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Global Existence and Blow-up of Solutions
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GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR NONLINEAR WAVE EQUATIONS

By

HENGLI JIAO

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ABSTRACT

GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR NONLINEAR WAVE EQUATIONS

By

HENGLI JIAO

This dissertation studies the global existence and finite-time blow-up of solutions to the nonlinear wave equation $u_{tt} - \Delta u = G(u, Du, D^2u)$ in high space dimensions. For $G(u, Du, D^2u) = |u|^p$, we establish the global existence of solutions with small initial data when $p > \frac{3+\sqrt{17}}{4}$ for $n = 5$, $p > \frac{2+\sqrt{10}}{3}$ for $n > 7$ respectively; we also prove that most solutions of the equation blow up when $1 < p < \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$ for $n \geq 3$. For $G(u, Du, D^2u) = |u_r|^p, |u_t|^p, |u_{rr}|^p$ or $|u_{tt}|^p$, we prove that most solutions of the equations blow up when $n \geq 2$ and $1 < p \leq \frac{n+1}{n-1}$; we also estimate the life-span of the solutions when $n \geq 2$ and $p = \frac{n+1}{n-1}$.

To: my wife and our parents

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Contents

Introduction

The existence of global solutions of initial value problems for nonlinear partial differential equations has been studied extensively. There are many partial but only few general results. In this dissertation, we are concerned with equations of the following type

$$\begin{aligned} &u_{tt} - \Delta u = G(u, Du, D^2 u), \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \end{aligned} \tag{0.1}$$

where $f, g \in C_0^\infty(\mathbb{R}^n)$.

We concentrate here on the following topics:

(a) Global existence of the solution, i.e., existence of solution u(x, t) for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ with smooth initial data.

(b) Finite-time blow-up of solutions, i.e., u or the derivatives of u of certain order tend to infinity as t tends to some point T_0 with $T_0 < \infty$.

(c) Life span of a solution, i.e., a solution that exists for all x up to a maximal time T_0, in the sense that u(x, t) can be defined at most on interval $[0, T_0)$.

Let $G(u, Du, D^2u) = |u|^p$ with $p > 1$. (0.1) becomes semilinear wave equation

$$\begin{aligned} &u_{tt} - \Delta u = |u|^p, \\ &u(x,0) = f(x), \quad u(x,0) = g(x). \end{aligned} \tag{0.2}$$

Equation (0.2) was first studied by John [6] who proved that for $n = 3$ and $1 < p < 1 + \sqrt{2}$, a global solution does not exist for any smooth non-trival data with compact support; and for $p > 1 + \sqrt{2}$, a global solution exists provided that the initial data are sufficiently small by the contraction mapping theorem. Later, Glassey [3], by using differential inequalities, has showed that for $n = 2$ and $1 < p < \frac{3+\sqrt{17}}{2}$ the solutions must blow up in finite time. Glassey [4] also proved that for $n = 2$, $p > \frac{3+\sqrt{17}}{2}$ a global solution exists for small data by estimating the solution and using contraction mapping theorem. Moreover, Glassey [4] conjectured that for $n \geq 2$, there exists a critical value $p_0(n)$ such that most solutions blow up in finite time when $1 < p \leq p_0(n)$, and a small global solutions exist for $p > p_0(n)$ in weighted L_∞ space with weight function

$$\phi(|x|, t) = (t + |x| + 2k)^{\frac{n-1}{2}}(t - |x| + 2k)^q$$

and $q = \frac{(n-1)p-n-1}{2}$, where $p_0(n)$ is the positive root of quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{0.3}$$

i.e.,

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)}.$$

Schaeffer [15] proved that for $n = 2$ and $p = p_0(n) = \frac{3+\sqrt{17}}{2}$, $n = 3$ and p=$p_0(3) = 1 + \sqrt{2}$, the solution must blow up in finite time. To obtain all the results mentioned above, the positivity of the fundamental solution for $n \leq 3$ was essential. By using some properties of special functions, Sideris [18] proved that for $n \geq 4$ and $1 < p < p_0(n)$, the solution blows up in finite time provided that some restrictions to the initial data are satisfied.

In Chapter 1, we are concerned with the blow-up of solutions to the semilinear wave equation for $n > 3$ in terms of p. Difficulty arises because the fundamental solution for the wave equation is no longer positive in higher dimensions ($n > 3$). Our approach to overcome the non-positivity of the fundamental solutions is to perform analysis near the wave front where the fundamental solution is positive. We will prove that for $n \geq 2$ and $1 < p < p_0(n)$, the solutions blow up provided that certain conditions on initial data are satisfied. In comparison with Sideris's proof, our proof is much simpler. Moreover, we will show that for $n \geq 3$ and $p = p_0(n)$, the solutions blow up in finite time provided that the initial data are large.

The question of existence of the small global solutions is open when $n > 3$ and $p > p_0(n)$. In Chapter 2 and 3, by modifying the method of Sideris [17] and Schaeffer [14] for quasilinear wave equations and using some technical estimates, we prove that when $p > p_0(5) = \frac{3+\sqrt{17}}{4}$ for $n = 5$, $p > \frac{2+\sqrt{10}}{3}$ for $n > 7$ respectively, there exists a global symmetric solution in weighted L_∞ space provided the initial data are small. However, the weight function is different from the one that Glassey conjectured.

Let $G(u, Du, D^2u) = |u_r|^p, |u_t|^p, |u_{tt}|^p$ or $|u_{rr}|^p$ in equation (0.1). Klainerman and Ponce [11] have shown that the equation has a small global solution if the initial data are small enough and p satisfies $(n-1)(p-1)^2 - 2p > 0$. However, it was conjectured that the critical value is $p_1(n) = \frac{n+1}{n-1}$ in this case.

Sideries [17] studied the case when $G(u, Du, D^2u) = |Du|^p$ and proved global existence of small radially symmetric solutions when $p > 2$ and finite-time blow-up of solutions for u_r^2. John [7] and Schaeffer [16] have shown finite-time blow-up of solutions with $|u_t|^3$ and $|u_r|^3$ for $n = 2$. By using the same method of [17], Schaeffer [14] has shown $p_1(5) = \frac{3}{2}$. By performing analysis on wave front, Rammaha [13] has shown finite time blow-up to the nonlinearity $|u_r|^p$ and $|u_t|^p$ provided that $p = \frac{n+1}{n-1}$ when n is odd and $1 < p < \frac{n+1}{n-1}$ when n is even.

In Chapter 4, by modifying Rammaha's method and using more technical estimates, we prove finite-time blow-up to the nonlinearity $|u_r|^p$ and $|u_t|^p$ with $p = \frac{n+1}{n-1}$ for any $n \geq 2$ and extend the results to the nonlinearity $|u_{rr}|^p$ and $|u_{tt}|^p$ with $p = \frac{n+1}{n-1}$ for any $n \geq 2$. By carefullyestimating the solution and using blow-up results from ODE, we will also obtain the life-spans of the solutions.

Let $G(u, Du, D^2u) = |u|^\alpha u_t$, or $|u_t|^\alpha u_{tt}$. John [8] has showed that for $n = 3$ and $\alpha = 1$, the solutions blow up in finite time with some restrictions on the initial data. John [7] has also proved that the finite-time blow-up of solutions to the nonlinearity $F'(u_t)u_{tt}$ with $n = 2$ and 3 for certain initial data. We also extend this kind of blow-up results to higher space dimensions in chapter 4.

Chapter 1

Finite-Time Blow-Up for Solutions of Semilinear Wave Equations

1.1 Preliminaries

Consider the semilinear wave equation

$$\begin{aligned} &u_{tt} - \Delta u = |u|^p, \qquad x \in \mathbb{R}^n, t \in \mathbb{R}, \\ &u(x,0) = 0, \quad u_t(x,0) = g(x), \end{aligned} \tag{1.1}$$

where g satisfies the following assumptions through this chapter

(a) $g \in C_0^\infty(\mathbb{R}^n)$, $\quad g(x) \geq 0$ and $supp\{g\} \subset \{x : |x| < k\}$,

(b) there exist positive constants k_0 and k_1 with $0 < k_0 < k_1 < k$ such that

(i) $g(x) \not\equiv 0$ on $\{x : k_0 < |x| < k_1\}$

(ii) k_0 is sufficiently close to k such that $\frac{k_0}{k} > z_0$, where z_0 satisfies that for all $z \in [z_0, 1)$, $P_n(z) \geq \frac{1}{2}$ and $T_n(z) \geq \frac{1}{2}$, P_n and T_n are the Legendre and Tschebyscheff polynomials of degree n , respectively.

We will prove the following blow-up theorems:

Theorem 1.1 *Let* $u \in C^2(\mathbb{R}^n \times [0,T))$ *with* $n \geq 2$ *be a solution of (1.1), where* $T > 0$ *is the life-span of* u *and* $1 < p < p_0(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$. *Then* $T < +\infty$. *That is,* u *blows up in finite time.*

Theorem 1.2 *Consider the equation*

$$\begin{aligned} &u_{tt} - \Delta u = |u|^p, \quad (x,t) \in \mathbb{R}^n \times [0,T), \ \ n \geq 2 \\ &u(x,0) = 0, \quad u_t(x,0) = \tfrac{1}{\varepsilon} g(x), \end{aligned} \tag{1.2}$$

where g satisfies (a) and (b) in Theorem 1.1, $p = p_0(n)$ and ε is small. Then $T < \infty$

Given $u \in C^2(\mathbb{R}^n \times [0,T)$ define $\bar{u}$ to be the spherical mean of u, i.e,

$$\bar{u}(r,t) = \frac{1}{\omega_n} \int_{|\omega|=1} u(r\omega, t) d\omega$$

for all $(r,t) \in \mathbb{R} \times [0,T)$. Then by Daboux's identity we have

$$(\bar{u})_{tt} - \Delta \bar{u} = \overline{(u_{tt} - \Delta u)} = \overline{|u|^p} = \frac{1}{\omega_n} \int_{|\omega|=1} |u(r\omega,t)|^p d\omega \geq |\bar{u}|^p .$$

Note that

$$\Delta \bar{u}(r,t) = \bar{u}_{rr} - \frac{n-1}{r} \bar{u}_r .$$

We are led to the following radially symmetric equation

$$\begin{aligned} &u_{tt} - \tfrac{n-1}{r} u_r - u_{rr} = G(u), \quad (r,t) \in \mathbb{R} \times (0, +\infty), \\ &u(r,0) = 0, \quad u_t(r,0) = g(r), \end{aligned} \tag{1.3}$$

where $G(u) \geq |u|^p$ (we used u instead of $\bar{u}$ to simplify notation).

It is well-known from Duhamel's principle that

Lemma 1.1 *[13] The solution of equation (1.3) has the following form if n is odd,*

$$u(r,t) = u^0(r,t) + C r^{\frac{1-n}{2}} \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^{\frac{n-1}{2}} P_{\frac{n-3}{2}} \left(\frac{\lambda^2 + r^2 - (t-\tau)^2}{2r\lambda} \right) G(u(\lambda,\tau)) d\lambda d\tau,$$

where

$$u^0(r,t) = C r^{\frac{1-n}{2}} \int_{|r-t|}^{r+t} \lambda^{\frac{n-1}{2}} P_{\frac{n-3}{2}} \left(\frac{\lambda^2 + r^2 - t^2}{2r\lambda} \right) g(\lambda) d\lambda,$$

and $C = C(n) > 0$.

If n is even,

$$\begin{aligned} &u(r,t) = u^0(r,t) + C r^{\frac{2-n}{2}} \int_0^t \int_0^{t-\tau} ((t-\tau)^2 - \rho^2)^{-\frac{1}{2}} \rho \\ &\int_{|r-\rho|}^{r+\rho} \lambda^{\frac{n}{2}} \left[(\lambda^2 - (r-\rho)^2)((r+\rho)^2 - \lambda^2) \right]^{-\frac{1}{2}} \\ &T_{\frac{n-2}{2}} \left(\frac{\lambda^2 + r^2 - (t-\tau)^2}{2r\lambda} \right) G(u(\lambda,\tau)) d\lambda d\rho d\tau , \end{aligned}$$

where

$$u^0(r,t) = Cr^{\frac{2-n}{2}} \int_0^t (t^2-\rho^2)^{-\frac{1}{2}} \rho \int_{|r-\rho|}^{r+\rho} \lambda^{\frac{n}{2}} \left[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)\right]^{-\frac{1}{2}}$$
$$T_{\frac{n-2}{2}}\left(\frac{\lambda^2+r^2-\rho^2}{2r\lambda}\right) g(\lambda)d\lambda d\rho.$$

In both cases, u^0 is the solution to linear wave equation

$$u_{tt} - \Delta u = 0 \ ,$$
$$u(r,0) = 0, u_t(r,0) = g(r).$$

Lemma 1.2 *Suppose that u and u^0 are same as in Lemma 1.1, then*

$$u(r,t) \geq u^0(r,t) + Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(u(\lambda,\tau)) d\lambda d\tau,$$
$$u^0(r,t) \geq Cr^{\frac{1-n}{2}} \int_{r-t}^{r+t} \lambda^{\frac{n-1}{2}} g(\lambda) d\lambda \tag{1.4}$$

for $t+k_0 < r < t+k_1$.

Proof of Lemma 1.2. If n is even, then

$$u^0(r,t) = Cr^{\frac{2-n}{2}} \int_0^t (t^2-\rho^2)^{-\frac{1}{2}} \rho \int_{|r-\rho|}^{r+\rho} \lambda^{\frac{n}{2}} \left[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2\right]^{-\frac{1}{2}}$$

$$T_{\frac{n-2}{2}}\left(\frac{\lambda^2+r^2-\rho^2}{2r\lambda}\right) g(\lambda)d\lambda d\rho.$$

Since $\rho \in [0,t]$, $\lambda \in (r-\rho, r+\rho) \cap [0,k]$ and $t+k_0 < r < t+k_1$, imply that

$$\frac{\lambda^2+r^2-\rho^2}{2r\lambda} \geq \frac{(r-\rho)^2+r^2-\rho^2}{2r\lambda} \geq \frac{r-\rho}{\lambda} \geq \frac{r-t}{k} \geq \frac{k_0}{k},$$

we have

$$T_{\frac{n-2}{2}}\left(\frac{\lambda^2+r^2-\rho^2}{2r\lambda}\right) \geq \frac{1}{2}$$

and

$$u^0(r,t) \geq Cr^{\frac{2-n}{2}} \int_0^t (t^2-\rho^2)^{-\frac{1}{2}} \rho \int_{|r-\rho|}^{r+\rho} \lambda^{\frac{n}{2}} \left[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)\right]^{-\frac{1}{2}} g(\lambda)d\lambda d\rho.$$

Changing order of the integral leads to

$$u^0(r,t) \geq Cr^{\frac{2-n}{2}} \int_{r-t}^{r+t} \lambda^{\frac{n}{2}} g(\lambda) d\lambda \int_{|\lambda-t|}^t [(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{-\frac{1}{2}} \rho (t^2-\rho^2)^{-\frac{1}{2}} d\rho.$$

Note that under the range of the integration

$(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)=(\rho^2-(r-\lambda)^2)(r+\lambda-\rho)(\lambda+r+\lambda)$, and

$\rho^2-(r-\lambda)^2\le t^2-(r-\lambda)^2, \quad r+\lambda-\rho\le r+\lambda-|\lambda-r|\le 3\lambda$

$r+\lambda+\rho\le r+(r+t)+t\le 2r+2t\le 4r.$

Hence

$$u^0(r,t)\ge Cr^{\frac{1-n}{2}}\int_{r-t}^{r+t}\lambda^{\frac{n}{2}}g(\lambda)d\lambda\int_{|\lambda-t|}^{t}(t^2-(r-\lambda))^{-\frac{1}{2}}\rho(t^2-\rho^2)^{-\frac{1}{2}}d\rho d\lambda\ .$$

Note that

$$\int_{|\lambda-t|}^{t}(t^2-(r-\lambda))^{-\frac{1}{2}}\rho(t^2-\rho^2)^{-\frac{1}{2}}d\rho=1,$$

we have then

$$u^0(r,t)\ge Cr^{\frac{1-n}{2}}\int_{r-t}^{r+t}\lambda^{\frac{n-1}{2}}g(\lambda)d\lambda,$$

$$Q(r,t)=Cr^{\frac{2-n}{2}}\int_0^{\tau}\int_0^{t-\tau}((t-\tau)^2-\rho^2)^{-\frac{1}{2}}\rho\int_{|r-\rho|}^{r+\rho}\lambda^{\frac{n}{2}}\left[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)\right]^{-\frac{1}{2}}$$

$$T_{\frac{n-2}{2}}\left(\frac{\lambda^2+r^2-(t-\tau)^2}{2r\lambda}\right)G(u(\lambda,\tau))d\lambda d\rho d\tau,$$

and

$$\frac{\lambda^2+r^2-\rho^2}{2r\lambda}\ge\frac{(r-\rho)^2+r^2-\rho^2}{2r\lambda}\ge\frac{r-\rho}{\lambda}\ge\frac{r-t+\tau}{\lambda}\ge\frac{\tau+k_0}{\tau+k}\ge\frac{k_0}{k}$$

then

$$T_{\frac{n-2}{2}}\left(\frac{\lambda^2+t^2-\tau^2}{2\lambda r}\right)\ge\frac{1}{2}$$

therefore

$$Q(r,t)\ge Cr^{\frac{2-n}{2}}\int_0^{\tau}\int_0^{t-\tau}((t-\tau)^2-\rho^2)^{-\frac{1}{2}}\rho\int_{|r-\rho|}^{r+\rho}\lambda^{\frac{n}{2}}\left[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)\right]^{-\frac{1}{2}}\ .$$

$$G(u(\lambda,\tau))d\lambda d\rho d\tau$$

Changing the order of the integration leads to

$$Q(r,t)\ge Cr^{\frac{2-r}{2}}\int_0^{t}d\tau\int_{r-t+\tau}^{r+t-\tau}\lambda^{\frac{n}{2}}G(u(\lambda,\tau))d\lambda\int_{|\lambda-r|}^{t-\tau}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2]^{-\frac{1}{2}}$$

$$\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}d\rho.$$

A similar argument to the estimate of u^0 yields

$$\int_{|\lambda-r|}^{t-\tau}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2]^{-\frac{1}{2}}\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}d\rho \geq C\lambda^{-\frac{1}{2}}r^{-\frac{1}{2}},$$

therefore

$$Q(r,t) \geq Cr^{\frac{1-n}{2}}\int_0^t\int_{r-t+\tau}^{r+t-\tau}\lambda^{\frac{n-1}{2}}G(u(\lambda,\tau))d\lambda d\tau.$$

□

1.2 Proof of the Theorems

Proof of Theorem 1.1. Define

$$F(t) = \int_{\mathbb{R}^n} u(r,t)dx,$$

The idea is to show that $F(t)$ blows up in finite time.

It is easy to see that

$$\dot{F}(t) = \int_{\mathbb{R}^n} u_t(r,t)dx,$$

$$\ddot{F} = \frac{d^2}{dt^2}\int_{\mathbb{R}^n} u(r,t)dx = \int_{\mathbb{R}^n} u_{tt}(r,t)dx = \int_{\mathbb{R}^n}(\Delta u(r,t) + G(u))dx,$$

by the divergence theorem and support property of u

$$\int_{\mathbb{R}^n}\Delta u(r,t)dx = 0.$$

Therefore for any $t > 0$,

$$\begin{aligned}\ddot{F}(t) &= \int_{\mathbb{R}^n} G(u(r,t)dx \geq \int_{\mathbb{R}^n}|u|^p(r,t)dx \\ &\geq \left|\int_{\mathbb{R}^n} u(r,t)dx\right|^p\left[\int_{|x|<t+k}dx\right]^{1-p} \\ &= |F(t)|^p[\omega_n(t+k)^n]^{1-p} \qquad (1.5)\end{aligned}$$

for any $t \geq 0$.

By assumptions on g, there exists an $r_0 \in (k_0, k_1)$ and a $\delta > 0$ such that $g(r) \neq 0$ for $r \in (r_0-\delta, r_0+\delta)$. Hence

$$u^0(r,t) \geq Cr^{\frac{1-n}{2}}\int_{r-t}^{r+t}\lambda^{\frac{n-1}{2}}g(\lambda)d\lambda \geq Cr^{\frac{1-n}{2}}\int_{r_0-\delta}^{r_0+\delta}\lambda^{\frac{n-1}{2}}g(\lambda)d\lambda \geq C(g)r^{\frac{1-n}{2}},$$

where

$$C(g) = C \int_{r_0-\delta}^{r_0+\delta} \lambda^{\frac{n-1}{2}} g(\lambda) d\lambda > 0.$$

From Lemma 1.2, we have $u(r,t) \geq u^0(r,t) > 0$ for $t+k_0 < r < t+k_1$. Therefore

$$\begin{aligned}
&\int_{t+k_0<r<t+k_1} |u(r,t)|^p \, dx \geq \int_{t+k_0<r<t+k_1} \left|u^0(r,t)\right|^p dx \\
&\geq \int_{t+k_0}^{t+k_1} (C(g))^p r^{\frac{(1-n)p}{2}} r^{n-1} \omega_n dr \\
&\geq (k_1-k_0)\omega_n (C(g))^p (t+k_0)^{\frac{(1-n)p}{2}+n-1}.
\end{aligned}$$

There exist a constant $A_0 > 0$ and a T_0 such that for $t > T_0$

$$\ddot{F}(t) \geq A_0 \omega_n (C(g))^p (t+k)^{\frac{(1-n)p}{2}+n-1}. \tag{1.6}$$

After two integrations, we have

$$F(t) \geq A_1 (C(g))^p (t+k)^{\frac{(1-n)p}{2}+n+1} + Bt + E \tag{1.7}$$

for t large enough.

Since $1 < p_0(n) < \frac{2n}{n-1}$, we have $n+1-\frac{n-1}{2} > n+1-\frac{n-1}{2}p_0 > n+1-\frac{n-1}{2}\frac{2n}{n-1} = 1$. Therefore if t is large enough, the linear term in (1.7) can be absorbed into first term. That is, there exists a $T_1 > T_0$ and an A_2 such that for $t > T_1$

$$F(t) \geq A_2 (C(g))^p (t+k)^{n+1-\frac{n-1}{2}p}. \tag{1.8}$$

Since $\ddot{F}(t) > 0$ for all t and $\dot{F}(0) > 0$, hence $\dot{F}(t) > 0$ for all t. Combining (1.3) and (1.8) yields

$$\begin{aligned}
&\dot{F}(t)\ddot{F}(t) \geq \dot{F}(t) F^p(t) [\omega_n (t+k)^n]^{1-p}, \\
&\dot{F}(t)\ddot{F}(t) \geq \dot{F}(t) F(t) [\omega_n (t+k)^n]^{1-p} A_2 (C(g))^p (t+k)^{(n+1-\frac{n-1}{2}p)(p-1)}.
\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\dot{F}^2(t) \geq A_3 (t+k)^{-\frac{n-1}{2}p^2+\frac{n+1}{2}p+1-2} \dot{F}(t) F(t)\,.$$

Let $\delta = -(-\frac{n-1}{2}p^2 + \frac{n+1}{2}p - 1) = (\frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1) + 2$, then $0 < \delta < 2$ for $n \geq 3$ and $1 < p < p_0(n)$.

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\dot{F}^2(t) &\geq A_3[(t+k)^{-\delta}\dot{F}(t)F(t)+\delta(t+k)^{-\delta-1}F^2(t)\\ &\geq \frac{A_3}{2}\frac{d}{dt}[(t+k)^{-\delta}F^2(t)],\end{aligned}$$

which implies that

$$\dot{F}(t)\geq A_3[(t+k)^{-\delta}F^2(t)-F^2(T_1)(T_1+k)^{-\delta}]+\dot{F}^2(T_1).$$

Since

$$\begin{aligned}F^2(t)(t+k)^{-\delta} &\geq C(t+k)^{2(n+1)-(n-1)p-2-(\frac{n-1}{2}p^2-\frac{n+1}{2}p-1)}\\ &\geq C(t+k)^{(n-1)(\frac{2n}{n-1}-p)}\to\infty \quad \text{as } t\to\infty,\end{aligned}$$

There exists $T_2>T_1$ such that for $t\geq T_2$,

$$\dot{F}^2(t)\geq\frac{1}{4}A_3F^2(t)(t+k)^{-\delta}.$$

Therefore

$$\frac{\dot{F}(t)}{F(t)}\geq[\frac{1}{4}A_3(t+k)^{-\delta}]^{\frac{1}{2}}$$

which implies that

$$lnF(t)-lnF(T_2)>\frac{1}{2}A_3^{\frac{1}{2}}\frac{1}{1-\frac{\delta}{2}}[(t+k)^{1-\frac{\delta}{2}}-(T_2+k)^{1-\frac{\delta}{2}}]\,.$$

There exists a $T_3>T_2$, such that for $t>T_3$

$$lnF(t)>A_4(t+k)^{1-\frac{\delta}{2}},$$

$$F(t)>e^{A_4(t+k)^{1-\frac{\delta}{2}}},$$

where $A_4=\frac{1}{4}A_3^{\frac{1}{2}}\frac{1}{1-\frac{\delta}{2}}$ and $1-\delta>0$. Hence, there exists a $T_4>T_3$ such that for $t>T_4$

$$F(t)>(t+k)^{2n}. \tag{1.9}$$

Then by (1.5) for $t>T_4$

$$\ddot{F}(t)\geq\omega_n^{1-p}(t+k)^{-n(p-1)}|F(t)|^{\frac{p-1}{2}}|F(t)|^{\frac{n+1}{2}}$$

$$\geq \omega_n^{1-p}(t+k)^{-n(p-1)}((t+k)^{2n})^{\frac{n-1}{2}}|F(t)|^{\frac{p+1}{2}},$$

i.e.

$$\ddot{F}(t) \geq \omega_n^{1-p}(F(t))^{\frac{p+1}{2}}. \tag{1.10}$$

Since $\dot{F}(t) > 0$ for any $t > 0$, we have

$$\dot{F}(t)\ddot{F}(t) \geq \omega_n^{1-p}\dot{F}(t)F^{\frac{p+1}{2}}, \qquad \text{for } t > T_4.$$

Integrating both sides once, we have

$$\frac{1}{2}(\dot{F}^2(t) - \dot{F}^2(T_4)) \geq \frac{\omega_{n-1}^{1-p}}{\frac{p+1}{2}+1}\left(F^{\frac{p+1}{2}+1}(t) - F^{\frac{p+1}{2}+1}(T_4)\right).$$

Therefore there exists a $T_5 > T_4$ such that for $t > T_5$

$$\dot{F}^2(t) > \frac{\omega_{n-1}^{1-p}}{\frac{p+1}{2}+1}F^{\frac{p+1}{2}+1}(t).$$

Hence for $t \geq T_5$,

$$\dot{F}(t) > \left(\frac{\omega_{n-1}^{1-p}}{\frac{p+1}{2}+1}\right)^{\frac{1}{2}} F^{\frac{p+3}{4}}(t),$$

i.e.,

$$\frac{\dot{F}(t)}{F^{\frac{p+3}{4}}} > \left(\frac{\omega_{n-1}^{1-p}}{\frac{p+1}{2}+1}\right)^{\frac{1}{2}}$$

which implies that

$$\frac{1}{\frac{p+3}{4}-1}\left(\frac{1}{F^{\frac{p-1}{4}}(T_5)} - \frac{1}{F^{\frac{p-1}{4}}(t)}\right) > \left(\frac{\omega_{n-1}^{1-p}}{\frac{p+1}{2}+1}\right)^{\frac{1}{2}}(t - T_5).$$

If $t \to \infty$, we have a contradiction. i.e. T must be finite. □

Proof of Theorem 1.2. We use the same argument in the proof of theorem 1.1 and note that $\delta = -(-\frac{n-1}{2}p^2 + \frac{n+1}{2} + 1 - 2) = 2$ in this case, then for $t > T_2$

$$\frac{\dot{F}(t)}{F(t)} \geq \frac{1}{2}A_3^{\frac{1}{2}}(t+k)^{-1}$$

which implies that

$$lnF(t) - lnF(T_2) \geq \frac{1}{2}A_3^{\frac{1}{2}}(ln(t+k) - ln(T_2+k)).$$

So there exists a $T_3' > T_2$ such that $t > T_3'$

$$lnF(t) \geq \frac{1}{4}A_3^{\frac{1}{2}}ln(t+k) = ln(t+k)^{\frac{1}{4}A_3^{\frac{1}{2}}}$$

i.e.,

$$F(t) \geq (t+k)^{\frac{1}{4}A_3^{\frac{1}{2}}}$$

Let

$$A_4 = \frac{1}{4}A_3^{\frac{1}{2}} = \frac{1}{4}\left[\omega_n^{1-p}A_2C^p(g)\right]^{\frac{1}{2}} = \frac{1}{4}\left[\omega_n^{1-p}A_2\frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta}\lambda^{\frac{n-1}{2}}g(\lambda)d\lambda\right]^{\frac{1}{2}}.$$

If ε is sufficiently small, then $A_4 > 2n$, consequently, (1.10) holds. The rest of the proof can be carried out using the exact argument in the proof of theorem 1.1. □

Chapter 2

Global Existence for Semilinear Wave Equations in Five Space Dimensions

In this chapter, we will study the existence of global solution for $n = 5$. We will prove that the global existence of solutions with small initial radially symmetric data for $p > \frac{3+\sqrt{17}}{4}$.

Theorem 2.1 *Consider the radially symmetric wave equation with* $n = 5$

$$\begin{aligned} &u_{tt} - \frac{4}{r}u_r - u_{rr} = G(u), \quad r \geq 0,\ t \geq 0, \\ &u(r,0) = f(r), \quad u_t(r,0) = g(r), \end{aligned} \tag{2.1}$$

where f and g are even functions, $f, g \in C_0^2(\mathbb{R})$, and $supp\{f,g\} \subset \{r : |r| \leq k\}$.
If G satisfies

$$|G(u)| \leq A|u|^p, \qquad |\frac{\partial G(u)}{\partial u}| \leq B|u|^{p-1}, \qquad A, B > 0,$$

then the equation has a unique global solution in the space $\Omega_{p,k}$ for provided that $p > p_0(5) = \frac{3+\sqrt{17}}{4}$ and $\sum_{i=0}^{2}(||f^{(i)}||_\infty + ||g^{(i)}||_\infty)$ is sufficiently small, where

$$\Omega_{p,k} = \left\{ \begin{array}{l} u \in C^0(\mathbb{R} \times [0,\infty)) \mid ru, (ru)_r \in C^0(\mathbb{R} \times [0,\infty)), \\ u \;\; is\ even\ and\ u(r,t) = 0\ if\ |r| > t + k, \\ \|u\| = sup(|\phi u| + |\phi(ru)| + |\phi(ru)_r|) < +\infty, \\ \phi(r,t) = (r+t+2k)(t-r+2k)^{2p-3} \end{array} \right\}.$$

We will prove the theorem by several lemmas.

For $u \in \Omega_{p,k}$, define the operator with $r > 0$

$$Tu(r,t) = u^0(r,t) + \frac{C_0}{r^3} T_0 u(r,t), \tag{2.2}$$

where

$$u^0(r,t) = C_0 r^{-3} \int_{|r-t|}^{r+t} \lambda(r^2 + \lambda^2 - t^2) g(\lambda) d\lambda,$$

$$+\frac{\partial}{\partial t}\left[C_0 r^{-3} \int_{|r-t|}^{r+t} \lambda(r^2 + \lambda^2 - t^2) f(\lambda) d\lambda \right] \tag{2.3}$$

$$T_0 u(r,t) = \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda(r^2 + \lambda^2 - (t-\tau)^2) G(u(\lambda,\tau)) d\lambda d\tau. \tag{2.4}$$

By Duhamel Principle, solving (2.1) is equivalent to finding a fixed point for the operator T in $\Omega_{p,k}$. The goal in this chapter is to prove T is a contraction map. Therefore We need to prove the following:

(i) $Tu(r,t)$ is even in r;

(ii) $Tu(r,t) = 0$ for $|r| \geq t + r$;

(iii) Tu, rTu and $(rTu)_r$ are continuous;

(iv) $||Tu|| < \infty$.

Since $\lambda(r^2 + \lambda^2 - (t-\tau)^2)G(u(\lambda,\tau))$ is a odd function in λ, therefore

$$\begin{aligned} T_0 u(r,t) &= \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda(r^2 + \lambda^2 - (t-\tau)^2) G(u(\lambda,\tau)) d\lambda \\ &= \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda(r^2 + \lambda^2 - (t-\tau)^2) G(u(\lambda,\tau)) d\lambda. \end{aligned}$$

Note that $g(\lambda)$ and $f(\lambda)$ are even functions, $\lambda(\lambda^2 + r^2 - t^2) f(\lambda)$ and $\lambda(\lambda^2 + r^2 - t^2) g(\lambda)$ are odd functions of λ, we have $u^0(-r,t) = u^0(r,t)$. Similarly $\frac{1}{r^3} T_0 u(r,t) = \frac{1}{(-r)^3} T_0 u(-r,t)$. Therefore $Tu(r,t) = Tu(-r,t)$.

From the expressions of u^0 and T_0u, we can obtain $Tu(r,t)=0$ for $|r|\geq t+k$. It remains to prove (iii) and (iv).

For $u\in\Omega_{p,k}$ and $0<r<t+k$, let $z(r)=min\{1,\frac{1}{r}\}$. Then we have

$$|u(r,t)|\leq\frac{||u||}{\phi(r,t)},\qquad |u(r,t)|\leq\frac{||u||}{r\phi(r,t)}.$$

Hence

$$|u(r,t)|\leq\frac{||u||z(r)}{\phi(r,t)},\tag{2.5}$$

$$|u_r(r,t)|\leq\frac{|(ru)_r|+|u|}{r}\leq\frac{2||u||}{r\phi(r,t)}.\tag{2.6}$$

In order to prove (iii) and (iv), we will prove the following Lemmas.

Lemma 2.1 *For* $p\in(\frac{3+\sqrt{17}}{4},2)$, $0<r<t+k$, $t_1=max\{0,\frac{1}{2}(t-r-k)\}$ *and* $t_2=max\{0,\frac{1}{2}(t+r-k)\}$, *there exists a constant* C_1 *such that*

$$\int_{t_1}^{t}\frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|}d\tau\leq\frac{C_1}{(t-r+2k)^{2p-2}},$$

$$\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}d\tau\leq\frac{C_1}{(t-r+2k)^{2p-2}}.$$

Proof of Lemma 2.1. Without loss of generality, we assume $t-r\geq0$. If $r-t+\tau>0$, i.e., $\tau>t-r$, then

$$\phi(r-t+\tau,\tau)=(r-t+\tau+\tau+2k)(t-r+2k)^{2p-3}=(2\tau+r-t+2k)(t-r+2k)^{2p-3}$$

$$\int_{t_1}^{t}\frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}d\tau=\int_0^r\frac{\lambda z^p(\lambda)}{(2\lambda+t-r+2k)^p(t-r+2k)^{p(2p-3)}}d\lambda$$

$$\leq(t-r+2k)^{-p(2p-3)}\left[\int_0^1\frac{\lambda}{(2\lambda+t-r+2k)^p}d\lambda+\int_1^\infty\frac{\lambda^{1-p}}{(2\lambda+t-r+2k)^p}d\lambda\right]$$

$$\leq(t-r+2k)^{-p(2p-3)}\left[(t-r+2k)^{-p}+(t-r+2k)^{-\frac{p}{2}}\int_1^\infty\lambda^{1-\frac{3}{2}p}d\lambda\right]$$

$$\leq(t-r+2k)^{-p(2p-3)-\frac{p}{2}}\left[(t-r+2k)^{-\frac{p}{2}}+\frac{1}{\frac{3}{2}p-2}\right]$$

$$\leq\left(k^{-\frac{p}{2}}+\frac{1}{\frac{3}{2}p-2}\right)(t-r+2k)^{-p(2p-3)-\frac{p}{2}}.$$

For $p > \frac{3+\sqrt{17}}{4}$, we have $-p(2p-3) - \frac{p}{2} < -2p+2$ and note that $t-r+2k > k$ then

$$(t-r+2k)^{-p(2p-3)-\frac{p}{2}} = (t-r+2k)^{-2p+2}(t-r+2k)^{-p(2p-3)-\frac{p}{2}+2p-2}$$

$$\leq k^{-p(2p-3)-\frac{p}{2}+2p-2}(t-r+2k)^{-2p+2}.$$

If $r-t+\tau < 0$, i.e., $\tau < t-r$, then $\phi(t-r-\tau,\tau) = (t-r+2k)(2\tau-(t-r)+2k)^{2p-3}$

$$\int_{t_1}^{t-r} \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau} d\tau = \int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

If $t-r \leq k$, then $t_1 = max\{0, \frac{1}{2}(t-r-k)\} = 0$

$$\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\leq \int_0^{t-r} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2(t-r)+2k)^{p(2p-30}} d\lambda$$

$$\leq (t-r+2k)^{-p}\int_0^k \frac{\lambda z^p(\lambda)}{(-(t-r)+2k)^{p(2p-3)}} \leq (t-r+2k)^{-p}\int_0^k \frac{\lambda}{k^{p(2p-3)}} d\lambda$$

$$\leq (t-r+2k)^{-p}k^{2-p(2p-3)}.$$

Since $p < 2$, we have $(t-r+2k)^{2-p} \geq k^{2-p}$. Therefore

$$\int_{t_1}^{t-r} \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau} d\tau \leq (t-r+2k)^{-2p+2}k^{-p(2p-3)+p}.$$

If $t-r > k$, then $t_1 = \frac{1}{2}(t-r-k) > 0$

$$\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda)^{p(2p-3)}} d\lambda$$

$$\leq \int_0^{t-r-t_1} \frac{\lambda^{1-p}}{(t-r+2k)^p(t-r-2\lambda)^{p(2p-3)}} d\lambda$$

$$\leq (t-r+2k)^{-p}\left[\int_0^{\frac{k}{2}} + \int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} + \int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}}\right] \frac{\lambda^{1-p}}{(t-r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\leq (t-r+2k)^{-p}(t-r+k)^{-p(2p-3)}\int_0^{\frac{k}{2}} \lambda^{1-p} d\lambda$$

$$+(t-r-2(\frac{t-r+2k}{4})+2k)^{-p(2p-3)}\int_{\frac{k}{2}}^{t-r+2k} \lambda^{1-p} d\lambda$$

$$+(\frac{t-r+2k}{4})^{-p+1}\int_{\frac{t-r+2k}{2}}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)} d\lambda$$

Since

$$\int_0^{\frac{k}{2}} \lambda^{1-p} d\lambda = \frac{1}{p-1}(\frac{k}{2})^{2-p}$$

$$\int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} \lambda^{1-p} d\lambda = (\frac{k}{2})^{1-p}\left(\frac{t-r+2k}{4} - \frac{k}{2}\right) \le (\frac{k}{2})^{1-p}\frac{t-r+2k}{4}$$

$$\int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)} \le \int_{-\infty}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)} d\lambda$$

$$= \frac{1}{2}\int_k^\infty \alpha^{-p(2p-3)} d\alpha = \frac{1}{2}\frac{k^{1-p(2p-3)}}{p(2p-3)-1}.$$

$t-r+k = \frac{t-r}{2} + \frac{t-r}{2} + k > \frac{t-r}{2} + \frac{3k}{2} > \frac{t-r+2k}{2}$ $t-r+2k > 3k$, $-p(2p-3) < -p+2$, and $-p(2p-3)+1 < -p+2$, therefore there exists a constant, say M_0, such that for $t-r \le k$

$$\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}} d\lambda \le M_0(t-r+2k)^{-2p+2}\ .$$

Combine all above, there exists a constant M_1 such that

$$\int_{t_1}^t \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\bigg|_{\lambda=|t-r+\tau|} d\tau \le \frac{M_1}{(t-r+2k)^{2p-2}}$$

Now we prove the second inequality. First we consider $t+r > k$, then $t_2 = \frac{1}{2}(t+r-k) > 0$ and $\phi(t+r-\tau,\tau) = (t+r+2k)(2\tau-(t+r)+2k)^{2p-3}$

$$\int_{t_2}^t \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\bigg|_{\lambda=t+r-\tau} d\tau = \int_r^{r+t-t_2} \frac{\lambda z^p(\lambda)}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\le \int_r^{r+t-t_2} \frac{\lambda^{1-p}}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\le (t+r+2k)^{-p}\left[\int_0^{\frac{k}{2}} + \int_{\frac{k}{2}}^{\frac{t+r+2k}{4}} + \int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}}\right] \frac{\lambda^{1-p}}{(t+r-2\lambda+2k)^{p(2p-3)}} d\lambda\ .$$

A similar argument used in the proof of first inequality, there exists a constant M_0' such that for $t+r > k$,

$$\int_{t_2}^t \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\bigg|_{\lambda=t+r-\tau} d\tau \le \frac{M_0'}{(t+r+2k)^{2p-2}} \le \frac{M_0'}{(t-r+2k)^{2p-2}}.$$

If $t+r \le k$, then $t_2 = 0$ and

$$\int_{t_2}^{t} \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau} d\tau \le \int_0^t \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau} d\tau \le \int_0^t \frac{\lambda^{1-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau} d\tau$$

$$= \int_0^t \frac{(t+r-\tau)^{1-p}}{(t+r+2k)^p(2\tau-r-t+2k)^{p(2p-3)}} d\tau$$

$$\le (t+r+2k)^{-p}k^{-p(2p-3)} \int_0^t (r+t-\tau)^{1-p} d\tau$$

$$= (t-r+2k)^{-p}k^{-p(2p-3)}\left[-\frac{1}{2-p}r^{2-p} + \frac{1}{2-p}(r+t)^{2-p}\right]$$

$$\le (t+r+2k)^{-p}k^{-p(2p-3)}\frac{1}{2-p}(r+t)^{2-p}$$

$$\le \frac{k^{-p(2p-3)}}{2-p}(t+r+2k)^{-2p+2}.$$

Therefore there exists a constant M_1' such that for $0 < r < t+k$

$$\int_{t_2}^{t} \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau \le \frac{M_1'}{(t+r+2k)^{2p-2}} \le \frac{M_1'}{(t-r+2k)^{2p-2}}.$$

$\square$

Lemma 2.2 *For $p \in (\frac{3+\sqrt{17}}{4}, 2)$, there exists a constant C_2 such that for $0 < r < t+k$*

$$\int_{t_1}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|} d\tau \le \frac{C_2}{(t-r+2k)^{2p-2}},$$

$$\int_{t_2}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau \le \frac{C_2}{(t-r+2k)^{2p-2}}.$$

Proof. Note $p \in (\frac{3+\sqrt{17}}{4}, 2)$, then there exists a number $\delta_0 > 0$ such that

$$p = \frac{3+\delta_0+\sqrt{(3+\delta_0)^2+8}}{4}.$$

For $r-t+\tau > 0$, i.e., $\tau > t-r$,

$$\int_{t-r}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau} d\tau = (t-r+2k)^{-p(2p-3)} \int_0^r \frac{\lambda^{2-p}}{(2\lambda+(t-r+2k)^p} d\lambda$$

$$\le (t-r+2k)^{-p(2p-3)}[(t-r+2k)^{-p}\int_0^1 \lambda^{2-p} d\lambda$$

$$+ \int_1^\infty \frac{\lambda^{2-p}}{(2\lambda+t-r+2k)^{(3-p-\delta_1 p)+p-(3-p-\delta_1 p)}} d\lambda],$$

where $0 < \delta_1 < min\{\delta_0, 2 - \frac{3}{p}\}$, it is easy to see $p - (3 - p + \delta_1 p) > 0$ and

$$\int_1^\infty \frac{\lambda^{2-p}}{(2\lambda + t - r + 2k)^{(3-p+\delta_1 p)+p-(3-p+\delta_1 p)}} d\lambda$$

$$\leq (t - r + 2k)^{-2p+3+\delta_1 p} \int_1^\infty \lambda^{-1-\delta_1 p} d\lambda$$

$$= \left[\frac{1}{p\delta_1}\right] (t - r + 2k)^{-2p+3+\delta_1 p}.$$

Therefore

$$\int_{t-r}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda, \tau)}\bigg|_{\lambda = r-t+\tau} d\tau$$

$$\leq (t - r + 2k)^{-p(2p-3)} \left[(t - r + 2k)^{-p} + \frac{1}{p\delta_1}(t - r + 2k)^{-2p+3+p\delta_1}\right]$$

$$= (t - r + 2k)^{-p(2p-3)-2p+3+p\delta_1} \left[(t - r + 2k)^{p-3-p\delta_1} + \frac{1}{p\delta_1}\right]$$

$$\leq \left[k^{p-3-p\delta_1} + \frac{1}{p\delta_1}\right] (t - r + 2k)^{-p(2p-3)-2p+3+p\delta_1}.$$

Since $p = \frac{3+\delta_0+\sqrt{(3+\delta_0)^2+8}}{4}$, we have $-p(2p-3) - 2p + 3 + p\delta_1 < -2p + 2$, i.e., $2p^2 - (3 + \delta_1)p - 1 > 0$, which requires $p > \frac{3+\delta_1+\sqrt{(3+\delta_1)^2+8}}{4}$. Therefore

$$\int_{t-r}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda, \tau)}\bigg|_{\lambda = r-t+\tau} d\tau \leq \left[k^{3-p-p\delta_1} + \frac{1}{p\delta_1}\right] (t - r + 2k)^{-2p+2} .$$

For $r - t + \tau < 0$, i.e., $\tau < t - r$,

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\phi^p(\lambda, \tau)}\bigg|_{\lambda = t-r-\tau} d\tau = \int_0^{t-r-t_1} \frac{\lambda^{2-p}}{(t - r + 2k)^p (t - r - 2\lambda + 2k)^{p-(2p-3)}} d\lambda.$$

If $t - r \leq k$, then $t_1 = 0$ and

$$\int_0^{t-r} \frac{\lambda^{2-p}}{(t - r + 2k)^p (t - r - 2\lambda + 2k)^{p(2p-3)}} d\lambda$$

$$\leq \int_0^{t-r} \frac{\lambda^{2-p}}{(t - r + 2k)^p (t - r - 2(t - r) + 2k)^{p(2p-3)}} d\lambda$$

$$\leq (t - r + 2k)^{-p} \int_0^k \frac{\lambda^{2-p}}{k^{p(2p-3)}} d\lambda \leq (t - r + 2k)^{-p} k^{3-p-p(2p-3)} .$$

Note that $(t-r+2k)^{2-p} \geq k^{2-p}$, we have

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau} d\tau \leq k^{1-p(2p-3)}(t-r+2k)^{-2p+2}$$

for all $t-r \leq k$. If $t-r > k$, then $t_1 = \frac{1}{2}(t-r-k) > 0$

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau} d\tau = \int_0^{t-r-t_1} \frac{\lambda^{2-p}}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\leq (t-r+2k)^{-p}[(t-r+k)^{-p(2p-3)} \int_0^{\frac{k}{2}} \lambda^{2-p} d\lambda$$

$$+(t-r-2(\frac{t-r+2k}{4})+2k)^{-p(2p-3)} \int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} \lambda^{2-p} d\lambda$$

$$+(\frac{t-r+k}{2})^{-p+2} \int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)} d\lambda]$$

Note that

$$\int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} \lambda^{2-p} d\lambda \leq \frac{1}{3-p}(\frac{t-r+2k}{4})^{3-p}$$

and $-p(2p-3)-p+3 \leq -p+2$, therefor there exists a constant N_0 such that

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau} d\lambda \leq N_0(t-r+2k)^{-2p+2}$$

for $t-r > k$. Combining all estimates gives

$$\int_{t_1}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|} d\lambda \leq C_2(t-r+2k)^{-2p+2} .$$

Now let us turn our attention to the second inequality in the lemma If $t+r \leq k$, the proof is similar to that of

$$\int_{t_2}^{t} \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau \leq C_1(t+r+2k)^{-2p+2}$$

and is omitted.

If $t+r > k$, then $t_2 = \frac{1}{2}(t+r+k) > 0$ and $\phi(t+r-\tau,\tau) = (t+r+2k)(2\tau-(t+r)+2k)^{2p-3}$

$$\int_{t_2}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau} d\tau$$
$$\int_{r}^{r+t-t_2} \frac{\lambda^{2-p}}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}} d\lambda,$$

$$\leq (t+r+2k)^{-p}\left[\int_0^{\frac{k}{2}} + \int_{\frac{k}{2}}^{\frac{t+r+2k}{4}} + \int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}}\right] \frac{\lambda^{2-p}}{(t+r-2\lambda+2k)^{p(2p-3)}} d\lambda$$

$$\leq (t+r+2k)^{-p}[(t+r+k)^{-p(2p-3)} \int_0^{\frac{k}{2}} \lambda^{2-p} d\lambda$$

$$+(\frac{t+r+2k}{2})^{-p(2p-3)} \int_{\frac{k}{2}}^{\frac{t+r+2k}{4}} \lambda^{2-p} d\lambda$$

$$+(\frac{t+r+k}{2})^{2-p} \int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}} (t+r-2\lambda+2k)^{-p(2p-3)} d\lambda]$$

$$\leq (t+r+2k)^{-p}[(t+r+k)^{-p(2p-3)} \left(\frac{k}{2}\right)^{-p(2p-3)+3-p} + \left(\frac{t+r+2k}{2}\right)^{3-p}$$

$$+\frac{k^{1-p(2p-3)}}{p(2p-3)-1} \left(\frac{t+r+k}{2}\right)^{2-p}] .$$

Note that $-p-p(2p-3) < -2p+2$ and $-p-(2p-3)+3-p < -2p+2$ (which requires $p > \frac{3+\sqrt{17}}{4}$), then we conclude there exists a constant N_1' such that for $t+r > k$

$$\int_{t_2}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau < N_1'(t+r+2k)^{-2p+2} < N_1'(t-r+2k)^{-2p+2} .$$

This conclude the proof of the lemma. □

Lemma 2.3 *There exists a constant C_3 such that for $p \in (\frac{3+\sqrt{17}}{4}, 2)$*

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda G(u(\lambda,\tau)) d\lambda d\tau \leq \frac{C_3 ||u||^p}{(t-r+2k)^{2p-2}} .$$

Proof. By assumptions on G and (2.5), we have

$$\lambda G(u(\lambda,\tau)) \leq \lambda A |u(\lambda,\tau)|^p \leq \frac{A\lambda z^p(\lambda)||u||^p}{\phi^p(\lambda,\tau)} \leq \frac{A||u||^p \lambda^{1-p}}{\phi^p(\lambda,\tau)},$$

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda G(u(\lambda,\tau)) d\lambda d\tau \leq A||u||^p \int_0^t \int_{|r-t+\tau|}^{t+r-\tau} \frac{\lambda^{1-p}}{\phi^p(\lambda,\tau)} d\lambda d\tau.$$

Let $\tau+\lambda=\beta$ and $\tau-\lambda=\alpha$, then

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda G(u(\lambda,\tau))d\lambda d\tau$$

$$\leq A||u||^p \int_{-k}^{t-r}\int_{t-r}^{t+r} \frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\alpha+2k)^{p(2p-3)}(\beta+2k)^p}\frac{1}{2}d\beta d\alpha$$

$$= A||u||^p[\int_{-k}^{\frac{t-r}{2}} (\alpha+2k)^{-p(2p-3)}d\alpha \int_{t-r}^{t+r} \frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}d\beta$$

$$+\int_{\frac{t-r}{2}}^{t-r} (\alpha+2k)^{-p(2p-3)}d\alpha \int_{t-r}^{t+r} \frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}d\beta]. \tag{2.7}$$

Now we estimate the following integral for $1-p-l<-2$.

Case 1: if $t-r-\alpha\leq 2$,

$$\int_{t-r}^{t+r} \frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}d\beta = \int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}} \frac{\lambda^{1-p}}{(2\lambda+\alpha+2k)^l}d\lambda$$

$$\leq \int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}} \frac{\lambda^{1-p}}{(2\lambda+k)^l}d\lambda \leq \int_0^\infty \frac{\lambda^{1-p}}{(2\lambda+k)^l}$$

$$\leq \frac{1}{k^l}\int_0^1 \lambda^{1-p}d\lambda + \int_1^\infty \frac{\lambda^{1-p}}{(2\lambda+k)^l}d\lambda = \frac{1}{k^l}\frac{1}{2-p}+\frac{1}{2^l}\frac{1}{p+l-2}$$

$$\leq \left(\frac{1}{k^l}\frac{1}{2-p}+\frac{1}{2^l}\frac{1}{p+l-2}\right)\frac{(2+k)^{p+l-2}}{(t-r-\alpha+k)^{p+l-2}},$$

where we used the fact $k<t-r-\alpha+k\leq 2+k$.

Case 2: If $t-r-\alpha>2$,

$$\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}} \frac{\lambda^{1-p}}{(2\lambda+k)^l}d\lambda \leq \frac{1}{2^l}\int_{\frac{t-r-\alpha}{2}}^\infty \lambda^{1-p-l}d\lambda$$

$$= \frac{1}{2^l(l+p-2)}\left(\frac{t-r-\alpha}{2}\right)^{2-l-p}.$$

Since $t-r-\alpha>2$, we have $\frac{k}{2}(t-r-\alpha)>k$. Therefore $(1+\frac{k}{2})(t-r-\alpha)>t-r-\alpha+k$ i.e., $t-r-\alpha>\frac{2}{k+2}(t-r-\alpha+k)$, then

$$\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}} \frac{\lambda^{1-p}}{(2\lambda+k)^l}d\lambda \leq \frac{(\frac{k+2}{2})^{p+l-2}}{2^{2-p}(l+p-2)}(t-r-\alpha+k)^{-p-l+2}.$$

From Cases 1 and 2, there exists a constant N_3 such that

$$\int_{t-r}^{t+r} \frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}d\beta \leq N_3(t-r-\alpha+k)^{2-p-l} .$$

Now we can estimate first integral in (2.7) and get

$$\int_{-k}^{\frac{t-r}{2}} (\alpha+2k)^{-p(2p-3)}d\alpha \int_{t-r}^{t+r} \frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}d\beta$$

$$\int_{-k}^{\frac{t-r}{2}} (\alpha+2k)^{-p(2p-3)}N_3(t-r-\alpha+k)^{2-p-p}d\alpha$$

$$\leq N_3\left(\frac{t-r}{2}+k\right)^{2-2p}\int_{-k}^{\frac{t-r}{2}} (\alpha+2k)^{-p(2p-3)}d\alpha$$

$$\leq N_3\left(\frac{t-r+2k}{2}\right)^{2-2p}\int_{-k}^{\infty} (\alpha+2k)^{-p(2p-3)}$$

$$= \frac{N_3k^{1-p(2p-3)}}{(p(2p-3)-1)2^{2-2p}}(t-r+2k)^{2-2p} .$$

The second integral in (2.7) can be estimated as follows

$$\int_{\frac{t-r}{2}}^{t-r} (\alpha+2k)^{-p(2p-3)}d\alpha \int_{t-r}^{t+r} \frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}d\beta$$

$$\leq (\frac{t-r}{2}+2k)^{-p(2p-3)}\int_{\frac{t-r}{2}}^{t-r} (t-r+2k)^{-p+l}d\alpha \int_{t-r}^{t+r} \frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}d\beta$$

$$\leq (\frac{t-r}{2}+2k)^{-p(2p-3)}(t-r+2k)^{-p+l}N_3\int_{\frac{t-r}{2}}^{t-r} (t-r-\alpha+k)^{2-l-p}d\alpha$$

$$\leq 2^{p(2p-3)}N_3(t-r+2k)^{-p(2p-3)-p+l}\int_{-\infty}^{t-r} (t-r-\alpha+k)^{2-l-p}d\alpha$$

$$\leq \frac{2^{p(2p-3)}N_3k^{3-l-p}}{l+p-3}(t-r+2k)^{-p(2p-3)-p+l} ,$$

where $l = 3-p+p\delta_1$ and $-p(2p-3)-p+l < -2p+2$.

From the estimates of first integral and second integral of (2.7), the result follows.□

Lemma 2.4 *Let $u \in \Omega_{p,k}$, there exists a constant C_4 such that for $0 < r < t+k$, $\alpha = 1$ and 2,*

$$\int_0^t (r+t-\tau)^\alpha |G(u(r+t-\tau,\tau))|d\tau \leq C_4\frac{||u||^p}{(t-r+2k)^{2p-2}},$$

$$\int_0^t |r-t+\tau|^\alpha |G(u(r-t+\tau,\tau))|d\tau \le C_4 \frac{||u||^p}{(t-r+2k)^{2p-2}},$$
$$\int_0^t (r+t-\tau)^2 |\frac{\partial G}{\partial r}(u(r+t-\tau,\tau))|d\tau \le C_4 \frac{||u||^p}{(t-r+2k)^{2p-2}},$$
$$\int_0^t (r-t+\tau)^2 |\frac{\partial G}{\partial r}(u(r-t+\tau,\tau))|d\tau \le C_4 \frac{||u||^p}{(t-r+2k)^{2p-2}}.$$

Proof. Note that

$$\lambda^\alpha |G(u(\lambda,\tau))| \le A\lambda^\alpha |u(\lambda,\tau)|^p \le A||u||^p \frac{\lambda^\alpha z^p(\lambda)}{\phi^p(\lambda,\tau)}$$

we have

$$\int_0^t (r+t-\tau)^\alpha |G(u(r+t-\tau,\tau)|d\tau = \int_{t_2}^t \lambda^\alpha |G(u(\lambda,\tau))|\Big|_{\lambda=r+t-\tau} d\tau$$

$$\le \int_{t_2}^t A||u||^p \frac{\lambda^\alpha z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau \le A||u||^p \int_{t_2}^t \frac{\lambda^\alpha z^p(\lambda)}{\phi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau} d\tau.$$

By Lemma 2.1 and Lemma 2.2, we get the first inequality.

Similarly, we can get the second inequality in the lemma.

Note that

$$\lambda^2 \left|\frac{\partial G}{\partial u}(u(\lambda,\tau))\right| = \lambda^2 \left|\frac{\partial G}{\partial r}\cdot\frac{\partial u}{\partial r}\right| \le \lambda^2 A|u|^{p-1}|u_r|$$

$$\le A\lambda^2 \left(\frac{z(\lambda)||u||}{\phi(\lambda,\tau)}\right)^{p-1} \frac{2||u||}{\lambda\phi(\lambda,\tau)} \le \frac{A\lambda^2 z^{p-1}(\lambda)||u||^p}{\lambda\phi^p(\lambda,\tau)}$$

$$\le A||u||^p \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)},$$

By Lemma 2.2, the third and fourth inequalities follow from lemma 2.2. □

Lemma 2.5 *Let $u \in \Omega_{k,p}$ and*

$$T_1u(r.t) = \frac{C_0}{r^3}T_0u(r,t) = \frac{C_0}{r^3}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda(\lambda^2+r^2-(t-\tau)^2)G(u(\lambda,\tau))d\lambda d\tau.$$

Then T_1u, rT_1u and $(rT_1u)_r$ are continuous, and the following inequalities hold

$$|T_1u(r,t)| \le C_5\frac{||u||^p}{\phi(r,t)},$$

$$|rT_1u(r,t)| \le C_5\frac{||u||^p}{\phi(r,t)},$$

$$|(rT_1u)_r(r,t)| \le C_5\frac{||u||^p}{\phi(r,t)}.$$

Proof. Since $\lambda|G(u(\lambda,\tau))| \leq A||u||^p \frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}$ and $\frac{\lambda z^p(\lambda)}{\phi^p(\lambda,\tau)}$ is integrable in λ. Also $\frac{\partial}{\partial r}(\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau)) = 2r\lambda G(u(\lambda,\tau))$ is integrable in λ.

Therefore

$$(T_0u)_r(r,t) = \int_0^t \int_{r-t+\tau}^{r+t-\tau} 2r\lambda G(u(\lambda,\tau))d\lambda d\tau$$
$$+\int_0^t \lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))\Big|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau} d\tau$$

is continuous

Note that

$$\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))\Big|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau} = 2r\lambda^2 G(u(\lambda,\tau))\Big|_{\lambda=r=t+\tau}^{\lambda=r+t-\tau} d\tau \ ,$$

then

$$(T_0u)_r(r,t) = 2rJu(r,t)$$

where

$$Ju(r,t) = \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda G(u(\lambda,\tau))d\lambda d\tau + \int_0^t \lambda^2 G(u(\lambda,\tau))\Big|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau} d\tau \ .$$

Therefore, Ju is continuous, and by Lemmas 2.3, 2.4 there exists a constant M such that

$$|Ju(r,t)| \leq M\frac{||u||^p}{(t-r+2k)^{2p-2}}. \tag{2.8}$$

Differentiate Ju with respect to r, we have

$$(Ju)_r(r,t) = \int_0^t [3(r+t-\tau)G(u(r+t-\tau,\tau)) - 3(r-t+\tau)G(u(r-t+\tau,\tau))$$
$$+(r+t-\tau)^2\frac{\partial}{\partial r}(G(u(r+t-\tau,\tau)) - (r-t+\tau)^2\frac{\partial}{\partial r}(G(u(r-t+\tau,\tau))].$$

By lemma 2.4, we can conclude that for $0 < r < t+k$

$$|(Ju)_r(r,t)| \leq 8C_4\frac{||u||^p}{(t-r+2k)^{2p-2}} \ . \tag{2.9}$$

Since $Ju(0,t) = 0$ and $T_0u(0,t) = 0$, we have, using (2.8) and (2.9)

$$|Ju(r,t)| = |\int_0^r (Ju)_r(s,t)ds| \leq \int_0^r |(Ju)_r(s,t)|ds$$

$$\leq 8C_4||u||^p\frac{r}{(t-r+2k)^{2p-2}} \tag{2.10}$$

$$T_0u(r,t) = \int_0^r (T_0u)_r(s,t)ds = \int_0^r 2sJu(s,t)ds. \tag{2.11}$$

Hence the following inequalities is true

$$T_1u(r,t)=\frac{C_0}{r^3}T_0u(r,t)=\frac{C_0}{r^3}\int_0^r 2sJu(s,t)ds=\frac{2C_0}{r^3}\int_0^r 2s\int_0^s (Ju)_s(\lambda,t)d\lambda ds,$$

which implies that T_1u, rT_1u are continuous.

If $t-r+2k>\frac{1}{2}(t+r+2k)$

$$\frac{2C_0}{r^2}\int_0^r |Ju(s,t)|ds\le\frac{2C_0}{r^2}\int_0^r 8C_4||u||^p\frac{s}{(t-s+2k)^{2p-2}}ds$$

$$\le\frac{16C_0C_4||u||^p}{r^2}\frac{r^2}{(t-r+2k)^{2p-2}}\le 32C_0C_4\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t-r+2k\le\frac{1}{2}(t+r+2k)$, i.e., $r\ge\frac{r+t+2k}{3}$, then

$$\frac{2C_0}{r^2}\int_0^r |Ju(s,t)|ds\le\frac{2C_0}{r^2}\int_0^r \frac{M||u||^p}{(t-r+2k)^{2p-2}}$$

$$\le 2C_0M\frac{1}{r}\frac{||u||^p}{(t-r+2k)^{2p-2}}\le\frac{8C_0M}{k}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}. \quad (2.12)$$

Now we estimate rT_1u

$$|rT_1u(r,t)|=\frac{C_0}{r^2}|T_0u(r,t)|\le\frac{C_0}{r^2}\int_0^r 2s|Ju(s,t)|ds\le\frac{2C_0}{r}\int_0^r |Ju(s,t)|ds \quad (2.13)$$

If $t-r+2k>\frac{1}{2}(t+r+2k)$, then

$$\frac{2C_0}{r}\int_0^r |Ju(s,t)|ds\le\frac{2C_0}{r}\int_0^r \frac{M||u||^p}{(t-s+2k)^{2p-2}}ds$$

$$\le 2C_0M\frac{||u||^p}{(t-r+2k)^{2p-2}}\le 4C_0M\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t-r+2k\le\frac{1}{2}(t+r+2k)$

$$\frac{2C_0}{r}\int_0^r\frac{M||u||^p}{(t-s+2k)^{2p-2}}ds=\frac{2C_0M}{(2p-3)r}\left(\frac{||u||^p}{(t-r+2k)^{2p-3}}-\frac{||u||^p}{(t+2k)^{2p-3}}\right)$$

$$\le\frac{2C_0M}{(2p-3)}\frac{||u||^p}{r(t-r+2k)^{2p-3}}\le\frac{8C_0M}{2p-3}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

For $(rT_1u)_r$, we have

$$(rT_1u)_r(r,t)=-\frac{3C_0}{r^3}T_0u(r,t)+\frac{C_0}{r^2}(T_0u)_r(r,t)$$

We have proved that the first term is continuous and satisfies the estimate. The secod term is also continuous, since

$$\frac{C_0}{r^2}(T_0 u)_r(r,t) = \frac{C_0}{r} Ju(r,t) = \frac{C_0}{r}\int_0^r (Ju)_r(s,t)ds.$$

If $t - r + 2k > \frac{1}{2}(t + r + 2k)$, then by (2.10)

$$\frac{2C_o}{r}|Ju(r,t)| \leq \frac{2C_0}{r}\frac{8C_4||u||^p r}{(t-r+2k)^{2p-2}} \leq 32C_0C_4\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t - r + 2k \leq \frac{1}{2}(t + r + 2k)$, then by (2.8)

$$\frac{2C_0}{r}|Ju(r,t)| \leq \frac{2C_0}{r}\frac{M||u||^p}{(t-r+2k)^{2p-2}} \leq \frac{8C_0 M}{k}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

Lemma 2.6 *Let $h \in C^2(\mathbb{R})$ be an even function with $h(r) = 0$ for $|r| \geq k$. Define*

$$v(r,t) = \int_{r-t}^{r+t} \lambda(r^2 + \lambda^2 - t^2)h(\lambda)d\lambda$$

and $||v||_\infty = sup\{|v(r,t)|; r \in \mathbb{R}, t \geq 0\}$ for $v \in C^0(\mathbb{R} \times [0,\infty))$. Then there exists a constant C_6 such that

$$||wr^{-3}v||_\infty + ||wr^{-3}v_t||_\infty + ||wr^{-2}v||_\infty + ||wr^{-2}v_t||_\infty + ||wr^{-2}v_r||_\infty + ||wr^{-2}v_{rt}||_\infty$$

$$\leq C_6(||h||_\infty + ||h'||_\infty + ||h''||_\infty)\ ,$$

where $w(r,t) = t + r + 2k$.

Proof. Without loss of generality, we assume $r > 0$, $t > 0$ and note that $v(r,t) \neq 0$ only when $t - k < r < t + k$.

Differentiate v with respect to r

$$v_r = \int_{r-t}^{r+t} 2\lambda r h(\lambda)d\lambda + 2r(r+t)^2 h(r+t) - 2r(r-t)^2 h(r-t) = 2rl(r,t),$$

where

$$l(r,t) = \int_{r-t}^{r+t} \lambda h(\lambda)d\lambda + (r+t)^2 h(r+t) - (r-t)^2 h(r-t).$$

Since h is even and vanishes outside $(-k,k)$, we have the following

$$|v_r(r,t)| \leq 2r\left[\int_{r-t}^{r+t} |\lambda h(\lambda)|d\lambda + (r-t)^2|h(r+t)| + (r-t)^2|h(r-t)|\right]$$

$$\leq 2r[2k^2||h||_\infty + 2k^2||h||_\infty] = 8k^2r||h||_\infty,$$

hence

$$|r^{-1}v_r(r,t)| \leq 8k^2||h||_\infty. \tag{2.14}$$

Differentiating $l(r,t)$ with respect to r yields

$$l_r(r,t) = 3(r+t)h(r+t) - 3(r-t)h(r-t) + (r+t)^2h'(r+t) - (r-t)^2h'(r-t),$$

hence

$$|l_r(r,t)| \leq 6k||h||_\infty + 2k^2||h'||_\infty$$

and

$$|l(r,t)| = |\int_0^r l_r(s,t)ds| \leq (6k||h||_\infty + 2k^2||h'||_\infty)r.$$

Therefore

$$|v_r(r,t)| = 2r|l(r,t)| \leq r^2(12k||h||_\infty + 4k^2||h||_\infty),$$

i.e.,

$$|r^{-2}v_r(r,t)| \leq 12k||h||_\infty + 4k^2||h'||_\infty. \tag{2.15}$$

Now let us estimate $|wr^{-2}v_r|$. If $t - r + 2k < \frac{1}{2}(r + t + 2k)$, i.e., $r > \frac{r+t+2k}{4}$, (2.14) implies

$$\left|\frac{r+t+2k}{r^2}v_r(r,t)\right| \leq \frac{4}{r}|v_r(r,t)| \leq 4|r^{-1}v_r(r,t)| \leq 32k^2||h||_\infty.$$

If $t - r + 2k > \frac{1}{2}(t + r + 2k)$, by $-k < t - r < k$ and (2.14)

$$\left|\frac{t+r+2k}{r^2}v_r(r,t)\right| \leq \left|\frac{2(t-r+2k)}{r^2}v_r(r,t)\right|$$

$$\leq 6k\left|\frac{1}{r^2}v_r(r,t)\right| \leq 6k(12k||h||_\infty + 4k^2||h'||_\infty).$$

In order to estimate $|wr^{-3}v|$, we need to estimate $|r^{-3}v|$ and $|r^{-2}v|$. Since $|v_r(r,t)| \leq r^2(12k||h||_\infty + 4k^2||h'||_\infty)$ and $v(0,t) = 0$,

$$|v(r,t)| \leq \int_0^r |v_r(\lambda,t)d\lambda \leq (12k||h||_\infty + 4k^2||h||_\infty)r^3,$$

i.e.,

$$|r^{-3}v(r,t)| \leq 12k||h||_\infty + 4k^2||h'||_\infty.$$

If $r > k$,

$$\begin{aligned}|v(r,t)| &\leq \int_{r-t}^{r+t} |\lambda(r^2+\lambda^2-t^2)h(\lambda)|d\lambda \\ &\leq ||h||_\infty \int_{-k}^{k} |\lambda|(k^2+|r-t|\cdot|r+t|)d\lambda \\ &\leq 2k^2||h||_\infty(k^2+k(2r+k)) \leq 8k^2||h||_\infty.\end{aligned}$$

So $|r^{-1}v(r,t)| < 8k^3||h||_\infty$, therefore $|r^{-2}v(r,t)| \leq ||\frac{1}{k}r^{-1}v(r,t)| \leq 8k^2||h||_\infty$.

If $r \leq k$,

$$|r^{-2}v(r,t)| \leq k|r^{-3}v(r,t)| \leq 12k^2||h||_\infty + 4k^3||h'||_\infty$$

Therefore

$$|r^{-2}v(r,t)| \leq 12k^2||h||_\infty + 4k^3||h'||_\infty.$$

$$|r^{-1}v(r,t)| \leq k|r^{-2}v(r,t) \leq 12k^3||h||_\infty + 4k^3||h'||_\infty.$$

If $t-r+2k > \frac{1}{2}(t+r+2k)$ and by using $|t-r| < k$, we have

$$|\frac{w}{r^3}v(r,t)| \leq |\frac{2(r+t+2k)}{r^3}v(r,t)|$$

$$\leq 6k|\frac{1}{r^3}v(r,t)| \leq 6k(12k||h||_\infty + 4k^2||h'||_\infty)$$

If $t-r+2k \leq \frac{1}{2}(r+t+2k)$, i.e., $r > \frac{r+t+2k}{4}$, then

$$|\frac{w}{r^3}v(r,t)| \leq |\frac{4}{r^2}v(r,t)| \leq 4(12k^2||h||_\infty + 4k^3||h||_\infty).$$

Therefore

$$|\frac{w}{r^3}v(r,t)| \leq 72k^2||h||_\infty + 24k^3||h'||_\infty.$$

Using similar arguments, we have

$$\left|\frac{w}{r^2}v(r,t)\right| \leq 72k^3||h||_\infty + 24k^3||h'||_\infty.$$

To estimate $|wr^{-2}v_{rt}|$, we need to estimate $|r^{-1}v_{rt}|$ and $|r^{-2}v_{rt}|$.

$$v_t = \int_{r-t}^{r+t} -2\lambda t h(\lambda)d\lambda + 2r(r+t)^2h(r+t) + 2r(r-t)^2h(r-t),$$

$$v_{rt} = 6r(r+t)h(r+t) + 6r(r-t)h(r-t)$$

$$+2r(r+t)^2h'(r+t) + 2r(r-t)^2h'(r-t).$$

Since h is even and vanises outside $(-k,k)$, we have

$$|r^{-1}v_{rt}(r,t)| \le 12k||h||_\infty + 4k^2||h'||_\infty,$$

$$\lim_{r\to 0} r^{-1}v_{r,t} = 0.$$

$$|\frac{\partial}{\partial r}(r^{-1}v_{rt}(r,t))| = |6h(r+t)+6h(r-t)+10(r+t)h'(r+t)$$
$$+10(r-t)h'(r-t)+2(r+t)^2h''(r+t)+2(r-t)^2h''(r-t)|$$
$$\le 12||h||_\infty + 20k||h'||_\infty + 2k^2||h''||_\infty,$$
$$|r^{-1}v_{rt}(r,t)| \le \int_0^r |\frac{\partial}{\partial z}(z^{-1}v_{rt}(z,t)|dz$$
$$\le (12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty)r.$$

Therefore

$$|r^{-2}v_{rt}(r,t)| \le 12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty. \tag{2.16}$$

A similar argument as before yields

$$|wr^{-2}v_{rt}(r,t)| \le 6k(12||h||_\infty + 20k||h''||_\infty + 4k^2||h''||_\infty).$$

Since $v_t(0,t)=0$, and by (2.16) we have

$$|v_t(r,t)| \le \int_0^r |v_{rt}(s,t)|ds \le (12||h||_\infty + 20||h'||_\infty + 4k^2||h''||_\infty)r^3,$$

i.e.,

$$|r^{-3}v_t(r,t)| \le 12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty$$

On the other hand

$$|v_t| \le \int_{r-t}^{r+t} |2sth(s)|ds + 2r(r+t)^2|h(r+t)| + 2r(r-t)^2|h(r-t)|$$

If $r>k$, then

$$|v_t(r,t)| \le \int_{-k}^{k} 2t|s|h(s)|ds + 4rk^2||h||_\infty$$
$$\le 4tk^2||h||_\infty + 4rk^2||h||_\infty \le 4(2r+k)k^2||h||_\infty \le 12rk^2||h||_\infty$$

i.e., for $r > k$

$$|r^{-1}v_t(r,t)| \le 12k^2||h||_\infty, \qquad |r^{-2}v_t(r,t)| \le 12k||h||_\infty.$$

If $r \le k$,

$$|r^{-1}v_t(r,t)| \le k^2(12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty),$$

$$|r^{-2}v_t(r,t)| \le k(12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty).$$

By a similar argument as before, there exists a constant $C(k)$ such taht

$$|wr^{-2}v_t| \le C(k)(||h||_\infty + ||h'||_\infty + ||h''||_\infty),$$

$$|wr^{-3}v_t| \le C(k)(||h||_\infty + ||h'||_\infty + ||h''||_\infty).$$

□

Lemma 2.7 *There exists a constant C_7 such that*

$$||u^0|| \le C_7(||f||_\infty + ||f'||_\infty + ||f''||_\infty + ||g||_\infty + ||g'||_\infty + ||g''||_\infty).$$

Proof. Recall $||u^0|| = sup\{|\phi u^0| + |\phi(ru^0)| + |\phi(ru^0)_r|\}$ and note $t - r + 2k \le 3k$ on the support of u^0, then

$$|\phi(r,t)u^0(r,t)| \le (3k)^{2p-3}\left|w(r,t)r^{-3}\left(v^g(r,t) + v^f_t(r,t)\right)\right|$$

$$|\phi(r,t)ru^0(r,t)| \le (3k)^{2p-3}\left|w(r,t)r^{-2}\left(v^g(r,t) + v^f_t(r,t)\right)\right|$$

$$|\phi(r,t)(ru^0)_r(r,t)| \le (3k)^{2p-3}|w(r,t)(-2r^{-3}v^g(r,t)$$

$$+r^{-2}v^g_r(r,t) - 2r^{-3}v^f_t(r,t) + r^{-2}v^f_{rt}(r,t))|,$$

where

$$v^g = \int_{r-t}^{r+t} \lambda(r^2 + \lambda^2 - t^2)g(\lambda)d\lambda,$$

$$v^f = \int_{r-t}^{r+t} \lambda(r^2 + \lambda^2 - t^2)f(\lambda)d\lambda.$$

Therefore, The lemma follows from lemma 2.6.

Proof of Theorem 2.1. By Lemma 2.5 we have

$$||Tu|| \leq ||u^0|| + ||T_1 u|| \leq ||u^0|| + 3C_5||u||^p$$

Let $\varepsilon = (6C_5)^{-\frac{1}{p-1}}$, hence $C_5 = \frac{1}{6}\varepsilon^{1-p}$.

Let

$$\Omega^{\varepsilon}_{p,k} = \{u \in \Omega_{p,k} \; ||u|| \leq \varepsilon\}.$$

Choose $\sum_{i=0}^{2}(||f^{(i)}||_\infty + ||g^{(i)}||_\infty)$ small enough such that $||u^0|| < \frac{1}{2}\varepsilon$, then

$$||Tu|| \leq ||u^0|| + 3C_5||u||^p \leq \frac{\varepsilon}{2} + 3 \cdot \frac{1}{6}\varepsilon^{1-p}\varepsilon^p = \varepsilon$$

Therefore T maps $\Omega^{\varepsilon}_{p,k}$ to itself. Exactly same estimate as before but replace G by $A(|u|^p - |v|^p)$ and note that

$$|\, |u|^p - |v|^p| \leq (|u| + |v|)^{p-1}|u - v|,$$

then

$$||Tu - Tv|| \leq 3C_5(||u|| + ||v||)^{p-1}||u - v|| \leq 3C_5(2\varepsilon)^{p-1}||u - v|| = 2^{p-2}||u - v||.$$

Since $p < 2$, hence $2^{p-2} < 1$, and T is a contraction map on $\Omega^{\varepsilon}_{p,k}$, therefore T is a contraction map on $\Omega^{\varepsilon}_{p,k}$. By contraction mapping theorem, there exists a unigue fixed point u of $\Omega^{\varepsilon}_{p,k}$ such that $Tu = u$, i.e., the global solution of the nonlinear wave equation exists. □

Theorem 2.2 *For $n = 5$ and $p \geq 2$, the global solution of the nonlinear wave equation exists in $\Omega_{p,k}$, where $\Omega_{p,k}$ defined as before except $\phi(r,t) = (t+r+2k)(t-r+2k)^{p-1}$*

Proof The proof is similar to the proof of Theorem 2.1,and is omitted here.

Chapter 3

Global Existence for Semilinear Wave Equations in Seven and Nine Space Dimensions

Consider the radially wave equation with $n = 7$ and 9

$$u_{tt} - \frac{n-1}{r}u_r - u_{rr} = F(u), \quad r \geq 0,\ t \geq 0\ , \tag{3.1}$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \tag{3.2}$$

where f and g are even functions, $f, g \in C_0^2(\mathbb{R})$, and $supp\{f,g\} \subset \{r : |r| \leq k\}$. Assume

$$|F(u)| \leq A|u|^p, \quad |\frac{\partial F(u)}{\partial u}| \leq B|u|^{p-1}, \quad A, B > 0.$$

Let

$$z(r) = \begin{cases} -k + \frac{k}{2}e^{(\frac{k}{2}+r)\frac{k}{2}} & \text{if } r \leq -\frac{k}{2}, \\ r & \text{if } |r| < \frac{k}{2}, \\ k - \frac{k}{2}e^{(\frac{k}{2}-r)\frac{2}{k}} & \text{if } r \geq \frac{k}{2}. \end{cases}$$

Define

$$\Omega_{p,k,n} = \left\{ \begin{array}{l} u \in C^0(\mathbb{R} \times [0,\infty)) \mid rz^{\frac{n-7}{2}}u, (rz^{\frac{n-5}{2}}u)_r \in C^0(\mathbb{R} \times [0,\infty)), \\ u \text{ is even and } u(r,t) = 0 \text{ if } |r| > t + k, \\ \|u\| = sup(|\phi(r,t)rz^{\frac{n-7}{2}}u| + |\phi(r,t)(rz^{\frac{n-5}{2}}u)_r|) < +\infty, \\ \phi(r,t) = (r+t+2k)^{\frac{n-3}{2}}(t-r+2k)^{\frac{n-1}{2}p-\frac{n+1}{2}} \end{array} \right\}.$$

If $u \in \Omega_{p,k}$, then there exists a constant C such that

$$|u(r,t)| \leq \frac{C\|u\|}{rz^{\frac{n-7}{2}\phi(r,t)}}, \qquad |u_r(r,t)| \leq \frac{C\|u\|}{rz^{\frac{n-5}{2}\phi(r,t)}}.$$

By using the similar method to the case $n = 5$ and more technical estimates, we obtain the following

Theorem 3.1 *The equation with $n = 7$ has a unique global solution in $\Omega_{p,k}$ provided that $p \in (\frac{2+\sqrt{10}}{3}, 2)$ and $\sum_{i=0}^{2}(\|f^i\|_\infty + \|g^i\|_\infty)$ is sufficiently small.*

Here we note that $p_1(7) = \frac{2+\sqrt{10}}{3}$ is greater than $p_0(7) = \frac{2+\sqrt{7}}{3}$.
In order to prove the theorem, we need the following lemmas which can be proved by the similar argument to the case $n = 5$. We omit the detailed proofs of most lemmas.

Lemma 3.1 *For $p \in (p_0(n), 2)$, there exist constant $C_1(n,k,p)$ and $C_2(n,k,p)$ such that*

$$\int_{t_1}^{t} \frac{\lambda^{\frac{n-3}{2}-p}}{z^{\frac{n-7}{2}p}\phi^p(\lambda,\tau)}|_{\lambda=|r-t+\tau|} \leq \frac{C_1(n,k,p)}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-3}{2}}},$$
$$\int_{t_2}^{t} \frac{\lambda^{\frac{n-3}{2}-p}}{z^{\frac{n-7}{2}p}\phi^p(\lambda,\tau)}|_{\lambda=r+t-\tau} \leq \frac{C_1(n,k,p)}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-3}{2}}},$$
$$\int_{t_1}^{t} \frac{\lambda^{\frac{n-1}{2}-p}}{z^{\frac{n-7}{2}p+1}\phi^p(\lambda,\tau)}|_{\lambda=|r-t+\tau|} \leq \frac{C_1(n,k,p)}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-1}{2}}},$$
$$\int_{t_2}^{t} \frac{\lambda^{\frac{n-1}{2}-p}}{z^{\frac{n-7}{2}p+1}\phi^p(\lambda,\tau)}|_{\lambda=r+t-\tau} \leq \frac{C_1(n,k,p)}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-1}{2}}}.$$

For $p \in (p_1(7), 2)$,

$$\int_{t_1}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}|_{\lambda=|r-t+\tau|}d\tau \leq \frac{C_2(n,k,p)}{(t-r+2k)^{3p-2}},$$

$$\int_{t_2}^{t} \frac{\lambda^{2-p}}{\phi^p(\lambda,\tau)}|_{\lambda=r+t-\tau}\, d\tau \leq \frac{C_2}{(t-r+2k)^{3p-2}},$$

$$\int_{t_1}^{t} \frac{\lambda^{3-p}}{z(\lambda)\phi^p(\lambda,\tau)}|_{\lambda=|r-t+\tau|}d\tau \leq \frac{C_2(n,k,p)}{(t-r+2k)^{3p-2}},$$

$$\int_{t_2}^{t} \frac{\lambda^{3-p}}{z(\lambda)\phi^p(\lambda,\tau)}|_{\lambda=r+t-\tau}d\tau \leq \frac{C_2(n,k,p)}{(t-r+2k)^{3p-2}}.$$

Lemma 3.2 *Let $u \in \Omega_{p,k}$ and $n = 7, 9$, there exists a constant $C_3(n,k,p)$ such that for $p \in (p_0(n), 2)$*

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^{\frac{n-5}{2}} |F(u(\lambda,\tau)| d\lambda d\tau \le \frac{C_3(n,p,k)\|u\|}{(t-r+2k)^{\frac{n-1}{2}p-3}} \,. \tag{3.3}$$

For $p \in (p_0(9), 2)$

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2 (t-\tau) |F(u(\lambda,\tau)| d\lambda d\tau \le \frac{C_3(9,p,k)\|u\|}{(t-r+2k)^{4p-3}} \,. \tag{3.4}$$

The first inequality can be proved similar to the case $n = 5$. For the second inequality, we will prove it later.

Lemma 3.3 *Let $u \in \Omega_{p,k}$, there exists a constant $C_4(n,k,p)$ such that for $p \in (p_0(n), 2)$*

$$\int_{t-1}^t |r-t+\tau|^{\frac{n-3}{2}} |F(u(r-t+\tau,\tau))| d\tau \le \frac{C_4(n,k,p)||u||^p}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-3}{2}}} \,,$$

$$\int_{t_2}^t (r+t-\tau)^{\frac{n-3}{2}} |F(u(r+t-\tau,\tau))| d\tau \le \frac{C_4(n,k,p)||u||^p}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-3}{2}}} \,,$$

$$\int_{t_1}^t |r-t+\tau|^{\frac{n-1}{2}} |\frac{\partial F}{\partial r}(u(r-t+\tau,\tau))| d\tau \le \frac{C_4(n,k,p)||u||^p}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-1}{2}}},$$

$$\int_{t_2}^t (r+t-\tau)^{\frac{n-1}{2}} |\frac{\partial F}{\partial r}(u(r+t-\tau,\tau))| d\tau \le \frac{C_4(n,k,p)||u||^p}{(t-r+2k)^{\frac{n-1}{2}p-\frac{n-1}{2}}}.$$

For $p \in (p_1(7), 2)$

$$\int_{t_1}^t |r-t+\tau|^3 |\frac{\partial F}{\partial r}(u(r-t+\tau,\tau))| d\tau \le \frac{C_4(7,k,p)||u||^p}{(t-r+2k)^{3p-2}},$$

$$\int_{t_2}^t (r+t-\tau)^3 |\frac{\partial F}{\partial r}(u(r+t-\tau,\tau))| d\tau \le \frac{C_4(7,k,p)||u||^p}{(t-r+2k)^{3p-2}}.$$

Lemma 3.4 *Define*

$$T_1 u(r,t) = \frac{C}{r^{n-2}} \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^{\frac{n-1}{2}} r^{\frac{n-3}{2}} P_{\frac{n-3}{2}} \left(\frac{\lambda^2 + r^2 - (t-\tau)^2}{2\lambda r} \right) |F(u(\lambda,\tau))| d\lambda d\tau,$$

there exists a constant $C_5(n,k,p)$ such that for $p \in (p_0(n), 2)$

$$|r z^{\frac{n-7}{2}} T_1 u(r,t)| \le C_5(n,p,k) \frac{(1 + (t-r+2k)^{\frac{n-5}{2}})||u||^p}{\phi(r,t)} \,,$$

$$|rz^{\frac{n-5}{2}}T_1u(r,t)| \le C_5(n,p,k)\frac{(1+(t-r+2k)^{\frac{n-5}{2}})\|u\|^p}{\phi(r,t)} .$$

For $p \in (p_1(7), 2)$

$$|rT_1u(r,t)| \le C_5(7,k,p)\frac{\|u\|^p}{\phi(r,t)} ,$$

$$|rz(r)T_1u(r,t)| \le C_5(7,k,p)\frac{\|u\|^p}{\phi(r,t)} .$$

Lemma 3.5 *Let* $h \in C^2(\mathbb{R})$ *be an even function with* $h(r) = 0$ *for* $|r| \ge k$. *Define*

$$v(r,t) = \int_{|r-t|}^{r+t} \lambda^{\frac{n-1}{2}} r^{\frac{n-3}{2}} P_{\frac{n-3}{2}}\left(\frac{\lambda^2+r^2-t^2}{2\lambda r}\right) h(\lambda)d\lambda ,$$

there exists a constant $C_6(n,k)$ *such that*

$$\|wr^{-\frac{n+1}{2}}v\|_\infty + \|wr^{-\frac{n+1}{2}}v_t\|_\infty + \|wr^{-\frac{n-1}{2}}v_r\|_\infty + \|wr^{-\frac{n+1}{2}}v_{r,t}\|_\infty$$
$$\le C_6(n,k)(\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty)$$

when $r < \frac{k}{2}$, *and*

$$\|wr^{-\frac{n+1}{2}}v\|_\infty + \|wr^{-\frac{n+1}{2}}v_t\|_\infty + \|wr^{-\frac{n-1}{2}}v_r\|_\infty + \|wr^{-\frac{n+1}{2}}v_{r,t}\|_\infty$$
$$\le C_6(n,k)(\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty)$$

when $r \ge \frac{k}{2}$, *where* $w(r,t) = (t+r+2k)^{\frac{n-3}{2}}$.

Lemma 3.6 *Let* u^0 *be the solution of the equation*

$$\begin{aligned} &u_{tt} - \frac{n-1}{r}u_r - u_{rr} = 0, \quad r \ge 0,\ t \ge 0 , \\ &u(r,0) = f(r), \quad u_t(r,0) = g(r), \end{aligned} \tag{3.5}$$

where f *and* g *are even functions,* $f, g \in C_0^2(\mathbb{R})$, *and supp* $\{f, g\} \subset \{r : |r| \le k\}$, *then there exists a constant* $C_7(n,k)$ *such that*

$$\|u^0\| \le C_7(n,k)(\sum_{i=0}^{2}(\|f^i\|_\infty + \|g^i\|_\infty) .$$

Proof of Inequality (3.4). If $t \le 12r$, then

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau \le 12r \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2|F(u(\lambda,\tau)|d\lambda d\tau$$

by (3.3), we are done.

If $t > 12r$

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau$$

$$= \int_0^{t-2r} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau + \int_{t-2r}^{t} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau$$

$$\le 2r \int_{t-2r}^{t} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2|F(u(\lambda,\tau)|d\lambda d\tau + \int_0^{t-2r} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau$$

$$\le 2r \int_0^{t} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2|F(u(\lambda,\tau)|d\lambda d\tau + \int_0^{t-2r} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau.$$

Now we only need to estimate the second integral. Note

$$\lambda^2(t-\tau)|F(u(\lambda,\tau))| \le A\lambda^2(t-\tau)\frac{\|u\|^p}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)},$$

and $t-\tau-r > t-(t-2r)-r = r > 0$, hence $|r-t+\tau| = t-\tau-r$ and

$$\int_0^{t-2r} \int_{|r-t+\tau|}^{r+t-\tau} \lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau$$

$$\le \int_0^{t-2r} \int_{t-\tau-r}^{r+t-\tau} A\lambda^2(t-\tau)\frac{\|u\|\lambda^2(t-\tau)}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda d\tau$$

$$\le A\|u\|^p \Big[\int_{t_1}^{t_2} \int_{t-\tau-r}^{\tau+k} \frac{\lambda^2(t-\tau)}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda d\tau$$

$$+ \int_{t_2}^{t-2r} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda^2(t-\tau)}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda d\tau \,\Big]. \tag{3.6}$$

We call the first integral and second integral I_1 and I_2 respectively in (3.6).

$$I_1 = \int_{t_1}^{t_2} \int_{t-\tau-r}^{\tau+k} \frac{\lambda^2(t-\tau)}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda d\tau$$

$$= \int_{t_1}^{t_2} \int_{t-\tau-r}^{\tau+k} \frac{\lambda^{2-p}(t-\tau)}{\lambda^p z^p(\lambda)(\tau+\lambda+2k)^{3p}(\tau-\lambda+2k)^{p(4p-5)}} d\lambda d\tau \ ,$$

But $\lambda > t-\tau-r > t-t_2-r > t-r-\frac{1}{2}(t+r-k) = \frac{t-3r+k}{2} > \frac{k}{2}$, by the definition of z, we have $z(\lambda) > \frac{k}{2}$. Note

$$\frac{t-\tau}{t-\tau-r} < \frac{t-(t-2r)}{t-(t-2r)-r} = 2\,,$$

i.e., $t-\tau < 2(t-\tau-r)$, then

$$I_1 \le \left(\frac{2}{k}\right)^p \int_{t_1}^{t_2}\int_{t-\tau-r}^{\tau+k} \frac{\lambda^{2-4p}(t-\tau)}{(\tau-\lambda+2k)^{p(4p-5)}} d\lambda d\tau$$

$$\le \left(\frac{2}{k}\right)^p \int_{t_1}^{t_2} 2(t-\tau-r)\int_{t-\tau-r}^{\tau+k} \frac{\lambda^{2-4p}}{(\tau-\lambda+2k)^{p(4p-5)}} d\lambda d\tau$$

$$\le M(t_2-t_1)(t-t_2-r)^{3-4p},$$

where $M = 2\left(\frac{2}{k}\right)^p (p(4p-5)-1)^{-1}k^{-p(4p-5)+1}$.

Since

$$\begin{aligned}
t_2-t_1 &= max\{0,\frac{1}{2}(t+r-k)\} - max\{0,\frac{1}{2}(t-r-k)\}\\
&= \frac{1}{4}[(t+r-k)+|t+r-k|] - \frac{1}{4}[(t-r-k)+|t-r-k|]\\
&= \frac{1}{4}[(t+r-k)-(t-r-k)+|t+r-k|-|t-r-k|]\\
&\le \frac{1}{4}(2r+|t+r-k-(t-r-k)| = r.
\end{aligned}$$

If $t_2 \neq 0$, then

$$\begin{aligned}
t-t_2-r &= t-\frac{1}{2}(t+r-k)-r\\
&= \frac{1}{2}t-\frac{3}{2}r+\frac{1}{2}k \ge \frac{1}{4}(t-r+2k)+\frac{1}{4}t-\frac{3}{2}r\\
&\ge \frac{1}{4}(t-r+2k)
\end{aligned}$$

$$I_1 \le 4^{4p-3}Mr(t-r+k)^{3-4p}$$

Let us estimate

$$I_2 = \int_{t_2}^{t-2r}\int_{t-\tau-r}^{t-\tau+r} \frac{\lambda^2(t-\tau)}{\lambda^p z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda d\tau\,.$$

Since $z(\lambda)$ is increasing function $\lambda > t-\tau-r$, $z(\lambda) > z(t-\tau-r) > 0$. Then we have

$$I_2 \le \int_{t_2}^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{\phi^p(\lambda,\tau)} d\lambda d\tau$$

$$\int_{t_2}^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

Let $L = \frac{3}{4}(t+r) - \frac{k}{2}$ and $t_2 > 0$. Then $t-2r > \frac{3}{4}(t+r) > L > \frac{1}{2}(t+r-k)$.

$$I_2 \le \int_{t_2}^{L} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$+ \int_{L}^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$= J_1 + J_2.$$

$$J_1 = \int_{t_2}^{L} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau \ .$$

Since $t-r-\tau \ge t-r-L = \frac{t-7r}{4} + \frac{k}{2} \ge \frac{k}{2}$, we have $z(t-\tau-r) \ge \frac{k}{2}$ and

$$J_1 \le \int_{t_2}^{L} \frac{(t-\tau-r)^{2-p}}{(\frac{k}{2})^p(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \lambda^{1-3p} d\lambda d\tau$$

$$\le 2r(\frac{2}{k})^p \int_{t_2}^{L} (t-\tau-r)^{3-4p}(2\tau-t-r+2k)^{-p(4p-5)} d\tau$$

$$\le 2(\frac{2}{k})^p r(t-L-r)^{3-4p} \int_{t_2}^{\infty} (2\tau-t-r+2k)^{-p(4p-5)} d\tau$$

$$= 2(\frac{2}{k})^p (p(4p-5)-1)^{-1} k^{-p(4p-5)+1} r(t-L-r)^{3-4p},$$

but $t-L-r = t-\frac{3}{4}(t+r)+\frac{k}{2}-r = \frac{1}{4}(t-7r+2k) \ge \frac{1}{16}(t-r+2k)+\frac{3t-28r}{16} > \frac{1}{16}(t-r+2k)$, we have

$$J_1 \le 2(\frac{2}{k})^p (p(4p-5)-1)^{-1} k^{-p(4p-5)+1} 16^{4p-3} r(t-r+2k)^{3-4p} \ .$$

$$J_2 = \int_L^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$\leq (2L-t-r+2k)^{-p(4p-5)} \int_L^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$\leq (\frac{1}{2})^{-p(4p-5)}(t-r+2k)^{-p(4p-5)} \int_L^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

If $r < \frac{k}{2}$, then $t-2r > t-r-\frac{k}{2}$ and

$$\int_L^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} d\tau \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda$$

$$= \int_L^{t-r-\frac{k}{2}} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} d\tau \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda$$

$$+ \int_{t-r-\frac{k}{2}}^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)} d\tau \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda = l_1 + l_2.$$

In l_1, $t-\tau-r > \frac{k}{2}$, then $z(t-\tau-r) > \frac{k}{2}$ and

$$l_1 \leq (\frac{2}{k})^p \int_L^{t-r-\frac{k}{2}} (t-\tau-r)^{2-p} d\tau \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda$$

$$\leq (\frac{2}{k})^p \int_L^{t-r-\frac{k}{2}} (t-\tau-r)^{2-2p} d\tau \int_{t-\tau-r}^{t-\tau+r} (\tau+\lambda+2k)^{1-2p} d\lambda$$

$$\leq (\frac{2}{k})^p \int_L^{t-r-\frac{k}{2}} (t-\tau-r)^{2-2p}(t-r+2k)^{1-2p} 2r d\tau$$

$$\leq (\frac{2}{k})^p (t-r+2k)^{1-2p} \int_{-\infty}^{t-r-\frac{k}{2}} (t-\tau-r)^{2-2p} d\tau$$

$$= (\frac{2}{k})^p \frac{1}{2p-3} (\frac{k}{2})^{3-2p} (r-t+2k)^{1-2p}.$$

In l_2, $t-\tau-r \le \frac{k}{2}$, $z(t-\tau-r)=t-\tau-r$.Hence

$$l_2 = \int_{t-r-\frac{k}{2}}^{t-2r} (t-\tau-r)^{2-2p} d\tau \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda$$

$$= \int_r^{\frac{k}{2}} \beta^{2-2p} d\beta \int_\beta^{\beta+2r} \frac{\lambda}{(t-r-\beta+\lambda+2k)^{3p}} d\lambda$$

$$\le 2(t-r+2k)^{-3p} r \int_r^{\frac{k}{2}} \beta^{2-2p}(\beta+r) d\beta$$

$$\le \frac{4}{4-2p} (\frac{k}{2})^{4-2p} r (t-r+2k)^{-3p}.$$

By the estimates of l_1 and l_2, note that $-p(4p-5)-2p+1 < -4p+3$ and $-p(4p-5)-3p < -4p+3$, then there exists a constant N_1 such that for $r < \frac{k}{2}$

$$J_2 \le N_1 r (t-r+2k)^{-4p+3}.$$

If $r \ge \frac{k}{2}$, then $t-2r < t-r-\frac{k}{2}$ and $t-\tau-r > t-(t-2r)-r = r \ge \frac{k}{2}$, $z(t-\tau-r) \ge \frac{k}{2}$ and

$$\int_L^{t-2r} \frac{(t-\tau-r)^{2-p}}{z^p(t-\tau-r)(2\tau-t-r+2k)^{p(4p-5)}} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$\le (\frac{k}{2})^p (2L-t-r+2k)^{-p(4p-5)} \int_L^{t-2r} (t-\tau-r)^{2-p} \int_{t-\tau-r}^{t-\tau+r} \frac{\lambda}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$\le (\frac{k}{2})^p (2L-t-r+2k)^{-p(4p-5)} \int_{\frac{1}{4}(t-7r+2k)}^{r} \beta^{2-p}(-d\beta$$

$$\int_\beta^{\beta+2r} \frac{1}{(t-r-\beta+\lambda+2k)^{3p}} d\lambda$$

$$\le (\frac{k}{2})^p (2L-t-r+2k)^{-p(4p-5)} \frac{1}{4}(t-7r+2k) \int_r^{\frac{1}{4}(t-7r+2k)} \beta^{2-p} d\beta$$

$$\int_\beta^{\beta+2r} \frac{\lambda}{(t-r-\beta+\lambda+2k)^{3p}} d\lambda$$

$$\leq (\frac{k}{2})^p(2L-t-r+2k)^{-p(4p-5)}\frac{3}{4}(t-7r+2k)\int_r^{\frac{1}{4}(t-7r+2k)} \beta^{-1-p\delta}\beta^{3+p\delta-p}$$

$$\int_\beta^{\beta+2r} \frac{1}{(t-r-\beta+\lambda+2k)^{3p}}d\lambda$$

$$\leq (\frac{k}{2})^p(2L-t-r+2k)^{-p(4p-5)}\frac{3}{4}(t-7r+2k)\int_r^{\frac{1}{4}(t-7r+2k)} \beta^{-1-p\delta}$$

$$\int_\beta^{\beta+2r} (t-r-\beta+\lambda+2k)^{-4p+3+p\delta}d\lambda$$

$$\leq (\frac{k}{2})^p(t-7r+2k)(2L-t-r+2k)^{-p(4p-5)}r(t-r+2k)^{-4p+3+p\delta}\int_{\frac{k}{2}}^\infty \beta^{-1-p\delta}d\beta$$

$$\leq (\frac{k}{2})^{p-p\delta}\frac{1}{p\delta}(t-7r+2k)(2L-t-r+2k)^{-p(4p-5)}r(t-r+2k)^{-4p+3+p\delta},$$

where δ is small enough such that $-p(4p-5)+1+p\delta<0$ which is possible by $p\in(p_0(9),2)$.

Note that $2L-t-r+2k=\frac{1}{2}(t+r+2k)>\frac{1}{2}(t-r+2k)$ and $t-7r+2k<t-r+2k$, then there exists a constant N_2 such that for $r\geq\frac{k}{2}$

$$J_2\leq N_2r(t-r+k)^{-p(4p-5)+1-4p+3+p\delta}\leq N_2r(t-r+2k)^{-4p+3}$$

Combining the estimates to J_1 and J_2, there a constant N_3 such that for $t_2>0$

$$I_2=J_1+J_2\leq N_3r(t-r+2k)^{-4p+3},$$

Combining the estimates to I_1 and I_2, there exists a constant N_4 such that

$$\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau\leq N_4r(t-r+2k)^{-4p+3}.$$

for $t_2>0$.

If $t_2=0$, i.e., $t+r\leq\frac{k}{2}$, then $I_1=0$. Therefore we only need to estimate I_2.

If $r<\frac{k}{2}$, then $t-2r>t-r-\frac{k}{2}$ and

$$I_2=\int_0^{t-r-\frac{k}{2}}\int_{t-\tau-r}^{t-\tau+r}\frac{\lambda^2(t-\tau)}{z^p(\lambda)\phi^p(\lambda,\tau)}d\lambda d\tau+\int_{t-r-\frac{k}{2}}^{t-2r}\int_{t-\tau-r}^{t-\tau+r}\frac{\lambda^2(t-\tau)}{z^p(\lambda)\phi^p(\lambda,\tau)}d\lambda d\tau.$$

The first integral can be estimated using similar method to I_1. The second is I_2. If $r > \frac{k}{2}$, then $t-\tau-r > t-(t-2r)-r = r > \frac{k}{2}$, we have $z(t-\tau-r) > \frac{k}{2}$. Since $\lambda < \tau + k$, then $\tau - \lambda + 2k > k$ and

$$I_2 = \int_0^{t-2r}\int_{t-\tau-r}^{t-\tau+r} \frac{\lambda^2(t-\tau)}{z^p(\lambda)\phi^p(\lambda,\tau)} d\lambda$$

$$\leq \frac{t}{k^{p(4p-5)}}\int_0^{t-2r}\int_{t-\tau-r}^{t-\tau+r} \frac{\lambda^{2-p}}{z^p(t-\tau-r)(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$I_2 \leq \frac{t}{k^{p(4p-5)}}\int_0^{t-2r}\frac{1}{(\frac{k}{2})^p}\int_{t-\tau-r}^{t-\tau+r} \frac{\lambda^{2-p}}{(\tau+\lambda+2k)^{3p}} d\lambda d\tau$$

$$\leq \frac{t}{k^{p(4p-5)}}(t-r+2k)^{-3p}(t+r)^{2-p}r(t-2r)$$

$$\leq \frac{k}{k^{p(4p-5)}}k^{2-p}kr(t-r+2k){-}3p$$

$$\leq k^{-4p^2+3p+4}r(t-r+2k)^{-3p} \leq k^{-4p^2+3p+4}r(t-r+2k)^{-4p+3}\ .$$

Combining all estimates, we conclude that there exists a constant $C_3(9,p,k)$ such that for $p \in (p_0(9), 2)$

$$\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda^2(t-\tau)|F(u(\lambda,\tau)|d\lambda d\tau \leq \frac{C_3(9,p,k)\|u\|}{(t-r+2k)^{4p-3}}.$$

Proof of 3.1: Similar to the proof of theorem 2.1

Theorem 3.2 *(almost global existence) Let* $S(r,t,c) = \{(r,t) \in \mathbb{R}^2 |\ r-t+2k \leq c$ *and* $r < t+k, r > 0\}$. *For small* $n = 7, 9$ *and* $\delta > 0$, *the equation (3.1) with Cauchy data (3.2) has a unique solution in* $\Omega_{p,k,9}(S(r,t,\left(\frac{1}{4C_5(n,p,k)\delta}-1\right)^{\frac{2}{n-5}})$ *provide that* $p > p_0(n)$ *and* $\sum_{i=0}^2(\|f^{(i)}\| + \|g^{(i)}\|) < \frac{\delta}{C_7(n,k)}$, *where* $C_5(n,p,k)$ *and* $C_7(n,k)$ *as in Lemma 3.4 and Lemma 3.6 respectively.*

Proof of theorem 3.2: Let

$$\Omega^\delta_{p,k,n} = \{u \in \Omega_{p,k,9}(S(r,t,\left(\frac{1}{4C_5(n,p,k)\delta^{p-1}}-1\right)^{\frac{2}{n-5}})|\ \|u\| < \delta\}.$$

Define

$$Tu = u^0 + T_1 u$$

then by Lemma 3.4, we have

$$\begin{aligned} \|T_1 u\| &\leq \|u^0\| + 2C_5(n,p,k)(1 + |t - r + 2k|^{\frac{n-5}{2}})\|u\|^p \\ &\leq C_7(n,k)(\sum_{i=0}^{2}(\|f^{(i)}\| + \|g^{(i)}\|) + 2C_5(n,p,k)\frac{1}{4C_5(n,p,k)\delta^{p-1}}\delta^p < \delta\,. \end{aligned}$$

Now the theorem follows from the standard contraction mapping theorem. $\square$

Chapter 4

Finite-time Blow-up and Life Span of Quasilinear Wave Equations

4.1 Blow-Up of the Solutions

Consider the quasilinear wave equation

$$u_{tt} - \frac{n-1}{r} u_r - u_{rr} = G(Du, D^2u) \,, \tag{4.1}$$

$$u(r,0) = 0, \quad u_t(r,0) = g(r). \tag{4.2}$$

We will prove that for $n \geq 2$, the solution of the equation blow up in finite-time for several classes of nonlinearities., we also obtain the life-span of the solutions.

Theorem 4.1 *Assume*

(a) $g \in C_0^\infty(\mathbb{R})$, $\quad g(r) \geq 0$ *and* $supp\{g\} \subset \{r : |r| < k\}$,

(b) there exist positive constants k_0 *and* k_1 *with* $0 < k_0 < k_1 < k$ *such that*

(i) $g(x) \not\equiv 0$ *on* $\{x : k_0 < |r| < k_1\}$

(ii) k_0 *is sufficiently close to* k *such that* $\frac{k_0}{k} > z_0$*, where* z_0 *satisfies that for all* $z \in [z_0, 1)$*,* $P_n(z) \geq \frac{1}{2}$ *and* $T_n(z) \geq \frac{1}{2}$*,* P_n *and* T_n *are Legendre and Tschebyscheff polynomials of degree n , respectively.*

Let $u \in C^2(\mathbb{R}^2 \times [0,T))$ *be a solution of radially wave equation (4.1) with Cauchy data (4.2) and* $G(Du, D^2u) = |u_t|^p, |u_r|^p, |u_{tt}|^p$ *or* $|u_{rr}|^p$*.* $T > 0$ *is the life span of* u *and* $1 < p \leq p_0(n) = \frac{n+1}{n-1}$*. Then* $T < +\infty$*. That is,* u *blows up in finite time.*

Lemma 4.1 *Let $n \geq 2$ and $u \in C^2(\mathbb{R} \times [0,T))$ be a solution of the Equation (4.1). Then there exists a constant $C > 0$ such that for $t + k_0 < r < t + k_1$,*

$$u(r,t) \geq Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(r-t+\tau,\tau)|^p d\tau \tag{4.3}$$

for $t > \frac{3}{2}(k-k_0) = 3\delta$, where $\delta = \frac{k-k_0}{2}$.

Proof: By Lemma 1.2, we have

$$u(r,t) \geq u^0(r,t) + Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(Du(\lambda,\tau), D^2u(\lambda,\tau))d\tau$$

$$u^0(r,t) \geq Cr^{\frac{1-n}{2}} \int_{r-t}^{r+t} \lambda^{\frac{n-1}{2}} g(\lambda)d\lambda > 0$$

Therefore

$$u(r,t) > Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(Du(\lambda,\tau), D^2u(\lambda,\tau))d\lambda d\tau \tag{4.4}$$

Note that $t > \frac{3}{2}(k-k_0)$ implies $r > k$, we can change the order of the integration (4.4), and obtain that

$$u(r,t) > Cr^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} \int_{\lambda-k}^{\lambda-r+t} G(Du, D^2u)d\tau d\lambda\ . \tag{4.5}$$

If $G(Du, D^2u) = |u_t|^p$, then

$$u(r.t) > Cr^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} \int_{\lambda-k}^{\lambda-r+t} |u_t|^p d\tau d\lambda\ .$$

By Holder's inequality,

$$\begin{aligned} u(r,t) &> Cr^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} \left| \int_{\lambda-k}^{\lambda-r+t} u_t(\lambda,\tau)d\tau \right|^p \left(\int_{\lambda-k}^{\lambda-r+t} d\tau \right)^{1-p} d\lambda \\ &\geq Cr^{\frac{1-n}{2}} (t-r+k)^{1-p} \int_k^r \lambda^{\frac{n-1}{2}} \left| \int_{\lambda-k}^{\lambda-r+t} u_t(\lambda,\tau)d\tau \right|^p d\lambda \\ &> Cr^{\frac{1-n}{2}} (k-k_0)^{1-p} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda,\lambda-r+t) - u(\lambda,\lambda-k)|^p d\lambda\ . \end{aligned}$$

By the support property of u, $u(\lambda, \lambda-k) = 0$, therefore

$$u(r,t) > Cr^{\frac{1-n}{2}} (k-k_0)^{1-p} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda,\lambda-r+t)|^p d\lambda\ . \tag{4.6}$$

If $G(Du, D^2u) = |u_{tt}|^p$,

$$u(r,t) > Cr^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} \int_{\lambda-k}^{\lambda-r+t} |u_{tt}|^p d\tau d\lambda \ .$$

By Holder's inequality

$$\left| \int_{\lambda-k}^{\lambda-r+t} (\lambda - r + t - \tau) u_{tt}(\lambda, \tau) d\tau \right|^p$$
$$\leq \left\{ \int_{\lambda-k}^{\lambda-r+t} |u_{tt}|^p d\tau \right\} \left\{ \int_{\lambda-k}^{\lambda-r+t} (\lambda - r + t - \tau)^q \right\}^{\frac{p}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$u(r,t) > Cr^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} \left| \int_{\lambda-k}^{\lambda-r+t} (\lambda - r + t - \tau) u_{tt}(\lambda, \tau) d\tau \right|^p$$
$$\left\{ \int_{\lambda-k}^{\lambda-r+t} (\lambda - r + t - \tau)^q d\tau \right\}^{\frac{p}{q}} d\lambda$$
$$\geq Cr^{\frac{1-n}{2}} \left[(t - r + k)^{q+1} \right]^{1-p}$$
$$\int_k^r \lambda^{\frac{n-1}{2}} \left[((\lambda - r + t - \tau) u_t(\lambda, \tau))|_{\tau=\lambda-k}^{\lambda-r+t} + u(\lambda, \tau)|_{\lambda-k}^{\lambda-r+t} \right]^p d\lambda$$
$$\geq Cr^{\frac{1-n}{2}} (k - k_0)^{(q+1)(1-p)} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda, \lambda - r + t)|^p d\lambda \tag{4.7}$$

Combining (4.6) and (4.7) leads to such that

$$u(r,t) > C(t+k)^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda, \lambda - r + t)|^p d\lambda \ . \tag{4.8}$$
$$> Ct^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda, \lambda - r + t)|^p d\lambda \tag{4.9}$$

for $t > \frac{3}{2}(k - k_0)$ Let $\lambda = \tau + r - t$, then (4.9) becomes

$$u(r,t) \geq Ct^{\frac{1-n}{2}} \int_{t-r+k}^t (\tau + r - t)^{\frac{n-1}{2}} |u(\tau + r - t, \tau)|^p d\tau$$
$$\geq Ct^{\frac{1-n}{2}} \int_{t-r+k}^t \tau^{\frac{n-1}{2}} |u(\tau + r - t, \tau)|^p d\tau$$
$$\geq Ct^{\frac{1-n}{2}} \int_{2\delta}^t \tau^{\frac{n-1}{2}} |u(\tau + r - t, \tau)|^p d\tau$$

where we used $t - r + k < k - k_0 = 2\delta$.

Note that

$$u(r,t) > Ct^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(Du, D^2u) d\lambda d\tau$$
$$\geq Ct^{\frac{1-n}{2}} \int_0^t (r-t+\tau)^{\frac{n-1}{2}} \int_{r-t+\tau}^{r+t-\tau} G(Du, D^2u) d\lambda d\tau$$
$$\geq Ct^{\frac{1-n}{2}} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} \int_{r-t+\tau}^{r+t-\tau} G(Du, D^2u) d\lambda d\tau$$

Since $r + t - \tau > r + t - (t - \delta) = r + \delta = r + \frac{k-k_0}{2} > \frac{t+r+k}{2} > t - \delta + k > \tau + k$, we have

$$u(r,t) \geq Ct^{\frac{1-n}{2}} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} \int_{r-t+\tau}^{\tau+k} G(Du, D^2u) d\lambda d\tau \tag{4.10}$$

If $G(Du, D^2u) \geq |u_r|^p$, By Holder's inequality

$$u(r,t) > Ct^{\frac{1-n}{2}} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} \left| \int_{r-t+\tau}^{\tau+k} u_r d\lambda \right|^p \left\{ \int_{r-t+\tau}^{\tau+k} d\lambda \right\}^{1-p} .$$
$$\geq Ct^{\frac{1-n}{2}} (t-r+k)^{1-p} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} |u(r-t+\tau, \tau)|^p d\tau$$
$$\geq Ct^{\frac{1-n}{2}} (k-k_0)^{1-p} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(r-t+\tau, \tau)|^p d\tau . \tag{4.11}$$

If $G(Du, D^2u) \geq |u_{rr}|^p$, by Holder's inequality and a similar argument to the case $|u_{tt}|^p$, we have

$$u(r,t) > Ct^{\frac{1-n}{2}} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} \left| \int_{r-t+\tau}^{\tau+k} (\tau + k - \lambda) u_{rr}(\lambda, \tau) d\lambda \right|^p$$
$$\left\{ \int_{r-t+\tau}^{\tau+k} (\tau + k - \lambda)^q d\lambda \right\}^{1-p} d\tau$$
$$\geq Ct^{\frac{1-n}{2}} (t-r+k)^{(1-p)(q+1)} \int_0^{t-\delta} \tau^{\frac{n-1}{2}} |u(r-t+\tau, \tau)|^p d\tau$$
$$\geq Ct^{\frac{1-n}{2}} (k-k_0)^{(1-p)(q+1)} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(r-t+\tau, \tau)|^p d\tau$$

Hence there exists a constant C such that for $t + k_0 < r < t + k_1$ and $t > 3\delta$, the following holds

$$u(r,t) > Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(\tau + r - t, \tau)|^p d\tau .$$

Lemma 4.2 *For any $n \geq 2$, if $t + k_o < r < t + k_1$ and $t > 3\delta$, then*

$$u(r,t) \geq \begin{cases} Ct^{\frac{1-n}{2}} \ln t & \text{for} \quad p = \frac{n+1}{n-1} , \\ Ct^{1-\frac{1-n}{2}} & \text{for} \quad 1 < p < \frac{n+1}{n-1} . \end{cases}$$

Proof: From lemma 4.1

$$u(r,t) > Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(\tau + r - t, \tau)|^p d\tau \ .$$

Since $\tau + k_0 < r - t + \tau < \tau + k_1$, lemma 1.2 implies

$$u(r - t + \tau, \tau) \geq u^0(r - t + \tau, \tau) > C\tau^{\frac{1-n}{2}}$$

therefore

$$\begin{aligned} u(r,t) &\geq Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{n-1}2 \left(Ct^{\frac{1-n}{2}}\right)^p d\tau \\ &\geq Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2} - \frac{n-1}{2}p} p d\tau \qquad (4.12) \end{aligned}$$

Note that $\frac{n-1}{2} - \frac{n-1}{2}p = -1$ when $p = \frac{n+1}{n-1}$, and $-1 < \frac{n-1}{2} - \frac{n-1}{2}p < 0$ when $1 < p < \frac{n+1}{n-1}$. (4.12) implies

$$u(r,t) \geq \begin{cases} Ct^{\frac{1-n}{2}} lnt & \text{for} \quad p = \frac{n+1}{n-1}, \\ Ct^{1-\frac{1-n}{2}} & \text{for} \quad 1 < p < \frac{n+1}{n-1}. \end{cases}$$

Lemma 4.3 *[13] Let $p = \frac{n+1}{n-1}$ for any $n \geq 2$, there exists a constant B such that for $T_m = B^{p^m} T_1$, $t + k_o < r < t + k_1$ and $t > T_m$*

$$u(r,t) \geq Ct^{\frac{1-n}{2}} (lnt)^{p^{m-1}} \ .$$

Lemma 4.4 *Let $1 < p < \frac{n+1}{n-1}$ for any $n \geq 2$. Then there exists a constant C and S_m such that, for $t + k_0 < r < t + k_1$, $t > S_m$*

$$u(r,t) \geq Ct^{\frac{1-n}{2}} t^{\alpha(\sum_{i=0}^{m-1} p^i)},$$

where $\alpha = \frac{n+1}{2} - \frac{n-1}{2}p$ ..

Proof: If $m = 1$, it is true because for $t + k_0 < r < t + k_1$

$$u(r,t) \geq u^0(r,t) \geq Ct^{\frac{1-n}{2}} \ .$$

. Now assume that it is true for $m > 1$. WE will prove it is true for $m + 1$ by induction. Let $t + k_0 < r < t + k_1$ and $t > S_m$, then by Lemma 4.1

$$u(r,t) \geq Ct^{\frac{1-n}{2}} \int_{2\delta}^{t-\delta} \tau^{\frac{n-1}{2}} |u(r - t + \tau, \tau)|^p d\tau$$

where $t > S_m + \delta$. By the induction assumption, we have

$$\begin{aligned} u(r,t) &\geq Ct^{\frac{1-n}{2}} \int_{S_m}^{t-\delta} \tau^{\frac{n-1}{2}} \left[C\tau^{\frac{1-n}{2}} \tau^{\alpha(\sum_{i=0}^{m-1} p^i)} \right]^p d\tau \\ &\geq Ct^{\frac{1-n}{2}} \int_{S_m}^{t-\delta} \tau^{\frac{n-1}{2} - \frac{n-1}{2}p + \alpha(\sum_{i=0}^{m-1} p^i)p} d\tau \\ &\geq Ct^{\frac{1-n}{2}} \tau^{\frac{n+1}{2} - \frac{n-1}{2}p + \alpha(\sum_{i=0}^{m-1} p^i)} |_{S_m}^{t-\delta} \\ &= Ct^{\frac{1-n}{2}} \tau^{\alpha(\sum_{i=0}^{m} p^i)} |_{S_m}^{t-\delta} . \end{aligned}$$

Therefore there exists a $S_{m+1} > S_m + \delta$ such that for $t > S_{m+1}$

$$(t-\delta)^{\alpha(\sum_{i=0}^{m} p^i)} - S_m^{\alpha p(\sum_{i=0}^{m} p^i)} > \frac{1}{2} t^{\delta(\sum_{i=0}^{m} p^i)} .$$

We have

$$u(r,t) \geq Ct^{\frac{1-n}{2}} t^{\alpha(\sum_{i=0}^{m-1} p^i)} .$$

Lemma 4.5 *Let $n \geq 2$ and $1 < p \leq \frac{n+1}{n-1}$. Then for any $l > 0$, there exist a constant C_l and an integer m such that if $t + k_0 < r < t + k_1$ and $t > max\{T_m, S_m\}$,*

$$u(r,t) > C_l t^l .$$

Proof: In [13], the case $P = \frac{n+1}{n-1}$ was proved for odd integer n. By using Lemmas 4.1, 4.2, 4.3 and 4.4, we can follow the simliar method in [13] to prove that the inequality holds for any integer. So we omit the proof.

For $1 < p < \frac{n+1}{n-1}$, note that for any $l > 0$ there exists an m such that

$$\frac{1-n}{2} + \alpha(\sum_{i=0}^{m} p^i) > l ,$$

Hence

$$u(r,t) > Ct^l$$

for $t \geq S_m$ **Proof of theorem 4.1.** Ramaha [13] proved that for n odd and $G(Du, D^2u) \geq |u_r|^p$ or $|u_t|^p$, the solutions must blow up when $p = \frac{n+1}{n-1}$. By the results of lemma 4.1, 4.2, 4.3, 4.4, and 4.5, we can unify both odd and even dimensions to show the blow up. The details are omitted. If $G(Du, D^2u) \geq |u_{tt}|^p$ or $|u_{rr}|^p$, we define

$$F(t) = \int_{\mathbb{R}^n} u(r,t)dx = \int_0^{t+k} \omega_n r^{n-1} u(r,t)dr$$

then

$$\ddot{F}(t) = \int_{\mathbb{R}^n} u_{tt}(r,t)dx = \omega_n \int_0^{t+k} r^{n-1} u_{tt}(r,t)dr$$

Case 1: $G(Du, D^2u) \geq |u_{tt}|^p$

$$\begin{aligned}
\dot{F}(t) - \dot{F}(0) &= \int_0^t \ddot{F}(s)ds = \int_0^t \int_{r<\tau+k} u_{tt}(r,\tau)dxd\tau \\
&= \omega_n \int_0^t \int_0^{\tau+k} r^{n-1} u_{tt}(r,\tau)drd\tau \\
&\leq \omega_n \int_0^t \left(\int_0^{t+k} r^{n-1}|u_{tt}(r,t)|^p dr\right)^{\frac{1}{p}} \left(\int_0^{r+t} r^{n-1}dr\right)^{\frac{1}{q}} d\tau \\
&\leq \omega_n^{\frac{1}{q}} \int_0^t (\ddot{F}(t))^{\frac{1}{p}}(t+k)^{\frac{n}{q}} d\tau \leq \omega_n^{\frac{1}{q}} \ddot{F}^{\frac{1}{p}}(t)(t+k)^{\frac{n}{q}+1}
\end{aligned}$$

therefore we have

$$\ddot{F}^{\frac{1}{p}}(t) \geq \omega_n^{-\frac{1}{q}}(t+k)^{-(\frac{n}{q}+1)}(\dot{F}(t) - \dot{F}(0)) \ ,$$

i.e.

$$\ddot{F}(t) \geq \omega_n^{-\frac{p}{q}}(t+k)^{-(\frac{n}{q}+1)p}(\dot{F}(t) - \dot{F}(0))^p \ . \tag{4.13}$$

On the other hand

$$\dot{F}(t) - \dot{F}(0) = \int_0^t \int_{\mathbb{R}^n} |u_{tt}|^p dxd\tau$$

$$\begin{aligned}
&\geq \omega_n \int_0^t \int_{t+k_0}^{t+k_1} r^{n-1}|u_{tt}|^p drd\tau \\
&\geq \omega_n \int_{t+k_0}^{t+k_1} \left(\int_{t-k}^t |u_{tt}|^p d\tau\right) r^{n-1}dr \\
&\geq \omega_n \int_{t+k_0}^{t+k_1} r^{n-1}|\int_{t-k}^t (t-\tau)u_{tt}(r,\tau)d\tau|^p \left(\int_{t-k}^t (t-\tau)^q d\tau\right)^{1-p} dr \\
&\geq \omega_n \int_{t+k_0}^{t+k_1} r^{n-1}|u(r,t)|^p (k^{q+1})^{1-p} dr \\
&\geq \omega_n (t+k_0)^{n-1} k^{(q+1)(1-p)} \int_{t+k_0}^{t+k_1} |u(r,t)|^p dr \ .
\end{aligned} \tag{4.14}$$

Combining this with lemma 4.5, for $l = \frac{1}{p}\left(n + \frac{3p-1}{p-1}\right)$, there exists a constant C_l and $T_{m(l)}$ such that for $t + k_0 < r < t + k_1$ and $t > T_{m(l)}$

$$u(r,t) \geq C_l t^l$$

Combining (4.14) this with leads to

$$\begin{aligned}\dot F(t)-\dot F(0) &\geq \omega_n k^{(q+1)(1-p)}(k_1-k_0)C_l(t+k_0)^{(n-1)+n+\frac{3p-1}{p-1}}\\ &\geq Ct^{2n+\frac{2p}{p-1}}. \end{aligned}\tag{4.15}$$

From (4.13) and (4.15), we have

$$\ddot F(t)\geq C(t+k)^{n-(n+1)p}(\dot F(t)-\dot F(0))^{\frac{p-1}{2}}(\dot F(t)-\dot F(0))^{\frac{p+1}{2}}.$$

Therefore there exists a T' such that for $t>T'$

$$\begin{aligned}\ddot F(t) &\geq Ct^{n-(n+1)p}t^{(2n+\frac{2p}{p-1})\frac{p-1}{2}}(\dot F(t)-\dot F(0))^{\frac{p+1}{2}}\\ &\geq C(\dot F(t)-\dot F(0))^{\frac{p+1}{2}}\geq C\dot F^{\frac{p+1}{2}}(t)\end{aligned}$$

Case 2: $G(Du,D^2u)\geq |u_{rr}|^p$.

By the support property of u and the Divergence Theorem, we have

$$\begin{aligned}\ddot F(t) &= \int_{\mathbb{R}^n} u_{tt}(r,t)dx=\int_{|x|<t+k}(\Delta u+|u_{rr}|^p)dx\\ &= \int_{|x|<t+k}|u_{rr}|^p dx=\omega_n\int_0^{t+k} r^{n-1}|u_{rr}|^p dr\\ &\geq \omega_n\left|\int_0^{t+k} r^{n+1}u_{rr}(r,t)dr\right|^p\left[\int_0^{t+k} r^{(n-1+2q}dr\right]^{1-p}\\ &\geq \omega_n\left|\int_0^{t+k} r^{n-1}u(r,t)dr\right|^p\left[(t+k)^{(n+2q)}\right]^{1-p}\\ &\geq \omega_n^{1-p}\left|\int_0^{t+k}\omega_n r^{n-1}u(r,t)dr\right|^p\left[(t+k)^{n+2q}\right]^{1-p}\\ &= \omega_n^{1-p}F^p(t)(t+k)^{(n+2q)(1-p)}\end{aligned}\tag{4.16}$$

On the other hand

$$\begin{aligned}\ddot F(t) &= \omega_n\int_0^{t+k} r^{n-1}|u_{rr}(r,t)|^p dr\\ &\geq \omega_n\int_{t+\frac{k_0+k_1}{2}}^{t+k} r^{n-1}|u_{rr}|^p dr\\ &\geq \omega_n t^{n-1}\left|\int_{t+\frac{k_0+k_1}{2}}^{t+k}(r-t-\frac{k_0+k_1}{2})u_{rr}(r,t)dr\right|^p\\ &\qquad\left(\int_{t+\frac{k_0+k_1}{2}}^{t+k}(r-t-\frac{k_0+k_1}{2})^q dr\right)^{1-p}\\ &\geq \omega_n t^{n-1}\left(k-\frac{k_0+k_1}{2}\right)^{(q+1)(1-p)}\left|u(t+\frac{k_0+k_1}{2},t)\right|^p\end{aligned}\tag{4.17}$$

Let $l' = \frac{2(n+2q)-(n+1)}{p}$, since $t+k_0 < t + \frac{k_0+k_1}{2} < t+k_1$, by Lemma 4.5 there exists an integer $m'(l')$, a constant $C_{l'}$, and $T_{m'(l')}$ such that $t > T_{m'(l)}$

$$u(t+\frac{k_0+k_1}{2},t) \geq C_{l'}t^{l'} \tag{4.18}$$

Combining (4.18 and (4.17) leads to

$$\ddot{F}(t) \geq Ct^{2(n+2q)-2}\,.$$

Then there exists a $T'' > T_{m'(l')}$ such that $t > T''$

$$F(t) \geq Ct^{2(n+2q)}\,. \tag{4.19}$$

From (4.16), there exists T_0 such that for $t > T_0$ the following holds

$$\begin{aligned} \ddot{F}(t) &\geq \frac{1}{2}\omega_n^{1-p}t^{(n+2q)(1-p)}F^{\frac{p-1}{2}}(t)F^{\frac{n+1}{2}}(t) \\ &\geq CF^{\frac{p+1}{2}} \end{aligned} \tag{4.20}$$

therefore $F(t)$ blows up. □

4.2 Life-span of the solutions

Theorem 4.2 *Let T be the life span of classic solution of the equation (4.1) with $G(Du, D^2u) \geq |u_t|^p$ or $|u_{tt}|^p$ and $g(r)$ replaced by $\varepsilon g(r)$ for $0 < \varepsilon < 1$. Then there exists a constant C such that*

$$T \leq exp(C\varepsilon^{1-p})$$

if $p = \frac{n+1}{n-1}$ for $n \geq 2$.

Lemma 4.6 *Let $\Omega = \{(r,t) \in \mathbb{R}^n \mid t+k_0 < r < t+k_1,\ t \geq 2\delta\}$ and u, $v \in C(\Omega)$ satisfy the following*

$$u(r,t) > ar^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}}|u(\lambda,\lambda-r+t)|^p d\lambda + br^{\frac{1-n}{2}} \tag{4.21}$$

$$v(r,t) \leq ar^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}}|v(\lambda,\lambda-r+t)|^p d\lambda + br^{\frac{1-n}{2}} \tag{4.22}$$

where a and b are positive constants. Then

$$u(r,t) > v(r,t) \qquad \text{for } (r,t) \in \Omega \tag{4.23}$$

Proof: we will prove this by contradiction. If this is not true, then $\Omega_1 = \{(r,t) \in \Omega \mid u(r,t) \leq v(r,t)\}$ is not empty and closed. Let $\Omega_2 = \{(r_0,t) \in \Omega_1 \mid$ such that (r_0,t) nearest to $r = k$ $\}$. Since

$$u(k,t) > br^{\frac{1-n}{2}} \geq v(k,t)$$

$(k,t) \notin \Omega_1$, hence $(k,t) \notin \Omega_2$. So $r_0 > k$. Let $r \in (k, r_0)$, then $(r,t) \notin \Omega_1$. Therefore $u(r,t) > v(r,t)$ for $k < r < r_0$, then we have the following

$$u(r_0,t) > ar^{\frac{1-n}{2}} \int_k^{r_0} \lambda^{\frac{n-1}{2}} |u(\lambda, \lambda - r + t)|^p d\lambda + br^{\frac{1-n}{2}} \tag{4.24}$$

$$v(r_0,t) \leq ar^{\frac{1-n}{2}} \int_k^{r_0} \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda + br^{\frac{1-n}{2}} \tag{4.25}$$

which contradicts to $(r_0,t) \in \Omega_1$. Therefore Ω_1 must be empty, for any $(r,t) \in \Omega$

$$u(r,t) > v(r,t) \qquad \text{for any } (r,t) \in \Omega \tag{4.26}$$

Lemma 4.7 *Let $S \in C^2([0, T+k))$ and satisfy*

$$S(x) = a \int_0^x \left(\frac{1}{s+k} - \frac{1}{x+k} \right) |S(s)|^p ds + b.$$

Then $h(z) = S(x)$ satisfies the following equations

$$\begin{aligned} &h''(z) + h'(z) = a|h(z)|^p\,, \\ &h(0) = b, \quad h'(0) = 0\,, \end{aligned} \tag{4.27}$$

where $z = ln\left(\frac{x+k}{k}\right)$.

Proof: Differentiating the equation $h(z) = S(x)$, we have the following

$$S'(x) = h'(z)\frac{1}{x+k}$$
$$h'(z) = (x+k)S'(x)$$

therefore

$$h''(z) = (x+k)S'(x) + (x+k)^2 S''(x)\,.$$

On the other hand,

$$S'(x) = \frac{a}{(x+k)^2} \int_0^x |S(s)|^p ds$$
$$S'' = \frac{-2a}{(x+k)^3} \int_0^x |S(s)|^p ds + \frac{a}{(x+k)^2}|S(x)|^p,$$
$$S''(x) = \frac{-2}{(x+k)} S'(x) + \frac{a}{(x+k)^2}|S(x)|^p,$$

Hence

$$h''(z) + h'(z) = 2(x+k)S'(x) + (x+k)^2\left(\frac{-2}{(x+k)}S'(x) + \frac{a}{(x+k)^2}|S(x)|^p\right)$$
$$= a|S(x)|^p = a|h(z)|^p$$

with $h(0) = S(0) = b$ and $h'(0) = kS'(0) = 0$.

Lemma 4.8 *[19] Let τ_1 be the life span of the solution*

$$\begin{aligned} &h''(z) + h'(z) = a|h(z)|^p, \\ &h(0) = b, \quad h'(0) = 0. \end{aligned} \tag{4.28}$$

Then 1) $h''(z) > 0$ for $0 \le z < \tau_1$. 2) $\tau_1 < +\infty$. 3) $h(z) \to +\infty$ as $z \to \tau_1$.

Lemma 4.9 *[19] Let $z(\eta)$ be the life span of the solution to the following equation*

$$\begin{aligned} &h''(z) + h'(z) = a|h(z)|^p, \\ &h(0) = b\eta, \quad h'(0) = 0. \end{aligned} \tag{4.29}$$

Then $z(\eta) \le \tau_1 \eta^{1-p}$ for $0 < \eta < 1$, where τ_1 is defined in Lemma 4.8.

Proof of theorem 4.2. Let $v(r,t) = S(r-k)r^{\frac{1-n}{2}}$ for $(r,t) \in \Omega$. Since $\lambda - r + t + k_0 < \lambda < \lambda - r + t + k_1$ and $\frac{n-1}{2} - \frac{n-1}{2}p = -1$, we have

$$\begin{aligned} a\int_k^r \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda + b &= a\int_k^r \lambda^{\frac{n-1}{2} - \frac{n-1}{2}p} |S(\lambda - k)|^p d\lambda + b\,. \\ &= a\int_0^{r-k} (y+k)^{-1} |S(y)|^p dy + b \\ &\ge a\int_0^{r-k} \left(\frac{1}{y+k} - \frac{1}{(r-k)+k}\right) |S(y)|^p dy + b \\ &= S(r-k), \end{aligned}$$

therefore

$$v(r,t) = S(r-k)r^{\frac{1-n}{2}} \le ar^{\frac{1-n}{2}} \int_0^{r-k} \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda + br^{\frac{1-r}{2}} \tag{4.30}$$

Note that there exists a constant $C_0(g)$ such that

$$u^0(r,t) \ge C_0(g)\varepsilon r^{\frac{1-n}{2}}$$

From (4.6) and (4.7) in the proof of Lemma 4.1, there exists a constant C_1 such that

$$u(r,t) \geq C_1 r^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} |u(\lambda, \lambda - r + t)|^p d\lambda + C_0(g)\varepsilon r^{\frac{1-n}{2}}$$

Let $a = C_1$ and $b = C_0(g)\varepsilon$ in (4.29). Then

$$S(x) = C_1 \int_0^x \left(\frac{1}{y+k} - \frac{1}{x+k} \right) |S(y)|^p dy + C_0(g)\varepsilon, \tag{4.31}$$
$$v(r,t) = C_1 r^{\frac{1-n}{2}} \int_k^r \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda + C_0(g)\varepsilon r^{\frac{1-n}{2}}$$

and by Lemma 4.6 we have

$$u(r,t) > v(r,t) \qquad \text{for } (r,t) \in \Omega.$$

Let $h(z) = h(ln(\frac{x+k}{k})) = S(x)$, then by Lemma 4.7

$$h''(z) + h'(z) = C_1 |h(z)|^p$$
$$h(0) = C_0(g)\varepsilon, \quad h'(0) = 0.$$

From Lemma 4.8, the life span $z(\varepsilon)$ of h satisfies

$$z(\varepsilon) \leq \tau_1 \varepsilon^{1-p}$$

where τ_1 is the life span of the solution of equation (4.27) with $a = C_1$ and $b = C_0(g)$. Therefore

$$ln(\frac{x+k}{k}) \leq \tau_1 \varepsilon^{1-p},$$

and

$$x \leq k exp\left(\tau_1 \varepsilon^{1-p}\right) - k.$$

Let $x = r - k > t + k_0 - k$, for ε small we have

$$t < \frac{1}{2} k exp\left(\tau_1 \varepsilon^{1-p}\right).$$

□

Theorem 4.3 *Let T be the life span of classic solution of the equation (4.1) with $G(Du, D^2u) \geq |u_r|^p$ or $|u_{rr}|^p$ and $g(r)$ is replace by $\varepsilon g(r)$ for $0 < \varepsilon < 1$. Then there exists a constant C such that*

$$T \leq exp(C\varepsilon^{p(1-p)})$$

if $p = \frac{n+1}{n-1}$ for $n \geq 4$.

Proof

Let

$$F(t) = \int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau) r^{\frac{n-3}{2}} u(r,\tau) dr d\tau$$

$$F''(t) = \int_{t+k_0}^{t+k} r^{\frac{n-3}{2}} u(r,t) dr$$

$$u(r,t) = u^0(r,t) + Qu(r,t)$$

For $t + k_0 \le r \le t + k_1$, $u^0(r,t) \ge C\varepsilon r^{\frac{1-n}{2}}$ and

$$Qu(r,t) \ge Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(Du, D^2u) d\lambda d\tau$$

Integrating $u(r,t)$ over $t + k_0 \le r \le t + k$ yields

$$\int_{t+k_0}^{t+k} r^{\frac{n-3}{2}} u^0(r,t) dr \ge C\varepsilon (t+k)^{-1}$$

exchanging the order of integration, we have for $t \ge k_2 = \frac{1}{2}(k - k_0)$,

$$\begin{aligned}
I &= \int_{t+k_0}^{t+k} r^{\frac{n-3}{2}} Qu(r,t) dr = \int_{t+k_0}^{t+k} r^{-1} \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda^{\frac{n-1}{2}} G(Du, D^2u) d\lambda d\tau dr \\
&= \int_0^{t-k_2} \int_{\tau+k_0}^{\tau+k} \lambda^{\frac{n-1}{2}} G(Du, D^2u) [\int_{t+k_0}^{\lambda+t-\tau} r^{-1} dr] d\lambda d\tau \\
&+ \int_{t-k_1}^{t} \int_{\tau+k_0}^{2t-\tau+k_0} \lambda^{\frac{n-1}{2}} G(Du, D^2u) [\int_{t+k_0}^{\lambda+t-\tau} r^{-1} dr] d\lambda d\tau \\
&+ \int_{t-k_1}^{t} \int_{2t-\tau+k_0}^{\tau+k} \lambda^{\frac{n-1}{2}} G(Du, D^2u) [\int_{\lambda-t+\tau}^{\lambda+t-\tau} r^{-1} dr] d\lambda d\tau \ ,
\end{aligned}$$

where $k_2 = \frac{1}{2}(k - k_0)$ First estimate the inner integrals

$$\begin{aligned}
&\int_{t+k_0}^{\lambda+t-\tau} r^{-1} dr \ge (\lambda + t - \tau)^{-1} (\lambda - \tau - k_0) \ge (t+k)^{-1} (\lambda - \tau - k_0) \\
&\ge (t+k)^{-2} (t-\tau)(\lambda - \tau - k_0),
\end{aligned}$$

where we used $\lambda - \tau \le k$ and $(t+k)^{-1}(t-\tau) \le 1$.

$$\begin{aligned}
&\int_{\lambda-t+\tau}^{\lambda+t-\tau} r^{-1} dr \ge (\lambda + t - \tau)^{-1} 2(t-\tau) \ge 2(t+k)^{-1}(t-\tau) \\
&\ge k_2^{-1} (t+k)^{-1} (t-\tau)(\lambda - \tau - k_0) \ge k k_2^{-1} (t+k)^{-2} (t-\tau)(\lambda - \tau - k_0),
\end{aligned}$$

where we used $\lambda - \tau - k_0 \le 2k_2$ and $k(t+k)^{-1} \le 1$.

Combining the above estimate, we have

$$I \ge C(t+k)^{-2} \int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)(\lambda - \tau - k_0) \lambda^{\frac{n-1}{2}} G(Du, D^2u) d\lambda d\tau \ .$$

If $G(Du, D^2u) = |u_r|^p$, then by Holder's inequality

$$\int_0^t \int_{\tau+k_0}^{t+k} (t-\tau)(\lambda-\tau-k_0)\lambda^{\frac{n-1}{2}}|u_r|^p d\lambda d\tau$$
$$\geq |\int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)(\lambda-\tau-k_0)\lambda^{\frac{n-3}{2}} u_r d\lambda d\tau|^p$$
$$[\int_0^t \int_{\tau+k_0}^{tau+k} (t-\tau)(\lambda-\tau-k_0)\lambda^{\frac{n-1}{2}-q} d\lambda d\tau]^{1-p}$$

Since $p = \frac{n+1}{n-1}$ and $q = \frac{n+1}{2}$, $\frac{n-1}{2} - q = -1$.

$$\int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)(\lambda-\tau-k_0)\lambda^{\frac{n-1}{2}-q} d\lambda d\tau \leq C(t+k)\int_0^t (\tau+k_0)^{-1} d\tau$$
$$\leq C(t+k)log(\frac{t+k}{k}).$$

Note $u(r,t) = 0$ for $r \geq t+k$, we have

$$|\int_0^t \int_{\tau+k_0}^{t+k} (t-\tau)(\lambda-\tau-k_0)\lambda^{\frac{n-3}{2}} u_r d\lambda d\tau|$$
$$= |-\int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)\lambda^{\frac{n-3}{2}} u(\lambda,\tau) d\lambda d\tau) d\lambda d\tau$$
$$-\frac{n-3}{2}\int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)\lambda^{\frac{n-5}{2}}(\lambda-\tau-k_0) u d\lambda d\tau|$$
$$\geq |\int_0^t \int_{\tau+k_0}^{\tau+k} (t-\tau)\lambda^{\frac{n-3}{2}} u(\lambda,\tau) d\lambda, d\tau) d\lambda d\tau| = F(t)\ .$$

Combining above estimates concludes that for $t \geq k_2$

$$F''(t) \geq CF^p(t)((t+k)log(\frac{t+k}{k}))^{1-p}(t+k)^{-2}. \tag{4.32}$$

Since

$$F''(t) \geq \int_{t+k_0}^{t+k} r^{\frac{n-3}{2}} u^0(r.t) dr \geq C\varepsilon(t+k)^{-1}$$

Integrating twice leads to

$$F(t) \geq C\varepsilon(t+k)log(\frac{t+k}{k}); \qquad t \geq k_2. \tag{4.33}$$

From (4.32) and (4.33), we have

$$F''(t) \geq C\varepsilon^p(t+k)^{-1}log(\frac{t+k}{k}), \qquad t > k_2 \tag{4.34}$$

Integrating two more times concludes the following

$$F(t) \geq C\varepsilon^p(t+k)[log(\frac{t+k}{k})]^2; \qquad t \geq k_3 = 2k\ . \tag{4.35}$$

From (4.32) and (4.35), we have the following

$$F''(t) \geq C\varepsilon^{p(1-p)}(t+k)^{-2}log^{p-1}(\frac{t+k}{k})F(t); \qquad t > k_3 . \tag{4.36}$$

Since $F'(t) > 0$, multiply (4.36) by $F'(t)$ and integrate from $k_4 \geq k_3$

$$\begin{aligned}&(F'(t))^2 \geq (F'(k_4))^2 + \\ &C\varepsilon^{p(1-p)}[(t+k)^{-2}log^{p-1}(\frac{t+k}{k})F^2 - (k_4+k)^{-2}log^{p-1}(\frac{k_4+k}{k})] .\end{aligned} \tag{4.37}$$

Since for $t_2 > t_1 \geq 0$,

$$F'(t_1) \leq \frac{F(t_2) - F(t_1)}{t_2 - t_1} \leq F'(t_2) ,$$

$F(0) = 0$, So $F(k_4) \leq k_4 F'(k_4)$ Choose k_4 such that

$$k_4^2 C\varepsilon^{p(1-p)}(k_4+k)^{-2}log^{p-1}(\frac{k_3+k}{k}) \geq 1 , \tag{4.38}$$

that is, $k_4 = O(exp(C\varepsilon^{p(1-p)}))$ as $\varepsilon \to 0$.

By (4.36),

$$\begin{aligned}&F'(t) \geq F'(k_4)^2 + (k_4^2 a_{\varepsilon,p}(t))^{-1}[a_{\varepsilon,p}(t)F^2(t) - a_{\varepsilon,p}(k_4)F^2(k_4)]. \\ &\geq \frac{a_{\varepsilon,p}(t)}{k_4^2 a_{\varepsilon,p}(k_4)}F^2(t)\end{aligned} \tag{4.39}$$

Since

$$\frac{a_{\varepsilon,p}(t)}{k_4^2 a_{\varepsilon,p}(t)} \geq C(t+k)^{-2}log^{p-1}(\frac{t+k}{k}) ,$$

by (4.37), we have

$$F'(t) \geq C(t+k)^{-1}log^{\frac{p-1}{2}}(\frac{t+k}{k})F(t) . \tag{4.40}$$

Integrating once leads to

$$log\frac{F(t)}{F(k_4)} \geq Clog^{\frac{p+1}{2}}(\frac{t+k}{k_4+k}); \qquad t \geq k_4 .$$

If $t > k_5 = Ck_4^2$, then

$$log\frac{F(t)}{F(k_4)} \geq \frac{2p+4}{p-1}log(\frac{t+k}{k}) . \tag{4.41}$$

By (4.35) and (4.41), we have

$$F(t) \geq C\varepsilon^p (t+k)^{\frac{2t+4}{p-1}}, \qquad t \geq k_5 \,. \tag{4.42}$$

From (4.32) we have

$$F''(t) \geq C(t+k)^{-1-p} log^{1-p}(\frac{t+k}{k}) F^{\frac{p+1}{2}}(t) F^{\frac{p-1}{2}}(t).$$

Combining this with (4.42) leads to

$$F''(t) \geq C\varepsilon^{\frac{p(p-1)}{2}} F^{\frac{p+1}{2}}(t), \qquad t \geq t_5 \,.$$

Multiplying $F'(t)$ to both sides and integrating once

$$F'(t)^2 \geq C\varepsilon^{\frac{p(p-1)}{2}} (F^{\frac{p+3}{2}}(t) - F^{\frac{p(p-1)}{2}}(k_5)).$$

Note that $F(t) \geq F'(k_5)(t-k_5) \geq F(k_5)(\frac{t-k_5}{k_5})$, if $t \geq k_6 = 5k_5$, we have

$$F^{\frac{p+3}{2}}(t) - F^{\frac{p+3}{2}}(k_5) \geq \frac{1}{2} F^{\frac{P+3}{2}}(t) \,.$$

Therefore

$$F'(t) \geq C\varepsilon^{\frac{p(p-1)}{4}} F^{\frac{p+3}{4}}(t), \qquad t \geq k_6.$$

One more integration gives

$$\frac{1}{F^{\frac{p-1}{4}}(k_6)} \geq C\varepsilon^{\frac{p(p-1)}{4}} t \,. \tag{4.43}$$

Hence the life span is finite.

Now we need to prove that the life span T satisfies $T \leq exp(C\varepsilon^{p(1-p)})$.

If $T \leq k_6 = Ck_4^2$, by $k_4 = O(exp(C\varepsilon^{p(1-p)})$, we are done.

If $T \geq k_6$, by (4.43) and (4.42), we have

$$\frac{1}{C^{\frac{p-1}{2}} \varepsilon^{\frac{p(p-1)}{2}} k_6^{p+2}} \geq C\varepsilon^{\frac{p(p-1)}{2}} k_6 \,,$$

which leads to $C\varepsilon^{\frac{3p(p-1)}{4}} k_6^{p+3} \leq 1$. This is a contradiction as $\varepsilon \to 0$. Therefore the life span T must satisfies

$$T \leq k_6 \leq exp(C\varepsilon^{p(1-p)})$$

when ε is small. $\square$

Theorem 4.4 *Let $u \in C^2(\mathbb{R} \times [0,T))$ be a solution of*

$$\begin{aligned} &u_{tt} - \Delta u = F'(u)u_t, or F'(u_t)u_{tt} \qquad (r,t) \in \mathbb{R} \times [0,T) \\ &u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r) , \end{aligned} \tag{4.44}$$

where F satisfies $F(x) \geq |x|^p$, $1 < p \leq \frac{n+1}{n-1}$ and $f, g \in C_0^\infty(\mathbb{R})$ satisfy the same conditions as g of Theorem 4.1, in addition, $g - F(g) \geq 0$. Then $T < +\infty$, especially, $T < exp(C\varepsilon^{1-p})$ for $p = \frac{n+1}{n-1}$.

Proof: We only prove the case $F'(u)u_t$. Define

$$v(r,t) = \int_0^t u(r,s)ds ,$$

then v satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}[\Box v - F(v_t)] = 0 , \\ &v(r,0) = 0, \quad v_t(r,0) = f(r). \end{aligned}$$

Therefore

$$\begin{aligned} &\Box v - F(v_t) = w(r,t) , \\ &v(r,0) = 0, \quad v_t(r,0) = f(r) , \end{aligned}$$

where $w(r,t) = F(v_t) + g(r) - F(g(r))$. Note that

$$v(r,t) = v^0(r,t) + Q(r,t)$$

By lemma 1.2, for $t + k_0 < r < t + k_1$ and $t \geq 2\delta$,

$$\begin{aligned} &v(r,t) = v^0(r,t) + Q(r,t) , \\ &v^0(r,t) \geq C(f)\varepsilon r^{\frac{1-n}{2}} , \\ &Q(r,t) \geq Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} w(\lambda,\tau) d\lambda d\tau \\ &\geq Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{1-n}{2}} (g(\lambda) - F(g(\lambda))) d\lambda d\tau \\ &+Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} F(v_t) d\lambda d\tau \\ &\geq Cr^{\frac{1-n}{2}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \lambda^{\frac{n-1}{2}} F(v_t) d\lambda d\tau \\ &\geq Cr^{\frac{1-n}{2}} \int_k^r \int_{r-t+k}^{r+t-\tau} \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda. \end{aligned}$$

Hence

$$v(r,t) \geq Cr^{\frac{1-n}{2}} \int_k^r \int_{r-t+k}^{r+t-\tau} \lambda^{\frac{n-1}{2}} |v(\lambda, \lambda - r + t)|^p d\lambda + C(f)\varepsilon r^{\frac{1-n}{2}} .$$

Now following the method of Theorem 4.1, we can prove $T < +\infty$ for $1 < p \leq \frac{n+1}{n-1}$. Following the method of Theorem 4.2, we can obtain $T < exp(C\varepsilon^{1-p})$ for $p = \frac{n+1}{n-1}$.

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