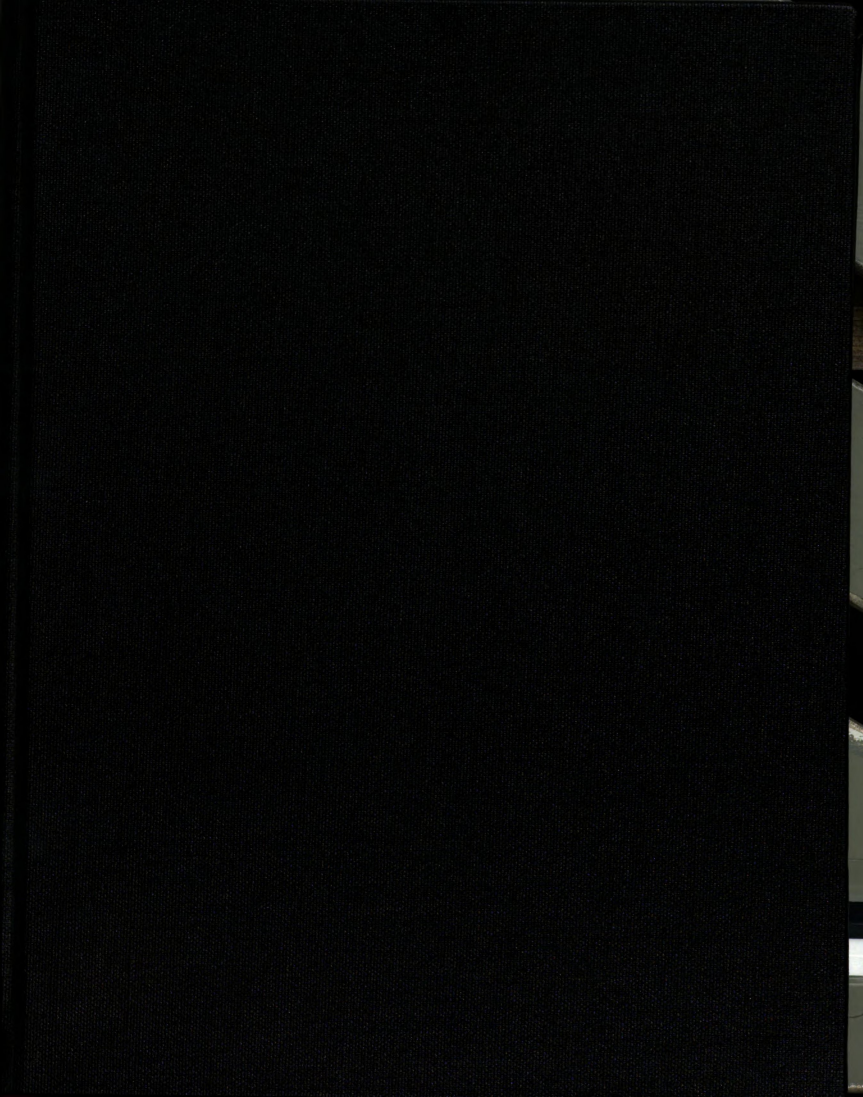






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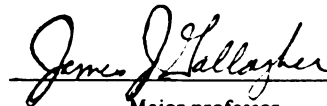
MATHEMATICAL KNOWLEDGE AND INDIVIDUAL EXPERIENCE
A STUDY OF CHILDREN'S MATHEMATICAL LEARNING
WITH THIRD GRADERS IN BRAZIL

presented by

Maria Bellini Alves Monteiro

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**MATHEMATICAL KNOWLEDGE AND INDIVIDUAL EXPERIENCE:
A STUDY OF CHILDREN'S MATHEMATICAL LEARNING
WITH THIRD GRADERS IN BRAZIL**

By

Maria Bellini Alves Monteiro

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ABSTRACT

MATHEMATICAL KNOWLEDGE AND INDIVIDUAL EXPERIENCE: A STUDY OF CHILDREN'S MATHEMATICAL LEARNING WITH THIRD GRADERS IN BRAZIL

By

Maria Bellini Alves Monteiro

This is a study about third grade students' mathematical reasoning both in mathematics classes and outside school. For three months, I worked with 30, 10-year-old children attending a school in a neighborhood of unskilled workers. I spent the time performing participant observation in the school, classroom and in the outside community--in stores, bakeries, butcher shops, as well as in the streets.

The students were asked to solve division problems during informal interviews and in the classroom context at the end of the period we had spent together. The problems were elaborated, based upon situations occurring in their daily experiences. They worked with eight, word problems involving division, at different levels of difficulty.

I found a variety of strategies those children used to solve division problems out of the school. Those strategies show the complexity of mathematical reasoning they were able to do. Out of school, students solved division problems by rounding numbers before operating them and they did estimation of possible results. The students also simplified calculations by doing successive additions and subtractions. They used some proportional reasoning and they worked with the remainder to balance the division. However, in their mathematic's class, their mathematical reasoning was limited to formal algorithms in solving the same problems. One important consequence of this difference in mathematical reasoning was their success in achieving a correct answer which was much higher in the out-of-school setting.

To conclude, I explored the differences between the strategies students used in and out of school to highlight the possible existence of two distinct views of mathematical knowledge. In doing that, I studied the roles of particular aspects of the students' social and cultural context in the construction of the notion of division. I identified the following three connections between the students' idea of division and some of the students' points of view on their cultural and social context: 1) division as separation of equal quantities, 2) division as an act of sharing, distributing and giving, and 3) division in a physical sense, that is splitting in two or more pieces or groups. Finally, based on my findings I have included some possible implications for the practice of mathematical education based on my findings, especially related with mathematical knowledge we teach and how we teach it.

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1994

To my mother

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CHAPTER I

INTRODUCTION

I developed this research with 30 third graders in an elementary school in Brazil. In its beginning two interesting events happened. I was exchanging journals with the students as a way of communicating and commenting on their experiences involving mathematics inside and outside school. When I asked them to show me how they had mathematics in their games, homes, or activities outside school, Francisca drew the picture that follows:



Figure 1 Student crying

On another day gathering data about the students' perspectives of their social milieu, Rosa, reacting to the picture showing the Brazilian President, said:

If I could talk to the president...

I was going to ask him to build more hospitals, more houses for the homeless... to stop increasing the prices of materials used in building houses. But he could do more. He could stop increasing the prices for rice, beans, sugar and the milk... He could provide more opportunities for employment for the homeless beggars, because they perhaps wanted to have one and cannot find it even though they look for it. They, the beggars ask him for food, but what is the point of giving food for one day? All the other days they will remain hungry because it won't be enough for every one of them. So, it is better to lower prices of things and provide for more employment opportunities.

This quote and the picture reflect two different sides of the students' way of approaching mathematics. In the picture, the child's tears seem to show her frustration and sadness while she is doing her mathematic work. The picture also shows the child doing calculations that are usually the kind of mathematics the student has at school.

On the other hand, the quote reveals a different perspective toward mathematics: It is the real situation where the child lives. Rosa is expressing the idea of division related to sharing and fairness in a context where not only quantitative relationships were considered

The picture and the quote reveal a paradox between the world of school mathematics and the world of the students' experiences related to their social and cultural milieu. That is the paradox I will be addressing in this dissertation.

The study site was a third-grade elementary school, mathematics class in the southeast part of Brazil. I worked with a group of 30 students, ten years old on average, who lived in a neighborhood of unskilled workers in a medium-sized town.

I began by observing students in mathematics classes. As I did so, I was looking for components of the student's daily experience surfacing during the process of classroom mathematical learning. I believed that events from the students' everyday life,

when brought into the situation of mathematics class, would reveal how both school mathematics and the students' social and cultural background interrelate in the learning process. In my second visit to the students in the classroom, I proposed to them to keep a journal with a general note saying that it would be the instrument they could use to tell me more individually about their experience with mathematics. They could write, make drawings, use pictures, or do anything else of their concern to establish the conversation with me. We exchanged these journals during the whole observation period. The period of observation consisted of three months of daily visits to the classroom and the school for at least three hours each day. Besides observing class, I interviewed the teacher responsible for the third grade with whom I was working one hour every week. In addition, we were together every day during lunch and break time where students were around in informal conversations. At school, I could interview the principal, supervisor, counselor, other teachers responsible for the library and gymnastics, and the secretary. My concern was to have a comprehensive view of the school life that in some way could affect students' learning in the classroom situation.

What I wrote in my personal journal after the first month of observations was a reflection of my perplexities. I did not perceive a clear manifestation of the encounter between school mathematics and the students' experiences outside school in the construction of mathematics learning. At that time I wrote:

... Why is it? What can I do to see better? Am I looking in a wrong direction?

What should I look for? If I am wrong and there is nothing there to see, I have to doubt whether students construct knowledge in the formal process of schooling. If they construct knowledge in the learning situation, what takes part in this construction? (Personal notes. Journal in September, 1990)

At that time I began interviewing students out of the classroom context. We talked freely about their everyday experiences in a variety of situations involving mathematics. Daily problems they faced came out easily, as well as their own ways of solving them.

Thus, I realized that I should pursue this side of data collection more deeply. I went to each store, grocery, bakery and butcher's shop near where students lived and I talked to the vendors and observed children buying and making change by themselves and with their parents. In addition, also observed children playing games on the streets. I talked to them about their activities involving mathematics and their ways of solving problems they face in their daily activities. Finally, I went to many of the childrens' homes to visit their parents. Those contacts were very rich and helped me to understand the children's cultural and social background.

Staying in the community was essential to uncover relevant themes that were part of the students' daily lives. I also felt that I had to look again more carefully at the way children were doing mathematics in the classroom, in contrast to what they were doing outside school. To do that, I planned an interview with a group of between two and four students, to explore more deeply two basic points: 1) the students' perception of their social and cultural milieu and 2) their way to solve mathematics problems involving division. Students were studying division in their mathematics classes and, thus, that topic became the focus of the research. In my last day of observation, I asked the children to solve the same problems in the classroom that they had solved in the interview. I did it in the same way the teacher used to do it with word problems in the classroom situation.

The results I found helped me to look at the paradox I mentioned in the beginning of this Introduction and to learn from it. The report of this dissertation is my account of interpreting this paradox and trying to illuminate the question that remains about the role that culture and social factors play in how students construct mathematical knowledge in elementary school.

Plan of the Dissertation

I will report this research starting with the problem I focused on and a description of four propositions that form the conceptual bases I used to analyze mathematical teaching and learning in this study. That framework guided my readings about mathematical content knowledge and context. Chapter One is a summary of those readings and a literature review about students' way of doing division.

In Chapter Two, I describe the children's perspectives about their social milieu and about mathematics. They are important to help us to understand children's ways of doing mathematics and how they relate their own experience within the school and in the community as a whole.

In Chapter Three, I explore children's experiences involving mathematics in two contexts. First I describe the formal situation of the classroom. The class about division shows how the teacher approaches the theme and how children react to the teaching situation. On the other hand, many situations that students have to face in their everyday life require mathematical reasoning. I describe some of these situations such as childrens buying things in stores, bakeries and in butcher shops, and children helping their parents at home or playing games.

In Chapter Four, I analyze how those children deal with mathematics, solving word problems. I will distinguish two contexts in doing the analysis: the classroom context and the outside-of-the-school context. The mathematical content involved in the situation problem was division. I took both contexts, in and out of the school, into consideration to analyze how children approach the problems and reason to solve them. I found important differences in the students' mathematical reasoning in dealing with word problems in the two situations.

Finally, in Chapter Five, I discuss the possibility of the existence of two different ways of doing mathematics in school and outside of school. To conclude this study, I consider the construction of the notion of division. I present three connections between

the students' perspectives of their social milieu and how they make sense out of the idea of division in mathematics.

CHAPTER II

THE PROBLEM

Introduction

This is a report about a study done in Brazil with 30, third graders about mathematical learning and its context. The study is a result of my reflections during my doctoral studies, since the fall of 1987, about cultural and social issues related to the educational process. During this time, I became interested in studying mathematical learning and social and cultural context. An approach that contextualizes mathematical learning has been the focus of attention of many researchers lately (Lerman, 1989; Bishop, 1988; Lave, 1984; Carraher, 1988). Drawing from this literature I have defined the problem in this study in terms of looking at the students' mathematical reasoning both in and outside the school context. I did it to find out how cultural and social components come together with mathematical knowledge as it is taught within the school, and to contrast students' use of mathematical knowledge in the two contexts. This chapter illustrates how this problem evolved from the very beginning until now, based on the literature I reviewed while looking for theoretical support to my questions.

My first approach was to look at how context has been focused on in the studies of teaching and learning mathematics. I became familiar with current research in mathematics education focusing on teachers' and students' understanding about mathematical knowledge and the culture of the classroom. From this research I learned that different approaches to mathematics shape different ways of teaching, with consequent implications for how students approach mathematical learning situations. The basic issue is the construction of meanings.

Then, I became puzzled about the idea of students constructing their own understanding of mathematical knowledge and the role of cultural and social bias on this

construction. Mathematical knowledge as a cultural phenomenon and its implication for the learning and teaching of mathematics is a concern within the field of ethnomathematics (Gerdes, 1988; D'Ambrosio, 1985; Ferreira, 1988). Mathematical knowledge as a cultural phenomenon constitutes the compelling ground upon which I based my research. My study was an initial approach to the role played by cultural context in how a group of students constructed their mathematical knowledge. Thus, studies of mathematical learning in different contexts were the main source in refining the basic questions for this study.

The following are the questions in this research:

1. How are students' daily experiences taken into account in mathematics classrooms?
2. How do components of students daily experiences surface during the process of classroom mathematical learning?
3. Which contrasts, conflicts, and confrontations emerge from mathematics teaching in a school setting?
4. How do individuals encounter mathematical knowledge in a classroom setting?
5. What role do cultural and social factors play in how students construct mathematical knowledge in elementary school?

Next, I will explain the conceptual framework which provides the basis of my understanding of mathematical teaching and learning.

Conceptual Framework for Analyzing Mathematical Teaching and Learning

From the studies I reviewed on mathematics and mathematics education, four propositions were derived.

Proposition One: Mathematical concepts are not “pure concepts” or “absolute truths” about physical and quantitative relations in the experiential world.

I will discuss proposition one by examining contributions both from the history of

mathematics and from the philosophical debate on the objective existence of mathematical abstractions. A full discussion of the arguments involved is impossible within the limits of this dissertation. Hence my purpose is to highlight some thoughts from scholars who have discussed this matter to shed light on the proposition. By doing that I will also outline the aspects of the proposition that I took into consideration.

Birnbaum (1987) conducted an extensive discussion of what mathematics is. In doing, that he analyzed different traditions which respond differently to the question of what mathematics is. From those traditions he concluded that what varies and makes them different is the nature of the answers regarding the acceptance or denial of the objective existence of mathematical abstractions. The question of what mathematics is moves towards the relation between mathematics and reality. Does mathematics describe the real world as it is? Does mathematics refer to “absolute truths” about the physical, quantitative relations in the universe? Is mathematics discovery or creation?

Machado (1987) examined the relationship between mathematics and reality. He took three, contemporaneous traditions (logicism, formalism, and intuitionism) to study that relationship. In his search for the relationship between those three traditions, he posited the argument that mathematics is embedded in the socio-historic process where it is produced. He presented his search for a way of looking at mathematics in the following terms:

Uma visão que explicita a situação da matemática como objeto da cultura, como ferramenta de trabalho, que revele com clareza o quanto a matemática esta inserida no processo histórico-social onde é produzida e que ela ajuda a produzir. Uma visão que logre a superação do mito da matemática hermética, ciência dos “eleitos” cuja função primordial, como a de outros mitos, e a justificação de privilégios de diferentes ordens através do elogio da técnica, ou de uma dimensão dela (Machado, 1987, pp. 16-17). (We shall search for a vision that makes explicit mathematics as an object of culture, a working tool, that clearly reveals how much it is inserted in the socio-historic process where it is produced. That vision should overcome the myth of an hermetic mathematics, science of the few “elected” whose main function, as it is the case of other myths, is the justification of privileges of various levels by

overvaluing technique as a whole or one of its dimensions).

Among intuitionists, logicians, and formalists, there is agreement over the idea that mathematics is manipulation of symbols in accordance with certain rules. Intuitionists differ from the other two by viewing mathematics as intuitive constructions of the formal manipulations of symbols. While logicians rely on refined logic to establish foundations for mathematics, intuitionists believe that the truth can be directly intuited (Kline, 1980).

Machado (1987) traced the formalist tradition back to Kant. According to Machado, formalists start with basic truth, from which they derive axioms. Using rules of inference, theorems are defined from the axioms. Thus, theorems are demonstrable elements of truth.

What all three traditions hold in common is the denial of objective existence of any mathematical reality external to the mathematician. This idealistic position considers mathematical concepts and theories as pure products of the mind with no relation to reality (Morozov, 1987).

Kline (1962) discussed the same question of what mathematics is in terms of the ontological status of its fundamental objects. In other words, the question is whether concepts, axioms, and theorems exist in some way in the objective world or whether they are entirely the result of human creation? He discussed these and other questions by using a cultural approach to the value of mathematics.

Kline's account of the non-Euclidian geometries is one example of his approach (Kline, 1962). He pointed out that the existence of the two geometries provided a significant moment in the construction of the mathematical body of knowledge. This was a moment of realization that mathematics does not offer definitive truth. Furthermore, he asserted that mathematical laws are "merely approximate descriptions and, however accurate, no more than men's way of understanding and viewing" (Kline, 1962 p. 576). Kline also discussed about the nature and values of mathematics. He

stated:

In those domains where it [mathematics] is effective it is all we have; if it is not reality itself, it is the closest to reality we can get. Mathematics then is a formidable and bold bridge between ourselves and the external world.

Though it is a purely human creation, the access it has given us to some domains of nature enables us to progress far beyond all expectations. Indeed it is paradoxical that abstractions so remote from reality would achieve so much. Artificial the mathematical account may be, a fairy tale perhaps, but one with a moral (Kline, 1962, p. 676).

The argument that mathematics is a human creation is reaffirmed in a discussion about the loss of certainty in the mathematical domain (Kline, 1990).

Reflecting on the literature I have presented so far, one may draw a conclusion that there is an objective, experiential world out there and that mathematical knowledge is a result of human creation dealing with that world. Thus, one may assume that mathematics does not hold “absolute truths” about physical and quantitative relations in the experiential world. As a consequence of that assumption, mathematics may be subjected to revisions and modifications according to different and new historical developments undergone by human beings. Also, one may state that mathematics is deeply embedded in the human culture. These conclusions bring me to the second proposition.

Proposition Two: Mathematical knowledge is culturally bounded.

This proposition is actually an extension of the first one. If it is assumed that mathematics is created, not discovered, and it is the result of human enterprise, the consequence is to admit that the very same cultural conditions of human existence are to be found in the mathematical knowledge humanity is creating. To state it in other terms, if people exist within a time and space frame that provides the conditions of their experience, then, all that they create, including the mathematical concepts they use, is also situated in the same space and time frame. What people create is usually referred to as culture. Therefore, I could state that mathematical knowledge is culturally bounded.

Historically, mathematics as a cultural product was shaped by social events and cultural differences. For instance, Bos (1984) pointed to those connections in a study of how navigation during the sixteenth and seventeenth centuries and the rise of merchant capitalism influenced the development of the Newton's Principia. In the same way, the development of ballistics has to do with Galileo's achievements on the parabola. In short, social movements can provoke and shape developments in science in general and in mathematics in particular (Kline, 1980; Machado, 1987).

Taken at large, culture can also be understood as a form of production. As pointed out by Giroux (1988), "Culture signifies the particular ways in which a social group lives out and makes sense of its 'given' circumstances and conditions of life" (Giroux, 1988, p. 193). However, one should note that these ways of life result from interchangeable relations between specific social behaviors and structures of the society at large. Therefore, they engender a historic-social process where culture can be both a product and a process of production (Freire, 1985).

Bishop (1988) supports the idea of relating culture and mathematics education. According to him mathematics is invented and created through social and interpersonal interactions. It is dependent on cultural context, as well as being a cultural product. He explains it by saying that there are six fundamental activities of the human being that are necessary for the development of mathematical knowledge: counting, locating, measuring, designing, playing and explaining. From these activities comes a symbolic technology, one of the elements of culture which functions to relate people to their environment and to other people. Other components of culture are ideological, sentimental and sociological ones. Different symbolic systems were developed in different cultures and different values. Western culture emphasizes the following values in mathematics: objectivity, security, control, and mystery. When these values are brought into the curriculum in mathematics education the emphasis falls on the idea of "absolute truth," objectivity and formalism in mathematic teaching. Bishop argues that,

in mathematics education, we should move toward progress, rationalism and openness. Students should construct their own meanings of mathematics content, rules of behavior, goals, and so on. They should develop mathematical knowledge by sharing meanings within the dynamic of classroom where the process of communication is fundamental.

D'Ambrozio (1985) relates culture and mathematical knowledge as he introduces the concept of ethnomathematics:

Marking a bridge between anthropologists and historians of culture and mathematicians is an important step towards recognizing that different modes of thought may lead to different forms of mathematics; this is the field which we may call ethnomathematics (D'Ambrosio, 1985, p. 44).

Ethnomathematics has been understood by some authors as the “study of mathematical ideas of non-literate peoples” (Ascher and Ascher, 1986) or as “mathematics incorporated in popular cultures (Ferreira and Imenes, 1986). In this sense mathematics can be of a kind that is different from the academic pattern of knowledge taught as mathematics. Ethnomathematics looks for “different mathematics” in daily life of “different cultures” to reconstruct their practices into methods and theories (Gerdes, 1989).

From the studies in the field of ethnomathematics two basic contributions might be highlighted. First is the attention given to the study of mathematical teaching and learning in relationship to the whole of social and cultural life (Gerdes, 1988). Second is the recognition of different modes of thought as having important implications for approaching construction of mathematical knowledge. From these contributions, a third proposition will be assumed.

Proposition Three: Learning implies constructing knowledge.

Learning is a result of interactions between individual and objects within social and cultural contexts. All three components, the subject, the objects, and their interaction, are active elements in the process of knowledge construction.

Von Glasersfeld (1985) sees knowledge as a conceptual construction. Therefore what is an object of knowledge should not be viewed as having a concrete existence, but rather a conceptual viability within the experiential world of a knowing subject. Here, the focus is on the activity of constructing. The knowledge that is created does not reveal an independent world outside the mind of the knower. Implied in this more radical constructivism is that different theories and conjectures about the world are equally valid until some criteria are established that allow for qualitative judgments and comparison among conjectures.

Lerman assumes that the rejection of certainty of knowledge, within constructivism, does not deny the existence of a real world. Rather he recognizes that there is no way of reaching absolute knowledge of the real world. Objectivity ultimately rests in the public nature of language. Through language, meanings are negotiated in a particular culture, time and place. What we know as theory is the result of our own attempts to organize our experiential world (Lerman, 1989).

Addressing constructivism in mathematics education, Confrey (1987) also agrees that knowledge is not “an accurate picture of the way things really are.” Because of that, according to her, knowledge is always subject to modification. She assumes a constructivist point of view where students are not merely receiving knowledge but constructing their own understanding of mathematical knowledge. The implication of this epistemological position is to see the learner and his/her social interactions as central elements in mathematics education.

Proposition Four: Mathematical knowledge and children’s experience have a unique encounter in the classroom setting.

This uniqueness could be described by taking into consideration the following aspects. First of all, schooling is an event which marks a new phase in the child’s life. It is organized, prepared to happen in such a way that expected outcomes will result from the school experience. These results are not defined by or expected from the student. In

fact, he/she does not have any participation in the decision making regarding what is going on within the school. On the other hand, the event of being at school becomes part of the child's life, imposed by norms and patterns of the modern society where we live. In addition, these school experiences might determine different directions, guidance and influences in his/her whole life.

Second, in the process of schooling, the child is not a passive element. While a student, he/she does not give up his/her condition of being a child. Within the classroom setting, life outside of school cannot be eliminated. In addition, constructing knowledge involves social construction where one of the elements is the individual him/herself. Therefore, the culture of the classroom and the cultural processes embedded in the child's experiences contribute to the uniqueness of this encounter.

Finally, in the encounter of individual experiences of students and school, mathematical knowledge in the classroom setting engenders not only idealized outcomes but also conflicts and confrontations. The study of this encounter is of great relevance to understanding the construction of mathematical knowledge. It may unveil not only its characteristics but also children's strategies of coping with the encounter.

Mathematical Content Knowledge and Context

In defining the four propositions in my theoretical framework, I have organized the literature I used into three groups: 1) studies with emphasis on student and teacher inside the classroom context; 2) studies with emphasis on cultural and social differences in and out of the classroom context and; 3) studies with emphasis on students' strategies in solving division problems independent of the classroom context.

Selected Studies of Classroom Learning in Mathematics

Several Studies can be included in the first group, varying from descriptions of teachers' and students' beliefs to, more comprehensive analyses of classroom social

organizations and interactions. Next, I will summarize the bibliography in the first group.

According to Ball (1988), teachers convey to students messages about their own beliefs and understanding of mathematical knowledge. This means that teachers' beliefs affect students. Since many teachers hold a view of mathematics as a body of skills to be mastered through drill, the practice in mathematics class might frequently result in teaching for mastery of procedures. Students learn by practicing and the teacher gives the necessary structural support to this practice. This might result in some students who are good at learning and following mathematics, algorithms and other students who do not have the necessary capability to get to the best level of learning. On the other hand, the view of mathematics as a way of abstractly working problems centered in real life context might result in a very different style of teaching.

Ball (1989) discusses teaching for understanding as an approach that empowers students to make sense of mathematical knowledge. According to Ball, students should be able to reason with and about mathematical ideas. Teachers should be able to guide this process through effective teaching based on knowledge of mathematics. Ball distinguishes two kinds of mathematical knowledge: knowledge “of” and knowledge “about” (see also Lampert, 1988). Knowledge “of” mathematics implies knowledge of mathematical concepts and procedures. Knowledge “about” embodies different dimensions of the subject matter: its nature, how it changes the way truth is established, pedagogical reasoning and subject matter in culture and society.

The latter view connects what students learn, and how they learn it, to the historical development and the growth of knowledge as a constructive process of continual invention and revision. Only “knowledge about” includes teachers’ understanding of subject matter. In addition, teachers’ subject matter knowledge interacts with their assumptions and beliefs about teaching, learning, students and context. These factors shape the ways in which they teach. Thus, good teaching is grounded in the knowledge

about students, beliefs, subject matter and context.

Authors who hold this line of thought believe that students must be actively involved in constructing their own understanding. Teachers help students to be independent learners when their teaching allows students to invent, create, and judge the validity of their own ideas and results. Lampert's research is an important example of this philosophy of. In her work with fifth grade teachers, she calls attention to the role that classroom culture plays in the social construction of mathematical knowledge (Lampert, 1988). Her research was conducted in a fifth grade classroom applying Lakatos' and Polya's ideas of valuing conscious guessing, inventiveness and risk taking. In her research, Lampert used reciprocal teaching to establish a social climate within the classroom between teacher and students, as well among students. The main goal was to establish a new model of social interaction in the classroom where meanings could be shared and students could create mathematical arguments instead of giving right answers. In addition, Lampert identified several patterns of the student's approach to learning situations including turning to the teacher as authority for verification and treating rules and formulas as if they were arguments.

Peterson, Carpenter and Fennema (1988) investigated teachers' knowledge of students' knowledge in mathematical problem solving and found a significant relationship between them. They found that more knowledgeable teachers use more active ways of teaching by questioning students about problem-solving processes. Less knowledgeable teachers use a more "passive way" of teaching by explaining to the students how to conduct the problem-solving processes. While investigating teacher's thinking about division Tirosh and Graeber (1991) found that pre-service elementary teachers present "misbeliefs" about the division process. The "misbeliefs" interfere with the way these teachers regard situation problems involving division (Ball, 1990; Tirosh and Graeber, 1990).

Kaput's work (1988) focused on students' understanding of multiplicative structures as ways of connecting concrete representations and mental processes. In other words, he looked for mechanisms bridging the formal mathematical system and "lived in

situations.” He went from the students’ understanding of mathematical structures to the construction of software “to facilitate the growth of this understanding” (Kaput, 1988).

Lanier and Anang (1982) conducted research in two ninth grade classes where the social organization of the classroom and the subject matter knowledge were the foci. They found that the relative position of the subject matter and the social organization in the two classes affected the students’ opportunities to learn. How teachers organized their interaction with students affected the students’ ability to ask questions, demonstrate their abilities and engage in public performance. The amount of teacher-student interaction also influenced students in learning basic skills, seeing the applicability of the mathematical knowledge, and enjoying the subject and the class. Ball and Wilcox (1989) also examined the interaction of context and content by comparing two in-service teacher education programs. They analyzed teachers’ points of view about their assumptions of what teachers should know and be able to do, and models of changing teachers’ practice. The authors found that both programs were similar in their intent but different in their context, curriculum, and their opportunities for teachers to learn.

Selected Studies of Social/Cultural Influences on Mathematics Learning

The second group of studies focuses on cultural and social influences on learning of mathematics. Stigler and Baraner’s article (1987) is illustrative of that focus. They argue that all cognition is highly contextual and domain specific. In this perspective, learning in general, and mathematics learning in particular, is an inherently cultural process. For the authors mathematics is a result of “culturally constructed symbolic representation and procedures for manipulating these representation” Culture, tools, practice, and institutions shape mathematical thinking and become part of the individual’s repertoire of mental representations. Stigler also demonstrates how characteristics of cultural belief systems may help explain differences in achievement in mathematics of learning among Japanese, Chinese, and American children.

Lester (1989) studied the use of mathematics in everyday situations in contrast with its use in mathematics classes. He found that in the everyday world people often use mathematics procedures and thinking processes that are quite different from those learned in schools. These differences include a focus on meaning, intentions, making sense, and natural language in contrast to a focus on syntax (symbols and rules), formal language, manipulation, and memorization.

Studies conducted in Brazil contributed to a belief that mathematical reasoning outside of the school context unveils new sides of the thinking processes not revealed by school mathematics (Carraher, Carraher and Schliemann, 1988). Based on Piaget's ideas they investigated everyday activities inside and outside of school looking at the implicit knowledge in the way people organize their actions to solve problems, and at the role of cultural contexts in this organization. The authors found that many students who fail in school mathematics do very well in their everyday activities involving mathematics. Children and teenagers construct complex models of mathematical reasoning when working with lottery games, in markets, etc.... (see also Schliemann and Acioly, 1989; Schliemann, 1990). According to Schliemann and Acioly (1989), "School experience does not play any role in the ability to solve problems at work." School experience seems to affect more academic questions or problems are outside of work.

Ginsburg and Allardice take a different position (1984). They studied children's difficulties with mathematics in the social context of school. They distinguished two systems of informal knowledge students use to solve problems. The first is tied to biological components and appears before children enter school, while the second is tied to cultural influences on students in pre-school years. Studying third and fourth grade students, the authors concluded that students have great ability to use informal knowledge and invented procedures. However, the students produced consistent errors, arbitrary results and misunderstanding of mathematical principles. The authors suggest that the

basic questions on which researchers should focus involve the cognitive processes of informal knowledge, invented procedures, and errors.

Onslow (1991) attributed students' difficulties with school mathematics to their lack of ability to construct links between the abstraction of school mathematics and its real world context. Students are knowledgeable in mathematics in everyday situations, but fail to understand the same basic principles when presented in the form of symbols.

Rogoff and Lave (1984) bring a special contribution to the discussion of the aspects of everyday cognition and its development in social context. They argue that thinking and context are interrelated and, thus, social context affects cognitive activity. Rogoff and Lave define context as physical and conceptual structures as well as the purpose of the activity and the social milieu in which it is embedded. According to Rogoff and Lave: "The person's interpretation of the context in any particular activity may be important in facilitating or blocking the application of skills developed in one context to a new one" (Rogoff and Lave, 1984, p. 3).

Studying the activity of grocery shopping, Rogoff and Lave (1984) looked at the nature of arithmetic activity related to problem-solving situations in the supermarket. They described several characteristics of arithmetic problem-solving in the supermarket in which procedures of solution were dialectically constituted. The operations used created circumstances of continual monitoring of results which might explain the virtual, error-free, arithmetic performance of shoppers in contrast with their results on formal testing situations.

The third group of studies on mathematics' teaching and learning is related to the students' strategies to solve division problems. The bibliography in this third group will be highlighted in the following section.

Selected Studies on Students Doing Division

The sequence of studies I will present next was chosen based upon two criteria. First, my research was conducted in a third grade class at an elementary school. During the time I observed mathematics classes, division was the unit being studied. Thus, I selected studies approaching mathematical teaching and learning in general and division in particular. Second, I was interested in learning more of what the literature had to say about students' strategies to solve division problems. Thus, I selected literature on research dealing with everyday cognition, focusing on a student's own way of dealing with those situations.

The study of the student's strategy for solving problems is not new. In 1981, a conference was held at Indiana University entitled "Issues and Directions in Mathematics Problem Solving Research." Several papers were presented and lines of research for mathematics education were discussed. Mayer (1981) argued the importance of establishing basic principles of learning and cognition to promote meaningful learning. He distinguished algorithmic procedures from representations used by the students in problem solving. The author suggested that problem solving involves a series of mental operations that transform knowledge representation rather than a series of learned behaviors.

In a paper about priorities for mathematics education research, Lesh and Akerstrom raise an important issue on applied problem solving. The issue is the adequacy of word problems when related to everyday situations that students deal with in the real world. The authors claim that word problems often differ from real world problems with respect to the degree of difficulty, the processes needed in the solutions, and errors most frequently made. These and other presentations reinforced the relevance of the theme of solving problems in future developments on mathematics education. The last decade has been particularly important in the development of studies involving multiplication and division in solving word problems. There is a general agreement on the constructivist approach to the knowledge acquisition, that underlies these studies.

The importance of the active role of the student in constructing mathematical knowledge is taken for granted. Two lines of development seem to emerge from that: 1) the recognition of the importance of the cultural and the cognitive aspects on learning process and 2) the relevance of the student's informal knowledge about mathematics. Common to both is the recent effort researchers have made to look for instances of mathematical reasoning outside the school setting. In addition, the relationship between mathematics taught at school and the mathematics students deal with in real world situations has become the focus of attention.

While investigating the levels of mathematical understanding shown by British secondary school students, Booth (1981) reports that children operate within a system of their own. They do not use the mathematics taught at school but, rather, the mathematics belonging to a universe different from that of formal mathematics.

There is no complete agreement about the origins of a personal system of operating. Borel (1987) uses the term "natural logic" to contrast the child's logic with forms that logicians know. The former is characterized as being real logic and natural thinking or naive logic and naive thinking. The latter is characterized as being formal abstraction, pure axiomatic or refined thinking, all of which are very specialized constructions of logicians.

In this regard, Fischbein's studies have been largely quoted in the discussion of ways children solve division word problem. Fischbein argues that children's work is mediated by implicit, unconscious, and intuitive models. He defines intuition as part of intelligent behavior with characteristics of immediacy, globalism and extrapolative capacity, among others. He further differentiates primary intuitions derived from the children's own experience without any formal instruction from secondary intuitions formed mainly at school. He believes that these intuitions of both types, being natural and basic or reflecting the way it was taught in schools, are resistant to change and intervene in a child's attempt to solve a problem. In one of his many publications,

Fischbein (1985) defines intuitive models associated with the four arithmetic operations. For division he distinguishes two models according to the structure of the problem. First, in partitive division, or a sharing-type of division problem, an object, or collection of objects is divided into equal segments or sub-collections. Second, the quotative division, or measurement, is defined as how many times a given quantity is contained in a larger quantity. According to Fischbein (1985), the model imposes the following four constraints on the search for a solution to the problem.

1. The dividend must be larger than the divisor,
2. The divisor must be a whole number,
3. The dividend must be larger than the operator,
4. The result must be smaller than the dividend.

While studying the invariance of the operation modeling a situation and the numbers involved in division and multiplication Greer (1988) examined Fishbein's arguments, he questioned why these intuitive models continue to affect students' thinking in more general domains. The author suggested that both the student's inadequate experiences and lack of attention to mathematical modes of thinking contributed to that. He also suggested that "hard, " or more difficult, numbers should be used routinely in problems, as well as creating cognitive conflicts and reducing single operation word problems.

Kouba (1989) and Bell et al. (1989) illustrated some of the limitations of Fischbein's theory of implicit intuitive models and present some contributions to this discussion. Kouba examined the intuitive models to derive a classification system for children's strategies by testing models of development of solution strategies for multiplication and division problems. She worked with more than one hundred children from the first, second and third grades and she concluded that "a more detailed definition of children's behavioral interpretation of division" was necessary to clarify Fischbein's intuitive models. Bell et al. studied more than 300 subjects ranging from 10 to 20 years

in age to review Fischbein's theory on multiplication problems and propose a new one. The authors claimed that there is insufficient consideration to pupils' numerical, rather than structural, perceptions in Fischbein's work. They developed the notion of "preferred multipliers" and they demonstrated how children and adolescents are sensible to the structural aspects of multiplicative problems by their success in estimating answers. Therefore, the numerical preferences and other factors dominate their choice of the operation to be used in a given situation. This study suggested that the same questions should be asked relating to division problems. In 1984, Bell et al. published a study about the effect of the structure of the problem and other factors on the choice of operation in verbal arithmetic problems. They worked with children 12 and 13 years old and observed that the children reasoned in a qualitative way about problems, but were unable to relate the numbers to the problem quantitatively. The students did not perceive, for a given context, the invariance of the relationship among the numbers despite their size. In addition, uncertainties about language and notation were identified.

Bell and his colleagues found that the type of numbers in the problem was a source of difficulty, especially when the operation of division or multiplication involved numbers less than one. They asserted that a hierarchy emerged related to the structure of the problem. Children moved from partition to fractional partition, to quotation, to fractional quotation and finally to rate. The authors found that the students made successive attempts to transform the problem into more accessible structures to be solved.

The influence of the context was also studied by O'Brien and Cabral (1989). They were interested in looking at the achievement of students in first, second and third grades on multiplication and division problems, in two different environments. Eighty-nine students were interviewed to solve problems mentally and using paper and pencil. They found that the environment was not a factor in determining achievement. Environment in this study was defined by the environmental condition within which students had to

solve problems: with or without paper and pencil. However, there were significant differences in performance, according to the structure of the problem and the students' grades.

Teule-Sensacq and Uinrich's study (1982) also focused on students' achievement. They suggested that working division problems within a situation where the students might face the new could result in more success and variation on solution procedures. This methodological procedure was described in terms of promoting dialectical discussions of student's actions facing the new.

Besides discussions about intuitive models, the interest of many researchers has been the variety of strategies students use to solve division problems. These strategies are rich in students' mathematical reasoning to solve problems which do not necessarily come out in the school context.

Generally speaking, it has been recognized that young children are good problem solvers (Moser and Carpenter, 1982; Gilbert and Leitz, 1982; Carraher et al. 1987; and others). Children facing a situation of problem solving come up with a variety of strategies by themselves and are persistent and creative in working out solutions. They make rational choices among strategies to solve particular problems and try to bring them to a more concrete situation, more directly related to their own experience. Research has been done to identify which factors influence their choices of strategies and to have a better understanding of the children's constructions (Zweng, 1964; Sowder, 1988 and 1989; Kalin, 1983; Weiland, 1985; Kouba, 1989; Boero, 1989; Schliemann, 1990; and others as reported before).

Sowder (1988) has a different view of children's strategies in solving problems. Studying about 70 students in middle school, he concludes that students "might be solving problems using strategies which are quite limited in applicability." They do so because of the emphasis on a computation-centered curriculum where the correct answer justifies the means. The author asserts that a correct answer is not a "safe indicator of

good thinking.” According to Sowder (1989), children’s solutions to story problems are immature strategies. He names three types of strategies: coping strategies, computation-driven strategies and strategies of limited usefulness. There is no complete agreement about the origins of a personal system of operating. He indicates that some of those strategies are guessing the operation to use, trying all the operations and choosing the most reasonable answer, and looking for isolated key words or phrases.

On the other hand, several other researchers have found very rich procedures among the strategies children use to solve problems and sometimes complex mental calculations, too. Among the more frequently named are: strategies they found sharing, counting, repeated additions and subtractions and recalled facts. In 1964, Zweng had already identified sharing and counting strategies among second grade students who were solving division problems. While working with second grade students using a set of blocks, Keranto (1984) found counting and sharing strategies. In partition division, children used trial and error procedures to try out the size of sub-groups, or give out one at a time to form sub-groups. More recently, among others, Kouba (1989) and Boero et al. (1989) described, the use of the same manipulative distribution strategy based on concrete material or drawings. Kouba discussed “double counting” and “transitional counting.” In the first situation, students dealt with two counting sequences. They kept a running count of the total number of objects in the groups. At the same time, they kept counting out the objects to form the group. In the second situation, students counted sequence based on multiples of a factor in the problem. Kouba also made reference to the fact that among students from the first, second and third grades, the majority in his study made use of a trial-and-error strategy. They guessed or estimated a number and checked it. If it didn’t work, they tried another number.

Boero and colleagues, working with six-to 11-year-old children also mentioned trial and error as an approach to use with successive addition and subtraction. Repeated addition and subtraction, as well as recalling of facts among others, are always present in

the strategies children use to solve division problems (Boero, 1989).

Kalin (1983) used the expression “multiplicative thinking strategy,” referring to recalled number facts by taking the multiplication as the inverse of division. Thus, 16 divided by two is equal to eight because eight times two is equal to 16. The author also adopted the term “solution strategy,” when a student takes the known fact to find out the unknown fact. Boero et al. (1989) described this strategy as being part of the distinct trial-and-error strategies, in partitive division problems. Kouba (1989) described recalled number facts as the maximum grade of abstract procedures children take to solve multiplication and division problems. The author argued that, in this strategy, childrens do not use physical materials. They get the answer by recalling the appropriate multiplication or division combination or derived fact.

Besides the strategies mentioned above, some authors refer to other strategies children use when solving division problems. Weiland (1985) describes children using distributive algorithm as shown in the following situation: 927 pennies to share with nine people.

900 pennies - 100 to each
27 pennies - 3 to each

Answer: 103 pennies each

or

$$\begin{aligned} 927 \div 9 &= (900 \div 9) + (27 \div 9) \\ &= 100 + 3 \\ &= 103 \end{aligned}$$

Finally, Carraher et al. (1987) studied written and oral mathematics to investigate the effect of the situation on the choice of procedures and efficiency in problem solving. The students were from third grade solved problems in three different situations: simulated store problems, word problems, and computation exercises. The authors found

that there was no uniform strategy for solving problems among the situations studied. Furthermore, the children had a good understanding of the decimal system, but the educational system led them to focus on algorithm calculations. In oral mathematics, they found two types of repeated grouping and decomposing. Children usually preferred to deal with hundreds, tens and units in the opposite direction to that used in algorithm. Also, they tended to work mentally with quantities ending in one or more zeros because they were easier to work with. During the computational procedures children kept monitoring the quantities with which they dealing.

Such diversity of strategies used by children solving problems raises the question about how students make sense of mathematics. In the next chapter, I will discuss some cultural and social components in children's sense making.

CHAPTER III

CHILDREN'S PERSPECTIVES

Introduction

How much and in what ways can cultural background and social environment shape students' understanding of mathematical concepts? This is one among many questions that could be posed when the subject is the social construction of mathematical knowledge. To get to the answer, I believe it is necessary to know more deeply how social and cultural forces affect learning of mathematical concepts. That is, how may cultural background and social influences help explain the students' construction of mathematical concepts?

In this research, my main purpose was to unveil some aspects of this encounter. To do this, I worked both inside and outside the school buildings. First, I tried to capture moments in which students dealt with mathematical situations where social connections could be made spontaneously by the students, or provoked in some way. This happened in a variety of situations within the school, in mathematics classes, in the journals we exchanged, in formal interviews, and in conversations at lunch time or other school activities. This happened also outside the school, in the students' houses, on the streets during our conversation, at play, and at the stores. I tried to keep my mind open to detect moments of the students' mathematical reasoning in which their social or cultural background made a difference in how students dealt with mathematical knowledge.

Second, I tried to create opportunities in which I could collect data about the students' perspectives on social themes and the mathematics they could relate to these themes. I used pictures from local newspapers showing scenes of the students' social context. Initially, I asked them to describe those scenes and comment on the picture. Then I asked each one of them what he or she could see, create or invent about

mathematics, related to the pictures.

Ten pictures were selected from those which showed scenes taken from the students' milieu. The selection was also based on the idea that the picture should challenge a student's mathematical thinking. All of the pictures should portray social events common in the students' daily life. The themes in the picture were not defined beforehand, but they were brought up during the interviews. The following are the scenes in the pictures used in the interviews:

- 1- A car passing a sign indicating 60km/h speed limit
- 2- Several men from a paving company repairing potholes in the street
- 3- Several musical instruments and pictures of two men talking
- 4- A saint's image being carried by four men and a crowd gathering in the church surrounding the image
- 5- A parking lot behind a building under construction where several men are working
- 6- An ex-president, running for senator of one of the states located in the north region of Brazil, leaving his campaign committee
- 7- A water reservoir with three pipes in it
- 8- A building with the word "*Greve*" (strike) painted on the glass door
- 9- A candidate running for governor of the state of *Minas Gerais* being interviewed
- 10- Brazil's current President in a meeting at the Presidential Palace

The students in my study were third graders in an urban school with 1100 students. They lived in a community of unskilled workers in a medium-sized town located in south east Brazil. The students ranged from nine to 14 years of age. Five of them were considered out of the expected range of age for this grade. The number of family members in those students' household averaged six people. The school was located 10 to 20 minutes away from their houses in the majority of the cases. There is no public

transportation specially provided for the children. They go to school on their own and by foot.

Children's Perspectives About Their Social Millieu

To give the reader a better understanding of the students' perspective on the milieu where they live, I will report here how they expressed themselves in the interviews, as they reacted to the pictures. As I said before, the pictures were used to stimulate students' thoughts and expression of themes related with the social and cultural context where they live.

When I asked what the picture was about, the students gave a detailed description of what they could see. They also added some kind of action or evaluation of it. For example, if students were talking about the car in picture #1, they would say that it was running, or the road was full of holes or dirty. It was through these kinds of comments that I identified important social themes as part of the students' daily experience and their way of expressing themselves.

Several themes were identified in the interviews including: religion, death, protection, politics, inflation, power, authority, salary, food, and work. I will describe them next in three major categories. I chose the categories based on the centrality of the theme in relation to others, as they were discussed by the children. For instance, poverty came up when students talked about the power of deciding about things in picture number 10 (the president of Brazil). Then "poverty" was subsumed under the theme "power."

The three major categories are: religion, security and power. Under the category of religion are the themes of death and protection. Security covers mainly the theme of violence. Power embraces several themes discussed by the children in the interviews, including: poverty, inflation, workers, jobs, salaries, strikes, shelter and elections.

Religion. Religion is one of the themes mentioned before. Traditionally, the Catholic church is predominant in this region. According to this tradition, there is

emphasis on the initiation of children in the Church's rituals such as baptism, first communion and others. At the age of seven or so, children go to the church school to be prepared to receive first communion. That is a special day in their lives. However, there is no emphasis afterward on continuing to perform this ritual, or even going to the church on Sundays as their parents usually do. Talking about picture #4, they expressed the idea of religion mixed with feelings of gratitude, protection, hope, sadness, and death. The strongest connection they made was with death. They mentioned it several times. The following are some examples.

Carlos: There is a coffin, a saint all illuminated.... They might be praying for the dead....

Laura:... praying for a friend of them because he used to do many good things to them.

Vera: I see a "*velorio*" (deathwatch), a saint, and a lot of people praying. They are praying because a relative is dead... to get salvation... it is sad....

Death was also a subject approached by the students in our conversation through journals. I asked them to tell me about some experience they had outside of school. They reported very traumatizing personal experiences, several of them involving the death of someone. The following journal entry is an example:

I remember it as if it was today. I was four years old when in a rainy day this happened. It was raining, and my brother who worked in the country club, he loved to work there, but that day was really raining and my mother did not want to wake him up to go to work. When he got up on his own he was mad at my mother because she did not wake him up to work. She said to him: 'Jorge don't go there today it is too rainy.' He answered: 'Mummy I like to work there, and besides when it rains we just keep chatting while it rains and go home at the end of the time with our money.' My mother insisted but he wanted to go anyway. By the lunch time, my little brother was in his cradle and my mother was fixing the lunch. My other brother came and asked for J. and just while my mother was answering that he was about to come home from his work my cousin rushed into our house. Something horrible happened, she said. And then she told my mother that my brother fell down from the *pinguela** when he was going to work. My mother was crying

desperately. They called the firemen and the police but they did not find the body. Even after the rain stopped and the waters lowered they did not find my brother. They never have found anything.

This kind of “drama” is part of their daily lives due to the poor conditions in which most of them live. Children and their families look to the Church for support. Because of that the Church assumes two roles: 1) a buffer between the people and the struggles they face, and 2) dispenser of protection and improvement for their lives. It was apparent to me that when children talked of the Church they were not referring to the institution, but to the power of changing things by praying to the saints. Rituals such as mass and processions are performed to mediate their hopes and requests to the saints.

This is indicated in the following piece extracted from one of the interviews:

Eduardo:... People are attending a mass... there is a saint and they are praying....

Neuza: I think they are praying to *Nossa Senhora* (Our Lady). I think they are praying to be protected... to improve life.

Silvio: Probably they are asking (Our Lady) to provide for food in their homes... and not let it run short for others, too....

In general, there is a promise made to the saint by those asking for grace whatever it might be. It means that a promise is made to the saint as way of “symbolic payment” or thanksgiving for things they receive. The following conversation illustrates this point.

Ruth:... a woman is married and her husband drinks a lot, so she goes there (to the saint) and asks him to make her husband stop drinking and then, he stops. When he stops, she... my mom has already done the promise... she took white lace to the saint.

Leda: Every year I go to Aparecida do Norte (a place where the saint is who is considered a national protector of Brazilian people). I, my grandmother, and my aunt...

* A *pinguela* is a connection between two hills made with a piece of wood, usually a cut tree. It is used most often to cross small rivers or flooded places during heavy tropical rains. It is not normally dangerous. However when it is wet, a *pinguela* is a deadly trap.

this year all my family is going to....

Interv: Do you know why your mother made the promise?

Ruth: I know. It is because she was sick and had to have medicine in the hospital. She asked the saint to cure her. When she got better she went there and gave her (the saint) white lace.

Security. In the picture #6 there are several cars in a parking lot, including a police car. This was enough to initiate a conversation about “security.” Children emphasized the role of the policeman in controlling crime and protecting people against robbery, assault, and other criminal acts. Children expressed themselves as follows:

Lucia: There is a police car, other cars and a wheel barrow... I think that there are men working there taking soil out of the excavation.

Interv: Why is there a police car?

William: The policeman is there to see if anyone exceeds the limit allowed by the traffic signal... to obey....

Fernanda: Do you know what I think about this? The police might be hunting some criminals or kidnappers....

L.E: There is a policeman there... might have been an assault....

In the large cities in Brazil, violence and crime are major problems and a great number of the population live in subhuman conditions without adequate food, clothing, and shelter. The city where this research was done is the third largest city in the state and is not far from two big capitals, only 300 km away. This proximity is one factor in facilitating the increase of crime in the region, because it has been a place for robbers and other criminals to hide. While I was collecting data, three events gained the attention of the whole population, one of them having national impact. All three happened in the school neighborhood. The first one was the kidnapping of a police officer by three prisoners. They kept the policeman in a house for days until the kidnappers surrendered. This caused change in the routine life of the whole city including the school. All school

activities were suspended for three days to preserve the children's security. The second event was the murder of a man, and the third, a rape of a fourth grader student from the school where I was collecting data. I heard from neighbors that this was the third rape that had occurred that year to a student from this school.

I should say that there is a police station adjacent to the school's main entrance. Every day, students pass by it as they enter the school. This station is open day and night but has few personnel. This limits the amount of assistance they can give to the neighborhood. Many times the police are called to deal with cases of alcoholism involving men beating women and other violence. The police deal with ongoing family violence, but their action is peripheral to enhancing the actual security of children and families, and community security is by no means guaranteed.

Power. These passages have shown how some specific pictures gave rise to various themes including religion, protection, security and others. I would like to stress that among the 10 pictures, seven brought up themes related mainly to politics, power, inflation, poverty, work, and salary. In addition, I would like to address the relationships of students' comments to these and other themes. The following Figure illustrates how I see the power relationships students perceive in the society as a whole.

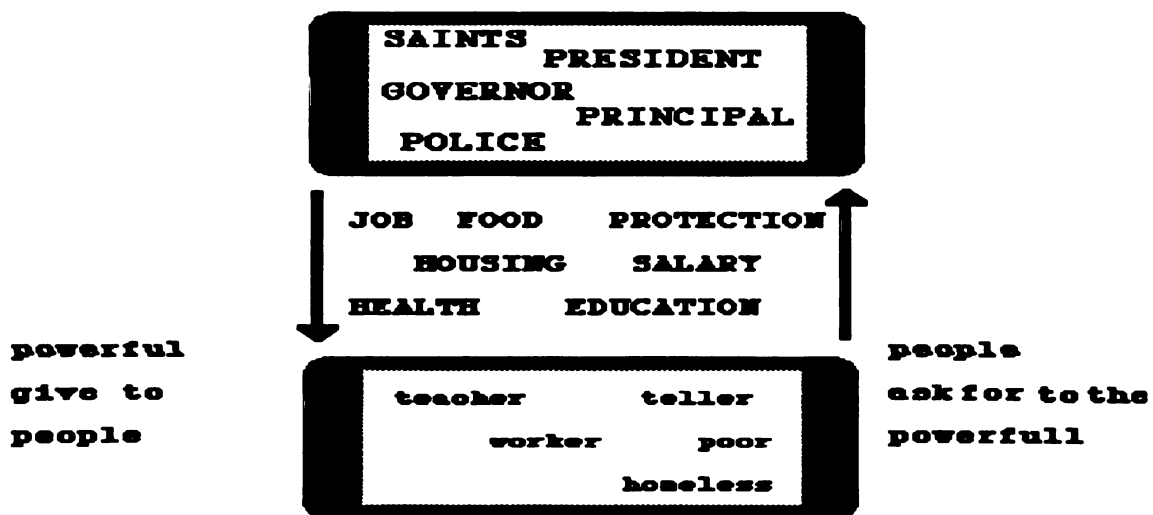


Figure 2 Power relationship students perceive in the society

Children associate the power to do things, give, decide, divide, take and so on with someone else above them such as the mayor, the president, the governor, a policeman, and saints. More than that, the power is not related to institutions, for example, the church or the government *per se*. Power is personalized in the figure that represents the institution at that moment. Power is not perceived as residing in the government. Instead, it resides in the official such as the governor. It is not the church as an institution but the saint who has the power. The saint, the governor, and the mayor have the authority and power to decide and solve problems. On the other side are those who do not have the power to decide, to have or to do anything, and who as a consequence, ask for and wait for those in power to act. The powerless are people in general like themselves, the workers, the poor. This condition of inequality is expressed often by the children when they talk about workers, salaries, jobs, and housing and other issues. The following are some examples:

Eduardo was talking about what he thought was not good in picture #2.

Interv: Eduardo:... people working for nothing.

Do you know someone who works without payment?

Eduardo: No, working for nothing... I am saying... doing a lot of effort and getting almost nothing.

Interv: And here (in the picture) who do you think has low earnings?

Eduardo: The workers (showing the men who were working throwing asphalt).

In another moment of the conversation in picture #4:

Eduardo: In this picture there are several people and there is a saint... they went to visit the saint.

Interv: Why did they go to visit the saint?

Eduardo: Because the saints are the protectors of the poor.

Nelio: Here there are several people. There is one... kind of window with some flowers, and there is a saint. I think they are praying to *Nossa Senhora* (Our Lady)

Aparecida. I think they are praying... to protect them and to improve their lives.

Rosa: They might be asking not to fall short of food in their homes... for other people not to fall short of food either.

In picture #5, about the former president:

Hilda:... he could help the poor, and not increase the prices too much.

In picture #8, about the former state governor:

Fatima: He is being interviewed by the press. To be a good governor he could help poor people and abandoned children.

Interv: If you were the governor what would you do?

William: I would give more food, more houses, and schools.

Laura: It is a bank on strike...

Inter: Why are they on strike?

Roberto: Because B. (the mayor) doesn't want to give them money.

Lucia: To get more money... to have more money....

Interv: Money for whom?

Lucia: Ah! For us! For everybody who works.

Interv: Why do you think the mayor doesn't want to give the money?

Roberto: Because he is selfish.

One important aspect of this picture is that it was the only one where children mentioned some kind of social movement to get something from the established power. The word strike was in the picture and they related it to several strike movements that the community had seen lately: workers in banks, teachers from public and private schools, and students and teachers from the University. In our conversations about strikes none of the students mentioned or referred to a strike as a movement of workers asserting rights. Some students mentioned dissatisfaction of workers with low salaries, like the teachers' salaries, or not having wage increases equal to the inflation rate. This might suggest that the students see a strike not as a movement for requesting something that is their right,

such as fair salary. Rather, the students see it as being a response, or a reaction to the unfair condition of low salary. In this sense, the worker stops working because he or she does not get a good salary.

Children expressed their understanding of a strike also as a result of dissatisfaction. Workers strike because they are dissatisfied with their salaries. As mentioned earlier, the children gave several reasons to justify the strike but the workers feeling inferior to their bosses was also present in their comments. Once more I noted the personification of the figure of power. Here are some examples:

Luiz: The bank is on strike. There are some people working and others outside.

Interv: Why was the bank on strike?

Luiz: Because they were tired.

Fabio: They were on strike because people in Brasilia (where the central government is) do not send money to pay people who work here... and then they go on strike. And then while they don't have this increase in their salary, and the same for the bank workers, they don't go to work.

William:... strike is not to work for some days...

Interv: Why?

Creuza: because the Mayor is paying low salaries....

William:... they are angry with their boss... because they pay low salaries, the same with teachers.

Interv: Why does a person go on strike?

Fatima: because the president doesn't give them better salaries.

Among the themes more often addressed by students during these discussions were low salary, poverty, inflation, housing and food. Picture #10, where the Brazilian President appeared to be talking to someone, provided a particularly rich opportunity for children to express their anxiety about these themes. Children showed that they believe in the person who is the president. They believe that, by being a president, he

could solve problems. The following conversation about important things students said the president should do exemplifies this and reinforces the point about power and inequality I discussed before.

Rosa:... he should build a village with a lot of houses and give them to the homeless to live. To give food to them....

Roberto:... ah Maria, to give them a job too.

Laura:... one who cannot buy anything, does not have anything to eat, too... money... because it is misery what they are paying... It has to improve... the government, Brazil. For example... can you imagine everybody... even doctors, electricians... everybody could have one hundred thousand per month?

Roberto: (surprised!) GOSH!

Laura: Ah, that is for everybody to be rich like them... Isn't he rich?

Do you think he earns 100 thousand?

Laura: Ah! more than that....

Interv: How much more?

(Everybody answered together in chorus): MORE than one billion!!!

Nobody talked about elections as a process of choosing their representatives, although they were in the middle of a political campaign. Within a month from the time I collected these data, elections would be held for the governor, congressman, and district representatives. In addition to that, there was a presidential election a few months earlier after 20 years of dictatorship. Both events seemed not to have introduced anything new to the children's lives. In fact, lately in this country, politics were seen as an opportunity for corruption.

Students' Perspectives about Mathematics.

To learn about the social and cultural factors in the construction of mathematical knowledge, I examined two dimensions: the students' social context and their

perceptions of school mathematical knowledge. This examination was necessary for two reasons: 1) to be able to identify them when together, and 2) to perceive their differences in the construction of mathematical concepts.

Having examined the students' context, I will now present some general findings about how students perceive mathematics. After that I will analyze the contrasts I see between occasions when students talk about mathematics and when students do mathematics in their daily activities.

First of all, I have to distinguish here two different aspects of the data: when students are "talking" about mathematics and when they are "doing" mathematics. These moments do not represent a sequence in the students' responses, and they are not identifiable as hierarchical levels. These moments are only different instances in student responses or behavior. To make this clear, "talking about" mathematics means students referring to mathematics as a third element in the conversation. They are not talking about themselves or about something related to them. They talk about something outside of themselves. "Doing mathematics" means students are reasoning in mathematical terms or analyzing any kind of mathematical situation without thinking about whether their activity is mathematics. Talking about mathematics typically happened when students and I were talking about mathematics classes, or in their journals, or talking about mathematics in the pictures I used in the interview, or about mathematics in their games and other activities. Doing mathematics, on the other hand, happened when students were doing things requiring mathematical reasoning, inside or outside school, such as organizing teams to play *queimada*, scoring the game set, solving school problems, cooking, or helping their parents.

Initially I address mathematics as students talked to me. Later in this report I will be referring to "doing mathematics" to contrast with the first one. On different occasions, we discussed this subject: in class, in their journals, during the interviews or in our conversation during their games, on the street, and so on.

Here are some of the definitions they gave me when I asked them, through their journals, what mathematics meant to them:

Mathematics is...

... number, calculations...

... calculator, computer, television...

... the number of things.

... things we learn... the numbers...

... calculation and problems we solve... divisions or things like that.

... number of persons, desk, shoes... written numbers also, ordinal numbers, cardinals, fractional, multiplication, etc...

... a subject matter to learn how to work as fare man, driver, teacher...

... a subject matter we learn at school which has calculations, division, multiplication, etc...

The message from these definitions is clear. According to the students' responses, mathematics is nothing but numbers and calculations. The meanings expressed here do not go beyond counting, calculations, and some elementary and meaningless operations. I say "meaningless" because students only named them without any context that could say more than numbers.

The idea of mathematics as being the expression of numbers became quite strong during the interviews. Using the pictures described earlier, showing scenes of the students' daily experience, I asked them to tell me what they could see, create or invent about mathematics related to those pictures. The result was astonishing: The majority of students pointed out to me almost exclusively the number of the picture. I have to explain that each picture was over a white sheet of paper showing numbers from one to 10 to identify each one of them. Many times, when I insisted that they take a careful look at the scene and tell me something more, they found a smaller number on the top of the picture, the date when it was taken. Sometimes I got answers such as the following:

Eduardo: There are several trees... two pipes, adding this one, are three...

Rita: One car adding another car is equal to two cars....

Sonia: There are six doors... they are divided in four parts... There is a man sitting on a chair with four legs....

Beto: The number 60... (in the picture showing the speed limit).

Nelio: There are seven men and the number seven is odd.

Edmar: There is the number of the license plate on the truck. Within the truck there is a watch, the thing that shows the velocity of the truck and has numbers in it.

Ana: The truck has four tires and the number four is mathematics.

Hilda: There is money they can put together, count, do things....

According to these students mathematics was numbers shown in the pictures and in the control panel of cars, numbers of people or things in the picture and, more rarely, some calculations they created based on the number of objects in the picture.

I found the same pattern of response when students and I were talking about mathematics in their activities outside of school. In this situation, the most common answer was to mention the time when they were doing homework at home or when they were playing school with their friends. Besides that, students were not able to identify mathematics related to the pictures or in their activities outside school or at home.

Exceptions could be made in the activity involving money, although students did not mention that in the discussion about the pictures. In the conversations we had, students made several references to the activity of dealing with money at the stores, in the bus or in others situations where they had to pay or receive money. Those situations were quoted as examples of opportunities where students were dealing with mathematics because they have to perform calculations and were dealing with numbers.

The following dialogue about mathematics between two students and the interviewer shows how difficult it was for students to relate mathematics to things other

than money and numbers. Part of this dialogue deals with a woodworker who is the father of one of the students and occasionally is assisted at his work by the student.

Cid: Mathematics helps you not to get in trouble. For example, if you are going to work as a *bancario* (teller in a bank), you're gonna divide money to give people or... as *trocador* (bus fare collector).

Beto:... or as a teacher.

Interv:... how is that?

Beto: If I were a teacher, how do I know how to divide to teach the students?

Interv: Does your father know mathematics?

Eduardo: Yes.

Inter: Does mathematics help him in his job?

Eduardo: No, he is a woodworker. He makes furniture.

Interv:... and mathematics does not help him to deal with his work?

Eduardo:... dividing nails, for example... There is a drawer full of nails. When the nails drop on the floor... if the drawer drops on the floor, he gonna know how to divide... no, not to divide like that..., no, no, it doesn't help anything.

Interv: That is interesting. This kind of mathematics that helps on the job but doesn't help your father....

Eduardo: Well..., it helps only if it is the *trocador* (fare collector) or the banker, and the teacher.

Beto: The *bancario*? (teller in a bank).

Eduardo: Yes, the *bancario*!

Interv: Ah, it is not the banker?

Eduardo: Banker is the owner of the bank and *bancario* is the one who works in the bank.

Interv: So, there is no mathematics in the wood workshop?

Eduardo: None. Only when his work is paid for, or when he is going to make

change.

Mathematics was identified with numbers by students also in their journals. I had explained in the beginning of my contact with them, that journals would be an instrument for us to establish a conversation about mathematics. I told them that I would like to have a conversation about their experiences with numbers in their games, at home, or at school. I asked them to show me with drawing how they saw mathematics in their games and other activities. Many of the students started copying and making calculations involving basic facts of division, multiplication, subtraction and addition. They copied exercises from books they had at home. Sometimes they just copied the table of results of the four fundamental operations. Other students expressed themselves through drawings like the following:

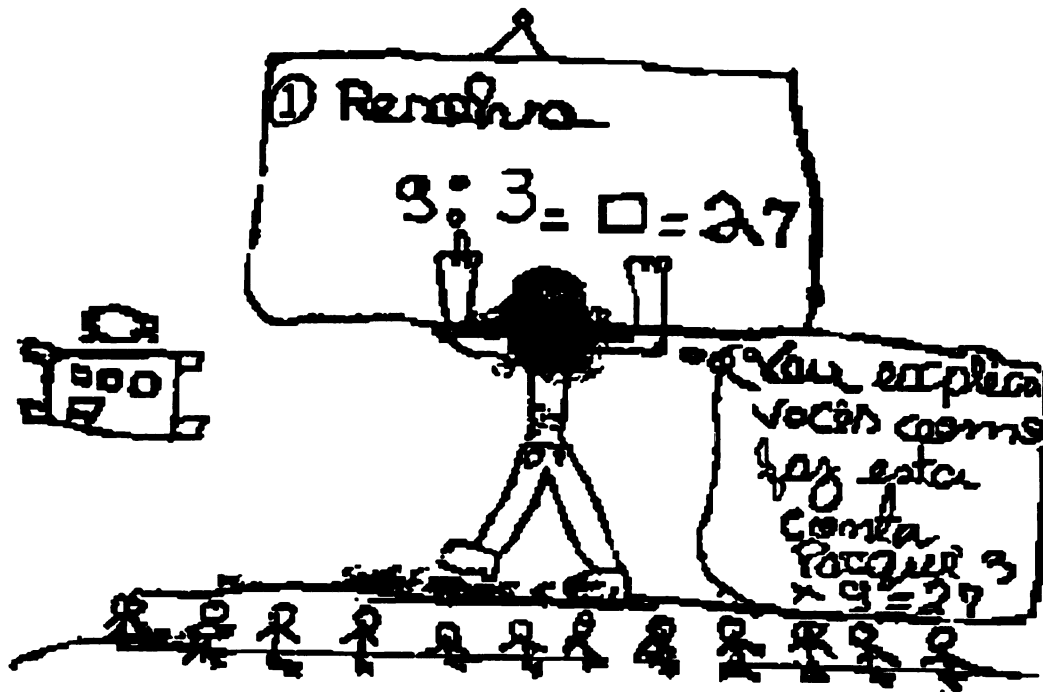


Figure 3 Mathematics class

She explained what the picture was about:

I see mathematics in the price of the balloons and in subtraction because the woman who is selling them had nine and she gave one to the little girl and kept eight.

In an informal conversation, Fatima and her brother told me about things they do to help their mother at home which involve mathematics. Part of the dialogue is as follows:

Fatima: I do the dishes and carry water from outside.

Inter: How much water do you bring each time?

Fatima: I bring two full gallons.

Interv: What mathematics can you see in carrying water?

Fatima: Carrying two and two, four.

Dair: I know. It is when one can is heavier than the other one.

In a variety of our interactions the students identified mathematics with numbers rather than the meanings they represented. As one can see, those children did not perceive mathematics as part of their everyday activities. Students did not relate mathematical knowledge they study at school with mathematical knowledge they deal with outside school.

Studying the students' perspectives on mathematics and social themes related to it, I learned that the context where students do mathematics is crucial to understand the kinds of mathematics the students use.

In the next chapter I will approach the contexts where the students face situations involving mathematics in and out of school. These were cases where students were reasoning in mathematical terms about situations happening in their daily life. However, for the most part, there was no evidence at all that children see school mathematics as related to their lives outside the school.

CHAPTER IV

CONTEXTS OF CHILDREN'S EXPERIENCES

INVOLVING MATHEMATICS

Introduction

I conducted my research in a urban public, elementary school in the same city of almost one million inhabitants, in southeast Brazil, where I work at the University. The school is supported by the city's Secretary of Education, which includes a centralized administrative and pedagogical system of control. The school is located in the neighborhood of the University and presents similar characteristics of other schools in the city. In addition, this school is a place for prospective teachers to practice. It is a medium-sized school with 1100 students, 50 teachers and an administrative staff. I spent three months there performing participant observation. I had access to all facilities and school activities. I was invited to participate in teachers' meetings, celebrations, or other events, in addition to everyday participation in the third grade mathematics class.

To collect data at school, I carried out participant observation in mathematics classes and interviewed students, teachers, supervisors, the counselor, the secretary, and the principal. Being at school every day, I could establish a good relationship with those in the school. After some time, my presence there was taken as commonplace. I took photographs and videotaped special events and some of the routines of the school day, such as lunch and break time. I did not videotape any classes, because the use of the camera could have disturbed the teacher's routine in conducting classes. The camera itself was very unusual for them. The use of a video-camera was not part of the daily routine of that community of working class families, small scale vendors, and construction workers.

The teacher I worked with was an undergraduate student teacher with a major in Language Arts. Linda, as I will call her from now on, is a young, active woman, very interested in music, politics, and education. She comes from the school neighborhood, and she knows each student as a neighbor. Linda has worked at this school for fifteen years. Linda was very accepting, and made me feel “at home” with her. The only concern Linda expressed to me when we first talked was that she did not have “different things to show me such as different materials and new methodology” (Field notes: August 6th, 1990). In the classroom, besides observing, I took field notes and audiotaped each mathematics class. After some time, because Linda insisted on having my active participation, I started helping her with some student activities. I usually followed the students doing mathematical problems and correcting homework, and explained to them when they came out with some doubt about Linda’s explanations of the subject matter. In this period of my study, Linda was teaching division. My participation in mathematics classes with the students was very important because I could both establish a close relationship with them and learn about their ways of solving mathematical problems in the classroom context.

In this section, I will present the school context where children’s experiences involving mathematics occurred. I will first portray the school as the arena where institutionalized experiences are planned to promote mathematical learning. Then I will describe the classroom setting where those experiences might actually occur under the supervision and guidance of the teacher. To do this, I will look for the routines, rules and main activities that occur in class, and the relationship that involves teacher and students. At the end, I will describe the actual activity of teaching, as I observed it, in a class about division.

The School Context

Within the school, the supervisors are responsible for planning the curriculum that is followed by teachers in their teaching. The supervisors' plan summarizes the content knowledge in each subject matter, its sequence, objectives, and suggested activities for the academic year. Each semester, students are given tests prepared by teachers under the supervisors' guidance. The general approaches for the elaboration of the curriculum are given by the city's Secretary. The orientation is based on units of teaching where certain phases should be followed in the teaching process. The first phase is an exploratory one. The teacher should make sure that the student is "ready" to learn the subject matter to be taught. Also, the teacher is urged to look for connections with concrete experiences of the student related to the topic. The next three phases are Presentation, Assimilation, and Organization. They correspond to the introduction of the theme to be studied in the classroom, the activities and practice with the new content knowledge, and the discovery of relations in order to organize what was learned. At the end of the process is the evaluation of the objectives achieved.

The pedagogical orientation given for third grade mathematics is that the theme of division comes after the teaching of the basic operations of addition, subtraction, and multiplication. Also, the teaching of division should follow some steps, according to levels of complexity in algorithmic procedures. The following is a summary of the recommended steps:

1. Division of whole numbers involving basic facts of division.

Ex.: $24 \div 3$, $18 \div 2$, $45 \div 5$, $30 \div 6$, and so on.

2. Division involving basic facts with remainder.

Ex.: $18 \div 4$, $45 \div 6$, $23 \div 4$, and so on.

3. Division of whole numbers without regrouping.

Ex.: $66 \div 2$, $63 \div 3$, $48 \div 2$, and so on.

4. Division of numbers with remainder without regrouping.

Ex.: $38 \div 3$, $47 \div 4$, $65 \div 3$, and so on.

5. Division of whole numbers with three digits.

Ex.: $369 \div 3$, $648 \div 2$, $468 \div 2$, and so on.

6. Division of numbers with three digits with remainder.

Ex.: $365 \div 3$, $645 \div 2$, $469 \div 2$, and so on.

Only one supervisor was responsible for the five, third grade teachers. Having to work with the five teachers as a group was difficult. The supervisor told me one day that she was having problems getting everybody together, because it was necessary to replace the teachers to let them be available for a meeting. There were no extra persons available at the school to give this support. In addition, everyone had other activities after class. Therefore, to have meetings it was necessary to cancel classes. During the three-month period that I was there, they met just once. Linda, the teacher I was working with, told me once that she was willing to have those meetings, because she wanted to discuss questions related to evaluation and content knowledge planned for her class.

I had the opportunity to observe one of the pedagogical meetings called *Conselho de Classe* (Class Council). As the supervisor explained to me, the idea was to deal with problems based on what each of the teachers knew about the specific students from previous years of teaching. The meeting was supposed to be a time for teachers to exchange personal experiences and knowledge in dealing with problematic students. I was told that this meeting is held every year, twice each semester. The supervisor also told me that they have tried to hold separate meetings of teachers for each grade. Finally she told me that the best situation is one where teachers from different subject matters in the same class get together, because every teacher is dealing with the same students. Her perspective was that in this case, there shouldn't be any problem keeping them interested in the discussion. This was not the case in the elementary school, because

there every teacher was teaching a different group of students.

At the meeting I attended, all teachers who worked in the same shift i.e. before noon were there. The principal, the school's counselor, a social worker who attends the school, and the supervisors were there, too. The first part of the meeting was a presentation of slides by the social assistant, about human relations on the job. The message was the importance of being a leader, regardless of the position you are in, and the effectiveness of working together. At the end, none of the teachers wanted to make any commentary about the sequence of slides.

The second part of the meeting was the group discussion of teachers in each grade. I stayed with the group of third grade teachers to follow the discussions, but, as the supervisor had predicted, the discussion was limited because each teacher had a different group of students. I asked myself how teachers should feel about getting so little done after so many hours of work. Reflecting on this and other events, I realized that the school witnesses many teachers' struggles. Some of them might remain hidden, whereas others might surface within the routine of the school.

The surfacing I observed happened naturally through the teachers' informal conversations among themselves in the teachers' lounge, the patio or the corridors. The following entry from my journal about my conversation with two elementary teachers during a break is an example:

Yesterday, I was talking with Linda and the teacher responsible for the library about the current situation of the teacher as professional. Both were upset with the situation, saying that being a teacher is not a good thing these days. Teachers don't have money to afford any leisure, they cannot give their children even minimum things such as money to have lunch at school. They cannot buy good clothes, have vacations, or even buy books that are important in their profession. Linda told us that she bought two books last month and, because of that, she is short of money and having problems. Linda used an expression that was sad but expressed exactly how teachers see their profession. She said: "I stay in line [waiting to get paid] more time than my salary stays with me." It means that they

stay in line for a long time to receive a small amount of money that they have to use immediately and in full to pay everything they have to pay. (From my Journal, September 28th, 1990).

Some further information can give us a better sense of the professional conditions under which teachers work. The initial salary for an elementary teacher, working from 7 a.m. through 11:30 a.m., without considering the extra time for official meetings, planning, and paper work was approximately three times the minimum salary at the time of my observations. However, when the high Brazilian rate of inflation was considered, the result was a significant devaluation of salaries. For instance, an elementary teacher who had a salary of \$250 in the beginning of this study, was receiving only \$170 three months later, once devaluation was taken into account, even though she was getting wage increases in local currency. Checking one's wages against a stable currency is a common comparison to assess its real value. Thus, the wages of this teacher were shrinking as time passed.

I also had a sense of the teachers' professional environment from other entries in my journal:

In another situation, teachers having coffee in the lounge during a break discussed when the problem about noise on the first floor. Students having lunch were causing the noise. A supervisor told the teachers that she was worried about this problem. Teachers working on the first floor could not work during this time because it was too noisy. She said that teachers should look after their students, keeping them all together as well as preventing them from playing during this time. Perhaps, she added, it would be better if only students who have school lunch go out at this time. The other teachers and students would continue the normal class. One of the teachers said: "but, if a teacher wants to have lunch, she has the right to do this." The other supervisor jumped into the conversation, saying that it was not possible to keep students inside the classroom without the teacher. Someone else suggested

changing the lunch time. Other teachers argued that, not having an exact time for the lunch break could have bad implications for conducting classes. She proceeded, saying that teachers would not know if they had time to begin a new subject or to finish previous subject. As a result, she concluded, they would lose a lot of teaching time. This discussion ended, with no solution to correct the noise during lunch time. (From my journal, September 28th, 1990).

The Classroom Context

Linda's third grade classroom was also a classroom for other students in the afternoon shift, and still other secondary students in the evenings. Simplicity was its basic characteristic, in its furniture, building construction, and appearance. Since the school accommodated students in three sequential shifts with different students in each, it was not uncommon to hear students complaining about things that were misplaced or had even disappeared. There were not enough personnel to guard school property, or even time between shifts to clean. There were 30 student desks, a blackboard, a teacher's desk, and a cupboard large enough for keeping the students' essential materials to be used in the class: books, notebooks, and chalk. What struck me the most was that there was nothing on the walls, or anywhere else, that could give any identity to the classroom or that would relate to the third graders.

When the school day started, the classroom was quickly transformed. There were close relationships between the students and the teacher, and among the students. Linda knew every one of the 30 students in the class. In our conversations, she talked to me about them with affection and concern.

Carlos is serious..., pays attention, asks question when has doubts....

Maura is calm, quiet, seems not to be enjoying school but seems not to have difficulties in learning.

William has some difficulties. His mother is concerned... it seems that he needs to

pay more attention....

Anita has a lot of difficulties in learning. She didn't know her mother... Her father is very concerned about her.

Linda was one of the few teachers who stayed with her class during break-time. Almost every day I saw her playing with the students. After break she used to sing with them in class, to calm them down before starting teaching again. At the end of the school day, she always received hugs and kisses from them when it was time to go home. The class routine started at seven in the morning, with a prayer that Linda used to say with the whole class. All of her students were caught up in an attitude of respect following her words, quiet and in silence.

Each day Linda corrected the homework from the day before. These were opportunities to repeat some points she considered important and to clarify doubts about any mistakes or student misunderstandings. Sometimes Linda called on students to solve problems or make calculations on the blackboard. As she mentioned to me once, she used this time to call on students she knew needed explanations, or the ones who could help the others. In fact, as I observed she took care that, at some point, everybody had an opportunity to do different things in the class.

After correcting the homework, Linda introduced the subject matter planned for the day. She always tried to get the students to share their experiences, asking questions and listening to them. After this, she explained the topic, using the blackboard to register the main points. At the end, the blackboard reflected the sequence of the class. Linda used drawings and examples to illustrate aspects of the content she was teaching. After her explanation, the students would copy what she had written on the blackboard, do the exercises she gave, and correct them using the same scheme as they did for homework correction. While students were practicing the exercises, Linda and I went around to the student's desks, looking at what they were doing and explaining problems that emerged from the exercises or from the problems to be solved. I found it interesting that when

students gave the right answer, they demanded that we grade their notebooks and leave a signal of our appreciation, such as “Very Good!” or “OK!”, or “Congratulations!”.

Sometimes Linda used mimeographed activities to give more time in class, but this was infrequent, because there was just one mimeograph machine for all of the teachers to use.

After finishing practice, it was time to copy the homework for the next day. If there was another class after this, the students would keep their materials for mathematics, and take out what was necessary for the next class. If not, they would say goodbye to us and go home.

There were no rules of behavior posted on the walls, but implicitly there were several rules that the children should follow expressed by the teacher’s words:

1. Pay attention when the teacher is explaining.
2. Do not talk while the teacher is explaining.
3. Do not disturb your classmates.
4. Do not copy while the teacher is explaining.
5. Do your homework.
6. You must have your material to work.
7. You may sit anywhere if you are well behaved.
8. You should ask questions at any time if you have a problem.

Despite these implicit rules, things did not always go well. There were moments when I observed a lot of tension between students and teacher, or between a student and another student. I remember a particular event that happened with one student I registered in my journal:

Fabio is an interesting case. Unfortunately he doesn’t disclose himself in his journal as does Anita. It is hard to get to know him because there are many moments he doesn’t speak about himself. In addition he seems to be very strict with himself and with the others. If the teacher complains to him about his bad behavior or because he is not correcting the exercises, he expresses anger in his face and doesn’t talk for while. After some time, he comes back as if nothing had happened. One day he began to sing in a loud voice while Linda was correcting

exercises. She asked him to stop singing because he was disrupting other students, and to pay attention. He continued singing. Linda said if he didn't stop she would send him out of the classroom. He answered saying that she wouldn't do this with him. She did. He went out and stayed there for about one hour. I was coming in when I saw him standing up outside of the classroom near the door. The whole class was coming outside to have lunch, and he asked Linda if he could come in. She told him he should ask his friends, because he had disturbed their learning. They went down to the first floor to have lunch, including Fabio. After lunch and the break time, when the whole class was back, Fabio insisted that Linda let him come into the classroom. Linda said that he could if he asked his classmates to excuse him for what he had done. (At this time I was there audio recording.) He came to the front and asked everybody to excuse him. His colleagues began to ask him many questions about the reasons why he had done that, and why he liked to bother everyone. He answered saying that he didn't know exactly why. He feels nervous sometimes, or angry, and he needed to shout or do something like he did. (My personal journal, October 10th 1990, p. 74.)

I have to say that I felt very uncomfortable with what was going on. While Fabio was standing up in front of the class, talking with his classmates the teacher was sitting in the back of the classroom. I imagined that Fabio would feel very bad to be exposed like that to his friends, but he didn't manifest this feeling. However, I also observed that Fabio felt nervous about talking about himself in school, especially if we were talking about bad performance and grades. On the other hand, he talked freely about his job after school. At this time, he was collecting *esterco* (cattle manure) and selling it to people to use in their gardens. He prepared the soil, planted, and took care of it after the planting.

Fabio was not an isolated case of tension in the classroom. In general, students paid more attention in the beginning of an explanation, but after some time they would start playing around and making noise. Sometimes, Linda would become angry with them and would start talking in a loud voice to overcome the children's noise. Sometimes this would cause the atmosphere of the classroom to become heavy and uncomfortable. I observed this happening most when students could not understand what the teacher had explained to them, or when she spent too much time explaining and

they were not very interested in the explanation. I noted that students had a repertoire of activities for these occasions. Some examples follow:

1. Students rub glue on their hands, leave it to dry and start peeling off the fine skin produced by the glue.

2. Students build *estalinhos* (a folded paper) and make noise with them by doing a brisk movement up and down. The device produces noise as the air is expelled out of the folded sheet of paper. I should remark that an *estalinho* is an ingenious piece of work. The paper must be folded in geometric forms so that some parts of it move, expelling the air and making the desired noise.

3. Students get up and sharpen their pencils, ask their friends for a colored pencil, or an eraser, or something else.

4. Students write notes to friends, making jokes or sending messages.

5. Students play with *figurinhas* (small pictures that come with bubble gum).

Linda would complain strongly only when these students' actions reached a point where other students would be disturbed by the noise. On several occasions, I observed her attitude regarding these activities. She would not take many of them seriously enough to consider that the students were breaking the rules of good behavior in class. Rather, she continued with her class as if nothing had happened. Usually the student or students involved in this action would eventually relax and go back to paying attention to the class. As I observed it, her strategy of overlooking these actions many times was effective in allowing students to relax, while assuring the continuity of the class.

Next, I will describe Linda's teaching and show her students in action as she teaches a class on the topic of division.

A Class About Division

It was before lunch when Linda started the lesson. She drew a line on the blackboard, on the left side, wrote the date and the name of the class (Mathematics,

August 14th), and started reviewing some basic concepts about adding, subtracting, and multiplying. She asked the students what each concept was about and how to perform each related operation. Adding was identified as “grouping elements,” subtracting as “taking out elements,” and multiplying as “making addition easier” because it showed how many times a number had been added. Then, Linda said: “We’re gonna learn a new operation today. It is division!”

What follows is a transcript of the main parts of the lesson in which Linda introduces the idea of division and the algorithm to divide. One student’s participation will be indicated by “St” followed by a number, (St1, for example), and “Sts” will denote various students in simultaneous participation

Introducing the Idea of Division:

Teacher: What does this word “division” remind you to do?

St1: Subtract

St2: Diminish

St3: Diminish the price of the... (many students talking together).

Teacher: Division reminds you of subtraction?! (with surprise).

St: No... it’s to distinguish, to divide (Students keep talking together and making a lot of noise.)

T: (Linda stops her teaching to complain about this). I’ll stop the class and I’ll wait. When you decide, we can keep going... (Students quiet down and she continues).

Teacher: R. (a student) remembered a word.... When I said division, she remembered setting apart. What more do you (all students) remember?

St1: To set apart...

St2: Division

St3: Basic facts (everybody talks together and Linda complains again).

Teacher: Shhh.... do you remember that I told you that we do things well, or we don't do anything?

Sts: Yes, yes...

Teacher: May I continue?

Sts: Yes, yes...

Teacher: R. told us.... I want to know what this word means to you. She told us that division reminds her of setting apart. Have you heard this term... have you used this word in some way? Do you?

St1: Yes

St2: Me, too

Sts: (students agreed)

Teacher: How?

S1: I remember... to divide....

St2: I know... I use that when I play soccer... to divide the team.

Teacher: Yes, that's it. Very good. He (the student) remembered using the word.

He divided the team: some for one side and some for the other side. And you ?

Several students suggested different examples of division, as for example: dividing a team to play *queimada* (a ball game), dividing a job, a salary, dividing a loaf of bread, a lunch, a biscuit, and others. I was impressed by the students' answers because they were expressing many situations from their daily activities.

After some time Linda continued:

Teacher: That's it. You see... if I ask... did you see? I asked what division was and everybody... only R. said "set apart"... You see how we are doing division all the time? I say... let's divide a blackboard... (as she had done when the class started). We are dividing all the time....

St1: Divide a head of the... (students laugh)

St2: Divide the ears...

Sts: (making noise)

Teacher: Shhh...

Teacher: Then, what would it be to divide?

St1: Separate

St2: Divide

St3: *Repartiar* (This is a way to say "share" in Portuguese. The correct word is *repartir* which means "share.")

Teacher: Ah! Here is a new word... *Repartiar*, share, separate....

St1: Share food... one dish for each one.

Teacher: We divide the time, see... before lunch we're gonna do this and after lunch we're gonna do that.

St2: Ah! Mrs. Linda, dividing a grain of rice... to cut in the middle...(students talk for a while).

Teacher: We're gonna start learning division... from the beginning. It will be division without a remainder. Later on, we'll see other kind of divisions, but only later... now... (pause). The first thing we'll try is to divide everything exactly. For example, the set of fingers on both of my hands. How many fingers do I have?

Sts: Ten

Teacher: Are they equally divided on each hand?

Sts: Yes.

Teacher: Let's suppose that there is a limit in my hand and (while she was talking she drew on the blackboard).

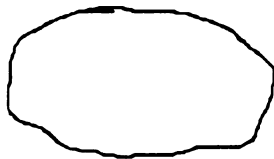


Figure 5 A set.

This is a set.

This is a set. In this case what would be my hand here and there.

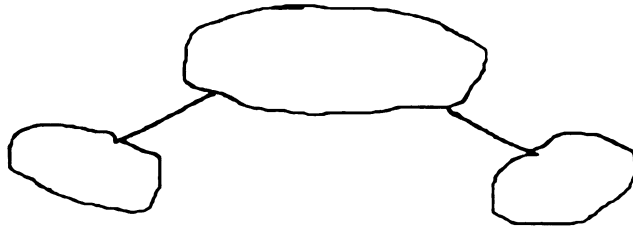


Figure 6 : A set and subsets.

St1: Set and....

St2: Set

Teacher: Set and set?! Set and....

St3: subset

Teacher: Ok, and what is the name of the subset that has the same amount, look (showing her hand)... unless a person has any deficiency... one, two, three, four, five... (counting on each hand). There are five fingers on each hand. There are the same number of....

St1: Fingers.

Teacher: Fingers... elements. Then what is the name of the set with the same number of elements.

Sts: E... e... (students try to remember the word "equipotent").

St1: Equipotent.

Several examples were given using the same structure of drawing limits as I described above, but varying from fingers to flowers, stars, candies, and other things. Also, Linda used different amounts: 24; 36, 12. Some students went to the blackboard to do the division based on what she had done earlier, using the same drawing. Then it was time for a break. When the students came back from the patio, they sang with Linda

as they usually do.

Linda restarted the lesson to relate division to multiplication.

Division and Multiplication

Linda drew set limits on the blackboard using the same scheme described earlier.

Using *tampinhas* (metal bottlecaps with which children like to play) she filled the whole set as follows:

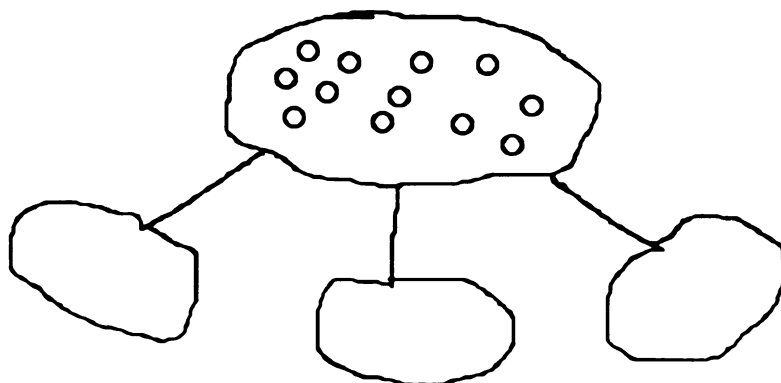


Figure 7 A set of *tampinhas*.

She proposed that the students divide the set of *tampinhas* into three equal subsets. As the students spoke, she did the division on the blackboard. She took the *tampinhas* one by one and transferred them to each subset.

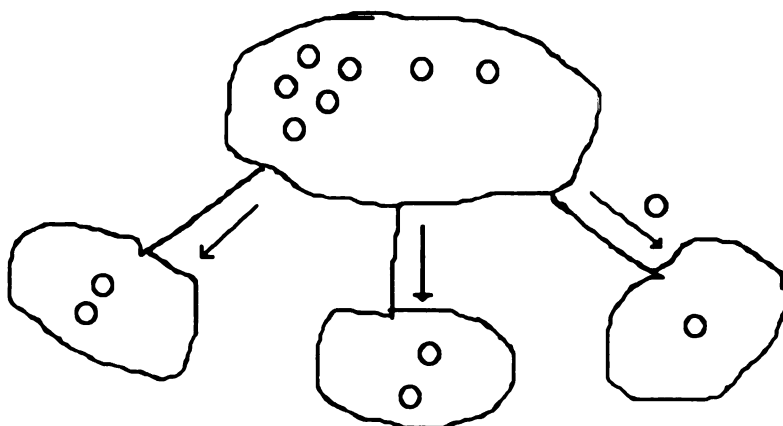


Figure 8 Dividing *tampinhas*.

At the end, Linda said to the students:

Teacher: Let's see if you can make some connections. There were 12: we divided them; pull them apart in three... how many elements do we have in each subset?

Sts: Four

Teacher: Can you make any connections? There were 12 divided by three, equal to four. (Students were making some noise. They did not answer her about the connections). She followed up the explanation by saying:

Teacher: Let's see if you did perceive... I'll write here two ways of representing that:

$$12 \div 3 = 4$$

(While she was writing, the students were repeating in chorus):

$$12 \div 3 = 4...$$

Teacher: What does 12 mean? It is...

St₁: Set

St₂: Element

St₃: Twelve?

Teacher: Set. Three are the....

St₁: Elements

St₂: Subsets

Teacher: Subsets. Look: one... two... three... subsets.

Teacher: And four, is what?

St: Elements

Teacher:... are the elements of the divided set. What can you get from this operation here? Can you connect this operation with another one? Everybody should think. Pay attention to this: $12 \div 3 = 4...$ everybody quiet, don't talk now. Do you remember something that....

St₁: I know, I know...

Teacher: Shhh. You're not to talk. You're to think! Look! We got exactly: four, four, four - three equal subsets (stressing). $12 \div 3 = 4$. Observe carefully and see if you can relate this with another thing we studied before... are you thinking?

Sts: Yes

St: $12 \div 4 = 3$

Teacher: Let's see what R. has to say. What did you notice?

R: $4 \times 3 = 12$ (while she was saying this, Linda wrote the same on the blackboard repeating after the student $4 \times 3 = 12$).

Teacher: Do you all agree?

Sts: Yes

In these sequences I described the teacher introducing the process of division and relating it with multiplication. Linda emphasized the division with the idea of setting apart equal number of elements from a whole set. The equal division sounded to me like a requirement to do division. This was reinforced by the connection made with multiplication. The ideas of division as sharing or separating, and the requirement of equal part was not discussed after that first moment of the class.

Linda gave a sequence to the class introducing the algorithm to do division.

The Algorithm

Linda explained that the use of drawings to divide flowers, stars or other things is not useful when the number of elements is big. She gave the example of dividing 72 elements by two. This would take too much time, she said:

Teacher:... if we know that four times three is equal to 12 or three times four is equal to 12, (...) another way we have to represent that 12 divided by 3 is: (and then she wrote on the board)

$$12 \overline{) 3}$$

structure used to
do division in many
Brazilian schools

Because we know multiplication, we don't need to draw or count anymore. We're gonna know that 12 divided by three will be...

St₁: Four (Linda wrote on the blackboard)

Teacher: Four. Why Four? Because four times 3...

St₂: Twelve

$$\begin{array}{r} 12 \overline{) 3} \\ 12 \quad 4 \end{array}$$

Teacher: Then, here we'll subtract. Are there any *tampinhas* left?

Sts: No (in chorus)

Teacher: We do that:

$$\begin{array}{r} 12 \overline{) 3} \\ \underline{- 12} \quad 4 \\ 00 \end{array}$$

(She repeated everything again based upon the algorithm.)

Teacher: Twelve divided by three is four because three times four is 12. Two minus two is... one minus one....

St₁: Zero

Teacher: This is the remainder. So, are there any elements left?

St: No

After this, Linda named each term, writing on the board:

$$\begin{array}{rcccl} \text{dividend} & \longrightarrow & 12 & \overline{) 3} & \longleftarrow & \text{divisor} \\ & & \underline{-} & & & \\ & & 12 & 4 & \longleftarrow & \text{quocient} \\ & & \underline{} & & & \\ & & 00 & & \longleftarrow & \text{remainder} \end{array}$$

She read each term, she wrote it, and repeated its meaning.

Teacher: Dividend. Why? It is the one to be... divided.

St₁: Divide by...

Teacher: Divisor... it is how many times we are dividing. Quotient, is how many elements we've got in each subset, and the remainder is what is left.

At the end Linda wrote some notes for the class on the board and told the students to copy them. It was a synthesis of what they had done before. The class ended. The students had almost 60 minutes of mathematics class.

Looking through how Linda taught this class, I observed some important aspects that I will highlight here.

First, the ways students expressed their ideas about division was impressive. They were able to draw upon a good repertoire of meanings for dividing from concrete aspects of their lives such as food, money, and games. However, these meanings that were contextualized in the students' experience outside of school were not absorbed in the class context. The examples were quoted and forgotten a few minutes later. The sequence of the class continued, and covered the topics for that day. One serious implication of this disconnection is that students might not have understood the symbolism that the teacher presented in referring to their experiences.

Second, the different meanings of division the teacher chose to discuss with the students were squeezed into the scheme of separating a set into two or more subsets. The students expressed at least three different ideas about division which were not taken into consideration: set apart, share and to allot. I will talk more about these ideas later. In addition to this, the scheme used did not resolve situations where division was done into unequal subsets.

Third, when the teacher introduced the algorithm for doing division, it came as a flat picture of the process of dividing. I am saying this because it was not presented as a symbolic representation of a dynamic process. The teacher said that the algorithm would simplify the

mechanics of doing division, but even in this case, the simplification came through a new, and much more complex, mechanism for doing division. Each step of the algorithm should have been clarified for the students, in terms of what it means and why it should be done. Later, I will explore some aspects and implications of this mechanization for developing the students' understanding of the process of doing division using the school algorithm.

Finally, I would like to go back to some words the teacher used in this class. She said at some point:

.... you see how we are doing division all the time?

.... we are dividing all the time !

And a few minutes later she said:

We're gonna start learning division... from the beginning.

... Later on, we'll see other kind of divisions, but only later... Now....

If division is something that students do "all the time," what exactly are they going to learn? Perhaps this is one of the questions mathematics educators should be asking when planning what to teach students.

Mathematics in Everyday Life

During the three months I spent collecting data for this study, I was able to spend several weekends with students on the streets. On these occasions I spent most of the time talking to them about their experiences and games, or at the stores talking to the vendors and observing childrens buying goods. I was interested in knowing how the students dealt with real-world situations where mathematical knowledge was required. My assumption was that this setting was a privileged place to learn about the social construction of mathematical knowledge. These situations were so rich in data to that I began going to each bakery, market and store in the neighborhood after class. In addition to the interviews and observations I went to several students' houses to talk to their parents and relatives. There I could learn more about the students' routines in their

daily lives and experiences. I used journals as an instrument to establish a conversation with students about themselves. The journal made it possible for me to get their own descriptions of a variety of games and activities that they engage in when they are not at school. It is important to remember here that the school day lasts only four hours.

The students' routine, in general, could be described as having the following main activities:

- going to school;
- playing at home, on the streets, or at a friend's house;
- watching television;
- helping their parents at home or at work; and
- buying things in small stores, bakeries, or at the butcher shop.

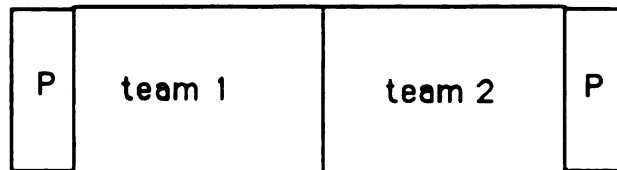
Among these activities, watching television was the least frequently mentioned, although it is general knowledge that almost every family in this community has a television set. The children did not emphasize that watching television was one of the main activities for them either in our conversations or in their journals. When it was mentioned, they noted that they enjoyed watching cartoons, soap operas, movies, and variety shows. No mathematical activity related to television programs was mentioned.

On the other hand, the activities that children mentioned the most during our conversations and in their journals were games and other group activities on the streets. The most frequent activities that children play in the street are games. There is a variety of them, each one rich in opportunity for mathematical reasoning. Many times the children's mathematical reasoning was expressed clearly during their explanations about these games, and other times they were implicit in the situation. I will report several games and other activities children described to me or those that I observed *in loco*.

Games

The games they enjoy playing the most are played with a ball. *Queimada* and *Bete* are Brazilian names for the most common ones. Playing *queimada* involves:

1. Definition of two symmetrical squares or rectangles to be the field where two teams play.
2. Definition of a smaller area at each end of the field to be the place for the prisoners of each team.



P = Prison

Figure 9 Playing field for *queimada*

3. Composition of the teams: In defining the composition of the teams, children take into consideration the number of childrens who want to play, how strong and agile each player is, and the equitable division of strength between the two teams. When there is no agreement about the definition of each team, childrens usually decide based on *adedanha*, a method used to decide who will be the first to choose a player. According to *adedanha*, the leaders from each team come together to decide who is going to be the first by choosing “odd” or “even” and hiding both hands behind their back. After both children have chosen and hidden their hands, they say “a... de... dannn... nha!” and he/she must show a number of fingers, at the same time as the other leader. They add the total number of fingers showed by both leaders and the winner is the one whose choice

(odd or even) matches the number of fingers. The winner chooses a player first, and both leaders take turns at choosing until everyone has a team.

Adedanha is used in several activities and other games, such as *Bete* and *Pique* among others. In this situation, what drew my attention was the fact that children bargain about the pairs even after the decision has been made during the *adedanha* dispute. They look for the best balance of strengths through negotiation among the leaders about their players. But there are other methods children use to attain a balance when defining the teams. When there is an odd number of childrens who want to play, or, when an equitable division of the kids' strengths is not possible, they apply the system of getting one extra "life." This means that the team having one less player than the other team can be hit twice by the ball before having one player sent to the prison.

4. The game is scored by counting the number of times each team takes all the opponents prisoner. The player is a prisoner when he/she is hit by the ball.
5. Definition of the winning team: the winner is the team with the higher score.

This game involves constructing two circles of a meter or less in diameter about 10 meters apart. Three short sticks are leaved together to form a target. Two pairs of children from opposing teams who take turns at pitching and hitting the ball. When a batter hits the ball he/she can run back and forth between circles to score points until the other team gets the ball. Winning can be done in two ways - either by knocking down the opponents' or reaching 100 points. Betriz describes the game in her journal with the following pictures and words.

"The game of *queimada* I can't explain, and also *bete*, but with *bete* I'll try to draw.



Figure 10 Playing *Bete* 1 (Student's drawing #1)

We draw that thing on the ground. On the people's hands are pieces of bamboo to hit the ball with. Inside the circle are three sticks that we arrange. This bat is for hitting the ball. The other two people throw the ball to tear down the three sticks. The ones that hold the bat are pairs. The ones that hold the ball are also pairs. The ones that hold the ball throw it and then I hit it and if I hit it far I get 100. If I get 100 I win one game and also have to trade bats like this. I'll draw it.

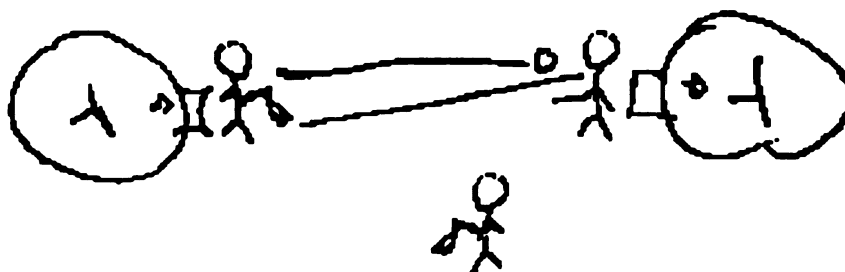


Figure 11 Playing *Bete 2* (Student's drawing #2)

Then the two keep trading places until they get to 100. And then trade until we win one game: like this:

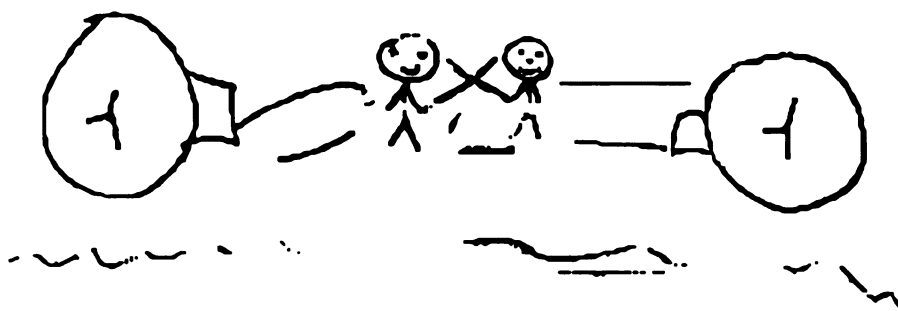


Figure 12 Playing *Bete 3* (Student's drawing #3)

Another very common game that children enjoy playing is called *pique*. The child who has the *pique* must run after other children and catch them. There are several

variations of this game:

-hide-and-seek: When children hide themselves in several different places from the person who has the *pique*. One particular point is defined as a neutral zone and the children cannot be caught there. If one child is caught out of this zone, he or she holds the *pique* and the game starts again.

-*pique-cola*: When a child is caught, he or she has to stay in that same place until another child frees him or her. While she or he is on hold, the child cannot run. When everybody is on hold, the *pique* starts again with another child in charge.

-*pique*-“freeze”: When a child is caught, he or she is “frozen” till another player frees him or her. It differs from “*pique-cola*” because in this case the child cannot make any movement or change his or her position. The child that is caught has to act as a statue.

-*pique*- flag: Played by two teams. Each team has a flag and has to defend it from the opponent. The field is defined as shown in Figure 13 and the flag stands behind where the players are:

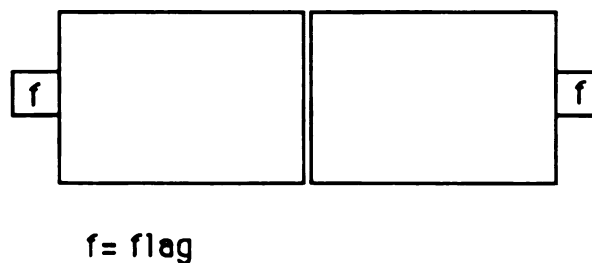


Figure 13 Playing Flag

The team who gets the enemy’s flag first scores one point, and the winner is the one who has more points at the end. The flag is only symbolically defined. It can be a piece of anything such as a wood stick, for example. The players must be agile and run fast. In this situation and others, where it is necessary to define two teams and to balance strengths, the whole process of decision-making described before takes place.

There are some table-games that children play with a friend, but they are not very

common in this community, except for checkers and cards. Several card games were mentioned, and some of them deal with specific values assigned to each card. But in general, they count the points they get one by one, and the winner is the one who gets more points than the other.

Another very popular game is called *bafinho*. This is a game in which two children bet and dispute stamps or cards organized in a pile on the floor. They do this by cupping one of their hands and hitting the pile with it in an attempt to turn over as many as cards as possible. Each player gets all cards that were turned over. If none, another player gets the turn. Each time a different kind of stamp is the object of dispute among children. It depends on the type that is most popular at a particular moment. The cards come with a chewing gum and there is a variety of them in the market. When these data were collected the most popular ones when these data were collected were stamps about animals in the Brazilian wetland, and different monsters. This game is more common among boys than girls, although not exclusively so. In general, they play until they get the card they want, or until they lose everything. In each game, a player can win or lose over a 100 cards. Sometimes children borrow some cards from their opponent in order to continue playing.

Another individual activity very common among boys is to roll an *arco* (hoop). This is a kind of game where children put an *arco* into motion with a stick. The *arco* might be made from metal, rubber or another material, and can be small or big. It doesn't matter. The important thing is to keep the *arco* running under complete control of the child who is rolling it. The more the child can do with the *arco* without touching it, the better he is considered. Children strive for complete control over the *arco*. The following figure illustrates how children play:

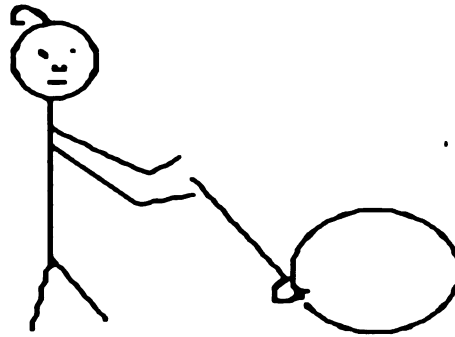


Figure 14: Playing *Arco*.

There are other ball games children play that are very popular in the Brazilian context and well known in the United States: soccer and volleyball are two of them.

The play activity that girls engage in, almost exclusively, is play-home or play-school, where in pairs or in groups they simulate a situation of house activity or school activity. These activities are interesting because children perform as adults do when at home or at school. The children described what they do when they are playing school. Most of the time, their friends are their students and they are required to behave and perform things as they do when they are in the real situation of a school day. In an informal conversation, Carlos told me how he does this: "I played with Anita that I was the teacher. I wrote on the blackboard, she copied and tried to solve it, and then I explained to her how to do it."

Fernanda also talked about how she plays school:

I play school with my cousins.... They have a blackboard I play like this: she gives me some calculations to do and I solve them. I make divisions and so on. I play alone, too. Sometimes I throw papers around on the floor while I am explaining.... Sometimes I complain to the students as if they were real students; I call the roll, apply tests, cry out about the test results, and even tear the tests up. Also, I have lots of books. I have mathematics books and I copy a lot of exercises on division and multiplication from them.

The girls expressed this activity through drawings where mathematics were

involved. Some of their drawings were inspired by books, and others created by themselves. Children work with mathematics as it is at school. They work with fundamental facts of basic operations, make calculations, and solve exercises, all within the same pattern that they experience at school, correcting and evaluating themselves at the end. In addition to this, the children clearly say that "they are playing school, having mathematics class."

This drawing by a child is an example:

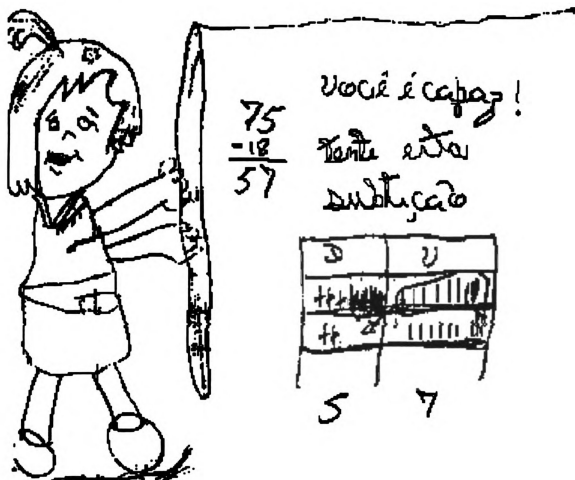


Figure 15 Playing school (Note: The child has written "you are able. Try this subtraction.")

Buying Things and Other Activities

Helping parents is another activity children perform as part of their daily routine. There are a variety of things childrens do within this category that are particularly rich in opportunities for mathematical reasoning, because of the nature of the activities they perform. This is particularly true because, when children help their parents, they deal with money in buying and making change, making measurements, estimating, and counting. Regarding experiences in buying and receiving change, Fernanda told me in our first interview:

When I went to buy a loaf of bread I had 50 *cruzeiros* and it was 16 for four loaves of bread. Four loaves of bread... 16, so I counted. It was note of one each, so 17, 18, 19, 20. There were four notes of one and three of 10, then it was 50.

Vera helps her parents buying groceries at the supermarket. She described how she does it:

On the day before, I checked out the prices and using a calculator I figured out the total. Then I went to the market and I verified whether the prices were higher than before I bought the things I want. To verify if the change was correct, I counted first the notes that are smaller and then the bigger ones.

One interesting aspect of the activity children do when they are buying things at the bakery, bar or small stores is the way they deal with situations where any mistake is found in the payment that the children make, or in the change they receive. Vendors are quick to say that, in general, all children are very smart in dealing with these situations.

Vendors also say that children are smart in trying to get some money to buy things for themselves. One strategy they use is to bargain the price. For instance, children may ask vendors to cut off the price so they can pay less and have some money left to buy goods they really want to buy. One of the vendors I interviewed told me:

Sometimes children ask me to give them the receipt including a candy. Another thing they do is to check if the price in another store is more expensive and then

they buy where it is less expensive, and tell their parents that they bought in the other store where it was more expensive.

Several vendors reported that, in general, children do not check to see if the change is right or not. They take the change and go out running. In contrast, children talk among themselves about situations where mistakes are made by vendors, and in some cases, how they take advantage of this. They reported some cases where they criticized vendors and pointed out what happened and how it happened. During my interviews with some of them, they explained to me what they did when they had to face this kind of situation.

Fabio: I went to the bakery to buy milk and I paid with 100 *cruzeiros*. I didn't check the change. At home, when I was going out, I checked the money that I had in my jacket and I saw that I got the wrong change. I had already more than 100 *cruzeiros*. I kept 50 for me and gave the rest to my mom.

Vera: The other day I went to make change with two notes of 50 *cruzeiros*; instead the vendor gave me three notes of 50. I checked the change and I saw the mistake. I told my father and he demanded me to go back and give the money to the vendor.

Dealing with money and being responsible for doing things by themselves gives these children a strong sense of the value of things. The following conversation among three students and the interviewer, about 1000 *cruzeiros*, shows their understanding (1000 *cruzeiros* was equal to one fifth of the minimum salary per month at the time the data was collected, which was equivalent to 10 American dollars):

Fabio: 1000 *cruzeiros* is a lot of money!

Interv: What can we buy with 1000 *cruzeiros*?

Eduardo: I can buy many things. I can buy a tire for my bicycle.

Interv: Is it enough to buy a tire for the bicycle?

Eduardo: It will not be enough for the special tire (a more expensive one), but for the regular type, it is.

Interv: How much does the tire cost?

Fabio: It costs 90, perhaps a little more....

Eduardo: It might cost around 900.

Interv: 90 or 900?

Fabio: 90

Eduardo: The special tire costs 2700.

Interv: This bill (a 1000 *cruzeiros* bill) would be enough to buy it?

Fabio: No! Are you kidding!

Interv: How much do you need to add to that?

Fabio: More 2 thou....

Eduardo: 1700....

Fabio: I can do a lot of things with this money....

Eduardo: I can fix my bicycle, and give the change to my mother to buy food.

Fabio: My gosh! I could eat a lot with this amount of money. I would do a lot of things....

Interv: How could you divide this amount?

Fabio: I would divide this money with them (other students who were participating in the interview). We would have a lunch buying a loaf of bread with salami (a variety of mortadella very appreciated by children there). It would be three loaves of bread with salami and three Coca Colas (Coke).

Interv: Would it be enough?

Fabio: Enough? (stressing) More than enough!

I showed them a 10 *cruzeiro* bill and asked:

Interv: and this, how much can it buy?

Sts. Almost nothing!

Fabio: It doesn't buy even a half of 100 grams of the salami.

William: It buys one gram.

Interv: How much does the salami cost?

Eduardo: 100 gram? 35 cruzeiros.

Interv: If the salami costs 35, how much could you buy with 10 *cruzeiros*?

Fabio: You have to add 25 more...

Interv: Yes, but 10 *cruzeiros* can buy how much salami?

William: It is not enough even to buy the white part of the salami... (referring to the small spots of fat that can be seen in a slice of mortadella).

Eduardo: It might be enough to buy around 20 gram.

Interv: Now, considering that you know how to spend the money, it's only necessary to get it....

Eduardo: Spend!? No way! Only next year. I broke my father's thermos and I'll get some money only next year.

(exerpt from first student's interviews, September 21, 1990).

Children go to stores to buy things for themselves such as candies, gum, and snacks, but they go mainly to buy things for their parents, such as loaves of bread, milk, cigaretts, and groceries. When children go to the store for their parents, they get money from their parents, who usually tell them if there will be change or not and how much it should be. When children were buying things for their parents, I observed that the majority of them did not count how much they had in hand to pay the amount they owed. They brought the money rolled up, and gave it to the vendor. The vendor checked it out to see that it was enough and, when there was change, they rolled it up and gave it to the child in the same way. If, for any reason, the amount of money was not enough, the vendor sent a note to the parents telling them how much was missing.

In some cases it did not happen this way. This occurred when vendors were dealing with older children around nine years of age. In these instances, the vendors told the children how much was missing and asked them to tell their parents. Sometimes they looked to see how many items the children brought up to buy, and sold them the amount

that the money paid for. This happened most when the child was younger, seven years old or less, and she or he was trying to buy some candy or something similar. Some children buy for their parents using a notebook, where the vendor writes what the children are buying and the specific price, instead of paying in cash. This system of buying using a notebook is very common in small communities in this region. The vendor writes down what is being sold, and the payment is made at the end of the week or the month, depending on how the worker receives her/his salary.

In one weekend that I spent observing children buying, I saw several cases where children paid more attention to the amount of money they were carrying when they were buying things for themselves. They did this when they didn't have one specific thing to buy; instead, they had to balance how much money they had with the price of what they wanted to buy. I saw many children coming to the store, looking around, asking for the prices of many different things, and finally asking for one that matched with the amount they had. The following entry in my journal illustrates this situation:

One child was buying loaves of bread for his parents. He came and asked the vendor the price of 12 loaves of bread. It was 54 *cruzeiros*. Then he asked for 50 *cruzeiros* of bread (he had 50 *cruzeiros* in his hands). I asked him how many loaves of bread he would get with 50 *cruzeiros*. He answered 10. I asked the vendor the price of each loaf. The vendor told me that it was 4.50 *cruzeiros*. I insisted with the child saying: If each loaf costs 4.50 *cruzeiros* and you have 50 *cruzeiros* will you be able to get 10 loaves? The child reaffirmed to me that he would get 10. Then I asked if he would get any change for paying with 50 *cruzeiros*. The child answered that he wouldn't get any change. He got the bread and left the place without knowing how many loaves of bread he bought. After he left the vendor told me he gave 12 loaves of bread to the child.

In another situation three children were buying for themselves. One of them came into the bakery, looked around and asked the vendor for the price of one bottle of soft drink. It was 30 *cruzeiros*. The child kept looking around for other products. A few minutes later he asked the price of another product and finally ordered a biscuit. He paid with two notes of 10 *cruzeiros* and waited for the change. After

receiving the change he checked how much he got and asked for the price of another product. Finally he bought a loaf of bread, kept the change he had got in the new purchase and went out of the store.

The second child was with his mother. He spent time looking around to find something to buy. He didn't ask the price of any product. He chose what he wanted after some negotiation with his mother.

The third child came and asked how much one kind of candy cost. It was two *cruzeiros* each. Then the child took some money out of her pocket and began to count: one... two... (notes of one *cruzeiro*)... one candy. Another note... another candy... until the end of the notes. The vendor asked her: how much money is it? She answered: a lot of money. The vendor counted the money and there were five *cruzeiros*. The girl said to the vendor: give me five candies. The vendor replied: but the price is two *cruzeiros* for each candy! The girl then asked him: how many candies can I have? The vendor gave her three candies and she went out, keeping the candies with her.

This same perception was expressed by a vendor where many children go to buy small things such as pencils, erasers, marbles, and inexpensive toys. She said:

I think that children are smarter than adults. They bargain on the price, ask if they can pay later.... Today children are very smart and clever. They look the merchandise over before buying it.... Sometimes they spend one week looking for the best deal before buying. They compare the prices...

Besides going to the stores, children help their parents doing housework or working with them in their jobs. It is interesting to note how different the way children make calculations is from school mathematics. Vera's father owns a butcher shop and she works there with her mother and her eight-year-old brother, a second grade student. I went to the butcher shop to observe Vera working, but she was not there. I did, however, interview her father and observe and talk to her brother, Pedro. He was helping his father, receiving money from the customers and giving change when it was necessary. I gave him some situation problems involving change, and he was able to solve almost every one. The following dialogue between us shows part of his reasoning to figure out

the customers' change. To start, I asked him the price for 100 grams of ground meat. He asked the same question to his father, who answered him that it was 28 *cruzeiros*. Then:

Interv: If I buy 100 grams and pay you with a 50 *cruzeiro* bill, how much will I get for change?

Pedro: 22.

Interv: What did you do? How did you do it so quickly in your mind?

Pedro: I counted that 28... 29... 30. Then it is missing 20 and 2, 22.

Interv: If your father tells you to give 43 *cruzeiros* of change to the costumer and he gives you a 100 bill, would you know how much he spent? How much will he be paying you?

Pedro: 57.

Interv: How did you know that?

Pedro: Ah! It's hard to tell you...

Interv: And, if you have to give him 25 *cruzeiros* of change for 100 *cruzeiros*, how much had he spent?

Pedro: 75.

Interv: What did you think to get this conclusion?

Pedro: I did like this 20... 30... 40... 50... 60... 70. Then I thought that it was missing 5, so 75.

When I talked to Vera later, she told me that she was not making change any more because she had made a mistake, giving two 500 *cruzeiros* bills instead of one.

At home, children help their parents doing housework such as washing clothes, sweeping, cleaning the house and cooking. Many of them are responsible for babysitting their younger brothers or sisters. Some of their reports show how they dealt with situations where measurements are required:

Ana: My mom tells me what I should do. It has to be a lot of *angu* (a dish prepared with corn flour) because my brother eats a lot and me, too. I have to put four

portions of flour. I take a spoon and fill it four times. If it is not enough I put more....

About the water, I put more or less to the middle of the pan. The pan is big and heavy! My stepmother fills it for me because I am afraid to spill the water.

To cook the rice, I put five cups into the pan. I bring the water to boil and add in the rice later. My sister is who defines the amount of water, and I do the rest.

The activities and games described above demonstrate that there are many situations involving children that require mathematical reasoning. My description also shows that the children deal with these situations solving problems using strategies that are not taught in school. I will explore these strategies in Chapter 5.

CHAPTER V

CHILDREN DOING MATHEMATICS

Introduction

To learn how students generally deal with mathematical situations, and particularly with division, I collected data in several different situations. First, I collected data from my observations in a mathematics classroom. There, I followed students doing mathematics in situations planned by Linda, the teacher. I observed the students when they were practicing exercises based on the teacher's explanations, in quizzes and tests they took, and in questions they asked in the classroom. In addition, I observed how they expressed their ideas in mathematics through their journals.

Outside school, I collected data by observing students solving problems *in loco*. The most significant part of the data, however, was collected in informal conversations and interviews with students. These interviews were very rich in data because I could explore in more detail the strategies students used to solve problems. Repeatedly, I asked them "What do you think that makes you say that? Why?"

Two experienced teachers and I interviewed each student twice. We conducted the first interview in the sixth week of classroom observation and the second interview during the eleventh and twelfth weeks of classroom observation, at the end of the data collection period. We conducted the interviews in a very friendly, open and informal climate. I invited the students to participate and they were free to accept or not. Each interview lasted about 40 minutes and was conducted in groups of two to four students. To be interviewed, the students had to come back to school in the afternoon, after the school day. Initially, I thought that this factor would limit the number of students who would participate, because they would lose part of their free time, but this was not the case. For the first interview, 24 of 30 students were present, and 26 were present for the

second interview. Only three students did not participate in any interview.

In both interviews one of the objectives was to learn how students deal with situations involving mathematical reasoning. However, the first interview was a more exploratory one. The conversation flowed from whatever the student said at a given moment. The strategy was to elicit evidence and a description of the student's experiences with mathematical reasoning outside school. As a result, the students described a variety of problem situations they face daily. This was important, because I used these situations to plan the problem-solving for the second interview. From these interviews, I was able to identify a variety of mathematical problems. Then I classified the problems from very easy to very difficult, according to the school mathematical knowledge that was required to solve them and the familiarity of the situation for the student. For example, problems involving division with a remainder and decimals were considered difficult or very difficult, because students were working on division of whole numbers in mathematics classes.

The following are some examples of the problems students worked with in the first interview:

1. Very difficult: How many grams of salami can be bought with 10 *cruzeiros* (Brazilian money), if 100 grams cost 35 *cruzeiros*?
2. Difficult: How much does one kilo of carrots cost if I buy two kilos for 85 *cruzeiros*?
3. Regular: With 23 *cruzeiros*, how many pieces of gum can you buy if the price of one piece is 5 *cruzeiros*?
4. Easy: How can you divide 10 *cruzeiros* among 3 friends?
5. Very easy: With 5 *cruzeiros* in change to be divided equally between you both, how much money will each one of you get?

Not every child had to solve all of the problems, but all of the children had an opportunity to discuss a problem and possible solutions during the first interview.

Different kinds of problems were objects of discussion. The situations in which they had to deal with money were the most frequent ones: figuring out change, the cost of vegetables, bread and candy, division and combination of bills, etc...

While the first interview was more exploratory, the second was directed toward solving specific problems involving division. For the second interview, I planned eight problem-solving situations. I based the situations on the results of the first interview, and on observations of the students in stores, supermarkets, and playing in the street. My interest was focused, among other things, on learning from them what their reasoning was in solving division problems. The focus of the interview was not on the solution itself, but, rather, on how and why the student arrived at a particular solution. After each student response, we posed questions like, "What was your thinking in saying that?"; "How do you know that is the answer?" or "Why have you done this or that?" The students had access to a piece of paper and pencil and if they wished, they were able to write, evaluate, or make drawings to find the solution to the problem. However, this was not encouraged, because I was interested in letting them express themselves beyond the limits of the mathematical algorithms they had learned. In addition, I was interested in comparing students' strategies for solving mathematical problems inside and outside of the classroom context.

I applied these same problems in the classroom. On the last day of my classroom participation, I asked the students to solve these same problems for me on a sheet of paper. The problems had been mimeographed, as Linda usually does, with some written exercises and quizzes. In examining their responses, it is important to remember that 26 of the 30 students had already solved these problems in the interview.

The following are the problems in the format I presented to the students in our interviews and in the classroom:

1. If you have 50 *cruzeiros*, how many loaves of bread could you buy if each loaf of bread costs 4.50?

2. If you spend 45 *cruzeiros* to buy milk and pay with a 50 *cruzeiro* bill, how many candies could you buy with the change?
3. Seven childrens want to play *queimada*. How could you divide the two teams?
4. You have two oranges to be divided with three friends. How much orange does each one get?
5. To organize a collection of 150 *figurinhas* in two albums, how many *figurinhas* should I put in each album?
6. There are 45 loaves of bread to be divided between two classes of first grade students. How can the canteen woman do this division?
7. If you use six sheets of paper to cover four shelves, how many sheets of paper will be necessary to cover just one shelf?
8. Ms. Luiza has 30 cups of milk to be divided among the third grade students. But only 20 students came to school today. How can Ms. Luiza divide the cups of milk equally among the students? How much will each student get?

All students had the opportunity to solve each problem inside and outside the classroom. In this chapter I will analyze how students did mathematics inside and outside of classroom context. I will describe the strategies students used to solve division problems and identify possible sources of difficulties they encountered, especially in situations in the classroom context where specific objectives are expected to be achieved.

Solving Problems in the Classroom Context

In the classroom, the students received a mimeographed paper with the same eight problems we had discussed in the interview. When I handed out the paper, I explained that they could answer the problems in any way they knew. A student asked me if it was necessary to use division, and I answered that it was not, that they should “solve the problem in the way they knew how to..” I did not interfere in what they were doing

unless they asked me a question. The traditional students' questions came right in the beginning:" Do I need to use this space?" "May I use the blank space at the end?" I let them make these decisions according to what they thought was the best way to do it. The one point on which I insisted was that they should try to show the work done.

Approaching the Problems

Analyzing the results, I found some general procedures related to how students approached the situation problems in the classroom. First, the students took the problems as if they were fixed situations requiring the application of one specific mode of calculation to find the answer. Almost immediately after I handed out the papers and students started reading the problems, they asked several questions about what operation they should use:

Nando: These problems are all in division?

Laura: ...is to divide?

Vanda: Is it division or multiplication?

Beto: ...Does it need two operations? Is it only division?

Ana: ...what I'm gonna do is to subtract

Even when it was recognized as a "known situation," they resisted the idea of solving the problem based on what they knew. Repeatedly they said: "But I didn't learn how to do this" or "We didn't learn to do division with these numbers." Observe that they referred to "doing division with these numbers," rather than "this kind of division" or "dividing in such way."

The second aspect I found relevant when students approached the problem situation is that they did a linear reading of the problem. They gave more importance to the words used in the problem than to the actions implied in it. The following short dialogue with a student illustrates this:

Elias: I don't understand

Interv: What do you not understand?

Elias: The question is here ... How much is

She read the question for me and added: What is the question? Where does it start?"

In another situation, a student asked me to explain the problem. I told her that she should read the problem with attention, and look for the information the problem gave her and the question that was asked. She immediately asked me, "Can it be subtraction?"

I also observed that, in some cases, when they seemed to understand a problem situation, they kept a focus on calculations and the algorithm rather than on actions implicit in the problem and their results. Tereza is one example. She told me that she didn't understand the problem. Then, I explained the problem to her emphasizing the situation of buying a loaf of bread in the bakery as she usually does every day. At the end she asked me "Is it necessary to do the calculations?" " Laura also said: "Mrs. Maria, this is difficult. How am I gonna divide this thing here by this thing here?" (She was showing me the numbers involved in the calculations.) Observe that she was referring to the amounts involved in the operation described in the problem. But, the focus was on the numbers alone not on the action suggested in the problem.

There are some discrepancies among the cases. They came out when students were solving problems about candies (Problem #2) and choosing teams to play *queimada*. (Problem #3). Roberto is one of them. He came to me and said:

"Mrs. Maria, you see this problem here? If you have 45 *cruzados* 5 *cruzeiros* is left, ok? How many candies is it possible to buy? I could buy a half of a candy, because the price for each candy is 10." Other students also made comments about the impossibility of buying candies because their cost was more than five *cruzeiros*, and that was the change they had.

A third aspect I observed is that students insisted on looking for confirmation of their procedures all the time. Questions such as: "Is this correct?" "Is this done in this way?" "Is this calculation correct?" came as long as the activity lasted. It seems to me

that the circumstances of being in the formal situation of a class works as a constraint on the students' decision-making. This idea is reinforced by the fact that students used the same pattern of resolving problems based on what was required in mathematics classes. Although I had left only a blank space after each problem for them to show their work, the students divided it as they usually did with Linda. Students divided the blank space into columns where they wrote the representation of the operation to be done, the calculations done, and the answer. This is how they solve problems in mathematics classes. I observed only three students making use of drawings to solve problems. It seems that, in the minds of the students, the non-traditional way to solve mathematical problems is not the recommended way to go in class.

The exception to following the traditional pattern of classroom way of doing things was the case of answering some particular problems. Several students were able to give answers that were completely unusual to their daily routine in mathematics classes. Students answer problems following the question as it is proposed. In the case of problem #2, for example, the question was "How many candies can you buy with the change?" Usually, students answer this kind of question as follows "I can buy x candies with the change." Instead, these are some of the answers students gave:

- "I could buy 1 loaf of bread with the change."
- "I could not buy anything."
- "It is missing 5 *cruzeiros* to buy one candy."

I also observed that students often based their answer to a problem on the result of some calculation, without considering whether the answer found was reasonable or not. This might explain partially why there are so many students who give completely absurd answers to simple problems. The data I collected are rich in this regard. The following case is particularly interesting. The problem asked how the canteen woman could divide 45 loaves of bread between two first grade classes.

Fatima solved the problem as follows:

$$45 \div 2 = 810$$

$$\begin{array}{r} 45 \\ \underline{4} \\ 05 \end{array} \quad \begin{array}{r} 2 \\ \hline 810 \end{array}$$

The answer to the problem was "*Poderia fazer esta divisão dividindo 5 por 2, 4 por 2 que da em 810.*" (The division could be done dividing 5 by 2, 4 by 2 resulting in 810.)

Observe that she did not know how to divide 45 by 2 following the algorithm. She probably knew that each digit should be divided (4 and 5), and she knew that the algorithm was developed by subtracting on the left side, and she knew that division and multiplication were related in some way. However, she probably did not know the nature of the relationship between division and multiplication. (I saw her make the same mistake in classroom activities.) Also, she could not relate the numbers on the left side of the algorithm and its function. By the way she wrote the algorithm, it seems that she did not connect what she had learned about place value to the mechanics of the division algorithm. Finally, based on the calculations she had made, she wrote the answer without considering that it was complete nonsense.

Solving the Problems

I have been describing the students' approach to mathematical problems in the classroom. Now I will describe some specific strategies they used to solve these problems. I should remind the reader that all problems involved the process of division, because this was the topic Linda was teaching at that time. She had already introduced the long algorithm procedures for doing division with whole numbers and division with remainder. She also had taught that division was related to multiplication, as described in Chapter 4, and that this could be used as a way of checking the correctness of the

division.

Generally speaking, the students presented a limited range of strategies for solving the problems. Basically, they chose one of the four operations they knew and applied its algorithm as it was taught in mathematics classes. Although the problems involved division, students used all four operations to solve them. However, in their applications of basic operations, it was not clear to me whether the students had made sense out of the problems. In some cases there was no sense at all. Only two students used drawings and they only did so for problems #4, #7, and #8.

Adding. Adding parts to get the whole and then counting how many parts are added could be one alternative for solving these problems. Based on the work showed on her paper, Ruth used this strategy. In the problem about buying loaves of bread (Problem #1), Ruth started adding the price of each loaf, 4.50, but stopped after the first two. By basing her answer on the result of this calculation, she made the mistake of saying two loaves of bread. This was wrong because the total amount of money was 50 *cruzeiros*, not 9 *cruzeiros*. Besides Ruth, five other students also added to find the solution of this and other problems involving the ideas of partition and measurement. However, the use of addition did not make any sense, because they added the part to the whole. Fernanda is one example. She added 4.50 (the price of each loaf) to 50 (the total amount of money available). In addition to this, there were mistakes in the application of the algorithm. Two students did not take into consideration that they were adding decimals and they did not pay attention to the use of the decimal point in their calculations.

Subtracting. Doing successive subtractions makes sense in solving any of these problems. The ones that involve the idea of measurement are easier to do. Since the whole and the size of each part are defined, students could subtract one part successively until everything was gone. In problem #1, 12 students took this option, apparently. I am saying “apparently” because none of them presented this reasoning in its complete form.

As was described in the first situation of addition, students did only the first step in subtraction. All of them made mistakes in the use of the algorithm for subtraction. As they had done before, they placed the numerals incorrectly. These mistakes in the use of the algorithm led five of the students to absurd answers, such as 50 *cruzeiros* buying 400 loaves of bread at 4.50 *cruzeiros* each.

There is one interesting aspect that bears consideration here, and this is the use of subtraction in a different sequence than they used in mathematics classes. Linda had taught that, in the representation of subtraction, the first term is the value that is bigger. However, some students represented subtraction in the same way they say it orally. They wrote $4.50 - 50$ instead of $50 - 4.50$.

For problems involving the idea of partition, the use of successive subtractions could have been one alternative, too. Students could have estimated the size of each part, and taken it successively from the whole, if it was an infinite set. Or, in the case of a finite set, they could take units one by one from the whole until it was gone, as Linda had done with the *tampinhas*, when she introduced division in mathematic class. But students would have faced two additional difficulties. If the estimation did not work well, they had to redistribute everything again, or make new adjustments. Or, in the second situation, they might have made additional calculations to get the results. What I observed in the students' work is that they did not make estimations in order to subtract. My understanding is that the students resisted the inclusion of a new value in the problem on which they were working. They tended instead to reason with the data they had. This was the case with problems #7 and 8. In Problem #8, about 30 cups of milk to be distributed among 20 students, the subtraction was $30 - 20 = 10$. Students did not go beyond this to figure out how 10 cups of milk could be distributed again among 20 students. The result was a wrong answer of 10 cups for each student. Only one student said one cup for each child.

In the problem about six sheets of paper to cover four shelves (Problem #7), the

student subtracted shelves from paper. She did $6 - 4$ and answered 2 sheets. It seems that she did not make any sense of the problem situation.

Multiplying. This is the one of the most difficult strategies to apply in solving division problems. It implies that the student has to estimate the size or the amount of each unit to be multiplied until one gets the whole set. When the problem has the unit defined, students have to estimate the number of times to get the whole set. In this case, the difficulties I mentioned before for subtracting can also be applied here. But, it seems to me that the major difficulty is the introduction of a new element in the problem situation, defined in classroom context. Not one of the six students who used multiplication made estimations. All 10 times that multiplication was used, I could not make any sense of their strategies. They multiplied seven children by two teams, 150 *figurinhas* by two albums, or 30 cups of milk by 20 children. In this context, the absurdity of answers was the common point among the students.

Dividing. Linda was teaching division at the same time I was collecting data. It was therefore natural that the students looked at the problems thinking about the division they were studying. The challenge, however, was to apply what they had learned about the process of dividing, to new steps in the application of the basic algorithm. Following the algorithm, as Linda had taught it in mathematics classes, the students were able to solve only the problem about division of two teams to play *queimada*. Seventeen students (63%) out of 27 solved this problem using the algorithm correctly. But this was the easiest problem they had to solve. Only two students could solve more difficult problems (Problems #6 and 7) using the algorithm they had learned in the classroom. Based on these results, it seems that the students did not make sense of the school algorithm for solving division problems. In spite of this, I found four students who were able to solve the problem using their own version of the algorithm. All four students took the total number and did the division in only one step. This is one example: 45 loaves of bread divided by two classes:

$$\begin{array}{r}
 \underline{} \quad 45 \quad \overline{) 2} \\
 \underline{44} \\
 01
 \end{array}$$

Some students tried to use the same scheme Linda had taught in mathematics class, but they made mistakes such as:

$$\begin{array}{r}
 \underline{} \quad 45 \quad \overline{) 2} \\
 \underline{4} \\
 05 \\
 \underline{} \\
 5 \\
 \underline{} \\
 0
 \end{array}$$

or

$$\begin{array}{r}
 \underline{} \quad 50 \quad \overline{) 4.50} \\
 \underline{50} \\
 00
 \end{array}$$

In these examples the students seemed to have no idea what the numbers on the left side meant, and how to get them. They probably knew that there was some relation to the number called the dividend. Also, in their prior practice, they had done a lot of matching cases where the numbers on the left side repeated the dividend. In the second example above, the student gave a partially correct answer. My guess is that she knew the result beforehand, at least partially, and then she made up the algorithm just to satisfy the condition of having an algorithm for division in the context of the classroom. I had the opportunity to observe this same student in other situations in mathematics class and outside school. She revealed herself as being a very smart girl with good initiative on solving problems. Except for this, the majority of the students could not solve the problem based on the application of the algorithm taught in mathematics class.

Solving Problems Outside the Classroom Context

Analyzing what the students did to solve division problems, I identified some general procedures in the student's strategies.

1. Rounding numbers to be worked in the problem.
2. Estimation of possible results when the answer was not easily found.
3. Simplifying calculations, particularly when the mathematical algorithm suggested by the problem situation was unknown to the student.
4. Use of some mathematical properties of operations to make calculations easier, such as decomposing big numbers into smaller numbers in order to deal with them.

To exemplify the above procedures and to present a variety of strategies students used to solve division problems, some situations will be described here as they came out during the interviews. The described situations refer only to the second interview, where the eight selected problems were presented to the students.

Situation 1: Buying

In this situation, there are two types of problems. The first type is a problem where a certain amount is defined in terms of cost. An example of this is the following problem: "If you have 50 cruzeiros, how many loaves of bread costing 4.50 each could you buy?"

The second type is a problem where the student has to find the available amount from previous expenses made to buy something else. Example: "If you spend 45 cruzeiros buying a liter of milk and you pay with a 50 cruzeiro note how much candy could you buy?" Note that on this particular problem, the cost of the candy is not defined purposely. I was interested in knowing how the students would react to this missing data in the problem. In this regard, some of the children asked how much the candy would cost, but the majority of them assumed different values, according to their own experience buying candy at stores near their houses. In this situation of "buying,"

students used a variety of strategies based on their own experiences that I will describe in this chapter.

Situation 2: Making distributions

In this situation, there are two types of problems: 1) the ones dealing with discontinuing division, as for example dividing people in two teams, and 2) the problems dealing with division of a continuum such as dividing oranges and loaves of bread and milk. I was interested in knowing how children dealt with these different situations and if, in their reasoning, they perceived them differently. None of the 27 students showed evidence of having a problem dividing under either of these circumstances. They approached the problems naturally, both in situations with continuous and discontinuous quantities involved. The students took this difference into consideration when they were solving the problem and looking for the answer. In the strategies I will describe next, I will identify how it occurred and whether it affected the way they proceeded. Generally speaking, in problems involving the situation of making distributions, two main strategies could be identified: 1) using the remainder to balance the division, and decomposing numbers.

Situation 3: Making measurements

Two problems were included in this category:

1. If you need six sheets of paper to cover four shelves in a cupboard, how much paper is necessary to cover only one shelf?
2. Mrs. Luiza will distribute 30 cups of milk among the students of the third grade, but only 20 students came to class today. How can Mrs. Luiza distribute equally the amount of milk among all 20 students? How much milk will each student get?

Defining this as making measurements does not imply that these problems involve the idea of measurement as defined in some of the Brazilian textbooks about division in

elementary school (Amaral and Castilho, 1990). This is to look for how many times a given quantity is contained in a larger quantity, or how many times a subset is contained in a larger set. The situation of making measurements here means that the children were not necessarily counting to solve the problem. Their attitude in approaching these problems, revealed by their answers, was related to size (Problem #1) or volume (Problem #2).

In the first problem, the size of the shelves is a fixed determinant for the solution. This means that the size of the shelf makes six sheets of paper necessary to cover four of them. The childrens had to find out how much paper was necessary to cover one shelf. The amount for one shelf was already determined when 6 was defined as necessary for the whole set.

In the second problem, the children had to find out how much (volume) milk one child could have if 30 cups were available for 20 children. In other words, they had to estimate the volume of milk, in cups, each child could get out of 30 cups. For this set of problems, the strategies the students presented were: 1) making successive approximations, 2) proportion, and 3) decomposing numbers.

Estimating from their own experience

In this section, I will describe the strategies I found in the students' solutions to the problems we discussed in the interview. This strategy could be seen in two senses. First of all, it was clear to me that students relied on their own experiences when they approached the problem. I took a lot of criticism from them about some of the prices I gave them. For example, they corrected me on different occasions because of the high inflation rate we have in Brazil; prices are increasing all the time. From the beginning to the end of my data collection, the price of a liter of milk and a loaf of bread had changed three times. The following student's answer about buying candy with a 5 *cruzeiro* bill illustrates how the approach they make to the problem is related to their

own experience:

I thought: 50 minus 45 remains five. Then, I ask how much the candy costs. If the vendor say 2.50, I can buy two because 2.50 with 2.50 is five. If the candy is 1.00 *cruzeiro*, I can buy five because five times one is five. If the candy costs 2.00 *cruzeiros*, it is enough for two candies because adding two and two is four, and four plus one is five.

Note that the price for each candy was not defined in the problem. The student answered the price she knew based on her own experience on buying candies in the markets. Also she took into consideration that the price was changeable, because of the current inflation.

In another sense, some students also relied on their own experience to reason about the situation problems. This was not always clear for many of the students, but the cases of William and Laura might illustrate how this happened. The problem we were discussing was about how many loaves of bread costing 4.50 each could be bought with 50.00 *cruzeiros*. William explained that he usually buys 10 loaves of bread for 40.00 *cruzeiros*. If the bread was . 50 more expensive and he had 50.00 *cruzeiros*, he concluded that he could buy 12 loaves of bread. His reasoning could be described as follows:

40 *cruzeiros* buy 10 loaves of bread

adding 10 *cruzeiros* buy two loaves of bread

50 *cruzeiros* might buy 12 loaves of bread

Thus, his answer was not correct, but he didn't give an absurd answer like the

others had when they were trying to use only algorithms they had learned at school.

Laura, for example, answered that she could buy 220 loaves of bread. She concluded the following:

$$4.50 \div 5 = 220 \quad \text{by doing:}$$

4.50	5	
— 4		220
— 05		
— 5		
— 0		

However, when Laura was interviewed, she answered almost immediately 11 loaves of bread. During the interview she explained what she had in mind. She said:

"I didn't think anything, I did it from memory because my grandmother buys 12 loaves of bread every day and she pays 50 *cruzeiros* for them...

Other examples can be found in this chapter in different situations I will be describing.

Decomposing Numbers

This strategy was very helpful and utilized often by the students in solving mathematical problems. Basically, it appeared in almost all situations. It is interesting to note that in the situation of making distributions and sharing, the use of decomposing numbers differs from the situation of measurement problems. The main difference is that for the latter, students used it mainly for establishing proportions in terms of prices and quantities, and for the former the students used decomposition of numbers to regroup operations to make the calculations easier, using a distributive property of the division.

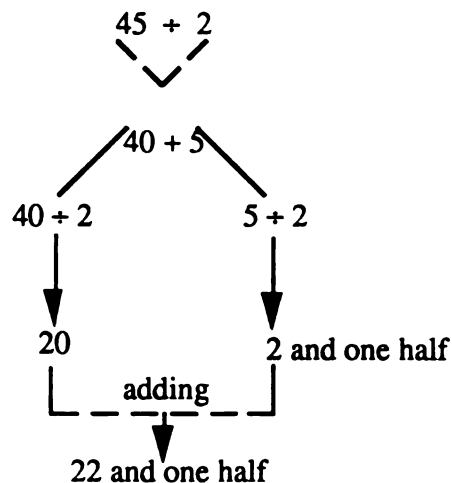
Here, they made successive divisions with smaller whole numbers than the one given in the problem, and then they made successive additions to find the exact answer. In my interpretation, this process involved many mental calculations that were not expressed verbally in the students' explanations. In addition to this, the students had to keep track of the partial results of the intermediary divisions in order to sum them up and find the final response. The following examples show some childrens making use of this strategy to solve the next problem:

There are 45 loaves of bread to be divided among two classes of first grade students. How could this division be done?

When the interviewer finished stating the problem, Marco answered 22.5 almost immediately, and he explained:

... 40 divided by 2 is 20 and 5 divided by 2 is 2 and one half.

His reasoning could be described as follows:



The solid line shows what the student verbalized about his reasoning. The dotted line shows what might be implicit, but was not expressed by the child in his explanation.

Or, it could be expressed like this:

$$\begin{aligned}
 45 \div 2 &= \\
 &= (40 \div 2) + (5 \div 2)
 \end{aligned}$$

$$= 20 + 2.5$$

$$= 22.5$$

To solve the same problem, another student tried to apply the algorithm he had learned in the classroom. When he was interviewed, he had learned the algorithm only for whole numbers without remainders. The result was what follows:

$$\begin{array}{r} 45 \quad | \quad 2 \\ \hline 22 \text{ } 1/2 \end{array}$$

And he explained to the interviewer: "Because 4 divided by 2 is 2; now 5 divided by 2 is ... 2 and a half plus 2 and a half ... 22 and a half."

A detail from my notes is important in order to understand what happened:

"Roberto was initially a bit confused when he was explaining his thinking to me about how to put together the number two, result of the first division with two and a half, result of the second division."

To me, it seems that he was confused because the way he was supposed to perform using the school algorithms did not fit his own way of thinking in solving the problem.

He might have thought:

$$\begin{array}{l} 45 : 2 = \\ \left| \begin{array}{l} 4 : 2 = 2 - - - \\ 5 : 2 = 2.5 + 2.5 \end{array} \right| \\ \quad \quad \quad | \\ \quad \quad \quad - - + 20 = 22.5 \end{array}$$

Another way to express this follows:

$$45 \div 2 =$$

$$= (40 + 5) \div 2$$

$$= (40 \div 2) + (5 \div 2)$$

$$=20 + 2.5 = 22.5$$

Another example could illustrate this same situation, where the student seems to use the school algorithms to start out, but mixes these up with his/her own way of thinking. It comes from Edina. I gave her the same problem and she said:

Edina: 45? (silence)

Interv: Tell me what you are thinking....

Edina: I thought like this: 45 divided by 2 is two times 2 equal 4, two times 2 equal 4, then I have 22. Two times 2, 4. 45 less 44 is equal 1.

Interv: And then, what conclusion can you make?

Edina: Each class will get 22 loaves of bread and there will remain one.

She described the first steps according to the algorithms she had learned in mathematics class:

Step 1 : 45 $\overline{)2}$ "45 divided by 2"

She had learned how to divide whole numbers and do basic division calculations. This means that doing divisions involving tens and units were not part of the students' skills taught in math classes yet.

Step 2 : 45 $\overline{)2}$ "is 2 times 2 equal 4, 2 times 2 equal 4"

4 2

At this point, the next step would be subtract four tens and divide five units like this:

$$\begin{array}{r} \underline{45} \overline{)2} \\ \underline{4} 22 \\ 05 \end{array}$$

But the student took the other way:

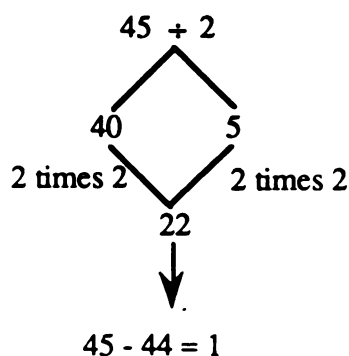
...2 times 2 is 4, then I have 22. 2 times 2...4.

45 less 44 is equal 1

or

$$\begin{array}{r} 45 \\ - 44 \\ \hline 1 \end{array} \quad \begin{array}{r} 2 \\ \hline 22 \end{array}$$

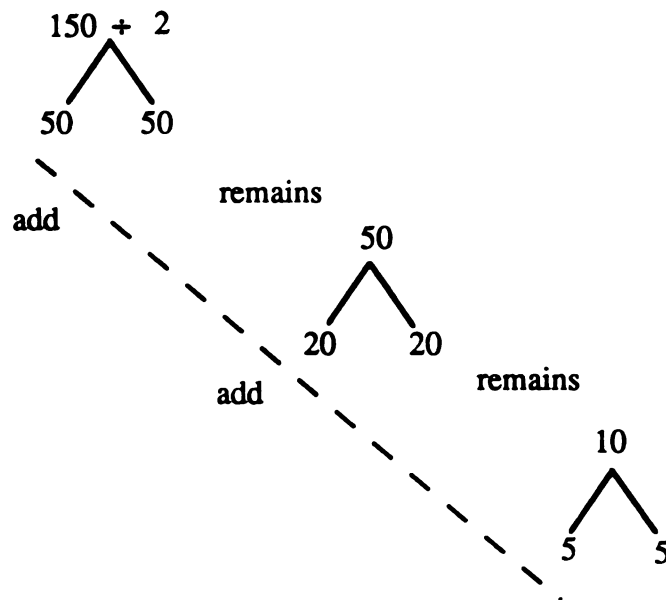
It seems that the student's mental calculation was:



In this case another aspect should be highlighted, and this is how the student dealt with multiplication to solve the problem. It seems that she did not use multiplication as repetitive additions. It seems that the idea of multiplication embedded in this reasoning is to look for how many times a specific number is within the other one. Thus, 22 is two times in 44, and two is two times in four.

In the problem of organizing 150 *figurinhas*. into two albums, I identified several examples of the strategy of decomposing numbers: Silvia explained how she could divide 150 into two albums: "I could do it like this: 50 for each, it would remain 50. Then, 20 for each and it would remain 10. And I divide five for each, it would be 75 for each one."

It could be expressed in this way:

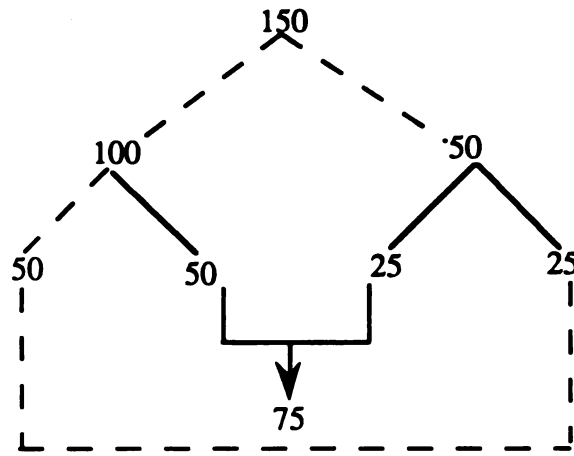


Or it could be expressed like this:

$$\begin{aligned}
 150 \div 2 &= \\
 &= (100 \div 2) + (50 \div 2) \\
 &= 50 + 20 + (10 \div 2) \\
 &= 70 + 5 \\
 &= 75
 \end{aligned}$$

Another example comes from Fatima, who gave the answer for this problem in 15 seconds. She explained her reasoning: "In 50 don't I divide 25 and 25? I took 50 from 100 and put 25 and 50."

According to what the student said, the following scheme could be illustrative:



Or:

$$\begin{aligned}
 150 \div 2 &= \\
 &= (100 \div 50) + (50 \div 2) \\
 &= 50 + 25 \\
 &= 75
 \end{aligned}$$

Another student, took less than 5 seconds to answer this problem. He explained his reasoning:

100 divided by 2 is 50 and 50 divided by 2 is 25.

Or:

$$\begin{array}{rcccl}
 & 150 & \div & 2 & \\
 \swarrow & & & & \searrow \\
 100 \div 2 & + & 50 \div 2 & & \\
 \downarrow & & \downarrow & & \\
 50 & + & 25 & = & 75
 \end{array}$$

The following examples are interesting, although the students were not able to find the correct answer by themselves. This is one case: "It is 53 ... because 200 divided by four is equal to 20 ... no, I mean 50, then 25 divided by two, I will have 55."

In trying to capture what is implicit in this expressed reasoning, some questions came out: Why did the student divide 200 by four? Why did he work with 200 instead of 150? Was dividing 25 by two a mistake? Or was he, in fact, dividing 50 by two and he made the mistake when he divided it twice? There are several possible answers to these questions, and to check each one would require further interviews with the student. This was impossible within the constraints of this study, but one hypothesis of explanation should be explored here. The student rounded the number to facilitate his calculations. In this study, rounding has been one common starting place to set up strategies for solving several problems.

Here, 150 could have been rounded to 200. Then, 200 was divided by four and not by two, possibly to compensate for the earlier rounding. The student probably chose four and not three because it would be easier for him to divide 200 by 4 instead of 200 by 3. He might have known that 200 divided by four is equal 50. In several informal conversations I had with some students, I observed them making this kind of calculation just to show that they knew it. Then, a possible representation could be:

50 for album one and 50 for album two

$50 \div 2 = 25$ for each and 50 he left out

With 50 *figurinhas* left, he had to divide again among the two albums. So, 50 divided by two is equal to 25. He might have made a mistake in saying 55.

The second example is from a student about 11 years old. She started her studies two years later than the usual school-age, that is seven in Brazil. During the interview, in solving the same problem about the *figurinhas*, she worked all the time with combinations of numbers to get the total of *figurinhas* she had or should have. But the interesting thing about this case was her refusal to accept that it was possible to divide 150 by two, being fair. She said that each album could have 50 *figurinhas*, but the 50 remainders could not be relocated because one album would have more *figurinhas* than the other one. She offered the following alternatives:

- a) 50 *figurinhas* could be kept
- b) 50 *figurinhas* could be taken out
- c) 10 *figurinhas* more should be bought to allow 80 on each album.
- d) to buy a third album and put 50 *figurinhas* on each one.

It seems that her difficulty was in finding out that 50 divided by two is 25. In the problem about 45 loaves of bread divided among two classes, this same student could not divide five by two. Her refusal was so strong in both cases that I started thinking about some particular difficulties she could have that would cause her to avoid these divisions. I raised two hypotheses as explanations. One possibility is that Anita was dealing only with whole number multiples of 10. She made combinations in such a way that she could operate easily, as for example:

30 and 20; or

20 and 20; or

30 and 20

To deal with multiples of 10 would be easier, because she could eliminate the zero and deal with numbers from zero to nine. In this case, dividing 50 by two would imply solving $5 \div 2$ and transferring the result 2.5 to 25, because she should be dealing with 50, not five. The constraint of having in mind whole numbers from one to 10 would make this transference difficult. However, this same student easily solved the problem about dividing 30 cups of milk among 20 children. She answered one and one-half cups for each child. She could not explain what she had done in her mind to find the solution. The difference between the two situations was that in the problem about *figurinhas*, the division was with discontinuous quantities. In the division with cups of milk, there was a continuum. This could be the reason for her difficulty in the previous problems.

Another possible explanation is that she could have been thinking in terms of the formal division she had learned in the classroom, where decimals had not been

introduced yet. Neither of these hypotheses could be verified, because the subjects of this study were not available after some time.

Finally, I would like to present an example of the decomposing numbers strategy within the situation of buying candies with change of five *cruzeiros* after paying 45 *cruzeiros* with a 50 *cruzeiros* bill (Problem #2). This case is particularly interesting because the student worked in two directions: going from 50 *cruzeiros* he had initially, and coming back to 45 *cruzeiros* he would need to pay for a liter of milk. The dialogue with the student follows:

Eduardo: I can buy two candies costing 2.50 each.

Interv: What did you think to conclude this?

Eduardo: 50 minus 2.50 is enough for two candies....

Interv: 50 minus....

Eduardo:...five, then I have 45.

45...50 minus 2... 50 minus... 2.50 will be 7.50...

Interv: Eh...take it easy! (sounding like I was not following his explanation) You've paid with a bill of...

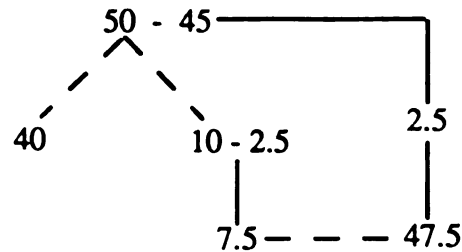
Eduardo: 50.

Interv: ...To pay for how much?

Eduardo: 45 for the milk and then, I'm gonna...10... 50 (he probably was looking for the amount to complete 50 *cruzeiros*) and, with the five that remains from all, 50 minus 2.50 will be 7.50...47.50 minus 2.50 will be 45.

He previously had 50 *cruzeiros*, and he knew that he could spend money until 45 *cruzeiros* remained, since this was the price of milk. He knew that five was left to buy candy. However, how could he get 7.50 out of 50 subtracting 2.50? It would be possible if he was thinking about 10, as he initially started to say, and not the whole 50. In this case,

one possible graphic representation could be:



Or: $50 - 2.50$

$$=(40+10)-2.50$$

$$=40+(10-2.50)$$

$$=40+7.50$$

$$=47.50$$

And: $47.50-2.50=$

$$=(40.00+7.50)-2.50$$

$$=40.00+(7.50-2.50)$$

$$=40.00 + 5.00$$

$$=45.00$$

Another possible explanation is that he knew the results and, when he was asked to explain his answer he tried to go back, “proving” he was right.

Using proportion and equivalence

Based on the data collected during the interview, I found some particular cases of students using proportion and equivalence. This happened in the situation of making measurement and, possibly, when they were reasoning about buying. Some children found the correct answer for the problem by establishing relations of proportion or equivalence among the data. One case was based on a drawing. This strategy was not used in formal terms as someone would do in mathematics class in higher levels. It was done based on concrete and simple situations of the everyday experiences students have outside school.

To solve the problem about 30 cups of milk to be divided among 20 students, Eduardo drew 30 cups. He put a mark point after each three cups while he was counting "three for one, for two, for three childrens until the tenth child. Immediately he realized that the answer was 1.5 cups for each child. I asked him what he had done to find out that the answer was not three cups for each child, but one-and-a-half per child. He answered: "With three by three there was not enough... it was 10 One-and-a-half plus one-and-a-half is three, as I had done. By three I got 10 children, since 1.50 is a half of three I could get 20."

I observed that Eduardo worked with counting by multiples initially, but he realized that he was wrong when he perceived at least two things. First of all, the relationship between the number of cups of milk and the number of children - more of cups for each child, less children can get milk. Second, the proportional relationship between three and one-and-a-half cups and 10 and 20 children. It could be described as follows:

3 cups for each child 10 children 30 cups

1.5 cups for each child 20 children 30 cups

It could be expressed as follows.

$$\frac{3 \text{ cups}}{1 \text{ child}} \cdot \frac{30 \text{ cups}}{10 \text{ children}} \quad \text{then} \quad \frac{1.5 \text{ cups}}{1 \text{ child}} \cdot \frac{15 \text{ cups}}{10 \text{ children}} \cdot \frac{x \text{ cups}}{20 \text{ children}}$$

Vera solved the same problem in a simpler way. She immediately knew that every child could have one cup and 10 cups would remain to be redistributed. To solve the problem, she related one cup for two children then, proportionally 10 cups for 20 children. The representation could be:

$$\frac{1 \text{ cup}}{2 \text{ children}} \cdot \frac{10 \text{ cups}}{20 \text{ children}}$$

The same thing happened when I gave this problem to a group of three students.

They all had the same reaction. They knew that each child could receive one cup of milk and 10 cups would remain. The major problem then became to find out how to divide 10 cups among 20 children. How they solved this part of the problem is described in the following conversation:

Interv: What can I do with these 10 cups?

Laura: Cut them in half.

Interv: Cutting them in half, the milk will be enough for everybody?

Roberto: Only if it is one-half for one and one-half for the other.

Laura: One-and-a-half for each one because there are 10 and there are 20 children, then one cup and a half for each.

Roberto: It is only a half because the remainder is only 10 cups, dividing by 20...(He had understood that Laura was saying one-and-half for each child in the second division.)

Laura: That's it. Divide one cup in two and it will be one-cup-and-a-half .

I reminded them that Laura was referring to the total that each child could get, because they had already divided one for each child before. And the conversation followed:

Interv: How have you thought to find out the answer of one-cup-and-a-half?

Roberto: I don't know.

Interv: You knew that 10 cups were excess.

Roberto: That's it, 10.

Interv: There were 20 children.

There are several interesting aspects to consider in this dialogue. But, for the purpose of exemplifying the strategy they used to solve the problem, I will highlight two points in the conversation:

1. The relationship was established when the child expressed that: "If 20 is the double of 10, has to be half to be right." It could be written this way:
"If 20 is the double of 10, then 10 is half of 20."

$$\text{or: } \frac{\text{one cup}}{20 \text{ children}} \cdot \frac{\text{half a cup}}{10 \text{ children}}$$

2. The relationship was established by Laura when she explained that if there were 10 cups and 20 children it would be necessary to divide one cup in two. And then each child could get 1.5 cup of milk. Or:

$$\frac{10 \text{ cups}}{20 \text{ children}} \cdot \frac{\text{one cup}}{\text{two children}} \cdot \frac{\text{half a cup}}{\text{one child}}$$

In the following examples, I could not distinguish clearly whether the student was thinking in terms of proportional amounts or if he/she was making successive additions. This doubt exists because several amounts were mentioned by the student and the interpretation may vary according to which amount is considered as the one which guided the student's reasoning. The problem that we were discussing was about buying candies with five *cruzeiros* of change (Problem #2). Eliana said that she could buy two candies and she would have one *cruzeiro* left because:

"... each candy costs two *cruzeiros*. Two plus two equals four, so... I can buy two candies; four plus two I can buy three candies, four plus four equals eight, so there's one *cruzeiro* left. I've paid four-and-a-half for the milk, and there's five left; so I buy two candies and there's one *cruzeiro* left."

Another way to say the above could be:

Each candy costs two *cruzeiros*;

$$2 + 2 = 4 \text{ buy 2 candies}$$

$$4 + 2 = 6 \text{ buy 3 candies}$$

$$4 + 4 = 8 \text{ buy 4 candies.}$$

Or, if thought in terms of proportion, it could be described as follows:

$$\frac{2 \text{ cruzeiros}}{1 \text{ candy}} \cdot \frac{4 \text{ cruzeiros}}{2 \text{ candies}} \cdot \frac{6 \text{ cruzeiros}}{3 \text{ candies}} \cdot \frac{8 \text{ cruzeiros}}{4 \text{ candies}}$$

My interpretation is that, if the student was adding successively, she probably would say:

$$2 + 2 = 4 \text{ buy 2 candies}$$

$$4 + 2 = 6 \text{ buy 3 candies}$$

$$6 + 2 = 8 \text{ buy 4 candies}$$

But instead, she might have realized that she could simplify her calculations by relating the amount of candy she could buy with the amount of money she was spending.

Then:

$$\frac{2 \text{ cruzeiros}}{1 \text{ candy}} \cdot \frac{4 \text{ cruzeiros}}{2 \text{ candies}} \cdot \frac{8 \text{ cruzeiros}}{4 \text{ candies}}$$

Another possibility is when she got to $4 + 2$, she added the next 2 to the old 2, and got 2 fours. Then:

$$2 + 2 = 4 \text{ (2)}$$

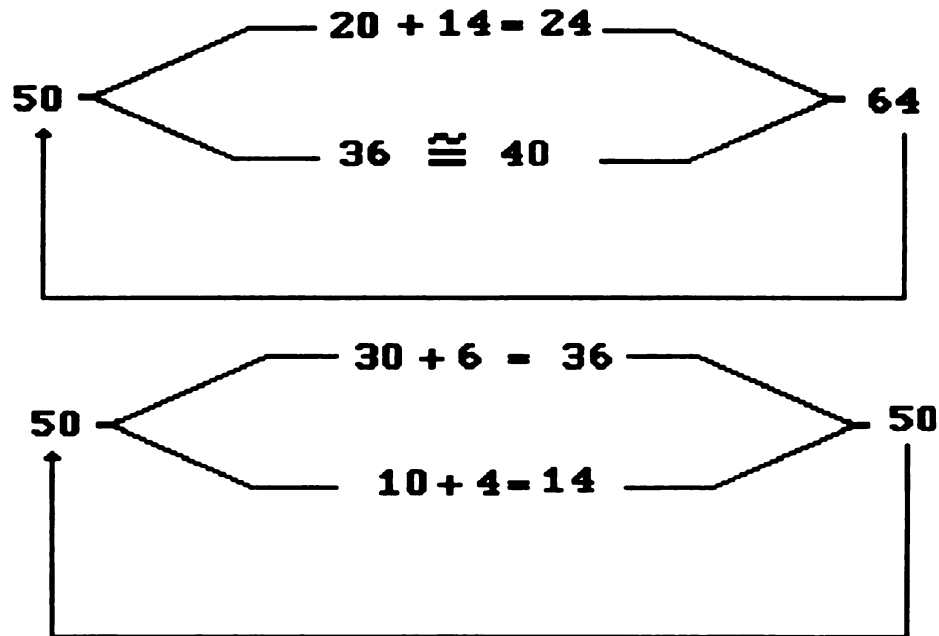
$$4 + 2 = 6 \text{ (3)}$$

$$4 + (2 + 2) = 4 + 4 = 8 \text{ (4)}$$

Another very interesting example is the one where the student tried to figure out the answer to the same problem by making the balance of different amounts. In this situation, the student had spent 36 *cruzeiros* buying milk with a 50 *cruzeiro* bill. He explained how he found out that the change would be 14 *cruzeiros*:

"I divided like this: it is not 20, if so, it would be 24, and I would have 64. It would be... I put 14 because I already had 30, so it would be 36... Then I decided for 14."

In trying to understand what he thought in order to say 14, at least two possibilities come out, and I will explain them using the following diagram:



Here, the student rounds the numbers to facilitate mental calculations. In this scheme, the student may have rounded 36 to 40 and worked with the remainder four, because it should be part of the answer. In this try he might have estimated, and the total he found was 64. This showed that he was wrong in working with 20. This explains why it didn't matter whether he said 64 or 54, because both were more than 50; consequently they were wrong anyway. In his second try, he may have decided to work with 10, but he had the remainder four from 36 in mind, which was part of the answer, so he said 14.

Another example of solving the problem of buying 50 *cruzeiros* of bread costing 4.50 each is from Mauro. He explained that with 10 *cruzeiros* he could buy two loaves of bread and there would be one *cruzeiro* left. Fifty *cruzeiros* could buy 10 loaves of

bread and five *cruzeiros* would be left. Five *cruzeiros* could be used to buy one more loaf and he would still have .Fifty left. In this case, I could say:

10 <i>cruzeiros</i> to pay 2 loaves of bread	1 <i>cruzeiro</i> left
50 <i>cruzeiros</i> to pay 10 loaves of bread	5 <i>cruzeiros</i> left
5 <i>cruzeiros</i> to pay 1 loaf of bread	50 <i>cruzeiros</i> left

Making successive approximations

To solve the same problem described in the previous situation about buying candies (Problem #2), two examples of successive approximations can be shown where students performed successive and implicit additions when solving the problem:

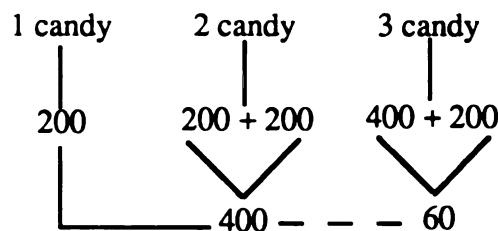
Anita: I don't know how much the milk costs.

Eduardo: 43.

Roberto: There's seven left... seven minus... How many candies can I buy?

Anita: four, no... each candy costs two, so I can buy... Let me see, one... two... three. (She was counting in her fingers while talking.)

This could be expressed as follows:



Anita was adding successively each amount she would pay for each candy she had added.

A second case shows students solving the same problem:

Silvia: I can buy two candies because I'll have five left.

Nélio: I think it's right because two plus two is four and there's 1 left.

Vera: There's five left, milk costs 45... To complete 50 it would be five left. So

the candy is two, plus two is four, there wasn't one to buy three candies, it has to be one left.

In the situation of buying loaves of bread costing 4.5 each (Problem #1), other examples of doing successive approximations appeared. Students were looking for the number that could be the dividend in the division of 50 by 4.5. But students did not complete their reasoning to find the answer. In the following example, it is important to note that the student had in his hand five 10 *cruzeiro* bills.

I thought like this... But I'll take out (showing a 10 bill) 50 cents: four plus four is eight, with this amount I can get two loaves of bread, and plus this (another 10 bill) I can buy four, this here buy six, and this (another 10 bill) buy eight, with this (another 10 bill) I buy 10.

It could be argued that the situation of having five 10 bills suggested successive additions to the student. Note also that the student tried to round the value of four-and-a-half to four to facilitate the calculations. Two other students did the same rounding. One of them, Laura, explained her reasoning: "I thought: 10 to two, remains one. Then, I put the same value for 10 loaves of bread and five remained. With five I can get one more loaf of bread."

Other solutions were described where students made successive approximations by adding to find the solution to the problems, such as:

$2 + 2 + 2 + 2 + 2 = 10$ loaves of bread. Or, in problem #2 about buying candies:

$1.50 + 1.50 = 3.00$ ——— 2 candies

adding 1.50 ——— 3 candies

adding 1.50 ——— 4 candies

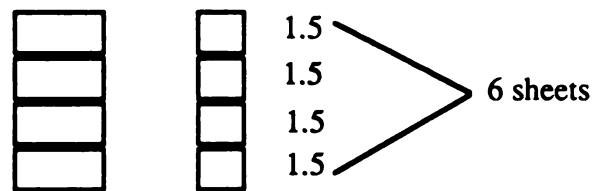
3.00 plus 3.00 is equal 6.00

6.00 plus 3.00 is equal 9.00

Then 10.00 buys 6 candies and 1.00 remains.

One very interesting resource for solving problems in informal situations was the use of drawings. Laura drew the situation described in problem #7 on paper, and, by

making successive approximations subtracting, she got the solution:



And explained what she thought: "I spent one here, one here, here and here (showing each one of the shelves). Four are gone. Two sheets are remainders. Then I cut them in halves."

Roberto had done the same. They both took out what was clear for them - one sheet for each shelf. The ratio of two sheets for four shelves became easier to perceive. They could easily take half out for each shelf again. Other students added one-and-a-half four times and checked to see if the total was six.

Edna spent less than 10 seconds to give the answer for the problem. She explained:

I thought that if I spent two sheets of paper on each shelf, it wouldn't be enough because two sheets for one shelf, two sheets for another shelf, would be four sheets, adding two would be six sheets, so it wouldn't be enough, then it will be one-and-one-half.

When the interviewer asked her why it would be okay being one-and-one half, she said:

"Because one-and-one-half one-and-one-half I get three sheets of paper. Adding one more (shelf) four-and-one-half and one (shelf) more it will be six (sheets)."

Using the remainder to balance the division.

When the problem presented was a non-exact division, the students might or might not leave the remainder in either of the situations. When they were making distributions

of loaves of bread among two classes of students, or organizing two teams with seven students, they realized that in both situations they would have a remainder. Some of the students answered leaving the remainder untouched, and others preferred not to. But the aspect that seems to make the difference in the decision whether to omit the remainder or not is the context of the problem. This can be shown in the following problems:

- 1) seven children want to play *queimada*. (a Brazilian game). How could you divide the two teams?
- 2) There are 45 loaves of bread to be divided among two classes of first grade students. How could this division be done?

Both of these problems have the same mathematical structure: There is a set to be divided into two subsets, the first one being represented by an odd number. In the first problem, the amount refers to persons, which cannot be divided in a continuum. Supposedly, this would require a division with a remainder.

In the second problem, although the amount is represented by an odd number, it refers to a loaf of bread, which could actually be divided. In the answer to this kind of problem, it is not necessarily required that the student leave the remainder out. The student could divide the one loaf of bread left into two pieces and have half for each class. However, the students in this study showed a different attitude related to these two problems. There was no apparent preference in continuing the division or leaving the remainder out in the second problem. They suggested both ways. But, in the one about the division of the two teams, the remainder was extensively discussed. They tried to find different solutions which would avoid the remainder, and at the same time, continue being fair in creating the teams. They did this by describing the same strategies they use in real situations when they are playing. The most common strategy was defining one extra life for the team with less players. In this case, the answer to the problem was to have one team with four children and the other team with three childrens and one extra life. This meant that one of the children could have the role of two childrens in the

game.

Another strategy was to balance the two teams, not in terms of the number of players, but in terms of the strength of the two teams. It is important to say that the strength of the children in this game is very important. This is because the best player is the one that can throw the ball with enough strength for it not to be caught by the enemy. Other strategies were: 1) one player would play on one side for half of the game and on the other side for the second half of the game, and 2) the remaining child would assume a different role in the game such as being the judge or catching the ball when it was out of the field.

The common point in the discussions about the remainder was to avoid having the child out of the game. The following conversation in a group of children illustrates several aspects discussed in this regard, particularly the desire of not having one out:

Fernanda: You cannot because it is necessary to have eight... only if one stays out to be chosen when someone loses.

Laura: Only if it is four players on one side and three on the other side....

Paulo: ... and one keeps one life.

Interv: What is another possibility?

Paulo: One stays out.

Fernanda: And if he does not accept this? He has to play if not he would mess the game up....

Paulo: Or then, the team which has two girls would have one more player on its side.

The same reaction related to the remainder occurred when they were dealing with two oranges to share with three friends. This problem presents the same mathematical structure that the others discussed before, as far as the idea of sharing is concerned. However, this problem presents one important, distinctive aspect. The set to be divided is smaller than the number of people with whom the oranges should be shared. The

children's preference was to divide the two oranges in two pieces each and give one piece to each friend. The remaining one-half would either be divided again in three small pieces, or it would be given to a fourth person defined by the children themselves. This could be the younger brother, the grandmother or the teacher, or the children themselves. One child's strategy to avoid the remainder was to divide each orange in three pieces and give two pieces for each person.

The following conversation among three students illustrates this:

Maura: I know. Cut the two oranges in three pieces each and give two pieces for each friend.

William: Then there will be a remainder of one

Maura: No

William: Yes, and you take the other part and divide again.

Maura: No. There are two pieces for each. You cut the two oranges in three pieces each and give two pieces for each friend.

Laura: Cut in two, and you have four. Then there will remain one and will be one for each.

William: Cut the two oranges. Give one piece for me, another for M., and another for....

Laura: The one remainder, cut it in three. So it will be one part and a half for each one.

From the cases and situations I have presented here, it is possible to see how much children can do in terms of mathematical reasoning in the everyday situations they face. This contrasts strongly with the emphasis on the application of algorithms when students are solving the same kind of problems in mathematics classes.

In the final chapter that follows, I will discuss the possibility that there exists two different ways of doing mathematics depending upon the context in which children find themselves.

CHAPTER VI

CONCLUSIONS

Are There Two Different Mathematics In Two Different Worlds?

Discussions with students about mathematics contrasted with my observations of them in situations where they used mathematics. The difference between our discussions and my observations reveals the distance between what students understand through mathematics and their experiences outside of school.

One of the key differences I will focus on is that students apply algorithms to school problems mechanically, without thinking about why, or whether, the resulting answers are reasonable, while they don't do this in "real world" mathematical situations. I will approach these differences or contrasts taking into account how: 1) the characteristics of students doing mathematics in school and out of school were shown to me, and 2) students performed in both situations. I will explore how these and contrasts reveal the possible existence of two different views of mathematical knowledge. To conclude, I will reflect on the roles of particular aspects of the students' social and cultural context in the construction of their notions of division.

The first sign that I might be dealing with two different things when approaching students' experiences with mathematics inside and outside school occurred in a conversation I had with Fernanda. She expressed a clear distinction between her own idea of division and what the teacher expected from her when doing division at school. In fact, she pointed out the two aspects that I would look at in this study the: 1) student's idea of division and 2) the division taught in school. Here is that segment of our conversation:

Fernanda: I thought that division was like this: 10 divided by five was five and five. At home, if I change 10 *cruzeiros*, I split five and five.

Interv: Is it a division?

Fernanda: Ah, I think it is, but when I have to do it in the notebook, it is not. If I do that, the teacher will consider it wrong.

As I was talking with Fernanda, I wondered why she thought that her way of doing division would not be division in the classroom context. By expressing her idea of division in two different ways, at home and at school, she suggested to me that there were two different views: 1) one coming from how she does things in her daily life, and 2) the other from her understanding of division from classroom teaching. Thus, I should distinguish “real world” from school mathematics. This might clarify the difference between how students perceive the nature of mathematical knowledge in the real world and in school.

As discussed earlier, the real world to students I worked with is a place where limits are defined according to different, concrete positions people have in different places. Decisions are made according to interests and the particular conditions of power of the decision maker, as well the limits of the situation where the decision has to be made. Consequently, there are always different sides to be considered in a decision-making process. The decision varies according to who is deciding and how it is made. The power of doing things--deciding, taking, or giving--is related to the ideas of authority that students recognize, and to whom they submit. Power is also related to good social and economic conditions, in contrast to poverty and inequality. According to these differences perceived by the children, there is a logic to what is possible for them and what is not.

This logic of the possible is defined in terms of what they realistically have access to or believe is attainable. This can be seen at different levels. For example, they know it is possible to vote in a presidential election, but it is not possible to interfere in

the president's decisions about what has to be done. It is possible for them to have a job, but not to decide about a fair salary. It is possible for them to have access to school education, but it is not possible for them to go to private school because they cannot pay for it. In sum, as Freire has pointed out earlier in his work with illiterate adults in Brazil (1970) the idea of what is possible is related to the power they have and their conditions of existence.

In contrast, school life represents a break in the students' daily experience outside school (Singer, 1988). This does not mean that what is going on outside school does not affect school life have an influence on it. The image that better represents what I observed in the school site is a filter. This means that students are taken from the limits of the real world and put in a place that is presented as different. This difference is defined as the school's social distance from real world events, and by the student's treatment in the school's context. The case of meningitis is an example. Meaning it is spread, and several cases had been reported in the school's neighborhood. While the entire community knew of the existence of the disease and, while insufficient food and housing and inadequate arrangements for controlling the disease were contributing factors, the school approached the situation as if it resulted from inadequate personal hygiene. To the students, the school's approach made it appear as if washing their cups and refraining from kissing one another would stop the disease.

In the same way, mathematics teachers presented numbers to children as being a world unto themselves. That is, numbers alone, without context, were assumed to have meaning. Therefore, students were being taught the mechanical manipulation of numbers without realizing that mathematics actually comes from life outside school and that their experience is full of mathematics. When students reported how they dealt with situations requiring mathematical reasoning inside and outside of school, they spoke of a different kind of mathematics.

School mathematics follows an approach within the parameters of a formalist tradition of mathematics (Machado, 1987). Students think of mathematics as a manipulation of numbers according to certain rules. By dealing with numbers, mathematics is abstract, value-free and exact. Numbers speak for themselves about truths within a field of centrality where it is possible to say exactly what is right and what is wrong. Thus, school mathematics operates within a fictional world, working according to its own rules, and where its logic is different from the logic of everyday life.

The following conversation shows how different the approach is when school mathematics is used. Three students and I were discussing some situations the students face in stores in the school's neighborhood. We had just talked about buying a liter of milk with 50 *cruzeiros* and candy with the change they got. Each one of them answered me, correctly, how much candy they could buy with the change. The figures given by each child were different according to the price they paid for the milk. After this, I asked them to solve the problem of how much candy I could buy with five *cruzeiros* at one-and-a-half *cruzeiros* for each candy. Here is how our conversation proceeded:

Fernanda: It's adding? What calculation could it be? Let me see how much this will be.

Rita: It's adding? (asking herself.)

Interv: Why are you adding?

Rita: Mine (the result) is already wrong. I will make every calculation: divide, multiply,

subtract and add... but adding didn't work.

Laura: I finished. The answer is 510.

Anita: But, buying 510 candies?!? (They laugh.)

Interv: It means that I go to the store with five *cruzeiros* and can buy 510 candies?! (with surprise.)

Rita: GOSH!! Only if it is with "fish man" (All laugh again.)

I proposed that they should not think in terms of pencil and paper answers and asked again, if I had five *cruzeiros*, approximately how much candy could I buy. They all answered “two” or “three.”

This situation shows some of the limitations the students encountered when they tried to solve the problem using school mathematics. First of all, the students did not approach the action of buying candy as a real situation. The evidence for this is the absurd answer (510) Laura gave to the problem in contrast with the estimation (three pieces of candy) she made at the end of the conversation. The first answer was the result of the wrong division. The answer she gave was only a number resulting from the calculations she had made. It was not a reasonable conclusion based on real events. To reach a conclusion about the plausibility of the answer, the student would have to think as if she had the five *cruzeiros* to buy the candy. She would then realize that it was only possible to buy two or three pieces of candy. Second, the emphasis was on the algorithm to be applied, not on the action of paying one-and-a-half for a certain amount of candy, which could not exceed five *cruzeiros*. The evidence for this is the fact that all three students were looking for one possible algorithm to get the answer; it could have been adding, subtracting, multiplying or dividing. Finally, Fernanda seems not to make any sense of the algorithm she used to divide. She wrote on the paper:

$$\begin{array}{r}
 1.50 \overline{) 5} \\
 \underline{-1} \\
 05 \\
 \underline{-5} \\
 00
 \end{array}$$

When I looked at what she had done in connection with my observations of her in the mathematics classes, I observed that she did not understand the application of

the algorithm for division. One example of this is the numbers on the left side that are only repetitions of the divisor or quotient.

Another situation occurred when Helena tried to invent something about mathematics based upon picture # 7:

Helena: One lake adding three of these pipes would be four.

Inter: Can you add a lake and pipes?

Helena: In mathematics, it can be done.

From this conversation, I realized that in saying “in mathematics it can be done” she was distinguishing two different meanings of “possible.” Possible here goes beyond her own sense of possibility in the real world. Adding lakes and pipes was possible in a fictional world with no real context and where explanation was based only in the correctness of the algorithm applied.

In other words, what is possible in the real world is different from what is possible in school mathematics. In school mathematics, a possible solution is the result of a mechanistic application of an algorithm. When it does not work, the result is regarded as impossible to obtain. When students are solving school problems, they apply the algorithms they have learned at school and they are supposed to find the correct answer. There is no need for reasoning in terms of plausibility, as is necessary in real situations. The problem buying five *cruzeiros* worth of candies costing one-and-one-half each was supposed to be real and meaningful because it was taken from a concrete situation (buying candy) in students’ experiences outside of the school. However, students approached it in the same mechanistic way described above. This is why Laura gave more than 500 pieces of candy as a possible answer with the school mathematics approach. When we discussed the same situation, there was no doubt in our conversation that five *cruzeiros* would not buy more than a few pieces of candy. The school mathematics which emphasizes algorithms and mechanistic working with problems does not leave room to reason about absurd answers in the same terms that children would do

in daily life situations. The text of the problem, or the words that are used to describe it, are ineffective in leading students to approach it as they would a real situation.

Decontextualization of school mathematics seems to play an important role in how students reason and solve problems in school. Tables 1, 2 and 3 summarize how students performed in both in and out of classroom situations when solving word problems.

Table 1

Results of Students Solving Problems in the Classroom Context

Problem	# of Students	Correct Answer	Wrong Answer	No Attempt	Correct Answer No Work
#1	27	4	20	3	2
#2	27	19	7	1	4
#3	27	19	8	-	5
#4	27	4	19	4	3
#5	27	8	14	5	2
#6	27	8	14	5	2
#7	27	6	18	3	4
#8	27	7	16	4	5

Table 2

Results of Students Solving Problems in the Real World Context

Problem	# of Students	Right Answer	Wrong Answer	No Attempt	Close Estimation
#1	26	8	4	1	13
#2	26	25	1	-	-
#3	22	22	-	-	-
#4	26	22	-	2	2
#5	23	12	5	2	4
#6	19	12	4	-	3
#7	21	4	12	1	4
#8	18	11	3	1	3

Table 3

Results of Students Solving Problems

Problems	Classroom Context			Real World Context		
	# of Students	Correct F	Solution %	# of Students	Correct F	Solution %
1	27	4	14.8	26	8	30.8
2	27	19	70.4	26	25	96.1
3	27	19	70.4	22	22	100.0
4	27	4	14.8	26	22	84.6
5	27	8	29.6	23	12	52.1
6	27	8	29.6	19	12	63.1
7	27	6	22.2	4	4	100.0
8	27	7	25.9	18	11	61.1

An example of my previous point is shown in problem #6 in both tables. This problem is about 45 loaves of bread being distributed between two classes of first grade students. In the classroom context, only eight out of 27 students (29.6%) solved this problem correctly. In addition, two of them gave only an answer, but did not show the work they did to arrive at it, and five students (18.5%) did not even try to solve the problem.

On the other hand, 12 out of 19 students (63.1%) were able to respond correctly to the same problem outside of the school context. In addition, three out of seven who could not solve it, gave reasonable approximations of the right answer. Thus, the likelihood of absurd answers diminished when students were solving problems in the interviews. As shown in the tables, when the problems were solved in the classroom context, the majority of the students either did not try to solve the problem, solved it incorrectly, or gave the answer without showing their work on the problem.

Table three summarized part of data in table one and two combining percentages of correct answers both in classroom and real world contexts. Table three shows evidence that children's level of correct answers is higher in the real world context as compared with classroom context.

Another important aspect to observe is the richness of the students' reasoning when they solved the problems outside the classroom. In problem #3, for instance, the discussion about the remainder when dividing seven kids into two teams did not occur in the classroom situation, though it did in the interview. Also, as I mentioned in Chapter 4, the children produced different strategies for solving the problem, as they considered various aspects involved in a balanced organization of the two teams. This became evident in both the discussion of the team's definition during the interview, and when they were playing on the street.

In the classroom, when students were solving problem #4 about three friends dividing two oranges, none of the students performed the algorithm correctly (dividing

two by three), although six students had described this as the operation to be done. One student used a drawing to find the answer. In the classroom, only three possible correct variations. Here is a case where different conditions and interests account for different ways of dividing two oranges among three friends. When facing the same problem in a classroom situation, students divided two by three, not two oranges among three friends, even though the words used in the problem were the same.

In terms of the strategies students used both inside and outside of the classroom context, the difference is far greater than I expected to find. In classroom situations, the students looked for one of the algorithms taught, and applied it to find the answer. They took the number resulting from the algorithm, whatever it was, and mechanically wrote it as the answer, without reflecting on the result at least to see whether it was reasonable. The teacher had told them to do this. She also has implied that if they performed the operation correctly, they would obtain the right answer. Therefore the students had faith in the teacher. Message and obeyed her command. Students did not use any strategies to facilitate calculations, nor did they take any initiative to bring their experiences from everyday life into the situation.

In contrast, when students were solving the same problems outside the classroom, they used a variety of strategies to reason their way through these situations. Rounding, making estimations, and simplifying were the most common ones that I found. Carraher et al. (1984) illustrate particularly the strategy of rounding. They argue that it is easier for children to deal with tens, to facilitate calculations. Almost all studies about children doing division outside the school context (Boero, 1989; Burton, 1983, Carraher, 1987 Kouba, 1989; Moser, 1982) mentioned. Other strategies I, too, had found among students, such as making successive additions and subtractions.

In my study, the children presented sophisticated processes of mentally calculating the answer. For example, some of the students reasoned in terms of proportion, made balances of different amounts, and used some properties of addition and subtraction.

Weiland (1985) mentioned strategies involving the distributive property of operations. Schlieman and Magalhaes (1990) studied proportional reasoning used in shopping, kitchens, laboratories and at school. All these strategies presented in these and others studies show the variety of ways students solve problems. On the other hand, the school's mathematics leads to students' reasoning based on fixed algorithms. When students do not understand the algorithms, the algorithms make it more difficult rather than easier to solve problems. When this happens, learning mathematics become mechanical, boring and lacks the rich capabilities already possessed by the students.

As I reflect on what the students said about mathematics and how they used it inside and outside school, I am convinced that we as mathematics educators have to review our practice, if we want: 1) students to learn with understanding and 2) to enhance the development of students' creative minds.

Constructing the Notion of Division

In this chapter I have been analyzing the contrasts and contradictions I found between students' perceptions of school mathematics and mathematics in their daily experience. As I mentioned, the three main sources for my premises have been: 1) social and cultural context, 2) mathematical knowledge, and 3) mathematical reasoning in and out of the classroom situation. Now, I would like to resume the initial frame work of this study and look at the basic points which have guided the study. The focus is on the construction of mathematical knowledge as a process in which the learner and his/her social interactions are central elements (Confrey, 1987). In this sense, the initial question of this work is how do individual student encounter mathematical knowledge in school? More specifically, what do students experience as they encounter mathematical knowledge in construction the idea of division?

I have already said that this encounter could not be seen when I examined information from school only. When I was in the classroom, it was impossible for me to

connect what students were learning about division in their mathematics class with their experiences outside school. Therefore, it was necessary to become familiar with both contexts, and then to look for connections. This procedure contrasted both contexts and unveiled the encounter I was looking for. It became apparent in how students talked about division, and particularly, in how they did division. This report is replete with examples of how they did division in a variety of situations both in and out of school.

In their journals, I asked what division meant to them and they responded:

- "It is a very important calculation."
- "It is the opposite of multiplication. It's to split numbers: it's the division of numbers."
- "It is to split, to distribute."
- "It is distribute, to share something."
- "It is the same as multiplying because four times five is equal to 20 and 20 divided by four is equal to five. Then, if someone does not know multiplication, the person also does not know about division."
- "It is an operation used by people."
- "It is sharing one thing with other people."
- "It is to split, to allot, distribute."
- "It is a number divided by another one."
- "It is when I have a biscuit and share with another boy who is with me."
- "It is sharing by two persons or more."
- "It is 10 divided by two is equal to five."
- "It is sharing with others."

Taking a careful look at these answers, I observed that, although some students had drawn from definitions they had heard in classroom situations, the majority of them expressed the idea of sharing, splitting, or both. Sharing was the most frequent idea of division students dealt with while I was studying them outside the classroom.

Sometimes they interpreted incorrectly the teacher's explanation. This was the case of Sandra, who answered that division is "the same thing as multiplication because four times five is equal to 20 and 20 divided by four is equal to five." In fact the teacher's explanation was that 20 divided by four is five because five times four is equal to 20. She was relating division and multiplication as a process of taking equal amounts apart (division) or putting them together (multiplication).

It is interesting to note that the guidelines teachers have to work with when teaching division are based on two main ideas: Partition and measurement (Amaral and Castilho, 1990). Sharing is seldom mentioned by authors when dealing with types of division, although Zweng (1964) considered sharing as one distinct aspect of partitive division.

I would like to call the reader's attention to several important aspects of division I identified in how students talk about and do division. First, students distinguished the division they did in numerical terms from division they did in broader terms, such as sharing or distributing. Division in numerical terms was the one in which they dealt with numbers, and where the emphasis was on numerical relationships or on quantities. This was the case when students were dividing numbers outside of a "real context," such as numerical problems in the classroom. In this situation they might have had 15 divided by three, and made calculations according to the school algorithm to find an answer. This kind of division started and ended strictly with manipulation of numbers, and the correct numerical result was what mattered. There were other times in the classroom when the problem contained a had a depiction of some situation from daily life outside school, but the division was guided by formal mathematics as it was presented in the school context. In this case the problem referred to elements of daily life outside of school, but the students did not assume this was the real situation. The following word problem used by the teacher, in an attempt to be more connected with the students' experiences is an example: "Jose has 39 coffee packets to arrange on three shelves. How many packets

does he need to put on each shelf?"

This kind of word problem talks about things students know well, because they deal with vendors at their stores frequently. In addition, this is the way vendors show the products they have to sell, exposing them on shelves. Also, coffee is one of the most common product sold, besides milk and bread.

When students saw this problem in the classroom, they had in mind only the numbers involved and the algorithm they used to find out the right answer. In the real situation, several aspects would be taken into consideration besides having an equal amount on each shelf. In a real situation, Mr. Jose would be considering the size of the shelf, its location, how much weight the shelf could support, whether he wanted to put all coffee packets on the same or on different shelves, and so on. Therefore, when the students have to solve problems within the classroom, they do not solve them as they would in their daily experiences. This point is extensively demonstrated in Chapter 4. When students think in numerical terms, the division of four by five, for example, is viewed as impossible to do because the students say that they "cannot take five from four. If it was four divided by four, it would be possible." This alleged "impossibility" is because the students simply have not learned the algorithm for this kind of operation yet. They were reasoning only in numerical terms, without considering the process of division as a real possibility, symbolized in numerals.

On the other hand, students did divide two by three when they solved the problem about the oranges in the interview. They found several ways to divide two oranges among three children. In this case, they were not working in numerical terms, dealing only with numbers. Students were dividing two oranges among three children as they would be doing in a real situation. What is missing in the example described above is the algorithm to put this operation in terms of school mathematics. In this sense, I have asked myself whether problem solving is taught in schools through a process of "domesticating" minds, (Freire, 1987) rather than stimulating creativity and freedom of

thought to find possible and intelligent solutions. In other words, do we teach mathematics in our schools in a way that puts constraints on children's minds, instead of promoting the development of children's intelligent reasoning? The evidence that leads me to this question comes from my observations of how the children solved problems outside school, compared with how they did in mathematics class. These students are far beyond where the school is. School is not teaching problem-solving strategies. Nor is school building on, acknowledging or even uncovering the intuitive, informal, but sophisticated and complex strategies students have constructed and use outside of school.

Coming back to the discussion of the students' ideas of division, the second idea I found in their definitions is exactly what I was talking about earlier. Students did division outside of school as an action of sharing, giving, or making a distribution. This means that students solved problems not in numerical terms, but as if they were actively dividing something. The action was of sharing, distributing or giving. The emphasis here was on the process of doing division, on the act of dividing. The correct numerical result was not that important; the "how to" came first. This idea of division was so strong among the children I studied that almost every time they talked about division, they referred to situations where they were sharing, distributing or giving.

To corroborate this interpretation, I should mention that when I asked students about things they could divide, they also answered by naming things that are part of their own life, such as personal objects, toys, candies, food, and so on. The following student's statement is an example: "I can divide my things. I cannot divide other people's things."

The most striking characteristic of this idea of division as an act of sharing, giving, or distributing is fairness. Being fair, in the sense that is used in this meaning of division, does not necessarily mean equal division. In the division of children into two teams to play a game, for example, several solutions were proposed for the purpose of being fair, and most of them did not just take into account the number of players on each team. In

fact, even when they had the same number of players on each side, other factors were considered, such as strength and weakness of each player.

In the situation of sharing, as for example having two oranges to be divided among three children, they showed clearly that being fair might not mean equal division. One student gave me the following solution to this problem: “The youngster gets one whole orange and the other two divide the second orange in the middle.” Or in another example from S.: “I will keep one for me, give another to her (a friend) and she (another girl) will get none.” This example again raises the issue of people’s interests when dividing. The use of different criteria to make divisions in numerical terms or as an action of sharing, distributing, or giving is never approached in mathematics class. In the classroom division are made into equal parts is taken for granted in all situations. The use of algorithms to make calculation is taught as the only aspect of problem solving that is important in finding the right solution. The relative importance of the algorithm is not discussed, nor its intelligent application to help with the solution of the problem. Once more, I should say that the school and mathematics class, as they exist today, do not offer the opportunity to discuss intelligently this kind of issue and to enrich the students’ experience.

The third idea of division I identified is division in a physical sense. This means division that implies splitting into two or more pieces, or groups. The physical aspect of pulling apart, or separating is prevalent. In this case it is meaningless to talk about fairness as in sharing, or in equal division as in numerical division. The point here is, the results of this kind of division are not necessarily equal parts or groups with the same amount. This is how this idea of division differs from the division in the classroom. In splitting, or dividing in a physical sense, it does not matter if an equal or fair division results. Students did mention this kind of division on different occasions when they described people in the church on both sides of the saint’s image, or different directions of the reflection of light (picture #4). In these cases, the results were not the same

amount on each side, nor was fairness the issue. The same comments came up when they identified the division of the glass door as separating the people in and outside of the building (picture #9), and the division of men working from men looking (picture #2). Once more, students were not referring to equal numbers as a result of the division. They were pointing out differences they could see in the pictures, when taken from two different sides. Another example of this kind of division was when students divided numbers as they were dealing with two pieces of one set. This was the case when students were trying to solve the problems about *figurinhas* (problem #5), or about buying candy with the change from 50 *cruzeiros* (problem #2). In those cases, students defined two sides or groups and tried to work out both to get the whole set. When students approached these problems in this perspective, they dealt with two groups, not necessarily equals, to make the whole set again.

In concluding, I would like to bring into this discussion some connections I found in reflecting upon these ideas of division, and the frame I constructed earlier of some of the students' views of their cultural and social context.

Reflections on the Students' Ideas of Division

The first connection I perceive here is between the idea of division as separation of parts, and the strong distinction students make between their own social condition and the condition of those who have the power to decide and do things to improve their lives. The polarization of rich and poor, those who work and those who pay, those who ask for and those who give, might generate in some way a distinction between "us" and "them," "ours" and "theirs," "we" and "they," and so on. The inequality the students perceive in their social context splits or sets apart two sides in the society where they live. In this way of understanding division, the requirement of equal parts has no meaning. In fact, it goes against their own perception of their reality, where equal division is not the rule.

In addition, fairness and equality are not requirements for division of this kind. This might have to do with the students' need to face unequal conditions of living that they are in, considering others they know and see. This condition of inequality opens the door to the discussion of equal and unequal division. Looking at things this way, I began to understand why students talked about being fair when dividing instead of being equal. Fairness is a requirement of their way of dividing, while equality is part of a fictional world which they do not have access to fully understand. This fictional world could be compared to life within the school. It is real because it is part of their lives, but it is also fiction as long as students cannot understand it.

The immediate implication of this is having students dealing with two different meanings of division at the same time, and having to make sense out of it. Students might think about division in school and in reality. In their social context, there is fairness and unfairness, and both are embedded in inequality. In the school context, the idea that is conveyed is of equality and equal, and both are embedded in inequality. These contradictions, between equal and equality embedded in inequality, might reinforce in students the approach to division they have in the school context. This explains why students do so differently when dividing in and out of school contexts. And, this could also be one possible explanation for the distance from which they approach mathematics themselves. The filtration process the school uses to approach mathematical knowledge may suggest to students a fictional world of mathematics where, again, they do not have access and they do not need to fully understand, but they do have to know and follow the rules of functioning.

The apparent passive acceptance of rules of functioning and conditions of inequality in the context of this study might be explained, at least partially, by the students' perspectives of power relationships. Students manifested their vision of power as being embodied in the figure of someone who can decide, do and change things.

However, they see themselves as, powerless. Perhaps this condition of the impossibility of doing things might explain why the students accept and refer to different ways of dividing: 1) the division they do, 2) the division others do, and 3) the division that is done within the school.

The second connection I see is the idea of division as an act of sharing, distributing and giving with the students' feelings of being part of a group. It seems to me that they saw themselves as participants identified with one side of a society that they perceived as divided. Their side is the one where people share and support one another, dividing things they have among each other when it is required or necessary. The action of dividing as sharing, distributing or giving away could be related to a feeling of common identity developed among those who are powerless. The mutual action of dividing could be understood as an act of self-protection and survival in a hostile environment of inequalities.

Finally, I see a third connection in the numerical idea of division and its identification by the children with the other side of a society perceived as divided. The connection is drawn from the parallel between two sides: 1) the artificiality and abstractness of school mathematics and 2) the distance of the sphere of decision, power, and control from their reality. Due to its artificiality and abstractness, the numerical idea of division relates to these children's inability to solve everyday problems at the level they could if they had the power that knowledge confers on them. In doing so, school mathematics does not serve students in their need to understand, explain and hopefully transform a reality they perceive as unequal. This sphere is outside of students' understanding, and thus their ability to explain. The teaching of mathematics based on these ideas reinforces and justifies their position and condition of inequality. Students take this inequality for granted within a fairy tale world and fantasy. Their apparent conformity can be explained once more as being a mechanism of survival and

self-defense. The equal division in numerical terms could be related to equal division where things are not real.

To conclude, I must say that these reflections of mine on the construction of the ideas of division are not final. My intent was to shed some light on the learning of the concept of division. I think that cultural and social connections could help teachers understand and deal with learning situations in mathematics classes. I also must say that I believe mathematics should be a channel to improve students' comprehension of life, and "an instrument in their action as subject," Freire's (1970).

Implications for Practice and Recommendations for Further Research

Findings in this research suggest that school mathematics as we teach it now in Brazilian schools should be reviewed in three main ways: 1) curriculum organization, 2) content knowledge we choose to teach, and 3) the way we teach it.

In the school where this research was conducted is similar to a great number of Brazilian schools. It takes the format of a set of disciplines, which in their turn, form blocks of content knowledge similar to bricks that can be used by children to organize the experience they might have at school. Therefore, teachers at schools seem to assume that children will build up their own knowledge as if they would be building up a wall of bricks.own knowledge as if they would be building up a wall of bricks.

The research done has shown that children use a variety of different strategies, always based upon their day by day experiential world, when doing their mathematical reasoning. This line of research has a main implication for those in charge of reviewing curriculum organization, especially in mathematics education.

My research suggests that mathematics as a science does not result from physical impressions of quantitative relations existing in the world within a child's mind. Also, it should not be taken as a one single person's lonely construction. Rather, knowledge gains its meaning from shared social interpretation which is its use (Lerman,1989). In this sense time and space, as well as social content and culture, become particularly

important in dealing with curricular organization. Being more specific, formal mathematics cannot lose sight of existing relationship between mathematical knowledge and children's comprehension of their existential space. The world at school, that is expressed, among other things, through the knowledge it imparts, should not be opposed to the world and experiential knowledge children have within their familiar, social and cultural life. My research has shown deep ruptures--between the world of school and the world of life--that review of the curriculum, such as I have suggested, would prevent from widening. Much of the needed review, of curriculum organization can be summarized as adopting a new vision of the role of schooling in the life of individuals and the society.

In addition to the review of curriculum organization, content to be taught also needs review. One should consider first that adopting a broader vision about the role of schooling, as indicated before, also requires a different perspective on the choice of what should be taught in mathematics and on the way it should be taught.

A commonly used method to teach mathematics in Brazilian schools today is to emphasize the use of memory, calculations and repetitions. This way of teaching is clearly connected to a vision of schooling where mathematics teaching is taken as transmitting mechanical processes of calculations, involving abstracts formulas that once put together are assumed to be the content knowledge of mathematics. Findings in my research suggest that there is a need to move toward a new direction, in school's mathematics where students actively participate in the construction of their understanding of mathematical knowledge. In addition, my findings also indicate that students' constructions of their understanding frequently takes place through interpersonal negotiation of meanings during their day-by-day experiences. An implication of these findings is the need for schools to find ways to integrate into the students' school life their sense making, achieved during those interpersonal negotiations happening in their daily life.

Accepting that as true has another implication for the way mathematics should be taught. A class of mathematics where construction of meanings by interpersonal negotiation is valued should be rather different from the current ones. The mathematics class should be transformed into a locus for collaborative and integrative learning and teaching. It should take into account the social and cultural context in and out of school as the ground for teaching and learning. Based on that, the students' activities of intellectual inquiry should be emphasized. They should incorporate discussions and collective dialogues exploring mathematical ideas as routine procedure in the classroom.

Another important aspect in mathematics classes that should be reviewed is how teachers teach computation. My research suggests that children would do it better, should the mathematical algorithm used to make the computations be the result of the student's effort to make explicit his own way of reasoning. The implication of these findings is that teachers should take advantage of the different strategies students use to make calculations and help them to build a bridge toward the formal algorithm. In addition, teachers could explore these different strategies to enlarge the student's understanding of mathematical principles and properties related to those strategies. For example, when students use successive additions to make divisions, teachers should relate these procedures with the associative property of addition. In another situation, students round numbers to facilitate calculations. It could be related with rounding numbers in solving problems where estimation of possible results are used when dealing with more complex situation problems. There are a variety of mathematical properties that teachers find it difficult to teach to children due to its abstractness. However, teaching procedures would be more effective if the strategies students use out of school could be used. They could be taken as linking points between the student's understanding of mathematical situations they deal with in their day-by-day life and meaningful mathematical properties they could systematize in their classes of mathematics. Summing up, student's mathematical reasoning in their daily experiences

out of school should be brought into the classroom situation, and be used as a reach source and resource to enrich the teacher's formal mathematics teaching strategies.

Finally, as implied this study, the following questions surfaced that should be more deeply explored in further research because such research would contribute to new developments in mathematics education.

- 1) How can students bring into the classroom situation the same autonomy they have in dealing with mathematics out of school?
- 2) How can students deal with mathematical situations out of school after completing their schooling process?
- 3) What can mathematics educators do to make school mathematics more effective in contributing to students' computation competency out of school?
- 4) Will the use of more complex strategies to solve mathematical problems at an early age improve the child's mind?
- 5) How can school mathematics empower the student to understand their social and cultural milieu?

APPENDICES

APPENDIX A

List of the scenes in the pictures used in the interviews:

- 1- A car passing a sign indicating 60km/h speed limit.
- 2- Several men from a paving company repairing potholes in the street.
- 3- Several musical instruments and pictures of two men talking.
- 4- A saint's image being carried by four men, and a crowd gathering in the church surrounding the image.
- 5- A parking lot behind a building under construction where several men are working.
- 6- An ex-president, running for senator of one of the states located in the north region of Brazil, leaving his campaign committee.
- 7- A water reservoir with three pipes in it.
- 8- A building with the world *greve* (strike) painted on the glass door.
- 9- A candidate running for Governor of the State of Minas Gerais being interviewed.
- 10- Brazil's current President in a meeting at the presidential palace.

APPENDIX B

List of the problems presented in the interviews:

- 1- If you have 50 *cruzeiros*, how many loaves of bread could you buy if each loaf costs 4.50?
- 2- If you spend 45 *cruzeiros* to buy milk and pay with a 50 *cruzeiro* bill, how many candies could you buy with the change?
- 3- Seven kids want to play *queimada*. How could you divide the two teams?
- 4- If you have two oranges to be divided with three friends, how much orange could each one get?
- 5- To organize a collection of 150 *figurinhas* in two albums, how many *figurinhas* should I put in each album?
- 6- There are 45 loaves of bread to be divided between two classes of first grade students. How can the canteen woman do this division?
- 7- If you use six sheets of paper to cover four shelves. how many sheets of paper will be necessary to cover just one shelf?
- 8- Ms. Luiza has 30 cups of milk to be divided among 30 students in the third grade, but only 20 students came to school today. How can Ms. Luiza divide the cups of milk equally among the students? How much will each student get?

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