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ANTICIPATIVE STOCHASTIC CALCULUS WITH RESPECT TO GAUSSIAN PROCESSES, STOCHASTIC KINAMATICS IN HILBERT SPACE AND TIME REVERSAL PROBLEM

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ABSTRACT

ANTICIPATIVE STOCHASTIC CALCULUS WITH RESPECT TO GAUSSIAN PROCESSES, STOCHASTIC KINEMATICS IN HILBERT SPACE AND TIME REVERSAL PROBLEM

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Let $\{X_t, t \in T\}$ be a Gaussian process with reproducing kernel Hilbert space (RKHS) $K(C)$ of its covariance C. We first define the Ogawa integral $\delta(u)$ of a function $u \in L_2((\Omega, \mathcal{F}, P); K(C))$ with respect to X and prove the relation

$$I^s(u) = \delta(u) - traceD^M u,$$

where I^s is the Skorohod integral with respect to X, defined by Mandrekar and Zhang and D^M is the Malliavin derivative. For the reader's convenience we recall the definition and important properties of Skorohod integral. We define, in a very general setup, the Itô-Ramer integral L, generalizing earlier work of Ramer and Kusuoka. The integral L can be considered with respect to a Gaussian process, under the assumption that the measure $P \circ X^{-1}$ on $\mathbf{R}^T$ is Radon. We obtain that

$$L(u) = \delta(u) - traceD^F u,$$

where now, D^F denotes the H-Fréchet derivative. This is done in Chapter 1, by using our generalization of a result of Gross. Our work has been used to obtain an extension of Girsanov's theorem to the general case of Gaussian processes by Gawarecki and Mandrekar.

In Chapter 2, we consider E. Nelson's construction of diffusion in infinite dimensional case. Nelson's work on finite dimensional diffusions has proved important in the study of stochastic kinematics and quantum theory models. Our generalization involves the choice of stochastic driving term and rigorous definition of stochastic integral with respect to it. We explain why the stochastic driving term for the motion is a cylindrical Brownian motion and we introduce an extension of the stochastic integral of Metivier and Pellaumail to cover the studied case. As a consequence we derive results parallel to those of Nelson.

In order to apply the above results to physical problems one needs to study time reversibility of stochastic processes. A preliminary research in this direction is presented in Chapter 3. We investigate transformations of the Skorohod integral under maps $R : T \to T$. With mild assumptions on the transformation R we obtain that

$$I^s_{X_{R(t)}}(u_{R(t)}) = I^s_{X_t}(u_t) \text{ and } (D^M)_s^{X_{R(t)}} u_{R(t)} = (D^M)^{X_t}_{R(s)} u_{R(t)},$$

where the first integral is with respect to the transformed Gaussian process and the second is with respect to the original process X and the same refers to the Malliavin derivative. We also investigate connections with the time reversal problem and Skorohod-type SDE's.

To Edyta

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Contents

Introduction

Anticipative Stochastic Differential Equations (SDE's) arise in some practical problems. In the Filtering Theory, a symmetric treatment of the problem with respect to the direction of the time flow was successfully applied by Pardoux [49]. This technique is known as the Time Reversal of diffusion processes and it was itself of interest of several authors: Föllmer [17], Haussmann and Pardoux [23], who gave conditions under which the time reversed process is again a diffusion and described its infinitesimal operator. Recently, by application of Skorohod stochastic integration and Malliavin calculus, the results were improved by Millet, Nualart and Sanz [34].

Studies of Boundary Value Problem for SDE's lead to anticipative solutions if the initial condition is a future dependent random variable (see Buckdahn and Nualart [8], Nualart and Pardoux [40]). Another type of anticipative SDE's arises if the coefficients of the equation are allowed to be anticipative, which was studied by Buckdahn [5]-[7].

Analysis of anticipative SDE's requires extension of Girsanov Theorem. There are two approaches to the problem. One, due to Ramer and Kusuoka, uses either the Ramer integral (see Ramer [51], Kusuoka [29]) or the Skorohod integral (see Nualart and Zakai [41], Buckdahn [5]-[7]). This raises the question about the relationship between the Itô-Ramer and Skorohod integrals, which we study here.

Our analysis involves the Ogawa integral (see Ogawa [43]-[45]) in a natural way. The other method is due to Bell [2]-[3] and Ustunel and Zakai [56]. It employs Malliavin calculus and the concept of Integration by Parts Operator (IPO) with respect to Gaussian measure, as a generalization of divergence operator. The statement of the theorem of interest in [3] is inaccurate and we give precise generalization of the divergence theorem of Goodman [20] for our setup. To carry out this program one needs to generalize a fundamental result of Gross [22]. Then we define the Itô-Ramer integral and we extend some work of Kusuoka [29] to the case of Gaussian processes. The first Chapter is devoted to the above problem. This work has been used to obtain an extension of Girsanov's theorem to the general case of Gaussian processes by Gawarecki and Mandrekar [18].

In the second Chapter we discuss the role of cylindrical Brownian motion in Nelson's Kinematic theory of stochastic motion (see [36]) in Hilbert space. First we explain why a Hilbert space valued Brownian motion can not be recovered by Nelson's technique ([36]) as a stochastic driving term from a diffusion satisfying Nelson's regularity conditions. Therefore we consider cylindrical stochastic processes. To construct a diffusion from Nelson's assumptions, which is driven by a cylindrical Brownian motion, one needs to introduce a class of integrable functions with respect to 2-cylindrical martingales, which is larger than that used by Metivier and Pellaumail. This requires modification of the work on stochastic integral of Metivier and Pellaumail [35].

Applications of the results of Chapter 2 to physics require a study of Time Reversal problem. We begin this in Chapter 3 for Skorohod-type SDE's. This handles the case of reversal in time and space and, in particular, relates backward and forward Brownian motion. The harder problem of determining whether time reversal of a (non-anticipating) diffusion is again a diffusion (see Fölmer [17], Haussmann

and Pardoux [23]) is presently under study but the results, being incomplete, are not presented here. However we show that the *Go and Return* problem of Ogawa [46] can be handled by our techniques. We also show some applications of our results on transformations of Gaussian processes to line integrals of Cairoli and Walsh [9] (see Example 3.2.1).

The Appendix contains a review of some notions used in this work. It is included to provide the reader with an easily available reference.

Chapter 1

Anticipative Integrals with respect to Gaussian Processes

1.1 Introduction

There are several goals in the development of the theory of stochastic integration. Two of them, very natural, are enlargement of the class of integrands and enlargement of the class of integrators. We are specifically interested in a generalization leading to anticipating integrands and Gaussian integrators.

Extension of the Itô integral to not necessarily non-anticipating integrands was first done by Itô [25] with the help of stochastic integration with respect to quasi-martingales. Generalization of the class of integrands to Gaussian processes was attempted for example by Cramér [13] and Cambanis and Huang [10]. Much of their approach was defining the integral via step functions.

We will however concentrate on different techniques. Ramer [51] introduced a stochastic integral on an Abstract Wiener Space using functional analysis approach. He recognized this integral as an abstract version of double centered stochastic integral of Itô, introduced by Shepp [54]. Ramer's integral, further referred to as the Itô-Ramer integral, proved to be much more general object which will be discussed later. A completely different technique, based on Wiener Chaos Decomposition, was used by Skorohod [55] to yield an integral with respect to a white noise random measure. This idea was further developed by Mandrekar and Zhang [33] who obtained an integral of not necessarily non-anticipating integrands with respect to any Gaussian process.

Another interesting attempt was made by Kuo and Russek [28], Ogawa [43], [44], [45] and Rosinski [52], who developed a stochastic integral, also without any special kind of measurability assumptions, with respect to a white noise random measure on an arbitrary set. This integral was defined in terms of random series of usual Wiener integrals. We further refer to this integral as the Ogawa integral.

In the next sections we present the above ideas with more details. We study the relationship between the Itô-Ramer and Skorohod integrals which unify the results on Girsanov-type theorems obtained by Ramer [51], Kusuoka [29], Nualart and Zakai [41] and Bell [2]-[3]. In particular we generalize the Itô-Ramer and Ogawa integrals (the latter appears in the course of the analysis) to the case of an arbitrary Gaussian integrand and, by extending a result of Gross [22], we carry out the work of Kusuoka [29] in our setup.

1.2 Preliminaries

We begin with some selected basic concepts to make this work more self-contained. The material concerning covariances, Reproducing Kernel Hilbert Spaces and Gaussian processes was taken from the book of Billingsley [4] and papers of Chatterji and Mandrekar [12],[30]. For further details we refer to the work of Aronszajn [1], Gross [21] and Kuo [27].

The ideas introduced in this section lead to a useful concept of the stochastic integral with respect to Gaussian processes defined in [21] and [31]. This stochastic integral was used in [31] and [33] to develop the theory of stochastic integration with respect to Gaussian processes for integrands not requiring any special measurability assumptions.

Let T be any set and let C be a real function on $T \times T$. C is called a **covariance** on T if $C(s,t) = C(t,s)$ and $\sum_{t,s\in i} a_t a_s C(t,s) \geq 0$ for all finite subsets $i \subset T$ and $\{a_s,\ s \in i\} \subset \mathbf{R}$. For a covariance C on T, there exists a unique Hilbert space H of real valued functions on T, called the **Reproducing Kernel Hilbert Space** (RKHS for short) of the covariance C, satisfying $\forall t \in T\ :\ C(\cdot,t) \in H$ and $\forall t \in T,\ f \in H\ :\ (f(\cdot), C(\cdot,t))_H = f(t)$. Here, for all $t \in T$, $C(\cdot,t)$ denotes the function of the first variable.

Notation. We denote a scalar product in a Hilbert space by $(\cdot,\cdot)$ with possible subscript if identification of the Hilbert space is ambiguous. For a Locally Convex Topological Vector Space (LCTVS for short), by $\langle\cdot,\cdot\rangle$ we denote duality between the space and its adjoint. Should any ambiguity arise, subscripts identifying the space are added.

With a covariance C on T we associate a centered Gaussian process $X = \{X_t,\ t \in T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, such that $E(X_s X_t) = C(s,t)$, where we will always take $\mathcal{F}$ to be the σ-field generated by the family $\{X_t,\ t \in T\}$. Without loss of generality we assume that all probability spaces considered here are complete. Denote by $H(X)$ the closed linear span of $\{X_t,\ t \in T\}$ in $L_2(\Omega, \mathcal{F}, P)$. Note that if $Y_1, Y_2, ..., Y_n \in H(X)$ then $(Y_1, Y_2, ..., Y_n)$ is a multivariate normal variable. Then the RKHS H of the process X is of the following form:

$$H = \{f : \ f(t) = E(X_t Y_f), \text{ for a unique } Y_f \in H(X)\}$$

Let $\pi : \ H \to H(X)$ be a map defined by: $\pi(f) = Y_f$. Then π is an isometry. In particular $\pi(C(\cdot, t)) = X_t$.

Definition 1.2.1 *(1) The isometry $\pi : \ H \to H(X)$ is called a* **stochastic integral** *with respect to Gaussian process X.*
(2) If K is a Hilbert space isometric to the RKHS H of the Gaussian process X under an isometry V then we define a **stochastic integral** *S of any $k \in K$ with respect to X by $S(k) = \pi(V(k))$.*

Several interesting examples of Gaussian processes, their RKHS's and stochastic integrals with respect to these processes can be found in [30] and [12]. We present here only those examples which we discuss later.

Example 1.2.1 *Gaussian processes.*

(a) Brownian motion. Let $T = \{(t_1, ..., t_n) = t \in \mathbf{R}^n, \ t_i \geq 0\}$ and define $C(t, t') = \prod_{i=1}^{n}(t_i \wedge t_i')$. Then the function C is a covariance on T . For $n = 1$, the associated Gaussian process is called the **Wiener-Lévy Brownian motion** or **Brownian motion** for short, and for $n > 1$ the associated Gaussian process is called the **Cameron-Yeh process.**

The RKHS H of the covariance C is given by

$$H = \{f : \ f(t) = \int_0^{t_n} ... \int_0^{t_1} g(u_1, ..., u_n) du_1 ... du_n, \ g \in L_2(\mathbf{R}^n)\}$$

with the scalar product

$$(f_1, f_2) = \int_0^{\infty} ... \int_0^{\infty} g_1(u) g_2(u) du$$

where du is Lebesgue measure on $\mathbf{R}^n$.

Denote the Brownian motion process by B and consider stochastic integral with respect to this process. For $h \in H$, $\pi(h) = \int_0^1 h'dB$, h' denoting the derivative of h and the last integral is the Wiener integral. Indeed, π is an isometry between H and $H(B)$ because the Wiener integral is an isometry between $L_2([0,1])$ and $H(B)$ ([27]). Also $\pi(C(\cdot,t)) = B_t = \int_0^1 1_{[0,t]}(s)dB_s$, $1_A(\cdot)$ being the characteristic function of set A. The RKHS H is isometric to the Hilbert space $L_2([0,1])$, with the Borel σ-field and Lebesgue measure, by an isometry $V(f)(t) = \int_0^t f(s)ds$, for $f \in L_2([0,1])$, $t \in [0,1]$. The stochastic integral S for functions from $L_2([0,1])$ is then just the Wiener integral, $S(f) = \pi(V(f)) = \int_0^1 fdB$.

(b) Gaussian white noise measure. Let (S, Σ, μ) be a σ-finite measurable space, $T = \{A \in \Sigma : \ \mu(A) < \infty\}$. For $A, A' \in \Sigma$ define $C(A, A') = \mu(A \cap A')$. Then the function C is a covariance on T.The associated Gaussian process is called **Gaussian white noise measure**. The RKHS of covariance C is given by

$$H = \{f : \ f(A) = \int_A f(u)\mu(du), \ f \in L_2(S, \Sigma, \mu)\}.$$

Now let us consider stochastic integral with respect to Gaussian white noise measure. The map $V : L_2(S, \Sigma, \mu) \to H$ given by $V(1_A)(\cdot) = \mu(\cdot \cap A) = C(\cdot, A)$ is an isometry. Then the stochastic integral $S : L_2(S, \Sigma, \mu) \to H(X)$ is defined by $S(f) = \pi(V(f))$. In case of $S = [0,1], \Sigma$ - the Borel σ-field and μ - Lebesgue measure, the stochastic integral S is the Wiener integral.

(c) Generalized Gaussian process. Let $T = C_0^\infty(G)$, the space of smooth (i.e. infinitely differentiable) functions with compact support in a bounded domain G with a smooth boundary in $\mathbf{R}^n$. For $\phi \in C_0^\infty(G)$, we denote $(D^\alpha \phi)(x) = \partial^{|\alpha|}\phi(x)/\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$, where $|\alpha| = \sum_{j=1}^n \alpha_j$, α_j are non-negative integers. If C is a covariance on T, then the associated Gaussian process is called a **generalized**

Gaussian process.

(c1) In the case of

$$C(\phi_1, \phi_2) = \int_G \phi_1(u)\phi_2(u)\mu(du),$$

where μ is Lebesgue measure on $\mathbf{R}^n$, the associated process is called **Gaussian white noise**. The RKHS of C is $L_2(G, \mu(du))$.

(c2) For

$$C(\phi_1, \phi_2) = \sum_{|\alpha| \leq m} \int_G (D^\alpha \phi_1(u))(D^\alpha \phi_2(u))\mu(du),$$

the associated Gaussian process is called **Gaussian white noise of order m**. The RKHS of C is the Sobolev space $H_0^m(G)$.

1.3 Skorohod Integral and Stochastic Differentiation

The Skorohod integral and stochastic differentiation as presented here was introduced by Mandrekar and Zhang and most of this section recalls results of [33]. For the original work of Skorohod we refer to [55].

1.3.1 Multiple Wiener Integrals

For the detailed construction of Multiple Wiener Integrals with respect to Gaussian processes we refer to [33]. For the original construction of Itô see [24].

Let C be a covariance defined on an arbitrary set T with the RKHS H and let p be a non-negative integer. Tensor product $H^{\otimes p}$ of p copies of RKHS's H consists

of all functions of p variables of the following form:

$$f(t_1, t_2, ..., t_p) = \sum_{\alpha_1, \alpha_2, ..., \alpha_p} a_{\alpha_1, \alpha_2, ..., \alpha_p} e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) ... e_{\alpha_p}(t_p)$$

with $\sum_{\alpha_1, \alpha_2, ..., \alpha_p} a^2_{\alpha_1, \alpha_2, ..., \alpha_p} < \infty$. Here $\{e_\alpha, \ \alpha = 1, 2, ...\}$ is an ONB of H and the summation is over all tuples $(\alpha_1, ..., \alpha_p)$. Furthermore, the scalar product of two functions $f, g \in H^{\otimes p}$ is defined as

$$(f, g)_{H^{\otimes p}} = \sum_{\alpha_1, \alpha_2, ..., \alpha_p} a_{\alpha_1, \alpha_2, ..., \alpha_p} b_{\alpha_1, \alpha_2, ..., \alpha_p},$$

if

$$\begin{aligned} f(t_1, t_2, ..., t_p) &= \sum_{\alpha_1, \alpha_2, ..., \alpha_p} a_{\alpha_1, \alpha_2, ..., \alpha_p} e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) ... e_{\alpha_p}(t_p), \\ g(t_1, t_2, ..., t_p) &= \sum_{\alpha_1, \alpha_2, ..., \alpha_p} b_{\alpha_1, \alpha_2, ..., \alpha_p} e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) ... e_{\alpha_p}(t_p). \end{aligned}$$

Notation. For $f \in H^{\otimes p}$ we denote by $\tilde{f}$ its symmetrization, which is defined as $\tilde{f} = \frac{1}{p!} \sum_s f(t_{s(1)}, t_{s(2)}, ..., t_{s(p)})$, where the sum is over all permutations s of the set $\{1, ..., p\}$. We denote by $H^{\odot p}$ the pth symmetrized tensor product of H, which is a Hilbert subspace of $H^{\otimes p}$ consisting of all symmetric functions in $t_1, ..., t_n$. Also, $H^{\otimes 0} = H^{\odot 0} = \mathbf{R}$ (real numbers). Note that if $f \in H^{\otimes p}$ then $\tilde{f} \in H^{\odot p}$.

For any $p = 0, 1, ...,$ Multiple Wiener integral, I_p is a linear map from $H^{\otimes p}$ to $L_2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the underlying probability space on which the Gaussian process X is defined. The integral is determined by the following properties:

(1) $I_0(f) = f$ for $f \in H^{\otimes 0} = \mathbf{R}$.

(2) $I_1(f) = \pi(f)$ for $f \in H^{\otimes 1} = H$.

(3) $I_{p+1}(fg) = I_p(f) I_1(g) - \sum_{k=1}^{p} I_{p-1}(f \underset{k}{\otimes} g)$ for $f \in H^{\otimes p}$, $g \in H^{\otimes 1}$ and $f \underset{k}{\otimes} g = (f(t_1, ..., t_k, ..., t_p), g(t_k))_H$.

(4) $\|I_p(f)\|^2_{L_2(\Omega)} = p! \|\tilde{f}\|^2_{H^{\otimes p}}$ for $f_p \in H^{\otimes p}$.

Below we list some other useful properties of Multiple Wiener Integral. For $f, g \in H^{\otimes p}$ and $h \in H^{\otimes 1}$, we have,

(5) $I_p(f) = I_p(\tilde{f})$.

(6) $E\{I_p(f)\} = 0$ and $E\{I_p(f)I_p(g)\} = p!(\tilde{f}, \tilde{g})_{H^{\otimes p}}$.

(7) $I_p(H^{\otimes p}) \perp_{L_2(\Omega)} I_k(H^{\otimes k})$ for all k < p.

(8) $I_{p+1}(fh) = I_p(f)I_1(h) - \sum_{k=1}^{p} I_{p-1}(f \underset{k}{\otimes} h)$ for $f \in H^{\otimes p}$, $h \in H^{\otimes 1}$ and $f \underset{k}{\otimes} h = (f(t_1, ..., t_k, ..., t_p), h(t_k))_H$.

Let $f \in H^{\otimes p}$, $f(t_1, t_2, ..., t_p) = e_{\alpha_1}(t_1)e_{\alpha_2}(t_2)...e_{\alpha_p}(t_p)$ where among $\alpha_1, ..., \alpha_p$ only n are different with repeats $p_1, p_2, ..., p_n$, $p_1 + p_2 + ... + p_n = p$. Denote the corresponding n different e_{α_i}'s as $u_1, u_2, ..., u_n$ and assume that they are orthonormal elements of H. Then $I_p(f) = \prod_{i=1}^{n} \mathcal{H}_{p_i}(I_1(u_i))$ where $\mathcal{H}_{p_i}$'s are Hermite polynomials normed in the following way: $\mathcal{H}_p(t) = \frac{1}{\sqrt{2^p}} H_p(\frac{t}{\sqrt{2}})$ with $H_p(t) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2}$.

A version of Wiener Chaos Decomposition is given in the next Lemma.

Lemma 1.3.1 $L_2(\Omega, \mathcal{F}, P) = \bigoplus_{p=0}^{\infty} I_p(H^{\odot p})$.

1.3.2 Malliavin Derivative

The Malliavin derivative defined in this section plays an analogous role in Skorohod stochastic calculus with respect to Gaussian processes as the Malliavin derivative in case of Skorohod calculus with respect to Brownian motion.

Let $u = \{u_t, \ t \in T\}$ be a measurable stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $u_\cdot(\omega)$ can be considered as an H-valued random variable in Bochner sense. We assume that $u_\cdot(\omega) \in L_2(\Omega, H)$, in particular, $E\|u\|_H^2 < \infty$. This condition implies that $u_t \in L_2(\Omega)$ for each $t \in T$ in view of the following inequality:

$$u_t^2(\omega) = (u_\cdot(\omega), C(\cdot, t))_H^2 \leq \|u_\cdot(\omega)\|_H^2 C(t,t).$$

By Lemma 1.3.1, for each $t \in T$, there exist unique $f_p^t(\cdot) \in H^{\odot p}$, $p = 0, 1, ...,$ $f_p^t(t_1, ..., t_p) = f_p(t_1, ..., t_p, t) \in H^{\otimes(p+1)}$, such that

$$u_t(\omega) = \sum_{p=0}^{\infty} I_p(f_p^t) = \sum_{p=0}^{\infty} I_p(f_p(t_1, ..., t_p, t)). \tag{1.1}$$

For computation of the $L_2(\Omega, H)$ norm of a stochastic process u it is very useful that for $u_t = I_p(f(\cdot, t))$ and $v_t = I_q(g(\cdot, t))$, where $f(\cdot, \cdot) \in H^{\otimes(p+1)}$, $g(\cdot, \cdot) \in H^{\otimes(q+1)}$ and for each fixed $t \in T$, $f(\cdot, t) \in H^{\odot p}$ and $g(\cdot, t) \in H^{\odot q}$, we have,

$$E\{(u_\cdot, v_\cdot)_H\} = \begin{cases} p!(f, g)_{H^{\otimes(p+1)}} & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}.$$

Now we recall definition of Malliavin derivative.

Definition 1.3.1 *Let $u \in L_2(\Omega, H)$. By the* **Malliavin derivative** $D_h^M u_t$ *for fixed $h \in H$ we understand a random variable in $L_2(\Omega)$, defined as a limit of the following series:*

$$\sum_{p=1}^{\infty} p I_{p-1}((f_p(t_1, ..., t_{p-1}, s, t), h(s)))$$

where u_t has the unique representation (1.1). If for fixed $t \in T$, the Malliavin derivative $D_h^M u_t$ exists for all $h \in H$ and the series

$$\sum_{p=1}^{\infty} p I_{p-1}(f_p(t_1, ..., t_{p-1}, s, t))$$

defines a random variable in $L_2(\Omega, H)$, then we define the **Malliavin derivative** $D_s^M u_t \in H$*, as a function of argument s, in the following way: $D_s^M u_t = \sum_{p=1}^{\infty} p I_{p-1}(f_p(t_1, ..., t_{p-1}, s, t))$. In this case $(D_s^M u_t, h(s))_H = D_h^M u_t$.*

We give sufficient and necessary conditions for existence of Malliavin derivative as well as for some regularity of this derivative in the following Lemma.

Lemma 1.3.2 *Let $u \in L_2(\Omega, H)$ where u_t has the representation (1.1).*
(1) Let $t \in T$ be fixed. $D_s^M u_t \in L_2(\Omega, H)$ exists iff

$$\sum_{p=1}^{\infty} pp! \|f_p(\cdot, t)\|^2_{H^{\otimes p}} < \infty$$

and in this case

$$D_s^M u_t = \sum_{=1}^{\infty} p I_{p-1}(f_p(\cdot, s, t)) \text{ and } E\|D^M_{\cdot} u_t\|^2_H = \sum_{p=1}^{\infty} pp! \|f_p(\cdot, t)\|^2_{H^{\otimes p}} < \infty.$$

(2) The Malliavin derivative $D_s u_t \in L_2(\Omega, H^{\otimes 2})$, i.e. it is a Hilbert-Schmidt operator, iff

$$\sum_{p=1}^{\infty} pp! \|f_p\|^2_{H^{\otimes (p+1)}} < \infty.$$

Example 1.3.1 *Malliavin derivative for Brownian motion.*

In the case of standard Brownian motion, Multiple Wiener Integrals I_p and consequently, the Malliavin derivative defined above, coincide with Multiple Wiener Integrals I_p^i and the Malliavin derivative D^i defined in [41]. More precisely,

$$I_p^i(f_p) = I_p(V^{\otimes p} f) \text{ and } D_s F = V(D^i F)(s)$$

for any $f_p \in L_2([0,1]^p)$ and $F \in L_2(\Omega)$ with the first equality in $L_2(\Omega)$ and the second in $L_2(\Omega, H)$. Here $V : L_2([0,1]) \to H$ is defined by: $Vf = \int_0^{\cdot} f(s)ds$ (clearly $V^{\otimes p} f_p \in H^{\otimes p}$ and $VD^i F \in L_2(\Omega, H)$).

1.3.3 Skorohod Integral

We are ready to recall definition of the Skorohod integral, which is based on the Wiener Chaos Decomposition of Lemma 1.3.1.

Definition 1.3.2 *Let $u \in L_2(\Omega, H)$ has the decomposition (1.1). If*

$$\sum_{p=0}^{\infty} I_{p+1}(f_p) = \sum_{p=0}^{\infty} I_{p+1}(\tilde{f}_p) \text{ converges in } L_2(\Omega)$$

then the sum is called the **Skorohod integral** *of u and is denoted by $I^s(u)$.*

Note that for $u \in L_2(\Omega, H)$, we have, $I^s(u) \in L_2(\Omega)$ iff $\sum_{p=1}^{\infty}(p+1)!\|\tilde{f}_p\|^2_{H^{\otimes(p+1)}}$ is finite and in this case, the $L_2(\Omega)$ norm of the Skorohod integral $I^s(u)$ coincides with the above sum. Furthermore, the domain of the Skorohod integral I^s consists of all $u \in L_2(\Omega, H)$ for which the above sum is finite. As we can see the measurability condition for the integrand in the Itô integral is replaced in the Skorohod integral by a "growth" condition.

Example 1.3.2 *Skorohod integral with respect to Brownian motion.*

As a continuation of Example 1.3.1, we have

$$I^i(u) = I^s(Vu)$$

for $u \in L_2(\Omega, L_2([0,1]))$ (clearly $Vu \in L_2(\Omega, H)$). Here $I^i(u)$ is the Skorohod integral defined in [41]. In the case when u is adapted to the natural filtration of Brownian motion, $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$, then the Skorohod and Itô integrals coincide: $I^s(Vu) = I^i(u) = \int_0^1 u_t dB_t$. If u is adapted to the future filtration $\mathcal{F}^t = \sigma\{B_1 - B_s, t \leq s \leq 1\}$ then the Skorohod and backward Itô integrals coincide (see [41]).

1.4 Extension of Ogawa Integral and its Relationship to Skorohod Integral

In this section we introduce the Ogawa integral with respect to Gaussian process $X = \{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For original definition and for properties of the Ogawa integral we refer to [43]-[45],[28],[52],[41].

Definition 1.4.1 *Let $u = \{u_t, t \in T\}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$, such that u is an H-valued random variable. Let $\{e_n\}_{n=1}^{\infty} \subset H$ be an ONB in H. Assume that $\mu(\|u\|_H^2 < \infty) = 1$.*

(1) A process u is called **Ogawa integrable with respect to the process** X **and ONB** $\{e_n\}_{n=1}^{\infty} \subset H$ *if the following series converges in probability:*

$$\delta_e^o(u) = \sum_{n=1}^{\infty} (u, e_n)_H \pi(e_n).$$

In this case $\delta_e^o(u)$ is called the **Ogawa integral** *of the process u with respect to X and ONB $\{e_n\}_{n=1}^{\infty}$.*

(2) If the limit in (1) exists with respect to all ONB's of H and does not depend on the choice of basis, then process u is called **universally Ogawa integrable** *with respect to X and $\delta^o(u)$ denotes its Ogawa integral.*

To obtain the relationship between the Ogawa and Skorohod integrals one only needs the following technical Lemma, which is an analogue of Proposition 3.5 [41].

Lemma 1.4.1 *Let $F \in L_2(\Omega)$ be such that its Malliavin derivative $D^M F \in L_2(\Omega, H)$ and let $f \in H$. Then,*

$$I^s(Ff) = I^s(f)F - (D^M_{\cdot} F, f(\cdot)). \tag{1.2}$$

Proof. Let $F = I_m(f_m)$. Then,

$$\begin{aligned}
I^s(Ff) &= I_{m+1}(f_m f) \\
&= I_m(f_m)I_1(f) - mI_{m-1}((f_m(t_1, ..., t_{m-1}, \cdot), f(\cdot))) \\
&= FI^s(f) - D_f^M F.
\end{aligned}$$

If $F = \sum_{m=0}^{\infty} I_m(f_m)$, then $Ff \in \mathcal{D}(I^s)$ because

$$\sum_{m=0}^{\infty}(m+1)!\|\frac{1}{m+1}\{f_m(t_1,...,t_m)f(t)$$

$$+\sum_{i=1}^{m} f_m(t_1,...,t_{i-1},t,t_{i+1},...,t_m)f(t_i)\}\|^2_{H^{\otimes(m+1)}}$$

$$\leq \sum_{m=0}^{\infty}\frac{m!}{m+1}(m+1)^2\|f_m f\|^2_{H^{\otimes(m+1)}}$$

$$= \sum_{m=1}^{\infty} m!(m+1)\|f_m\|^2_{H^{\otimes m}}\|f\|^2_H < \infty,$$

since $E(\|D^M F\|^2_H) = \sum_{m=0}^{\infty} m!m\|f_m\|^2_{H^{\otimes m}} < \infty$. Because

$$E(|D^M_f F|^2) = E(|(D^M F, f(x))|^2) \leq E\|D^M F\|^2_H\|f\|^2_H < \infty,$$

we have

$$\sum_{m=1}^{N} mI_{m-1}((f_m(t_1,...,t_{m-1},\cdot), f(\cdot))) \to D^M_f F$$

in $L_2(\Omega)$ as $N \to \infty$ and therefore $\sum_{m=1}^{N} I_m(f_m)I_1(f)$ converges in $L_2(\Omega)$ to $I^s(Ff) - D^M_f F$ and (1.2) is valid for $F = \sum_{m=0}^{\infty} I_m(f_m)$, i.e. for any $F \in L_2(\Omega)$.

□

Proposition 1.4.1 *Let $u \in L_2(\Omega, H)$ and assume that the Malliavin derivative $D^M u(\omega)$, of u exists and for every $\omega \in \Omega$ it is a Hilbert-Schmidt operator on H, with $E\|D^M u\|^2_{H^{\otimes 2}} < \infty$. Assume furthermore that $D^M u(\omega)$ is even a trace class operator on H for every $\omega \in \Omega$. Then,*

$$u \in \mathcal{D}(I^s) \cap \mathcal{D}(\delta^o) \text{ and } \delta^o(u) = I^s(u) + trD^M u \quad \mu \text{ a.e.}$$

Proof. The statement, $u \in \mathcal{D}(I^s)$, follows from Theorem 3.1 in [33]. Let $P_N \in \mathcal{P}(H)$, $P_N = \sum_{k=1}^{N} h_k \otimes h_k$ where $\{h_k\}_{k=1}^{\infty}$ is an ONB in H. Compute the

expression for $I^s(P_N u)$. Begin with

$$\begin{aligned} I^s((u,h_k)h_k) &= (u,h_k)I^s(h_k) - (D^M((u,h_k)),h_k) \\ &= (u,h_k)I^s(h_k) - (D^M_{h_k}u,h_k) \\ &= (u,k_k)I^s(h_k) - (D^M u(h_k),h_k). \end{aligned}$$

The first equality is a consequence of Lemma 1.4.1. The second can be justified as follows. Let $u = \sum_{m=0}^{\infty} I_m(f_m)$ be Wiener chaos expansion of u with $f_m \in H^{\otimes(m+1)}$, symmetric with respect to the first m variables, as proved to be possible in Lemma 3.1 [33]. Since u is given by an $L_2(\Omega, H)$ convergent series $\sum_{m=0}^{\infty} I_m(f_m)$, we have,

$$\sum_{m=0}^{n} I_m((f_m,h_k)) = (\sum_{m=0}^{n} I_m(f_m),h_k) \to (u,h_k)$$

in $L_2(\Omega)$. This means that (u,h_k) has the following series representation:

$$(u,h_k) = \sum_{m=0}^{\infty} I_m((f_m,h_k))$$

with $(f_m,h_k) \in H^{\odot m}$. Consequently,

$$\begin{aligned} (D^M(u,h_k),h_k) &= (\sum_{m=1}^{\infty} mI_{m-1}((f_m,h_k)),h_k) \\ &= (D^M_{h_k}u,h_k) \\ &= (D^M u,h_k \otimes h_k) \end{aligned}$$

Finally, we get,

$$I^s(P_N u)(\omega) = \sum_{k=1}^{N}(u(\omega),h_k)I^s(h_k)(\omega) - \sum_{k=1}^{N}(D^M u(\omega)h_k,h_k) \tag{1.3}$$

i.e.

$$I^s(P_N u) = \delta^o(P_N u) - tr(P_N D^M u).$$

Next we want to show that $I^s(P_N u) \to I^s(u)$ in $L_2(\Omega)$. It is enough to prove that

$$\begin{aligned} E \parallel P_N u - u \parallel_H^2 &\to 0 \\ E \parallel D^M P_N u - D^M u \parallel_{H^{\otimes 2}}^2 &\to 0 \end{aligned}$$

as $N \to \infty$ (see Theorem 3.1 in [33]).
Because $P_N \to I_H$ strongly (I_H denotes the identity operator on H),
we have,

$$\parallel P_N u(\omega) - u(\omega) \parallel_H \to 0 \ \forall \omega \in \Omega.$$

Therefore $E \parallel P_N u - u \parallel_H^2 \to 0$ by the Lebesgue dominated convergence theorem.
Also,

$$D^M P_N u = D^M (\sum_{k=1}^N (u, h_k) h_k) = \sum_{k=1}^N D^M (u, h_k) h_k$$

implies

$$\begin{aligned} \parallel D^M P_N u - D^M u \parallel_{H^{\otimes 2}}^2 &= \sum_{j=1}^{\infty} \{ \sum_{k=1}^N (D^M (u, h_k), h_j)(h_k, h_j) - (D^M u(h_j), h_j) \}^2 \\ &= \sum_{j=N+1}^{\infty} (D^M u(h_j), h_j)^2 \to 0 \end{aligned}$$

as $N \to \infty$, because $\parallel D^M u \parallel_{H^{\otimes 2}} < \infty$.
Since

$$\parallel D^M P_N(u) \parallel_{H^{\otimes 2}}^2 = \sum_{j=1}^N (D^M u(h_j), h_j)^2 \leq \parallel D^M u \parallel_{H^{\otimes 2}}^2,$$

we obtain, again by the Lebesgue dominated convergence theorem, that

$$E \parallel D^M P_N u - D^M u \parallel_{H^{\otimes 2}}^2 \to 0.$$

This proves that $I^s(P_N u) \to I^s(u)$ in $L_2(\Omega)$. Since $tr P_N D^M u(\omega) \to tr D^M u(\omega)$ for every $\omega \in \Omega$ we have $\sum_{k=1}^N (u, h_k) \pi(h_k)(\omega)$ converges in probability, independently of the choice of the ONB in H.

$\square$

1.5 Extension of Itô-Ramer integral

In this section we extend the Itô-Ramer integral and give some properties of this extended integral, which are parallel to those stated in Ramer [51] and Kusuoka [29]. In these papers the Itô-Ramer integral was defined in two slightly different ways. The main objective of both authors was to give a solution of the problem of absolute continuity of non-linear transformations of a Gaussian measure on a Banach space, which was first considered by Cameron and Martin [11]. Our work on the Itô-Ramer integral is inspired by the ideas developed in [51] and [29].

Let $(\imath, H, B)$ be an abstract Wiener space ([21]) and μ be standard Wiener measure on B i.e. the measure induced by isonormal cylindrical measure on H by $\imath$. Ramer and Kusuoka considered a transformation $T = I + F$, where $F : B \to H$ was such that DF, the Gateaux derivative of F in the direction of H, existed and for each $x \in B$, $DF(x)$ was a Hilbert-Schmidt operator on H. Then under certain conditions on T and F, the authors showed that,

$$\frac{d(\mu \circ T)}{d\mu}(x) = d_c(I_H + DK(x)) \exp\{-\text{"} < Kx, x > -trDK(x)\text{"} - \frac{1}{2} |Kx|_H^2\}.$$

Here, "$< Kx, x > -$ tr $DK(x)$" was called the Itô-Ramer Integral. We extend this integral for a very general setup as follows.

1.5.1 Definition of Itô-Ramer Integral

Let $\{X_t, \ , t \in T\}$ be a Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$. We consider Kolmogorov functional representation of the process X, i.e. the probability space $\mathbf{R}^T$, with the σ-field $\mathcal{R}^T$ generated by cylinder sets, and probability measure μ, such that the finite dimensional distributions of the canonical process $x(t) \in \mathbf{R}^T$ coincide with the finite dimensional distributions of the process X. $\mathbf{R}^T$

becomes a LCTVS when equipped with the product (Tihonov) topology. We assume that the measure μ on $(\mathbf{R}^T, \mathcal{R}^T)$ is Radon and we denote its support by $\mathcal{X}$ (see Proposition 3.4 in [57] and for an example of non-existence of a support in case of a non-Radon measure see [14]). Then H, the RKHS of X is separable. The measure μ is Gaussian and $\mathcal{X} = \overline{H}^{\mathbf{R}^T} \subset \mathbf{R}^T$ (the closure of H in the topology of $\mathbf{R}^T$). The triple $(\imath, H, \mathcal{X})$ is not necessarily an Abstract Wiener Space (AWS, see Appendix) but the following relation holds:

$$\mathcal{X}^* \underset{\imath^*}{\hookrightarrow} H \underset{\imath}{\hookrightarrow} \mathcal{X}$$

where $\imath^*$ is the conjugate map to $\imath$. Both $\imath^*$ and $\imath$ are continuous, dense embeddings.

Example 1.5.1 *Stochastic integral of a linear functional.*

The stochastic integral, introduced in Definition 1.2.1, of $e \in \mathcal{X}^*$ is given by $\pi(e)(x) = e(x)$ a.e. $\mu(dx)$.

We consider a triple $(\imath, H, Z)$ where $(\imath, H, Z) = (\imath, H, B)$ is an AWS or $(\imath, H, Z) = (\imath, H, \mathcal{X})$ is the triple associated with some Gaussian process. Let E be a real Banach space and $L(H, E)$ denote the space of bounded linear operators from H to E.

Definition 1.5.1 *(1) A map $f : \mathbf{R} \to E$ is called* **absolutely continuous** *if for any $-\infty < a < b < \infty$ and $\varepsilon > 0$, there exists some $\delta(\varepsilon, a, b) > 0$ such that $\sum_{i=1}^n \|f(t_i) - f(s_i)\|_E < \varepsilon$ holds for any integer n and $a \le t_1 < s_1 \le t_2 < s_2 ... t_n < s_n \le b$, $\sum_{i=1}^n |t_i - s_i| < \delta(\varepsilon, a, b)$.*
(2) A map $f : \mathbf{R} \to E$ is called **strictly absolutely continuous** *if it is continuous, strongly differentiable almost everywhere and it satisfies that $\int_a^b \|(df/dt)(t)\|_E dt$*

is finite and $f(b)-f(a) = \int_a^b (df/dt)(t)dt$ *for any* $-\infty < a, b < \infty$, *where* $(df/dt)(t)$ *denotes strong derivative of* f *at* t.

We note that given a map $f : \mathbf{R} \to E$, f is absolutely continuous if it is strictly absolutely continuous. In the case of reflexive Banach space E absolute continuity of $f : \mathbf{R} \to E$ implies its strict absolute continuity.

Definition 1.5.2 *A strongly measurable map (in the sense of Bochner)* $F : Z \to E$ *is said to be* **Stochastic Gateaux H-Differentiable** *(SGD) if there exists a strongly measurable map* $D^G F : Z \to L(H, E)$ *such that*

$$\frac{1}{t}\langle \varphi, F(z+th) - F(z)\rangle \to \langle \varphi, D^G F(z)h\rangle$$

in probability μ *as* $t \to 0$ *for every* $\varphi \in E^*$ *and* $h \in H$. $D^G F$ *is called the* **Stochastic Gateaux H-Derivative** *of* F.

Definition 1.5.3 *A strongly measurable map* $F : Z \to E$ *is called* **Ray Absolutely Continuous** *(RAC) if for every* $h \in H$ *there exists a strongly measurable map* $F_h : Z \to E$ *such that* $\mu(F_h = F) = 1$ *and* $F_h(z+th)$ *is strictly absolutely continuous in* t *for each* $z \in Z$.

Definition 1.5.4 *A map* $F : Z \to E$ *belongs to class* $H^1(Z \to E; d\mu)$ *if* F *is SGD and RAC.*

Notation. For K, a linear subspace of H, we denote by $\mathcal{P}(K)$ the set of all finite dimensional projections of H with range in K.

Now we define the Itô-Ramer integral with respect to a Gaussian process X.

Definition 1.5.5 *A map $F : Z \to H$ is said to belong to $\mathcal{D}(L)$, the domain of the Itô-Ramer integral, if the following conditions are satisfied :*
(1) $F \in H^1(Z \to H; d\mu)$.
(2) $D^G F(z) \in H^{\otimes 2}$ μ a.e.
(3) there exists a measurable function $LF : Z \to \mathbf{R}$ such that

$$L_P F(z) := \langle PF(z), z\rangle - trPD^G F(z) \to LF(z)$$

in probability μ as $P \to Id_H$, $P \in \mathcal{P}(Z^*)$.

Remark 1.5.1 *(1) In the definition of the Itô-Ramer integral we consider only finite dimensional projections $P \in \mathcal{P}(Z^*)$. If the triple $(\imath, H, Z)$ is an AWS $(\imath, H, B)$ then the above definition coincides with the definition of Itô-Ramer integral given in [29] for projections in $\mathcal{P}(H)$ with ranges in B^*.*
(2) If assumption (1) in Definition 1.5.5 is replaced by the requirement that F be continuously Gateaux H-differentiable, then Definition 1.5.5 coincides with the one given in Lemma 4.2 [51].

From now on we will concentrate on the case when $(\imath, H, Z) = (\imath, H, \mathcal{X})$ is the triple associated with Gaussian process X. We will return to the case $(\imath, H, Z) = (\imath, H, B)$ in examples on Brownian motion.

1.5.2 Preliminary Results

In order to study the domain of the Itô-Ramer integral with respect to Gaussian processes we need some general results. We begin with an extension of Fubini-type theorem (Remark 2.2 in Gross [22]). The result in [22] is justified with help of AWS arguments. Our reasoning is based on Karhünen-Loéve representation ([33]).

Proposition 1.5.1 *Let $(\imath, H, \mathcal{X})$ be as in Section 1.5.1 and μ be a Gaussian measure supported by $\mathcal{X}$. Let $K \subseteq \mathcal{X}$ be a finite-dimensional linear subspace with $K \subseteq \mathcal{X}^*$ and $\{k_1, k_2, \ldots, k_n\} \subset K$, its orthonormal basis (ONB). Let $L = \bigcap_{j=1}^{n} ker(k_j)$ (a closed complement of K in $\mathcal{X}$) and denote by P_K and P_L, the projections of $\mathcal{X}$ onto K and L resp. Define $\mu_K = P_K\mu$, $\mu_L = P_L\mu$, the image measures under P_K and P_L. Then we have the following equalities:*

$$\begin{aligned}\int_{\mathcal{X}} f(x)\mu(dx) &= \int_{L\times K} f(x+\tilde{x})\mu_L(dx)\otimes\mu_K(d\tilde{x}) \qquad (1.4)\\ &= \int_{L\times R^n} f\left(x+\sum_{j=1}^{n} x_j k_j\right)\mu_L(dx)\otimes\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\exp\left(-\frac{1}{2}\sum_{j=1}^{n} x_j^2\right)dx_1\ldots dx_n,\end{aligned}$$

for any measurable function $f : \mathcal{X} \to \mathbf{R}_+$.

Remark 1.5.2 *The formulation of the above proposition is correct. $\mathcal{X}$ is a Hausdorff LCTVS and for any Hausdorff LCTVS if K is its finite dimensional subspace, then K is closed and $L \subset \mathcal{X}$ defined as above is its closed complement, $L \oplus K = \mathcal{X}$. If $x \in \mathcal{X}$ then x can be decomposed in a unique way into $x = x_L + x_K$ with $x_L \in L$ and $x_K \in K$. Projections P_L and P_K are linear and continuous (in our case $P_K(x) = x_K = \sum_{j=1}^{n} k_j(x)k_j$ and $P_L(x) = x - x_K$).*

Proof. (of the Proposition) Because P_L, P_K are linear and continuous, the image measures $P_L\mu, P_K\mu$ are Gaussian measures on L and K respectively. We want to prove that $\mu = \mu_L \otimes \mu_K$ on $L \times K = \mathcal{X}$. First we will prove that $\mu_L \otimes \mu_K$ is a Gaussian measure and then, that functionals on $\mathcal{X}$ can be decomposed into a sum of two independent (with respect to the measure μ) Gaussian random variables related to subspaces L and K.

Claim 1. $\mu_L \otimes \mu_K$ is a Gaussian measure.

Proof(of Claim 1) $\forall \varphi \in \mathcal{X}^*, \varphi \circ P_L, \varphi \circ P_K$ are independent Gaussian random variables with respect to $\mu_L \otimes \mu_K$ on $\mathcal{X}$. This is because

$$\varphi \circ P_L(x) = \varphi \circ P_L(x_L + x_K) = \varphi(x_L)$$

hence, $\varphi \circ P_L|_L = \varphi|_L \in L^*$ is normally distributed with respect to μ_L.
Thus $\varphi \circ P_L$ on $\mathcal{X}$ is also normally distributed with respect to $\mu_L \otimes \mu_K$ since values of this functional are independent of the component belonging to K. By the same argument $\varphi \circ P_K$ and $(\varphi \circ P_L, \varphi \circ P_K)$ are Gaussian. Independence follows from the equalities below,

$$\begin{aligned}
&\int_{\mathcal{X}} \varphi \circ P_L(x) \varphi \circ P_K(x) \mu_L \otimes \mu_K(dx) \\
&= \int_{L \times K} \varphi \circ P_L(x_L + x_K) \varphi \circ P_K(x_L + x_K) d\mu_L \otimes \mu_K \\
&= \int_{L \times K} \varphi|_L(x_L) \varphi|_K(x_K) d\mu_L \otimes \mu_K \\
&= \int_L \varphi|_L(x_L) d\mu_L \int_K \varphi|_K(x_K) d\mu_K \\
&= \int_{\mathcal{X}} \varphi \circ P_L(x) d\mu_L \otimes \mu_K \int_{\mathcal{X}} \varphi \circ P_K(x) d\mu_L \otimes \mu_K
\end{aligned}$$

Finally $\varphi = \varphi \circ P_L + \varphi \circ P_K$ is a Gaussian random variable with respect to $\mu_L \otimes \mu_K$ on $\mathcal{X}$. This completes the proof of Claim 1.

Claim 2. $\varphi \circ P_L, \varphi \circ P_K$ are independent Gaussian random variables relative to μ on $\mathcal{X}$.

Proof(of Claim 2). $\forall \varphi \in \mathcal{X}^*$, $\varphi(x) = \sum_{i=1}^{\infty} (\varphi, e_i) \pi(e_i)(x) = \sum_{i=1}^{\infty} (\varphi, e_i) e_i(x)$, for ONB $\{e_i\}_{i=1}^{\infty}\, in\, H$, where $\{e_i\}_{i=1}^{\infty} \subset \mathcal{X}^*$ and $e_i = k_i$, $(i = 1, ..., n)$ (see [33]). Therefore we can express compositions of functional φ with projections P_K and P_L as follows:

$$\varphi \circ P_K(x) = \sum_{i=1}^{n} (\varphi, e_i) e_i(x)$$

$$\begin{aligned}\varphi \circ P_L(x) &= \sum_{i=1}^{\infty}(\varphi, e_i)e_i(x - \sum_{j=1}^{n} e_j(x)e_j) \\ &= \sum_{i=1}^{\infty}(\varphi, e_i)e_i(x) - \sum_{i=1}^{n}(\varphi, e_i)e_i(x) = \sum_{i=n+1}^{\infty}(\varphi, e_i)e_i(x)\end{aligned}$$

Because $\{e_i(x) = \pi(e_i)(x)\}_{i=1}^{\infty}$ and $\{e_i(x)\}_{i=n+1}^{\infty}$ are independent families of random variables with respect to μ, also $\varphi \circ P_L$ and $\varphi \circ P_K$ are independent. Claim 2 is proved.

Now to prove that $\mu = \mu_L \otimes \mu_K$ we compare characteristic functionals of these measures.

$$\begin{aligned}\mu_L \widehat{\otimes} \mu_K(\varphi) &= \int_{\mathcal{X}} \exp\{i\varphi(x)\}\mu_L \otimes \mu_K(dx) \\ &= \int_{\mathcal{X}} \exp\{i(\varphi \circ P_L(x) + \varphi \circ P_K(x))\}\mu_L \otimes \mu_K(dx) \\ &= \int_{L} \exp\{i\varphi|_L(x_L)\}\mu_L(dx_L) \int_{K} \exp\{i\varphi|_K(x_K)\}\mu_K(dx_K) \\ &= \int_{\mathcal{X}} \exp\{i\varphi \circ P_L(x)\}\mu(dx) \int_{\mathcal{X}} \exp\{i\varphi \circ P_K(x)\}\mu(dx) \\ &= \int_{\mathcal{X}} \exp\{i\varphi(x)\}\mu(dx) = \hat{\mu}(\varphi).\end{aligned}$$

Because $\mu = \mu_L \otimes \mu_K$, we get,

$$\int_{\mathcal{X}} f(x)\mu(dx) = \int_{L \times K} f((x_L + x_K))\mu_L \otimes \mu_K(dx_L, dx_K)$$

for any measurable function $f : \mathcal{X} \to R_+$. Now, $L = \bigcap_{j=1}^{n} kerk_j$, therefore the random vector $(k_1, ..., k_n)$ has the same distribution under both measures μ_K and $\mu = \mu_L \otimes \mu_K$, that is n-dimensional standard normal. Hence equation (1.4) follows.

□

Next we extend the results of Kusuoka, contained in paragraph 4 of [29], that are relevant to our work. Kusuoka was concerned the setup of AWS while we are interested in a more general situation of the triple $(\imath, H, \mathcal{X})$ associated with a Gaussian process.

Definition 1.5.6 *Let $A \subset \mathcal{X}$ be any subset. Define function* $\rho(\cdot\,;A) : \mathcal{X} \to [0, +\infty]$ *by*

$$\rho(x; A) = \begin{cases} inf\{\|h\|_H : x + h \in A\} & if (A - x) \cap H \neq \emptyset \\ +\infty & otherwise \end{cases}$$

Next Proposition can be proved in the same way as Proposition 4.1 in [29].

Proposition 1.5.2 *(1) If subsets A and A' of $\mathcal{X}$ satisfy $A \subset A'$ then $\rho(x; A) \geq \rho(x; A')$ $\forall x \in \mathcal{X}$.*
(2) $\forall A \subset \mathcal{X}$, $h \in H$, $x \in \mathcal{X}$, $\rho(x + h; A) \leq \|h\|_H + \rho(x; A)$.
(3) Let $\{A_n\}_{n=1}^{\infty}$ be an increasing sequence of subsets of $\mathcal{X}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $\forall x \in \mathcal{X}$ $\rho(x; A_n) \searrow \rho(x; A)$ as $n \to \infty$.

Theorem 1.5.1 *(1) If K is a compact subset of $\mathcal{X}$, then $\rho(\cdot; K) : \mathcal{X} \to [0, +\infty]$ is lower semi-continuous.*
(2) If G is a σ-compact subset of $\mathcal{X}$, then $\rho(\cdot; G) : \mathcal{X} \to [0, +\infty]$ is measurable.

Proof. Since (2) is a consequence of (1) and Proposition 1.5.2 (3), it is enough to prove (1). We follow the idea of proof given in [29].

Define $A_a = \{x \in \mathcal{X} : \rho(x, K) \leq a\}$, $B(a)$ - the closed ball of radius a, centered at 0, in H. We want to show that $A_a = K + B(a)$. The inclusion $A_a \supset K + B(a)$ is clear. For the opposite inclusion, take $x \in A_a$. Then $\exists \{h_n\}_{n=1}^{\infty} \subset (K - x) \cap H$ such that $\|h_n\| \leq a + \frac{1}{n}$. Being norm bounded, the sequence $\{h_n\}_{n=1}^{\infty}$ contains a weakly convergent subsequence $\{h_{n_k}\}_{k=1}^{\infty}$. Let $h \in H$ denotes its limit. Since $\mathcal{X}^* \subset H$ and $\forall t \in T$ $x_t(h) = h(t)$ (point evaluation) is an element of $\mathcal{X}^*$ we also have $h_{n_k} \to h$ in $\mathcal{X}$ (convergence in $\mathcal{X}$ is a pointwise convergence). Also,

$$\|h\|_H = \sup\{|\langle h, x\rangle| \,; x \in \mathcal{X}^*, \|x\|_H \leq 1\}$$

$$\begin{aligned} &= \sup\{\lim_{k\to\infty} |\langle h_{n_k}, x\rangle| \,; x \in \mathcal{X}^*, \|x\|_H \le 1\} \\ &\le \overline{\lim}_{n\to\infty} \|h_n\|_H \le a. \end{aligned}$$

Thus $h \in B(a)$. Since $K \subset \mathcal{X}$ is compact and $h_{n_k} \to h$ in $\mathcal{X}$ with $x + h_{n_k} \in K$, $(k = 1, 2, ...)$, also $x + h \in K$ and therefore $x \in K + B(a)$. Thus $A = K + B(a)$. We claim that $B(a) \subset \mathcal{X}$ is closed. Indeed, we have the following:

Lemma. *Let X be a reflexive Banach space and Y be a LCTVS. Let $T : X \to Y$ be linear and continuous. Then $T(B_X(0,1)) \subset Y$ is closed, where $B_X(0,1)$ is a closed unit ball centered at 0 in X.*

Proof(of Lemma). $T : X \to Y$ is linear and continuous, hence $T : X_\omega \to Y_\omega$ is linear and continuous (ω - means weak topology). This is because if $\{x_\alpha\}$ is a net in X with $x_\alpha \to x$ in X_ω, then $\forall y^* \in Y^*$, $y^*(Tx_\alpha) = (y^*T)x_\alpha \to (y^*T)x = y^*(Tx)$, for $(y^*T) \in X^*$.

Because $X^{**} \cong X$ by the canonical isomorphism κ, we get,$T \circ \kappa^{-1} : X^{**}_\omega \to Y_\omega$ is linear and continuous and further, $T \circ \kappa^{-1} : X^{**}_{\omega-*} \to Y_\omega$ is linear and continuous (where $\omega - *$ denotes the $\omega - *$ topology). The latter holds because reflexivity of X implies reflexivity of X^*. Now, the closed unit ball $B_{X^{**}}(0,1)$ is $\omega - *$ compact by Alaoglu-Banach theorem. That means $\kappa(B_X(0,1))$ is $\omega - *$ compact in X^{**}, hence $T \circ \kappa^{-1}(\kappa(B_X(0,1)) = T(B_X(0,1))$ is ω closed in Y. Because Y and Y_ω have the same closed, convex sets, $T(B_X(0,1))$ is closed in the topology of Y and the lemma is proved.

Thus $\imath(B(a)) \subset \mathcal{X}$ is closed, therefore $A_a = K + B(a) \subset \mathcal{X}$ is closed.

□

Next theorem can be proved as in [29] with obvious modifications.

Theorem 1.5.2 *Let E be a separable, reflexive Banach space and $F : \mathcal{X} \to E$ be a measurable map and suppose that there exists a constant $c > 0$ such that*

$\forall x \in \mathcal{X}, h \in H$, $\|F(x+h) - F(x)\|_E \leq c\|h\|_H$. Then there exists a measurable subset $\mathcal{D}_0$ of $\mathcal{X}$ and a map $DF : \mathcal{X} \to L(H, E)$ such that :

(1) $\mu(\mathcal{D}_0) = 1$.

(2) $\lim_{t \to 0} \frac{1}{t}(F(x+th) - F(x)) = DF(x)h$, $\forall x \in \mathcal{D}_0$, $h \in H$.

(3) $DF(\cdot)h : \mathcal{X} \to E$ is measurable $\forall h \in H$.

In particular, if $DF : \mathcal{X} \to L(H, E)$ is strongly measurable then $F \in H^1(\mathcal{X} \to E; d\mu)$.

Corollary 1.5.1 *Let G be a σ-compact subset of $\mathcal{X}$ and ϕ be a smooth function with compact support in R. Then $g(\cdot) = \phi(\rho(\cdot\,; G)) : \mathcal{X} \to R$,with the convention that $\phi(\infty) = 0$, belongs to $H^1(\mathcal{X} \to R; d\mu)$ and*

$$\|D^G g(x)\|_H \leq sup\{|\frac{d\phi}{dt}| : t \in R\}$$

for μ a.e. x.

Proof. First we observe that by Theorem 1.5.1, (2) g is measurable. Also,

$$\begin{aligned} \|g(x+h) - g(x)\| &\leq \sup\{\left|\frac{d\phi}{dt}(t)\right|; t \in R\}\,(\rho(x+h; G) - \rho(x, G)) \\ &\leq \sup\{\left|\frac{d\phi}{dt}(t)\right|; t \in R\}\,\|h\|_H \end{aligned}$$

by Proposition 1.5.2, (2), (with the convention $\infty - \infty = 0$, note that $\rho(x + h; G) = \infty$ iff $\rho(x; G) = \infty$). Therefore, assumptions of Theorem 1.5.2 are satisfied. $D^G g(\cdot)h$ can be thought of as $h(D^G g(\cdot))$ with $h \in H^{**}$, $D^G g(\cdot) : \mathcal{X} \to H^*$. Thus we have $D^G g : \mathcal{X} \to H^*$ is weakly measurable (by (3) of Theorem 1.5.2) and therefore it is strongly measurable in view of separability of H. The inequality at the end of Corollary is obvious.

□

1.5.3 The Domain of Itô-Ramer Integral

In this section we specify two subclasses of the domain of the Itô-Ramer integral. We begin with generalization of Ramer's result, Lemma 4.2 in [51]. As a tool we use the inequality given by Ramer in Lemma 4.1, [51] but we need it for functions in $H^1(R^n \to R^n, \gamma_n)$, where γ_n is the standard Gauss measure on R^n (see [29]).

Note. By $H^{\otimes 2}$ we denote the tensor product $H \otimes H$ which is identified with the Hilbert space of Hilbert-Schmidt operators on H.

Lemma 1.5.1 *Let $f : R^n \to R^n$ be an $H^1(R^n \to R^n, \gamma_n)$ function. Assume $\|f\|_{R^n}$ and $\|D^G f\|_{(R^n)^{\otimes 2}} \in L_2(R^n, \gamma_n)$ (the latter norm is the Hilbert-Schmidt norm). Then,*

$$\int_{R^n} ((f(x), x) - tr D^G f(x))^2 \gamma_n(dx) \leq \int_{R^n} (\|f(x)\|^2_{R^n} + \|D^G f(x)\|^2_{(R^n)^{\otimes 2}}) \gamma_n(dx).$$

Theorem 1.5.3 *Let $F \in H^1(\mathcal{X} \to H; d\mu)$ and assume that $D^G F(x) \in H^{\otimes 2}$ for μ a.e. x and $F \in L_2(\mathcal{X}, H)$, $D^G F \in L_2(\mathcal{X}, H^{\otimes 2})$. Then $F \in \mathcal{D}(L)$ and*

$$\int_{\mathcal{X}} |LF(x)|^2 \mu(dx) \leq \int_{\mathcal{X}} (\|F(x)\|^2_H + \|D^G F(x)\|^2_{H^{\otimes 2}}) \mu(dx).$$

Proof. We use Proposition 1.5.1 to extend Lemma 4.2 in [51] to our case. Any $P_n \in \mathcal{P}(\mathcal{X}^*)$ with $\dim P_n(H) = n$ can be written as follows:

$$P_n = \sum_{i=1}^{n} e_i \otimes e_i, \quad e_i \in X^*, \ \{e_i\}_{i=1}^n \ \text{ONB in } H.$$

We will first show that $\{L_{P_n} F\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathcal{X})$.

$$\begin{aligned}
&\int_{\mathcal{X}} (L_{P_l} F(x) - L_{P_m} F(x))^2 \mu(dx) \\
&= \int_{\mathcal{X}} \{\langle (P_l - P_m) F(x), x \rangle - tr(P_l - P_m) D^G F(x)\}^2 \mu(dx)
\end{aligned}$$

We can apply Proposition 1.5.1, to get that the last expression is equal to

$$\int_L \int_{R^l} \{\langle (P_l - P_m)F(\sum_{i=1}^{l} \alpha_i e_i + x_L), \sum_{i=1}^{l} \alpha_i e_i + x_L \rangle \tag{1.5}$$
$$+tr(P_l - P_m)DF(\sum_{i=1}^{l} \alpha_i e_i + x_L)\}^2 \mu(dx_L) d\gamma_l$$

where we assume that $l \geq m$, $K = span\{e_1, ..., e_l\}, L = \bigcap_{i=1}^{l} ker\ e_i$ (the closed complement of K in $\mathcal{X}$ from Proposition 1.5.1), $x_L = P_L x$, P_L is the projection of $\mathcal{X}$ onto L and γ_l denotes the standard Gauss measure on $\mathbf{R}^l$.

Using Lemma 1.5.1 we can bound the last expression by

$$\int_{\mathcal{X}} \{\|(P_l - P_m)F(x)\|_H^2 + \|(P_l - P_m)D^G F(x)\|_{H^{\otimes 2}}^2\} \mu(dx).$$

Both components converge to zero as $l, m \to \infty$. Indeed,

$$\begin{aligned} \int_{\mathcal{X}} \|(P_l - P_m)F(x)\|_H^2 \mu(dx) &= \int_{\mathcal{X}} \| \sum_{i=m+1}^{l} (e_i, F(x))_H e_i \|_H^2 \mu(dx) \\ &\leq \int_{\mathcal{X}} \sum_{i=m+1}^{\infty} (e_i, F(x))_H^2 \mu(dx) \to 0 \end{aligned}$$

since $F \in L_2(\mathcal{X}, H, d\mu)$.

Similar argument shows that the second component converges to zero. Resuming, we proved that,

$$\begin{aligned} P_n F \to F &\in L_2(\mathcal{X}, H, d\mu) \text{ and} \\ P_n D^G F \to D^G F &\in L_2(\mathcal{X}, H^{\otimes 2}, d\mu). \end{aligned}$$

We also obtained the following estimate :

$$\int_{\mathcal{X}} |L_{P_K} F(x)|^2 \mu(dx) \leq \int_{\mathcal{X}} \{\|P_K F(x)\|_H^2 + \|P_K D^G F(x)\|_{H^{\otimes 2}}^2\} \mu(dx) \tag{1.6}$$

for $P_K \in \mathcal{P}(\mathcal{P}^*)$.

Furthermore, the $L_2(\mathcal{X})$ limit, LF, does not depend on the choice of the sequence of projections. Indeed, let $\{P_n\}_{n=1}^{\infty}$, $\{Q_n\}_{n=1}^{\infty}$ be two sequences in $\mathcal{P}(\mathcal{X}^*)$ converging to Id_H. Then we have,

$$\begin{aligned}
&\|L_{P_n}F - L_{Q_m}F\|^2_{L_2(\mathcal{X})} \\
&\leq \|(P_n - Q_m)F(x)\|^2_{L_2(\mathcal{X},H)} + \|(P_n - Q_m)D^G F(x)\|^2_{L_2(\mathcal{X},H^{\otimes 2})} \\
&\leq 2\{\|P_n F - F\|^2_{L_2(\mathcal{X},H)} + \|P_n D^G F - D^G F\|^2_{L_2(\mathcal{X},H^{\otimes 2})}\} \\
&+ 2\{\|Q_m F - F\|^2_{L_2(\mathcal{X},H)} + \|Q_m D^G F - D^G F\|^2_{L_2(\mathcal{X},H^{\otimes 2})}\}
\end{aligned}$$

with the RHS converging to zero as $m, n \to \infty$.

The inequality,

$$\int_{\mathcal{X}} |LF(x)|^2 \mu(dx) \leq \int_{\mathcal{X}} (\|F(x)\|^2_H + \|D^G F(x)\|^2_{H^{\otimes 2}})\mu(dx)$$

follows from (1.6).

□

Now, Theorem 5.2 in [29] can be extended to our case.

Theorem 1.5.4 *Let $F \in H^1(\mathcal{X} \to H; d\mu)$ and ω be a positive weight function, i.e. $\omega : \mathcal{X} \to \mathbf{R}$ is measurable, $\omega(x) > 0$ $\forall x \in \mathcal{X}$ and $\omega(x + \cdot) : H \to \mathbf{R}$ is continuous $\forall x \in \mathcal{X}$. Assume that $D^G F(x) \in H^{\otimes 2}$ for μ a.e. x and that*

$$\int_{\mathcal{X}} (\|F(x)\|^2_H + \|D^G F(x)\|^2_{H^{\otimes 2}})\omega(x)\mu(dx) < \infty.$$

Then $F \in \mathcal{D}(L)$. Furthermore, there exists a positive, measurable function $k : \mathcal{X} \to R$, depending only on ω, such that

$$\int_{\mathcal{X}} |LF(x)|^2 k(x)\mu(dx) \leq \int_{\mathcal{X}} (\|F(x)\|^2_H + \|D^G F(x)\|^2_{H^{\otimes 2}})\omega(x)\mu(dx) < \infty.$$

Proof. The proof in [29] applies if instead of references to Corollary to Theorem 4.2 and to Theorem 5.1 [29], references to Corollary 1.5.1 and to Theorem 1.5.3 are made.

□

Definition 1.5.7 *A measurable map $F : Z \to H$ is said to be in class $H - C^1$ if the following conditions are satisfied :*
(1) $\forall z \in Z \; \exists DF(z) \in H^{\otimes 2}$ such that
$\|F(z+h) - F(z) - DF(z)h\|_H = o(\|h\|_H)$ *as* $\|h\|_H \to 0$.
(2) $\forall z \in Z$, $DF(z+\cdot) : H \to H^{\otimes 2}$ is continuous.

Now we obtain the following Corollary from Theorem 1.5.4.

Corollary 1.5.2 *$H - C^1 \subset \mathcal{D}(L)$.*

Proof. Clearly $\omega(x) = \{1 + \|F(x)\|_H^2 + \|DF(x)\|_{H^{\otimes 2}}^2\}^{-1}$ is a weight function for $F \in H - C^1$. Also $H - C^1 \subset H^1(\mathcal{X} \to H, d\mu)$. Indeed, first, $F \in H - C^1$ implies Fréchet differentiability of F which is stronger than SG-Differentiability. Also F is strongly measurable in view of separability of H. Further we need strong measurability of the H-Fréchet derivative of F, $DF : \mathcal{X} \to L(H)$. We have,

$$\forall x \in \mathcal{X}, g \in H, \;\; \frac{1}{t}((F(x+th) - F(x)), g) \to (DF(x))(h \otimes g).$$

The LHS of above is measurable, therefore the RHS, as a limit, is measurable. Thus $DF : \mathcal{X} \to H^{\otimes 2}$ is weakly measurable. Furthermore, the inclusion $H^{\otimes 2} \hookrightarrow L(H)$ is continuous and $H^{\otimes 2}$ is a separable subspace of $L(H)$, giving that DF is separably valued and weakly measurable as a function with range in $L(H)$. Hence, it is a strongly measurable map from $\mathcal{X}$ to $L(H)$.

To prove the RAC condition note that, by (2) in Definition 1.5.7

$$\sum_{i=1}^{n} \|F(x+t_{i+1}h) - F(x+t_ih)\|_H \leq \sup_{\alpha\in[a,b]} \|DF(x+\alpha h)\|_{H^{\otimes 2}} \|h\|_H(b-a)$$

where $a = t_1 \leq t_2 \leq \ldots \leq t_{n+1} = b$ is a partition of an interval $[a, b]$. Now, $F(x+th)$ is an absolutely continuous H valued function, so that it is RAC.

□

Since both, the Itô-Ramer and Ogawa integrals involve a series expansion of the integrand with respect to one dimensional Wiener integrals, one can expect to have a connection between these two types of integration. We give our result in the next Proposition.

Proposition 1.5.3 *Let $u \in H - C^1$ and assume that the H-Fréchet derivative of u, $D^Fu(x)$, is a trace class operator on H for every $x \in \mathcal{X}$. Then $u \in \mathcal{D}(L)$ and u is Ogawa integrable with respect to all ONB's $\{e_n\}_{n=1}^{\infty} \subset \mathcal{X}^*$ of H and $\delta_e^o(u) = L(u) + trD^Fu$ μ a.e.*

Proof. By Corollary 1.5.2 we already know that $u \in \mathcal{D}(L)$. Since Lu exists we can choose any sequence $\{P_N\}_{N=1}^{\infty} \subset \mathcal{P}(\mathcal{X}^*)$ of finite dimensional projections of the form: $P_N = \sum_{k=1}^{N} h_k \otimes h_k$ with $\{h_k\}_{k=1}^{\infty} \subset \mathcal{X}^*$ being an ONB in H and we have $L_{P_N}u \to Lu$ in probability. Compute the expression for $L_{P_N}(u)$

$$L_{P_N}(u)(x) = \sum_{k=1}^{N}(u(x), h_k)h_k(x) - \sum_{k=1}^{N}(D^Fu(x)h_k, h_k) \tag{1.7}$$

(recall that $h_k(x) = I_1(h_k)(x)$). The sum defining the Ogawa integral converges in probability because the two other expressions in (1.7) converge.

□

1.5.4 Comparison of Itô-Ramer and Skorohod Integration

Let us first consider the problem of relationship between stochastic differentiation in the sense of Gateaux and Malliavin. It will be used in comparing different types of stochastic integration. This question was raised by Mandrekar and Zhang in the concluding remarks of their paper [33].

Proposition 1.5.4 *Let $F \in L_2(\mathcal{X})$ and assume that $\frac{1}{t}(F(x+th) - F(x))$, $h \in H$, converges in $L_2(\mathcal{X})$ as $t \to 0$. Then the Malliavin derivative $D_h^M F$ exists and coincides with the above limit.*

Proof. We can apply the method of proof of Proposition 2.2. in [41]. The following formula is valid for functions $f_m \in H^{\odot m}$, $m = 0, 1, \ldots$:

$$I_m(f_m)(x+\varepsilon h) = \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \varepsilon^{m-i}$$
$$I_i((f_m(t_1, \ldots, t_i, t_{i+1}, \ldots, t_m), h(t_{i+1}) \ldots h(t_m)))(x).$$

This can be justified first for functions f_m of the form: $f_m(t_1, \ldots, t_m) = e(t_1) \ldots e(t_m)$, $e \in H$, $\|e\|_H = 1$ and then for functions $f_m(t_1, \ldots, t_m)$ being a symmetrization of $e_1(t_1) \ldots e_1(t_{p_1}) e_2(t_{p_1+1}) \ldots e_k(t_{p_1+\ldots+p_k})$ with $p_1 + \ldots + p_k = m$ and $e_1, \ldots, e_k$ orthonormal vectors in H. Finally one can use a convergence argument to get the above formula for all $f \in H^{\odot m}$.

From this point we can proceed as in [41] with obvious changes.

□

As an example for equivalence of SG and Malliavin differentiation consider elementary processes ([41]).

Definition 1.5.8 *A stochastic process $u = \{u_t;\ t \in T\}$ on $\mathcal{X}$ is called elementary if u is of the following form:*

$$u_t(x) = \sum_{j=1}^{N} \psi_j(e_1(x), \ldots, e_N(x)) e_j(t)$$

where $e_1, \ldots, e_N \in \mathcal{X}^*$ and are orthonormal in H, $\psi_j : R^N \to R^N$ $(j = 1, \ldots, N)$ are smooth functions with all derivatives of polynomial growth.

Note. An elementary process can be considered as an H-valued random variable on $\mathcal{X}$.

Corollary 1.5.3 *Let u be an elementary process. Then the Malliavin and SG and Fréchet derivatives of u coincide,*

$$D^G u(x)(h) = D_h^M u(x) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial \psi_j}{\partial x_k}(\bar{e}(x))(h, e_k) e_j.$$

Here $\bar{e}(x) = (e_1(x), ..., e_N(x))$.

Proof. The reminder in the form of Lagrange in Taylor series expansion of each ψ_j is bounded above by a polynomial in $\|x\|$, multiplied by a factor independent of x and converging to zero as the increment converges to zero. Therefore the reminder converges to zero in $L_2(\mathcal{X})$, hence Proposition 1.5.4 is applicable to each of the random variables $\psi_j(\bar{e}) \in L_2(\mathcal{X})$. Assertion for process u follows.

□

Corollary 1.5.4 *Let $u \in L_2(\mathcal{X}, H)$ be an H-valued, SG-Differentiable random variable. Assume that the following condition is satisfied:*

$$\forall k, h \in H \quad \frac{1}{\varepsilon}((u(x + \varepsilon k), h) - (u(x), h)) \qquad (G - M)$$

converges in $L_2(\mathcal{X})$ *as* $\varepsilon \to 0$.

Then the Malliavin derivative $D_k^M(u,h)(x)$ *exists and equals to* $(D^G u(x)k, h)$ μ *a.e. If SG-derivative* $D^G u(x)$ *is a trace class operator, then,*

$$trD^G u(x) = \sum_{n=1}^{\infty} D_{e_n}^M(u, e_n)(x)$$

μ *a.e., for any* $\{e_n\}_{n=1}^{\infty}$ *an ONB in* H.

Proof. Existence of SG-derivative $D^G u$ implies that for all $k, h \in H$,

$$\frac{(u(x+\varepsilon k) - u(x), h)}{\varepsilon} - (D^G u(x)(k), h) \to 0$$

in probability μ. It follows by Proposition 1.5.4 that under condition *(G-M)* the Malliavin derivative $D_k^M(u,h)$ exists and

$$(D^G u(x)(k), h) = D_k^M(u,h)(x)$$

outside $N_{k,h} \subset \mathcal{X}$ with $\mu(N_{k,h}) = 0$.

Now the last statement of the Corollary follows because H is separable.

□

Note. We do not claim that the Malliavin derivative of u exists or that it is a trace class operator.

In view of the Corollary 1.5.4 we propose the following notion:

Definition 1.5.9 *An* H*-valued random variable* $u \in L_2(\mathcal{X}, H)$ *will be called weakly (G-M) differentiable if* u *is SG differentiable and it satisfies condition (G-M).*

We obtain our result on relationship between the Itô-Ramer and Skorohod integrals as a conclusion from the above work.

Theorem 1.5.5 *Let $u \in L_2(\mathcal{X}, H)$ and assume that the Malliavin derivative $D^M u$ exists and u is weakly (G-M) differentiable. Then the Malliavin derivative and SG-Derivative of u coincide. If in addition, $u \in H - C^1$ then the Malliavin and H-Fréchet derivatives of u are the same. Consequently, if for every $x \in \mathcal{X}$, $D^M u(x)$ is a Hilbert-Schmidt operator on H, with $E \parallel D^M u \parallel^2_{H^{\otimes 2}} < \infty$ and $u \in H - C^1$ then $Lu = I^s(u)$ a.e. $d\mu$.*

Proof. Since for any $k, h \in H$,

$$(D^M u(k), h) = (D^M_k u)(h) = D^M_k(u, h) = (D^G u(k), h) \quad \mu \text{ a.e.}$$

we get the equality of derivatives. Equality of integrals follows from (1.3) of Proposition 1.4.1 and (1.7) of Proposition 1.5.3,

$$I^s(P_N u) = L_{P_N}(u) \quad \mu \text{ a.e.}$$

with $I^s(P_N u) \to I^s(u)$ in $L_2(\mathcal{X})$ and $L_{P_N}(u) \to Lu$ in probability.

□

1.5.5 Itô-Ramer Integral as an Integration by Part Operator

The idea to exploit an integration by parts formula in the problem of transformations of Gaussian measures on a Banach space was used by Bell in [2], [3] and by Ustunel and Zakai in [56]. The question of absolute continuity of the original and transformed measures was considered. Therefore it is of an interest to know whether the Itô-Ramer integral operator L satisfies the integration by parts formula.

Let us recall definition of Integration by Parts Operator (IPO) as in Bell [2], however we use a different class of test functions.

Definition 1.5.10 *A linear operator $\mathcal{A} : \mathcal{D} \to L_2(\mathcal{X}, d\mu)$, where $\mathcal{D} \subset H^{\mathcal{X}}$, is called an Integration by Parts Operator (IPO) on $\mathcal{D}$ for μ if the relation*

$$\int_{\mathcal{X}} (D^F\phi(x), u(x))\mu(dx) = \int_{\mathcal{X}} \phi(x)(\mathcal{A}u)(x)\mu(dx)$$

holds for all C^1_{poly} functions $\phi : \mathcal{X} \to R$ (continuously Fréchet H-Differentiable functions with the Fréchet derivative D^F of polynomial growth in directions of H), and all $u \in \mathcal{D}$ for which either side of the above exists.

It will be useful to note that the Itô-Ramer operator L is continuous in the norm

$$\|u\|^2 = E\|u\|^2_H + E\|D^G u\|^2_{H^{\otimes 2}}. \tag{1.8}$$

This follows directly from the proof of Theorem 1.5.3.

Theorem 1.5.6 *The Itô-Ramer operator L is an IPO on the closure in the norm (1.8), of the linear space of elementary processes in $\mathcal{D}(L)$.*

Proof. We need to show that for any C^1_{poly} function $\phi : \mathcal{X} \to R$ and any $u \in \mathcal{D}$,

$$\int_{\mathcal{X}} \phi(x)Lu(x)\mu(dx) = \int_{\mathcal{X}} (D^F\phi(x), u(x))\mu(dx) \tag{1.9}$$

where $\mathcal{D}$ is the closure in the norm (1.8) of the set of elementary processes in $\mathcal{D}(L)$.

Let $P_N = \sum_{k=1}^N f_k \otimes f_k \in \mathcal{P}(\mathcal{X}^*)$, where $\{f_k\}_{k=1}^\infty \subset \mathcal{X}^*$ is an ONB in H. We begin with an elementary process

$$u_t^M(x) = \sum_{i=1}^M \psi_i(e_1(x)...e_M(x))e_i(t).$$

Note that in the case of elementary processes,the Fréchet derivative and SG-derivative coincide. We have the following expression for $L_{P_N}(u^M)(x)$:

$$\begin{aligned}
L_{P_N}(u^M)(x) &= \sum_{k=1}^{N}(u_{\cdot}(x), f_k(\cdot))_H f_k(x) - \sum_{k=1}^{N}(D^G u_t(x)(f_k), f_k)_H \\
&= \sum_{k=1}^{N}\sum_{i=1}^{M}\psi_i(\bar{e}(x))(e_i, f_k)_H f_k(x) \\
&- \sum_{k=1}^{N}\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial\psi_i}{\partial x_j}(\bar{e}(x))(e_i, f_k)_H(e_j, f_k)_H
\end{aligned}$$

where $\bar{e}(x) = (e_1(x), ..., e_N(x))$ for short.

We want to find the Itô-Ramer integral of u^M, that is we need to find an $L_2(\mathcal{X})$ limit of $L_{P_N}(u^M)(x)$ as $N \to \infty$. We have the following:

$$\begin{aligned}
&E\{\sum_{k=1}^{N}\sum_{i=1}^{M}\psi_i(\bar{e})(e_i, f_k)_H f_k - \sum_{i=1}^{M}\psi_i(\bar{e})e_i\}^2 \\
&= E\{\sum_{i=1}^{M}\psi_i(\bar{e})\sum_{k=N+1}^{\infty}(e_i, f_k)_H f_k\}^2 \\
&\leq M^2\sum_{i=1}^{M}E\{\psi_i(\bar{e})\sum_{k=N+1}^{\infty}(e_i, f_k)_H f_k\}^2.
\end{aligned}$$

Each of the M terms in the latter sum converges to zero in $L_2(\mathcal{X})$. Also,

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial\psi_i}{\partial x_j}(\bar{e}(x))(e_i, f_k)_H(e_j, f_k)_H - \sum_{i=1}^{M}\frac{\partial\psi_i}{\partial x_i}(\bar{e}(x)) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial\psi_i}{\partial x_j}(\bar{e}(x))\sum_{k=1}^{N}((e_i, f_k)_H(e_j, f_k)_H - (e_i, e_j)_H)
\end{aligned}$$

converges to zero in $L_2(\mathcal{X})$ since it is a finite sum of square integrable random variables multiplied by non-random factors converging to zero. Therefore,

$$L(u^M) = \sum_{i=1}^{M}\psi_i(\bar{e})e_i - \sum_{i=1}^{M}\frac{\partial\psi_i}{\partial x_i}(\bar{e}).$$

On the other hand,

$$(D^F\phi(x), u^M(x)) = \sum_{i=1}^{M}(D^F\phi(x), \psi_i(\bar{e}(x))e_i).$$

Thus it is enough to show the following equality:

$$\int_{\mathcal{X}} \phi(x)\{\psi_i(\bar{e}(x))e_i(x) - \frac{\partial \psi_i}{\partial x_i}(\bar{e}(x))\}\mu(dx)$$
$$= \int_{\mathcal{X}} (D^F\phi(x), \psi_i(\bar{e}(x))e_i)\mu(dx).$$

By Proposition 1.5.1, for $K = span\{e_1, ..., e_M\}$ and $\tilde{K}$, its closed complement in $\mathcal{X}$ from the Proposition, we have,

$$\int_{\mathcal{X}} \phi(x)\{\psi_i(\bar{e}(x))e_i(x) - \frac{\partial \psi_i}{\partial x_i}(\bar{e}(x))\}\mu(dx)$$
$$= \int_{\tilde{K}\times K} \phi(x_{\tilde{K}} + \sum_{j=1}^{M} e_j(x_{\tilde{K}} + x_K)e_j)\{\psi_i(\bar{e}(x_{\tilde{K}} + x_K))e_i(x_{\tilde{K}} + x_K)$$
$$- \frac{\partial \psi_i}{\partial x_i}(\bar{e}(x_{\tilde{K}} + x_K))\}\mu_{\tilde{K}}(dx_{\tilde{K}}) \otimes \mu_K(dx_K)$$
$$= \int_{\tilde{K}} \int_{R^M} \phi(x_{\tilde{K}} + \sum_{j=1}^{M} \alpha_j e_j)\{\psi_i(\alpha_1, ..., \alpha_M)\alpha_i$$
$$- \frac{\partial \psi_i}{\partial x_i}(\alpha_1, .., \alpha_M)\}\gamma_M(d\bar{\alpha})\mu_{\tilde{K}}(dx_{\tilde{K}})$$

where $\bar{\alpha} = (\alpha_1, ..., \alpha_N) \in R^M$, $x_K = \sum_{i=1}^{M} e_i(x)e_i$, $\;x_{\tilde{K}} = x - x_K$ and γ_M is the M dimensional Gauss measure on R^M. Using the fact that the divergence operator is an IPO on $\mathbf{R}^M$ ([20]) we get, that the last expression equals to,

$$\int_{\tilde{K}} \int_{R^M} (D^F\phi(x_{\tilde{K}} + \sum_{j=1}^{M} \alpha_j e_j), \psi_i(\alpha_1, ..., \alpha_M)e_i)\gamma_M(d\bar{\alpha})\mu_{\tilde{K}}(dx_{\tilde{K}})$$
$$= \int_{\mathcal{X}} (D^F\phi(x), \psi_i(e_1(x), ..., e_M(x))e_i)\mu(dx).$$

Hence we have formula (1.9) for the Itô-Ramer integral L and the class of elementary processes. Next, if $u^M \to u$ in the norm (1.8), then because the operator L is continuous in this norm we get $Lu^M \to u$ in $L_2(\mathcal{X})$. This implies,

$$\int_{\mathcal{X}} \phi L u^M d\mu \to \int_{\mathcal{X}} \phi L u d\mu \text{ and } \int_{\mathcal{X}} (D^F\phi, u^M)d\mu \to \int_{\mathcal{X}} (D^F\phi, u)d\mu.$$

This completes the proof.

□

The problem is to show that the closure, in the norm (1.8), of elementary processes in $D(L)$ coincides or contains the class of processes introduced in Theorem 1.5.3. We do not investigate this question here, however we have one simple Corollary (due to the IPO property of the Skorohod integral ([33])).

Corollary 1.5.5 *L is an IPO on the class of processes $u \in L_2(\mathcal{X}, H)$ satisfying assumptions of Theorem 1.5.5 when the class of test functions is restricted to elements φ for which $D^M\varphi = D^G\varphi$.*

Notice, that the IPO property of the Skorohod integral involves the Malliavin derivative D^M which is the adjoint operator to the integral operator of Skorohod. Our work indicates that, in the sense of Theorem 1.5.6, the adjoint operator to the Itô-Ramer integral L is the SG-Derivative D^G.

1.6 Examples

1. Elementary processes. Let u be an elementary process (see Definition 1.5.8) of the form: $u_t = \sum_{j=1}^N \psi_j(\bar{e})e_j(t)$. Then u is Itô-Ramer integrable and

$$Lu = \sum_{j=1}^N \psi_j(\bar{e})e_j - \sum_{j=1}^N \frac{\partial\psi_j}{\partial x_j}(\bar{e})$$

(see the proof of Theorem 1.5.6). The Ogawa integral of u also exists, $\delta^o(u) = \sum_{j=1}^N \psi_j(\bar{e})e_j$.

2. Brownian motion. Let $\{B_t, t \in [0,1]\}$ be the standard Brownian motion of Example 1.2.1. Then the process $u_t = \int_0^t B_s ds$ is an H valued stochastic process, where H is the RKHS of Brownian motion. The Ogawa integral of u is given by

$$\delta^o(u) = \sum_{n=1}^\infty (u, e_n)\pi(e_n) = \sum_{n=1}^\infty (B, e_n')_{L_2([0,1])} \int_0^1 e_n' dB = \frac{1}{2}B_1^2 \quad (e_n' = \frac{de_n}{dt})$$

as proved in Example 3.2 [52] (the above formula is correct for any ONB $\{e_n\}_{n=1}^\infty$ in H).

Compute the Gateaux derivative of u in the direction of $h \in H$. Since $u(x+h)-u(x)=\int_0^\cdot h(s)ds$, we get that $D^G u(x)h=\int_0^\cdot h(s)ds$ independently of $x \in \mathcal{X}$. Operator $D^G u(x)$ on H is Hilbert-Schmidt because it has the same Hilbert-Schmidt norm as the kernel operator on $L_2([0,1])$ given by a kernel $1_{[0,t]}(s)$. Thus u is Itô-Ramer integrable by Theorem 1.5.3. By Example 3.2 in [52]

$$tr(P_N D^G u(x)) = \sum_{n=1}^N \Big(\int_0^\cdot e_n(s)ds, e_n(\cdot)\Big)_H$$

converges as $N \to \infty$ independently of the choice of an ONB $\{e_n\}_{n=1}^\infty \subset H$. For the ONB consisting of indefinite integrals of the Haar functions the result is easy to obtain and it is equal to $\frac{1}{2}$. Thus $Lu=\frac{1}{2}B_1^2-\frac{1}{2}$. Itô-Ramer integrability of u also follows from Corollary 1.5.2 since $Du(x+\cdot): H \to H^{\otimes 2}$ is continuous (it does not depend on h). Hence, $u: \mathcal{X} \to H$ is an $H-C^1$ map. The Skorohod integral $I^s(u)$ is the same as the Itô integral $\int_0^1 B_t dB_t$ (see Example 1.3.2). Hence,

$$I^s(u) = L(u) = \frac{1}{2}B_1^2 - \frac{1}{2}.$$

Notice that all the above could be also justified in a similar way by use of Ramer's and Kusuoka's results in view of the relationship given in Examples 1.3.2 and 1.3.1.

3. Reversed Brownian motion. Let us now consider the reversed Brownian motion process B_{1-t}. By Theorem 1.5.3 and arguments as in **(2)** above, the process $u_t=\int_0^t B_{1-s}ds$ is Itô-Ramer integrable. Also the Skorohod integral $I^s(u)$ exists. The Malliavin and Gateaux derivatives of process u coincide and are given by $Du(h)=\int_0^\cdot h(1-s)ds$. Theorem 1.5.5 implies that $Lu=I^s(u)$.

Example in [52] shows that u is not universally Ogawa integrable in the sense of [52]. Note that convergence in [52] is in $L_2(\Omega)$.

4. Ogawa non-integrable process. The following example in [44] shows that given any ONB in H one can construct a process that is not Ogawa integrable with respect to this basis. Let

$$u_t = \sum_{n=1}^{\infty} \frac{1}{n^p} e_n(t) sign(\pi(e_n)) \quad (\frac{1}{2} < p \leq 1),$$

then $u \in L_2(\Omega, H)$, but the series defining $\delta^o(u)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} sign(\pi(e_n))\pi(e_n) = \sum_{n=1}^{\infty} \frac{1}{n^p} |\pi(e_n)|$$

diverges a.e.

5. An example of a process with an infinite Wiener Chaos expansion. Consider the general case of a Gaussian process $X = \{X_t,\ t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and the associated triple $(\imath, H, \mathcal{X})$. Let

$$\begin{aligned} u_t &= \sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} I_n(e_1(t_1)...e_1(t_n))e_n(t) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} \mathcal{H}_n(\pi(e_1))e_n(t) \end{aligned}$$

where $\{e_n\}_{n=1}^{\infty} \subset H$ is an ONB of H, $\mathcal{H}_n$'s are Hermite polynomials normalized as in Section 1.3.1.

(a) $u \in L_2(\Omega, H)$ iff $p > \frac{1}{2}$, since

$$E\|u\|_H^2 = E\sum_{n=1}^{\infty} \frac{1}{n^{2p}} \frac{1}{n!} \mathcal{H}_n^2(\pi(e_1)) = \sum_{n=1}^{\infty} \frac{1}{n^{2p}}.$$

(b) u is Ogawa integrable with respect to ONB $\{e_n\}_{n=1}^{\infty}$ if $p > \frac{1}{2}$.

We have

$$\sum_{n=1}^{\infty} (u, e_n)\pi(e_n) = \sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} \mathcal{H}_n(\pi(e_1))\pi(e_n).$$

We need to check when this series converges in probability. Since (excluding the first term) the series consists of centered, integrable random variables adapted to the filtration $\mathcal{F}_n = \sigma\{\pi(e_1), \pi(e_2), ..., \pi(e_n)\}$, $(n \geq 1)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, it converges P-a.e. on the following set (see Proposition IV.6.2 in [38]):

$$\begin{aligned}\Omega_0 &= \{\sum_{n=1}^{\infty} E\{(\frac{1}{n^p}\frac{1}{\sqrt{n!}}\mathcal{H}_n(\pi(e_1))\pi(e_n))^2|\mathcal{F}_{n-1}\} < \infty\} \\ &= \{\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\frac{1}{n!}\mathcal{H}_n^2(\pi(e_1)) < \infty\}.\end{aligned}$$

But

$$E\{\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\frac{1}{n!}\mathcal{H}_n^2(\pi(e_1))\} = \sum_{n=1}^{\infty}\frac{1}{n^{2p}} < \infty,$$

therefore $P(\Omega_0) = 1$, and

$$\delta_e^o(u) = \sum_{n=1}^{\infty}\frac{1}{n^p}\frac{1}{\sqrt{n!}}\mathcal{H}_n(\pi(e_1))\pi(e_n).$$

(c) $u \in \mathcal{D}(I^s)$ iff $p > \frac{1}{2}$. We need to show $L_2(\Omega)$ convergence of the series defining the Skorohod integral of u. This can be proved as follows:

$$\begin{aligned}&\sum_{n=1}^{\infty}(n+1)!\|\frac{1}{n^p}\frac{1}{\sqrt{n!}}(e_1(t_1)...e_1(t_n)e_n(t))\tilde{}\|^2_{H^{\otimes(n+1)}} \\ &= \sum_{n=1}^{\infty}\frac{n+1}{n^{2p}}\|\frac{1}{n+1}(e_1(t_1)...e_1(t_n)e_n(t) \\ &\qquad + \sum_{i=1}^{n} e_1(t_1)...e_1(t_{i-1})e_1(t)e_1(t_{i+1})...e_1(t_n)e_n(t_i))\|^2_{H^{\otimes(n+1)}} \\ &= 2 + \sum_{n=2}^{\infty}\frac{1}{n^{2p}}\|e_1...e_1e_n\|^2_{H^{\otimes(n+1)}} = 2 + \sum_{n=2}^{\infty}\frac{1}{n^{2p}}.\end{aligned}$$

Second equality follows from orthogonality of components under norm. Also, by property (3) of Multiple Wiener Integrals, we get

$$I^s(u) = \sum_{n=1}^{\infty}\frac{1}{n^p}\frac{1}{\sqrt{n!}}\mathcal{H}_n(\pi(e_1))\pi(e_n) - 1 = \delta_e^o(u) - 1.$$

(d) Malliavin derivative.

(d.1) $u_t \in \mathcal{D}(D^M)$ with $D^M u_t \in L_2(\mathcal{X}, H)$ for t fixed, if $p \geq \frac{1}{2}$.

For the above to be true the following series must converge.

$$\sum_{n=1}^{\infty} nn!\|\frac{1}{n^p}\frac{1}{\sqrt{n!}}\{e_1...e_1\}e_n(t)\|^2_{H^{\odot n}} = \sum_{n=1}^{\infty}\frac{1}{n^{2p-1}}e_n^2(t) < \infty,$$

because $\sum_{n=1}^{\infty} e_n(t)^2 < \infty$ ($e_n(t)$ are the Fourier coefficients of $C(\cdot,t)$ in H).

(d.2) $u \in \mathcal{D}(D^M)$ with $Du \in L_2(\Omega, H^{\otimes 2})$ iff $p > 1$.

Indeed

$$\sum_{n=1}^{\infty} nn!\|\frac{1}{n^p}\frac{1}{\sqrt{n!}}e_1...e_1e_n\|^2_{H^{\otimes(n+1)}} = \sum_{n=1}^{\infty}\frac{1}{n^{2p-1}} < \infty.$$

Thus for $\frac{1}{2} < p \leq 1$, $u \in \mathcal{D}(I^s)$ and can be Ogawa integrable with respect to some basis while $Du \notin L_2(\mathcal{X}, H^{\otimes 2})$.

(e) $u \in \mathcal{D}(D^G)$ if $p > \frac{3}{4}$.

Now we have to restrict our considerations from general probability space $(\Omega, \mathcal{F}, P)$ to the triple $(\imath, H, \mathcal{X})$ and Kolmogorov functional representation of the process X. This is because the Itô-Ramer integral requires linear and topological structures on a probability space. Moreover, let us assume that $\{e_n\}_{n=1}^{\infty} \subset \mathcal{X}^* \subset H$.

Fix $x \in \mathcal{X}$. Let $r \in (-1,1)$. Denote

$$u_t^n(r) = \frac{1}{n^p}\frac{1}{\sqrt{n!}}\mathcal{H}_n(e_1(x+rh))e_n(t).$$

We will use the following estimate for Hermite polynomials (see 7.125 [53]):

$$\begin{aligned}\mathcal{H}_{2n}(t) &= (-1)^n(2n-1)!!e^{t^2}[cos\sqrt{2n+\frac{1}{2}}t + O(1/\sqrt[4]{n})]\\ \mathcal{H}_{2n+1}(t) &= (-1)^n(2n-1)!!\sqrt{2n+1}e^{t^2}[sin\sqrt{2n+\frac{3}{2}}t + O(1/\sqrt[4]{n})]\end{aligned}$$

where $O(t)$ denotes a quantity such that $\frac{O(t)}{t} = O(1)$ and $O(1)$ denotes a quantity which is bounded as $t \to \infty$. Note that expressions:

$((2n-1)!!)^2/(2n)!$ and $((2n-1)!!)^2(2n+1)/(2n+1)!$ are identical and are of the order $n^{-\frac{1}{2}}$ as $n \to \infty$. Hence,

$$\|\sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} \mathcal{H}_n(e_1(x) + re_1(h))e_n(t)\|_H^2 \leq O(1) \sum_{n=1}^{\infty} \frac{1}{n^{2p+\frac{1}{2}}}.$$

Thus $u^n(\cdot) : (-1,1) \to H$ and $\sum_{n=1}^{\infty} u^n(r) \to u(x+rh)$ in H for every $r \in (-1,1)$ and $p > \frac{1}{4}$.

By properties of Hermite polynomials, a.e. $[\mu]$,

$$\begin{aligned} D^G(\frac{1}{n^p} \frac{1}{\sqrt{n!}} \mathcal{H}_n(e_1)e_n)(x+rh)(h) &= \frac{du^n(r)}{dr} \\ &= \frac{1}{n^p} \frac{1}{\sqrt{n!}} n\mathcal{H}_{n-1}(e_1(x) + re_1(h))(e_1, h)e_n \end{aligned}$$

so that $du^n(r)/dr : (-1,1) \to H$ is continuous. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} \frac{1}{n!} n^2 \mathcal{H}_{n-1}^2(e_1(x) + re_1(h))(e_1,h)^2 \leq O(1) \sum_{n=1}^{\infty} \frac{1}{n^{2p-1+\frac{1}{2}}}$$

and the last series converges for $p > \frac{3}{4}$. Hence the series $\sum_{n=1}^{\infty} du^n(r)/dr$ converges uniformly for $r \in (-1,1)$. Therefore, with the same proof as for a series of real valued functions, a.e. $[\mu]$,

$$D^G u(x)h = \sum_{n=1}^{\infty} \frac{du^n(r)}{dr}|_{r=0} = \sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} n\mathcal{H}_{n-1}(e_1(x))(e_1,h)e_n$$

for $p > \frac{3}{4}$. Note that for $p > \frac{3}{4}$, a.e. $[\mu]$,

$$(D^G u(x)h)(t) = D^M u_t(x)h = \sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} n\mathcal{H}_{n-1}(e_1(x))(e_1,h)e_n(t).$$

(f) By Theorem 1.5.3 $u \in \mathcal{D}(L)$ for $p > 1$.

Indeed, $u \in L_2(\mathcal{X}, H)$ (proved in **(a)**) and $D^G u \in L_2(\mathcal{X}, H^{\otimes 2})$ follows from

$$\begin{aligned} E\{\|D^G u\|_{H^{\otimes 2}}^2\} &= E\{\sum_{n=1}^{\infty} \frac{1}{n^{2p}} \frac{1}{n!} n^2 \mathcal{H}_{n-1}^2(e_1)\} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2p-1}} < \infty. \end{aligned}$$

Now assume that $p > \frac{3}{4}$. Then $D^G u \in H^{\otimes 2}$ a.e. $[\mu]$. Indeed, $\|D^G u\|_{H^{\otimes 2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \frac{1}{n!} n^2 \mathcal{H}_{n-1}^2(e_1(x))$ converges as in **(d.2)**. Also condition *(G-M)* holds, because, as noticed in **(e)**, we even have equality of derivatives. Finally, $D_{\cdot}(u,h)_H \in L_2(\mathcal{X}, H)$, in view of the following equalities:

$$\begin{aligned} E\|D_{\cdot}(u,h)\|_H^2 &= E\|\sum_{n=1}^{\infty} \frac{1}{n^p} \frac{1}{\sqrt{n!}} n \mathcal{H}_{n-1}(e_1(x))(e_n, h)_H e_1\|_H^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2p-1}} (e_n, h)_H^2 \end{aligned}$$

Also $u \in \mathcal{D}(I^s)$. As it can be seen from (1.3) in the proof of Theorem 1.4.1 and (1.7) in the proof of Proposition 1.5.3, $u \in \mathcal{D}(L)$. The above reasoning is more refined than that in [59].

Chapter 2

Kinematics of Hilbert Space Valued Stochastic Motion

2.1 Introduction

In this Chapter we study two topics. First, we want to adopt Nelson's intuitive ideas on kinematics of stochastic motion to Hilbert space valued stochastic motion. The results we obtain show strong relation between Nelson's regularity assumptions for a diffusion and some properties of Doléans measure of some martingale associated with the diffusion. Next we can see that the Brownian motion process that arises in this analysis plays similar role as its finite dimensional counterpart in the analysis of finite dimensional stochastic motion. For this, it is necessary to modify the stochastic integral of Metivier and Pellaumail [35] with respect to 2-cylindrical martingales. The properties of Doléans measure are used here extensively, which again emphasizes the role of Nelson's conditions.

2.2 Hilbert-Schmidt and Trace Class Operators on Hilbert Space

In the previous Chapters we identified tensor product of Hilbert spaces, $H^{\otimes 2}$, with Hilbert-Schmidt operators T on H by $(Th, g)_H = (T, h \otimes g)_{H^{\otimes 2}}$. Now we want to discuss an analogous identification connected with trace class (or nuclear) operators on H.

Let us think of H as a unitary space and take $H \otimes H$, a unitary space, with the usual scalar product defined by $(h \otimes g, k \otimes l)_{H \otimes H} = (h, k)_H (g, l)_H$. Now $H^{\otimes 2}$, as a tensor product of Hilbert spaces, identified with Hilbert-Schmidt operators on H, is the completion of $H \otimes H$ in this usual scalar product.

For any continuous, linear operator T on H with an N dimensional range ($N = 1, 2, \ldots$), there exist orthonormal bases $\{e_n\}_{n=1}^{\infty}, \{f_n\}_{n=1}^{\infty} \subset H$ such that $\forall h \in H$,

$$Th = \sum_{n=1}^{N} \lambda_n (h, e_n)_H f_n, \qquad \lambda_n > 0, n = 1, \ldots, N.$$

Let us identify such an operator T with the element $\sum_{n=1}^{N} \lambda_n (e_n \otimes f_n) \in H \otimes H$ and let us define a norm in $H \otimes H$ by

$$\| \sum_{n=1}^{N} \lambda_n (e_n \otimes f_n) \|_1 = \|T\|_1 = \sum_{n=1}^{N} \lambda_n$$

where $\|T\|_1$ denotes the trace class norm of the operator T. The Banach space $H \otimes_1 H$ is the completion of the unitary space $H \otimes H$ in the norm $\| \cdot \|_1$. Since the completion in the trace class norm of the space of continuous, linear operators on H with finite dimensional ranges is precisely the space of trace class operators on H, any element of $H \otimes_1 H$ can be uniquely identified with a trace class operator. Note that for $g \otimes h \in H \otimes_1 H$ we have

$$\|g \otimes h\|_1 = \|g \otimes h\|_{H^{\otimes 2}} = \|g\|_H \|h\|_H \tag{2.1}$$

and in general $\|\cdot\|_{L(H)} \leq \|\cdot\|_{H^{\otimes 2}} \leq \|\cdot\|_1$. For more details we refer to [19] and [35].

Identification of the spaces $H \otimes_1 H$ and $H^{\otimes 2}$ with subspaces of the space of linear operators on a Hilbert space H allows to define symmetric and positive elements of $H \otimes_1 H$ and $H^{\otimes 2}$ (see [35]).

Definition 2.2.1 *We say that an element $b \in H \otimes_1 H$, or $b \in H^{\otimes 2}$, is symmetric (positive) if the associated linear operator is self-adjoint (positive), that is if $(bh, g)_H = (h, bg)_H$, i.e. $b = b^*$ where b^* denotes the adjoint operator, $((bh, h)_H \geq 0)$ $\forall h, g \in H$.*

2.3 Kinematics of Stochastic Motion

Let H be a separable, real Hilbert space. We assume that $\{X_t\}_{t\in I}$, $I = [0, T]$, $T > 0$, is an H-valued stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ and adapted to an increasing family of σ-fields $\{\mathcal{F}_t\}_{t\in I}$, where $\mathcal{F}_t \subset \mathcal{F}$, $\forall t \in I$. For simplicity we always assume that $X_0 = 0$. Let us recall that a stochastic process $\{X_t\}_{t\in I}$ is an H-valued martingale with respect to an increasing family of σ-fields $\{\mathcal{F}_t\}_{t\in I}$ if $\forall t \in I$, $X_t \in L_1(\Omega, \mathcal{F}_t, H)$ and $\forall s \leq t$, $E(X_t|\mathcal{F}_s) = X_s$ P-a.e.

We introduce, as in Nelson ([36]), the following regularity assumptions on the stochastic motion X_t and, mostly using Nelson's techniques, we study their consequences.

(R_0) $t \mapsto X_t$ is continuous from I to $L_1(\Omega, H)$

(R_1) Condition (R_0) holds and

$$DX_t = \lim_{\Delta \searrow 0} \{E(\frac{X_{t+\Delta} - X_t}{\Delta}|\mathcal{F}_t)\}$$

exists in $L_1((\Omega, \mathcal{F}_t), H)$ and $t \mapsto DX_t$ is continuous from I to $L_1(\Omega, H)$.

Note. DX_t defined in condition (R_1) can be interpreted as the **mean forward velocity**.

With an (R_1) process X we will associate the following process:

$$Y_t = X_t - \int_0^t DX_s ds, \qquad t \in I. \tag{2.2}$$

We will introduce one more regularity condition for a process Y, which may or may not be associated with an (R_1) process X.

$(R_{2.1})$

$$\sigma^2(t) = \lim_{\Delta \searrow 0} E\{\frac{(Y_{t+\Delta} - Y_t)^{\otimes 2}}{\Delta} | \mathcal{F}_t\}$$

exists in $L_1(\Omega, \mathcal{F}_t, H \otimes_1 H)$ and $t \mapsto Y_t^{\otimes 2}$, $t \mapsto \sigma^2(t)$ are continuous mappings from I to $L_1(\Omega, H \otimes_1 H)$.

$(R_{2.2})$ Same as condition $(R_{2.1})$ but with $H^{\otimes 2}$ replacing $H \otimes_1 H$.

We will now show that the mean forward velocity has a similar property as its analogues: the velocity in a physical phenomenon of motion and the mean forward velocity in stochastic motion in a finite dimensional space. The latter was investigated by Nelson in [36].

Theorem 2.3.1 *Let $\{X_t\}_{t\in I}$ be an (R_1) process. Then for any $u \leq v$ with $u, v \in I$*

$$E\{X_v - X_u | \mathcal{F}_u\} = E\{\int_u^v DX_s ds | \mathcal{F}_u\}.$$

Proof. Note that by assumption $t \mapsto X_t$ and $t \mapsto DX_t$ are continuous mappings from I to $L_1(\Omega, H)$ and so is $t \mapsto \int_0^t DX_s ds$.

Let $\varepsilon > 0$ be arbitrary. We will prove that

$$\begin{aligned}\mathcal{J} = \{t \in [u, v] : \ \forall u \leq s \leq t \ \|E\{X_s - X_u | \mathcal{F}_u\} \\ -E\{\int_u^s DX_r dr | \mathcal{F}_u\}\|_{L_1} \leq \varepsilon(s-u)\}\end{aligned}$$

is a closed subinterval of $[u, v] \subset I$.

Clearly $\mathcal{J} \neq \emptyset$ ($u \in \mathcal{J}$) and $\mathcal{J}$ is an interval. Denote $T_m = \sup\{t \in \mathcal{J}\}$ and we need to show that $T_m \in \mathcal{J}$. Let $\varepsilon_1 > 0$ be arbitrary. Then $\exists \delta > 0$ such that $\forall T_m - \delta \leq s \leq T_m$,

$$\|X_{T_m} - X_s\|_{L_1} < \frac{\varepsilon_1}{2} \quad \text{and} \quad \|\int_s^{T_m} DX_r dr\|_{L_1} < \frac{\varepsilon_1}{2},$$

by continuity of $t \mapsto X_t$ and $t \mapsto \int_0^t DX_r dr$.

Hence, for $s \in \mathcal{J}$,

$$\begin{aligned}
&\|E\{X_{T_m} - X_u|\mathcal{F}_u\} - E\{\int_u^{T_m} DX_r dr|\mathcal{F}_u\}\|_{L_1} \\
&\quad \leq \|E\{X_{T_m} - X_u|\mathcal{F}_u\} - E\{X_s - X_u|\mathcal{F}_u\}\|_{L_1} \\
&\qquad + \|E\{X_s - X_u|\mathcal{F}_u\} - E\{\int_u^s DX_r dr|\mathcal{F}_u\}\|_{L_1} \\
&\qquad + \|E\{\int_s^{T_m} DX_r dr|\mathcal{F}_u\}\|_{L_1} \\
&\quad \leq \frac{\varepsilon_1}{2} + \varepsilon(s-u) + \frac{\varepsilon_1}{2} \leq \varepsilon_1 + \varepsilon(T_m - u).
\end{aligned}$$

Since the above holds for arbitrary $\varepsilon_1 > 0$, we get $T_m \in \mathcal{J}$.

Now it is enough to show that T_m can not be smaller than v. If $T_m < v$ then $\exists \eta > 0$, $T_m + \eta \leq v$, such that

$$\begin{aligned}
&\|E\{X_{T_m+\Delta} - X_{T_m}|\mathcal{F}_u\} - E\{\Delta DX_{T_m}|\mathcal{F}_u\}\|_{L_1} \\
&\quad \leq \|E\{X_{T_m+\Delta} - X_{T_m}|\mathcal{F}_{T_m}\} - E\{DX_{T_m}\}\|_{L_1} < \frac{\varepsilon}{2}\Delta
\end{aligned}$$

and

$$\|\Delta DX_{T_m} - \int_{T_m}^{T_m+\Delta} DX_r dr\|_{L_1} < \frac{\varepsilon}{2}\Delta$$

for $0 < \Delta \leq \eta$.

First estimate follows from definition of mean forward derivative and contractivity of conditional expectation (Theorem 4, Chapter V, [15]). Second estimate is a consequence of the fundamental theorem for Bochner integral (Theorem 9, Chapter II, [15]), in view of continuity of the mapping $t \mapsto DX_t$.

Therefore

$$\|E\{X_{T_m+\Delta} - X_u|\mathcal{F}_u\} - E\{\int_u^{T_m+\Delta} DX_r dr|\mathcal{F}_u\}\|_{L_1}$$
$$\leq \|E\{X_{T_m+\Delta} - X_{T_m}|\mathcal{F}_u\} - E\{\int_{T_m}^{T_m+\Delta} DX_r dr|\mathcal{F}_u\}\|_{L_1}$$
$$+\varepsilon(T_m - u) \leq \varepsilon[(T_m + \Delta) - u].$$

Thus $T_m < T_m + \eta \in \mathcal{J}$ which can not happen.

□

Next theorem is an immediate consequence of Theorem 2.3.1.

Theorem 2.3.2 *Let $\{X_t\}_{t\in I}$ has property (R_1), then $Y_t = X_t - \int_0^t DX_s ds$ is an H-valued martingale.*

We also have the following connection between the processes Y and σ^2.

Theorem 2.3.3 *Let Y be as in Theorem 2.3.2 and has either property $(R_{2.1})$ or $(R_{2.2})$. Let $u \leq v$, $u, v \in I$, then*

$$E\{(Y_v - Y_u)^{\otimes 2}|\mathcal{F}_u\} = E\{\int_u^v \sigma^2(t)dt|\mathcal{F}_u\}.$$

Proof. Because of the martingale property of Y,

$$E\{(Y_v - Y_u)^{\otimes 2}|\mathcal{F}_u\} = E\{Y_v^{\otimes 2} - Y_u^{\otimes 2}|\mathcal{F}_u\}$$

and we are exactly in the same situation as in Theorem 2.3.1, namely $t \mapsto Y_t^{\otimes 2}$, $t \mapsto DY_t^{\otimes 2} = \sigma^2(t)$ and $t \mapsto \int_0^t \sigma^2(s)ds$ are all continuous mappings from I to either $L_1(\Omega, H \otimes_1 H)$ or $L_1(\Omega, H^{\otimes 2})$. Hence the assertion follows from the proof of Theorem 2.3.1.

$\square$

Corollary 2.3.1 *Let Y and σ^2 be as in Theorem 2.3.3 but we restrict ourselves to the case of $(R_{2.1})$ processes only. Then $\{\|Y_t\|_H^2\}_{t\in T}$ is an (R_1) process with*

$$D\|Y_t\|_H^2 = tr\sigma^2(t).$$

Hence

$$E\{\|Y_v\|_H^2 - \|Y_u\|_H^2|\mathcal{F}_u\} = E\{\int_u^v tr\sigma^2(t)dt|\mathcal{F}_u\},$$

for $u \leq v$, $u, v \in I$.

Proof. With an element $h \otimes h \in H \otimes H$ we associated a nuclear operator b and the following holds:

$$|tr(h\otimes h)| = |trb| \leq \|b\|_1 = \|h\otimes h\|_1 = \|h\|_H^2,$$

the inequality being valid for any trace class operator. Therefore

$$\begin{aligned}
&E\{\|E\{\frac{(Y_{t+\Delta}-Y_t)^{\otimes 2}}{\Delta}|\mathcal{F}_t\} - \sigma^2(t)\|_1\} \\
&\geq E\{|tr(E\{\frac{(Y_{t+\Delta}-Y_t)^{\otimes 2}}{\Delta}|\mathcal{F}_t\} - \sigma^2(t))|\} \\
&= E\{|E\{\frac{\|Y_{t+\Delta}-Y_t\|_H^2}{\Delta}|\mathcal{F}_t\} - tr\sigma^2(t)|\} \\
&= \|E\{\frac{\|Y_{t+\Delta}\|_H^2 - \|Y_t\|_H^2}{\Delta}|\mathcal{F}_t\} - tr\sigma^2(t)\|_{L_1(\Omega)}
\end{aligned}$$

The last equality follows from martingale property of Y. Since Y is an $(R_{2.1})$ process the last expression converges to zero as $\Delta \searrow 0$. Also, because $|trT| \leq \|T\|_1$ for a trace class operator T, the mapping $t \mapsto D\|Y_t\|_H^2 = tr\sigma^2(t)$ is continuous from I to $L_1(\Omega)$. Finally, the mapping $t \mapsto \|Y_t\|_H^2$ from I to $L_1(\Omega)$ is continuous because

$$\begin{aligned}
E\{|\ \|Y_t\|_H^2 - \|Y_s\|_H^2|\} &= E\{|\ \|Y_t^{\otimes 2}\|_1 - \|Y_s^{\otimes 2}\|_1|\} \\
&\leq E\{\|Y_t^{\otimes 2} - Y_s^{\otimes 2}\|_1\}
\end{aligned}$$

and the mapping $t \mapsto Y_t^{\otimes 2}$ is assumed to be continuous from I to $L_1(\Omega, H \otimes_1 H)$. The last assertion in the Corollary follows from Theorem 2.3.1.

$\square$

One can show that Nelson's requirement on existence of the process σ^2 can be expressed directly in terms of process X, namely one can assume convergence of $E\{\frac{1}{\Delta}(X_{t+\Delta} - X_t)^{\otimes 2}|\mathcal{F}_t\}$ either in $L_1(\Omega, H \otimes_1 H)$ or $L_1(\Omega, H^{\otimes 2})$. Because in this case, both $X^{\otimes 2}$ and $Y^{\otimes 2}$ are integrable and $\|h \otimes h\|_{H\otimes_1 H} = \|h\|_{H^{\otimes 2}} = \|h\|_H^2$ (by equality (2.1)), adding an extra assumption on DX_t to be in $L_2(\Omega, H)$ is reasonable. Let us formulate our assertion.

Theorem 2.3.4 *Assume that $\{X_t\}_{t\in I}$ and $\{Y_t\}_{t\in I}$ satisfy assumption ($R_{2.1}$), where $Y_t = X_t - \int_0^t DX_s ds$. We also require that the mapping $t \mapsto DX_t$ be continuous from I to $L_2(\Omega, H)$. The the following statements are equivalent:*

$$(1) \qquad E\{\frac{(X_{t+\Delta} - X_t)^{\otimes 2}}{\Delta}|\mathcal{F}_t\} \to \beta(t) \quad \text{in } L_1(\Omega, H \otimes_1 H) \text{ as } \Delta \searrow 0.$$

$$(2) \qquad E\{\frac{(Y_{t+\Delta} - Y_t)^{\otimes 2}}{\Delta}|\mathcal{F}_t\} \to \sigma^2(t) \quad \text{in } L_1(\Omega, H \otimes_1 H) \text{ as } \Delta \searrow 0.$$

If the limits exist then $\beta(t) = \sigma^2(t)$ P-a.e. The above remains correct if the assumption ($R_{2.1}$) and the space $L_1(\Omega, H \otimes_1 H)$ are replaced with assumption ($R_{2.2}$) and space $L_1(\Omega, H^{\otimes 2})$.

Proof. Consider

$$\begin{aligned}
&(Y_{t+\Delta} - Y_t)^{\otimes 2} \\
&\quad = (X_{t+\Delta} - X_t)^{\otimes 2} - (X_{t+\Delta} - X_t) \otimes \int_t^{t+\Delta} DX_s \\
&\quad - \int_t^{t+\Delta} DX_s ds \otimes (X_{t+\Delta} - X_t) + (\int_t^{t+\Delta} DX_s)^{\otimes 2}.
\end{aligned}$$

Regardless of (1) and (2),

$$\frac{1}{\Delta}(\int_t^{t+\Delta} DX_s ds)^{\otimes 2} = \Delta(\frac{1}{\Delta}\int_t^{t+\Delta} DX_s ds)^{\otimes 2} \to 0$$

in $L_1(\Omega, H \otimes_1 H)$ as $\Delta \to 0$. Indeed,

$$E\{\|\Delta(\frac{1}{\Delta}(\int_t^{t+\Delta} DX_s ds)^{\otimes 2}\|_{H\otimes_1 H}\} = \Delta E\{\|\frac{1}{\Delta}\int_t^{t+\Delta} DX_s ds\|_H^2\} \to 0$$

since the mapping $t \to DX_t$ is continuous from I to $L_2(\Omega, H)$. Now consider the second term in the expansion of $(Y_{t+\Delta} - Y_t)^{\otimes 2}$. Under the assumption $(R_{2.1})$ on X we have by equality (2.1)

$$\begin{aligned}
&(E\{\|(X_{t+\Delta} - X_t) \otimes (\frac{1}{\Delta}\int_t^{t+\Delta} DX_s ds)\|_{H\otimes_1 H}\})^2 \\
&\leq E\{\|X_{t+\Delta} - X_t\|_H^2\} E\{\|\frac{1}{\Delta}\int_t^{t+\Delta} DX_s ds\|_H^2\} \\
&= E\{\|(X_{t+\Delta} - X_t)^{\otimes 2}\|_{H\otimes_1 H}\} E\{\|\frac{1}{\Delta}\int_t^{t+\Delta} DX_s ds\|_H^2\}
\end{aligned}$$

and, as $\Delta \searrow 0$, the first factor converges to zero in view of the assumption $(R_{2.1})$ while the second factor converges to $E\{\|DX_t\|_H^2\}$ because the mapping $t \mapsto DX_t$ is continuous from I to $L_2(\Omega, H)$. The same argument works for the third term in the expansion of $(Y_{t+\Delta} - Y_t)^{\otimes 2}$. Hence the equivalence of (1) and (2) follows by contractivity of conditional expectation. The last assertion of the Theorem follows from equality (2.1).

□

2.4 Stochastic Integration in Hilbert Space

Stochastic integration in Hilbert and Banach spaces is a subject of the monograph [35]. We recall here the isometric integral and we explain why it is not a sufficient

tool for solving the problem of recovering the noise from stochastic motion in Hilbert space. Then we recall the concept of 2-cylindrical martingales and, using the main ideas in [35], Chapter 16, we introduce a stochastic integral with respect to an H-valued $(R_{2.1})$ martingale. This stochastic integral admits a wider class of processes as integrands than the isometric integral and the cylindrical integral of [35], Chapter 16. We use the results of this section to give a partial answer to the question of the role of Brownian motion in stochastic motion in Hilbert space.

2.4.1 General Assumptions and their Consequences

In what follows we always assume that the filtration $\{\mathcal{F}_t\}_{t\in I=[0,T]}$ satisfies *usual conditions*, i.e. that:

(1) The filtration is right-continuous, i.e. $\forall t \in I$, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$.

(2) The probability space $(\Omega, \mathcal{F}_T, P)$ is complete and $\forall t \in I$, $\mathcal{F}_t$ contains all sets of P-measure zero, which belong to $\mathcal{F}_T$.

Two processes X and Y are said to be **P-equivalent** if $P(\{\omega : \exists t, X_t(\omega) \neq Y_t(\omega)\}) = 0$.

A stochastic process X is called **cadlag** (in French: continue à droite et admet une limite à gauche) if $\forall \omega \in \Omega$ the sample path $t \mapsto X_t(\omega)$ is right continuous and has left limits.

Definition 2.4.1 . *We define $\mathcal{M}_I^2$, the space of H-valued, cadlag, square integrable martingales (i.e. $E\{\|M_T\|_H^2\} < \infty$, which implies that $\sup_{t\in I} E\{\|M_t\|_H^2\} < \infty$) and we identify P-equivalent processes. In the case of $H = \mathbf{R}$ we will write $\mathcal{M}_I^2(\mathbf{R})$ to avoid a possible confusion.*

As it is shown in [35], Section 10.1, $\mathcal{M}_I^2$ is a Hilbert space with a scalar product defined by $(M,N)_{\mathcal{M}_I^2} = E\{(M_T, N_T)_H\}$, because there is a one-to-one cor-

respondence (up to P-equivalence) between H-valued, cadlag, square integrable martingales and elements of $L_2(\Omega, \mathcal{F}_T, H)$.

Remark 2.4.1 *If $Y_t = X_t - \int_0^t DX_s ds$ with X an (R_1) process then there exists a version Y' of the process Y (i.e. $\forall t \in I$, $P(Y_t = Y_t') = 1$) which is cadlag. Moreover, if $E\{\|Y_T\|_H^2\} < \infty$ then $Y' \in \mathcal{M}_I^2$ and Y and Y' are P-equivalent.*

Indeed, Y is an $L_1(\Omega, H)$ continuous martingale and therefore it has a cadlag version by Theorem D-6 in [50].

In view of the last Remark, from now on we assume that $Y_t = X_t - \int_0^t DX_s ds$ is a cadlag martingale and if Y is an $(R_{2.1})$ or $(R_{2.2})$ process then $Y \in \mathcal{M}_I^2$ follows from equality (2.1).

2.4.2 Doléans Measure of $(R_{2.1})$ Elements of $\mathcal{M}_I^2$

First we recall basic definitions and properties of Doléans measure in general. Let us recall our general assumption that $\{X_t\}_{t \in I=[0,T]}$ is an H-valued stochastic process, where H is a separable Hilbert space.

A set $A = F \times (s, t] \subset \Omega \times I$, where $F \in \mathcal{F}_s$ is called a **predictable rectangle** and the collection of predictable rectangles is denoted here by $\Re$. The σ-field generated by $\Re$ is called the σ-field of **predictable sets** and denoted by $\mathcal{P}$. A stochastic process is called **predictable** if it is $\mathcal{P}$ measurable.

Assume that for a process $\{X_t\}_{t \in I}$, $X_t \in L_1(\Omega, H)$, $\forall t \in I$. For each $A = F \times (s, t] \in \Re$ define

$$\alpha(A) = E\{1_F(X_t - X_s)\}.$$

If α extends to a σ-additive, H-valued measure on $\mathcal{P}$, then it is called **Doléans**

measure of process X. The following results on Doléans measure are proved in [35], Sections 2.6 and 14.3.

Theorem 2.4.1 *Let $M \in \mathcal{M}_I^2$, i.e. it is a cadlag, square integrable martingale with values in a separable Hilbert space H. Then:*

(1) $\{\|M_t\|_H^2\}_{t\in I}$ has Doléans measure, which will be denoted by $\alpha_{\|M\|}$.

(2) $\{M_t^{\otimes 2}\}_{t\in I}$ has Doléans measure with values in the set of positive, symmetric elements of $H \otimes_1 H$ (see Definition 2.2.1). We will denote this measure by α_M. Moreover,

$$\alpha_{\|M\|} = tr\alpha_M = |\alpha_M|$$

where $|\cdot|$ denotes variation of a measure.

(3) There exists a unique, up to $\alpha_{\|M\|}$ equivalence, predictable $H\otimes_1 H$-valued process Q_M, such that

$$\alpha_M(G) = \int_G Q_M d\alpha_{\|M\|}, \quad \forall G \in \mathcal{P}.$$

The process Q_M takes its values in the set of positive, symmetric elements of $H \otimes_1 H$ and

$$trQ_M(\omega,t) = \|Q_M(\omega,t)\|_{H\otimes_1 H} = 1 \quad a.e.\ \alpha_{\|M\|}.$$

Now we will see how Nelson's regularity assumptions interfere with properties of Doléans measure. Because Doléans measure of an $\mathcal{M}_I^2$ martingale takes its values in trace class operators on H, from now on we restrict our considerations to $(R_{2.1})$ processes only.

Theorem 2.4.2 *Let X be an (R_1) process and $Y_t = X_t - \int_0^t DX_s ds$ be an $(R_{2.1})$ element of $\mathcal{M}_I^2$. Then, with the notation of previous sections:*

(1) There exists a jointly $\mathcal{F} \otimes \mathcal{B}(I)$ measurable versions of $D\|Y\|_H^2$ and σ^2.

Let us further consider these jointly measurable versions and let us denote these versions by the same symbols. Also let us denote by $E_{P\otimes\lambda}$ expectation with respect to the measure $P \otimes \lambda$, where λ is Lebesgue measure on I.

(2) Doléans measure $\alpha_{\|Y\|}$ of the process $\|Y\|^2$ has density $E_{P\otimes\lambda}\{D\|Y\|_H^2|\mathcal{P}\} = E_{P\otimes\lambda}(tr\sigma^2|\mathcal{P}\}$ with respect to the measure $P \otimes \lambda$,

$$d\alpha_{\|Y\|} = E_{P\otimes\lambda}\{tr\sigma^2|\mathcal{P}\}d(P \otimes \lambda).$$

(3) Doléans measure α_Y of the process $Y^{\otimes 2}$ has density $E_{P\otimes\lambda}\{\sigma^2|\mathcal{P}\}$ with respect to the measure $P \otimes \lambda$,

$$d\alpha_Y = E_{P\otimes\lambda}\{\sigma^2|\mathcal{P}\}d(P \otimes \lambda).$$

and its density, Q_Y, with respect to the measure $\alpha_{\|Y\|}$ satisfies the following equation:

$$\sigma^2 = Q_Y tr\sigma^2 \quad a.e. \ P \otimes \lambda.$$

Proof. (1) Note that the mapping $t \mapsto \sigma^2(t)$ is continuous from I to $L_1(\Omega, H \otimes_1 H)$ and hence, the mapping $t \mapsto tr\sigma^2(t)$ is continuous from I to $L_1(\Omega)$. Therefore (1) follows by Theorem 1.2 in [16].

(2) For predictable rectangles $F \times (s,t] \in \Re$ we have

$$\begin{aligned} \alpha_{\|Y\|}(F \times (s,t]) &= E\{1_F(\|Y_t\|_H^2 - \|Y_s\|_H^2)\} \\ &= E\{1_F \int_s^t tr\sigma^2(r)dr\} \\ &= \int_{F\times(s,t]} tr\sigma^2 d(P \otimes \lambda) \end{aligned}$$

The expressions at the beginning and at the end of the equality, both extend to measures on $\mathcal{P}$ and these extensions agree on generators, $\Re$, of $\mathcal{P}$, hence they are identical.

(3) An analogous equality as in the proof of (2) holds also here and the same extension argument can be applied in this case to yield

$$d\alpha_Y = E_{P\otimes\lambda}\{\sigma^2|\mathcal{P}\}d(P\otimes\lambda).$$

Now, we have the following situation: $\alpha_Y \ll \alpha_{\|Y\|} \ll P\otimes\lambda$ and therefore

$$\frac{d\alpha_Y}{d(P\otimes\lambda)} = \frac{d\alpha_Y}{d\alpha_{\|Y\|}}\frac{d\alpha_{\|Y\|}}{d(P\otimes\lambda)}.$$

Indeed, let $\{h_n\}_{n=1}^{\infty}$ be a dense subset of H. Denote

$$\theta = \frac{d\alpha_Y}{d(P\otimes\lambda)}, \quad \kappa = \frac{d\alpha_Y}{d\alpha_{\|Y\|}}\frac{d\alpha_{\|Y\|}}{d(P\otimes\lambda)}$$

and define

$$A_+ = \bigcup_{n,m}\{((\theta-\kappa)h_n, h_m)_H > 0\}, \qquad A_- = \bigcup_{n,m}\{((\theta-\kappa)h_n, h_m)_H < 0\}.$$

Because

$$\int_{A_+}((\theta-\kappa)h_n, h_m)_H d(P\otimes\lambda) = \int_{A_-}((\theta-\kappa)h_n, h_m)_H d(P\otimes\lambda) = 0$$

we get that $P\otimes\lambda$ a.e. $\forall n, m = 1, 2..., ((\theta-\kappa)h_n, h_m)_H = 0$, implying $\theta = \kappa$ $P\otimes\lambda$ a.e.

$\square$

2.4.3 Inadequacy of the Isometric Stochastic Integral

We recall the isometric stochastic integral of Metivier and Pellaumail [35] with respect to martingales from $\mathcal{M}_T^2$. We begin with defining the class of integrands, however we will restrict ourselves to processes with values only in linear operators on H. For a slightly more general case of operators from one Hilbert space H to another Hilbert space K we refer to [35].

According to Theorem 2.4.1, with a martingale $M \in \mathcal{M}_I^2$, we can uniquely associate predictable, $H \otimes_1 H$-valued process Q_M. We can think about Q_M as taking its values in the space of trace class operators on H using the usual identification of Section 2.2. Since the values of Q_M are actually self-adjoint, positive operators, there exists a square-root, denoted by $Q_M^{\frac{1}{2}}$, which is a Hilbert-Schmidt operator and is defined by $Q_M^{\frac{1}{2}} \circ Q_M^{\frac{1}{2}} = Q_M$.

Definition 2.4.2 *Let $M \in \mathcal{M}_I^2$. We call $L^*(H, \mathcal{P}, M)$ the space of processes X, the values of which are (possibly non-continuous) linear operators on H, with the following properties:*
(1) For every $(\omega, t) \in \Omega \times I$, the domain $\mathcal{D}(X(\omega, t))$ of $X(\omega, t)$ contains $Q_M^{\frac{1}{2}}(H)$.
(2) For every $h \in H$ the H-valued process $X \circ Q_M^{\frac{1}{2}}(h)$ is predictable.
(3) For every $(\omega, t) \in \Omega \times I$, $X(\omega, t) \circ Q_M^{\frac{1}{2}}(\omega, t)$ is a Hilbert-Schmidt operator and

$$\int_{\Omega \times I} \| X \circ Q_M^{\frac{1}{2}} \|^2_{H^{\otimes 2}} < \infty.$$

Proposition 2.4.1 *For every $X, Y \in L^*(H, \mathcal{P}, M)$ the process $X \circ Q_M \circ Y^*$ takes its values in trace class operators on H, it is predictable and*

$$\int_{\Omega \times I} tr(X \circ Q_M \circ Y^*) d\alpha_{\|M\|} < \infty.$$

The bilinear form $(X, Y) \mapsto \int_{\Omega \times I} tr(X \circ Q_M \circ Y^*) d\alpha_{\|M\|}$ is a scalar product on $L^*(H, \mathcal{P}, M)$ and for this scalar product this space complete.

A process X is called **elementary** if it is of the following form:

$$X(\omega, t) = \sum_{i=1}^{n} u_i 1_{A_i}(\omega, t) \tag{2.3}$$

where $u_i, i = 1, ..., n$ are continuous, linear operators on H and $\{A_i\}_{i=1}^n \subset \Re$.

Note. We will always assume that if $A_i = F_i \times (s_i, t_i]$, $A_j = F_j \times (s_j, t_j]$ and $i \neq j$, then $(s_i, t_i] \cap (s_j, t_j] = \emptyset$ by taking more refined partition of I if necessary.Observe that elementary processes are in $L^*(H, \mathcal{P}, M)$.

Notation. The closure of the space of elementary processes in $L^*(H, \mathcal{P}, M)$ will be denoted by $\Lambda^2(H, \mathcal{P}, M)$. Let $M \in \mathcal{M}_I^2$. The **isometric stochastic integral** is the unique isometric linear mapping from $\Lambda^2(H, \mathcal{P}, M)$ into $\mathcal{M}_I^2$, such that the image of $X = 1_{F\times(s,r]}u$, for every predictable rectangle $F \times (s, r]$ and continuous linear operator $u \in L(H)$, is the martingale $\{1_F[u(M_{r\wedge t}) - u(M_{s\wedge t})]\}_{t\in I}$.

We conclude this section with an example motivating extension of the isometric stochastic integral. Nelson's idea to recover the noise from stochastic motion described by a process X was to compute

$$W_t = \int_0^t \sigma^{-1}(s)dY_s.$$

Under some conditions the process W turned out to be a Brownian motion. Also the stochastic motion X would satisfy the following stochastic integral equation

$$X_t = X_0 + \int_0^t DX_s ds + \int_0^t \sigma(s)dW_s$$

(see Paragraph 11 in [36]). In our case, if σ^{-1} existed and were an admissible process for the isometric stochastic integral then we would get $W_t = \int_0^t \sigma^{-1}(s)dY_s \in \mathcal{M}_I^2$. However this does not happen and we give an example explaining why the isometric stochastic integral is not a sufficient tool for Nelson's technique.

Let us make some regularity assumptions about the process σ^2.

Definition 2.4.3 *We will call the process σ^2* **regular** *if:*
(1) σ^2 is a predictable process which takes its values in positive, self-adjoint elements of $H \otimes_1 H$.

(2) $\forall(\omega,t)\in\Omega\times I$ *all eigenvalues* $\lambda_n(\omega,t)$ *of* $\sigma^2(\omega,t)$ *are strictly positive,* $\lambda_n(\omega,t)>0,\ n=1,2...$

Thus for a regular process σ^2, for every $(\omega,t)\in\Omega\times I$, there exists an orthonormal basis $\{h_n\}_{n=1}^{\infty}\subset H$ such that

$$\sigma^2(\omega,t)(h)=\sum_{n=1}^{\infty}\lambda_n(\omega,t)(h,h_n(\omega,t))_H h_n(\omega,t),\quad \forall h\in H$$

with $\lambda_n(\omega,t)>0,\ n=1,2...,\ \sum_{n=1}^{\infty}\lambda_n(\omega,t)=\|\sigma^2(\omega,t)\|_1$.

Also, there exists the square-root of σ^2, denoted by σ, which is a Hilbert-Schmidt operator (see [35]) and has the following representation:

$$\sigma(h)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}(h,h_n)_H h_n,\quad \forall h\in H$$

(we will usually drop the dependence on (ω,t)).

Note. From now on we will always assume that the process σ^2 is regular.

The generalized inverse of σ ([32]), denoted by σ^-, is defined by a composition $P_{[Ker(\sigma)]^\perp}\circ\sigma^{-1}\circ P_{cl(Ran(\sigma))}$, where $P_{[Ker(\sigma)]^\perp}$ and $P_{cl(Ran(\sigma))}$ are respectively projections on the orthogonal complement of the kernel space and on the closure of the range of σ and σ^{-1} is the inverse relation to the operator σ. Note that because σ^2 is regular, $cl(Ran(\sigma))=H$. Then σ^- takes the following form:

$$\sigma^-(h)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}(h,h_n)_H h_n\quad \forall h\in Ran(\sigma).$$

It follows from Theorem 2.10, Corollary 2.13 in [32] and regularity of σ^2 that σ and σ^- are predictable processes. By Theorem 2.4.2 we have

$$Q_Y=\frac{\sigma^2}{tr\sigma^2}.$$

Hence, $\mathcal{D}(\sigma^-(\omega,t)) \supset Ran(\sigma) = Q_Y^{\frac{1}{2}}(H)$.

Moreover, $\sigma^- \circ Q_Y^{\frac{1}{2}}$ is a predictable process, so that requirements (1) and (2) of Definition 2.4.2 are satisfied. However (see (2.12) in [32]),

$$\sigma^- \circ Q_Y^{\frac{1}{2}} = \frac{1}{\sqrt{tr\sigma^2}}\sigma^- \circ \sigma = \frac{1}{\sqrt{tr\sigma^2}}P_{[Ker(\sigma)]^\perp} = \frac{1}{\sqrt{tr\sigma^2}}Id_H$$

is not a Hilbert-Schmidt operator unless H is finite dimensional. Thus $\sigma^- \notin \Lambda^2(H,\mathcal{P},Y)$.

2.4.4 Cylindrical Stochastic Integration

We learned in Section 2.4.3 that the requirements imposed on integrands by the isometric integral are too restrictive if one wishes to recover the noise from a stochastic motion in a Hilbert space by using Nelson's technique. We want to preserve an $(R_{2.1})$ martingale as an integrator, therefore our primary goal is to increase the class of integrands, to include the process σ^-. Failure of the isometric stochastic integral in Nelson's procedure is due to non-existence of a standard H-valued Brownian motion. In order to realize a Brownian motion process with covariance associated with an identity operator on H one has to abandon H-valued processes. It turns out that one can solve this problem with help of cylindrical processes. It is enough for our purposes to study 2-cylindrical H-martingales, with H - a separable Hilbert space. For the full theory we refer to [35]. Even though we are mainly interested in stochastic integration with respect to an $(R_{2.1})$ martingale, now treated as a 2-cylindrical H-martingale, eventually we want to be able to integrate with respect to cylindrical Brownian motion. Therefore we recall the definition of stochastic integral in full generality.

Definition 2.4.4 *(1) A 2-cylindrical $L_2(\Omega,\mathcal{F})$-valued H-random element $\tilde{U}$ is a continuous linear mapping from H to $L_2(\Omega,\mathcal{F})$.*

(2) We call $\{\tilde{M}_t\}_{t\in I}$ a 2- cylindrical H-martingale if each $\tilde{M}_t$ is a 2-cylindrical, $L_2(\Omega, \mathcal{F}_t)$-valued H-random element and $\forall h \in H$ the real valued process $\{\tilde{M}_t(h)\}_{t\in I}$ is a martingale relative to $\{\mathcal{F}_t\}_{t\in I}$.

Note. The space of 2-cylindrical H-martingales can be identified with the space $L(H, \mathcal{M}_I^2(\mathbf{R}))$.

Next we will recall definition of the quadratic Doléans measure of a 2-cylindrical H-martingale.

Definition 2.4.5 *For a 2-cylindrical H-martingale $\tilde{M}$, the* **quadratic Doléans function** $d_{\tilde{M}}$ *is the additive, $(H \otimes_1 H)^*$-valued function on $\Re$ defined by*

$$\langle b, d_{\tilde{M}}(F \times (s,t])\rangle = E\{1_F(\tilde{M}_t \otimes \tilde{M}_t(b) - \tilde{M}_s \otimes \tilde{M}_s(b))\}$$

where, for every $t \in I$, $\tilde{M}_t \otimes \tilde{M}_t$ denotes the continuous linear mapping from $H \otimes_1 H$ into $L_1(\Omega, \mathcal{F}_t)$ which is the linear continuous extension of the mapping $b = h \otimes g \mapsto \tilde{M}_t(h)\tilde{M}_t(g)$. Also, above, $b \in H \otimes_1 H$, $F \in \mathcal{F}_t$, $s, t \in I$, $s \le t$.

If $d_{\tilde{M}}$ extends to a σ-additive measure on $\mathcal{P}$ then the extension is called **quadratic Doléans measure** *of the 2-cylindrical H-martingale $\tilde{M}$ and will be denoted by $\alpha_{\tilde{M}}$.*

A simple condition for the existence of quadratic Doléans measure for a 2-cylindrical H-martingale $\tilde{M}$ is that for all $h \in H$, $\tilde{M}(h)$ had a cadlag version ([35]). Note that it assures existence of Doléans measure for a 2-cylindrical martingale associated with a martingale $M \in \mathcal{M}_I^2$ by $\tilde{M}_t(h) = (\dot{M}_t, h)_H$, $\forall h \in H$.

Example 2.4.1 *Cylindrical Brownian motion.*

Let us recall (see Proposition 4.11 in [35]) that the H-valued Brownian motion W has covariance $C \in H \otimes_1 H$, i.e. $\forall h \otimes g \in H \otimes H$,

$$(E\{W_t^{\otimes 2}\}, h \otimes g)_{H^{\otimes 2}} = tC(h \otimes g).$$

C, being a trace class operator, cannot be an identity on infinite dimensional Hilbert space H. If one wants to have a Brownian motion with $C = Id_H$, one has to consider cylindrical processes.

We say that a process $\{\tilde{W}_t\}_{t\in I}$ is a **cylindrical Brownian motion** if:

(1) $\forall h \in H$, $\{\tilde{W}_t(h)\}_{t\in I}$ is a Brownian motion.

(2) $\forall h, g \in H$, $t \in I$, $E\{\tilde{W}_t(h)\tilde{W}_t(g)\} = tC(h,g)$ where C is a continuous bilinear form on $H \times H$.

Note, that given any continuous bilinear form C on $H \times H$, there exists cylindrical Brownian motion with C as its covariance - see Paragraph 15.4 in [35]. In the case of $C(h,g) = (h,g)_H$, C is associated with an identity operator Id_H and we call the cylindrical Brownian motion **standard**. Note that standard cylindrical Brownian motion cannot be associated with any ordinary sense H-valued process.

Now we recall definition of cylindrical stochastic integral. We begin with a proposition which is an analogue of Theorem 2.4.1.

Proposition 2.4.2 *Let $\tilde{M}$ be a 2-cylindrical H-martingale and $\alpha_{\tilde{M}}$ its quadratic Doléans measure with bounded variation $|\alpha_{\tilde{M}}|$. There exists a process $Q_{\tilde{M}}$ with values in the set of positive elements of $(H \otimes_1 H)^*$ (i.e. $Q_{\tilde{M}}(h \otimes h) \geq 0$, $\forall h \in H$), such that for every $b \in H \otimes_1 H$ the real process $\langle b, Q_{\tilde{M}}\rangle$ is measurable for the $|\alpha_{\tilde{M}}|$-completion of the σ-field $\mathcal{P}$, it is defined up to $|\alpha_{\tilde{M}}|$-equivalence and has the property*

$$\langle b, \alpha_{\tilde{M}}(A)\rangle = \int_A \langle b, Q_{\tilde{M}}(\omega,t)\rangle\, |\alpha_{\tilde{M}}|(d\omega, dt) \tag{2.4}$$

$\forall b \in H \otimes_1 H$, $A \in \mathcal{P}$.

Now, if X is an elementary process of the form (2.3), we define for every $h \in H$,

$$(\int X d\tilde{M})_t(h) = \sum_{i=1}^{n} 1_{F_i}\{(\tilde{M}(u_i^*(h)))_{t_i \wedge t} - (\tilde{M}(u_i^*(h)))_{s_i \wedge t}\} \tag{2.5}$$

where u^* denotes the adjoint operator. The integral, $(\int X d\tilde{M})$, is a 2-cylindrical H martingale and for every $h \in H$ the real valued square integrable martingale $(\int X d\tilde{M})(h) \in \mathcal{M}_I^2(\mathbf{R})$ has norm given by (see 16.2.2 in [35])

$$\|(\int X d\tilde{M})(h)\|^2_{\mathcal{M}_I^2(\mathbf{R})} = \int \langle X^*(h) \otimes X^*(h), Q_{\tilde{M}} \rangle \, d|\alpha_{\tilde{M}}|. \tag{2.6}$$

Definition 2.4.6 **(A)** *$\tilde{L}(\tilde{M}, H)$ is the set of processes X with the following properties:*

(1) $\forall (\omega, t) \in \Omega \times I$, $X(\omega, t)$ is a linear operator on H with domain $\mathcal{D}(X(\omega, t))$ dense in H.

(2) Denoting by $X^*(\omega, t)$ the adjoint of $X(\omega, t)$, the linear form

$$\langle X^*(\omega, t)(h) \otimes X^*(\omega, t)(g), Q_{\tilde{M}}(\omega, t) \rangle$$

has $|\alpha_{\tilde{M}}|$-a.e. a unique continuous extension to $H \times H$ which results in a predictable process.

(3)

$$\mathcal{N}(X) = \sup_{\|h\| \le 1} \{\int_{\Omega \times I} \langle X^*(h) \otimes X^*(h), Q_{\tilde{M}}(\omega, t) \rangle \, d|\alpha_{\tilde{M}}|\}^{\frac{1}{2}} < \infty.$$

We define $\tilde{\Lambda}(\tilde{M}, H)$ - the closure of the class of elementary processes of the form (2.3) in the space $\tilde{L}(\tilde{M}, H)$.

(B) *The unique extension of the isometric mapping $X \mapsto (\int X d\tilde{M})$ given by (2.5), from the space of elementary processes into the space of 2-cylindrical H-martingales, to the isometric mapping from $\tilde{\Lambda}(\tilde{M}, H)$ into the space of 2-cylindrical H-martingales is called the* **stochastic integral** *and is denoted again by $X \mapsto (\int X d\tilde{M})$.*

Now we want to take advantage of the fact that the integrator in the cylindrical stochastic integral, which we consider, is actually a square integrable martingale. We therefore are able to express quadratic Doléans measure, its variation as well as the process $Q_{\tilde{M}}$ associated with the integrator in terms of its Doléans measure and the process Q_M, which are simpler objects. If we apply this analysis to an $(R_{2.1})$ integrator, the situation simplifies even more by use of our results from Section 2.4.2. Note that again, this is a consequence of Nelson's regularity assumptions on the stochastic motion.

Lemma 2.4.1 *The Doléans measure of a martingale $M \in \mathcal{M}_I^2$ and quadratic Doléans measure of $\tilde{M}$ coincide as $(H \otimes_1 H)^*$-valued measures on $\mathcal{P}$.*

Proof. Indeed, first note that $M_t^{\otimes 2} \in L_1((\Omega, \mathcal{F}_t); (H \otimes_1 H)^*)$. This is because if $T \in H \otimes_1 H$ then $T(h \otimes g) = (Th, g)_H$ extends uniquely to an element of $(H \otimes_1 H)^*$ with $\|T\|_{(H \otimes_1 H)^*} \leq \|T\|_1$ (see Paragraph 14.2 (2) in [35]). Therefore $M_t^{\otimes 2}$ is integrable. Also,

$$h \otimes g \mapsto M_t^{\otimes 2}(h \otimes g) = (M_t, h)_H (M_t, g)_H = \tilde{M}_t \otimes \tilde{M}_t(h \otimes g).$$

Hence $M_t^{\otimes 2} = \tilde{M}_t \otimes \tilde{M}_t$ as elements of $L_1((\Omega, \mathcal{F}_t); (H \otimes_1 H)^*)$. Therefore $\forall b \in H \otimes_1 H$, $F \in \mathcal{F}_t$, $s, t \in I$, $s \leq t$, we have,

$$\begin{aligned} \langle b, d_{\tilde{M}}(F \times (s,t]) \rangle &= E\{1_F(\tilde{M}_t \otimes \tilde{M}_t(b) - \tilde{M}_s \otimes \tilde{M}_s(b))\} \\ &= E\{1_F \left\langle b, M_t^{\otimes 2} - M_s^{\otimes 2} \right\rangle\} \\ &= \left\langle b, E\{1_F(M_t^{\otimes 2} - M_s^{\otimes 2})\} \right\rangle \\ &= \langle b, \alpha_M(F \times (s,t]) \rangle . \end{aligned}$$

Note that α_M is an $H \otimes_1 H$-valued measure and can be treated as an $(H \otimes_1 H)^*$-valued measure. Because $\tilde{M}$ is a 2-cylindrical martingale associated with $M \in \mathcal{M}_I^2$,

$d_{\tilde{M}}$ on the LHS of the above expression extends to $\alpha_{\tilde{M}}$, therefore $\alpha_{\tilde{M}} = \alpha_M$ as $(H \otimes_1 H)^*$-valued measures on $\mathcal{P}$.

$\square$

Now we explain how a cylindrical integral with respect to a square integrable martingale can be computed using only the Doléans measure and the associated process Q_M. In the case of a square integrable martingale we know that $\alpha_{\tilde{M}} = \alpha_M$ as $(H \otimes_1 H)^*$-valued measures. Thus

$$\langle b, \alpha_{\tilde{M}}(A) \rangle = \langle b, \alpha_M(A) \rangle = \left\langle b, \int_A Q_M d\alpha_{\|M\|} \right\rangle = \int_A \langle b, Q_M \rangle \, d\alpha_{\|M\|},$$

(we denote an element of $H \otimes_1 H$ and its extension to $(H \otimes_1 H)^*$ by the same symbol) because

$$\begin{aligned} \left\langle h \otimes g, \int_A Q_M d\alpha_{\|M\|} \right\rangle &= (\int_A Q_M d\alpha_{\|M\|}(h), g)_H \\ &= \int_A (Q_M(h), g)_H d\alpha_{\|M\|} \\ &= \int_A \langle h \otimes g, Q_M \rangle \, d\alpha_{\|M\|} \end{aligned}$$

so that operation of extension to element of $(H \otimes_1 H)^*$ and integration are interchangeable. Also

$$|\alpha_{\tilde{M}}|_{(H\otimes_1 H)^*} = |\alpha_M|_{(H\otimes_1 H)^*} \leq |\alpha_M| = \alpha_{\|M\|},$$

where, to avoid a confusion, we denoted by $|\cdot|_{(H\otimes_1 H)^*}$ the variation of an $(H \otimes_1 H)^*$ valued measure.

On the other hand if $\|T\|_{(H\otimes_1 H)^*} = 0$ then, by uniqueness of the extension, also $\|T\|_1 = 0$. This gives that if $|\alpha_{\tilde{M}}|(A) = 0$ then $|\alpha_M|(A) = \alpha_{\|M\|}(A) = 0$. Thus we arrived at the conclusion that

$$|\alpha_{\tilde{M}}|_{(H\otimes_1 H)^*} \equiv \alpha_{\|M\|}.$$

Further,

$$\langle b, \alpha_{\tilde{M}}(A)\rangle = \int_A \langle b, Q_{\tilde{M}}\rangle \, d|\alpha_{\tilde{M}}| = \int_A \langle b, Q_{\tilde{M}}\rangle \frac{d|\alpha_{\tilde{M}}|}{d\alpha_{\|M\|}} d\alpha_{\|M\|}$$

so that we can choose

$$Q_{\tilde{M}} = \frac{d\alpha_{\|M\|}}{d|\alpha_{\tilde{M}}|} Q_M$$

to be a predictable process.

We conclude that if in Definition 2.4.6 we replace the process $Q_{\tilde{M}}$ with Q_M and the measure $|\alpha_{\tilde{M}}|$ with $\alpha_{\|M\|}$ to get its condition (2) to hold for

$$\langle X^*(\omega,t)(h) \otimes X^*(\omega,t)(g), Q_M(\omega,t)\rangle$$

and measure $\alpha_{\|M\|}$ it will not change the space $\tilde{L}(\tilde{M}, H)$. Moreover, the seminorm

$$\mathcal{N}(X) = \sup_{\|h\|\le 1} \{ \int_{\Omega\times I} \langle X^*(h) \otimes X^*(h), Q_M(\omega,t)\rangle \, d\alpha_{\|M\|} \}^{\frac{1}{2}},$$

the space of integrable processes $\tilde{\Lambda}(\tilde{M}, H)$ together with the stochastic integral, all remain unchanged. Thus we can integrate processes from the space $\tilde{\Lambda}(\tilde{M}, H)$ with respect to an element $M \in \mathcal{M}_I^2$ in the sense of cylindrical stochastic integration.

2.4.5 An Example Motivating Modification of the Cylindrical Stochastic Integral

In Section 2.4.3 we have seen that $\sigma^- \notin \Lambda^2(H, \mathcal{P}, Y)$. The problem of non-admissibility of σ^- extends to the cylindrical case. Recall that σ^2 (see Definition 2.4.3) is assumed to be regular.

Lemma 2.4.2 *For an $(R_{2.1})$ process $Y \in \mathcal{M}_I^2$ we have $\sigma^- \in \tilde{L}(\tilde{Y}, H)$.*

Proof. We need to verify conditions **(1)-(3)** in part **(A)** of Definition 2.4.6. Condition **(1)** is satisfied easily since $\mathcal{D}(\sigma^-) \supset Ran(\sigma)$.

For condition **(2)** let us notice that $\forall h, g \in \mathcal{D}(\sigma^-)$ we have,

$$(g, \sigma^-(h))_H = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} (h, h_n)_H (g, h_n)_H = (\sigma^-(g), h)_H.$$

Therefore $\mathcal{D}(\sigma^-) \subset \mathcal{D}((\sigma^-)^*)$. Now $\forall (g, h) \in \mathcal{D}(\sigma^-) \times \mathcal{D}(\sigma^-)$ we obtain

$$\begin{aligned} (g,h)_H \quad &\mapsto \quad \left\langle (\sigma^-)^*(g) \otimes (\sigma^-)^*(h), Q_Y \right\rangle = \left\langle \sigma^-(g) \otimes \sigma^-(h), \frac{\sigma^2}{tr\sigma^2} \right\rangle \\ &= \quad \frac{1}{tr\sigma^2} (\sigma^2(\sigma^-(g)), \sigma^-(h))_H = \frac{1}{tr\sigma^2} (g, h)_H \end{aligned}$$

clearly extends continuously to $H \times H$ and this extension is predictable in view of predictability of σ^2.

For justification of condition **(3)** let us compute

$$\begin{aligned} \mathcal{N}(\sigma^-)^2 \quad &= \quad \sup_{\|h\| \le 1} \{ \int_{\Omega \times I} \left\langle (\sigma^-)^*(h) \otimes (\sigma^-)^*(h), Q_Y \right\rangle d\alpha_{\|Y\|} \} \\ &= \quad \sup_{\|h\| \le 1} \{ \int_{\Omega \times I} \|h\|_H^2 \frac{1}{tr\sigma^2} tr\sigma^2 d(P \otimes \lambda) = \lambda(I) < \infty. \end{aligned}$$

□

As a consequence of regularity of σ^2 we obtain,

Corollary 2.4.1 *Let $\sigma_N^-(h) = \sum_{n=1}^{N} \frac{1}{\sqrt{\lambda_n}} (h_n, h)_H h_n$. Then, $\sigma_N^- \in L_2((\Omega \times I, \mathcal{P}, \alpha_{\|Y\|}); L(H)) \subset \tilde{\Lambda}(\tilde{Y}, H)$.*

Proof. We have

$$\begin{aligned} \int_{\Omega \times I} \|\sigma_N^-\|_{L(H)}^2 d\alpha_{\|Y\|} \quad &\le \quad \int_{\Omega \times I} (\sup_{n \le N} \{\frac{1}{\lambda_n}\}) \sum_{n=1}^{\infty} \lambda_n d(P \otimes \lambda) \\ &\le \quad \int_{\Omega \times I} (1 + tr\sigma^2) d(P \otimes \lambda) < \infty \end{aligned}$$

since Doléans measure of a square integrable martingale has bounded variation. The assertion follows because $L_2((\Omega \times I, \mathcal{P}, \alpha_{\|Y\|}); L(H)) \subset \tilde{\Lambda}(\tilde{Y}, H)$ in view of Proposition in Paragraph 16.3 of [35].

□

Example 2.4.2 $\sigma^- \notin \tilde{\Lambda}(\tilde{Y}, H)$.

First let us note that

$$\mathcal{N}(\sigma_N^- - \sigma^-) = \sup_{\|h\|\leq 1} \Big(\int_{\Omega\times I} \sum_{i=N+1}^{\infty} (h_i, h)^2 d(P\otimes\lambda)\Big)^{\frac{1}{2}} = \lambda(I)^{\frac{1}{2}}.$$

Thus σ_N^- does not converge to σ^- in the seminorm $\mathcal{N}$.

Assume that $\sigma^- \in \tilde{\Lambda}(\tilde{Y}, H)$, so that there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of elementary processes with $\mathcal{N}(X_n - \sigma^-) \to 0$ as $n \to \infty$. Denote by P_N the orthogonal projection in H on the $span\{h_1, h_2, ..., h_N\}$. We have the following:

$$\begin{aligned}
&\mathcal{N}(X_n \circ P_N - \sigma_N^-) \\
&= \sup_{\|h\|\leq 1} \Big\{\int_{\Omega\times I} \left\langle (X_n \circ P_N - \sigma_N^-)^*(h) \otimes (X_n \circ P_N - \sigma_N^-)^*(h), \frac{\sigma^2}{tr\sigma^2} \right\rangle \\
&\qquad\qquad \times tr\sigma^2 d(P\otimes\lambda)\Big\} \\
&= \sup_{\|h\|\leq 1} \Big\{\int_{\Omega\times I} \|\sigma \circ (X_n \circ P_N - \sigma_N^-)^*(h)\|_H^2 d(P\otimes\lambda)\Big\} \\
&= \sup_{\|h\|\leq 1} \Big\{\int_{\Omega\times I} \sum_{i=1}^{N} [\sqrt{\lambda_i}(h_i, X_n^*(h))_H - (h_i, h)_H]^2 d(P\otimes\lambda)\Big\} \\
&\leq \sup_{\|h\|\leq 1} \Big\{\int_{\Omega\times I} \sum_{i=1}^{\infty} [\sqrt{\lambda_i}(h_i, X_n^*(h))_H - (h_i, h)_H]^2 d(P\otimes\lambda)\Big\} \\
&= \sup_{\|h\|\leq 1} \Big\{\int_{\Omega\times I} \left\langle (X_n - \sigma^-)^*(h) \otimes (X_n - \sigma^-)^*(h), Q_Y \right\rangle d\alpha_{\|Y\|} \\
&= \mathcal{N}(X_n - \sigma^-) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

We used the fact that for any $h, g \in \mathcal{D}(\sigma^-)$,

$$\begin{aligned}&\Big\langle (X_n - \sigma^-)^*(h) \otimes (X_n - \sigma^-)^*(g), Q_Y \Big\rangle \\ &\quad = \frac{1}{tr\sigma^2}((\sigma \circ X_n - Id_H)^*(h), (\sigma \circ X_n - Id_H)^*(g))_H\end{aligned}$$

and the bilinear form on the LHS extends uniquely to the continuous bilinear form on the RHS which is well defined on all of $H \times H$. Note that we proved the following inequality:

$$\mathcal{N}(X_n \circ P_N - \sigma_N^-) \leq \mathcal{N}(X_n - \sigma^-) \quad \forall n, N = 1, 2...$$

Next we will prove that for any $n = 1, 2...$, $\mathcal{N}(X_n \circ P_N \to X_n)$ as $N \to \infty$,

$$\begin{aligned}&\sup_{\|h\|\leq 1} \{ \int_{\Omega\times I} \langle (X_n \circ P_N - X_n)^*(h) \otimes (X_n \circ P_N - X_n)^*(h), Q_Y \rangle \, d\alpha_{\|Y\|} \\ &= \sup_{\|h\|\leq 1} \{ \int_{\Omega\times I} \| \sum_{i=1}^{N} \sqrt{\lambda_i}(h_i, X_n^*(h))_H h_i - \sum_{i=1}^{\infty} \sqrt{\lambda_i}(h_i, X_n^*(h))_H h_i \|_H^2 d(P \otimes \lambda) \} \\ &= \sup_{\|h\|\leq 1} \{ \int_{\Omega\times I} \sum_{i=N+1}^{\infty} \lambda_i (h_i, X_n^*(h))_H^2 d(P \otimes \lambda) \} \\ &\leq \|X_n^*\|_{L(H)}^2 \int_{\Omega\times I} \sum_{i=N+1}^{\infty} \lambda_i d(P \otimes \lambda) \to 0\end{aligned}$$

by monotone convergence theorem. Finally,

$$\begin{aligned}\mathcal{N}(\sigma_N^- - \sigma^-) &\leq \mathcal{N}(\sigma_N^- - X_n \circ P_N) + \mathcal{N}(X_n \circ P_N - X_n) + \mathcal{N}(X_n - \sigma^-) \\ &\leq 2\mathcal{N}(X_n - \sigma^-) + \mathcal{N}(X_n \circ P_N - X_n).\end{aligned}$$

For any given ε, we can choose an n, such that $\mathcal{N}(X_n - \sigma^-) < \varepsilon$ and then $\exists N_0$ $\forall N > N_0$, $\mathcal{N}(X_n \circ P_N - X_n) < \varepsilon$. But this is in contradiction with what we proved in the beginning of this Example. Hence $\sigma^- \notin \tilde{\Lambda}(\tilde{Y}, H)$.

2.5 Extension of the Cylindrical Stochastic Integral and Application to Nelson's problem

As we have seen in Section 2.4.3, σ^- failed to be an admissible process for the isometric integral. We need to find a larger space of which σ^-, assumed regular, is an element. Motivation for further studies comes from the following Lemma.

Lemma 2.5.1 *For every $h \in H$,*

$$\int_{\Omega\times I} \left\langle (\sigma_N^- - \sigma^-)(h) \otimes (\sigma_N^- - \sigma^-)(h), Q_Y \right\rangle d\alpha_{\|Y\|} \to 0 \text{ as } N \to \infty.$$

Proof. $\forall (\omega, t) \in \Omega \times I$,

$$(tr\sigma^2 \left\langle (\sigma_N^- - \sigma^-)(h) \otimes (\sigma_N^- - \sigma^-)(h), Q_Y \right\rangle)(\omega, t) = \sum_{i=N+1}^{\infty} (h, h_i(\omega, t))_H^2 \to 0$$

and is bounded by $\|h\|_H^2$ independently of (ω, t).

□

Now we consider an extension of the cylindrical stochastic integral.

Definition 2.5.1 *Let M be a 2-cylindrical H-martingale with the Doléans measure of finite variation.*

(A) *Define $\tilde{L}^w(\tilde{M}, H)$ as the set of processes satisfying conditions (1) and (2) of Part* **(A)** *of Definition 2.4.6 and the following condition:*

(3) $\forall h \in H$, $\mathcal{N}_h^w(X) = [\int_{\Omega\times I} \langle X^*(h) \otimes X^*(h), Q_{\tilde{M}} \rangle \, d|\alpha_{\tilde{M}}|]^{\frac{1}{2}} < \infty$.

For every $h \in H$, $\mathcal{N}_h^w$ is a seminorm and we say that a sequence $\{X_n\}_{n=1}^{\infty} \subset \tilde{L}^w(\tilde{M}, H)$ converges to $X \in \tilde{L}^w(\tilde{M}, H)$ if $\forall h \in H$ $\mathcal{N}_h^w(X_n - X) \to 0$. We will denote this convergence by $X_n \Rightarrow X$.

(B) *We denote by* $\tilde{\Lambda}^w(\tilde{M},H)$ *the closure in* $\tilde{L}^w(\tilde{M},H)$ *of the class of elementary processes in the topology of convergence "* $\Rightarrow$ *" defined in* **(A)-(3)**.

For every $X \in \tilde{\Lambda}^w(\tilde{M},H)$, $h \in H$ *we define* $(\int X d\tilde{M})^w(h)$ *as a limit of* $(\int X_n d\tilde{M})(h)$ *in* $\mathcal{M}_I^2(\mathbf{R})$, *where* $X_n \Rightarrow X$ *and* X_n *are elementary processes. We call* $(\int X d\tilde{M})^w \in L(H, \mathcal{M}_I^2(\mathbf{R}))$ *the* **stochastic integral**.

Note. We need to justify correctness of Part **(B)** of the above Definition. First, for any sequence $\{X_n\}_{n=1}^{\infty}$ of elementary processes, such that $X_n \Rightarrow X$ we have $\forall h \in H$,

$$\begin{aligned} \|(\int X_n d\tilde{M})(h) - (\int X_m d\tilde{M})(h)\|_{\mathcal{M}_I^2(\mathbf{R})} &= \|(\int (X_n - X_m) d\tilde{M})(h)\|_{\mathcal{M}_I^2(\mathbf{R})} \\ &= \mathcal{N}_h^w(X_n - X_m) \end{aligned}$$

by equality (2.6). Therefore whenever $X_n \Rightarrow X$, then $\forall h \in H$, $\{X_n\}_{n=1}^{\infty}$ is a Cauchy sequence for $\mathcal{N}_h^w$ and hence, $(\int X_n d\tilde{M})(h)$ converges in $\mathcal{M}_I^2(\mathbf{R})$ to a square integrable real valued martingale.

Now, the mappings $h \to (\int X_n d\tilde{M})(h)$ from H to $\mathcal{M}_I^2(\mathbf{R})$ are linear, continuous and for every $h \in H$ there exists a limit, which we denote by $(\int X d\tilde{M})^w(h)$. Therefore, by Banach-Steinhaus theorem $(\int X d\tilde{M})^w(h) \in L(H, \mathcal{M}_I^2(\mathbf{R}))$ - that means $(\int X d\tilde{M})^w$ is a 2-cylindrical H-martingale.

We conclude our considerations on Nelson's ideas with an analogue of Theorem 11.6 in [36]. As we proved in Lemma 2.5.1, $\sigma_N^- \Rightarrow \sigma^-$ with $\sigma_N^- \in \tilde{\Lambda}(\tilde{Y},H) \subset \tilde{\Lambda}^w(\tilde{Y},H)$ (see Corollary 2.4.1). Therefore $\sigma^- \in \tilde{\Lambda}^w(\tilde{Y},H)$.

Theorem 2.5.1 *Let* X *be an* (R_1) *process and let* $Y_t = X_t - \int_0^t DX_s ds$ *be an* $(R_{2.1})$ *process. Assume that* σ^2 *is regular. Then there exists a 2-cylindrical* H*-martingale*

$\tilde{W}$, *such that for every* $h, g \in H$,

$$E\{(\tilde{W}_t(h) - \tilde{W}_s(h))(\tilde{W}_t(g) - \tilde{W}_s(g))|\mathcal{F}_s\} = (t-s)(h,g)_H$$

and

$$X_t = \int_0^t DX_s ds + (\int \sigma d\tilde{W})_t.$$

The above equality is in the following sense, $\forall h \in H$, $(X_t - \int_0^t DX_s ds, h)_H = (\int \sigma d\tilde{W})_t(h)$ *in* $\mathcal{M}_I^2(\mathbf{R})$. *In particular,* $\int \sigma d\tilde{W}$ *is a 2-cylindrical* H*-martingale associated with an ordinary* H*-valued martingale.*

Proof. We define $\tilde{W} = (\int \sigma^- d\tilde{Y})^w$. Let us first prove that for $X \in \tilde{\Lambda}^w(\tilde{Y}, H)$ we have

$$E\{[((\int Xd\tilde{Y})_t^w - (\int Xd\tilde{Y})_s^w)(h)][((\int Xd\tilde{Y})_t^w - (\int Xd\tilde{Y})_s^w)(g)]|\mathcal{F}_s\} \qquad (2.7)$$
$$= E\{\int_s^t \left\langle X_r^*(h) \otimes X_r^*(g), \sigma^2(r)\right\rangle dr|\mathcal{F}_s\} \quad \forall h \in H.$$

Recall that by condition **(2)** of Definition 2.4.6 and because $Q_Y = \sigma^2/tr\sigma^2$ and $d\alpha_{\|Y\|} = tr\sigma^2 d(P \otimes \lambda)$, the expression

$$\left\langle X^*(h) \otimes X^*(g), \sigma^2\right\rangle = \langle X^*(h) \otimes X^*(g), Q_Y\rangle\, tr\sigma^2$$

is well defined on $H \times H$.

We first obtain equality (2.7) for elementary processes of the form (2.3).

$$E\{[(\int Xd\tilde{Y})_t^w(h) - (\int Xd\tilde{Y})_s^w(h)][(\int Xd\tilde{Y})_t^w(g) - (\int Xd\tilde{Y})_s^w(g)]|\mathcal{F}_s\}$$
$$= E\{[\sum_{i=k}^n 1_{F_i}((Y_{t_i}, u_i^*(h))_H - (Y_{s_i}, u_i^*(h))_H)]$$
$$\times [\sum_{j=k}^n 1_{F_j}((Y_{t_j}, u_j^*(g))_H - (Y_{s_j}, u_j^*(g))_H)]|\mathcal{F}_s\}$$

where we assume (by refining the partition of I if necessary) that $s_k = s$ and $t_n = t$. The components with factors for which $i \neq j$ will give zero by the martingale property of Y. For $i = j$ we compute for each component

$$\begin{aligned}
&E\{1_{F_i}\left\langle u_i^*(h) \otimes u_i^*(g), (Y_{t_i} - Y_{s_i})^{\otimes 2}\right\rangle |\mathcal{F}_{s_i}\} \\
&= \left\langle 1_{F_i} u_i^*(h) \otimes u_i^*(g), E\{\int_{s_i}^{t_i} \sigma^2(r) dr |\mathcal{F}_{s_i}\}\right\rangle \\
&= E\{\int_s^t \left\langle 1_{F_i \times (s_i, t_i]} u_i^*(h) \otimes u_i^*(g), \sigma^2(r)\right\rangle dr |\mathcal{F}_{s_i}\}.
\end{aligned}$$

By taking conditional expectation with respect to $\mathcal{F}_s \subset \mathcal{F}_{s_i}$ and summing all terms from $i = k$ to n we get the desired result.

Next,

$$\begin{aligned}
&E\{\int_s^t |(\sigma \circ X_n^*(h), \sigma \circ X_n^*(g))_H - (\sigma \circ X^*(h), \sigma \circ X^*(g))_H| d\lambda\} \\
&\leq (E\{\int_s^t \|\sigma \circ (X_n - X)^*(h)\|_H^2 d\lambda\})^{\frac{1}{2}} (E\{\int_s^t \|\sigma \circ X^*(g)\|_H^2 d\lambda\})^{\frac{1}{2}} \\
&+(E\{\int_s^t \|\sigma \circ (X_n - X)^*(g)\|_H^2 d\lambda\})^{\frac{1}{2}} (E\{\int_s^t \|\sigma \circ X_n^*(h)\|_H^2 d\lambda\})^{\frac{1}{2}} \\
&\leq \mathcal{N}_h^w(X_n - X)\mathcal{N}_g^w(X) + \mathcal{N}_g^w(X_n - X)\mathcal{N}_h^w(X_n).
\end{aligned}$$

Therefore convergence $X_n \Rightarrow X$ implies that, $\forall h \in H$,

$$E\{\int_s^t \left\langle X_n^*(h) \otimes X_n^*(g), \sigma^2\right\rangle d\lambda |\mathcal{F}_s\} \to E\{\int_s^t \left\langle X^*(h) \otimes X^*(g), \sigma^2\right\rangle d\lambda |\mathcal{F}_s\}$$

in $L_1(\Omega)$ by contractivity of conditional expectation. Convergence $X_n \Rightarrow X$ implies also that, $\forall h \in H$,

$$(\int X_n d\tilde{Y})^w(h) \to (\int X d\tilde{Y})^w(h) \text{ in } \mathcal{M}_I^2(\mathbf{R})$$

which, in turn, implies

$$\begin{aligned}
&E\{[((\int X_n d\tilde{Y})_t^w - (\int X_n d\tilde{Y})_s^w)(h)][((\int X_n d\tilde{Y})_t^w - (\int X_n d\tilde{Y})_s^w)(g)] |\mathcal{F}_s\} \\
&\quad \to E\{[((\int X d\tilde{Y})_t^w - (\int X d\tilde{Y})_s^w)(h)][((\int X d\tilde{Y})_t^w - (\int X d\tilde{Y})_s^w)(g)] |\mathcal{F}_s\}
\end{aligned}$$

in $L_1(\Omega)$, $\forall h \in H$. This concludes the proof of equality (2.7).

Using (2.7) we can now prove that

$$
\begin{aligned}
&E\{[\tilde{W}_t(h) - \tilde{W}_s(h)][\tilde{W}_t(g) - \tilde{W}_s(g)]|\mathcal{F}_s\} \\
&\quad = E\{[(\int \sigma^- d\tilde{Y})^w_t(h) - (\int \sigma^- d\tilde{Y})^w_s(h)] \\
&\qquad \times [(\int \sigma^- d\tilde{Y})^w_t(g) - (\int \sigma^- d\tilde{Y})^w_s(g)]|\mathcal{F}_s\} \\
&\quad = E\{\int_s^t \left\langle (\sigma^-)^*(h) \otimes (\sigma^-)^*(g), \sigma^2 \right\rangle d\lambda|\mathcal{F}_s\} \\
&\quad = E\{\int_s^t (h,g)_H d\lambda|\mathcal{F}_s\} = (t-s)(h,g)_H
\end{aligned}
$$

(for the third equality recall the proof of Lemma 2.4.2).

To show the last assertion of the theorem let us prove that $\sigma \in \tilde{\Lambda}(\tilde{W}, H)$ and that for an elementary process X_n of the form (2.3),

$$
(\int X_n d\tilde{W}) = (\int X_n \circ \sigma^- d\tilde{Y})^w \tag{2.8}
$$

(implicitly $X_n \circ \sigma^- \in \tilde{\Lambda}^w(\tilde{Y}, H)$). Indeed,

$$
\int_{\Omega \times I} \|\sigma\|^2_{L(H)} d|\alpha_{\tilde{W}}| \leq \int_{\Omega \times I} \|\sigma\|^2_{H \otimes_1 H} d|\alpha_{\tilde{W}}| = \int_I E\{\|\sigma\|^2_{H \otimes_1 H}\} d\lambda < \infty.
$$

Last equality follows from the fact that

$$
\begin{aligned}
\langle h \otimes g, \alpha_{\tilde{W}}(F \times (s,t]) \rangle &= E\{E\{1_F(\tilde{W}_t(h)\tilde{W}_t(g) - \tilde{W}_s(h)\tilde{W}_s(g)|\mathcal{F}_s\}\} \\
&= P(F)\lambda((s,t])(h,g)_H
\end{aligned}
$$

for $F \in \mathcal{F}_s$, $h, g \in H$ and $s \leq t$, $s, t \in I$. Therefore $\alpha_{\tilde{W}} = (P \otimes \lambda)tr$ as $tr \in (H \otimes_1 H)^*$ is the extension of $h \otimes g \mapsto (h,g)_H$. Hence, $|\alpha_{\tilde{W}}| = P \otimes \lambda$.
Finiteness of $\int_I E\{\|\sigma\|^2_{H \otimes_1 H}\} d\lambda$ follows from the property $(R_{2.1})$ of Y. Also, $\sigma \in \tilde{\Lambda}(\tilde{W}, H)$ (see the proof of Corollary 2.4.1).

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of elementary processes. First we will establish that $X_n \circ \sigma^- \in \tilde{\Lambda}^w(\tilde{Y}, H)$. The domain $\mathcal{D}(X_n \circ \sigma^-)$ is dense in H and

$$\left\langle (X_n \circ \sigma^-)^*(h) \otimes (X_n \circ \sigma^-)^*(g), Q_Y \right\rangle = \frac{1}{tr\sigma^2}(Id_{\mathcal{D}(\sigma^-)} \circ X_n^*(h), Id_{\mathcal{D}(\sigma^-)} \circ X_n^*(g))_H$$

extends uniquely to a continuous bilinear form $(1/tr\sigma^2)(X_n^*(h), X_n^*(g))_H$. Further, $\forall h \in H$ (using the above extension),

$$\int_{\Omega\times I} \left\langle (X_n \circ \sigma^-)^*(h) \otimes (X_n \circ \sigma^-)^*(h), Q_Y \right\rangle d\alpha_{\|Y\|} = \int_{\Omega\times I} \|X_n^*(h)\|_H^2 d(P \otimes \lambda) < \infty.$$

Therefore, $X_n \circ \sigma^- \in \tilde{L}(\tilde{Y}, H) \subset \tilde{L}^w(\tilde{Y}, H)$.

Since $\sigma^- \in \tilde{\Lambda}^w(\tilde{Y}, H)$, there exists a sequence $\{Z_m\}_{m=1}^{\infty}$ of elementary processes, such that $Z_m \Rightarrow \sigma^-$ as $m \to \infty$. Now, for every $h \in H$,

$$\begin{aligned}
&\int_{\Omega\times I} \langle [(X_n \circ Z_m)^* - (X_n \circ \sigma^-)^*](h) \otimes [(X_n \circ Z_m)^* - (X_n \circ \sigma^-)^*](h), Q_Y \rangle d\alpha_{\|Y\|} \\
&\quad = \int_{\Omega\times I} \left\langle (Z_m - \sigma^-)^*(X_n^*(h)) \otimes (Z_m - \sigma^-)^*(X_n^*(h)), Q_Y \right\rangle d\alpha_{\|Y\|} \to 0.
\end{aligned}$$

Since $\{X_n \circ Z_m\}_{m=1}^{\infty}$ is a sequence of elementary processes, $X_n \circ \sigma^- \in \tilde{\Lambda}^w(\tilde{Y}, H)$.

Next, we will prove that if $X_n \to \sigma$ in $\tilde{\Lambda}(\tilde{W}, H)$ then $X_n \circ \sigma^- \Rightarrow Id_H$ in $\tilde{\Lambda}^w(\tilde{Y}, H)$ as $n \to \infty$. Clearly $Id_H \in \tilde{\Lambda}(\tilde{Y}, H) \subset \tilde{\Lambda}^w(\tilde{Y}, H)$. Observe that

$$\begin{aligned}
&\left\langle (X_n \circ \sigma^- - Id_H)^*(h) \otimes (X_n \circ \sigma^- - Id_H)^*(g), \frac{\sigma^2}{tr\sigma^2} \right\rangle \\
&\quad = \frac{1}{tr\sigma^2}((Id_{\mathcal{D}(\sigma^-)} \circ X_n^* - \sigma)(h), (Id_{\mathcal{D}(\sigma^-)} \circ X_n^* - \sigma)(g))_H
\end{aligned}$$

extends uniquely to a continuous bilinear form on $H \times H$, namely, to

$$\frac{1}{tr\sigma^2}((X_n^* - \sigma)(h), (X_n^* - \sigma)(g))_H.$$

For every $h \in H$, and for this extension, we have

$$\begin{aligned}
&\int_{\Omega\times I} \left\langle (X_n \circ \sigma^- - Id_H)^*(h) \otimes (X_n \circ \sigma^- - Id_H)^*(h), Q_Y \right\rangle d\alpha_{\|Y\|} \\
&\quad = \int_{\Omega\times I} \langle (X_n - \sigma)^*(h) \otimes (X_n - \sigma)^*(h), tr \rangle \, d|\alpha_{\tilde{W}}| \to 0
\end{aligned}$$

Since convergence in $\tilde{\Lambda}^w$ implies convergence of stochastic integrals in $\mathcal{M}_I^2(\mathbf{R})$, we conclude that $\forall h \in H$,

$$(\int X_n \circ Z_m d\tilde{Y})(h) \to (\int X_n \circ \sigma^- d\tilde{Y})^w(h) \qquad \text{as } m \to \infty$$

and

$$(\int X_n \circ \sigma^- d\tilde{Y})^w(h) \to (\int Id_H d\tilde{Y})(h) = (Y, h)_H \qquad \text{as } n \to \infty$$

in $\mathcal{M}_I^2(\mathbf{R})$. Equality (2.8) can be proved as follows:

$$\begin{aligned}
&(\int X_n d\tilde{W})_t(h) \\
&= \sum_{i=1}^n 1_{F_i}[(\int \sigma^- d\tilde{Y})^w_{t_i\wedge t}(u_i^*(h)) - (\int \sigma^- d\tilde{Y})^w_{s_i\wedge t}(u_i^*(h))] \\
&= \sum_{i=1}^n 1_{F_i}[\mathcal{M}_I^2(\mathbf{R}) - \lim_{m\to\infty}[(\int Z_m d\tilde{Y})_{t_i\wedge t}(u_i^*(h)) - (\int Z_m d\tilde{Y})_{s_i\wedge t}(u_i^*(h))]] \\
&= \sum_{i=1}^n 1_{F_i}[\mathcal{M}_I^2(\mathbf{R}) - \lim_{m\to\infty}[(\int u_i \circ Z_m d\tilde{Y})_{t_i\wedge t}(h) - (\int u_i \circ Z_m d\tilde{Y})_{s_i\wedge t}(h)]] \\
&= \mathcal{M}_I^2(\mathbf{R}) - \lim_{m\to\infty} \sum_{i=1}^n 1_{F_i}[(\int u_i \circ Z_m d\tilde{Y})_{t_i\wedge t}(h) - (\int u_i \circ Z_m d\tilde{Y})_{s_i\wedge t}(h)] \\
&= \mathcal{M}_I^2(\mathbf{R}) - \lim_{m\to\infty} (\int X_n \circ Z_m d\tilde{Y})_t(h) \\
&= (\int X_n \circ \sigma^- d\tilde{Y})^w_t(h).
\end{aligned}$$

Because, $\forall h \in H$,

$$(\int X_n d\tilde{W})(h) \to (\int \sigma d\tilde{W})(h)$$

in $\mathcal{M}_I^2(\mathbf{R})$, we get

$$(X_t - \int_0^t DX_s ds, h)_H = (Y_t, h)_H = (\int \sigma d\tilde{W})_t(h),$$

in $\mathcal{M}_I^2(\mathbf{R})$. This concludes the proof.

$\square$

Let us mention that the assertion of the last theorem states in particular that $\tilde{W}$ is a 2-cylindrical standard Brownian motion, provided $\tilde{W}_t(h)$ has continuous sample paths in t for every $h \in H$. The following Proposition gives some regularity of the process $\tilde{W}$.

Proposition 2.5.1 *Under the assumptions of Theorem 2.5.1, if $X : I \to H$ is continuous, then $\forall h \in H$ the real valued martingale*

$$\tilde{W}(h) = (\int \sigma^- d\tilde{Y})^w(h)$$

has P-a.e. continuous paths.

Proof. The proof is based on the following Lemma:

Lemma([35]). *Let $\{M^n\}_{n=1}^{\infty} \subset \mathcal{M}_I^2$ be a sequence of H-valued martingales which converges in $\mathcal{M}_I^2$ toward M. Then there exists a subsequence $\{M^{n_k}\}_{k=1}^{\infty}$ with the following property: for P-almost all $\omega \in \Omega$, the paths $t \mapsto M^{n_k}(t,\omega)$ converge uniformly on I to the paths $t \mapsto M(t,\omega)$.*

Since $t \mapsto DX_t$ is continuous from I to $L_1(\Omega, H)$ we can choose its jointly measurable version in (t,ω) (see [16], Theorem 1.2). Hence, $t \mapsto \int_0^t DX_s ds$ is continuous from I to H. This, together with continuity of $X : I \to H$ gives continuity of $Y : I \to H$. Since for an elementary process of the form (2.3) the stochastic process

$$(\int X_n d\tilde{Y})(h) = \sum_{i=1}^{n} 1_{F_i}[Y_{t_i \wedge t}(u_i^*(h)) - Y_{s_i \wedge t}(u_i^*(h))]$$

has continuous sample paths then, by choosing $X_n \Rightarrow \sigma^-$, we get,

$$\forall h \in H \quad (\int X_n d\tilde{Y})(h) \to (\int \sigma^- d\tilde{Y})^w(h)$$

in $\mathcal{M}_I^2(\mathbf{R})$, which completes the proof.

□

Chapter 3

Anticipative Stochastic Differential Equations

3.1 Introduction

Anticipative stochastic integration naturally leads to development of the theory of anticipative Stochastic Differential Equations. This allows for analysis of non-adapted processes as solutions of these equations. Anticipative SDE's were considered by several authors. In particular Skorohod-type SDE's were studied by Buckdahn,Nualart Ocone and Pardoux ([5]-[8], [39],[40],[42]). Another approach was presented by Ogawa, [46]-[47], where the author used his concept of stochastic integration.

A natural way to obtain an anticipative SDE is to impose a boundary condition to be a future-dependent random variable. In particular one can consider equations with boundary condition of the type $X_0 = \psi(X_1)$. An interesting presentation of Ogawa-type SDE's is given in [46]. As a nice example the author considers *Go and Return* problem. Recently, in [47] Ogawa studied multidimensional stochastic

integral equations, which were linear of Fredholm-type. This seems to be a strong application of anticipative calculus.

In this Chapter we consider a Gaussian process $\{X_t, t \in T\}$ with arbitrary index set T and we study consequences of transformations of the index set T on the Skorohod integral with respect to X. We obtain applications to time and space reversal in case of Brownian motion and Brownian Sheet. Even though we consider here general transformations of the parameter set our motivation came from the Time Reversal Problem of a diffusion process and applications of this method to the problem of filtering. In the case of Skorohod linear diffusions we obtain existence and uniqueness of the solution for the reversed equation (a problem considered in [42]). As an example we formulate and solve *Go and Return* problem for Skorohod linear diffusions. Further applications of anticipative stochastic calculus and kinematics of Hilbert space valued stochastic motion to Time Reversal Problem and Filtering Theory will be a subject of future research.

3.2 Skorohod Integral under Transformation of a Parameter Set

Assume that $\{X_t\}_{t\in T}$ is a centered Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an arbitrary parameter set T. The covariance function of X will be denoted by C_X and the RKHS of C_X will be denoted by $H(C_X)$.

Definition 3.2.1 *A map $R : T \to T$ will be called a non-degenerate transformation of the parameter set T if*

$$cl(span\{X_t, t \in T\}) = cl(span\{X_{R(t)}, t \in T\})$$

where "cl" denotes closure in $L_2(\Omega, \mathcal{F}, P)$.

For any transformation $R : T \to T$ let $T_1 \subset T$ be such a set for which $\forall t \in T$, $T_1 \cap R^{-1}(t)$ is a single element of T. Thus $R : T_1 \to R(T_1)$ is a bijection. In particular if $R : T \to T$ is bijective then $T_1 = T$. There are possibly many choices of T_1. Now we consider behavior of the Skorohod integral under non-degenerate transformations.

Proposition 3.2.1 *Let $\{X_t\}_{t \in T}$ be a Gaussian process and R be a non-degenerate transformation on T. Denote by I^s_X the Skorohod integral with respect to X and by $I^s_{X^R}$, the Skorohod integral with respect to (a Gaussian process) $X^R = \{X_{R(t)}\}_{t \in T_1}$. Then:*

(1) A map $f_p \mapsto f^R_p = f(R(t_1), ..., R(t_p))|_{T_1^p}$ is an isometry from $H(C_X)^{\otimes p}$ onto $H(C^{\otimes p}_{X^R})$.

(2) If $u \in \mathcal{D}(I^s_X)$ then $u^R = \{u_{R(t)}, t \in T_1\} \in \mathcal{D}(I^s_{X^R})$ and

$$I^s_X(u) = I^s_{X^R}(u^R). \tag{3.1}$$

Moreover, denote by D^X and D^{X^R} the Malliavin derivatives with respect to X and X^R respectively.

(3) If $u_t \in \mathcal{D}(D^X), \quad t \in T_1$, then $u^R_t \in \mathcal{D}(D^{X^R})$ and

$$D^{X^R}_s u^R_t = D^X_{R(s)} u_{R(t)}, \quad s, t \in T_1, \ P\text{-a.e.} \tag{3.2}$$

The equality is in the sense of $H(C_{X^R})$, with $s \in T_1$ as the variable. Also $D_s u_t \in H(C_X)^{\otimes 2}$ $(s, t \in T)$ implies $D^{X^R}_s u^R_t \in H(C_{X^R})^{\otimes 2}$ $(s, t \in T_1)$ and the equality of norms, $\|D_s u_t\|_{L_2(\Omega, H(C_X)^{\otimes 2})} = \|D^{X^R}_s u^R_t\|_{L_2(\Omega, H(C_{X^R})^{\otimes 2})}$.

If $v \in L_2(\Omega, H(C_{X^R}))$ then,

(4) $v = u^R$ for some $u \in L_2(\mathcal{X}, H(C_X))$ and $\|v\|_{L_2(\Omega, H(C_{X^R}))} = \|u\|_{L_2(\Omega, H(C_X))}$.

Moreover, $v \in \mathcal{D}(I^s_{X^R})$ implies $u \in \mathcal{D}(I^s_X)$ and $v_t \in \mathcal{D}(D^{X^R})$ implies $u_t \in \mathcal{D}(D^X)$ with $D^{X^R}_{R(s)} v_t = D^X_s u_t$ for $s, t \in T_1$.

In the case of $D_s^{X^R} v_t \in H(C_{X^R})^{\otimes 2}$ $(s,t \in T_1)$ also $D_s u_t \in H(C_X)^{\otimes 2}$ $(s,t \in T)$ and the H-S norms of these derivatives are equal.

Proof. (1) Let us denote $f^R(t_1, ..., t_n) = f(R(t_1), ..., R(t_n))$ for $(t_1, ..., t_n) \in T_1^p$. Thus $f_p^R(t_1, ..., t_p, t) = f_p(R(t_1, ..., R(t_p), R(t)), (t_1, ..., t_p, t) \in T_1^{p+1}$. Denote $H(X) = cl(span\{X_t, t \in T\}) = cl(span\{X_t^R, t \in T_1\})$. Let $f(t) \in H(C_X)$. Then $f(t) = E(X_t \pi^X(f))$ with $\pi^X(f) \in H(X)$ and, for any $t \in T_1$,

$$f^R(t) = f(R(t)) = E(X_{R(t)} \pi^X(f)) = E(X_t^R \pi^X(f))$$

i.e. $f^R \in H(C_{X^R})$ and $\pi^{X^R}(f^R) = \pi^X(f)$.
Also, if $g \in H(C_{X^R})$ then, for $t \in T_1$,

$$g(t) = E(X_t^R \pi^{X^R}(g)) = E(X_{R(t)} \pi^{X^R}(g)).$$

But, $\pi^{X^R}(g) \in H(X)$, then $f(t) = E(X_t \pi^{X^R}(g))$ defines an element of $H(C_X)$. Hence, $g(t) = f(R(t)), \quad t \in T_1$ and

$$\|g\|_{H(C_{X^R})} = \|\pi^{X^R}\|_{L_2(\Omega, \mathcal{F}, P)} = \|f\|_{H(C_X)}.$$

Now (1) follows for any p in view of the form of ONB and scalar product in the tensor product of RKHS's.

(2) In order to obtain (2) we will first prove that for every p the following equality holds:

$$I_X^p(f_p) = I_{X^R}^p(f_p^R). \tag{3.3}$$

Note that we have already proved the above for $p = 1$, as $I^1 = \pi$ by definition. For $p = 0$ equality (3.3) is obvious.

Every $f_p \in H(C_X)^{\otimes p}$ can be represented as a following series (see Section 1.3.1):

$$f(t_1, t_2, ..., t_p) = \sum_{\alpha_1, \alpha_2, ..., \alpha_p} a_{\alpha_1, \alpha_2, ..., \alpha_p} e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) ... e_{\alpha_p}(t_p)$$

with $\sum_{\alpha_1,\alpha_2,...,\alpha_p} a^2_{\alpha_1,\alpha_2,...,\alpha_p} < \infty$ and $\{e_\alpha, \alpha = 1, 2, ...\}$ an ONB of $H(C_X)$. Then $f_p^R \in H(C_{X^R})^{\otimes p}$ by (1). It is enough to prove equation (3.3) for functions of the form: $e_{\alpha_1}(t_1)e_{\alpha_2}(t_2)...e_{\alpha_p}(t_p)$, because for arbitrary $f_p \in H(C_X)$ we will have

$$\begin{aligned} I_X^p(f_p) &= lim_{n_1,...,n_p\to\infty} I_X^p((\sum_{\alpha_1=1}^{n_1} ... \sum_{\alpha_p=1}^{n_p} a_{\alpha_1,...,\alpha_p} e_{\alpha_1}...e_{\alpha_p}\tilde{)}) \\ &= lim_{n_1,...,n_p\to\infty} I_{X^R}^p((\sum_{\alpha_1=1}^{n_1} ... \sum_{\alpha_p=1}^{n_p} a_{\alpha_1,...,\alpha_p} e^R_{\alpha_1}...e^R_{\alpha_p}\tilde{)}) \\ &= I_{X^R}^p(lim_{n_1,...,n_p\to\infty}(\sum_{\alpha_1=1}^{n_1} ... \sum_{\alpha_p=1}^{n_p} a_{\alpha_1,...,\alpha_p} e^R_{\alpha_1}...e^R_{\alpha_p}\tilde{)}). \end{aligned}$$

We used properties (5) and (6) of Multiple Wiener Integrals (see Section 1.3.1) as well as a simple fact that the operations $f \mapsto f^R$ and symmetrization "~" commute.

In view of (1) we have, $\sum_{\alpha_1,\alpha_2,...,\alpha_p} a_{\alpha_1,\alpha_2,...,\alpha_p} e^R_{\alpha_1} e^R_{\alpha_2}...e^R_{\alpha_p} \to f_p^R$ in $H(C_{X^R})^{\otimes p}$ and hence $(\sum_{\alpha_1,\alpha_2,...,\alpha_p} a_{\alpha_1,\alpha_2,...,\alpha_p} e^R_{\alpha_1} e^R_{\alpha_2}...e^R_{\alpha_p}\tilde{)} \to (f_p^R\tilde{)}$ in $H(C_{X^R})^{\otimes p}$. Thus,

$$I_X^p(f_p) = I_{X^R}^p((f_p^R\tilde{)}) = I_{X^R}^p(f_p^R).$$

Let us now prove equation (3.3) for functions of the form $e_{\alpha_1}(t_1)e_{\alpha_2}(t_2)...e_{\alpha_p}(t_p)$ for $p > 1$. We can use property (8) of MWI and we only need to show that for $(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p) \in T_1^{p-1}$, one gets

$$[(f_p \underset{k}{\otimes} g_1)^X]^R(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p) = (f_p^R \underset{k}{\otimes} g_1^R)^{X^R}(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p),$$

where the superscripts X and X^R outside the brackets indicate that the operation "$\underset{k}{\otimes}$" is taken in $H(C_X)$ and $H(C_{X^R})$ respectively.

We have

$$\begin{aligned} &(f_p \underset{k}{\otimes} g_1)^X(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p) = \\ &\quad = (f_p(t_1, ..., t_k, ..., t_p), g_1(t_k))_{H(C_X)} \end{aligned}$$

$$= E\{\pi^X(f_p(t_1, ..., t_{k-1}, \cdot, t_{k+1}, ..., t_p))\pi^X(g_1(\cdot))\}$$
$$= E\{\pi^{X^R}(f_p^{R(k)}(t_1, ..., t_{k-1}, \cdot, t_{k+1}, ..., t_p))\pi^{X^R}(g_1^R(\cdot))\}$$
$$= (f_p^{R(k)}(t_1, ..., t_k, ..., t_p), g_1^R(t_k))_{H(C_{X^R})}$$
$$= (f_p^{R(k)} \underset{k}{\otimes} g_1^R)^{X^R}(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p)$$

where $R(k)$ transforms only the k^{th} coordinate with $t_1, ..., t_{k-1}, t_{k+1}, ..., t_p$ fixed. But the above implies that

$$\begin{aligned}
&[(f_p \underset{k}{\otimes} g_1)^X]^R(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p) \\
&\quad = (f_p \underset{k}{\otimes} g_1)^X(R(t_1), ..., R(t_{k-1}), R(t_{k+1}), ..., R(t_p)) \\
&\quad = (f_p^{R(k)} \underset{k}{\otimes} g_1^R)^{X^R}(R(t_1), ..., R(t_{k-1}), R(t_{k+1}), ..., R(t_p)) \\
&\quad = (f_p^R \underset{k}{\otimes} g_1^R)^{X^R}(t_1, ..., t_{k-1}, t_{k+1}, ..., t_p).
\end{aligned}$$

Thus,

$$I_p^X((f_p \underset{k}{\otimes} g_1)^X) = I_p^{X^R}([(f_p \underset{k}{\otimes} g_1)^X]^R) = I_p^{X^R}((f_p^R \underset{k}{\otimes} g_1^R)^{X^R})$$

which allows us to use the inductive relation (8) for Multiple Wiener Integrals to complete the proof of equality (3.3).

Now if $u \in \mathcal{D}(I_X^s)$ and $u_t = \sum_{p=0}^{\infty} I_p(f_p(t_1, ..., t_p, t))$ then, for $t \in T_1$,

$$u_{R(t)} = \sum_{p=0}^{\infty} I_p^X(f_p(\cdot, R(t))) = \sum_{p=0}^{\infty} I_p^{X^R}(f_p^R(\cdot, t))$$

hence,

$$\begin{aligned}
I_X^s(u) &= \sum_{p=0}^{\infty} I_{p+1}^X(\tilde{f}_p) = \sum_{p=0}^{\infty} I_{p+1}^{X^R}((\tilde{f}_p)^R) \\
&= \sum_{p=0}^{\infty} I_{p+1}^{X^R}((\tilde{f_p^R})) = I_{X^R}^s(u^R)
\end{aligned}$$

proving (2).

(3) Let $u \in \mathcal{D}(D^X)$. Then, for $s, t \in T_1$,

$$\begin{aligned} D^X_{R(s)} u_{R(t)} &= \sum_{p=1}^{\infty} p I^X_{p-1}(f_p(\cdot, R(s), R(t))) \\ &= \sum_{p=1}^{\infty} p I^{X^R}_{p-1}(f^R_p(\cdot, s, t)) = D^{X^R}_s u^R_t. \end{aligned}$$

The equality of norms claimed in (3) follows by Lemma 1.3.2, (2) and by (1) of this Proposition.

(4) To prove (4), let $v \in L_2(\mathcal{X}, H(C_{X^R}))$. Then for $t \in T_1$ we have

$$v_t = \sum_{p=0}^{\infty} I^{X^R}_p(g_p(\cdot, t)) = \sum_{p=0}^{\infty} I^{X^R}_p(f^R_p(\cdot, t)),$$

as by (1), for any $g \in H(C_{X^R})^{\otimes(p+1)}$ there exists $f \in H(C_X)^{\otimes(p+1)}$ with $g = f^R$. Hence, for $t \in T_1$,

$$v_t = \sum_{p=0}^{\infty} I^{X^R}_p(f^R_p(\cdot, t)) = \sum_{p=0}^{\infty} I^X_p(f_p(\cdot, R(t))) = u_{R(t)}.$$

According to (1), $u_t = \sum_{p=0}^{\infty} I^X_p(f_p(\cdot, t)) \in L_2(\mathcal{X}, H(C_X))$ and equality of norms claimed in (4) is satisfied. The last part of assertion (4) follows from (1),(2) and (3) since failure to satisfy any stated condition by u implies violation of this condition by v.

□

Example 3.2.1 *Transformations of parameter set and Skorohod integral.*

1. Brownian motion and Time Reversal. Let $\mathcal{F}_t = \sigma\{B_s, s \le t\}$ and $\{u_t, t \in [0,1]\}$ be $(\mathcal{F}_t)_{t\in[0,1]}$ adapted stochastic process, such that $u \in L_2(\Omega, L_2[0,1])$. Then $\{\tilde{B}_t = B_1 - B_{1-t}, t \in [0,1]\}$ is also a Brownian motion and $\{\bar{u}_t = u_{1-t}, t \in [0,1]\}$ is adapted to filtration $\tilde{\mathcal{F}}^t = \sigma\{\tilde{B}_1 - \tilde{B}_s, t \le s \le 1\}$. Denote $\bar{B}_t = B_{1-t}$. We have

$$\int_0^1 u_t dB_t = I^s_B\Big(\int_0^{\cdot} u_r dr\Big) = I^s_{\bar{B}}\Big(\int_0^{1-\cdot} u_r dr\Big). \tag{3.4}$$

By the same method as in the proof of Theorem 3.2.1 we can show that

$$I^s_{\tilde{B}}((\int_0^{\cdot} u_r dr\tilde{)}) = I^s_B(\int_0^{\cdot} u_r dr)$$

with $(\int_0^{\cdot} u_r dr\tilde{)} = \int_0^1 u_r dr - \int_0^{1-\cdot} u_r dr$. Hence, we get,

$$\int_0^1 u_t dB_t = I^s_{\tilde{B}}((\int_0^{\cdot} u_r dr\tilde{)}) = I^i_{\tilde{B}}(\bar{u}) = \int_0^1 \bar{u}_t * d\tilde{B}_t$$

where I^i is the Skorohod integral defined in [41] (see Example 1.3.2) and " $*$ " denotes the backward Itô integral. We obtained the relation: $I^i_B(u) = I^i_{\tilde{B}}(\bar{u})$ given in [42]. In particular $\bar{u} \in \mathcal{D}(I^i_{\tilde{B}})$. Note that in [42], the process $B_{1-t} - B_1 = -\tilde{B}_t$ was used as an integrator. But it is true that $I^s_X = -I^s_{(-X)}$, which is easy to check using recursive properties of Multiple Wiener Integrals.

Note also that $\bar{B}_t$ is not a Brownian motion process and the Equation 3.4 is reversed pathwise in H. In the case of Brownian motion, we also have,

$$I^s_{\tilde{B}}(\int_0^{1-\cdot} u_s ds) = I^s_{\tilde{B}}((\int_0^{\cdot} u_s ds\tilde{)}).$$

Indeed,

$$\begin{aligned} I^s_{\tilde{B}}(\int_0^{1-\cdot} u_s ds) &= I^s_B(\int_0^{\cdot} u_s ds) = I^i_B(u) = I^i_{\tilde{B}}(\bar{u}) = I^s_{\tilde{B}}(\int_0^{\cdot} u_{1-s} ds) \\ &= I^s_{\tilde{B}}(\int_0^1 u_s ds - \int_o^{1-\cdot} u_s ds). \end{aligned}$$

2. Brownian Sheet. Let $T = [0,1]^2$ and let us think of a point $(x,t) \in T$as the space-time parameter. Let $W(x,t)$ be a Brownian Sheet ([58]), that is, a Gaussian process $\{W_t, t \in T\}$ defined by covariance $C_W((x,t),(y,s)) = (x \wedge y)(t \wedge s)$, i.e. $C_W = C_B \otimes C_B$ where C_B is the covariance of Brownian motion. In this case we also have that $H(C_W)$, the RKHS of C_W, is the tensor product of the RKHS's $H(C_B)$ of C_B: $H(C_W) = H(C_B)^{\otimes 2}$.

(a) Time Reversal. Let $R(x,t) = (x, 1-t)$, then,

$$I^s_{W(x,t)}(u(x,t)) = I^s_{W(x,1-t)}(u(x,1-t)).$$

(b) Space Reversal. This is the case of $R(x,t) = (1-x,t)$.

3. Generalized Processes (see Example 1.2.1 (c) and [19]). Let $T = C_0^\infty(R)$ and consider Generalized Wiener Process $\{B_\varphi, \varphi \in T\}$ given by covariance function $C(\varphi,\psi) = \int (x \wedge y)\varphi(x)\psi(y)dxdy$. Consider $R : T \to T$ a non-singular transformation of the form: $R(\varphi)(x) = \varphi(r(x)) \in T$. Let $\{u_\varphi\}_{\varphi\in T} \in \mathcal{D}(I_B^s)$ then $\{u_\varphi^R\}_{\varphi\in T} \in \mathcal{D}(I_{B^R}^s)$ and $I_{B^R}^s(u^R) = I_B^s(u)$.

In the particular case of $R(\varphi)(t) = \varphi(-t)$, R is non-singular and we have the following "time reversal":

$$I_{B^R}^s(u^R) = I_{B_{\varphi(-t)}}^s(u_{\varphi(-t)}) = I_{B_{\varphi(t)}}^s(u_{\varphi(t)}) = I_B^s(u).$$

4. Ogawa Line Integral Let $\{X_t, t \in T\}$ be a Gaussian process and $\gamma : S \to T$ be a bijective parametrization. Let $Y_s = X_{\gamma(s)}$, then

(i) $C_X(\gamma(s_1),\gamma(s_2)) = C_Y(s_1,s_2)$

(ii) $H(C_X)$ and $H(C_Y)$ are isometric under the mapping $f \mapsto f\circ\gamma \in H(C_Y)$ for $f \in H(C_X)$.

(iii) $\pi^X(f) = \pi^Y(f\circ\gamma)$ for $f \in H(C_X)$.

Thus, $\delta^o(u) = \delta^o(v)$, for $v_s = u_{\gamma(s)}$, provided either of the integrals exists.

Consider the Brownian Sheet $\{W_{(x,t)}, (x,t) \in [0,1]^2\}$. One can define **Ogawa line integral**, $\Gamma - \delta^o$, over a curve $\Gamma \subset [0,1]^2$ with respect to $\{W_{(x,t)}, (x,t) \in \Gamma\}$ in a usual way. Now assume that Γ can be parametrized by a function $\gamma : [a,b] \to \Gamma$, $0 \le a \le b \le 1$ and $\gamma(s) = (\gamma_1(s),\gamma_2(s))$ with both coordinates non-decreasing and such that the map

$$\tilde\gamma^{-1}(\gamma_1(s),\gamma_2(s)) = \gamma_1(s)\gamma_2(s)$$

is bijective from Γ to $S = [\gamma_1(a)\gamma_2(a), \gamma_1(b)\gamma_2(b)]$. Then $\tilde\gamma : S \to \Gamma$ is a bijective parametrization and the process $B_s = W_{\tilde\gamma(s)}$ is a Brownian motion. Hence,

$$\Gamma - \delta_W^o(u) = \delta_B^o(v) = \int_S v_s \circ dB_s$$

where $v_s = u_{\tilde{\gamma}(s)}$ and the last integral is in the sense of Fisk and Stratonovich (see [45] for a definition) and is assumed to exist. In particular if $u_{(x,t)} = f(W_{(x,t)})$ with $f \in C^2$ then

$$\Gamma - \delta_W(f'(W)) = \int_S f'(B_s) \circ dB_s = f(W(\gamma_1(b), \gamma_2(b)) - f(W(\gamma_1(a), \gamma_2(a)).$$

Thus in this case, the Ogawa line integral shares the property of the Lebesgue integral. Properties of the Ogawa line integral and its relation to line integrals of Cairoli and Walsh [9] will be a subject of further investigation.

3.3 Skorohod-Type Linear Stochastic Differential Equations

The class of Skorohod Linear SDE's was considered by Buckdahn in [6] where the author proved existence and uniqueness of the solution. We give a short review of this result.

Assume that $\{B_t, t \in [0,1]\}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Here $\Omega = C_0([0,1])$. We consider the following Skorohod Linear SDE:

$$Z_t = \eta + \int_0^t b(s)Z(s)ds + I^i(\sigma Z 1_{[0,t]}), \quad 0 \leq t \leq 1 \tag{3.5}$$

where $b \in L_2([0,1], L_\infty(\Omega)), \eta \in L_\infty(\Omega)), \sigma \in L_{1,\infty} = L_2([0,1], D^{1,\infty})$. The space $D^{1,\infty}$ is defined as follows. Let

$$\mathcal{S} = \{F = f(B_{t_1}, ..., B_{t_n}), n \geq 1, t_1, ..., t_n \in [0,1], f \in C_b^\infty(\mathcal{R}^n)\},$$

where $C_b^\infty(\mathcal{R}^n)$ denotes the space of C^∞ functions which are bounded with all their derivatives. Recall the Malliavin derivative D^i of [41] (see Example 1.3.1). Denote by $D^{1,2}$ the closure of $\mathcal{S}$ in the following norm: $\|F\|_{1,2} = \|F\|_{L_2(\Omega)} +$

$\|D^i F\|_{L_2(\Omega, L_2([0,1]))}$. Then $D^{1,\infty}$ is the restriction of $D^{1,2}$ to these random variables for which $\|F\|_{1,\infty} = \|F\|_\infty + \| \; \|D^i F\|_{L_2([0,1])}\|_\infty < \infty$. The stochastic integral I^i is again as in Example 1.3.2.

The main result on Skorohod Linear SDE's in [6] is the following.

Theorem 3.3.1 *Suppose $\sigma \in L_{1,\infty}, b \in L_2([0,1], L_\infty(\Omega)), \eta \in L_\infty(\Omega)$. Denote by $\{T_t, t \in T\}$ the family of transformations associated with σ as follows:*

$$T_t\omega = \omega + \int_0^{t\wedge \cdot} \sigma_s(T_s\omega)ds, \quad \omega \in C_0([0,1]).$$

Let A_t be the inverse to T_t and L_t, the density $dP \circ T_t^{-1}/dP$, where P is the Wiener measure on $C_0[0,1]$. Then the process X defined by

$$X_t = \eta(A_t) exp\{\int_o^t b_s(T_s A_t)ds\} L_t, \quad t \in [0,1] \tag{3.6}$$

belongs to $L_1([0,1]\times\Omega)$, $\sigma X 1_{[0,t]} \in \mathcal{D}(I^i) \quad \forall t \in [0,1]$ and it verifies equation (3.5). Conversely, if $Y \in L_1([0,1] \times \Omega)$ is such that $\sigma Y 1_{[0,t]} \in \mathcal{D}(I^i) \quad \forall t \in [0,1]$ and verifies equation (3.5) and if moreover $\sigma, b \in L_\infty([0,1] \times \Omega)$ and the Malliavin derivative $D^i\sigma \in L_\infty([0,1]^2 \times \Omega)$, then Y_t is of the form (3.6) $\forall t \in [0,1]$.

Our purpose is to reverse equation (3.5). We begin with a supporting Lemma.

Lemma 3.3.1 *Let $\{u_s\}_{s\in[0,1]}$ be such that $u_s 1_{[0,t]}(s) \in \mathcal{D}(I^i_B) \quad \forall t \in [0,1]$. Then for the time reversed process, $\bar{u}_s = u_{1-s}$, we have $\bar{u}_s 1_{[0,t]}(s) \in \mathcal{D}(I^i_{\tilde{B}}) \quad \forall t \in [0,1]$ and if we denote $X_t = I^i_B(1_{[0,t]}(s)u_s)$ then,*

$$X_{1-t} - X_1 = -I^i_{\tilde{B}}(1_{[0,t]}(s)\bar{u}_s).$$

Here, $\tilde{B}_t = B_1 - B_{1-t}$.

Proof. Because for $v_s \in \mathcal{D}(I^i_B)$ we know that $\bar{v}_s \in \mathcal{D}(I^i_{\tilde{B}})$ (as pointed out in Example 3.2.1), by linearity of the domain of the Skorohod integral, we conclude that

$$\begin{aligned} X_{1-t} - X_1 &= I^i_B(-1_{[1-t,1]}(s)u_s) = -I^i_{\tilde{B}}(1_{[1-t,1]}(1-s)u_{1-s}) \\ &= -I^i_{\tilde{B}}(1_{[0,t]}(s)\bar{u}_s). \end{aligned}$$

In particular $1_{[0,t]}(s)\bar{u}_s \in \mathcal{D}(I^i_{\tilde{B}})$.

□

Now we can easily derive a result about time reversal for Linear Skorohod SDE's.

Theorem 3.3.2 *Assume that the coefficients of the linear Skorohod SDE satisfy assumptions of Theorem 3.3.1. If $\{Z_t\}_{t\in[0,1]}$ is the solution of equation (3.5) then the time reversed process $\bar{Z}_t = Z_{1-t}$, is the unique solution in $L_1([0,1]\times\Omega)$ of the time reversed equation*

$$X_t - \bar{Z}_0 = \int_0^t -\bar{b}(s)X_s ds + I^i_{\tilde{B}}(-1_{[0,t]}\bar{\sigma}X) \tag{3.7}$$

where $\bar{b}(t) = b(1-t), \bar{\sigma}(t) = \sigma(1-t)$ and $\tilde{B}_t = B_1 - B_{1-t}$.

Proof. We need to prove uniqueness only. Let $Y_t \in L_1([0,1]\times\Omega)$ be another solution of the Equation (3.7). Then $\bar{Z}_t - Y_t \in L_1([0,1]\times\Omega)$ is also a solution of (3.7) with vanishing initial condition. Now all the assumptions of the Theorem 3.3.1 are satisfied. Hence $\bar{Z}_t - Y_t = 0$.

□

Example 3.3.1 *Linear diffusions.*

Applying Theorem 3.3.2 to the following diffusion equation:

$$X_t = x_0 + \int_0^t b(s)X_s ds + \int_0^t \sigma(s)X_s dB_s$$

we get

$$\bar{X}_t = \bar{X}_0 + \int_0^t -\bar{b}(s)\bar{X}_s ds + \int_0^t -\bar{\sigma}(s)\bar{X}_s * d\tilde{B}_s$$

where the last integral is the backward Itô integral. Thus we obtain the result in [42] in this special case of linear equation.

Example 3.3.2 *"Go and Return" problem.*

Let $X(t,x), Y(t,y)$ be the unique solutions of the following equations:

$$X(t,x) = x + \int_0^t b(s)X(s,x)ds + \int_0^t \sigma(s)X(s,x) \circ dB_s \tag{G}$$

$$Y(t,y) = y - \int_t^1 b(s)Y(s,y)ds - \int_t^1 \sigma(s)Y(s,y) \circ dB_s \tag{R}$$

The solutions X_t, Y_t are adapted to $\sigma\{B_s, s \leq t\}$ and $\sigma\{B_1 - B_s, s \geq t\}$ respectively.

Ogawa ([46]) proves that $Y(t, X(1,x))$ solves a modified Equation (R) with y replaced with $X(1,x)$ and the stochastic integral changed to the Ogawa integral δ_H^o with respect to the system of Haar functions. Moreover the following equality holds P-a.e., $\forall x \in R$,

$$Y(0, X(1,x)) = x \tag{G - R}$$

Note that the described above *"Go and Return"* problem is meaningless, unless it is stated with help of anticipative calculus.

Let us now consider the *"Go and Return"* problem in terms of Itô and Skorohod SDE's. Since the rules of integration here are different from those for the

Stratonovich and Ogawa integrals, one cannot expect that $Y(t, X(1,x))$ will be a solution for the Skorohod equation corresponding to the Equation (R) if X_t is a solution of the Itô or Skorohod equation corresponding to the Equation (G). Also the *"Go and Return"* relation (G-R) may be violated. Indeed, let us examine the following example:

$$X_t = x + \int_0^t X_s dB_s \tag{G^I}$$

$$Y_t = y - \int_t^1 Y_s * dB_s \tag{R^I}$$

The solutions are given by

$$\begin{aligned} X_t &= x \exp\{B_t - \frac{1}{2}t\} \\ Y_t &= y \exp\{-(B_1 - B_t) - \frac{1}{2}(1-t)\} \end{aligned}$$

In this case we have, $Y(0, X(1,x)) = xe^{-1}$ and $Y(t, X(1,x))$ does not satisfy

$$Y(t, X(1,x)) = X(1,x) - I^i(1_{[t,1]}(s)Y(s, X(1,x))) \tag{R^s}$$

which is easy to check by simply comparing expectations of both sides.

Because of the above example we state the *"Go and Return"* problem for Skorohod equations in the following way:

$$X(t,x) = x + \int_0^t b(s)X(s,x)ds + I_B^i(1_{[0,t]}(s)\sigma(s)X(s,x)) \tag{G^s}$$

$$\begin{aligned} Y(t, X(1,x)) = X(1,x) &- \int_t^1 b(s)Y(s, X(1,x))ds \\ &- I_B^i(1_{[t,1]}(s)\sigma(s)Y(s, X(1,x)) \end{aligned} \tag{R^s}$$

$$Y(0, X(1,x)) = x \tag{$G-R^s$}$$

where the first equation is either an Itô or a Skorohod equation and we impose conditions on the coefficients b and σ sufficient for uniqueness and existence of solutions for the Equation (G^s). Clearly

$$X(t,x) - X(1,x) = -\int_1^t b(s)X(s,x)ds - I_B^i(1_{[t,1]}(s)\sigma(s)X(s,x)).$$

Thus $Y(t, X(1,x)) = X(t,x)$ satisfies Equations (R^s) and $(G - R^s)$. Note that in the case when (G^s) is an Itô equation, this solution is adapted to the natural filtration of Brownian motion.

Let us now consider the following equation:

$$\begin{aligned} Y(t, X(1,x)) = X(1,x) &- \int_0^t b(1-s)Y(s, X(1,x))ds \\ &- I_{\tilde{B}}^i(1_{[0,t]}(s)\sigma(1-s)Y(s, X(1,x)). \end{aligned} \tag{R_1^s}$$

If process X_t describes "motion" of a particle then process Y_t can serve as a model for motion of a particle with reversed "velocity" $\bar{b}$ and under reversed random forces $\bar{\sigma}\tilde{B}$ (see Chapter 2 for more detailed discussion of kinematic properties of a random motion).

By Theorem 3.3.2 $\bar{X}(t,x)$ satisfies

$$\bar{X}(t,x) = X(1,x) - \int_0^t \bar{b}(s)\bar{X}(s,x)ds - I_{\tilde{B}}^i(1_{[0,t]}(s)\bar{\sigma}(s)\bar{X}(s,x).$$

Hence $Y(t, X(1,x)) = \bar{X}(t,x)$ solves the Equation (R_1^s), $Y(0, X(1,x)) = X(1,x)$, and

$$Y(1, X(1,x)) = x. \tag{$G - R_1^s$}$$

Moreover, under the smoothness assumptions of Theorem 3.4 on b and σ, process $\bar{X}(t,x)$ is the unique solution in $L_1([0,1] \times \Omega)$ of the Equation (R_1^s).

Finally, let us note that equations (R_1^s) and (R^s) are equivalent,

$$\begin{aligned} &Y(t, X(1,x)) - X(1,x) \\ &\quad = -\int_t^1 b(s)Y(s, X(1,x))ds - I_B^i(1_{[t,1]}(s)\sigma(s)Y(s, X(1,x))) \end{aligned}$$

$$= -\int_0^{1-t} b(1-s)Y(1-s, X(1,x))ds - I_{\tilde{B}}^i(1_{[t,1]}(1-s)\sigma(1-s)$$
$$\times Y(1-s, X(1,x))),$$

which is equivalent to

$$\bar{Y}(t, X(1,x)) = X(1,x) - \int_0^t \bar{b}(s)\bar{Y}(s, X(1,x))ds - I_{\tilde{B}}^i(1_{[0,t]}(s)\bar{\sigma}(s)\bar{Y}(s, X(1,x))).$$

Thus also the Equation (R^s) have a unique solution in $L_1([0,1] \times \Omega)$ since otherwise the Equation (R_1^s) would have different solutions.

Appendix

A Abstract Wiener Space

Our framework concerned a structure related to general Gaussian process, i.e. $(\imath, H, \mathcal{X})$ with $\mathcal{X}$ a LCTVS. However, it seems desirable to review construction of an Abstract Wiener Space (AWS) in order to have better general background and understanding in the case when the Gaussian process is a Brownian motion.

Here, H always denotes a Hilbert space and B a Banach space. Both spaces are real and separable. H^* and B^* denote the dual spaces. We will always identify H^* with H.

A subset $\mathcal{C}$ of a Banach space B is called a **cylinder** if it can be written in the following form:

$$\mathcal{C} = \{e \in B : \ ((e, y_1), ..., (e, y_n)) \in A\}$$

where $\{y_1, ..., y_n\} \subset B^*$ and A is a Borel subset of $\mathbf{R}^n$. By $Cyl(B)$ we will denote the collection of all cylinders in B.

Note that, for example, any $\mathcal{C} \in Cyl(H)$ can be written in the form: $\mathcal{C} = \{h \in H, \ Ph \in A\}$, where $P \in \mathcal{P}(H)$ (a finite dimensional projection on H), and A is a Borel subset of $P(H)$.

The (canonical) **Gauss measure** γ_σ, with parameter $\sigma > 0$, on a Hilbert space

H, is a function on cylinder subsets of H defined by

$$\gamma_\sigma(\mathcal{C}) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_A exp\{-\frac{\|x\|_H^2}{2\sigma^2}\}dx,$$

for $\mathcal{C} \in Cyl(H)$, $\mathcal{C} = \{h \in H : \ Ph \in A\}$, $P \in \mathcal{P}(H)$. Here $n = dimP(H)$, $\|\cdot\|_H$ is the norm in H and dx denotes the Lebesgue measure on $P(H)$. We will write γ for γ_1.

The Gauss measure γ has no σ-additive extension from $Cyl(H)$ unless the Hilbert space H is finite dimensional (see [27] for the proof). To obtain a σ-additive extension one constructs a Banach space B containing H and studies σ-fields on B.

A seminorm $\|\cdot\|$ on a Hilbert space H is called a **measurable seminorm** if

$$\forall \varepsilon > 0 \ \exists P_0 \in \mathcal{P}(H) \ \forall P \perp P_0, \ P \in \mathcal{P}(H) : \ \gamma(\|Ph\| > \varepsilon) < \varepsilon.$$

Let $\|\cdot\|$ be a measurable norm on a Hilbert space H and define B to be a completion of H with respect to the norm $\|\cdot\|$. Then B is a Banach space and the following relation holds ([21]):

$$B^* \underset{\imath^*}{\hookrightarrow} H \underset{\imath}{\hookrightarrow} B$$

where $\imath$ is the natural embedding and $\imath^*$ is its conjugate: $\imath^*(e^*)(h) = e^*(i(h))$, with $e^* \in B^*$, $h \in H$. Both embeddings, $\imath$ and $\imath^*$ are continuous and have dense ranges.

Let μ_σ be a function on cylinder subsets of B, induced by the Gauss measure on H, that is,

$$\mu_\sigma(\mathcal{C}) = \gamma_\sigma(\mathcal{C} \cap H),$$

for any $\mathcal{C} \in Cyl(B)$. We will also write μ for μ_1. Note that the above definition of μ_σ is correct, since $B^* \underset{\imath^*}{\hookrightarrow} H$.

Next theorem ([21]) provides the σ-additive extension of γ, the following (Theorem 4.2, [27]) identifies the cylindrical σ-field with the Borel σ-field on B.

Theorem A.1 *The set function μ defined on $Cyl(B)$ induced by the Gauss measure γ on H has a σ-additive extension (denoted further also by μ) to the σ-field generated by $Cyl(B)$.*

Theorem A.2 *The σ-field generated by $Cyl(B)$ is the Borel σ-field of the Banach space B.*

The triple $(\imath, H, B)$ is called an **Abstract Wiener Space**. The measure μ on B of Theorem A.1 is called the **Wiener measure**. The measure μ_σ, a σ-additive extension of the set function μ_σ from $Cyl(B)$, is called the **Wiener measure with variance σ^2**.

Example A.1 *Standard AWS.*

Let $C_0 = C_0[0,1]$ be the Banach space of continuous functions on $[0,1]$ vanishing at zero, endowed with the supremum norm. Let C' be the Hilbert space of absolutely continuous functions in C_0 with square integrable derivatives, with respect to the scalar product $(f,g) = \int_0^1 f'(t)g'(t)dt$. Then the triple $(\imath, C', C_0)$ is an AWS (see [27] for the proof).

Now recall the Brownian motion process of Example 1.2.1 (a). There exists a version of this process with continuous sample paths (Theorem 37.1, [4]). This means that the Banach space C_0 can be considered as the set of sample paths of the continuous version of Brownian motion. As mentioned in Example 1.2.1 (a), C' is the RKHS of this process. The Wiener measure μ on C_0 is the extension of the Gauss measure on C'. For any $\mathcal{C} \in Cyl(C_0)$ of the form: $\mathcal{C} = \{B \in C_0 : (B(t_1), B(t_2), ..., B(t_n)) \in A\}$, A - a Borel subset of $\mathbf{R}^n$, the Wiener measure $\mu(\mathcal{C})$ can be expressed as follows:

$$\mu(\mathcal{C}) = [(2\pi)^n t_1(t_2 - t_1)...(t_n - t_{n-1})]^{-\frac{1}{2}}$$
$$\times \int_A exp\{-[\frac{u_1^2}{t_1} + \frac{(u_2 - u_1)^2}{t_2 - t_1} + ... + \frac{(u_n - u_{n-1})^2}{t_n - t_{n-1}}]/2\}du_1...du_n.$$

This is the probability of the event $\{(B(t_1), B(t_2), ..., B(t_n)) \in A\}$ for Brownian motion $B = \{B_t,\ t \in [0,1]\}$.

B Backward Itô and Fisk–Stratonovich Integrals

With Brownian motion $\{B_t, t \in [0,1]\}$, we associate two filtrations, $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$ and $\mathcal{F}^t = \sigma\{B_1 - B_s, s \geq t\}$, with the convention $\mathcal{F}_0 = \mathcal{F}^1 =$ the trivial σ-field. The first filtration is increasing as t increases and the second is increasing as t decreases. The forward Itô integral is defined for processes adapted to the natural filtration, $\{\mathcal{F}_t\}_{t\in[0,1]}$. The **Backward Itô integral** can be defined for processes $\{u_t, t \in [0,1]\}$, adapted to the "backward" filtration $\{\mathcal{F}^t\}_{t\in[0,1]}$, satisfying condition $E\int_0^1 u_s^2 ds < \infty$.

Let $u_k \in L_2(\Omega, \mathcal{F}^{t_k})$, $k = 0, 1, ..., n$, u_{n+1} an $\mathcal{F}^1$ measurable random variable. Assume that the sum

$$\sum_{k=0}^{n-1} u_{k+1} 1_{[t_k, t_{k+1})} + u_{n+1} 1_{\{1\}}$$

converges to u in $L_2(\Omega \times [0,1])$. Here, $0 = t_0 < t_1 < t_2 < ... < t_n = 1$. Then, the **backward Itô integral** of u is defined as the limit in $L_2(\Omega)$ of the sum

$$\sum_{k=0}^{n-1} u_{k+1}(B_{t_{k+1}} - B_{t_k})$$

and denoted by $\int_0^1 u_t * dB_t$. This definition is not ambiguous because the limit defining the backward Itô integral does not depend on the choice of the sequence approximating u in $L_2(\Omega \times [0,1])$.

Note, that the backward Itô integral of u coincides with the forward Itô integral of the process $\{\bar{u}_t = u_{1-t}, t \in [0,1]\}$ with respect to Brownian motion $\{B_1 - B_{1-t}, t \in [0,1]\}$. For more extensive discussion and applications we refer to the work of Kunita [26].

A random process $\{u_t, t \in [0,1]\}$, such that $P(\int_0^1 u_t^2 dt < \infty) = 1$ is said to be **Fisk–Stratonovich integrable**, if the following limit:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{(u_{t_{k+1}} - u_{t_k})}{2} (B_{t_{k+1}} - B_{t_k})$$

exists in probability for any sequence of partitions $0 = t_0 < t_1 < t_2 < ... < t_n = 1$, with $\max\{t_{k+1} - t_k, k = 0, ..., n-1\} \to 0$ as $n \to \infty$ and the limit is independent of the choice of the sequence of partitions. The Fisk–Stratonovich integral of u is denoted by $\int_0^1 u_t \circ dB_t$. For further properties and references see [45].

C Hilbert–Schmidt and Trace Class Operators on Hilbert Space

Let H be a separable Hilbert space. A linear operator $T : H \to H$ is called **Hilbert–Schmidt** if it admits a representation of the form

$$Th = \sum_{n=1}^{\infty} \lambda_n (h, h_n)_H e_n$$

where $h \in H$, $\{e_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty}$, are orthonormal sets in H, $\lambda_n > 0$, $n = 1, 2, ...$ and $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

Equivalently, T is a Hilbert–Schmidt operator on H if for some (hence for any) orthonormal basis $\{e_n\}_{n=1}^{\infty} \subset H$, $\sum_{n=1}^{\infty} \|Th_n\|_H^2 < \infty$.

With the above notation, the Hilbert–Schmidt norm of a Hilbert–Schmidt operator T is defined as follows:

$$\|T\|_2 = (\sum_{n=1}^{\infty} \|Th_n\|_H^2)^{\frac{1}{2}} = (\sum_{n=1}^{\infty} \lambda_n^2)^{\frac{1}{2}}.$$

The middle sum above is independent of the choice of the orthonormal basis.

The collection of Hilbert–Schmidt operators on H, with the norm $\| \; \|_2$ is a Hilbert space, denoted here by $H^{\otimes 2}$. The scalar product of two Hilbert–Schmidt operators $T, S \in H^{\otimes 2}$ is given explicitly by $(T,S)_{H^{\otimes 2}} = \sum_{n=1}^{\infty}(Te_n, Se_n)_H$, where $\{e_n\}_{n=1}^{\infty} \subset H$ is an orthonormal basis.

A linear operator $T : H \to H$ is called **trace class** if

$$\|T\|_1 = \sup \sum_{n=1}^{\infty} |(Th_n, e_n)_H| < \infty,$$

where the supremum is taken over all orthonormal systems of vectors $\{e_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty} \subset H$. The quantity $\|T\|_1$ is called the trace class norm of T.

Trace class operators on H, with the trace class norm $\| \; \|_1$, form a Banach space.

Every trace class operator is automatically a Hilbert–Schmidt operator and the following relation holds:

$$\|T\|_1 \geq \|T\|_2 \geq \|T\|$$

where the latter norm is the operator (supremum) norm.

In the conclusion, let us recall the notion of a tensor product of unitary spaces (i.e. linear spaces with scalar products). Let H, K be unitary spaces with bases $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ respectively. The tensor product $H \otimes K$ of the spaces H and K is the linear space, whose basis is formed by the pairs (e_i, f_j), denoted by $e_i \otimes f_j$. With every pair $x = \sum \alpha_i e_i \in H$, $y = \sum \beta_j f_j \in K$, we associate an element $x \otimes y = \sum \alpha_i \beta_j e_i \otimes f_j \in H \otimes K$.

The linear space $H \otimes K$ can be made into a unitary space by defining the scalar product as follows:

$$(x_1 \otimes y_1, x_2 \otimes y_2)_{H\otimes K} = (x_1, x_2)_H (y_1, y_2)_K.$$

In particular, $H^{\otimes 2}$, the space of Hilbert–Schmidt operators on H, is the completion of the unitary space $H \otimes H$ in the Hilbert–Schmidt norm under the identification of Section 2.2.

The role of tensor product is emphasized by the fact that there exists a bilinear map $\varphi : H \times K \to H \otimes K$, such that given any linear space L and a bilinear map $b : H \times K \to L$, there exists a linear map $l : H \otimes K \to L$, replacing b, in the sense, that $b = l \circ \varphi$.

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