}(a_1)$ will guarantee that as long as $\xi(t) \in \Omega_{a_1}$, $z(t) \in \Omega_{a_2}$. From Assumption 3.5 we have $\forall\ V \ge \beta(\mu)$

$$\dot{V} \le -\phi_2 \circ \beta_2^{-1}(V)$$

Hence, $\forall\ \beta(\mu) < a_1$ we have

$$\dot{V} < 0,\ \forall\ (\xi, z, \nu) \in \Omega_{a_1} \times \Omega_{a_2} \times D_1 \text{ and } V = a_1 \qquad (3.32)$$

From Assumption 3.4 and the choice of a_2 we have

$$\dot{W} < 0,\ \forall\ (\xi, z, \nu) \in \Omega_{a_1} \times \Omega_{a_2} \times D_1 \text{ and } W = a_2 \tag{3.33}$$

From (3.32) and 3.33) we conclude that the set $\Omega_{a_1} \times \Omega_{a_2}$ is positively invariant. Now, we choose μ such that $\beta(\mu) \leq b_1 < a_1$. This is needed for output feedback in order to guarantee that whenever $\xi \in \Omega_{b_1}$, the estimate of ξ, i.e., $\hat{\xi}$ belongs to Ω_{a_1}. Again, we can show that the set

$$\mathcal{R}_\mu \stackrel{def}{=} \Lambda_\mu \times \Gamma_\mu$$

is positively invariant, where $\Gamma_\mu = \{W \leq \alpha_2 \circ \gamma \circ \beta_1^{-1}(\beta(\mu))\}$; (see Chapter 2, pp 19 for more details). To show that the trajectories $(\xi(t), z(t))$ will enter the set $\mathcal{R}_\mu$ in finite time, observe that $\forall\ V \geq \beta(\mu)$ we have

$$\dot{V} \leq -\phi_2 \circ \beta_2^{-1} \circ \beta(\mu)$$

and $\forall\ W \geq \alpha_2 \circ \gamma \circ \beta_1^{-1} \circ \beta(\mu)$ we have

$$\dot{W} \leq -\phi_1 \circ \gamma \circ \beta_1^{-1} \circ \beta(\mu)$$

From now on, an argument identical to the one used in the proof of Lemma 2.1 will complete the proof.

Assumption 3.6 *There exists a compact positively invariant set $\mathcal{S}_\mu \subset \Lambda_\mu$ such that $\xi(t)$ will enter $\mathcal{S}_\mu$ in finite time and inside $\mathcal{S}_\mu$ the control component $\tilde{\varphi}(\xi, \zeta, \nu)$ takes the form*

$$\tilde{\varphi} = K_0\xi + f_2(\zeta, \nu)$$

where f_2 is, in general, a nonlinear function of ζ and ν that satisfies $f_2(\lambda_0(\nu), \nu) = L_2\mathcal{V}$.

Lemma 3.1 *Suppose that Assumptions 3.2 and 3.6 hold, then there exists a* $q \times q$ *matrix* L *such that the set* $\{\sigma = L\mathcal{V},\ e = 0,\ \zeta = \lambda_0(\nu)\}$ *is an integral manifold of the closed-loop system (3.23)-(3.26).*

Proof: It was shown in [23], using results from [49] and [13], that there exists a unique matrix $L_{q\times q}$ that solves the following two matrix equations:

$$\Gamma = -(K_{01}L + L_2),\ LS = SL$$

The rest of the proof follows by direct substitution. •

Our purpose from now on is to design controllers, using only measurement of the tracking error, to establish regional as well as semi-global asymptotic convergence of the trajectories to this set. An obvious consequence of this is that $e_1(t) \to 0$ as $t \to \infty$.

To study attractivity of the zero-error manifold, let $\tilde{\sigma} = \sigma - L\mathcal{V}$. Writing the closed-loop system in terms of the shifted variables $\tilde{\sigma}$ and $\tilde{z}$, we get

$$\begin{aligned} \dot{\tilde{\xi}} &= \mathcal{A}\tilde{\xi} + \mathcal{B}[f_1(e,\tilde{z},\nu) \\ &+ g_1(e,\tilde{z},\nu)\varphi_1(\tilde{\sigma} + L\mathcal{V}, e, \tilde{z}, \nu)] \qquad (3.34) \\ \dot{\tilde{z}} &= \phi_0(e,\tilde{z},\nu) \qquad (3.35) \end{aligned}$$

where $\tilde{\xi} = (\tilde{\sigma}^T, e^T)^T$. For convenience, we rewrite (3.34)-(3.35) as

$$\dot{\eta} = \tilde{h}_1(\eta, \mathcal{V}) \qquad (3.36)$$

where $\eta = (\tilde{\xi}^T, \tilde{z}^T)^T$ and $\tilde{h}_1(.)$ can be easily defined.

We point out that the origin is an equilibrium point of (3.36). In the following assumption we impose another requirement on the state feedback controller; namely, we require it to be locally asymptotically stabilizing.

Assumption 3.7 *The origin of the system $\dot{\eta} = \tilde{h}_1(\eta, \mathcal{V})$, is locally asymptotically stable.*

Implicit in this assumption is the requirement that the system be locally minimum phase which is guaranteed under Assumption 3.4. This point will be made clear in Example 1. The following lemma is an obvious consequence of the requirements we imposed on the state feedback control.

Lemma 3.2 *Suppose Assumptions 3.1 through 3.4 are satisfied, then under any stabilizing state feedback control that satisfies Assumptions 3.5, 3.6 and 3.7, and $\forall\ \nu \in D_1$, all states will remain bounded and $e_1(t) \to 0$ as $t \to \infty$.*

3.4 Recovering State Feedback Performance

In this section we want to study output feedback implementation of the class of state feedback controllers described in the previous section. In other words, suppose we were successful in designing a state feedback control that satisfies Assumptions 3.5, 3.6 and 3.7, the question is: can we recover the asymptotic properties of the state feedback control using an observer. With this understanding and since we have only measurements of the tracking error, we will consider an estimator with the objective to recover the properties of the state feedback control, vis a vis convergence of the trajectories to a residual set first, then attractivity of the manifold inside this set. Consider the following estimator [40]

$$\left.\begin{aligned} \dot{\hat{e}}_i &= \hat{e}_{i+1} + \tfrac{\alpha_i}{\epsilon^i}(e_1 - \hat{e}_1), \ i = 1, \cdots, n-1 \\ \dot{\hat{e}}_n &= \tfrac{\alpha_n}{\epsilon^n}(e_1 - \hat{e}_1) \end{aligned}\right\} \tag{3.37}$$

where ϵ is a design parameter to be specified, and the positive constants α_i are to be chosen such that the roots of

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

are in the open left half plane.

This is a high-gain observer that is known to exhibit peaking in the estimates $\hat{e}$. To eliminate this peaking phenomenon, we saturate the control outside a compact set of interest that will be made more precise in the sequel; see [12, 44, 36, 50].

Define the scaled estimation error

$$\chi_i = \frac{1}{\epsilon^{n-i}}(e_i - \hat{e}_i), \ 1 \leq i \leq n \tag{3.38}$$

then the closed-loop system, under output feedback, will take the following singularly perturbed form

$$\begin{aligned}
\dot{\xi} &= \mathcal{A}\xi + \mathcal{B}[f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi(\sigma,\hat{e},\zeta,\nu)] \\
&\stackrel{def}{=} \bar{h}_1(\xi,\zeta,\chi,\nu) && (3.39) \\
\dot{\zeta} &= A_2\zeta + B_2\varphi(\sigma,\hat{e},\zeta,\nu) && (3.40) \\
\dot{\chi} &= (1/\epsilon)A_1\chi + B[f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi(\sigma,\hat{e},\zeta,\nu)] \\
&\stackrel{def}{=} (1/\epsilon)A_1\chi + h_2(\xi,\zeta,\chi,\nu) && (3.41)
\end{aligned}$$

where $\hat{e} = e - D(\epsilon)\chi$, $D(\epsilon)$ is a diagonal matrix with ϵ^{n-i} as the i-th diagonal element, A_1 is an $n \times n$ Hurwitz matrix, $\bar{h}_1(\xi,\zeta,0,\nu) = h_1(\xi,\zeta,\nu)$ and $h_2(.)$ is defined in an obvious way. To answer the question posed at the beginning of this section we first demonstrate that there exists a finite time T_0, the value of which can be made arbitrarily small by choosing ϵ small enough, such that $\forall\, t \geq T_0$ the estimation error

χ will be of $O(\epsilon)$, i.e., $\|\chi\| \le k_0\epsilon$.

Before delving into this analysis, we need to saturate our control $\varphi(\xi, \zeta, \nu)$ over the compact set $\Omega_{a_1} \times \Omega_{a_2} \times D_1$. Let $S_1 = \max |\varphi(\xi, \zeta, \nu)|_{\zeta = T_1^{-1}(e + L_1\nu, z, \nu)}$, for all $(\xi, z, \nu) \in \Omega_{a_1} \times \Omega_{a_2} \times D_1$. Then the globally bounded control will be taken as

$$\varphi^s(\xi, \zeta, \nu) = S_1 \text{ sat } \left(\frac{\varphi(\xi, \zeta, \nu)}{S_1} \right)$$

where sat(.) is the saturation function defined in (2.50). Replace φ in the closed-loop system (3.39)-(3.41) by φ^s.

Let $V(\xi_0) < b_2 < b_1 < a_1$, where b_2 is chosen such that $\Omega_{b_2} \subset \Omega_{b_1}$. Then there exists a finite time T_1, independent of ϵ, such that

$$V(\xi(t, \epsilon)) \le b_1, \ \forall \ t \in [0, T_1)$$

Now, we look at the solution of (3.41) over this time period. By repeating previous analysis which was carried out in Chapter 2 (page 30) using the same Lyapunov function candidate for the fast dynamics, we conclude that $\forall \ T_0 \le t < T_1$,

$$\|\chi(t)\| \le k_0\epsilon, \ k_0 > 0$$

Next, we look at the effect of output feedback on Inequality (3.29). Calculating the derivative of V with respect to the system (3.39), it is easily shown that

$$\dot{V} \le -\phi_2(\|\xi\|) + k_5\epsilon, \ \forall \ V(\xi) \ge \beta(\mu), \ \forall \ t \in [T_0, T_1), \ k_5 > 0$$

Since ϕ_2 vanishes only at the origin, it is strictly positive in the set $\{\xi : \beta(\mu) \le V(\xi) \le a_1\}$. Hence, $(1/2)\phi_2(\|\xi\|) \ge c_3 > 0$ and it follows that $\forall \ \epsilon < \frac{c_3}{k_5}$, $\dot{V} \le -(1/2)\phi_2(\xi)$, $\forall \ V(\xi) \ge \beta(\mu)$. This shows that the set Λ_μ is positively invariant.

Notice that $\Lambda_\mu \subset \Omega_{b_1}$. Again, repeating previous arguments, we can show that the set $\Omega_{b_1} \times \Omega_{a_2} \times \Omega_\epsilon$ is a positively invariant set, and all trajectories inside $\Omega_{b_1} \times \Omega_{a_2} \times \Omega_\epsilon$ will reach the positively invariant set $\mathcal{R}_\mu \times \Omega_\epsilon$ in finite time. From this we conclude that $(\xi(t), z(t), \chi(t)) \in \Omega_{b_1} \times \Omega_{a_2} \times \Omega_\epsilon$, $\forall\ t \geq T_0$ and $\|\chi\| \leq k_0\epsilon$, $\forall\ t \geq T_0$. This implies that the set $\Omega_{b_1} \times \Omega_{a_2} \times \Omega_\epsilon$ has no finite exit time, *i.e.*, $T_1 = \infty$.

In terms of the shifted variables, the closed-loop under output feedback will take the form

$$\begin{aligned} \dot{\tilde{\xi}} &= \mathcal{A}\tilde{\xi} + \mathcal{B}[f_1(e,\tilde{z},\nu) + g_1(e,\tilde{z},\nu)\varphi_1(\hat{\xi},\tilde{z},\nu)] && (3.42) \\ \dot{\tilde{z}} &= \phi_0(e,\tilde{z},\nu) && (3.43) \\ \dot{\chi} &= (1/\epsilon)A_1\chi + B[f_1(e,\tilde{z},\nu) \\ &\quad + g_1(e,\tilde{z},\nu)\varphi_1(\hat{\xi},\tilde{z},\nu)] && (3.44) \end{aligned}$$

where $\hat{\xi} = [\sigma\ \hat{e}]^T$. For convenience, we rewrite (3.42)-(3.44) as

$$\begin{aligned} \dot{\eta} &= \tilde{h}_3(\eta,\chi,\mathcal{V}) && (3.45) \\ \dot{\chi} &= (1/\epsilon)A_1\chi + \tilde{h}_2(\eta,\chi,\mathcal{V}) && (3.46) \end{aligned}$$

Notice that for $\chi \equiv 0$, we obtain the state feedback system and $\tilde{h}_3|_{\chi=0} = \tilde{h}_1$. To study the system (3.45)-(3.46), we need some interconnection conditions between the slow (η) and the fast (χ) variables to hold.

Assumption 3.8 *There exists a C^1 function $\tilde{V} : R^{n+q+m} \to R_+$ that satisfies*

$$\frac{\partial \tilde{V}}{\partial \eta}\tilde{h}_3(\eta,0,\mathcal{V}) \leq -q_0\phi_3(\eta), \quad q_0 > 0 \qquad (3.47)$$

where $\phi_3(\eta)$ is continuous and positive definite in η. Moreover the following interconnection conditions are satisfied

$$\|\tilde{h}_2(\eta, 0, \mathcal{V})\| \leq \alpha_0 \phi_3^a(\eta), \;\; \alpha_0 > 0, \;\; 0 < a \leq 1/2 \tag{3.48}$$

$$\frac{\partial \tilde{V}}{\partial \eta}[\tilde{h}_3(\eta, \chi, \mathcal{V}) - \tilde{h}_3(\eta, 0, \mathcal{V})] \leq \bar{\alpha}\phi_3^b(\eta)\|\chi\|^c \tag{3.49}$$

$0 < b < 1$, $c = \frac{1-b}{a}$ *and* $\bar{\alpha} > 0$. $\forall\ (\xi, z, \chi, \mathcal{V}) \in S_\mu \times \Gamma_\mu \times \Omega_\epsilon \times D_2$

We remark that the existence of a C^1 function satisfying (3.47) follows, in light of Assumption 3.7, from The Converse Lyapunov Theorem; see [31, Theorem 4.7]. The additional element here is that we require this function together with ϕ_3 to satisfy (3.47)-(3.49) simultaneously.

To motivate Assumption 3.8 it is important to note that for a singularly perturbed system with asymptotically stable reduced system and exponentially stable boundary layer model, it is not true, in general, that the composite system will be asymptotically stable for sufficiently small ϵ. To guarantee asymptotic stability of the full system, in this case, some interconnection conditions similar to (3.48)-(3.49) will be needed. These conditions are adopted from [36]. They are less restrictive than the quadratic-type Lyapunov conditions used in [38]. To illustrate the need for such conditions, consider the singularly perturbed system

$$\left.\begin{array}{rcl} \dot{\eta} & = & -\eta^5 + \eta^2\chi \\ \epsilon\dot{\chi} & = & -\chi + \epsilon\eta^2 \end{array}\right\} \tag{3.50}$$

Observe that the origin is an equilibrium point of (3.50). Let $\tilde{V}(\eta) = \eta^{2n}$. It can be verified that with $\phi_3(\eta) = \eta^{2n+4}$ and $q_0 = 2n$, Inequalities (3.48) and (3.49) are satisfied with $a = \frac{2}{2n+4}$, $b = \frac{2n+1}{2n+4}$ and $c = 1$. However, $1 - b \neq a$. Hence (3.50) violates Assumption 3.8. The point that we want to make here is that the origin of

(3.50) is not locally asymptotically stable. To assert this we use the Center Manifold Theorem to show that the reduced system is given by

$$\dot{\eta} = -\eta^5 + \epsilon\eta^4 + O(|\eta|^6) \tag{3.51}$$

Clearly, the origin of (3.51) is not locally asymptotically stable for all $\epsilon > 0$ since the second term dominates near the origin. Hence, the origin of the full system (3.50) is not locally asymptotically stable.

To show that we can recover asymptotic stability under output feedback, we first let $\mathcal{W}_0 = \tilde{V}(\eta) + (\chi^T P_1 \chi)^\gamma$ be a Lyapunov function candidate for the system (3.45)-(3.46), where $\gamma = (1/2a) \geq 1$. Then it can be shown, using (3.47) and Assumption 3.8, that

$$\begin{aligned}
\dot{\mathcal{W}}_0 \;\leq\; & -q_0\phi_3(\eta) + \bar{\alpha}\phi_3^b(\eta)\|\chi\|^c - \frac{\gamma\lambda_{min}^{\gamma-1}(P_1)}{\epsilon}\|\chi\|^{2\gamma} \\
& +2\gamma\lambda_{max}^{\gamma-1}(P_1)\|P_1\|\alpha_0\phi_3^a(\eta)\|\chi\|^{2\gamma-1}
\end{aligned} \tag{3.52}$$

$\forall\ (\xi, z, \chi, \mathcal{V}) \in S_\mu \times \Gamma_\mu \times \Omega_\epsilon \times D_2$.

We apply (2.41) to the second and last term of (3.52) with $p = 1/b$ and $p = 2\gamma$ respectively, to obtain

$$\begin{aligned}
\dot{\mathcal{W}}_0 \;\leq\; & -q_0\phi_3(\eta) + \frac{\bar{\alpha}}{\epsilon_0}\phi_3(\eta) + \bar{\alpha}(\epsilon_0)^{p_0}\|\chi\|^{2\gamma} \\
& + \frac{2\gamma\alpha_0\lambda_{max}^{\gamma-1}(P_1)\|P_1\|}{\gamma_0}\phi_3(\eta) + 2\gamma\alpha_0\lambda_{max}^{\gamma-1}(P_1)\|P_1\|(\gamma_0)^{p_1}\|\chi\|^{2\gamma} \\
& - \frac{\gamma\lambda_{min}^{\gamma-1}(P_1)}{\epsilon}\|\chi\|^{2\gamma}
\end{aligned}$$

where $\epsilon_0 > 0$, $\gamma_0 > 0$, $p_0 = b/(1-b)$ and $p_1 = 1/(2\gamma - 1)$. Choose ϵ_0, γ_0 large enough such that

$$\frac{\bar{\alpha}}{\epsilon_0} + \frac{2\gamma\alpha_0\lambda_{max}^{\gamma-1}(P_1)\|P_1\|}{\gamma_0} \leq \frac{1}{2}$$

then choose ϵ small enough such that

$$\bar{\alpha}(\epsilon_0)^{p_0} + 2\gamma\alpha_0\lambda_{max}^{\gamma-1}(P_1)\|P_1\|(\gamma_0)^{P_1} \leq \frac{\gamma\lambda_{min}^{\gamma-1}(P_1)}{2\epsilon}$$

$$\Longrightarrow \dot{W}_0 \leq -\frac{1}{2}\left(\phi_3(\eta) + \frac{\gamma\lambda_{min}^{\gamma-1}(P_1)}{\epsilon}\|\chi\|^{2\gamma}\right)$$

This implies, that $\eta(t), \chi(t) \to 0$ as $t \to \infty$. Hence $e_1(t) \to 0$ as $t \to \infty$. The main result of this section is summarized in the following theorem

Theorem 3.1 *Suppose that Assumptions 3.1 through 3.8 are satisfied. Consider the closed-loop system consisting of the extended system (3.3), with $v = \varphi^s$, together with the observer (3.37). Suppose $(\xi_0, z_0) \in \Omega_{b_2} \times \Omega_{a_2}$ and $\hat{\xi}(0)$ is bounded, then there exists μ^* such that $\forall\ \mu \in (0, \mu^*]$ there is $\epsilon^* = \epsilon^*(\mu) \leq 1$ such that $\forall\ \epsilon \in (0, \epsilon^*]$ all state variables are bounded and $e_1(t) \to 0$ as $t \to \infty$.*

At this point we recall the remark, following Theorem 2.1, concerning the dependence of the region of attraction in the original coordinates on the exogenous signals. Again, to require uniformity of the region of attraction will further restrict the size of admissible exogenous signals. As a consequence of Theorem 3.1, we have the following semi-global result:

Corollary 3.1 (Semi-global Tracking) *Suppose $U = R^{n+m}$, also suppose Assumptions 3.1, 3.2, 3.6, 3.7 and 3.8 are satisfied and that Assumptions 3.4, 3.5 are satisfied with W,V which are radially unbounded, then for any compact set $\mathcal{N} \subset R^{n+m}$ and for all initial states $(x(0), \zeta(0)) \in \mathcal{N}$ there exists $\mu^* > 0$ such that $\forall\ \mu \in (0, \mu^*]$ there is $\epsilon^* \leq 1$ such that $\forall\ \epsilon \in (0, \epsilon^*]$ the states of the closed-loop system consisting of (3.3), (3.37) , and the control φ^s, are bounded and $e_1(t) \to 0$ as $t \to \infty$.*

Proof: For the class of state feedback controllers satisfying Assumptions 3.5, 3.6, 3.7 and 3.8, we showed earlier in this section that we achieved tracking under output feedback for all initial states in $\Omega_{b_2} \times \Omega_{a_2}$ where $b_2 < b_1 < a_1$. Clearly, Assumption 3.3 will be satisfied with $N_1 = R^n$ and $N_2 = R^m$. Therefore, $X_1 = R^{q+n}$. Hence, a_1, a_2 can be chosen arbitrarily large . It follows that b_2, a_2 can be made arbitrarily large and the inverse image of the region of attraction can also be made arbitrarily large. •

3.5 Examples of Controllers

In this section we give two examples of stabilizing state feedback controllers that satisfy Assumptions 3.5, 3.6, 3.7 and 3.8. We first require our system to satisfy a local assumption. Let

$$\begin{aligned}
\delta_0(e,\zeta,\nu) &= f(e,\zeta,\nu) - g(e,\zeta,\nu)g_0^{-1}(e,\zeta,\nu)f_0(e,\zeta,\nu) \\
&- g(e,\zeta,\nu)g_0^{-1}(e,\zeta,\nu)g_0(0,\lambda_0(\nu),\nu)g^{-1}(0,\lambda_0(\nu),\nu)f(0,\lambda_0(\nu),\nu) \\
&+ g(e,\zeta,\nu)g_0^{-1}(e,\zeta,\nu)f_0(0,\lambda_0(\nu),\nu) \qquad (3.53)
\end{aligned}$$

then substitute for $\zeta = T_1^{-1}(e + L_1\nu, z, \nu)$ and $\lambda_0(\nu) = T_1^{-1}(L_1\nu, \lambda(\nu), \nu)$ in the right-hand side of (3.53). Denote the resulting expression by $\delta(e, z, \nu)$.

Local Assumption: *Suppose there exists a C^1 function $\tilde{W}$ which is decrescent and positive definite in $\tilde{z}$ and a continuous function ψ that is also positive definite in $\tilde{z}$ such that $\forall\ \nu \in D_1$, they satisfy*

$$\frac{\partial \tilde{W}}{\partial \tilde{z}}\tilde{\phi}_0(0,\tilde{z},\nu) \le -\alpha_0\psi(\tilde{z}),\ \alpha_0 > 0 \qquad (3.54)$$

$$|\delta(0,z,\nu)| \le k_1\psi^a(\tilde{z}) \qquad (3.55)$$

where $0 < a \leq 1/2, \;\; k_1 > 0$

$$\frac{\partial \tilde{W}}{\partial \tilde{z}}[\tilde{\phi}_0(e, \tilde{z}, \nu) - \tilde{\phi}_0(0, \tilde{z}, \nu)] \leq k_2 \psi^b(\tilde{z}) \|e\|^c \tag{3.56}$$

where $0 < b < 1, \;\; c = \frac{1-b}{a}, \;\; k_2 > 0.$

Inequality (3.54) by itself follows from assumption 3.4. However, what we require in this assumption is the existence of a C^1 function such that (3.54)-(3.56) are simultaneously satisfied.

3.5.1 Example 1

Consider the nonlinear system (3.27)-(3.28). Let $V(\xi) = \xi^T P \xi$, be a Lyapunov function candidate for the system (3.27), where $P = P^T > 0$ is the solution of Lyapunov equation

$$P(\mathcal{A} + \mathcal{B}K) + (\mathcal{A} + \mathcal{B}K)^T P = -I$$

Notice that we added and subtracted the term $K\xi$ to (3.27) where K is chosen such that $\mathcal{A} + \mathcal{B}K$ is Hurwitz.

Suppose the following two inequalities hold

$$\left| f(e, \zeta, \nu) - g(e, \zeta, \nu) g_0^{-1}(e, \zeta, \nu) f_0(e, \zeta, \nu) - K\xi \right| \leq \rho(\xi, \zeta)$$

$$g(e, \zeta, \nu) g_0^{-1}(e, \zeta, \nu) \geq k > 0$$

$\forall \; (\xi, \zeta, \nu) \in X_1 \times U_2 \times D_1$. Observe the dependence of the function ρ on the state ζ. Also, we emphasize that what is required here is the knowledge of ρ and k since their existence is always guaranteed. Using the min-max control

$$\varphi = g_0^{-1}(-f_0 - \frac{1}{k}\eta \eta_s)$$

where

$$\eta_s = \begin{cases} \frac{s}{|s|}, & \text{if } \eta\,|s| > \mu > 0 \\ \eta\frac{s}{\mu}, & \text{if } \eta\,|s| \leq \mu \end{cases} \tag{3.57}$$

with $s{=}2\mathcal{B}^T P\xi$, $\eta(\xi,\zeta) \geq \rho(\xi,\zeta)$ and μ a design parameter, it can be shown that

$$\dot{V}(\xi) \leq -\frac{1}{2}\|\xi\|^2 \stackrel{def}{=} -\phi_2(\|\xi\|),\ \forall\ V(\xi) \geq \frac{\mu}{2}\lambda_{min}(P)$$

Thus Assumption 3.5 is satisfied with $\Lambda_\mu = \{\xi : V(\xi) \leq \frac{\mu}{2}\lambda_{min}(P)\}$. Let $S_\mu = \{\xi : \eta\,|s| \leq \mu\} \cap \Lambda_\mu$. Then we can exploit the technique used in Chapter 2 (page 21) to show that $\xi(t)$ will converge to S_μ in finite time. Inside S_μ the control will assume the structure $\varphi = g_0^{-1}(-f_0 - (\eta_0^2/k)\frac{s}{\mu})$. Notice that $\tilde{\varphi}$ is linear in ξ and independent of ζ inside S_μ, hence Assumption 3.6 is satisfied with $K_0 = -(2\eta_0^2/k\mu)\mathcal{B}^T P$ and $f_2 = 0$. Furthermore, using $\tilde{V}(\eta) = \tilde{W}(\tilde{z}) + \lambda(\tilde{\xi}^T P\tilde{\xi})^\gamma$, $\gamma = 1/2a$, $\lambda > 0$ as a Lyapunov function candidate for the system (3.34)-(3.35), it can be shown that (see Lemma 2.2 for details)

$$\dot{\tilde{V}}(\eta) \leq -\frac{\bar{\alpha}}{2c_0}(\psi^a(\tilde{z}) + \|\tilde{\zeta}\|)^{2\gamma} \stackrel{def}{=} -q_0\phi_3(\eta),\ \bar{\alpha}, c_0 > 0$$

To verify Inequality (3.48), note that

$$\tilde{h}_2|_{x=0} = f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi^s(\xi,\zeta,\nu)$$

Define $F(\sigma,e,\zeta,\nu) = f(e,\zeta,\nu) + g(e,\zeta,\nu)\varphi^s(\sigma,e,\zeta,\nu)$. Then it can be shown that inside the set $\Lambda_\mu \times \Gamma_\mu$ we have

$$F(L\mathcal{V},0,\zeta,\nu)|_{\zeta=T_1^{-1}(L_1\nu,z,\nu)} = \delta(0,z,\nu)$$

Hence by adding and subtracting this term to $F(\sigma,e,\zeta,\nu)|_{\zeta=T_1^{-1}(e+L_1\nu,z,\nu)}$, we obtain

$$|F(\sigma,e,\zeta,\nu)|_{\zeta=T_1^{-1}(e+L_1\nu,z,\nu)} \leq k_1\psi^a(\tilde{z}) + a_0\|\tilde{\xi}\| \leq \alpha_0\phi_3^a(\eta) \text{ with } a = 1/2\gamma$$

To verify Inequality (3.49), we know that $\tilde{V}(\eta) = W(\tilde{z}) + \lambda(\tilde{\xi}^T P \tilde{\xi})^\gamma$. With a straight-forward calculations we can show that it will be satisfied with $\bar{\alpha} = 2\lambda\gamma \|P\| \lambda_{max}^{\gamma-1}(P) b_0$ where b_0 is a Lipschitz constant, and $b = (2\gamma - 1)/2\gamma$. Hence with $\gamma = 1$, Inequalities (3.47), (3.48) and (3.49) will be satisfied.

3.5.2 Example 2

Instead of using a min-max controller, we can use the nonlinear high-gain controller

$$\varphi = g_0^{-1}(-f_0 - \frac{\eta^2(\xi, \zeta)s}{k\mu})$$

Again, using $V(\xi) = \xi^T P \xi$ as a Lyapunov function candidate for the system (3.27), it is easily shown that

$$\dot{V}(\xi) \leq -\frac{1}{2}\|\xi\|^2 = -\phi_2(\xi), \ \forall \ V(\xi) \geq \frac{\mu}{2}\lambda_{min}(P)$$

Again, using the same technique of Chapter 2 (page 21) we can ensure that $\eta(.)$ becomes constant inside Λ_μ. We note that in this case $S_\mu = \Lambda_\mu$. Using $\tilde{V}(\eta)$ of Example 1, we can establish the same result obtained in Example 1 with the same ϕ_3 function.

3.6 The Class of Nonlinear Systems

Since the models of most nonlinear systems are specified in terms of state space models, in this section we give conditions under which they can be transformed to the input-output model (3.1). Consider the system

$$\left.\begin{array}{rcl} \dot{x}_0 & = & f(x_0) + g(x_0)u + q(x_0, w_0(t)) \\ y & = & h(x_0) \end{array}\right\} \qquad (3.58)$$

where $w_0(t) \in \Omega_0 \subset R^{p_0}$; Ω_0 is compact, $y, u \in R$ and x_0 belongs to a domain $U_0 \subset R^n$. Notice that in (3.58) $q(.)$ is possibly a nonlinear function of w_0.

We assume the following

(1) the system has a uniform relative degree ρ_0.

(2a) $L_q L_f^i h(x_0) = 0,\ 0 \leq i \leq \nu_0 - 2,\ \forall\ x_0 \in U_0$

(2b) $L_q L_f^{\nu_0 - 1} h(x_0) \neq 0$, for some $w_0(t) \in \Omega_0$, some $x_0 \in U_0$

where $\nu_0 \leq \rho_0$. Suppose we can choose the functions ϕ_j such that

$$< d\phi_j, g >= 0,\ \forall x_0 \in U_0,\ j = 1, \cdots, n - \rho_0$$

and define a change of coordinates:

$$\bar{z} = \begin{bmatrix} \bar{z}_i \\ \bar{z}_{\rho_0 + j} \end{bmatrix} = \begin{bmatrix} L_f^{i-1} h(x_0) \\ \phi_j(x_0) \end{bmatrix} \stackrel{def}{=} T_0(x_0) \tag{3.59}$$

$i = 1, \cdots, \rho_0$, $j = 1, \cdots, n - \rho_0$. Then, in the new coordinates, we have

$$\begin{aligned}
\dot{\bar{z}}_i &= \bar{z}_{i+1},\ i = 1, \cdots, \nu_0 - 1 \\
\dot{\bar{z}}_{\nu_0+i} &= \bar{z}_{\nu_0+i+1} + L_q L_f^{(\nu_0-1+i)} h(x_0)|_{x_0 = T_0^{-1}(\bar{z})},\ i = 0, \cdots, \rho_0 - \nu_0 - 1 \\
&\stackrel{def}{=} \bar{z}_{\nu_0+i+1} + \gamma_{\nu_0+i}(\bar{z}, w_0(t)) \\
\dot{\bar{z}}_{\rho_0} &= L_f^{\rho_0} h(x_0) + L_g L_f^{\rho_0-1} h(x_0) u + L_q L_f^{\rho_0-1} h(x_0)|_{x_0 = T_0^{-1}(\bar{z})} \\
&\stackrel{def}{=} f_1(\bar{z}) + g_1(\bar{z}) u + \gamma_{\rho_0}(\bar{z}, w_0(t)) \\
\dot{\bar{z}}_{\rho_0+j} &= L_f \phi_j(x_0) + L_q \phi_j(x_0)|_{x_0 = T_0^{-1}(\bar{z})},\ j = 1, \cdots n - \rho_0 \\
&\stackrel{def}{=} \Phi_j(\bar{z}, w_0(t)) \\
y &= \bar{z}_1
\end{aligned}$$

Note that $\gamma_1, \cdots, \gamma_{\nu_0 - 1} \equiv 0$ (follows from (2a)). Assume that

$$d(L_q L_f^i h) \in \{dh, \cdots, d(L_f^i h)\}, \; i = \nu_0 - 1, \cdots, \rho_0 - 1$$

then we have

$$
\begin{aligned}
y^{(i)} &= \bar{z}_{i+1}, \; i = 0, \cdots, \nu_0 - 1 && (3.60)\\
y^{(\nu_0+i)} &= \bar{z}_{\nu_0+i+1} + \psi_i(\bar{z}_1, \cdots, \bar{z}_{\nu_0+i}, w_0, \cdots, w_0^{(i)}) && (3.61)\\
&\quad i = 0, \cdots, \rho_0 - \nu_0 - 1 &&\\
y^{(\rho_0)} &= \psi_{\rho_0-\nu_0}(\bar{z}_r, \bar{z}_1, \cdots, \bar{z}_{\rho_0}, w_0, \cdots, w_0^{(\rho_0-\nu_0)}) + g_1(\bar{z})u && (3.62)\\
y^{(\rho_0+i)} &= \psi_j(\bar{z}_r, \bar{z}_1, \cdots, \bar{z}_{\rho_0}, w_0, \cdots, w^{(\rho_0-\nu_0+i)}, u, \cdots, u^{(i-1)}) &&\\
&\quad + g_1(\bar{z})u^{(i)} && (3.63)\\
&\quad i = 1, \cdots, n - \rho_0 - 1, \; j = \rho_0 - \nu_0 + i, \; \bar{z}_r = (\bar{z}_{\rho_0+1}, \cdots, \bar{z}_n)^T &&
\end{aligned}
$$

Observability Assumption: *Suppose that equations (3.63) are explicitly solvable for $\bar{z}_r$ in terms of y, u, w_0 and their derivatives.*

This assumption together with the fact that the states $\bar{z}_1, \cdots, \bar{z}_{\rho_0}$ can be explicitly solved for in terms of y, w_0 and their derivatives, implies that the system is observable in the sense of [52]. From this we obtain

$$
\begin{aligned}
y^{(n)} &= \bar{f}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(n-\rho_0-1)}, w_0, \cdots, w_0^{(n-\nu_0)})\\
&\quad + \bar{g}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(n-\rho_0-1)}, w_0, \cdots, w_0^{(n-\nu_0)})u^{(n-\rho_0)}
\end{aligned}
$$

Let $m = n - \rho_0$ and $w(t) = (w_0^T, \cdots, (w_0^{(n-\nu_0)})^T)^T$, to obtain the system (3.1).

3.7 A Special Case: Nonlinear Systems with Polynomial Nonlinearity

In this section we give an example of a class of systems for which the transformation (3.5) is explicitly known and also Assumption 3.2(b), which is fundamental to this work, is satisfied. Consider the following system

$$y^{(n)} = f(y, \cdots, y^{(n-1)}, w(t)) + \sum_{i=0}^{m} l_i \theta^{(i)} \tag{3.64}$$

where

- f is a polynomial nonlinearity in its arguments.
- The polynomial $l_m s^m + \cdots + l_0$ with $l_m \neq 0$ is Hurwitz, i.e., all its zeros have negative real parts.
- $\theta = \sigma_0(y)u$ and $\sigma_0(y) \neq 0$ for all $y \in U_1 \subset R$
- $w(t) \in \Omega \subset R^p$; Ω is compact

It can be shown that the class of systems characterized by geometric conditions in [41] is transformable into the input-output model (3.64). This is a minimum phase system where the zero dynamics can be expressed, in suitable coordinates, as a linear Bounded-Input-Bounded-State (BIBS) system. Define

$$\left.\begin{array}{rcl} x_{i+1} &=& y^{(i)},\ i = 0, \cdots, n-1 \\ \zeta_{i+1} &=& \theta^{(i)},\ i = 0, \cdots, m-1 \\ v &=& \theta^{(m)} \end{array}\right\} \tag{3.65}$$

then the extended system will be given by

$$\left.\begin{array}{rcl}\dot{x}_i &=& x_{i+1},\ i=1,\cdots,n-1\\ \dot{x}_n &=& f(x,w(t))+B_1\zeta+l_m v\\ \dot{\zeta}_i &=& \zeta_{i+1},\ i=1,\cdots,m-1\\ \dot{\zeta}_m &=& v\\ y &=& x_1\end{array}\right\} \tag{3.66}$$

where $B_1=(l_0,l_1,\cdots,l_{m-1})$. The change of coordinates

$$z_i=\zeta_i-\frac{x_{n-m+i}}{l_m},\ i=1,\cdots,m$$

takes the system (3.66) into the normal form

$$\left.\begin{array}{rcl}\dot{x}_i &=& x_{i+1},\ i=1,\cdots,n-1\\ \dot{x}_n &=& f(x,w(t))+B_1\zeta+l_m v\\ \dot{z}_i &=& z_{i+1},\ i=1,\cdots,m-1\\ \dot{z}_m &=& -(1/l_m)B_1 z-f_1(x,w(t))\\ y &=& x_1\end{array}\right\} \tag{3.67}$$

where $f_1(x,w(t))=(1/l_m)f(x,w(t))+(1/l_m^2)B_1\bar{x}$ and $\bar{x}=(x_{n-m+1},\cdots,x_n)^T$. Clearly, the zero dynamics are BIBS stable. Let $v=l_{m0}^{-1}(-f_0-B_0\zeta+\tilde{v})$ where l_{m_0}, f_0 and B_0 are the nominal models of l_m, f and B, respectively. Then on the zero-error manifold we get

$$l_m^{-1}l_{m_0}f(0,\nu)+(l_m^{-1}l_{m_0}B_1-B_0)\zeta-f_0(0,\nu)+\tilde{v}=0$$

$$\begin{array}{rcl}\dot{z}_i &=& z_{i+1},\ i=1,\cdots,m-1\\ \dot{z}_m &=& -(1/l_m)B_1 z-f_1(0,\nu)\end{array}$$

and $\zeta = z + (1/l_m)\bar{\nu}$ where $\bar{\nu} = (r^{(n-m)}, \cdots, r^{(n-1)})^T$.

Since z is a solution of a linear system that is driven by an input which has a finite number of modes, it will be a linear combination of those modes. Hence, ζ will also be a linear combination of the same modes. Thus, we conclude that Assumption 3.2(b) is satisfied. Assumption 3.2(a), on the other hand, will not hold automatically for this class of systems and its satisfaction will be problem dependent.

3.8 A Design Example

Consider the second order system

$$\ddot{y} = -y + ay^3 - \dot{y} + u - w + b\dot{u} = \bar{f} + b\dot{u} \tag{3.68}$$

where $a \in [-0.1, 0.1]$, $b \in [8, 12]$ and w is a constant disturbance input such that $w \in [-1, 1]$. The nominal values of a_0, b_0 are $0, 10$ respectively. Notice that in (3.68) $\bar{f}$ is a polynomial function of y, $\dot{y}$, u and w. The extended system is given by

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + ax_1^3 - x_2 + x_3 - w + bv \\ \dot{x}_3 &= v \end{aligned}\right\} \tag{3.69}$$

Let $e_1 = x_1 - r_0 \sin(t)$, $e_2 = x_2 - r_0 \cos(t)$ and $z = x_3 - \frac{x_2}{b}$ where $r_0 \sin(t)$ is the reference signal. Then, in the new coordinates, the system will be in the normal form:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -e_1 + a(e_1 + r_0 \sin(t))^3 - (1 - 1/b)(e_2 + r_0 \cos(t)) + z - w + bv \\ \dot{z} &= -(1/b)z + (1/b)[e_1 + r_0 \sin(t) - a(e_1 + r_0 \sin(t))^3 \end{aligned}$$

$$+(1-1/b)(e_2+r_0\cos(t))+w]$$

Let $v = \varphi = g_0^{-1}(-f_0+\tilde{\varphi})$. With $g_0 = 10$, $f_0 = 0$ we have $\varphi = 0.1\tilde{\varphi}$. On the zero-error manifold we have

$$ar_0^3\sin^3(t)-(1-1/b)r_0\cos(t)+\lambda(\nu)-w+0.1b\tilde{\varphi}=0 \tag{3.70}$$

$$\dot{z}=-(1/b)z+(1/b)[r_0\sin(t)-a(r_0\sin(t))^3+(1-1/b)r_0\cos(t)+w]$$

With the given disturbance and reference signals we have

$$S_0=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and } \nu=\begin{bmatrix}r_0\sin(t)\\ r_0\cos(t)\\ w\end{bmatrix}$$

Let

$$\begin{aligned}\lambda(\nu) &= a_1\nu_{11}+a_2\nu_{21}+a_3\nu_{11}^2+a_4\nu_{21}^2+a_5\nu_{11}\nu_{21}+\nu_{31}+a_6\nu_{11}^3\\ &\quad +a_7\nu_{21}^3+a_8\nu_{11}^2\nu_{21}+a_9\nu_{21}^2\nu_{11}\end{aligned} \tag{3.71}$$

then we use Assumption 3.2 and equate coefficients of equal powers of the components of ν to obtain $a_3 = a_4 = a_5 = 0$. Since the exact knowledge of the matrix Γ, which will be a function of the parameters a and b, is not needed in the controller design we now substitute for $\lambda(\nu)$ in (3.70) its expression in (3.71) to obtain the vector $\mathcal{V}$, *i.e.*,

$$ar_0^3\sin^3(t)-(1-1/b)r_0\cos(t)+\lambda(\nu)-w=\Gamma\begin{bmatrix}\sin(t)\\ \cos(t)\\ \sin(3t)\\ \cos(3t)\end{bmatrix}$$

where

$$\Gamma_{11} = 0.75ar_0^3 + a_1 r_0 + 0.75a_6 r_0^3 + 0.25a_9 r_0^3,$$

$$\Gamma_{12} = 0.9r_0 + a_2 r_0 + 0.75a_7 r_0^3 + 0.25a_8 r_0^3,$$

$$\Gamma_{13} = -0.25ar_0^3 - 0.25a_6 r_0^3 + 0.25a_9 r_0^3,$$

and

$$\Gamma_{14} = 0.25a_7 r_0^3 - 0.25a_8 r_0^3$$

where $b = b_0 = 10$ is used. From this the matrix S will be given by

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

we take $J = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$. Notice that (S, J) is controllable.
Choosing the gain matrix K as

$$K = \begin{bmatrix} -3.4959 & -2.789 & 4.4719 & 0.0462 & -8.8743 & -5.2677 \end{bmatrix}$$

we found that s is given by

$$s = \begin{bmatrix} 0.4117 & 0.201 & -0.4498 & 0.2443 & 0.8104 & 0.3437 \end{bmatrix} \xi$$

and the function ρ is given by

$$\rho(\xi, x_3) = 0.1\|\xi\|^3 + 0.3\|\xi\|^2 + (10.9644 + 0.3r_0^2)\|\xi\| + |x_3| + 1 + r_0 + 0.1r_0^3$$

Notice the dependence of ρ on the extended state x_3. Setting the initial states of the servo-compensator to zero and selecting compact sets of initial conditions for x_1, x_2 and x_3 as $|x_1(0)| \leq 0.5$, $|x_2(0)| \leq 0.5$ and $|x_3(0)| \leq 0.5$, we found that $\xi(0)$ belongs to the set $\xi(0)^T P \xi(0) \leq c$ with $c = 1.5529$. Now, using the state feedback control

$$\varphi(\xi, x_3) = \begin{cases} -0.1\rho\frac{s}{|s|}, & \text{if } \rho\,|s| > \mu \\ -0.1\rho^2\frac{s}{\mu}, & \text{if } \rho\,|s| \leq \mu \end{cases} \tag{3.72}$$

we found that it will saturate at ± 15. Hence the output feedback control will be given by

$$\phi^s = 15 \text{ sat}\left(\frac{\varphi(\hat{\xi}, x_3)}{15}\right)$$

where

$$\hat{\xi} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \hat{e}_1 & \hat{e}_2 \end{bmatrix}$$

and $\hat{s} = \begin{bmatrix} 0.4117 & 0.201 & -0.4498 & 0.2443 & 0.8104 & 0.3437 \end{bmatrix} \hat{\xi}$.
With $\alpha_1 = 3$, $\alpha_2 = 2$ and after scaling the observer states such that $q_1 = \hat{e}_1$ and $q_2 = \epsilon\hat{e}_2$, we obtain the observer model:

$$\begin{aligned} \epsilon\dot{q}_1 &= q_2 + 3(e_1 - q_1) \\ \epsilon\dot{q}_2 &= 2(e_1 - q_1) \end{aligned}$$

With $\mu = 0.1$ and $\epsilon = 0.01$, the simulations were performed for $a = 0.0$, $b = 10$ and $r_0 = 0.1$. The tracking error and the control are shown in Figure 3.1 which clearly shows asymptotic tracking. The performance of the observer is shown in Figure 3.2. Notice the peaking in $\hat{e}_2$. In Figure 3.3 the control is shown to saturate at -15 during the same period in which $\hat{e}_2$ peaks. To study the effect of pushing ϵ small, simulations were carried out for two values of ϵ and the behavior of both the control ϕ^s and $\hat{e}_2$ was recorded. This is shown in Figure 3.4. Observe that for $\epsilon = 0.001$ the period

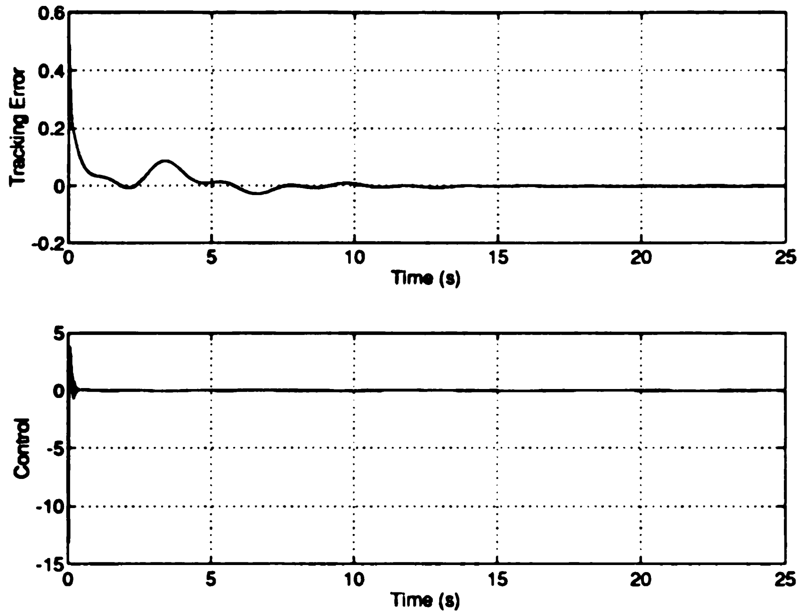

Figure 3.1. Tracking error e_1 and the control ϕ^s

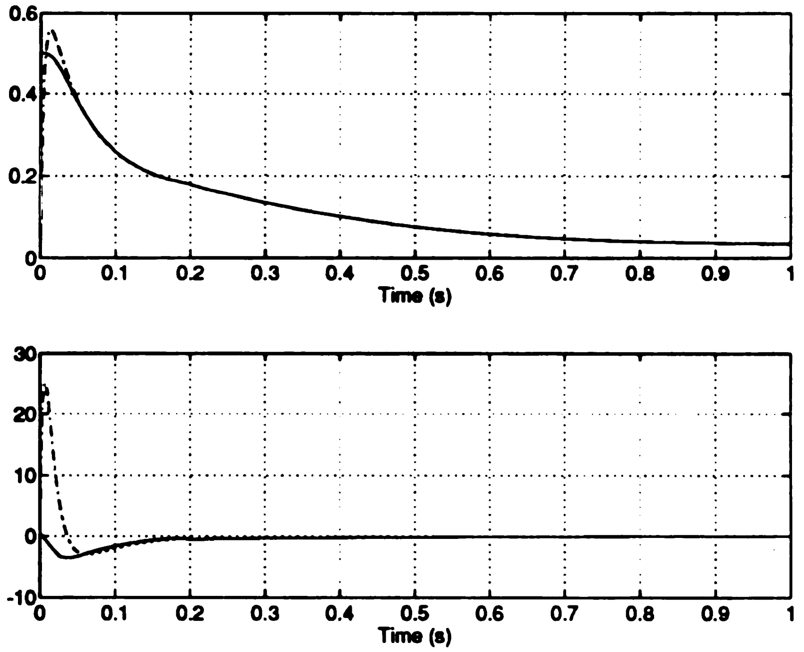

Figure 3.2. Observer performance; top: e_1 (solid) and its estimate; bottom: e_2 (solid) and its estimate

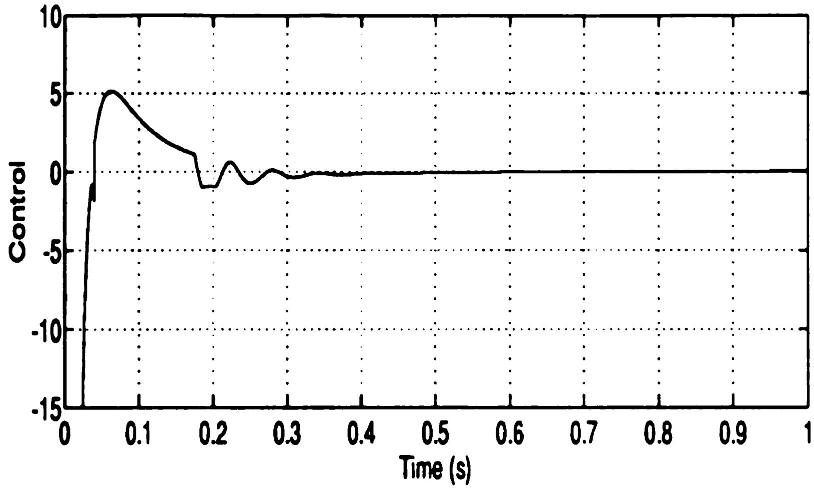

Figure 3.3. Saturation of the control ϕ^s

over which the control saturates decreases while the peaking in $\hat{e}_2$, over the same period, increases. To investigate the robustness of the control, simulations were also performed for two sets of parameters, *i.e.*, $a = 0.1,\ b = 12$ and $a = -0.1,\ b = 8$. The tracking error for these two sets is shown in Figures 3.5 and 3.6 respectively, which clearly indicates that asymptotic tracking is achieved.

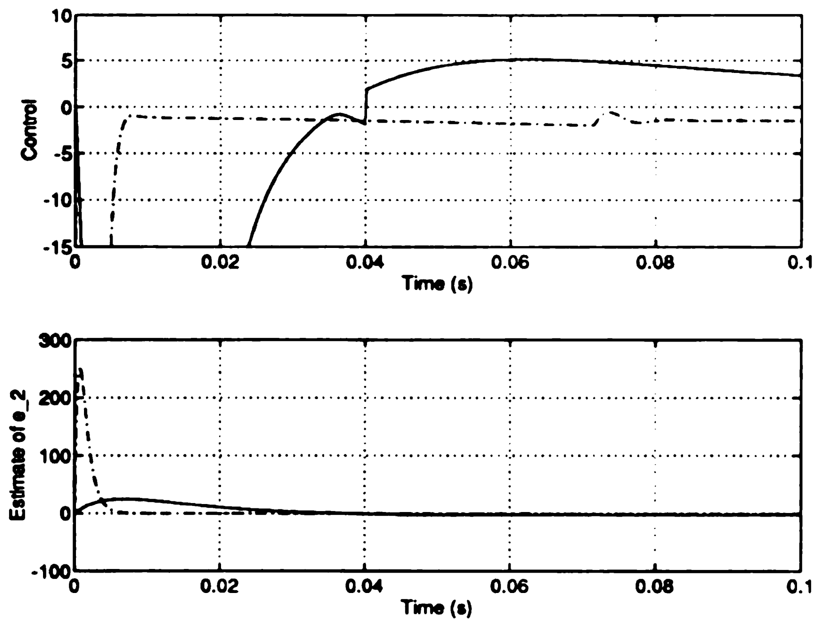

Figure 3.4. Control ϕ^s and $\hat{e}_2$ for $\epsilon = 0.01$ (solid) and $\epsilon = 0.001$

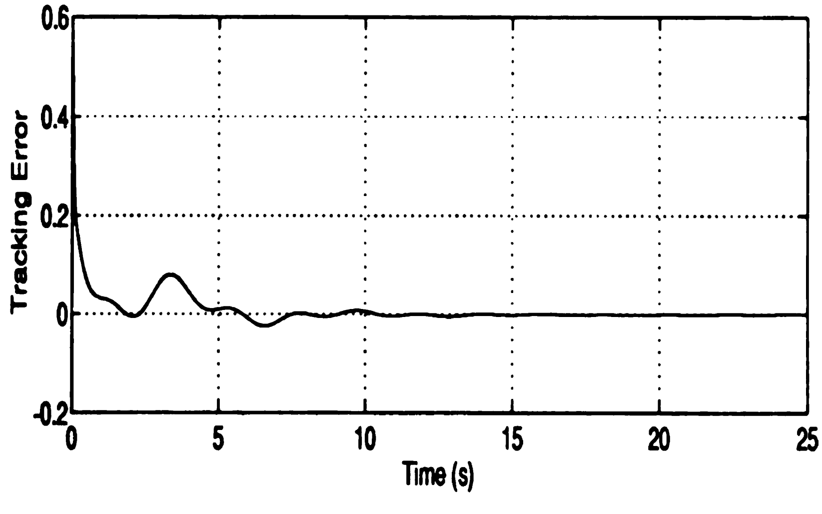

Figure 3.5. Tracking error when $a = 0.1$ and $b = 12$

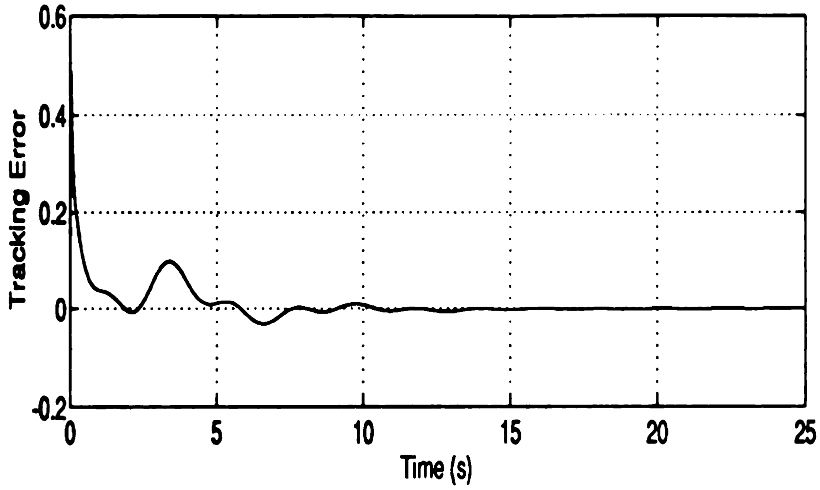

Figure 3.6. Tracking error when $a = -0.1$ and $b = 8$

3.9 Conclusions

In this paper we have extended the nonlinear servomechanism theory to include systems that are represented by input-output models. We allowed a time-varying exogenous signals that are not necessarily small. This aspect of the exogenous signals is considered an extension over our work in Chapter 2.

In this work instead of designing a specific state feedback control, we presented a characterization of a class of stabilizing controllers. The introduction of m integrators at the input, made the state ζ available for feedback. With this we avoided the more restrictive partial state feedback approach. Implementing the controller, using a linear high-gain observer, we showed that for a class of locally Lipschitz, globally bounded state feedback control, we can recover the asymptotic properties achieved under state feedback. With globally bounded control, only semi-global results are possible which, from a practical point of view, is not a serious limitation. The design procedure was illustrated on an example that we worked out all through including simulations. the results of those simulations are in good agreement with the predicted behavior of the system.

CHAPTER 4

Output Regulation of a Field-Controlled DC Motor

4.1 Introduction and Problem Definition

As an illustration, we will study the nonlinear model of a field-controlled DC motor using two design methods. In the first method we will follow the procedure of Chapter 2 where we augment an integrator driven by the tracking error, then we design a controller to stabilize the augmented system. In the second method we follow the procedure of chapter 3 where we first extend the original system by adding one integrator at the input, then we identify the internal model for the type of reference and disturbance signals present in the system. For this case study we will consider uncertain constant parameters. Hence the internal model is simply an integrator. However, we will show that the presence of an integrator at the input will exclude the necessity of adding an extra integrator. Then we design a stabilizing controller. Consider the nonlinear model

$$
\begin{aligned}
\dot{x}_1 &= -\theta_1 x_1 - \theta_2 x_2 u + \theta_3 \\
\dot{x}_2 &= -\theta_4 x_2 + \theta_5 x_1 u
\end{aligned}
$$

$$y = x_2$$

where $\theta_1 = \frac{R}{L}$, $\theta_2 = \frac{K_1}{L}$, $\theta_3 = \frac{V}{L}$, $\theta_4 = \frac{B}{J}$, $\theta_5 = \frac{K_1}{J}$, x_1 is the armature current (A), x_2 is the angular velocity (rad/s), V is a fixed voltage (V) applied to the armature circuit and u is the field winding voltage (V) and represents the control variable. The constants R, L and K_1 are, respectively, the resistance, the inductance and the torque constant of the armature circuit, while the parameters J and B are the load's moment of inertia and the viscous damping coefficient. This system was studied in [53] using measurement of the state x, while here we assume that we have only measurement of the speed x_2. Our desire is to get the output of the motor, in this case the speed, to follow a given constant reference in the presence of uncertain parameters. The following nominal values of the parameters, taken from [53], will be used in the simulations: $R = 7\Omega$, $L = 0.12H$, $V = 5(V)$, $B = 6.04 \times 10^{-6} N - m - s/rad$, $J = 1.06 \times 10^{-6} N - m - s^2/rad$ and $K_1 = .0141\ N - m/A$. To investigate the robustness of the controllers, the following changes in the parameters are allowed: $\theta_{01} \pm 0.15\theta_{01}$, $\theta_{02} \pm 0.9\theta_{02}$, $\theta_{03} \pm 0.2\theta_{03}$, $\theta_{04} \pm 0.15\theta_{04}$, $\theta_{05} \pm 0.2\theta_{05}$, where θ_{0i}, $i = 1, \cdots, 5$, denote the nominal values.

4.2 Method 1

Let ν be the desired constant operating velocity. Let (I_a, V_F) be the corresponding equilibrium values for (x_1, u). Then we can find V_F from the equation

$$x^2 - bx + 1 = 0 \tag{4.1}$$

where $x = K_1 V_F / \sqrt{BR}$, $b = V/(\nu\sqrt{BR})$ and I_a is given by

$$I_a = \frac{V/R}{1 + x^2} \tag{4.2}$$

Thus Assumption 2.1 is satisfied. Also, the above system has a relative degree one provided $x_1 \neq 0$. Thus, we will impose the constraint: $x_1 \geq I_{min}$ and define the set $U_\theta = \{x : x_1 \geq I_{min}\}$ to be our domain of interest. For (4.1) to have real roots, we must restrict ν to the admissible range $\nu \in [0, \frac{V}{2\sqrt{BR}}] \stackrel{def}{=} \Gamma$. Choosing $x^2 < 1$ implies that $I_a > V/2R$. Hence we choose $I_{min} = V/2R$ such that only one solution of (4.1) will be associated with the domain of validity U_θ. With this we ensure the uniqueness of the equilibrium point in U_θ. The change of variables $\Psi(x, \nu)$ is given by

$$
\begin{aligned}
e &= x_2 - \nu \\
z &= \frac{K_1}{J}x_1^2 + \frac{K_1}{L}x_2^2 - \frac{K_1}{J}I_a^2 - \frac{K_1}{L}\nu^2
\end{aligned}
$$

This change of variables will shift the desired equilibrium point to the origin. Its choice is motivated by our desire to transform the system into the normal form. Before proceeding with the design, we augment an integrator driven by the tracking error with the plant to obtain the matrices $\mathcal{A}$ and $\mathcal{B}$ and then investigate the validity of our assumptions.

$$
\begin{aligned}
\dot{\sigma} &= e \\
\dot{e} &= -\frac{B}{J}(e+\nu) + \frac{K_1}{J}\sqrt{\frac{J}{K_1}[z - \frac{K_1}{L}(e+\nu)^2 + \frac{K_1}{J}I_a^2 + \frac{K_1}{L}\nu^2]}\; u \\
\dot{z} &= -2\frac{R}{L}z + 2(\frac{RK_1}{L^2} - \frac{K_1 B}{LJ})(e+\nu)^2 - 2\frac{R}{L}(\frac{K_1}{J}I_a^2 + \frac{K_1}{L}\nu^2) \\
&\quad +2\frac{K_1 V}{JL}\sqrt{\frac{J}{K_1}[z - \frac{K_1}{L}(e+\nu)^2 + \frac{K_1}{J}I_a^2 + \frac{K_1}{L}\nu^2]} \\
&\stackrel{def}{=} \phi_0(e, z, d)
\end{aligned}
$$

The constraint $x_1 \geq I_{min}$ implies that

$$
z \geq \frac{K_1}{L}(e+\nu)^2 - \frac{K_1}{J}I_a^2 - \frac{K_1}{L}\nu^2 + \frac{K_1}{J}I_{min}^2
$$

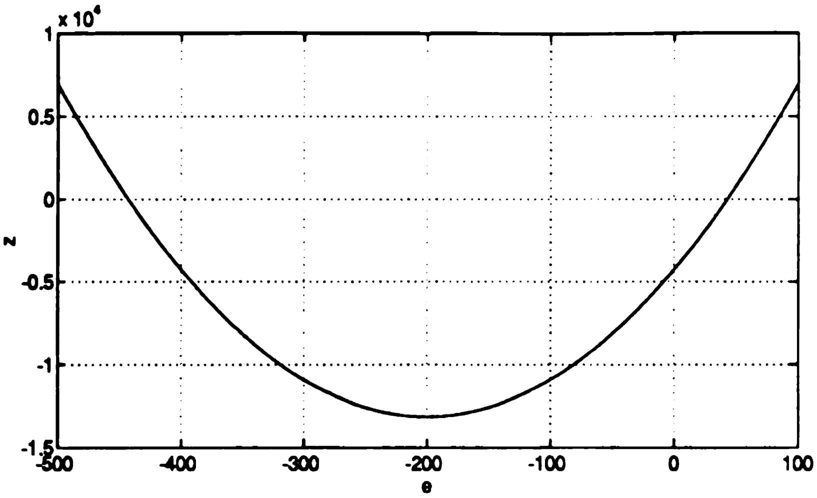

Figure 4.1. The domain N_d in the (e, z)-coordinates

From this we can define our domain of interest N_d that will be valid for all admissible values of the parameters. This is shown in Figure 4.1. It can be verified that choosing $I_a > I_{min}$ guarantees that N_d will contain the origin for all values of the parameters and the reference. From Figure 4.1 we identify the sets: $\mathcal{S} = R \times (-442.8, 42.8)$, $\mathcal{U} = (-4237, \infty)$, $\mathcal{S}_0 = R \times (-42.8, 42.8)$ and $\mathcal{U}_0 = (-4237, 4237)$.

The function $-\phi_0(0, z, d)$, for the nominal system, is shown in Figure 4.2. It is a first-quadrant third-quadrant nonlinearity over a certain domain and it will retain this property for all possible values of the parameters. Using $W(z) = -\int_0^z \phi_0(0, y, d)dy$ as a Lyapunov function candidate for the system $\dot{z} = \phi_0(0, z, d)$ it can be shown that the origin is asymptotically stable with a region of attraction, that is valid for all allowable values of the parameters, given by $\{z : z \geq -4237\}$. Also when $e \neq 0$, it can be shown, after some algebraic manipulations, that

$$\dot{W} \leq -\frac{1}{2}\phi_0^2(0, z, d), \ \forall \ |z| \geq \frac{2k_0}{m_0}\gamma_0(|e|)$$

$\forall\ z \in \mathcal{U}_0$, where $\gamma_0(|e|) = e^2 + 2|\nu||e|$, $k_0 = 2[|\frac{RK_1}{L^2} - \frac{K_1 B}{LJ}| + \frac{K_1 V}{L^2 I_{min}}]$ and $m_0 = 2\frac{|\frac{V}{L} - \frac{R}{L}|}{I_a + \sqrt{I_a^2 - 0.26544}}$. Hence Assumption 2.4 is satisfied. Furthermore, a linearization around the origin will reveal that this equilibrium point is exponentially stable, this implies that Assumption 2.6 is also satisfied. We proceed now with the design. The

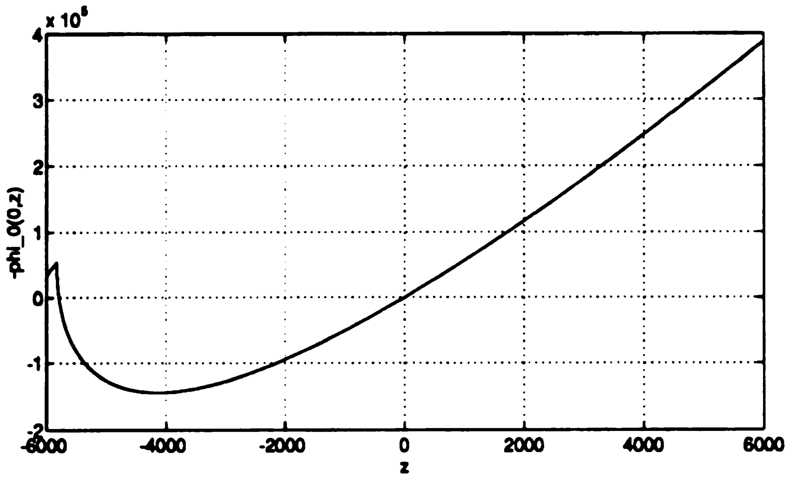

Figure 4.2. $-\phi_0(0,z)$; (nominal)

matrix K is chosen as $K = \begin{bmatrix} -3.1623 & -\theta_{04} \end{bmatrix}$, we then obtain the matrix P

$$P = \begin{bmatrix} 1.2662 & 0.1581 \\ 0.1581 & 0.1155 \end{bmatrix}$$

and $s = \begin{bmatrix} 0.3162 & 0.231 \end{bmatrix}\xi$. We take $f_0 = 0$, $g_0 = 2 \times 10^4$ and the function ρ is found to be

$$\rho = (3.1623 + 0.15\theta_{04})\|\xi\| + 1.15\theta_{04}|\nu|$$

with $\xi = (\sigma, e)$. Setting the initial conditions of $(\sigma, x_1, x_2) = (0, 0.5805, 300)$, we find that $\xi(0) = \begin{bmatrix} 0 & 100 \end{bmatrix}$, and it belongs to Ω_{c_1} with $c_1 = 1.155 \times 10^3$, the set Ω_{c_2} is given by $\{z : |z| \leq 4237\}$. With $\mu = 0.5$, the control is given by

$$u = \begin{cases} -g_0^{-1}\rho(\xi)\frac{s}{|s|}, & \text{if } \rho(\xi)\,|s| > 0.5 \\ -2g_0^{-1}\rho^2(\xi)s, & \text{if } \rho(\xi)\,|s| \leq 0.5 \end{cases} \tag{4.3}$$

The simulations were performed for the nominal system with a desired reference $\nu = 200\ rad/s$ starting from a steady state speed of 300 rad/s. Figure 4.3 shows the tracking error and the control, while Figure 4.4 shows the states σ and x_1 over a one second period. Figures 4.5 and 4.6 show the same quantities over an extended period of time which clearly indicates that asymptotic regulation is achieved and all other states converge to their respective calculated equilibria.

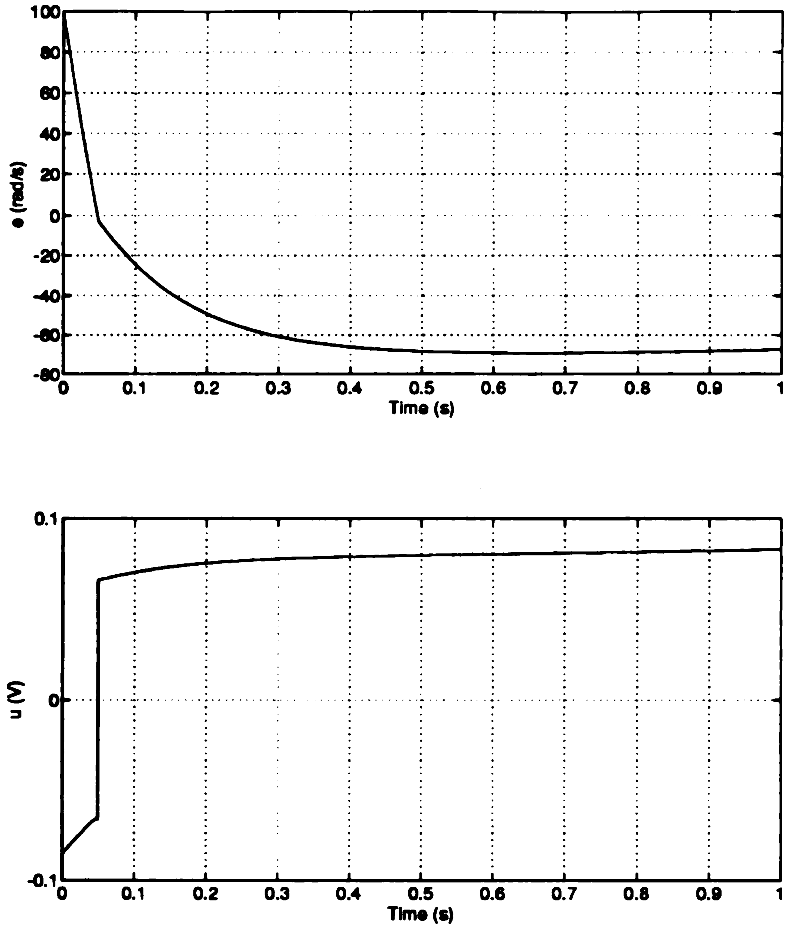

Figure 4.3. Tracking error and control $(0 - 1\ \mathrm{s})$

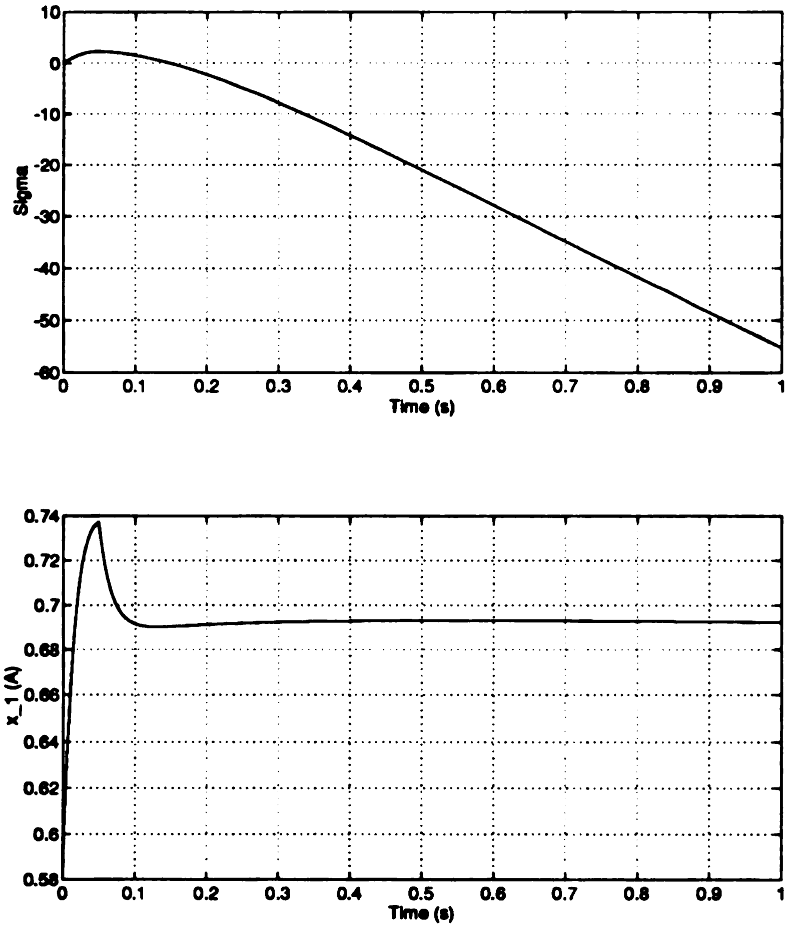

Figure 4.4. The states σ and x_1 $(0 - 1\ \mathrm{s})$

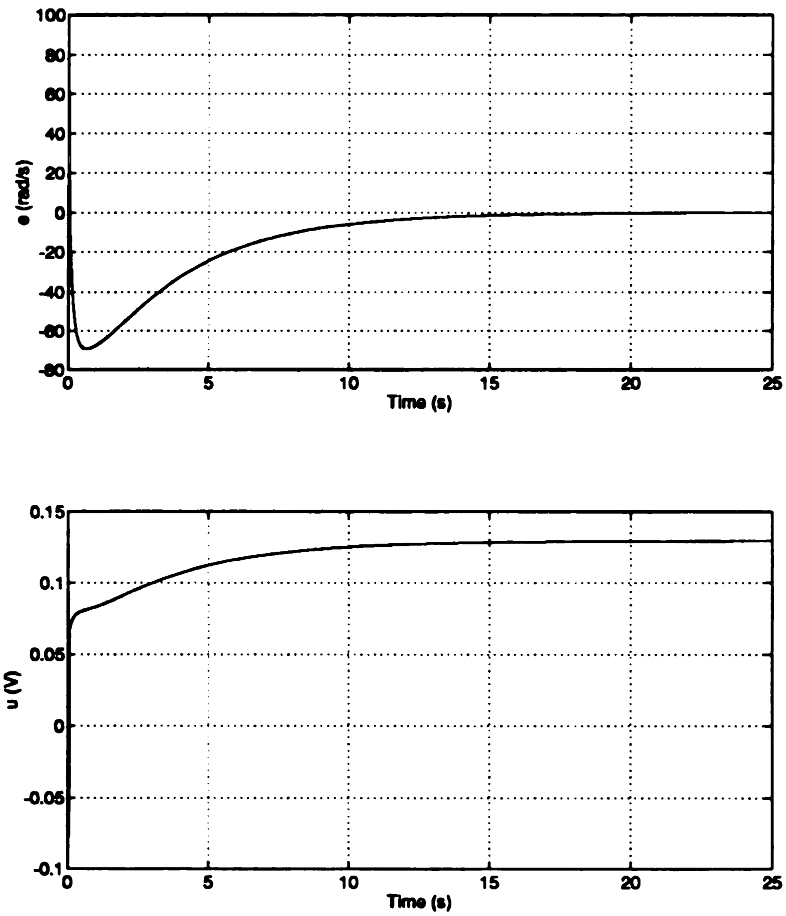

Figure 4.5. Tracking error and control (0 – 20 s)

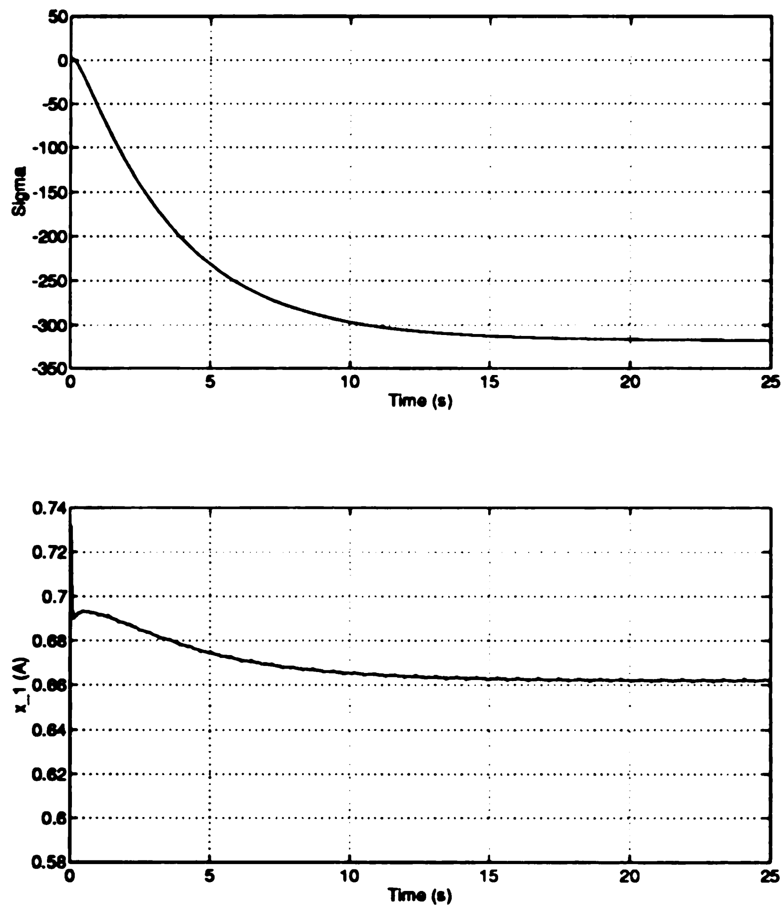

Figure 4.6. The states σ and x_1 $(0 - 20\ \mathrm{s})$

To show the robustness of the controller, we recorded the tracking error for two different sets of parameters, namely when the parameters take their highest and lowest values; this is shown in Figures 4.7 and 4.8 respectively.

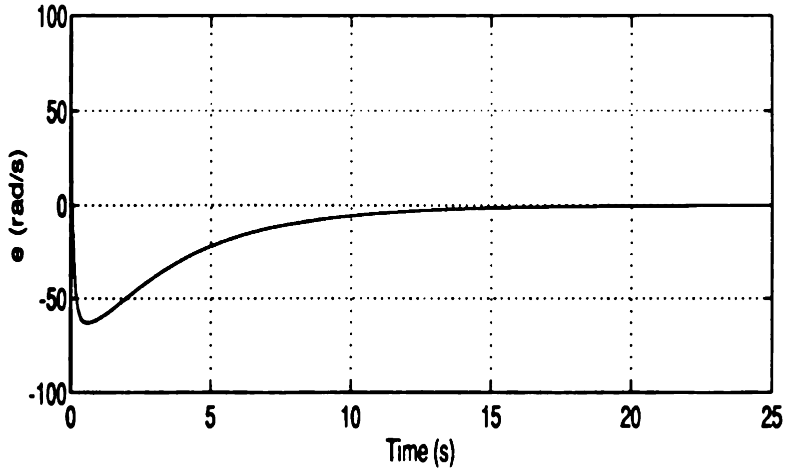

Figure 4.7. The tracking error when the parameters take their highest values

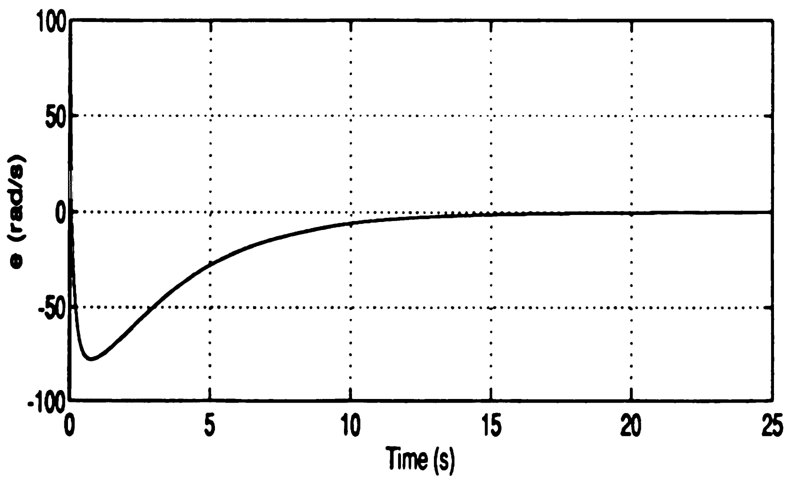

Figure 4.8. The tracking error when the parameters take their lowest values

4.3 Method 2

In this section we illustrate the design procedure of Chapter 3. The input-output model, with the speed as an output, can be obtained from the given state model as

$$\begin{aligned}\ddot{y} &= -(\theta_1+\theta_4)\dot{y} - \theta_1\theta_4 y + \theta_3\theta_5 u \\ &\quad -\theta_2\theta_5 y u^2 + \frac{\dot{y}+\theta_4 y}{u}\dot{u}\end{aligned}$$

Let $x_1 = y,\ x_2 = \dot{y},\ x_3 = u$ and $v = \dot{u}$, then the extended system is given by

$$\left.\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\theta_1+\theta_4)x_2 - \theta_1\theta_4 x_1 + \theta_3\theta_5 x_3 \\ &\quad -\theta_2\theta_5 x_1 x_3^2 + \tfrac{x_2+\theta_4 x_1}{x_3} v \\ \dot{x}_3 &= v\end{aligned}\right\} \tag{4.4}$$

The system (4.4) has a uniform relative degree of 2 on the domain

$$U = \{x : x_3 \neq 0,\ x_2 + \theta_4 x_1 \neq 0\}$$

This will be taken to be our region of interest. The following change of variables will transform (4.4) into the normal form and shift the equilibrium point of interest to the origin.

$$\left.\begin{aligned}e_1 &= x_1 - \nu \\ e_2 &= x_2 \\ z &= \tfrac{x_3}{x_2+\theta_4 x_1} - \tfrac{\bar{x}_3}{\theta_4 \nu}\end{aligned}\right\} \tag{4.5}$$

where ν is a constant reference speed and $\bar{x}_3$ is the value of x_3 at equilibrium which can be found from (4.4)

$$\bar{x}_3 = \frac{V}{2K_1\nu}\left[1 \pm \sqrt{1 - \frac{4RJ\nu^2}{V^2}}\right] \tag{4.6}$$

It can be verified that the root with the negative sign is the one associated with the minimum phase zero dynamics [53]; hence, we use it in (4.5). In the new coordinates the system (4.4) will be given by

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -(\theta_1+\theta_4)e_2 - \theta_1\theta_4(e_1+\nu) + \theta_3\theta_5[(z+\frac{\bar{x}_3}{\theta_4\nu})(e_2+\theta_4(e_1+\nu))] \\
&\quad -\theta_2\theta_5(e_1+\nu)(z+\frac{\bar{x}_3}{\theta_4\nu})^2(e_2+\theta_4(e_1+\nu))^2 + \frac{\theta_4\nu}{\theta_4\nu z+\bar{x}_3}v \\
\dot{z} &= \theta_1(z+\frac{\bar{x}_3}{\theta_4\nu}) + \theta_2\theta_5(e_1+\nu)(z+\frac{\bar{x}_3}{\theta_4\nu})^3(e_2+\theta_4(e_1+\nu)) \\
&\quad -\theta_3\theta_5(z+\frac{\bar{x}_3}{\theta_4\nu})^2
\end{aligned}$$

We identify the matrices $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We choose $K = \begin{bmatrix} -2.2361 & -3.0777 \end{bmatrix}$. The function s is found to be $s = \begin{bmatrix} 0.4472 & 0.4702 \end{bmatrix} e$, where $e = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$.

With $\alpha_1 = 3$, $\alpha_2 = 2$ and after scaling the observer states such that $q_1 = \hat{e}_1$ and $q_2 = \epsilon\hat{e}_2$, we obtain the observer model

$$\begin{aligned}
\epsilon\dot{q}_1 &= q_2 + 3(e_1 - q_1) \\
\epsilon\dot{q}_2 &= 2(e_1 - q_1)
\end{aligned}$$

Using $\mu = 0.5$, $g_0 = 10^4$, $\rho = 10$, and the control

$$v = -g_0^{-1}\rho^2 \frac{s}{\mu}$$

then assuming $|x_1(0)| \leq 300$, $|x_2(0)| \leq 0.5$ and $0.024244 \leq x_3(0) \leq 1.29721$, we find that the initial state $e(0)$ belongs to Ω_c with $c = 1.2162 \times 10^4$ and that the control will saturate at ± 10. Hence we end up with the controller

$$\phi^s(\hat{s}) = 10 \text{ sat } \left(\frac{-v(\hat{s})}{10} \right)$$

where $\hat{s} = \begin{bmatrix} 0.4472 & 0.4702 \end{bmatrix} \hat{e}$ and $\hat{e} = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 \end{bmatrix}^T$.
Setting $\epsilon = 0.01$, simulations were performed for the nominal system, using the same data we used earlier, with a desired speed of 200 rad/s and starting at the initial condition $(x_1,\ x_2,\ x_3,\ \hat{e}_1,\ \hat{e}_2) = (300,\ 0.5,\ 1.2,\ 0,\ 0)$ which corresponds to a steady state speed of 300 rad/s. The tracking error and the control are shown in Figure 4.9 where, clearly, asymptotic regulation is achieved and the control converges to zero which is the equilibrium value of v in (4.4). Also, the states x_2 and x_3 are shown, in Figure 4.10, to converge to their respective equilibria. Recall that x_3 corresponds to the original control variable, i.e., the field voltage and it is shown here to converge to the same value that the control u converged to in Method 1.

To study the performance of the observer, we simulated the nominal system over a tenth second period. The results are shown in Figures 4.11 and 4.12. Also the control ϕ^s is shown in Figure 4.13 over the same period of time. Notice that there is a one to one correspondence with respect to the period of time over which the peaking in $\hat{e}_2$ occurs and the period during which the control saturates.

To investigate the effect of making ϵ small on $\hat{e}_2$ and the control ϕ^s, we simulated the nominal system for two different values of ϵ; namely, $\epsilon = 0.01$ and $\epsilon = 0.005$.

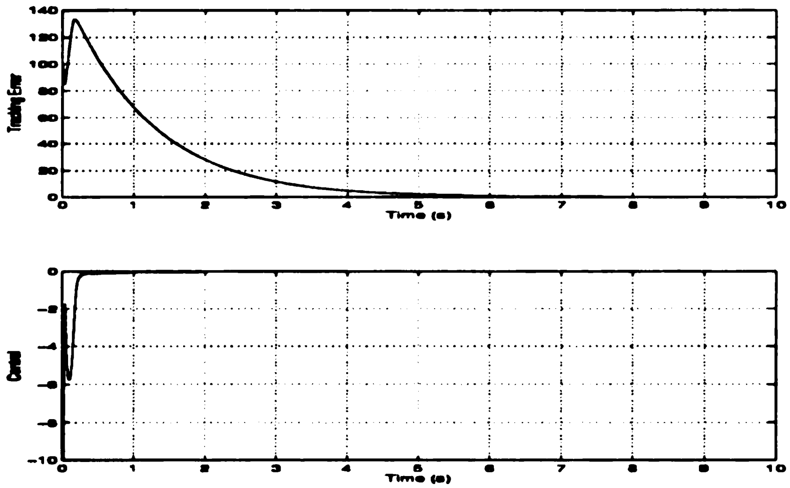

Figure 4.9. Tracking error and control of the nominal system

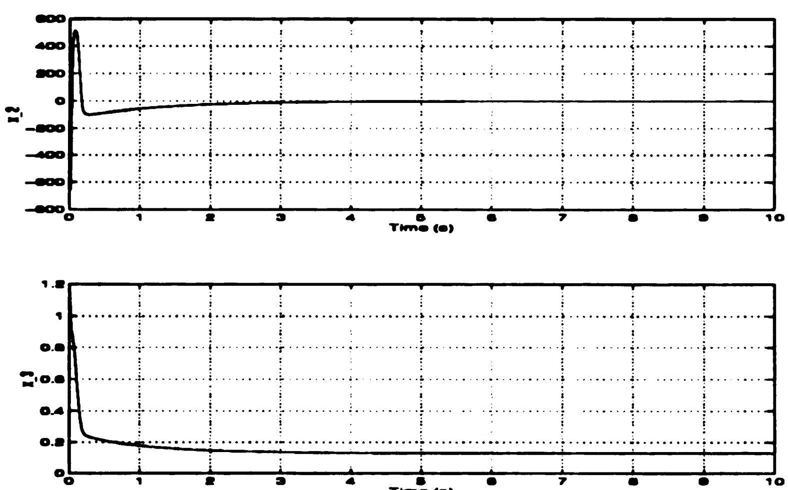

Figure 4.10. The states x_2 and x_3

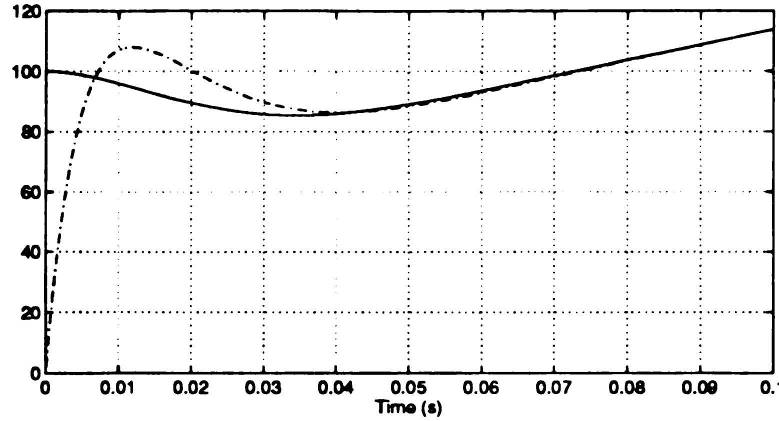

Figure 4.11. e_1 (solid) and its estimate $\hat{e}_1$

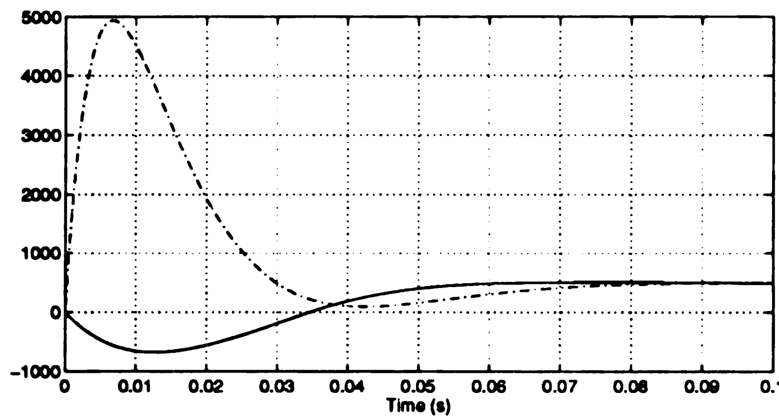

Figure 4.12. e_2 (solid) and its estimate $\hat{e}_2$

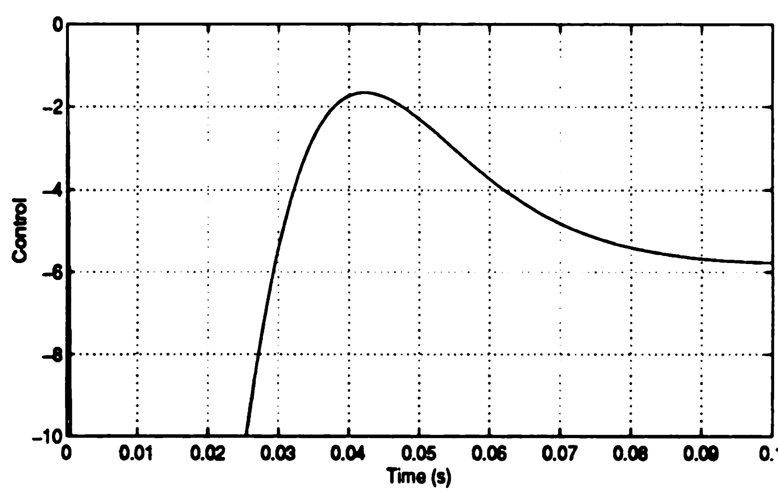

Figure 4.13. Saturation of the control ϕ^s

The results are shown in Figure 4.14 where the solid line represents the case when $\epsilon = 0.01$. Observe that the period over which saturation (peaking) occurs is smaller for smaller ϵ.

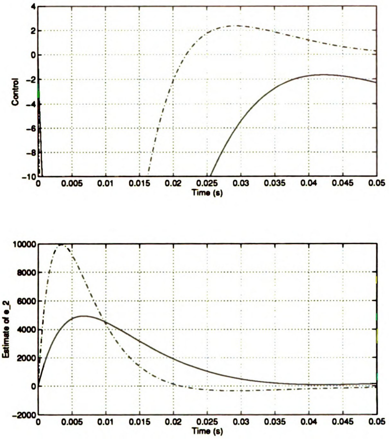

Figure 4.14. The control ϕ^s and $\hat{e}_2$ for $\epsilon = .01$ (solid) and $\epsilon = .005$

To investigate the robustness of the controller, we simulated the system for two different sets of parameters; namely, when the parameters assume , during the course of operation, their highest and lowest values. The tracking error for both cases is shown in Figures 4.15 and 4.16 respectively which clearly indicates that asymptotic regulation is achieved.

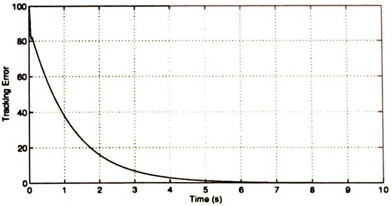

Figure 4.15. The tracking error when the parameters take their upper limits

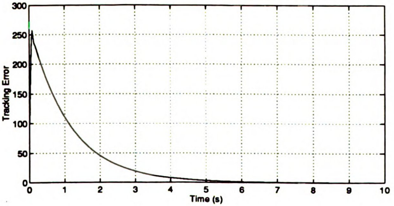

Figure 4.16. The tracking error when the parameters take their lower limits

4.4 Conclusions

In this case study, we demonstrated the application of the tools presented in designing robust nonlinear controllers to solve a physical problem. The control strategies used proved to be effective in regulating the speed of the field-controlled DC motor in the presence of uncertain parameters. All parameters of the motor were allowed to vary and the simulation results are quite encouraging.

CHAPTER 5

Conclusions and Future Work

5.1 Conclusions

In the work presented here we provided a solution to one of the fundamental problems in control theory for a class of nonlinear systems. The introduction of integral control in Chapter 2 to achieve robust asymptotic regulation for a class of feedback linearizable systems regionally as well as semi-globally is the major contribution of this chapter. The tools used to establish the results are standard Lyapunov techniques familiar to most control engineers. Our solution of the corresponding stabilization problem is shown to relax many of the restrictions imposed on the system under study as it is documented in some of the available literature. The integral control is shown to work, although locally, for a class of time varying signals.

In Chapter 3 we laid the foundation for the development of internal models for nonlinear systems. In this work we identified a subclass of input-output linearizable systems for which the internal model is known and linear. In conjunction with that we presented a class of state feedback stabilizing controllers that can be used to stabilize the zero-error manifold.

The case study treated in Chapter 4 indicates that the control schemes devised in this work can be successfully implemented to solve practical engineering problems.

5.2 Future Work

In this section we go over some of the topics that need further investigation and could be pursued in the future.

5.2.1 Multi-Input Multi-Output Systems

In our work here we dealt with SISO nonlinear systems only. Extending the results of Chapters 2 and 3 to the more general MIMO nonlinear systems could be done after carefully analyzing the relevant assumptions to determine the necessary modifications needed in order to handle the MIMO case. We conjecture that if Assumptions 2.1-2.3 hold for a MIMO system, Inequality (2.23) holds as a norm inequality and Inequality (2.24) is modified to account for the MIMO nature of the system, i.e.,

$$\|G_1(e,z,d)G_0^{-1}(e,\nu) - I\| \leq k_1 < 1$$

where G_1 is an $m \times m$ input matrix, then the analysis leading to the boundedness of the state ζ can be repeated with little modifications. We may assume that the MIMO system is square, i.e., the number of inputs equal to the number of outputs$=m$. Again, using $\bar{V} = (1/2)s^T s$, we can show attractivity of the boundary layer. Note that Assumption 2.4 is independent of the nature of the input or output. To prove asymptotic stability we need an assumption similar to Assumption 2.6 with the necessary modification on Inequality (2.40), i.e., replacing f_1, g_1 by F_1, G_1 respectively, where F_1 is an $m \times 1$ vector. The system (2.6) was given as a special case of the class of systems (2.1) that can be transformed uniformly in the parameters into a normal form via transforming the system into a strict feedback form first. Developing a MIMO version of (2.7) is yet to be done. Although partial results are available, the issue deserves further investigation to establish necessary and sufficient geometric conditions that will characterize a class of systems. We think that recovering the asymptotic properties using a high-gain observer can be achieved with little or no difficulty.

In Chapter 3, we conjecture that all relevant assumptions can be modified after a careful study. It is our intention to pursue this issue along with the derivation of geometric conditions that will characterize the class of systems that are transformable into a MIMO strict feedback form.

5.2.2 Unmatched Uncertainties

Due to the feedback linearizability property of the system, the uncertain terms will end up satisfying the matching condition. The tracking and disturbance rejection problems for systems with unmatched uncertainties need further study. We expect our results to hold for sufficiently small uncertainties. The issue then is to find bounds for which the results hold.

5.2.3 Internal Model

In linear servomechanism theory, the internal model is given and this provides a complete solution to the asymptotic tracking and disturbance rejection problem. Because this will specify the servo-compensator, then stabilization follows using any of the techniques available for linear systems design. This is not the case for nonlinear systems due to the harmonics that might be generated by the nonlinearities present in the system. Aside from the work presented in Chapter 3 which provides the internal model for a subclass of systems, identifying the internal model for any nonlinear system remains a challenging and open problem.

APPENDICES

APPENDIX A

Proof of Inequality 2.47

Need to show that there exists a constant $c > 0$ such that

$$(a+b)^\gamma \leq c(a^\gamma + b^\gamma), \ \ \gamma \geq 1$$

where a, b are positive constants. Hõlder's Inequality states that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \ \ \frac{1}{p} + \frac{1}{q} = 1$$

Set $f = (a, b)$ and $g = (1, 1)^T$, then

$$a \cdot 1 + b \cdot 1 \leq (a^p + b^p)^{1/p} \cdot (1^q + 1^q)^{1/q}$$

i.e.,

$$a + b \leq 2^{1/q}(a^p + b^p)^{1/p}$$

Set $p = \gamma$, $q = \frac{p}{p-1} = \frac{\gamma}{\gamma-1}$. Hence

$$(a+b)^\gamma \leq 2^{\gamma-1}(a^\gamma + b^\gamma)$$

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BIBLIOGRAPHY

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