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ELASTIC AND THERMAL PROPERTIES OF PARTICULATE COMPOSITES WITH INHOMOGENEOUS INTERPHASES

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Wei Wang

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Master degree in Mechanics

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ELASTIC AND THERMAL PROPERTIES OF PARTICULATE COMPOSITES WITH INHOMOGENEOUS INTERPHASES

by

Wei Wang

A THESIS

Submitted To
Michigan State University
in partial fulfillment of the requirements
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1994

ABSTRACT

ELASTIC AND THERMAL PROPERTIES OF PARTICULATE COMPOSITES WITH INHOMOGENEOUS INTERPHASES

by

Wei Wang

A particulate composite that consists of three phases: inclusion, interphase and matrix, is considered. A mathematical procedure that is based on the composite spheres assemblage method (CSA) and the generalized self-consistent method (GSC) is used to evaluate the effective moduli and the thermal expansion coefficients of the composite. The inclusion and matrix are assumed to be homogeneous and isotropic, and the interphase is isotropic and inhomogeneous. The perfect bonding conditions are assumed at both the inclusion-interphase and interphase-matrix interfaces. The power, linear and cubic functions are chosen to simulate the variation in the Young's modulus, Poisson's ratio and the thermal expansion coefficient in the interphase. In the analysis, the CSA method is employed to calculate the effective bulk modulus and thermal expansion coefficient of the composite, and the GSC method to calculate the effective shear modulus. The solution is obtained either in a closed form or in a form of infinite series.

The numerical results are presented, which illustrate the effects of interphase. It is shown that inhomogeneity of the interphase in particle reinforced composite does have an influence the on effective properties of composite.

DEDICATION

To my wife, Yu Kui, my sister, my brother in law and my parents

ACKNOWLEDGMENTS

I would like to express my gratitude to Dr. Iwona Jasiuk, my advisor, for her guiance, encouragement, and support throughout the course of this research work. I thank her for the excellent advice that I received in my research field.

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Finally, I am very grateful to those who have made my work become reality.

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NOMENCLATURE

RVE .... Representative volume element

CSA .... Composite sphere assemblage model

GSC .... Generalized self-consistent model

r, θ, φ .... Spherical coordinates

x, y, z .... Cartesian coordinates

u_r, u_θ, u_φ .... displacements in spherical coordinates

u_x, u_y, u_z .... displacements in Cartesian coordinates

p, l, m, e ...... superscripts which refer to inclusion(particle), interphase(layer), matrix and equivalent homogeneous phase

C_n^i .... unknown constants in bulk moduli calculations

D_n^i .... unknown constants in shear moduli calculations

C_{tn}^i .... unknown constants in thermal expansion coefficients

i .... superscript which refers to p, l, m, e

n .... subscripts which refer to numbers 1, 2, 3......

t .... subscript which refers to thermal expansion coefficients

t .... thickness of interphase

a, b, c .... radii of inclusion, interphase and matrix

f .... volume fraction

PB .... perfect bond

E .... Young's modulus

ν .... Poisson's ratio

G .... shear modulus

K .... bulk modulus

λ .... Lame constant

σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{\varphi\varphi}$ .... normal stresses

$\tau_{r\theta}$, $\tau_{\theta\varphi}$, $\tau_{\varphi r}$ .... shear stresses

ε_{rr}, $\varepsilon_{\theta\theta}$, $\varepsilon_{\varphi\varphi}$ .... normal strains

$\gamma_{r\theta}$, $\gamma_{\theta\varphi}$, $\gamma_{\varphi r}$ .... shear strains

W .... strain energy

P_n .... constants in interphase simulation

Q_n .... constants in interphase simulation

R .... constants in interphase simulation

S .... constants in interphase simulation

1. INTRODUCTION

In a particle reinforced composite, the particle usually bears the major part of the load. However, the important factor that will influence the reinforcement mechanism is the interface. The interface, serving as a load transfer between the matrix and inclusions, is known to have an important effect on the composite response [see Hughes (1991), Kerans et al. (1989), Wright (1990) and Jayaraman (1993), for example.]

Many composites are made up of inclusions which are perfectly bonded to a surrounding matrix. This model has been studied extensively [for reviews, see Christensen (1979), Hashin (1983), Hine et al. (1993), for example]. However, in many composites, there is an interface layer between inclusions and matrix. This interface layer exists as a natural consequence of composites processing or is intentionally introduced into the composite to improve the properties of composite. It is a three dimensional region between the inclusion and matrix and is usually called an interphase. The interphase can be very thin as is the interphase between tungsten wire reinforcement and steel, or relatively thick as in vapor deposited boron fiber in epoxy [Bogan and Hinder (1993)]. Although its volume fraction is small as compared with the inclusions and matrix volume fractions, its existence is important and it strongly affects both the local field and the overall properties of composites [Theocaris (1986)]. The structure of the interphase is very complicated [see Hull (1981), Drzal (1983), Chawla (1987), Date et al (1993), for example]. For simplicity, in the theoretical investigations many researchers modeled the interphase as a homogeneous region. These studies include those of Agarwal (1974), Mikata and Taya (1986), Pagano (1988), Jasiuk and Tong (1989), Benveniste et al. (1989), Chen et al. (1990), Pagano and Tandon (1990), Tong and Jasiuk (1990), Maurer (1990), Sullivan and Hashin

(1990), Qiu and Weng (1991), Dasgupa and Bhandarkar (1992), Bostroin et al (1992), Theocaris and Demakos (1992), Herve and Zaoui (1993), Gregory (1993), and many others.

The advantage of the homogeneous interphase model is that it gives relatively simple mathematical analysis. However, it may not give a good representation of the complex state at the interface that may be due to a chemical reaction or diffusion, for example. In order to account for this complex microstructure, recently, a number of researchers studied the inhomogeneous interphase. These publications include those of Theocaris et al. (1985), Theocaris (1986), Sideridis (1988), Papanicolaou et al. (1989), Sottos et al. (1989), Jayaraman et al. (1991), Jayaraman and Reifsnider (1992a,b), Jasiuk and Kouider (1993), Bogan and Hinder (1993), Nassehi et al. (1993), and other.

Most of the above papers discussed the fiber reinforced composite. In this thesis, we evaluate the effective bulk modulus, shear modulus and thermal expansion coefficient of particle reinforced composite. We assume that the particles are spherical in shape and that there is an inhomogeneous interphase between each particle and the matrix. We follow the classical elasticity methods to solve these problems and use some simplifying assumptions

1. The composite is modeled as a representative volume element (RVE), Fig. 1.2 and Fig. 1.3, [see, Hashin (1983)].

2. The three phases: particle, interphase and matrix are linearly elastic.

3. The particle and matrix are homogeneous and isotropic.

4. The interphase is isotropic and inhomogeneous with the Young's modulus, Poisson's ratio and thermal expansion coefficient having a radial variation.

5. There is perfect bonding between the phases.

6. The temperature changes uniformly in all the phases.

To determine the effective elastic moduli of a multiphase composite, we follow Hashin (1983) and subject the composite to homogeneous boundary conditions

$$u_i(s) = \varepsilon_{ij}^0 x_j \tag{1.1}$$

where ε_{ij}^0 are constant strain tensors. Then by the average strain theorem

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij}^0 \tag{1.2}$$

where the overbars denote volume average over composite. According the linearity of the equations of elasticity, the average stresses $\bar{\sigma}_{ij}$ and strains $\bar{\varepsilon}_{ij}$ must obey following relation

$$\bar{\sigma}_{ij} = C_{ijkl}^c \bar{\varepsilon}_{kl} = C_{ijkl}^c \varepsilon_{kl}^0 \tag{1.3}$$

where C_{ijkl}^c is defined as the effective elastic stiffness of a composite.

In our analysis, we evaluate the effective bulk modulus and thermal expansion coefficient by using the composite spheres assemblage model (CSA)[Hashin and Rosen (1964)]. In the CSA model we have three concentric spheres having radii a, b and c (Fig. 1.1), which correspond to the radius of particle, the outer radius of the interphase, and the outer radius of matrix, respectively. The microstructure of composite is represented by a collection of such various sizes of composite spheres which completely fill the space (Fig. 1.2). The ratios of radii a/b and a/c are taken to be constants for each composite sphere. The major advantage of this model is that we can evaluate the effective bulk modulus by using the single composite sphere (Fig. 1.1), and its result represents the effective bulk modulus of composite spheres assemblage model (CSA) [Christensen (1979, p. 50)].

We evaluate the shear modulus by using the generalized self -consistent model (GSC) [Christensen and Lo (1979)]. In the GSC method, a single composite sphere (Fig. 1.1) is embedded in the infinitely extended medium of yet unknown effective properties. This geometric model is shown in Fig. 1.3. In both methods the volume fraction f of particles is defined by $f = a^3/c^3$. In this thesis, the superscripts p, l, m, e denote the inclusion (particle), interphase (layer), matrix, and effective medium, respectively.

This thesis is an extension of previous theoretical works which dealt with the subject of inhomogeneous interphases and which addressed the local fields [Sottos et al (1989), Jayaraman et al. (1991, 1992a,b)], and the effective elastic properties [Theocaris (1987), Jasiuk and Kouider (1993)].

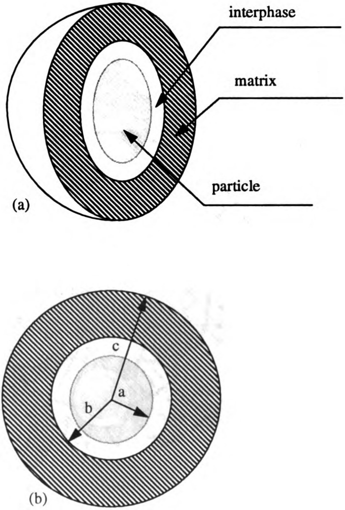

Fig. 1.1
(a) Single composite sphere
(b) Its cross section

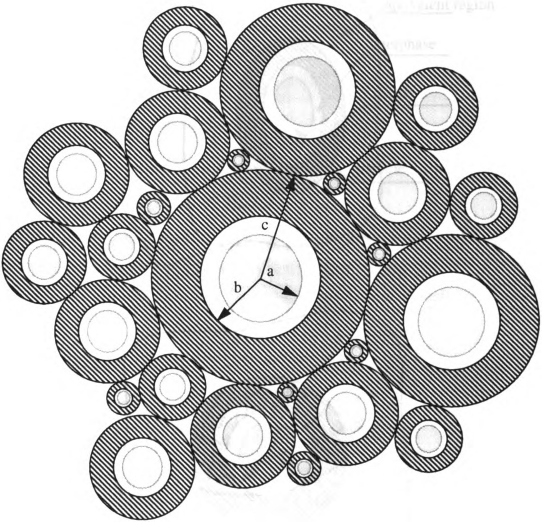

Fig. 1.2
Composite spheres assemblage model

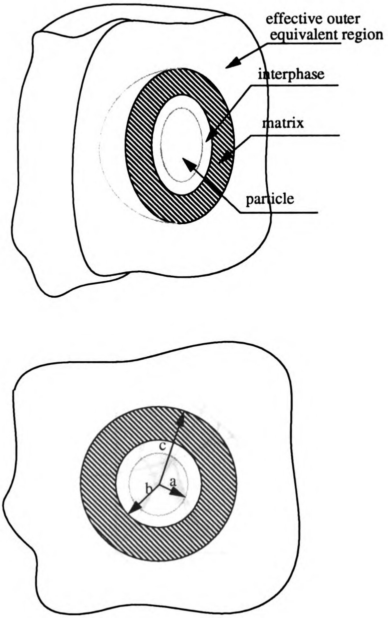

Fig. 1.3
Generalized self-consistent model

2. FORMULATION AND SOLUTION METHOD

2.1. EFFECTIVE BULK MODULUS

We use composite spheres assemblage model (CSA) to evaluate the bulk modulus of particulate reinforced composite. We subject a single composite sphere to the homogeneous surface displacements, which in Cartesian coordinate system are given by

$$\begin{aligned} u_x &= \varepsilon^0_{xx} x \\ u_y &= \varepsilon^0_{yy} y \\ u_z &= \varepsilon^0_{zz} z \end{aligned} \tag{2.1}$$

where

$$\varepsilon^0_{xx} = \varepsilon^0_{yy} = \varepsilon^0_{zz} = \varepsilon^0_{rr} \tag{2.2}$$

[see Hashin (1983)]. It is convenient for us to use the spherical coordinate system (see Fig. 2.1) which is related to the Cartesian system as given in Table 2.1,

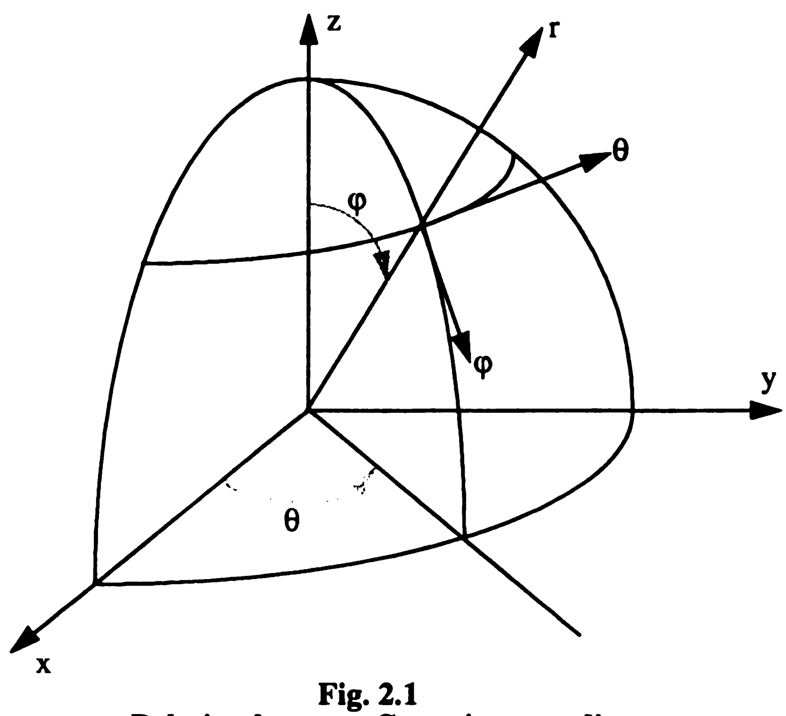

Fig. 2.1
Relation between Cartesian coordinates and spherical coordinates

	x	y	z
r	$\sin\varphi\cos\theta$	$\sin\varphi\sin\theta$	$\cos\varphi$
φ	$\cos\varphi\cos\theta$	$\cos\varphi\sin\theta$	$-\sin\varphi$
θ	$-\sin\theta$	$\cos\theta$	0

Table2.1
Direction cosines between Cartesian and spherical coordinates

and the Cartesian coordinates can be expressed by using spherical coordinates as

$$\begin{aligned} x &= r\sin\varphi\cos\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\varphi \end{aligned} \tag{2.3}$$

Using Table 1, we express the displacements in the spherical coordinates as

$$\begin{aligned} u_r &= u_x\sin\varphi\cos\theta + u_y\sin\varphi\sin\theta + u_z\cos\varphi \\ u_\varphi &= u_x\cos\varphi\cos\theta + u_y\cos\varphi\sin\theta - u_z\sin\varphi \\ u_\theta &= -u_x\sin\theta + u_y\cos\theta \end{aligned} \tag{2.4}$$

Thus, when we substitute eqns. (2.3) into eqns. (2.1) and then put them into eqn. (2.4), we get

$$\begin{aligned} u_r &= \varepsilon^0_{rr} r \\ u_\varphi &= 0 \\ u_\theta &= 0 \end{aligned} \tag{2.5}$$

Due to the spherically symmetric nature of the problem, the displacements in the bulk modulus calculations are in the form

$$\begin{aligned} u_r &= u_r(r) \\ u_\varphi &= 0 \\ u_\theta &= 0 \end{aligned} \tag{2.6}$$

that is, the displacement in r direction depends on r only and the displacement components in φ and θ direction are zero. Next, we use this displacement formulation approach to determine the unknown displacement $u_r(r)$.

The governing equations that we use to calculate the displacements are given below:

1) Strain-displacement Relations

$$
\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{\cot\varphi}{r}u_\varphi \\
\varepsilon_{\varphi\varphi} &= \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} \\
\gamma_{r\theta} &= \frac{1}{r\sin\varphi}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\
\gamma_{\theta\varphi} &= \frac{1}{r}\frac{\partial u_r}{\partial\varphi} - \frac{\cot\varphi}{r}u_\theta + \frac{1}{r\sin\varphi}\frac{\partial u_\varphi}{\partial\theta} \\
\gamma_{\varphi r} &= \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\varphi}
\end{aligned}
\tag{2.7}
$$

2) Constitutive Equations (Stress-strain Relations)

$$
\begin{aligned}
\sigma_{rr} &= 2G\left(\varepsilon_{rr} + \frac{\nu}{1-2\nu}e\right) & \tau_{r\theta} &= G\gamma_{r\theta} \\
\sigma_{\theta\theta} &= 2G\left(\varepsilon_{\theta\theta} + \frac{\nu}{1-2\nu}e\right) & \tau_{\theta\varphi} &= G\gamma_{\theta\varphi} \\
& & \tau_{\varphi r} &= G\gamma_{\varphi r} \\
\sigma_{\varphi\varphi} &= 2G\left(\varepsilon_{\varphi\varphi} + \frac{\nu}{1-2\nu}e\right) & e &= \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}
\end{aligned}
\tag{2.8}
$$

3) Equilibrium Equations (with no body forces)

$$
\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r}\sin\varphi + \frac{1}{r}\left(2\sigma_{rr}\sin\varphi - \sigma_{\theta\theta}\sin\varphi - \sigma_{\varphi\varphi}\sin\varphi + \tau_{r\varphi}\cos\varphi + \frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{\varphi r}}{\partial\varphi}\sin\varphi\right) = 0 \\
&\frac{\partial\tau_{r\theta}}{\partial r}\sin\varphi + \frac{1}{r}\left(3\tau_{r\theta}\sin\varphi + \frac{\partial\tau_{\theta\varphi}}{\partial\varphi}\sin\varphi + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\tau_{\theta\varphi}\cos\varphi\right) = 0 \\
&\frac{\partial\tau_{r\theta}}{\partial r}\sin\varphi + \frac{1}{r}\left(3\tau_{\varphi r}\sin\varphi + \frac{\partial\tau_{\theta\varphi}}{\partial\theta} + (\sigma_{\varphi\varphi} - \sigma_{\theta\theta})\cos\varphi + \frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}\sin\varphi\right) = 0
\end{aligned}
\tag{2.9}
$$

where G is shear modulus and ν is Poisson's ratio. In the isotropic material, they have the following relation

$$G = \frac{E}{2(1+\nu)} \tag{2.10}$$

where E is the Young's modulus.

Substituting eqns. (2.6) into eqns. (2.7) we get the expressions for strains in terms of displacements. Substituting the strain expressions into eqns. (2.8) we get the stresses in terms of displacements, and then substituting them into equilibrium eqns. (2.9), we get the equilibrium equation which can be used to find displacement u_r as follows

$$\begin{aligned} Er^2(\nu-1)(2\nu-1)(1+\nu)\frac{\partial^2 u_r}{\partial r^2} + r(2E + \frac{\partial E}{\partial r}r - 2r\nu\frac{\partial E}{\partial r} - r\nu^2\frac{\partial E}{\partial r} + 2r\nu^3\frac{\partial E}{\partial r} \\ + 4E\nu r\frac{\partial \nu}{\partial r} - 2E\nu^2 r\frac{\partial \nu}{\partial r} - 4E\nu - 2E\nu^2 + 4E\nu^3)\frac{\partial u_r}{\partial r} + (4E\nu + 2E\nu^2 - 4E\nu^3 \\ + 2\nu r\frac{\partial E}{\partial r} - 2\nu^2 r\frac{\partial E}{\partial r} - 4\nu^3 r\frac{\partial E}{\partial r} + 4E\nu^2 r\frac{\partial \nu}{\partial r} - 2Er + 2Er\frac{\partial \nu}{\partial r})u_r = 0 \end{aligned} \tag{2.11}$$

In eqn. (2.11), Young's modulus E and Poisson's ratio ν are taken as the function of radius r, that is $E = E(r)$ and $\nu = \nu(r)$. The procedure to obtain eqn. (2.11) is given in the appendix A.

Since we assume that the matrix and inclusion are homogeneous and isotropic, the Young's Modulus and Poisson's ratio are constant in these phases. If we let E and ν to be constant in eqn. (2.11) and simplify these equations, we get the following governing equation for displacement u_r in the matrix and inclusion.

$$r^2\frac{\partial^2 u_r}{\partial r^2} + 2r\frac{\partial u_r}{\partial r} - 2u_r = 0 \tag{2.12}$$

Eqn. (2.12) is a Euler equation. The general solution to this equation is

$$u_r = \frac{C_1 r^3 + C_2}{r^2} \tag{2.13}$$

where C_1 and C_2 are constants. Adopting eqn. (2.13) to the inclusion phase and matrix, it gives us

$$u_r^p = C_1^p r$$
$$u_r^m = C_1^m r + \frac{C_2^m}{r^2} \tag{2.14}$$

where C_1^p, C_1^m, C_2^m are constants. Superscripts p and m denote the inclusion (particle) and matrix. The coefficient C_2^p in the particle phase is taken as zero in order to avoid the singularity at the origin, and the corresponding stresses can be determined by eqns. (2.7) and (2.8) as

$$\sigma_{rr}^p = \left(\frac{E^p}{1-2\nu^p}\right)C_1^p$$
$$\sigma_{rr}^m = \frac{E^m C_1^m}{1-2\nu^m} - \frac{2E^m C_2^m}{1+\nu^m} \tag{2.15}$$

For the inhomogeneous isotropic interphase, eqn. (2.11) is the general governing equation to determine the displacements in the interphase. Eqn. (2.11) can be expressed in a more general form as

$$\frac{\partial^2 u_r}{\partial r^2} + f(r)\frac{\partial u_r}{\partial r} + g(r)u_r = 0 \tag{2.16}$$

where

$$f(r) = \frac{A(r)}{Er^2(\nu-1)(2\nu-1)(1+\nu)} \tag{2.17}$$

$$g(r) = \frac{B(r)}{Er^2(\nu-1)(2\nu-1)(1+\nu)} \tag{2.18}$$

and

$$A(r) = r(2E + \frac{\partial E}{\partial r}r - 2r\nu\frac{\partial E}{\partial r} - r\nu^2\frac{\partial E}{\partial r} + 2r\nu^3\frac{\partial E}{\partial r}$$
$$+4E\nu r\frac{\partial \nu}{\partial r} - 2E\nu^2 r\frac{\partial \nu}{\partial r} - 4E\nu - 2E\nu^2 + 4E\nu^3)$$

$$B(r) = 4E\nu + 2E\nu^2 - 4E\nu^3 + 4E\nu r\frac{\partial\nu}{\partial r}$$
$$-2E\nu^2 r\frac{\partial\nu}{\partial r} - 4E\nu - 2E\nu^2 + 4E\nu^3$$

are functions of the radial coordinate r only.

In general, the solution to the above equation cannot be found in a closed form, but under certain conditions of coefficients $f(r)$ and $g(r)$, the solution can be obtained in a form of infinite series in terms of r. If $f(r)$ and $g(r)$ both have Taylor series expansion at r_0, then a convergent series solution of eqn. (2.16) exists of the form $u_r = \sum_{n=0}^{\infty} a_n (r - r_0)^n$. For more details about series solution, see Appendix B and refer to publications of Johnson and Johnson (1982), Jayaraman et al. (1991), Jayaraman and Reifsnider (1992a,b), Jasiuk and Kouider (1993), Lutz and Ferrari (1993).

To simulate the inhomogeneity in the elastic constants of the interphase, we use three alternate variations:

case 1. Power variation

$$E^l = P_1 \left(\frac{r}{a}\right)^{Q_1} \qquad \nu^l = \text{constant} \tag{2.19}$$

case 2. Linear variation

$$E^l = P_2 \left(\frac{r}{a}\right) + Q_2 \qquad \nu^l = \text{constant} \tag{2.20}$$

case 3. Cubic variation

$$E^l = P_3 \left(\frac{r}{a}\right)^3 + Q_3 \left(\frac{r}{a}\right)^2 + R\left(\frac{r}{a}\right) + S \qquad \nu^l = P'\left(\frac{r}{a}\right)^3 + Q'\left(\frac{r}{a}\right)^2 + R'\left(\frac{r}{a}\right) + S' \tag{2.21}$$

where, P, Q, R, S are constants

Substituting eqn. (2.19) into eqn. (2.11), the governing equation is reduced to

$$r^2\frac{\partial^2 u_r}{\partial r^2} + r(Q+2)\frac{\partial u_r}{\partial r} + \left(\frac{2\nu^l Q}{1-\nu^l} - 2\right)u_r = 0 \tag{2.22}$$

This is the Euler equation. The solution is given by

$$u_r^l = C_1^l r^{s1} + C_2^l r^{s2} \tag{2.23}$$

where

$$s1 = -\frac{1}{2}\left(1 + Q + \sqrt{9 + Q^2 + \frac{10\nu^l - 2Q}{\nu^l - 1}}\right) \tag{2.24}$$

$$s2 = -\frac{1}{2}\left(1 + Q - \sqrt{9 + Q^2 + \frac{10\nu^l - 2Q}{\nu^l - 1}}\right) \tag{2.25}$$

C_1^l and C_2^l are constants. The superscript l represents the interphase (layer).

Substituting eqn. (2.23) into eqn. (2.7) and eqn. (2.8), we obtain the expression for stress σ_{rr}^l.

For the cases 2 [eqn. (2.20)] and 3 [eqn. (2.21)], we follow the same procedure. First, we get the series solution for the displacement in the interphase and then substitute it into eqns. (2.7) and (2.8) to get the stresses. Thus, we know the expressions for displacements and the stresses in the inclusion, matrix and interphase. These expressions include five unknown constants C_1^p, C_1^m, C_2^m, C_1^l and C_2^l. In order to evaluate these constants, we use the boundary conditions. If we assume perfect bonding between the phases, we have at $r = a$

$$\begin{aligned} \sigma_{rr}^p &= \sigma_{rr}^l \\ u_r^p &= u_r^l \end{aligned} \tag{2.26}$$

and at $r = b$

$$\begin{aligned} \sigma_{rr}^l &= \sigma_{rr}^m \\ u_r^l &= u_r^m \end{aligned} \tag{2.27}$$

In addition, on the outer surface we have

$$u_r(c) = \varepsilon_{rr}^0 c \tag{2.28}$$

as given in eqn. (2.5).

The bulk modulus is usually defined as [Mase and Mase (1992), Frederick and Chang (1972)]

$$K = \frac{p}{\varepsilon_{ii}} = \frac{\sigma_{ii}}{3\varepsilon_{ii}} \tag{2.29}$$

by which we infer that the bulk modulus K relates the pressure p to the volume change given by the cubical dilatation ε_{ii}. According to summation convention, we have $\sigma_{ii} = \sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{\theta\theta}$ and $\varepsilon_{ii} = \varepsilon_{rr} + \varepsilon_{\varphi\varphi} + \varepsilon_{\theta\theta}$. In our case, from eqns. (2.7) and (2.8), we find $\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{\theta\theta}$, $\varepsilon_{rr} = \varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta}$. So, eqn. (2.29) can be written as

$$K = \frac{\sigma_{rr}}{3\varepsilon_{rr}}. \tag{2.30}$$

Combining eqns. (1.3) and (2.30), the bulk modulus of the composite [Hashin (1983), Jasiuk and Kouider (1993)], can be expressed as

$$K^e = \frac{\bar{\sigma}_{rr}}{3\varepsilon^0_{rr}} = \frac{\sigma^m_r(c)}{3\varepsilon^0_{rr}} = \frac{\left(\frac{2C^m_1(1+\nu^m)}{1-2\nu^m} - \frac{4C^m_2}{c^3}\right)G^m}{3\varepsilon^0_{rr}} \tag{2.31}$$

where $\bar{\sigma}_{rr}$ is the average stress in the composite sphere, which is equal to the stress in the matrix on the outer boundary $\sigma^m_{ij}(c)$ by using CSA model. The superscripts e and m represent the effective composite and the matrix, respectively.

It is worthwhile to mention that, if when the GSC model (Fig. 1.3) is used to evaluate the effective bulk modulus, it gives the identical result to that of above CSA model [Christensen (1979)], but it is more complicated to use.

2.2. EFFECTIVE SHEAR MODULUS

In this section, we evaluate the effective shear modulus of particulate composite. The CSA model gives us a simple mathematical formalism in the bulk modulus evaluation, but we cannot use it to evaluate the shear modulus uniquely. The reason is as follows. In the case where the simple shear type displacements are prescribed on the surface of the composite sphere, the resulting boundary stresses are not those corresponding to a state of simple shear stress. Correspondingly, when the simple shear stresses are prescribed on the boundary, the resulting surface deformation state is not that of a simple shear deformation. Accordingly, CSA model can only give the bounds on the shear modulus. Therefore, as an alternate method, we choose the generalized self-consistent model (GSC) to evaluate the effective shear modulus. In this method a single composite sphere (Fig. 1.1) is embedded in the infinitely extended medium of yet unknown effective properties of the composite.

Again we first use the Cartesian coordinate system first, and we subject this composite system to the remote displacement boundary conditions

$$\begin{aligned} u_x &= \varepsilon^0_{xy} y \\ u_y &= \varepsilon^0_{xy} x \\ u_z &= 0 \end{aligned} \tag{2.32}$$

Substituting eqn. (2.3) into eqn. (2.32) and then substituting them into eqn. (2.4), we get

$$\begin{aligned} u_r &= \varepsilon^0_{xy} r (\sin\varphi)^2 \sin 2\theta \\ u_\theta &= \varepsilon^0_{xy} r \sin\varphi \cos 2\theta \\ u_\varphi &= \frac{1}{2} \varepsilon^0_{xy} r \sin 2\varphi \sin 2\theta \end{aligned} \tag{2.33}$$

Guided by the eqn. (2.33), which represents the deformation of a homogeneous medium, a general solution for the heterogeneous problem is in the form

$$\begin{aligned} u_r &= U_r(r)\ (\sin\varphi)^2 \sin 2\theta \\ u_\theta &= U_\theta(r)\ \sin\varphi \cos 2\theta \\ u_\varphi &= U_\varphi(r)\ \sin 2\varphi \sin 2\theta \end{aligned} \tag{2.34}$$

Following the same procedures as in the bulk modulus calculations, we substitute eqn. (2.34) into eqns. (2.7), (2.8) and the equilibrium eqns. (2.9). Consequently, each equation can be separated into the sum of two parts

$$H(r) \sin^2\varphi + J(r) = 0 \tag{2.35}$$

where $H(r)$ and $J(r)$ can be expressed as

$$\begin{aligned} H(r) &= h(E, \nu, u_r, u_\theta, u_\varphi) \\ J(r) &= j(E, \nu, u_r, u_\theta, u_\varphi) \end{aligned} \tag{2.36}$$

We equate to zero the coefficients of $\sin^2\varphi$ and the term independent of φ, that is

$$\begin{aligned} H(r) &= 0 \\ J(r) &= 0 \end{aligned} \tag{2.37}$$

The three equilibrium equations give six equations, but only three of them are independent. These three independent governing equations are given as follows

$$L_1 \frac{\partial U_r}{\partial r} - L_2 U_r + E \frac{\partial^2 U_\varphi}{\partial r} - L_3 \frac{\partial U_\varphi}{\partial r} + L_4 U_\varphi = 0 \tag{2.38}$$

where

$$L_1 = \left[\frac{E}{r(2\nu - 1)}\right]$$

$$L_2 = \left[\frac{-2\frac{\partial E}{\partial r}\nu^2 r - E\frac{\partial \nu}{\partial r} r + \frac{\partial E}{\partial r} r - \frac{\partial E}{\partial r}\nu r + 4E - 4E\nu^2 + 2E\frac{\partial \nu}{\partial r}\nu r}{(1+\nu)(-1+2\nu) r^2}\right]$$

$$L_3 = \left[\frac{-\frac{\partial E}{\partial r}\nu r - 2E\nu - 2E + E\frac{\partial \nu}{\partial r} r - \frac{\partial E}{\partial r} r}{(1+\nu) r}\right]$$

$$L_4 = \left[\frac{-12E\nu^2 - E\frac{\partial \nu}{\partial r}r - \frac{\partial E}{\partial r}\nu r + 12E - 2\frac{\partial E}{\partial r}\nu^2 r + \frac{\partial E}{\partial r}r + 2E\frac{\partial \nu}{\partial r}\nu r}{(1+\nu)\,(-1+2\nu)} \right]$$

$$L_5\frac{\partial^2 U_r}{\partial r} - L_6\frac{\partial U_r}{\partial r} + L_7 U_r + L_8\frac{\partial U_\varphi}{\partial r} - 3L_9 U_\varphi = 0 \tag{2.39}$$

where

$$L_5 = \left[\frac{(\nu-1)\,E}{-1+2\nu}\right]$$

$$L_6 = \left[\frac{C(r)}{(1+\nu)\,(-1+2\nu)^2 r}\right]$$

$$L_7 = \left[\frac{-2\frac{\partial E}{\partial r}\nu^2 r - 5E + 13E\nu + 2E\frac{\partial \nu}{\partial r}r + 4E\frac{\partial \nu}{\partial r}\nu^2 + 2E\nu^2 - 4\frac{\partial E}{\partial r}\nu^3 r + 2\frac{\partial E}{\partial r}\nu r - 16E\nu^3}{(1+\nu)\,(-1+2\nu)^2 r^2}\right]$$

$$L_8 = \left[\frac{3E}{(-1+2\nu)\,r}\right]$$

$$L_9 = \left[\frac{-4\frac{\partial E}{\partial r}\nu^3 r + 2\frac{\partial E}{\partial r}\nu r - 8E\nu^3 - 2\frac{\partial E}{\partial r}\nu^2 + 4E\frac{\partial \nu}{\partial r}\nu^2 r + 2E\nu^2 + 7E\nu + 2E\frac{\partial \nu}{\partial r}r - 3E}{(1+\nu)\,(-1+2\nu)^2 r^2}\right]$$

$$\begin{aligned} C(r) &= -\frac{\partial E}{\partial r}r + 4E\nu + 2E\nu^2 - 4E\frac{\partial \nu}{\partial r}\nu r + 2\frac{\partial E}{\partial r}\nu r \\ &\quad + \frac{\partial}{\partial r}(E)\nu^2 r - 2\frac{\partial E}{\partial r}\nu^3 - 4E\nu^3 + 2E\frac{\partial \nu}{\partial r}\nu^2 r - 2E \end{aligned}$$

and

$$(U_\theta - 2U_\varphi) = 0 \tag{2.40}$$

A computer program given in Appendix C was written by using MAPLE to simplify the above algebraic manipulations. Next, we solve the differential equation system (2.38), (2.39) and (2.40) to determine U_r, U_θ and U_φ.

For the homogeneous matrix and inclusion

$$E = \text{constant} \qquad \nu = \text{constant} \tag{2.41}$$

Then, eqns. (2.38), (2.39) and (2.40) give us

$$-r\frac{\partial U_r}{\partial r} - 4(1-\nu)U_r - (1-2\nu)r^2\frac{\partial^2 U_\varphi}{\partial r} \\ -2(1-2\nu)r\frac{\partial U_\varphi}{\partial r} + 12(1-\nu)U_\varphi = 0 \tag{2.42}$$

$$-(1-\nu)r^2\frac{\partial^2 U_r}{\partial r} - 2(1-\nu)r\frac{\partial U_r}{\partial r} + (5-8\nu)U_r \\ +3r\frac{\partial U_\varphi}{\partial r} - 3(3-4\nu)U_\varphi = 0 \tag{2.43}$$

$$U_\theta - 2U_\varphi = 0 \tag{2.44}$$

They are the same as in Christensen and Lo (1979). The method to solve the above differential system is to introduce an operator

$$r^n\frac{\partial^n U}{\partial r} = d(d-1)\ldots(d-n+1)U \tag{2.45}$$

So, we change the differential system to the general linear algebraic system. We can solve the algebraic system easily, and the final solution of the differential system depends on the roots of the algebraic system; again this technique to solve above differential system is standard [for example, see Wyie and Barret (1982) p. 209)]. We will discuss it later in more details in eqns. (2.55)-(2.57). Finally, the displacements in the homogeneous medium are given as

$$u_r = (D_1 r - \frac{6\nu}{1-2\nu}D_2 r^3 + 3D_3\frac{1}{r^4} + \frac{(5-4\nu)}{(1-2\nu)}D_4\frac{1}{r^2})(\sin\varphi)^2\sin 2\theta \tag{2.46}$$

$$u_\theta = (D_1 r - \frac{(7-4\nu)}{1-2\nu}D_2 r^3 - 2D_3\frac{1}{r^4} + 2D_4\frac{1}{r^2})\sin\varphi\cos 2\theta \tag{2.47}$$

$$u_\varphi = \frac{1}{2}(D_1 r - \frac{(7-4\nu)}{1-2\nu}D_2 r^3 - 2D_3\frac{1}{r^4} + 2D_4\frac{1}{r^2})\sin 2\varphi\sin 2\theta \tag{2.48}$$

where D_1, D_2, D_3 and D_4 are constants.

Guided by above equations, we can get displacements for different phases.

in the inclusion, $0 < r < a$:

$$u_r^p = \left(D_1^p r - \frac{6\nu^p}{1-2\nu^p} D_2^p r^3\right)(\sin\varphi)^2 \sin 2\theta$$
$$u_\theta^p = \left(D_1^p r - \frac{(7-4\nu^p)}{1-2\nu^p} D_2^p r^3\right)\sin\varphi \cos 2\theta \tag{2.49}$$
$$u_\varphi^p = \frac{1}{2}\left(D_1^p r - \frac{(7-4\nu^p)}{1-2\nu^p} D_2^p r^3\right)\sin 2\varphi \sin 2\theta$$

in the matrix, $b < r < c$:

$$u_r^m = \left(D_1^m r - \frac{6\nu^m}{1-2\nu^m} D_2^m r^3 + 3D_3^m \frac{1}{r^4} + \frac{(5-4\nu^p)}{(1-2\nu^p)} D_4^m \frac{1}{r^2}\right)(\sin\varphi)^2 \sin 2\theta$$
$$u_\theta^m = \left(D_1^m r - \frac{(7-4\nu^p)}{1-2\nu^p} D_2^m r^3 - 2D_3^m \frac{1}{r^4} + 2D_4^m \frac{1}{r^2}\right)\sin\varphi \cos 2\theta \tag{2.50}$$
$$u_\varphi^m = \frac{1}{2}\left(D_1^m r - \frac{(7-4\nu^p)}{1-2\nu^p} D_2^m r^3 - 2D_3^m \frac{1}{r^4} + 2D_4^m \frac{1}{r^2}\right)\sin 2\varphi \sin 2\theta$$

and in the equivalent homogeneous phase, $c < r < \infty$:

$$u_r^e = \left(D_1^e r + 3D_3^e \frac{1}{r^4} + \frac{(5-4\nu^e)}{(1-2\nu^e)} D_4^e \frac{1}{r^2}\right)(\sin\varphi)^2 \sin 2\theta$$
$$u_\theta^e = \left(D_1^e r - 2D_3^e \frac{1}{r^4} + 2D_4^e \frac{1}{r^2}\right) \sin\varphi \cos 2\theta \tag{2.51}$$
$$u_\varphi^e = \frac{1}{2}\left(D_1^e r - 2D_3^e \frac{1}{r^4} + 2D_4^e \frac{1}{r^2}\right) \sin 2\varphi \sin 2\theta$$

Some terms in the above equations are missing in order to avoid the unbounded solution. The coefficient D_1^e is specified by the displacement boundary conditions on the remote outer boundary given in eqn. (2.33), that is $D_1^e = \varepsilon_{xy}^0$.

The expressions for the corresponding stresses σ_{rr}^p, $\sigma_{\theta\theta}^p$, $\sigma_{\varphi\varphi}^p$, $\tau_{r\theta}^p$, $\tau_{\theta\varphi}^p$, $\tau_{\varphi r}^p$, σ_{rr}^m, $\sigma_{\theta\theta}^m$, $\sigma_{\varphi\varphi}^m$, $\tau_{r\theta}^m$, $\tau_{\theta\varphi}^m$, $\tau_{\varphi r}^m$, σ_{rr}^e, $\sigma_{\theta\theta}^e$, $\sigma_{\varphi\varphi}^e$, $\tau_{r\theta}^e$, $\tau_{\theta\varphi}^e$ and $\tau_{\varphi r}^e$ can be obtained by substituting the above displacements (2.49), (2.50) and (2.51) into eqns. (2.7) and (2.8).

Next, we consider the inhomogeneous interphase region. Eqns. (2.38), (2.39) and

(2.40) are the general governing equations to determine the displacements in the interphase. If we assume that the Young' modulus and Poisson's ratio have a radial variation, the closed form solution usually cannot be obtained easily for this differential system unless we consider a special case. In order to solve these equations we consider the power variation case, that is $E^l = P(r/a)^Q$ and $\nu^l = \text{constant}$.

Then, eqns. (2.38) - (2.40) are reduced to

$$-r\frac{\partial U_r}{\partial r} + (2Q\nu^l - 4 - Q + 4\nu^l)\,U_r + r^2(-1+2\nu^l)\frac{\partial^2 U_\varphi}{\partial r^2}$$
$$+r(Q+2)(-1+2\nu^l)\frac{\partial u_\varphi}{\partial r} + (12 - 12\nu - 2Q\nu + Q)\,u_\varphi = 0 \tag{2.52}$$

$$r^2(-1+\nu^l)\frac{\partial^2 U_r}{\partial r} + r(Q+2)(-1+\nu^l)\frac{\partial U_r}{\partial r} + (-8\nu^l + 5 - 2Q\nu^l)\,U_r$$
$$+3r\frac{\partial U_\varphi}{\partial r} + (-9 + 12\nu^l + 6Q\nu^l)\,U_\varphi = 0 \tag{2.53}$$

$$U_\theta - 2U_\varphi = 0 \tag{2.54}$$

If we introduce the operator eqn. (2.45), the differential system of eqns. (2.52) and (2.53) changes to

$$(-d + 4\nu^l + 2Q\nu^l - 4 - Q)\,U_r$$
$$+(2\nu^l d^2 + 2\nu^l d - d^2 - d - Qd + 2dQ\nu^l - 2Q\nu^l + Q + 12 - 12\nu)\,U_\varphi = 0 \tag{2.55}$$

$$(-d^2 - d + \nu^l d^2 + \nu^l d + dQ\nu^l - dQ - 8\nu^l + 5 - 2Q\nu^l)\,U_r$$
$$+(3d + 12\nu^l - 9 + 6Q\nu^l)\,U_\varphi = 0 \tag{2.56}$$

Eliminating U_φ from eqn. (2.55) and eqn. (2.56), we finally get

$$(-1+\nu^l)\,d^4 + (-2Q + 2\nu^l + 2Q\nu^l - 2)\,d^3$$
$$+(Q^2\nu^l - Q^2 - 13\nu - Q\nu^l - Q + 13)\,d^2$$
$$+(15Q - 17Q\nu^l - 14\nu^l + Q^2 - 3Q^2\nu^l + 14)\,d$$
$$+4Q - 4Q\nu^l + 24\nu^l - 4Q^2\nu^l = 0 \tag{2.57}$$

Eqn. (2.57) is a characteristic equation to determine U_r, which will give us four roots d_1,

d_2, d_3 and d_4. The calculation steps are given in the Appendix D.

For the our specified problem, the roots of characteristic equation usually follow two cases:

case 1. all roots are real and unequal.

case 2. there are two different complex conjugate roots.

For the case 1, the displacements in the interphase are given as

$$\begin{aligned} u_r^l &= (D_1^l r^{d_1} + D_2^l r^{d_2} + D_3^l r^{d_3} + D_4^l r^{d_4}) \sin\varphi^2 \sin 2\theta \\ u_\theta^l &= 2\,(b_1 r^{d_1} + b_2 r^{d_2} + b_3 r^{d_3} + b_4 r^{d_4}) \sin\varphi \cos 2\theta \\ u_\varphi^l &= (b_1 r^{d_1} + b_2 r^{d_2} + b_3 r^{d_3} + b_4 r^{d_4}) \sin 2\varphi \sin 2\theta \end{aligned} \tag{2.58}$$

For the case 2, if the complex roots are

$$\begin{aligned} d_1 &= m_1 + i n_1 \\ d_2 &= m_1 - i n_1 \\ d_3 &= m_2 + i n_2 \\ d_4 &= m_2 - i n_2 \end{aligned} \tag{2.59}$$

the displacements are given as

$$\begin{aligned} u_r^l = [&D'_1 \cos(n_1 lnr)\, r^{m_1} + D_2^l \sin(n_1 lnr)\, r^{m_1} + D_3^l \cos(n_2 lnr)\, r^{m_2} \\ &+ D'_4 \sin(n_2 lnr)\, r^{m_2}] \sin\varphi^2 \sin 2\theta \end{aligned} \tag{2.60}$$

$$\begin{aligned} u_\theta^l = 2[&b_1 \cos(n_1 lnr)\, r^{m_1} + b_2 \sin(n_1 lnr)\, r^{m_1} + b_3 \cos(n_2 lnr)\, r^{m_2} \\ &+ b_4 \sin(n_2 lnr)\, r^{m_2}] \sin\varphi \cos 2\theta \end{aligned} \tag{2.61}$$

$$\begin{aligned} u_\varphi^l = [&b_1 \cos(n_1 lnr)\, r^{m_1} + b_2 \sin(n_1 lnr)\, r^{m_1} + b_3 \cos(n_2 lnr)\, r^{m_2} \\ &+ b_4 \sin(n_2 lnr)\, r^{m_2}] \sin\varphi^2 \sin 2\theta \end{aligned} \tag{2.62}$$

The constants b_1, b_2, b_3, b_4 in eqns. (2.61)-(2.62) can be expressed in terms of unknown constants D_1^l, D_2^l, D_3^l, D_4^l. This is shown in the Appendix E. After we find $u_r^l, u_\theta^l, u_\varphi^l$, we can substitute them into eqns. (2.7) and (2.8) to obtain the expressions for σ_{rr}^l, $\sigma_{\theta\theta}^l$, $\sigma_{\varphi\varphi}^l$, $\tau_{r\theta}^l$, $\tau_{\theta\varphi}^l$ and $\tau_{\varphi r}^l$.

Till now, we have the unknown coefficients D_1^p, D_2^p, D_1^l, D_2^l, D_3^l, D_4^l, D_1^m, D_2^m, D_3^m, D_4^m, D_3^e, D_4^e.

Although the perfect bond conditions produce eighteen equations at $r = a$, $r = b$ and $r = c$, only twelve equations are independent. These are

$$\begin{array}{lllll} u_r^p = u_r^l & u_\varphi^p = u_\varphi^l & \sigma_{rr}^p = \sigma_{rr}^l & \tau_{r\theta}^p = \tau_{r\theta}^l & \text{at } r = a: \\ u_r^l = u_r^m & u_\varphi^l = u_\varphi^m & \sigma_{rr}^l = \sigma_{rr}^m & \tau_{r\theta}^l = \tau_{r\theta}^m & \text{at } r = b: \\ u_r^m = u_r^e & u_\varphi^m = u_\varphi^e & \sigma_{rr}^m = \sigma_{rr}^e & \tau_{r\theta}^m = \tau_{r\theta}^e & \text{at } r = c: \end{array} \qquad (2.63)$$

Next we apply the Eshelby formula, derived by Eshelby (1956), to evaluate the strain energy stored in the configuration of Fig. 1.3 [Christensen and Lo (1979)]. According to this formula, we have the elastic strain energy W under the applied displacement conditions at infinity

$$W_{comp} = W_0 + \frac{1}{2}\int_S (\sigma_{ij}^e n_j u_i^0 - \sigma_{ij}^0 n_j u_i^e)\, dS \qquad (2.64)$$

where S is the surface of the composite sphere defined by r=c, σ_{ij}^0 and u_i^0 are stresses and displacements in the composite in the absence of inclusion, and σ_{ij}^e and u_i^e are stresses and displacement disturbances in the effective medium. W_{comp} is the elastic strain energy stored in the model of composite given in Fig. 1.3, W_0 is the strain energy in the sphere having the effective properties of composite. Since the outer equivalent region has the properties of composite, which we want to evaluate, we can conclude that

$$W_{comp} = W_0 \qquad (2.65)$$

Combining eqn (2.64) and eqn (2.65), we obtain

$$\int_S (\sigma_{ij}^e n_j u_i^0 - \sigma_{ij}^0 n_j u_i^e)\, dS = 0 \qquad (2.66)$$

For our problem, eqn. (2.66) yields

$$\int_S (\sigma^0_{rr}u^e_r + \tau^0_{r\varphi}u^e_\varphi + \tau^0_{r\theta}u^e_\theta - \sigma^e_{rr}u^0_r - \tau^e_{r\varphi}u^0_\varphi - \tau^e_{r\theta}u^0_\theta)\, dS = 0 \tag{2.67}$$

where

$$dS = c^2 \sin\varphi d\theta d\varphi \tag{2.68}$$

Since this composite is subjected to the displacement boundary conditions at infinity, the stresses and displacements in the composite without inclusion are given as

$$\begin{aligned}
u^0_r &= D^e_1 r(\sin\varphi)^2 \sin 2\theta \\
u^0_\theta &= D^e_1 r \sin\varphi \cos 2\theta \\
u^0_\varphi &= \frac{1}{2} D^e_1 r \sin 2\varphi \sin 2\theta \\
\sigma^0_r &= 2G^e D^e_1 r(\sin\varphi)^2 \sin 2\theta \\
\tau^0_{r\varphi} &= 2G^e D^e_1 r \sin\varphi \cos 2\theta \\
\tau^0_{r\theta} &= G^e D^e_1 r \sin 2\varphi \sin 2\theta
\end{aligned} \tag{2.69}$$

where $D^e_1 = \varepsilon^0_{xy}$. The displacements u^e_r, u^e_φ, u^e_θ are given in eqn. (2.51). By using eqns. (2.7) and (2.8), we can find σ^e_{rr}, $\tau^e_{r\varphi}$ and $\tau^e_{r\theta}$. Now substituting all of these stresses and displacements into eqn. (2.67), we obtain the condition that $D^e_4 = 0$. We find the effective shear modulus by solving the twelve equations (2.63) for the twelve unknowns first and then setting $D^e_4 = 0$. The expression $D^e_4 = 0$ involves the effective shear modulus G^e, which we solve for. Appendix F illustrates the procedure how to determine G^e. In this equation, the effective Poisson's ratio ν^e does not appear and thus we can solve for the unknown G^e explicitly.

2.3 EFFECTIVE THERMAL EXPANSION COEFFICIENTS

In this chapter, we evaluate the effective thermal expansion coefficient of particulate composite. Many papers were published on this subject and on thermal stresses [Levin (1967), Rose and Hashin (1970), Takao and Taya (1985), Jasiuk et al. (1988), Tong and Jasiuk (1990), Benveniste and Dvorak (1990), Adams (1991), Lee et al. (1991), Siboni and Benveniste (1991), Li et al. (1992), Sutcu (1992), Kouider and Jasiuk (1993), and others]. But most of these publications either consider the case of no interphase or assume that the interphase is homogeneous. Some papers discussed the inhomogeneous interphase situation, but they didn't predict the thermal expansion coefficient of particulate composite with the inhomogeneous interphase. Our present study will demonstrate the effect of inhomogeneous interphase on thermal expansion coefficient of composite. In the analysis, we use composite spheres assemblage model (CSA) to evaluate the thermal expansion coefficient.

Considering the Cartesian coordinates, we let a composite sphere shown in Fig. 1.1 be subjected to a uniform temperature change. Then, the corresponding volumetric average strain is given by

$$\bar{\varepsilon}_{ij} = \alpha_{ik}\Delta T\delta_{kj} \tag{2.70}$$

where over bar represents the volumetric average, ΔT is the temperature change, α_{ij} are the unknown thermal expansion coefficients and δ_{ij} is Kronecker delta defined as

$$\delta_{ij} = 0 \text{ if } i \neq j$$

$$\delta_{ij} = 1 \text{ if } i = j$$

and by using the strain displacement relations, we find the displacements in the Cartesian spherical coordinate systems as

$$\begin{aligned} u_x &= \bar{\varepsilon}_{xx}x \qquad & u_r &= \bar{\varepsilon}_{rr}r \\ u_y &= \bar{\varepsilon}_{yy}y \qquad & u_\theta &= 0 \\ u_z &= \bar{\varepsilon}_{zz}z \qquad & u_\varphi &= 0 \end{aligned} \tag{2.71}$$

where $\bar{\varepsilon}_{xx} = \bar{\varepsilon}_{yy} = \bar{\varepsilon}_{zz} = \bar{\varepsilon}_{rr}$

Eqn. (2.71) is similar to eqn. (2.1) and eqn. (2.5), and the analysis is similar to the one for the effective bulk modulus. Again, since the loading is radial, the displacements are of the form

$$\begin{aligned} u_r &= u_r(r) \\ u_\theta &= 0 \\ u_\varphi &= 0 \end{aligned} \tag{2.72}$$

Here we have the same strain-stress relations and equilibrium equation as eqn. (2.7) and eqn. (2.9), but the constitutive equations are different. Note for thermo-elastic problem, the general constitutive equations are

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \alpha_{kn}\Delta T\delta_{nl}) \tag{2.73}$$

where $i, j = r, \theta, \varphi$

Therefore, for our problem eqn. (2.8) can be modified as

$$\begin{aligned} \sigma_{rr} &= 2G\left(\varepsilon_{rr} - \alpha\Delta T + \frac{\nu}{1-2\nu}e\right) \\ \sigma_{\theta\theta} &= 2G\left(\varepsilon_{\theta\theta} - \alpha\Delta T + \frac{\nu}{1-2\nu}e\right) \\ \sigma_{\varphi\varphi} &= 2G\left(\varepsilon_{\varphi\varphi} - \alpha\Delta T + \frac{\nu}{1-2\nu}e\right) \\ \tau_{r\theta} &= G\gamma_{r\theta} \\ \tau_{\theta\varphi} &= G\gamma_{\theta\varphi} \\ \tau_{\varphi r} &= G\gamma_{\varphi r} \end{aligned} \tag{2.74}$$

where $e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} - 3\alpha\Delta T$

Again the same approach is followed as in the bulk modulus evaluation and the governing equation that can be used to determine u_r is given as

$$\begin{aligned}
&Er^2(\nu-1)(2\nu-1)(1+\nu)\frac{\partial^2 u_r}{\partial r^2}+r(2E+\frac{\partial E}{\partial r}r-2r\nu\frac{\partial E}{\partial r}-r\nu^2\frac{\partial E}{\partial r}+2r\nu^3\frac{\partial E}{\partial r}\\
&\quad+4E\nu r\frac{\partial \nu}{\partial r}-2E\nu^2 r\frac{\partial \nu}{\partial r}-4E\nu-2E\nu^2+4E\nu^3)\frac{\partial u_r}{\partial r}+(4E\nu+2E\nu^2-4E\nu^3\\
&\qquad+2\nu r\frac{\partial E}{\partial r}-2\nu^2 r\frac{\partial E}{\partial r}-4\nu^3 r\frac{\partial E}{\partial r}+4E\nu^2 r\frac{\partial \nu}{\partial r}-2Er+2Er\frac{\partial \nu}{\partial r})u_r\\
&\qquad+r^2(1+\nu)^2(2\alpha\nu\frac{\partial E}{\partial r}+2E\nu\frac{\partial \alpha}{\partial r}-\alpha\frac{\partial E}{\partial r}-2E\alpha\frac{\partial \nu}{\partial r}-E\frac{\partial \alpha}{\partial r})T=0
\end{aligned} \tag{2.75}$$

where $E = E(r)$, $\nu = \nu(r)$ and $\alpha = \alpha(r)$. For the homogeneous media: the inclusion and the matrix, eqn. (2.75) was reduced to eqn. (2.12). The displacements in the inclusion and the matrix, are given in eqn. (2.14). Substituting eqn. (2.14) into eqn. (2.57), the stresses are

$$\begin{aligned}
\sigma_{rr}^p &= \frac{E^p C_{t1}^p}{1-2\nu^p}-\frac{\alpha^p E^p \Delta T}{1-2\nu^p}\\
\sigma_{rr}^m &= \frac{E^p C_{t1}^m}{1-2\nu^m}-\frac{2E^m C_{t2}^m}{(1+\nu^m)r^3}-\frac{\alpha^m E^m T}{1-2\nu^m}
\end{aligned} \tag{2.76}$$

For the inhomogeneous interphase case, we choose power, linear and cubic functions to simulate E^l, α^l and ν^l. A special case is given as

$$E^l = P\left(\frac{r}{a}\right)^Q \qquad \alpha^l = M\left(\frac{r}{a}\right)^N \qquad \nu^l = \text{constant} \tag{2.77}$$

If we substitute (2.77) into eqn. (2.75) and simplify it, we get

$$r^2\frac{\partial^2 u_r}{\partial r^2}+r(Q+2)\frac{\partial u_r}{\partial r}+\left(\frac{2\nu^l Q}{1-\nu^l}-2\right)u_r+\frac{r(1+\nu^l)(Q+N)\Delta TM\left(\frac{r}{a}\right)^N}{-1+\nu} = 0 \tag{2.78}$$

The closed form solution can be obtained for the displacement u_r^l from eqn. (2.78) (see Appendix G) as

$$u_r^l = C_{t1}^l\xi(r,\nu^l,Q,N)+C_{t2}^l\beta(r,\nu^l,Q,N)+\eta(r,\nu^l,Q,N,M,\Delta T,a) \tag{2.79}$$

where ξ, β and η are functions in terms of $r, \nu^l, Q, N, M, \Delta T, a$, which are given in

Appendix F. For other cases, we can use series solution method to determine u^l.

After determining the displacements in the interphase, we substitute them into eqns. (2.7) and (2.74), and the corresponding stresses can be found. The next step is to evaluate the five unknowns C^p_{t1}, C^l_{t1}, C^l_{t2}, C^m_{t1} and C^m_{t2}. Similarly as in the bulk modulus analysis, we assume that perfect bonding conditions at the particle-interphase and interphase-matrix interfaces are applied to our problem which give us eqns. (2.26) and (2.27). Since our CSA model is only subjected to a uniform temperature change, the outer surface is traction free, so we have the additional equation given as

$$\sigma^m_{rr}(c) = 0 \tag{2.80}$$

Combining eqns. (2.26), (2.27) and (2.80), we can evaluate the five unknowns.

The definition of thermal expansion coefficient is given in eqn. (2.70) that is

$$\alpha_{ij} = \frac{\bar{\varepsilon}_{ik}}{\Delta T}\delta_{kj} \tag{2.81}$$

Since only $\bar{\varepsilon}_{rr}$ exists in the composite, see eqn. (2.71), the thermal expansion coefficient of composite can be expressed as

$$\alpha_{rr} = \frac{\bar{\varepsilon}_{rr}}{\Delta T} \tag{2.82}$$

where $\bar{\varepsilon}_{rr}$ is average strain in radial direction. From eqn. (2.71) and using the CSA model, the definition of the average strain can be expressed as

$$\bar{\varepsilon}_{ij} = \frac{u_r}{r} = \frac{u^m_r(c)}{c} \tag{2.83}$$

So eqn. (2.82) is written as

$$\alpha^e = \frac{u^m_r(c)}{c\Delta T} = \frac{C^m_{t1} + \dfrac{C^m_{t2}}{c^3}}{\Delta T} \tag{2.84}$$

where the superscript e denotes effective composite.

3. NUMERICAL ANALYSIS

3.1 BULK MODULUS ANALYSIS

In this thesis, we consider that the inclusion and matrix are homogeneous and isotropic, and we assume an idealized model of interphase, in which the interphase is inhomogeneous but isotropic and has Young's modulus and Poisson's ratio varying spatially in a radial coordinate, eqns (2.19)-(2.21). Variations of E are illustrated in Fig. 3.1a,b

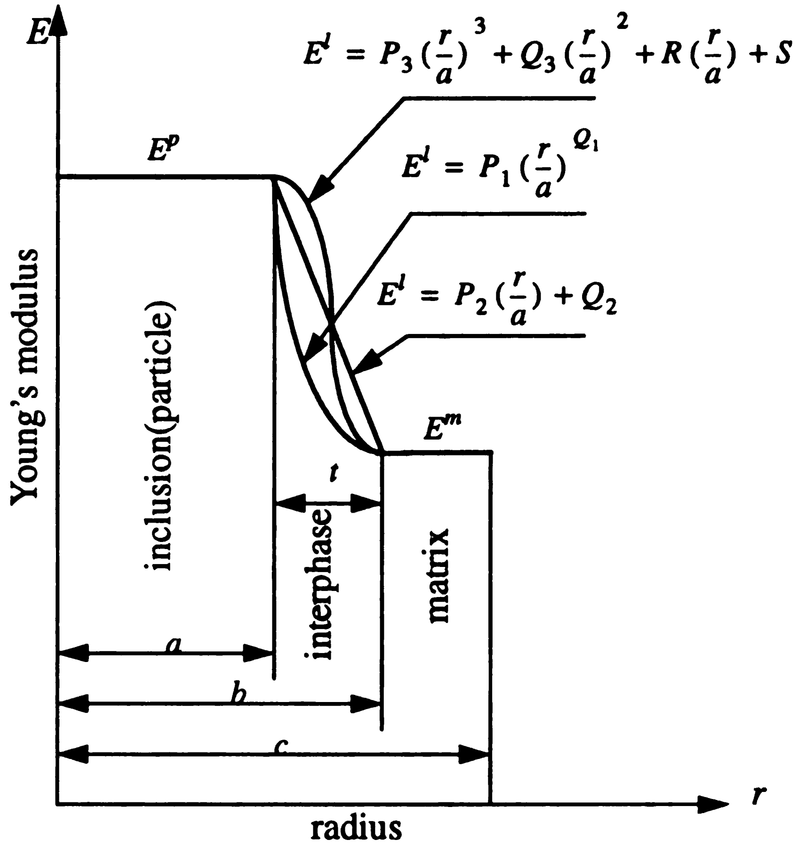

Fig. 3.1a
Schematic variation of Young's modulus in the interphase

In our calculation, we keep the values of E^p, E^m constant and change the thickness t of the interphase. When $r=a$, from the relation in Fig. 3.1, we have $E^i = E^p$, then we can get $P_1 = E^p$ from $E^i = P_1(r/a)^{Q_1}$. When $r = b$, we have $E^i = E^m$, and finally we get

$$Q_1 = [ln(E^m/E^p)]/[ln(b/a)] \qquad P_1 = E^p \tag{3.1}$$

Analogous to the power variation case, for linear variation, we have

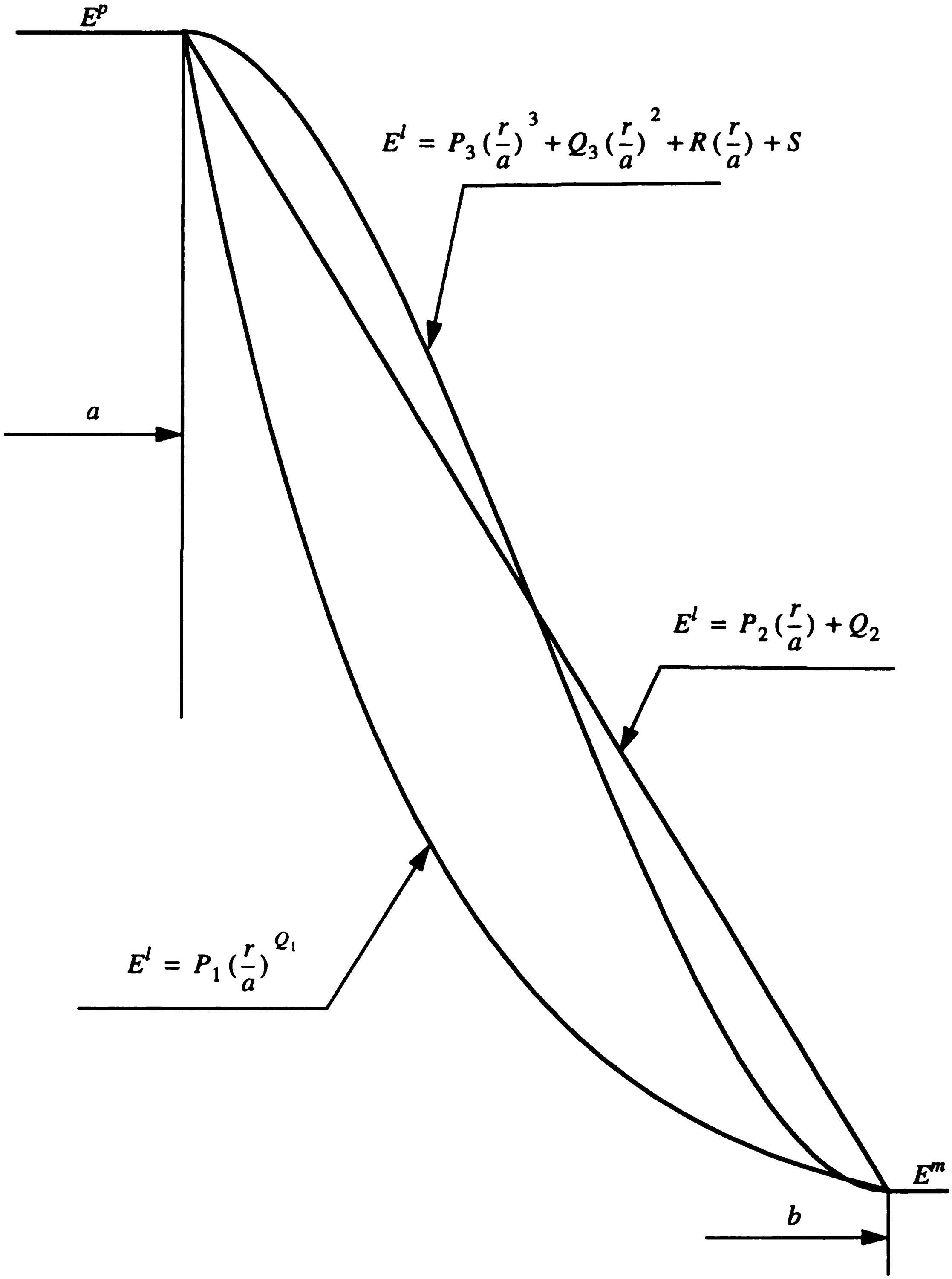

Fig. 3.1b
Variation of Young's modulus in the composite's interphase

$$Q_2 = \frac{bE^p - E^m a}{b-a} \qquad P_2 = \frac{a(E^m - E^p)}{b-a} \tag{3.2}$$

and for cubic variation case

$$\begin{aligned} P_3 &= \frac{2a^3 b (E^m - E^p)}{2b^4 - 3b^3 a - 3b^2 a^2 + 7a^3 b - 3a^4} \\ Q_3 &= \frac{3a^3 (bE^p - bE^m - aE^m + aE^p)}{2b^4 - 3b^3 a - 3b^2 a^2 + 7a^3 b - 3a^4} \\ R &= \frac{6a^4 (E^m - E^p)}{2b^4 - 3b^3 a - 3b^2 a^2 + 7a^3 b - 3a^4} \\ S &= \frac{2b^4 E^p - 3b^3 a E^p - 3b^2 a^2 E^p + 6a^3 b E^p + a^3 b E^m - 3a^4 E^m}{2b^4 - 3b^3 a - 3b^2 a^2 + 7a^3 b - 3a^4} \end{aligned} \tag{3.3}$$

where we assume that when $r=a$, $\frac{d}{dr}E^I(r) = 0$; $r=a$, $\frac{d}{dr}E^I(r) = 0$. We have similar procedure for ν^I. It should be emphasized that we can change the parameters P_3, Q_3, R, S, P', Q', R' and S' in eqn. (2.21) to obtain different variations of E^I and ν^I in the interphase. For example, if we let $P_3 = 0$, $Q_3 = 0$ and $R \neq 0$, $S \neq 0$, we obtain linear variation in Young' modulus. When $P_3 = 0$ and $Q_3 \neq 0$, $R \neq 0$, $S \neq 0$, we get the quadratic variation of Young'modulus in the interphase.

Also, when $Q_1 = 0$, $P_1 \neq 0$, eqn. (2.19) will give us a homogeneous interphase situation, and when $Q_1 = 0$, $P_1 = E^m$, eqn. (2.19) yields no interphase situation. Now, in the numerical examples presented in this thesis, we first consider the following elastic properties:

$$\begin{array}{ll} \nu^p = 0.3 & \nu^I = 0.3 \\ G^p = 25GPA & \nu^m = 0.4 \\ E^I = P\left(\frac{r}{a}\right)^Q & G^m = 1.0GPA \end{array} \tag{3.4}$$

and other elastic properties can be obtained directly from the following relationships

$$E = \frac{G(3\lambda + 2G)}{\lambda + G} \qquad G = E/(2(1+\nu))$$
$$\qquad K = 3/(E(1-2\nu))$$
$$\nu = \lambda/(2(\lambda + G)) \qquad K = \lambda + \frac{2}{3}G \tag{3.5}$$
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad E = 2G(1+\nu)$$

Now, we start the numerical analysis by considering different variations in the interphase. First, we consider the inhomogeneous interphase case where the Young's modulus has power variation. Substituting eqn. (3.4) into eqn. (3.1), we can determine the constant P. If we choose the different thicknesses t as follows

$$\begin{array}{rl} t = 0.01a & b = 1.1a \\ t = 0.05a & b = 1.05a \\ t = 0.02a & b = 1.02a \\ t = 0.008a & b = 1.008a \\ t = 0.005a & b = 1.005a \\ t = 0 & b = a \end{array} \tag{3.6}$$

and substitute these t's into eqn. (3.1), we can calculate the values of constant Q. We substitute a, b, c, Q as well as eqn. (3.4) into eqns. (2.26), (2.27) and (2.28), solve for the unknown coefficients, and substitute them into eqn. (2.31). We can obtain the effective bulk modulus in terms of volume fraction $f = (a^3/c^3)$ and the elastic properties of inclusion, interphase and matrix. We plot it for different values of thickness t, (see Fig. 3.4). In order to compare the power variation result with the homogeneous interphase assumption, we introduce the following relationship

$$E_1^I = \frac{\int_a^b P_1 \left(\frac{r}{a}\right)^{Q_1} dr}{b-a} \tag{3.7}$$

Eqn. (3.7) gives the same "area" average of Young's modulus as the power variation in interphase. Following the above steps and setting $Q_1 = 0$, $P_1 = E_1^I$ we obtain the homogeneous interphase case (Fig. 3.5). Finally, we take $P_1 = E^m$, $Q_1 = 0$ and plot the no interphase case as Fig. 3.3, which agrees with Christensen and Lo's (1979) result as expected. Observing Fig. 3.4 and Fig. 3.5, we find how the bulk modulus depends on the thickness t, and we observe that the thicker is the interphase, the higher is the bulk modulus, and the effect of interphase is higher at the larger volume fraction of particles. We choose $t = 0.1a$ and $t = 0.05a$ cases from Fig3.4 and compare with the same thickness situation in Fig. 3.5, and then compare with perfect bond situation. This comparison is given in Fig. 3.6. We are also considering the following two situations for the bulk modulus case. One is when the interphase is softer than the matrix and has a power variation in E^I and ν^I is constant, and the second is when the properties of the particles and the matrix are interchanged. Then the corresponding homogeneous interphase assumptions are considered too, the results of comparison are similar to the above analysis. These results are displayed in Fig. 3.7 - Fig. 3.12. We find that the power variation interphase model gives the lower prediction for the effective bulk modulus than the homogeneous interphase model and the higher prediction than the perfect bonding case. This agrees with Jasiuk and Kouider (1993). In this case, the homogeneous interphase model overestimates the effective bulk modulus.

Before we study the linear and cubic variations in Young's modulus in the interphase, we should discuss the series solution and test its convergence. Here, we choose one case from eqn. (3.6), with $b = 1.05a$ for our analysis. When we fix the relation $b = 1.05a$, from eqn. (3.1), we find that the Young's modulus in the power variation has the relation

$$E^I = 65\left(\frac{r}{a}\right)^{-64} \tag{3.8}$$

Now, we use the cubic variation to simulate this power variation as shown in Fig. 3.2
So, the approximate expression can be achieved as

$$E^I = -541314.1362\left(\frac{r}{a}\right)^3 + 1697798.147\left(\frac{r}{a}\right)^2 - 1775237.386\left(\frac{r}{a}\right) + 618818.3794 \tag{3.9}$$

by using the curve fitting method. Substituting eqn. (3.9) and eqn. (3.4) into eqn. (2.16),

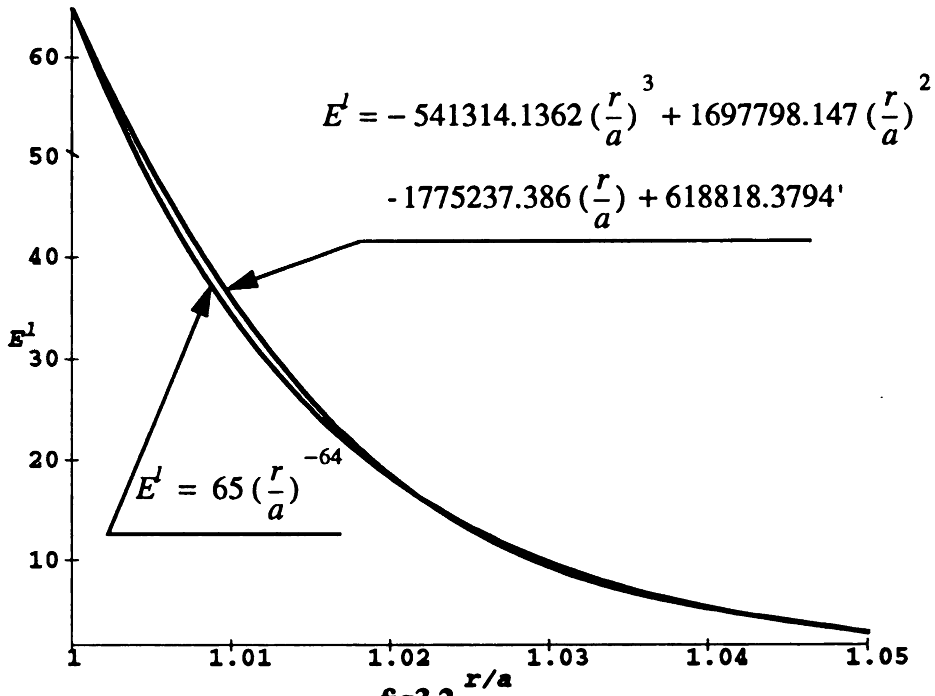

fig3.2
cubic variation simulating power variation

we obtain

$$\frac{\partial^2 u_r}{\partial r^2} + f(r)\frac{\partial u_r}{\partial r} + g(r)\,u_r = 0 \tag{3.10}$$

where

$$f(r) = \frac{-2706570.679\frac{r^3}{a^3} + 6791192.590\frac{r^2}{a^2} - 5325712.126\frac{r}{a} + 1237636.750}{r\left(-541314.136\frac{r^3}{a^3} + 1697798.147\frac{r^2}{a^2} - 1775237.386\frac{r}{a} + 618818.379\right)} \tag{3.11}$$

$$g(r) = \frac{-309322.364\frac{r^3}{a^3} - 485085.186\frac{r^2}{a^2} + 2028842.729\,(\frac{r}{a}) - 1237636.750}{r^2\left(-541314.136\frac{r^3}{a^3} + 1697798.147\frac{r^2}{a^2} - 1775237.386\frac{r}{a} + 618818.379\right)} \tag{3.12}$$

$f(r)$ and $g(r)$ both have the Taylor series expansion as

$$f(r) = \sum_{n=0}^{\infty} f_n (r - r_0)^n \tag{3.13}$$

$$g(r) = \sum_{n=0}^{\infty} g_n (r - r_0)^n \tag{3.14}$$

If $r_0 = 0$ is a regular singular point, then according to Fuchs' theorem, we can find a convergent series solution as

$$u_r = \sum_{n=0}^{\infty} a_n r^{n+k} \tag{3.15}$$

But $r_0 = 0$ is located at the center of inclusion and not in the interphase. One can also show that the other regular singular points lie outside the interphase region $a < r < b$, so we choose ordinary point r_0 in the interphase. It is also more convenient to choose $a \le r_0 \le b$, since this will require a smaller radius of convergence and thus a number of terms in the series will be smaller than for the case when r_0 is outside this interval. The series solution is expressed as follows

$$u_r = \sum_{n=0}^{\infty} a_n (r - r_0)^n \tag{3.16}$$

Eqns. (3.16), (3.14) and (3.13) are substituted into the differential eqn. (3.10) and coefficients of same powers of $(r - r_0)$ are set equal to zero to determine a_n. Since the center point in the interphase gives a faster covergence than the boundary point, we take $r_0 = (a+b)/2$ and this is an ordinary point in our series solution. In this case, $r_0 = 1.025a$. The MAPLE computer program was written to solve this problem. Finally, we get solution of eqn. (3.10) in the form

$$u_r = C_1^l \sum_{n=0}^{\infty} b_n (r - 1.025a)^n + C_2^l \sum_{n=0}^{\infty} c_n (r - 1.025a)^n \tag{3.17}$$

where b_n and c_n are known, and C_1^l and C_2^l are the unknown constants. After we have the expression given in eqn. (3.17), the effective bulk modulus can be evaluated easily following the previous steps.

The bulk moduli for the interphase properties given in Fig. 3.2 are shown in Fig. 3.13. These two curves almost overlap with each other, and the small difference comes mainly from the small difference between the cubic polynomial fit and the power variation. We must mention that when the series solution includes over sixty terms, the curve 1 in Fig. 3.13 does not change when the number of terms is increased. This implies that sixty terms series solution will give convergent solution for this particular case. We have also tested the convergence of our series by using the ratio test that is

$$\lim_{n \to \infty} \left| \frac{a_{n+1} (r - r_0)^{n+1}}{a_n (r - r_0)^n} \right| < 1 \tag{3.18}$$

We found that our series solutions satisfy this condition

Repeating the above procedures, we can derive bulk moduli for cases given in eqns. (2.20) and (2.21). First, we assume that ν^I is constant and take different variations of E^I. The comparison of linear variation of E^I, cubic variation of E^I, uniform interphase and no interphase case are given in Fig. 3.14. We observe that the homogeneous interphase yields the highest modulus. Again, the tendency of this results is the same as in Jasiuk and Kouider (1993). Then, we let E^I be constant and compare different variations of ν^I (Fig. 3.15). We also let E^I be the linear variation and compare the different variations of ν^I (3.16). We find that when E^I is fixed, power variation of ν^I gives highest value in effective bulk modulus and uniform case gives lowest result. This tendency is opposite to the one for varying E^I.

Observing Fig. 3.14, we may infer that the smoother variation of Young's modulus in the interphase gives the lower estimation of the effective bulk modulus, while the rougher variation of ν^I gives the lower estimation of the effective bulk modulus.

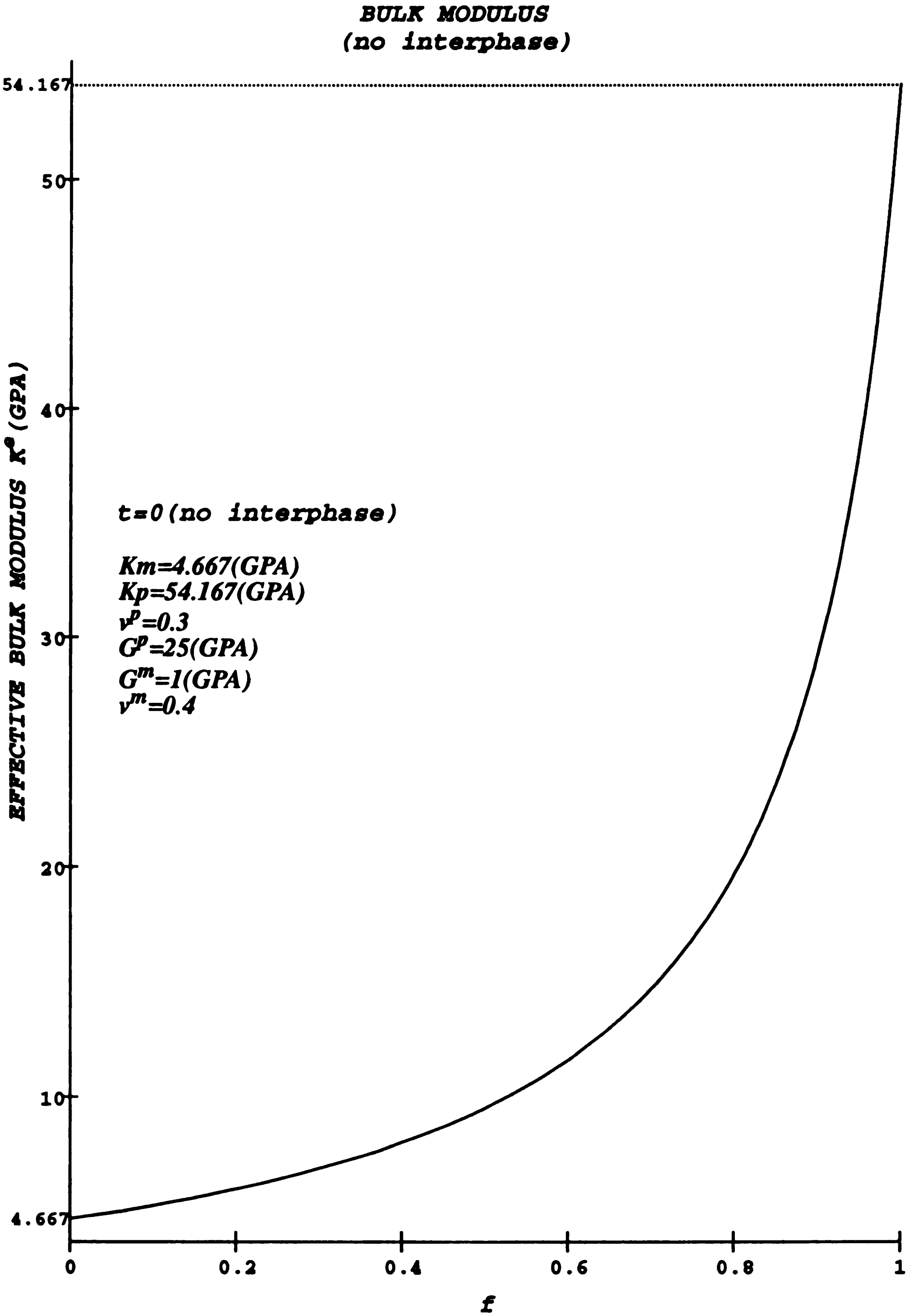

Fig. 3.3 Effective bulk modulus vs. volume fraction f for the no interphase case.

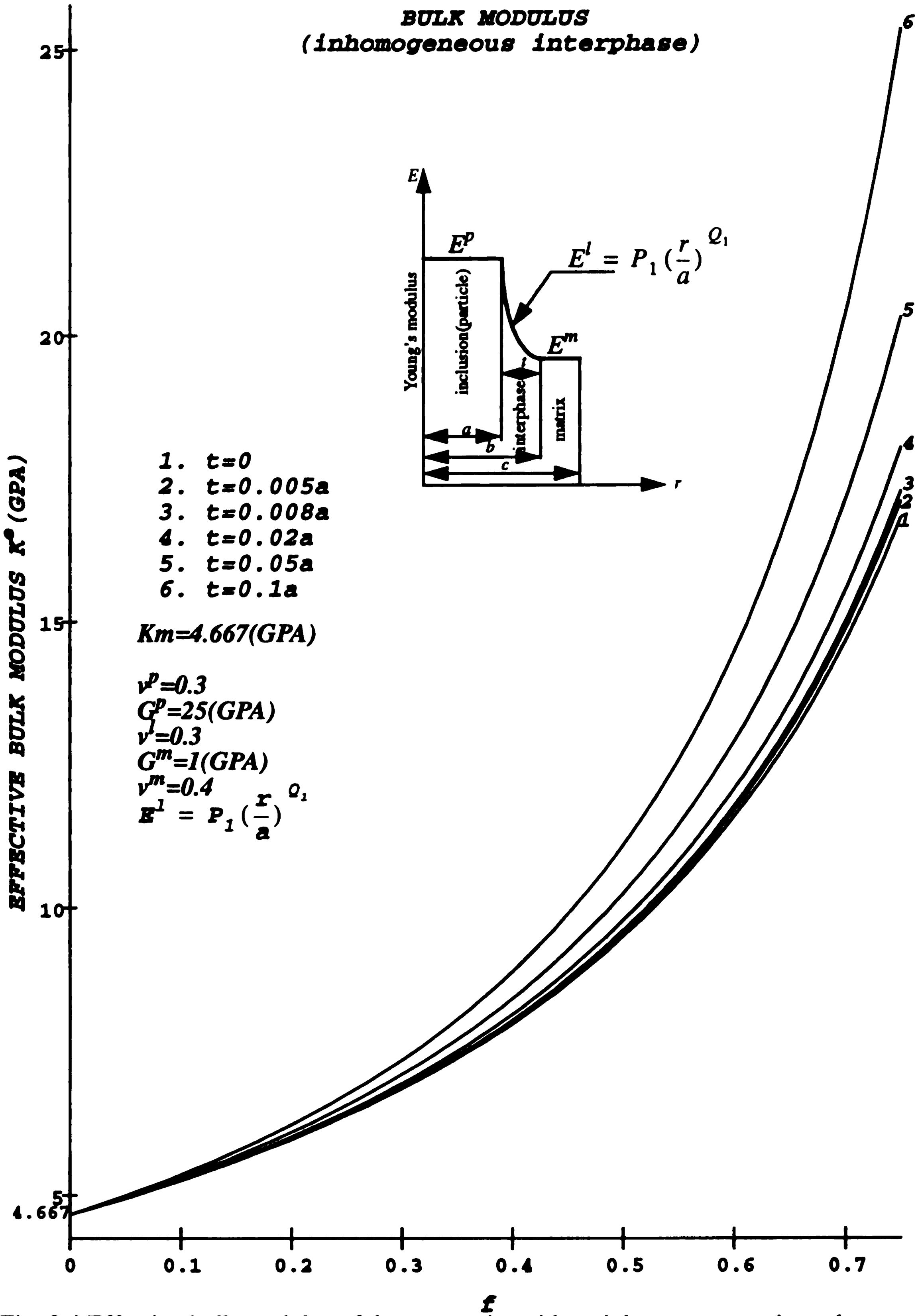

Fig. 3.4 Effective bulk modulus of the composite with an inhomogeneous interphase.

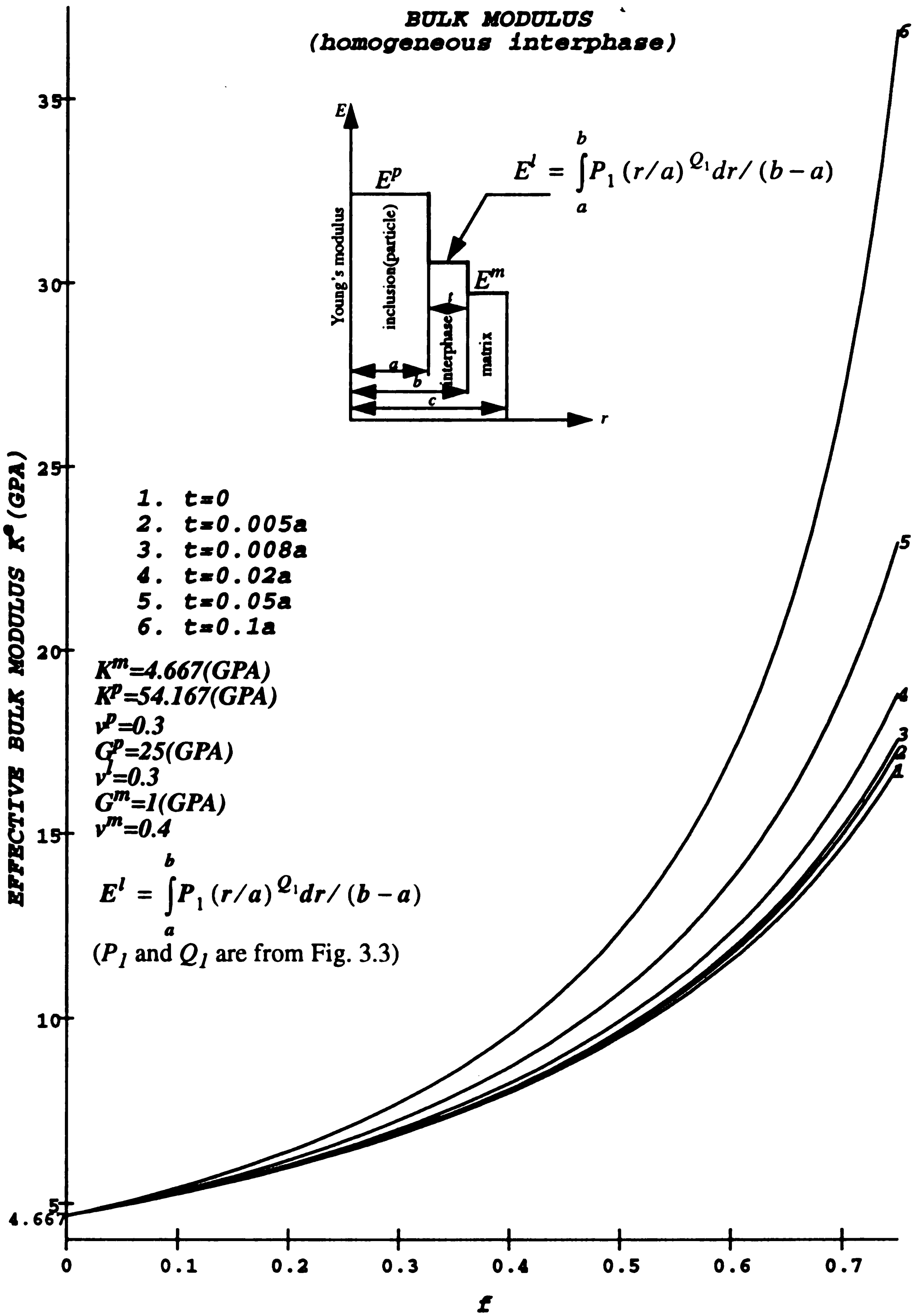

Fig. 3.5 Effective bulk modulus of the composite with a homogenous interphase.

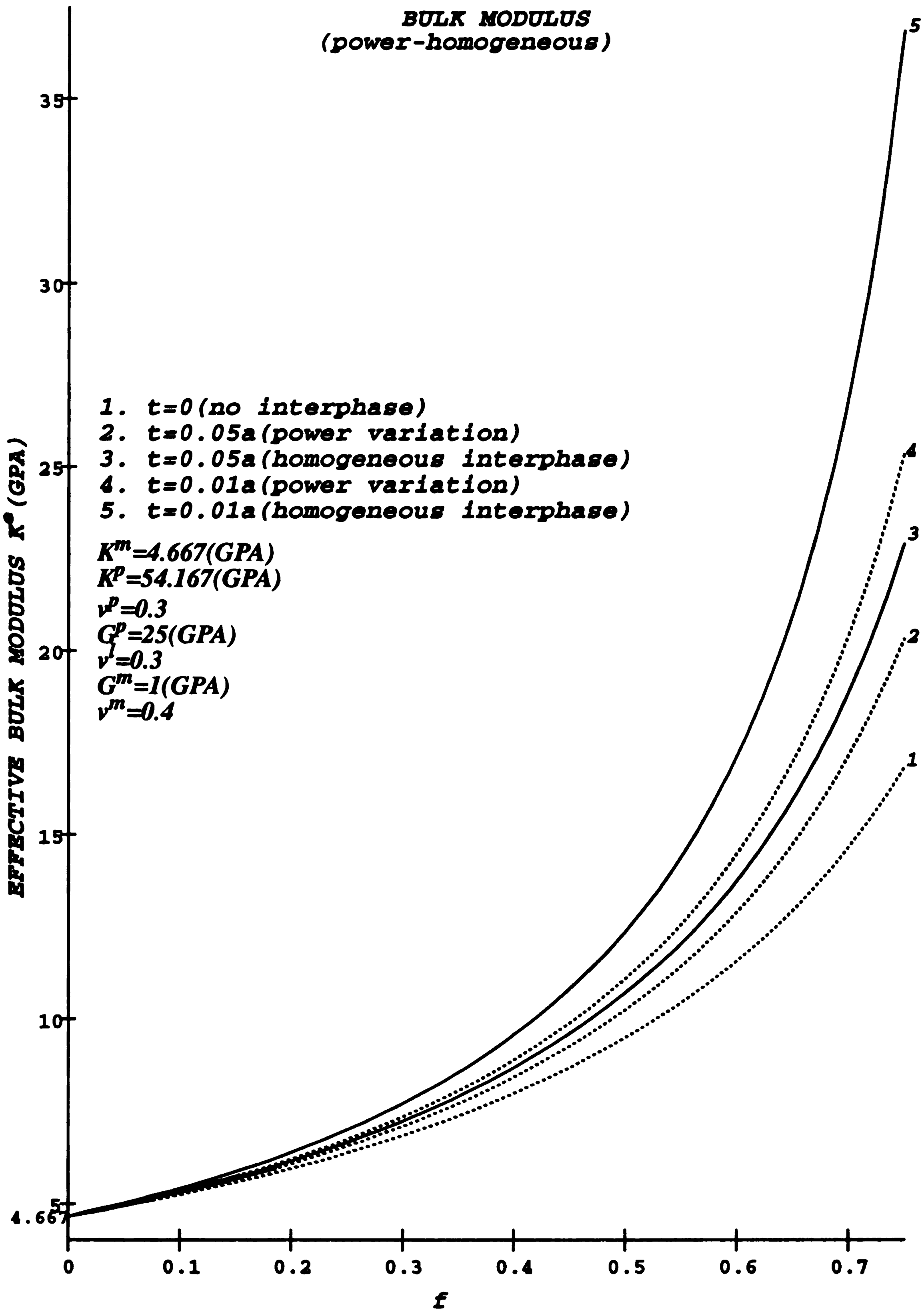

Fig. 3.6 Comparison between the inhomogeneous and homogeneous interphase cases.

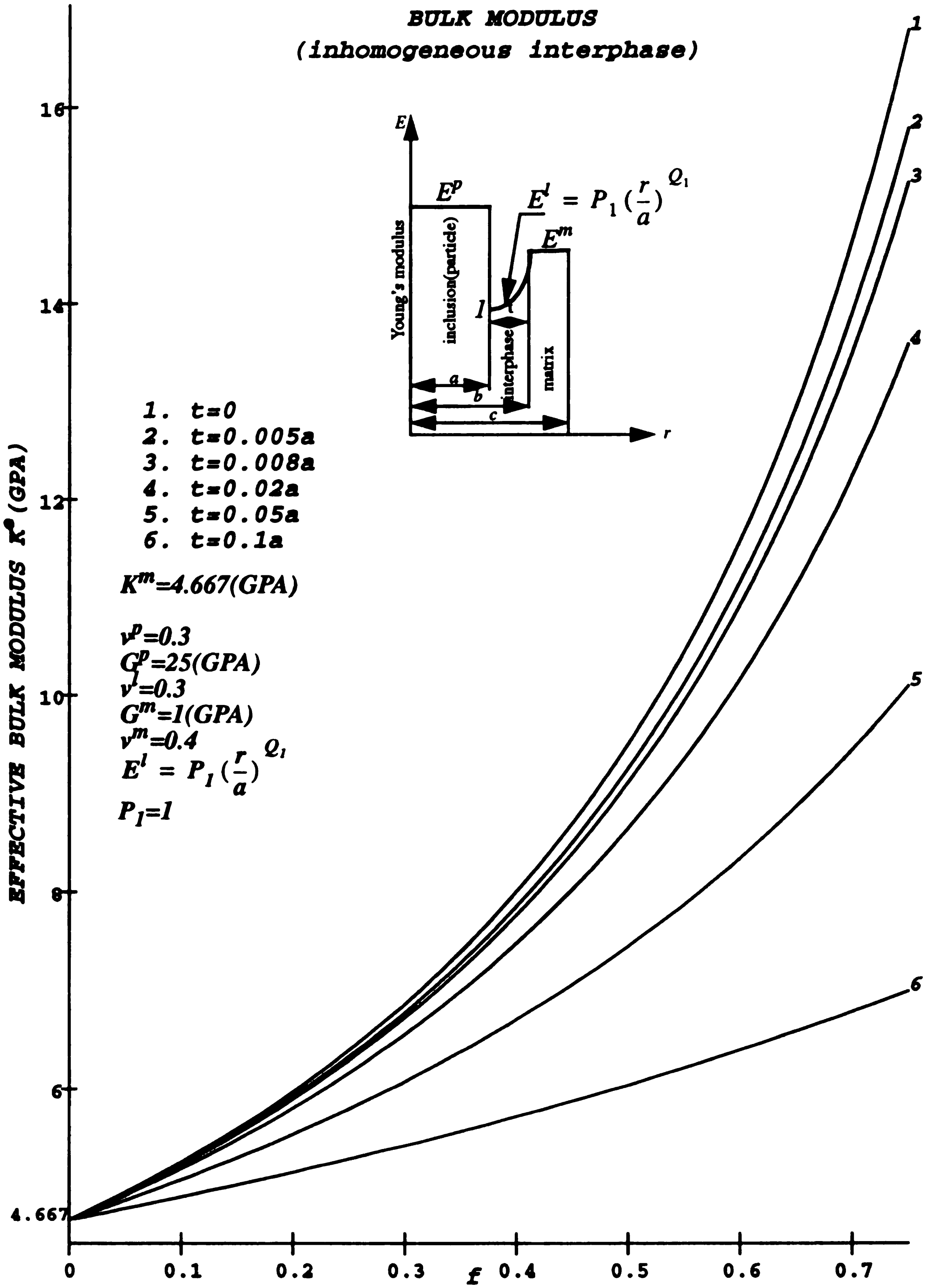

Fig. 3.7 Effective bulk modulus vs. f for an inhomogeneous interphase softer than matrix; Interphase has a power variation in Young's modulus and a constant Poisson's ratio.

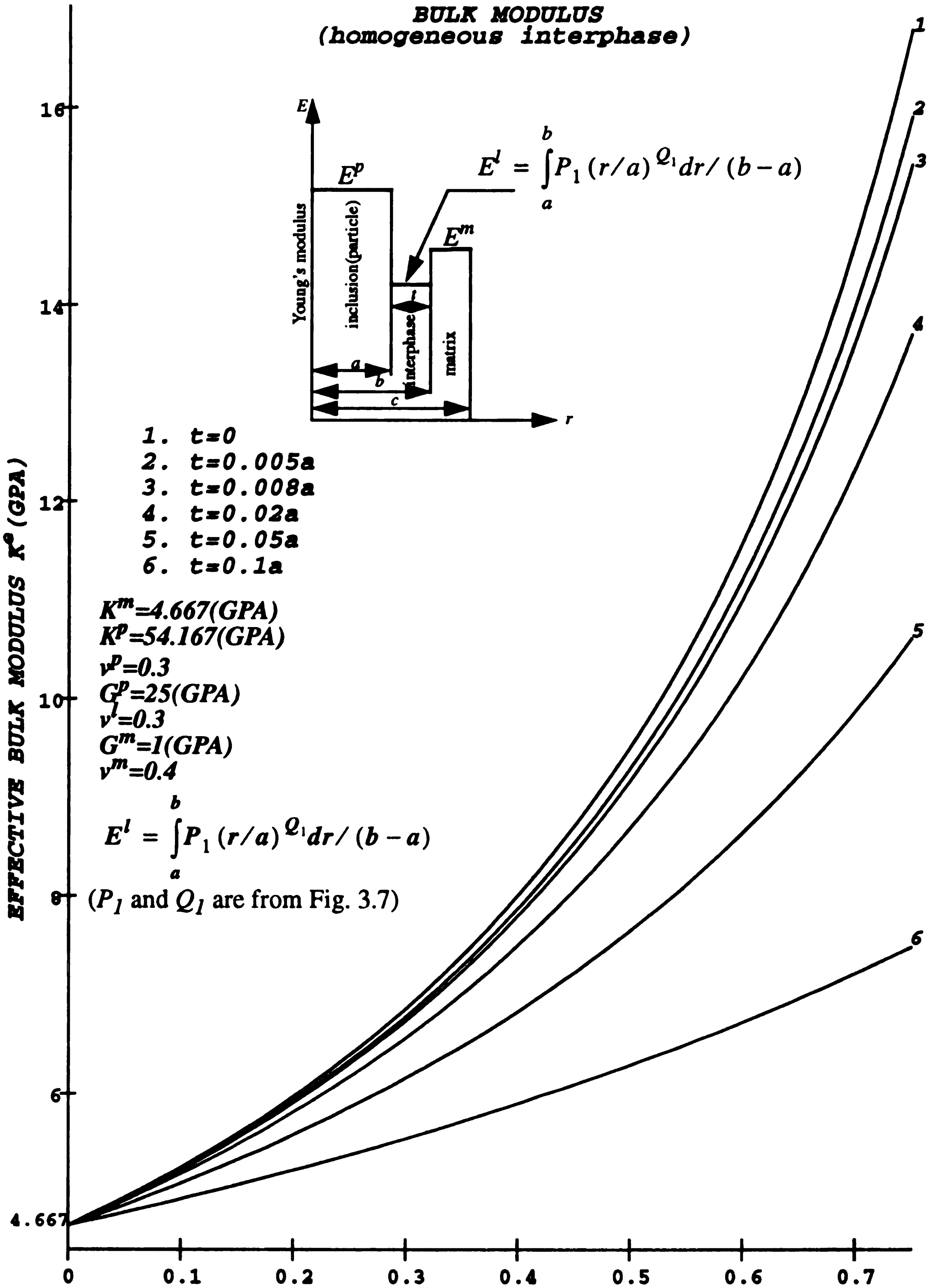

Fig. 3.8 Effective bulk modulus vs. f for a homogeneous interphase softer than matrix; Interphase has a constant Young's modulus and a constant Poisson's ratio.

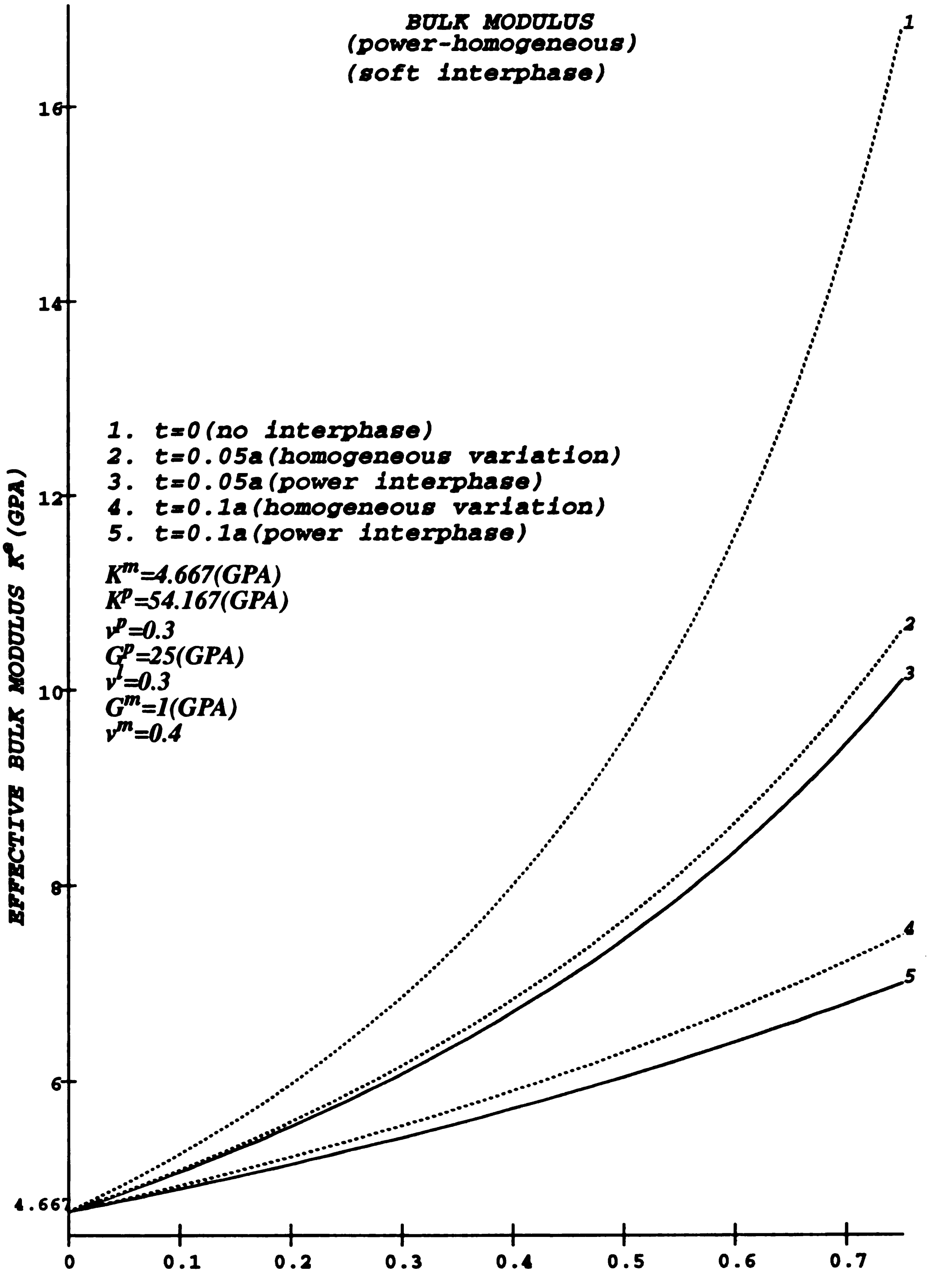

Fig.3.9 The comparison between Fig. 3.7 and Fig. 3.8.

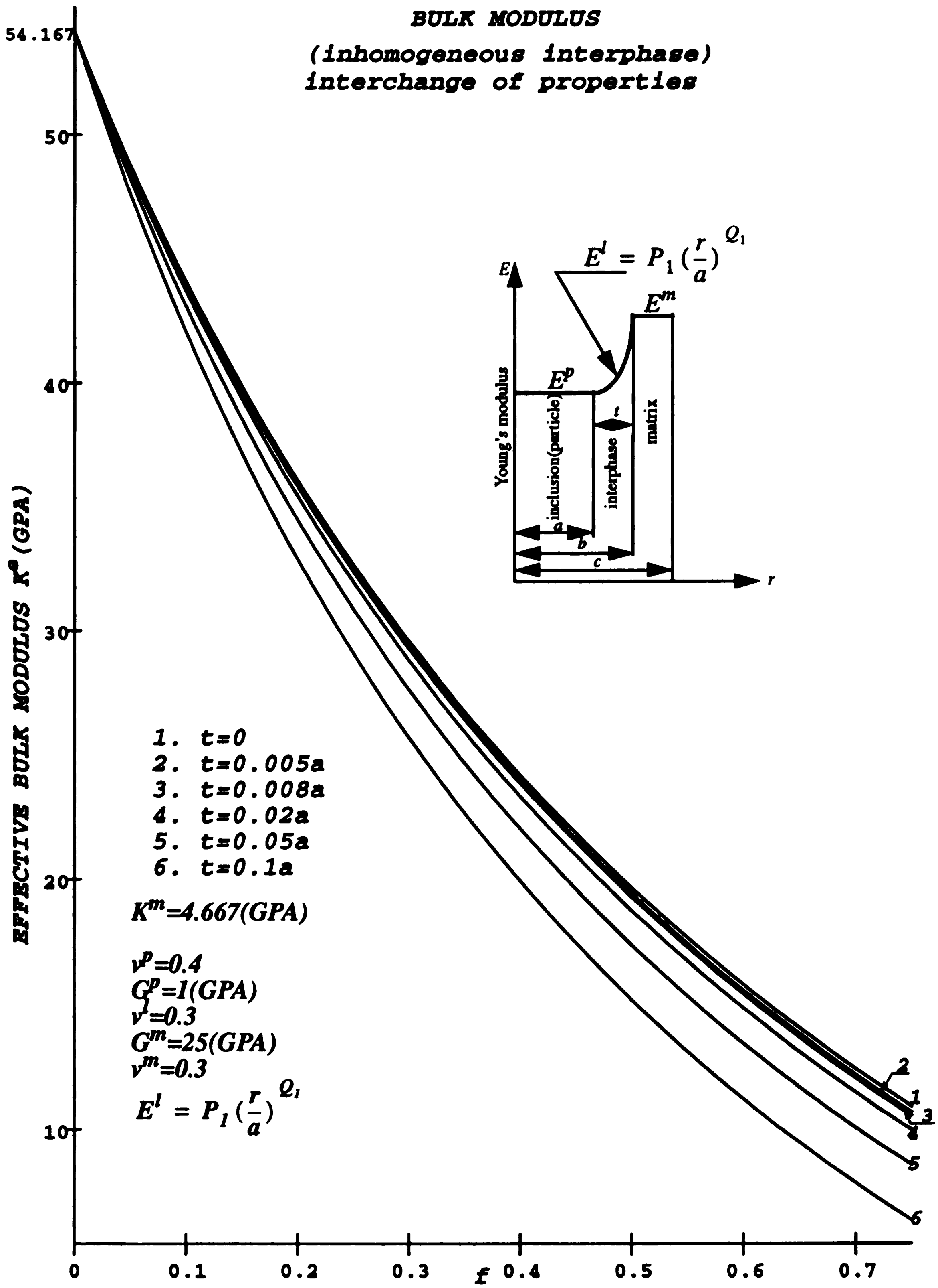

Fig. 3.10 Effective bulk modulus vs. f for an inhomogeneous interphase with interchange of the properties.

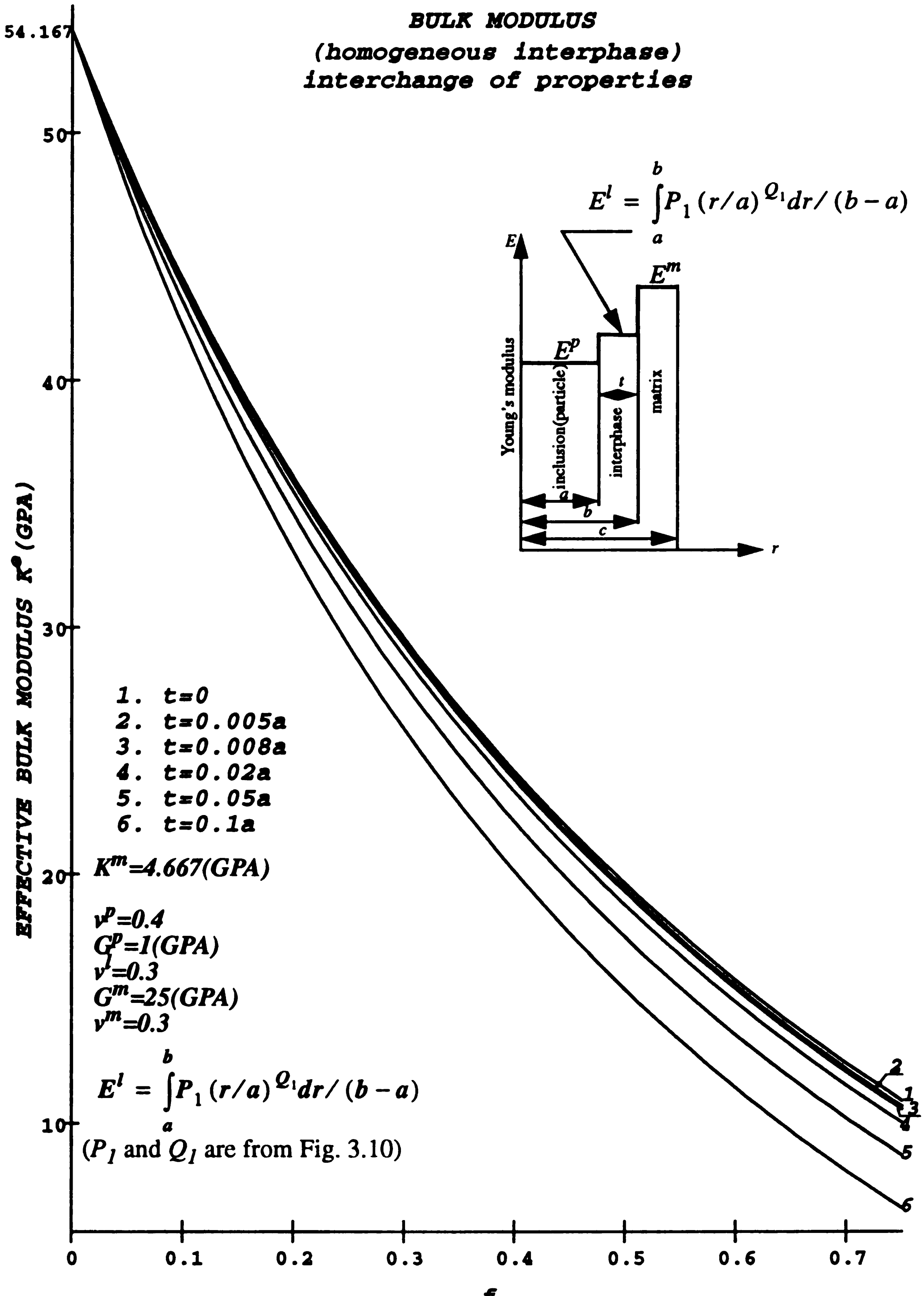

Fig. 3.11 Effective bulk modulus vs. f for a homogeneous interphase with interchange of the properties.

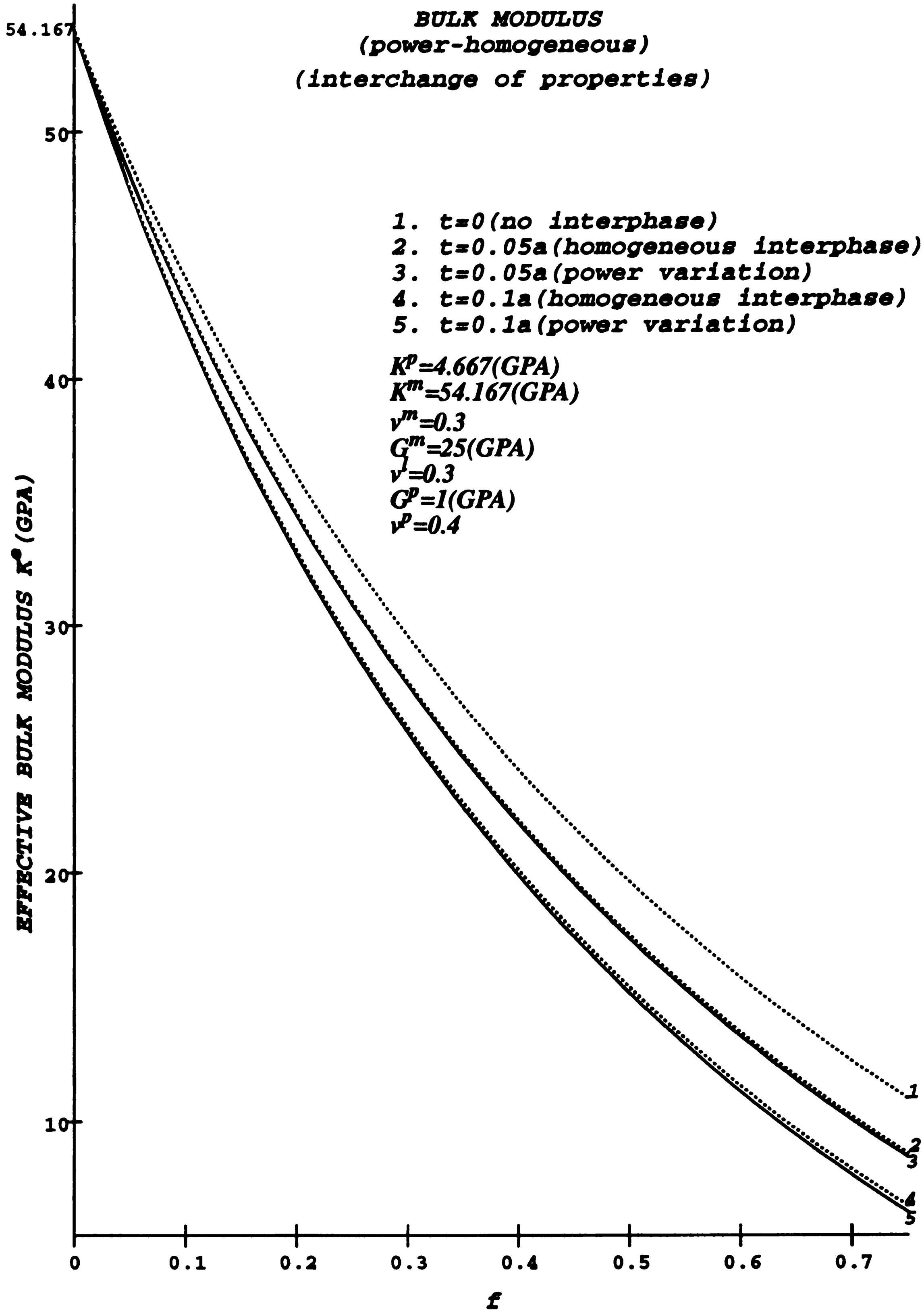

Fig. 3.12 The comparison between Fig. 3.10 and Fig. 3.11.

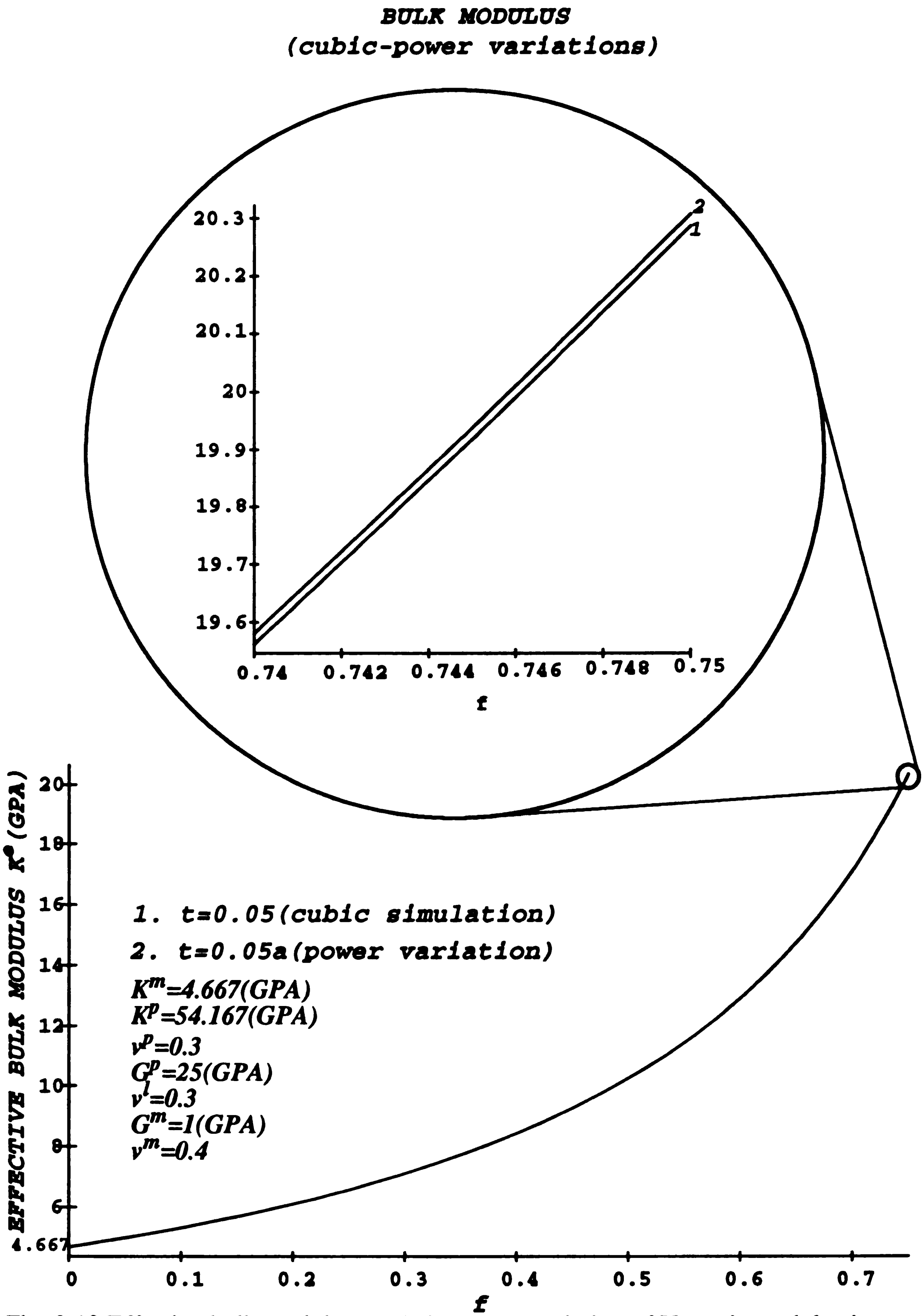

Fig. 3.13 Effective bulk modulus vs. *f*; the power variation of Young's modulus in interphase is simulated by cubic variation.

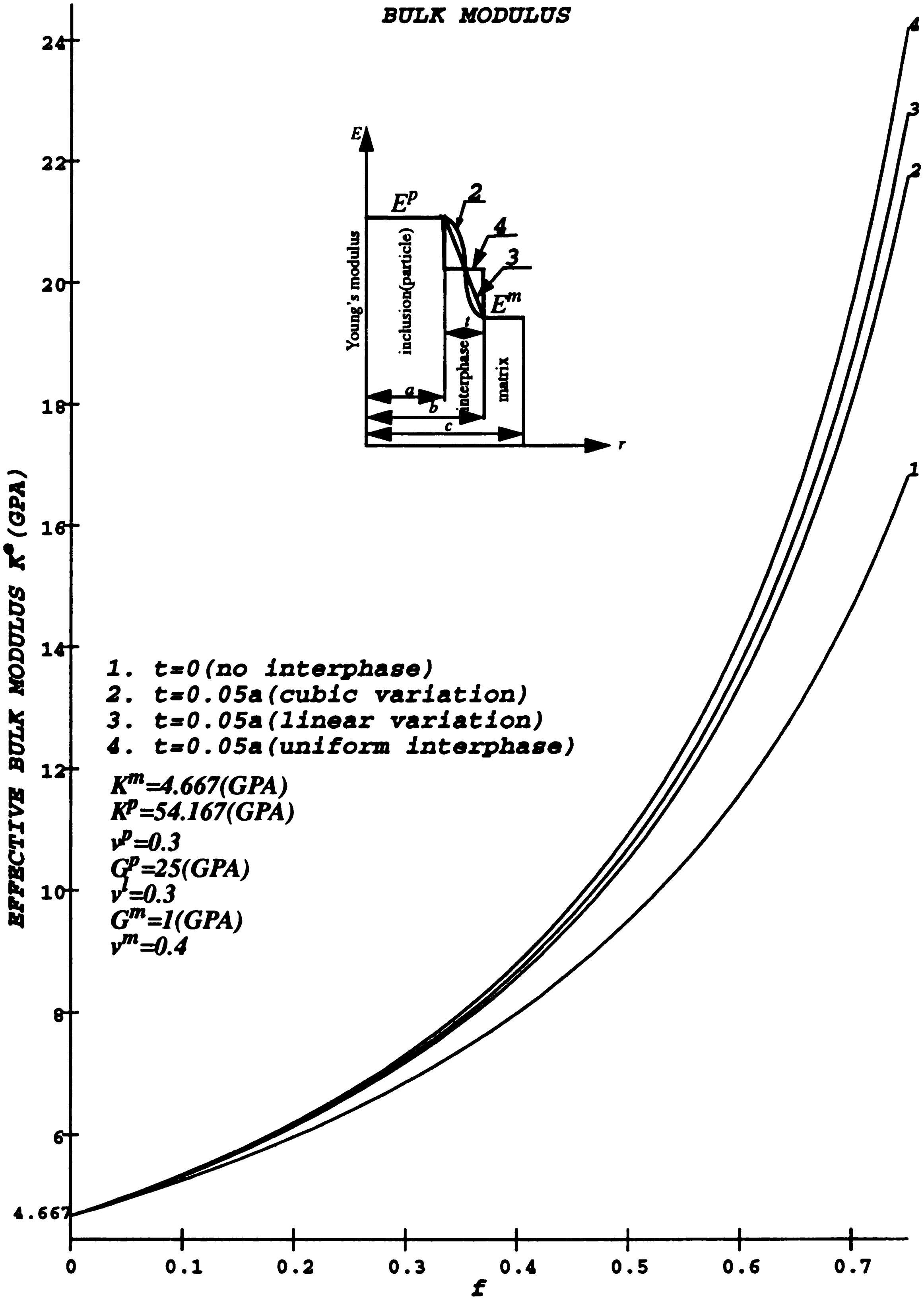

Fig. 3.14 Comparison of effective bulk moduli from different variations of Young's moduli in the interphase

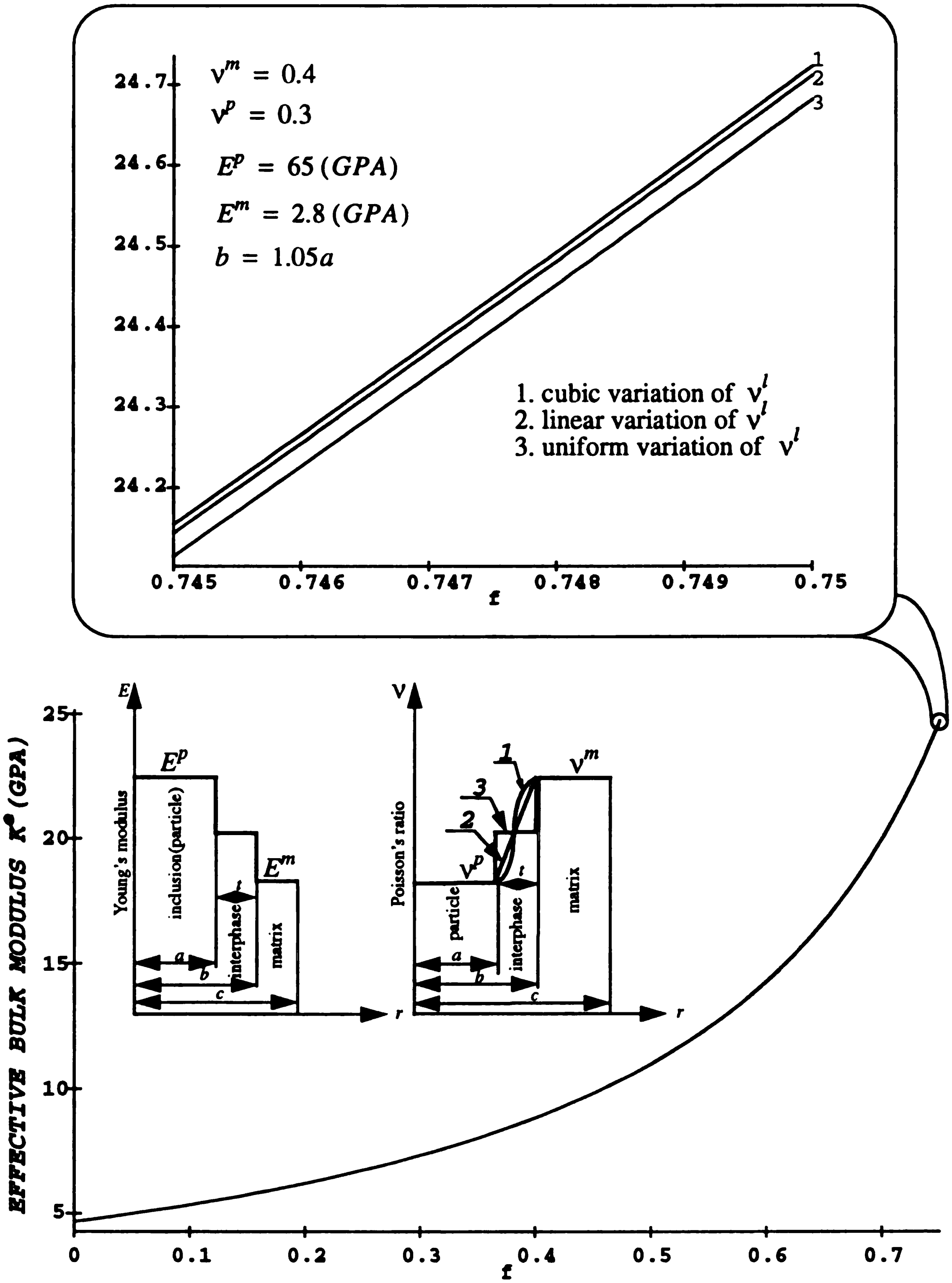

Fig. 3.15 Bulk moduli of composite with different variations of Poisson's ration and uniform variation of Young's modulus in the interphase.

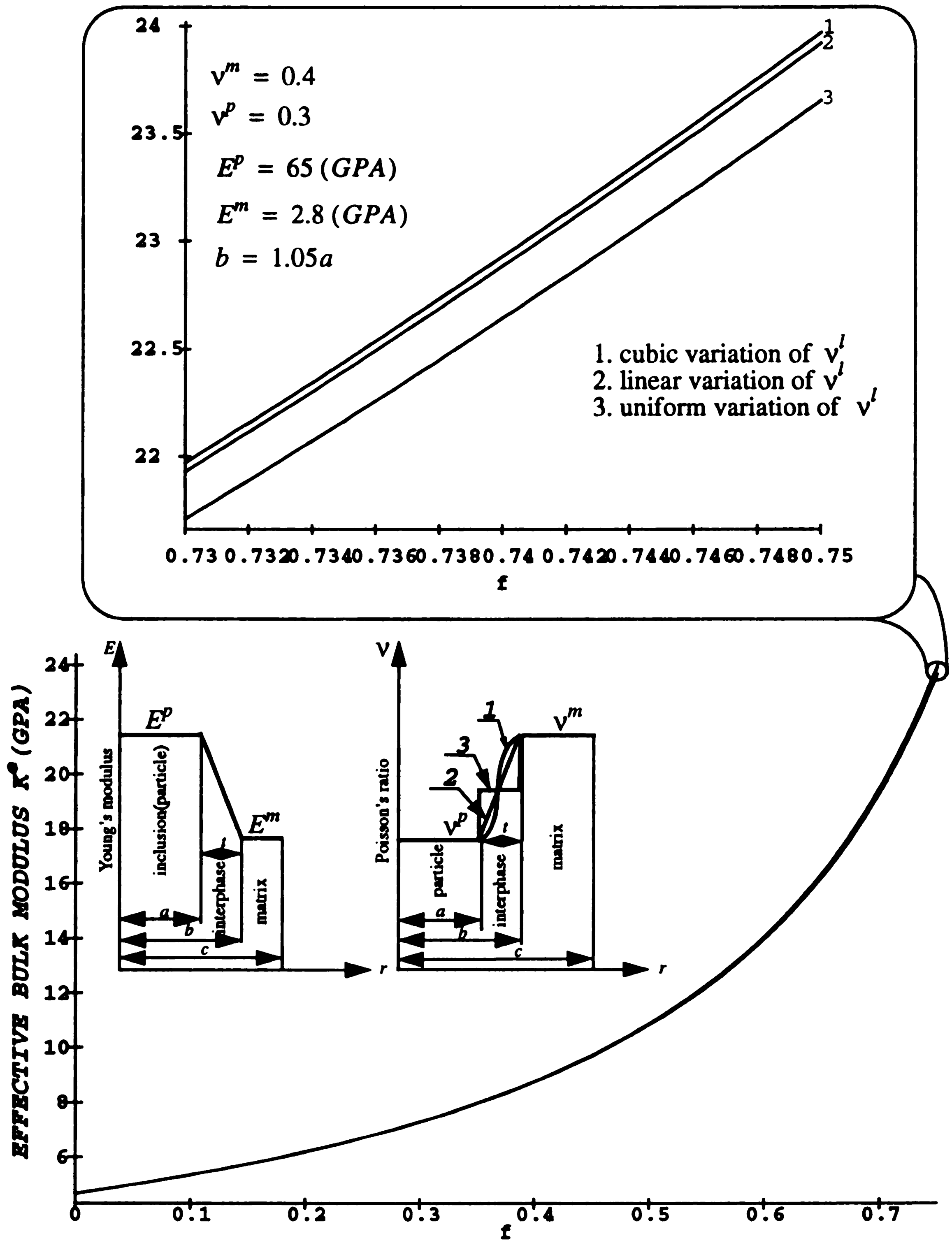

Fig. 3.16 Bulk moduli of composite with different variations of Poisson's ratio and linear variation of Young's modulus in the interphase.

3.2. SHEAR MODULUS ANALYSIS

In this section, we conduct the numerical analysis of the effective shear modulus. We assume that the interphase between the matrix and inclusion is isotropic and inhomogeneous, and that the Young's modulus of interphase has the elastic properties changing with the radial distance from the inclusion boundary, and the Poisson's ratio is constant. We illustrate our analytical results by considering a particulate composite, with the properties $K^m = 4.667GPA$, $K^p = 54.167GPA$, $G^m = 1.0GPA$, $G^p = 25.0GPA$, $\nu^m = 0.4$, $\nu^p = 0.3$ and $\nu^l = 0.3$, which has an interphase region with spatially varying properties given in eqn. (2.19). In our example we assume that $E^l(a) = E^p$ and $E^l(b) = E^m$ (Fig. 3.1a.b); constants P_1 and Q_1 are found from these two constraints [eqn. (3.1)]. For comparison we also consider a homogeneous interphase case such that $E^l = \int_a^b P(r/a)^Q dr/(b-a)$ which can be obtained from eqn. (2.19) by setting $Q_1 = 0$ and $P_1 = E^l$. We can also obtain the perfect bonding case by setting $Q_1 = 0$ and $P_1 = E^m$. We substitute the properties of the composite [eqn.(3.4)] into eqn. (2.57), and solve this fourth order equation. We get two complex conjugate roots as in eqn. (2.59). So we take the expressions (2.60), (2.61) and (2.62) as our displacements results. By using eqns. (2.63) and the remote boundary condition (2.33), we can obtain a set of linear equations to be solved for the unknown coefficients. Then we set the coefficient $D_4^e = 0$ and we can obtain a quadratic equation for G^e. We neglect the negative root of the quadratic equation and finally, we achieve the relation

$$G^e = F\left(\frac{a^3}{c^3}\right) \tag{3.19}$$

where F represents function in terms of volume fraction and depends on the properties of inclusion, interphase and matrix, and superscripts e symbolizes the effective composite region. We plot this power variation case in Fig. 3.18. The results that shows the same tendency as the bulk modulus results. We see that the thickness of interphase has an important influence on the effective shear modulus. The effect of interphase is more pronounced at the higher volume fractions. The homogeneous interphase case is given in Fig. 3.19 and the no interphase case is shown in Fig. 3.17. Similarly as in the bulk modulus analysis, we compare the power variation case and homogeneous interphase model case in Fig. 3.18. Again the perfect bonding condition yields lowest results, the power variation gives the intermediate values and the homogeneous interphase model yields the highest shear modulus. This agrees with the effective bulk modulus results when the same comparison is made. Comparing to Fig. 3.7 - Fig. 3.11 in the bulk modulus study, the analogous consequences are obtained for softer interphase case (Fig. 3.21 - Fig. 3.23) and case when there is an interchange of properties between the matrix and inclusion (Fig. 3.24 - Fig. 3.26). As in the bulk modulus analysis, we would like to have calculated other variations of interphase [eqn.(2.20), eqn.(2.21)]. However since these involve much complicated mathematical manipulations, we will leave it for the future study.

Summarizing the effective bulk modulus and shear modulus results we find that the homogeneous variation of Young's modulus in interphase overestimates the effective elastic moduli in our study. This agrees with the observation in Jasiuk and Kouider (1993), while homogeneous variation of Poisson's ratio in interphase underestimates the effective bulk modulus.

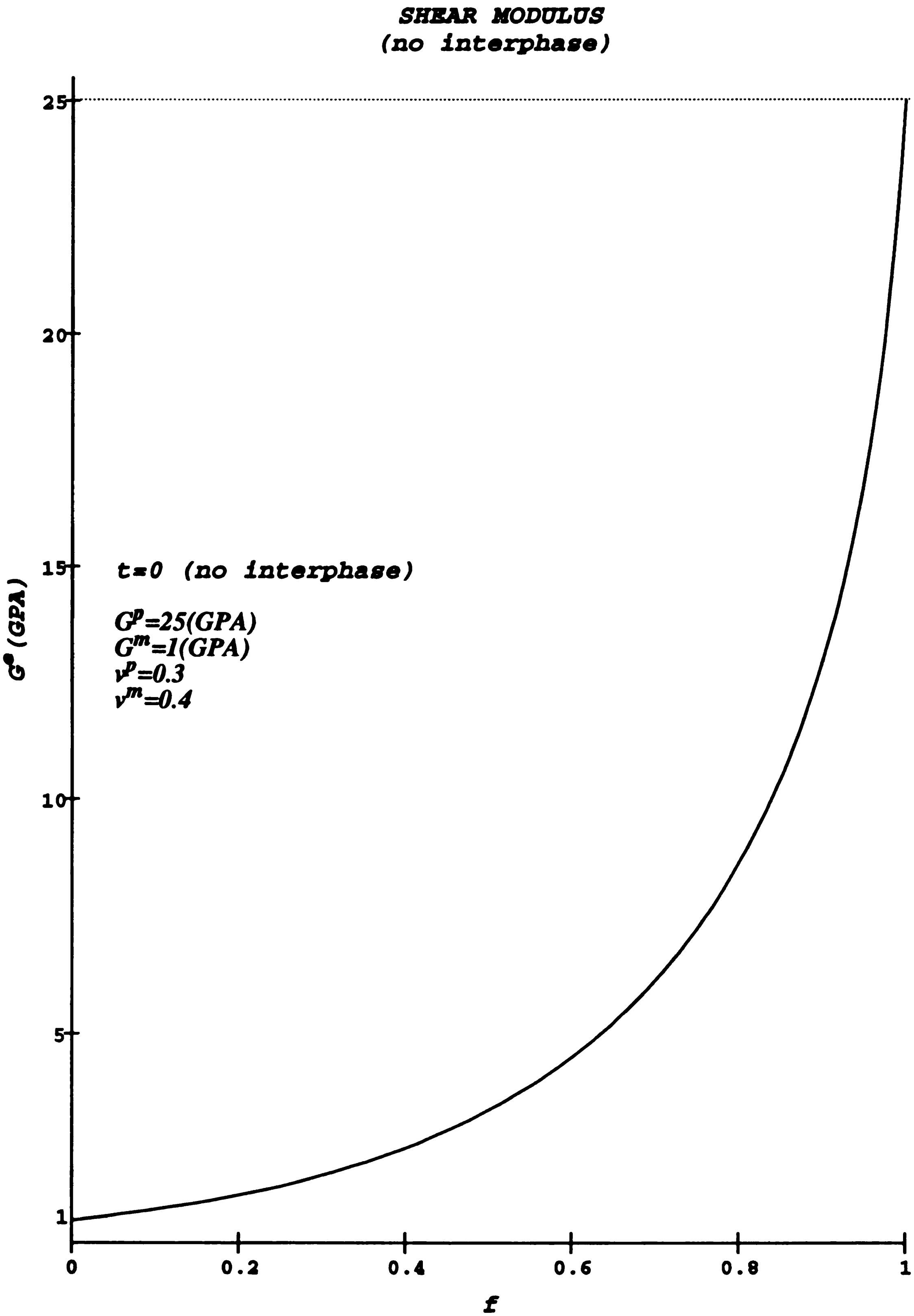

Fig. 3.17 Effective shear modulus vs. f for the no interphase case

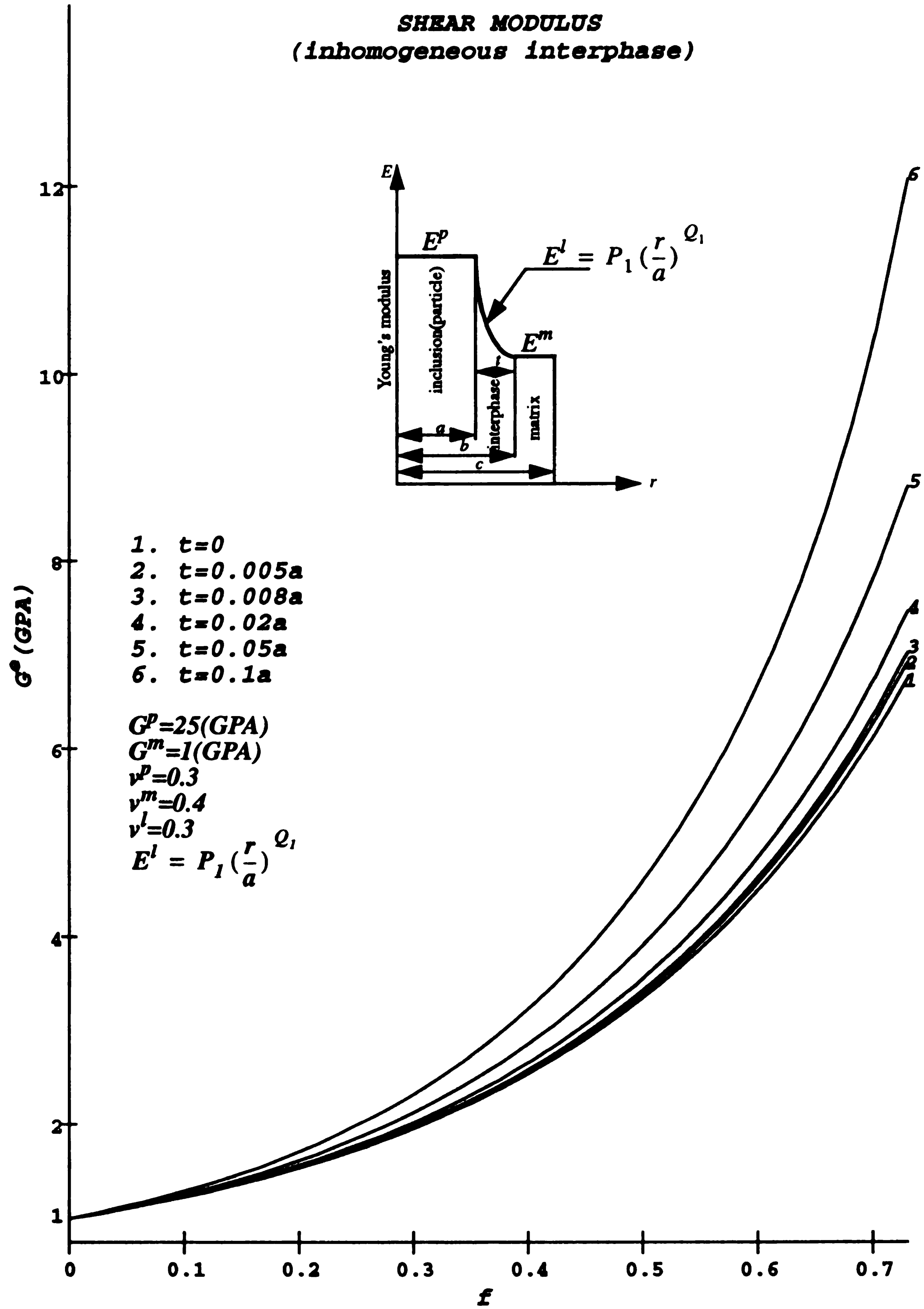

Fig. 3.18 Effective shear modulus vs. f for an inhomogeneous interphase stiffer than matrix; interphase has a power variation Young's modulus and a constant Poisson's ratio.

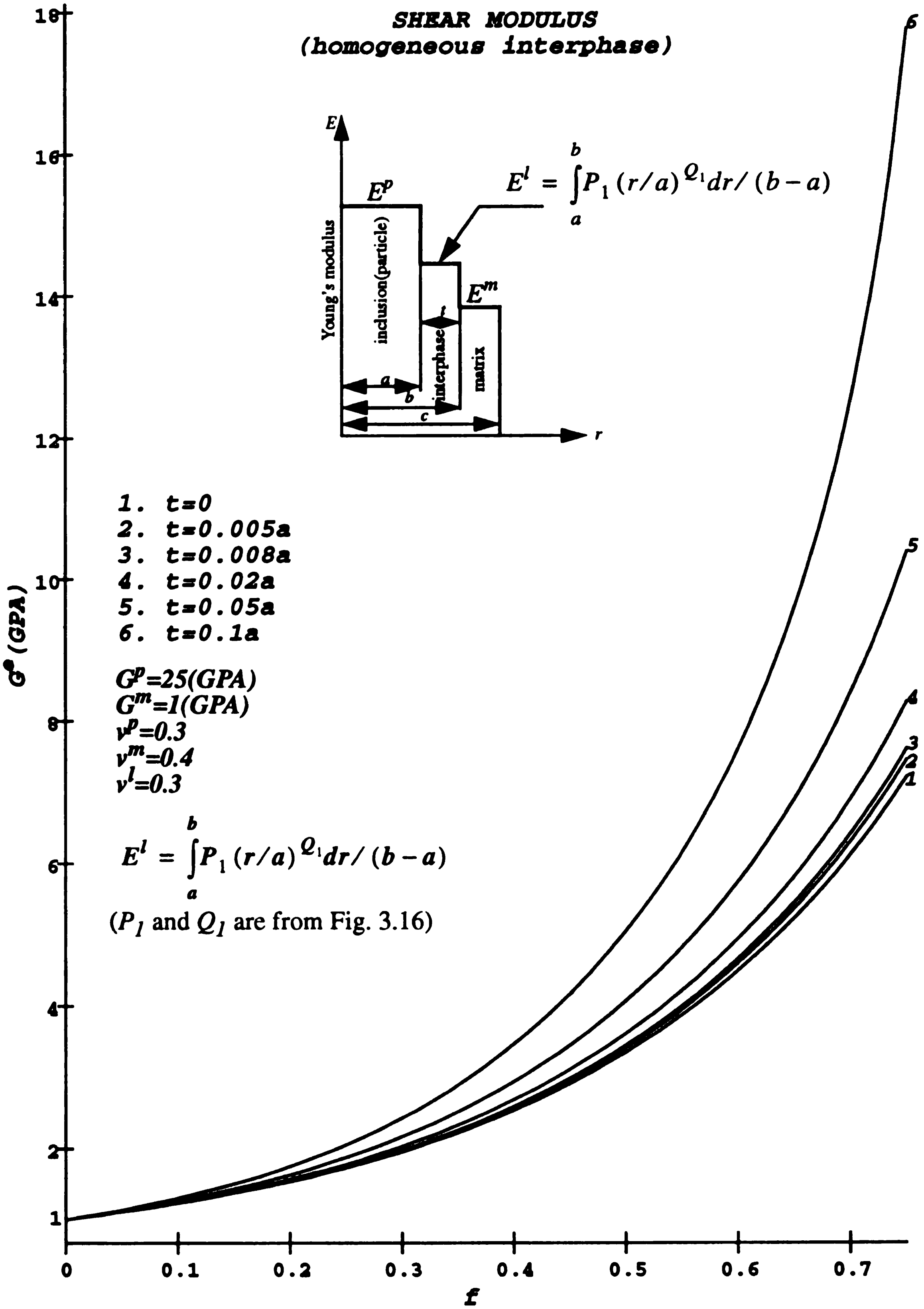

Fig. 3.19 Effective shear modulus vs. f for a homogeneous interphase stiffer than matrix; Interphase has a constant Young's modulus and a constant Poisson's ratio.

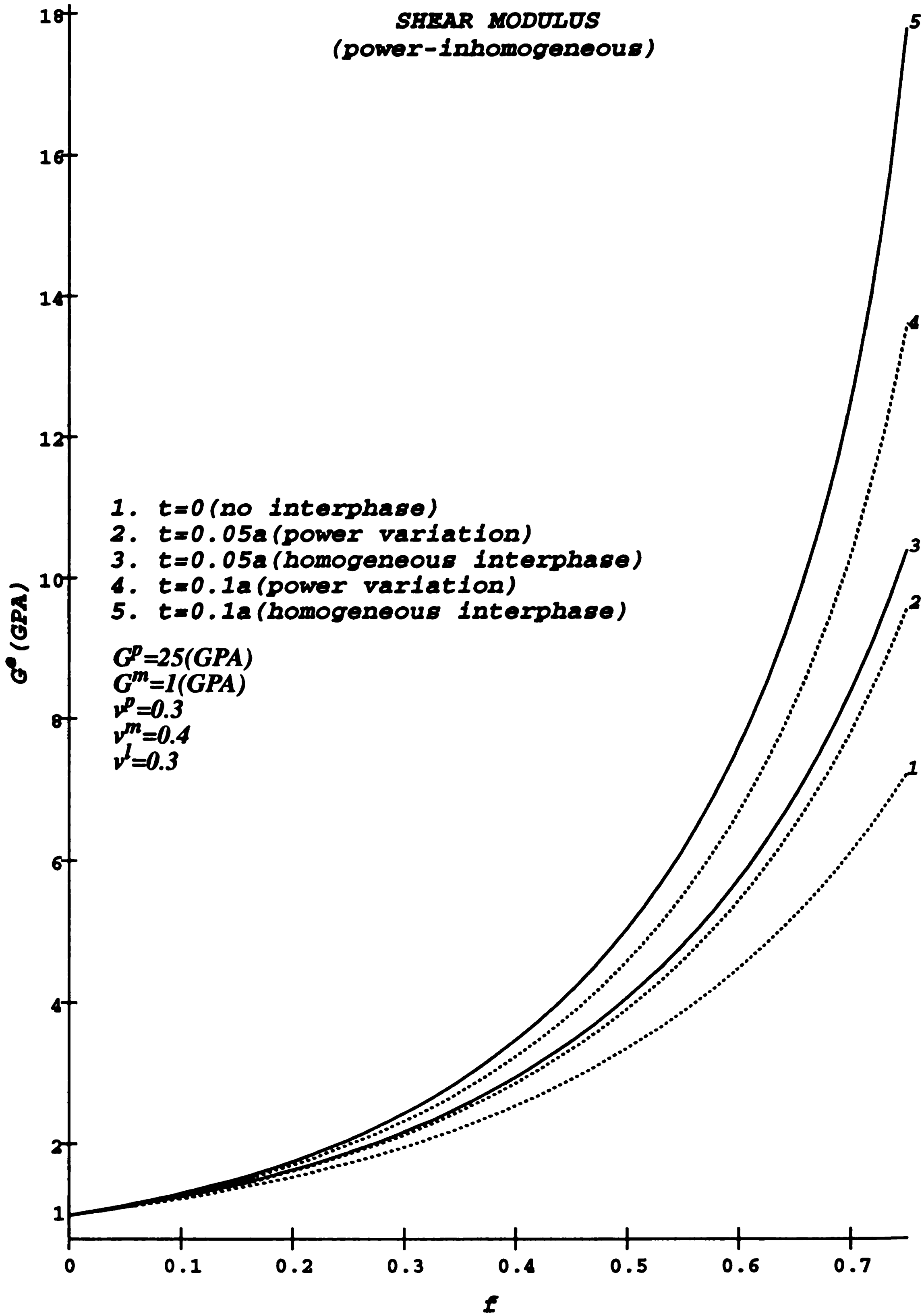

Fig. 3.20 Comparison between Fig. 3.18 and Fig. 3.19.

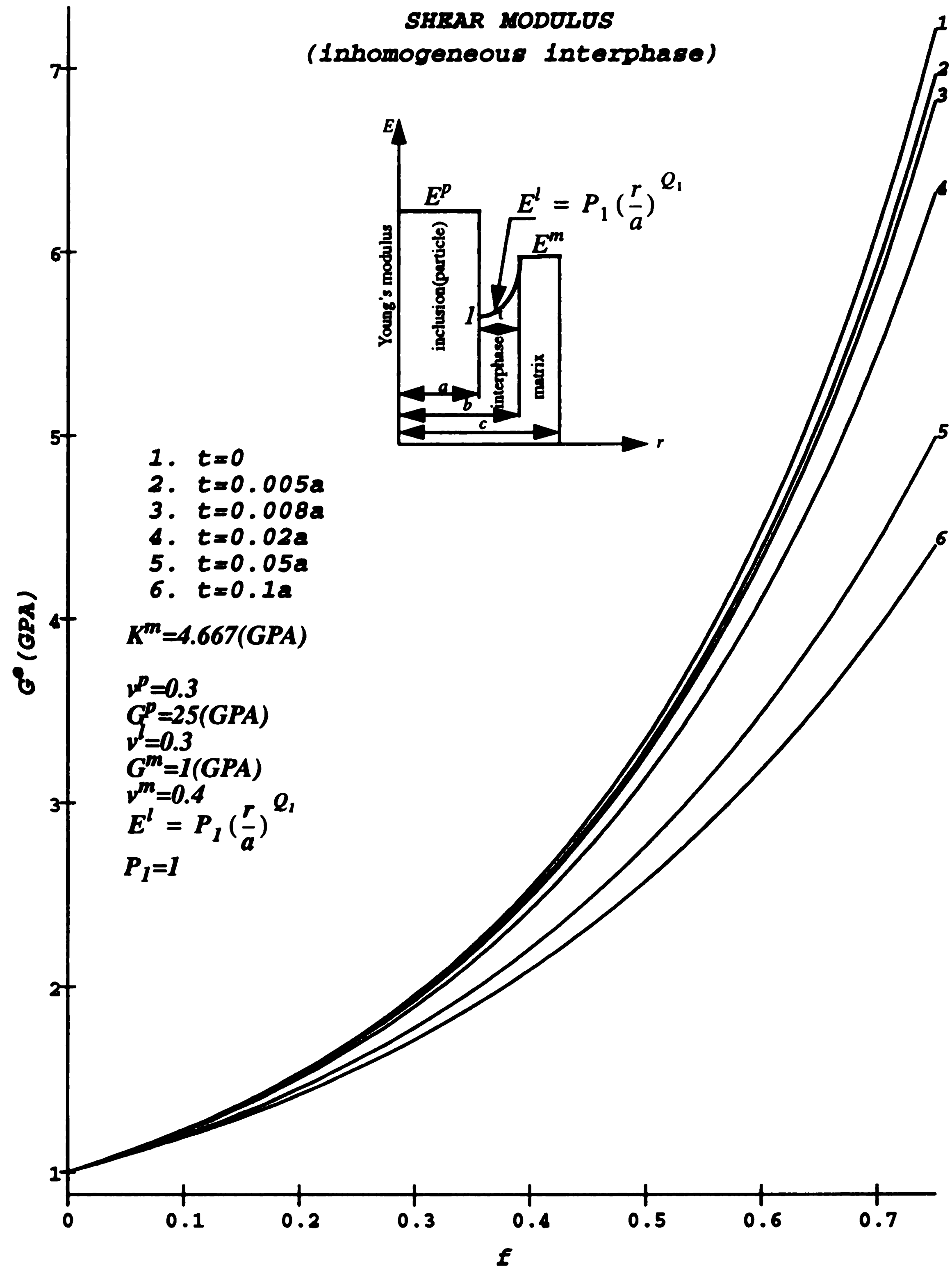

Fig. 3.21 Effective shear modulus vs. *f* for an inhomogeneous interphase softer than matrix; Interphase has a power variation in Young's modulus and a constant Poisson's ratio.

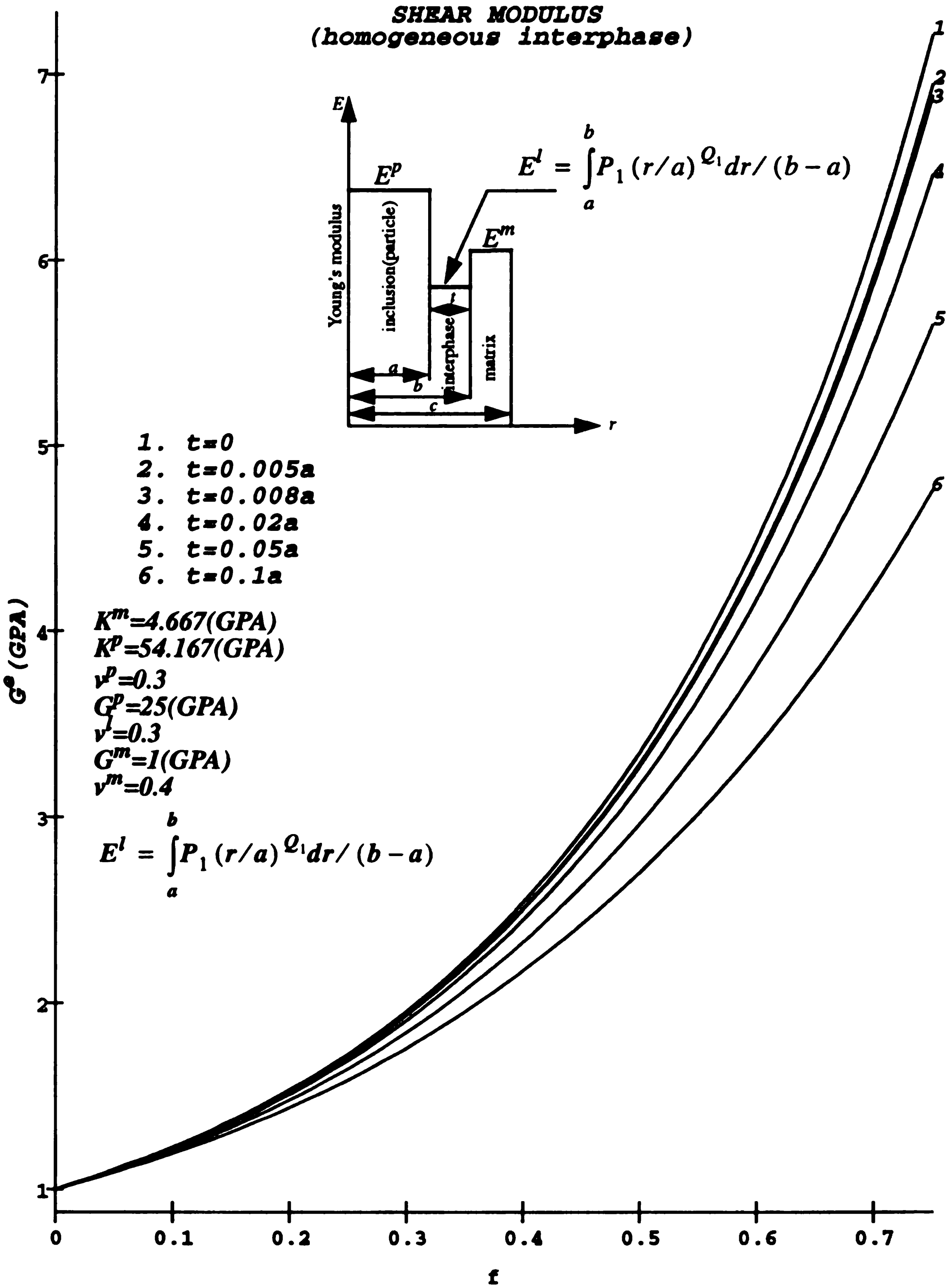

Fig. 3.22 Effective shear modulus vs. f for a homogeneous interphase softer than matrix; interphase has a constant Young's modulus and a constant Poisson's ratio.

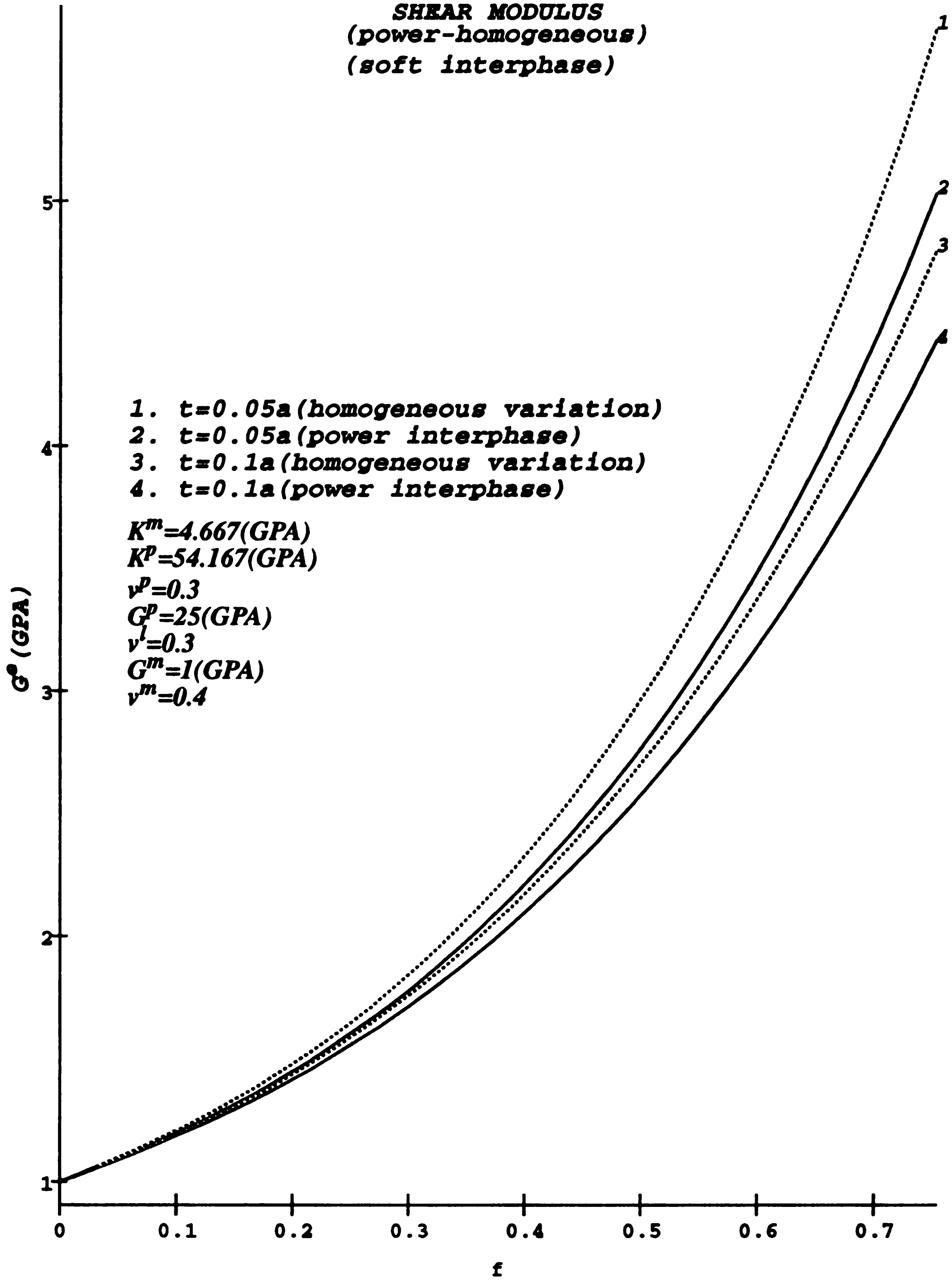

Fig. 3.23 The comparison between Fig. 3.21 and Fig. 3.22

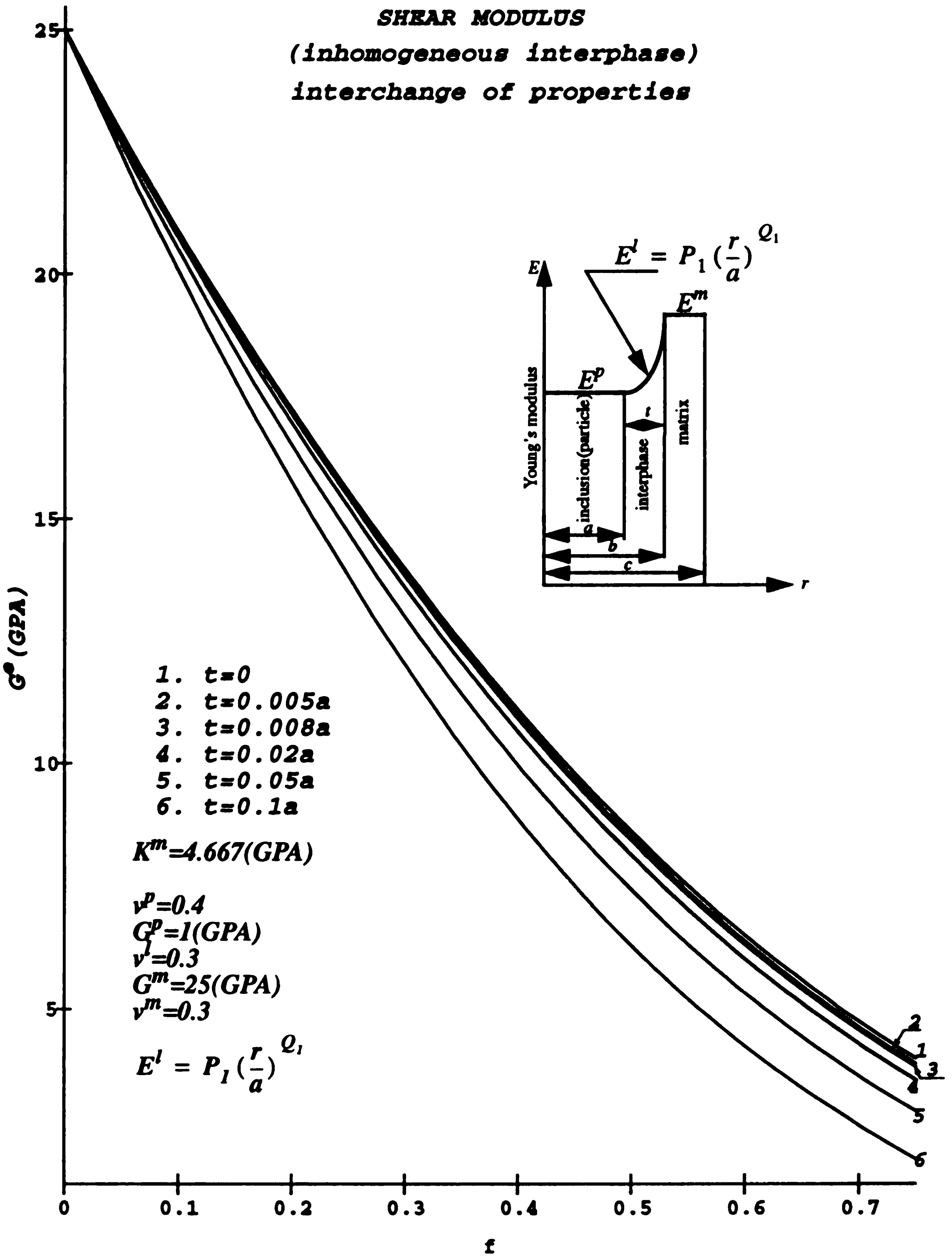

Fig. 3.24 Effective shear modulus vs. *f* for an inhomogeneous interphase with the interchange of properties.

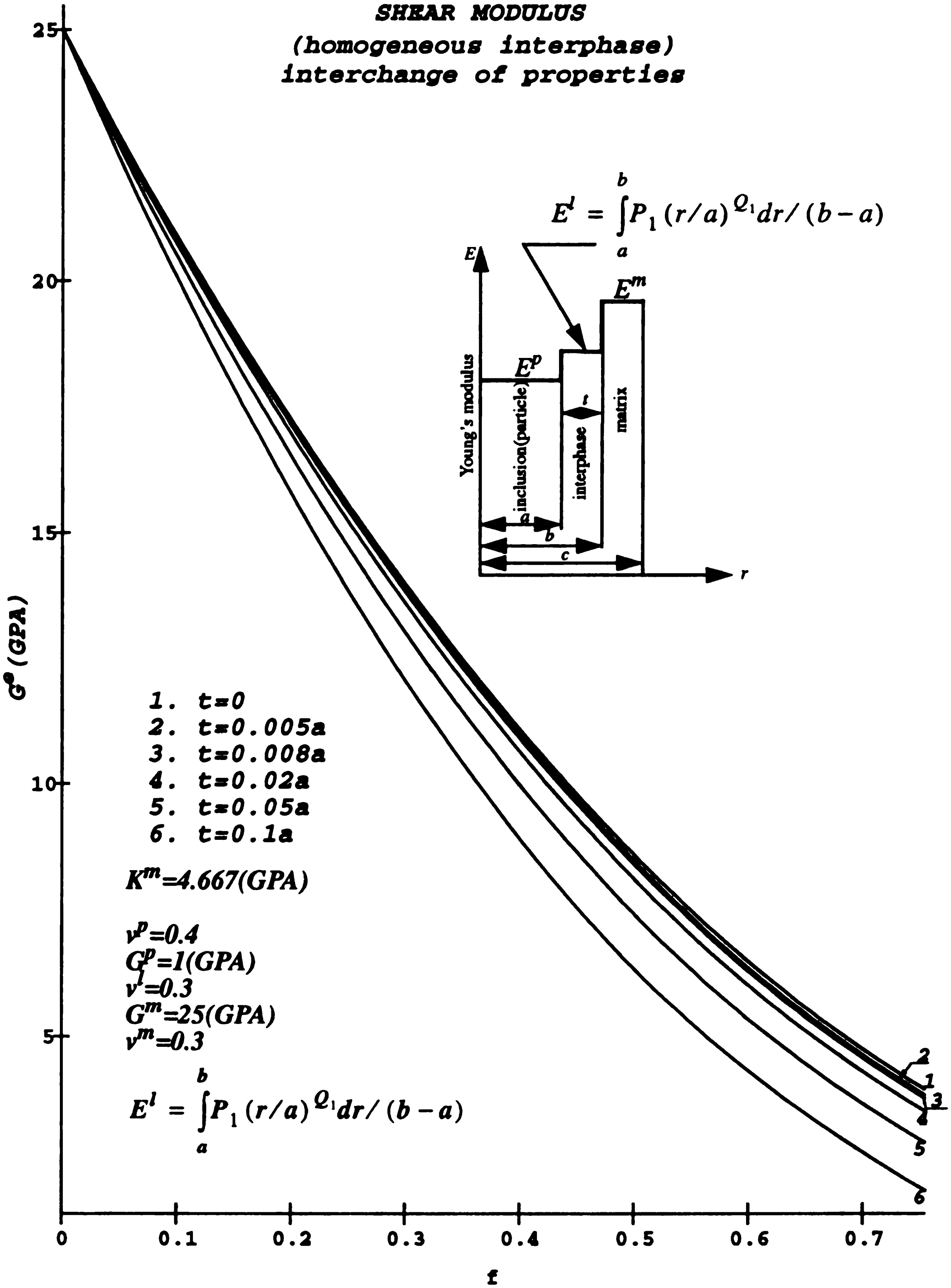

Fig. 3.25 Effective shear modulus vs. f for a homogeneous interphase with the interchange of the properties

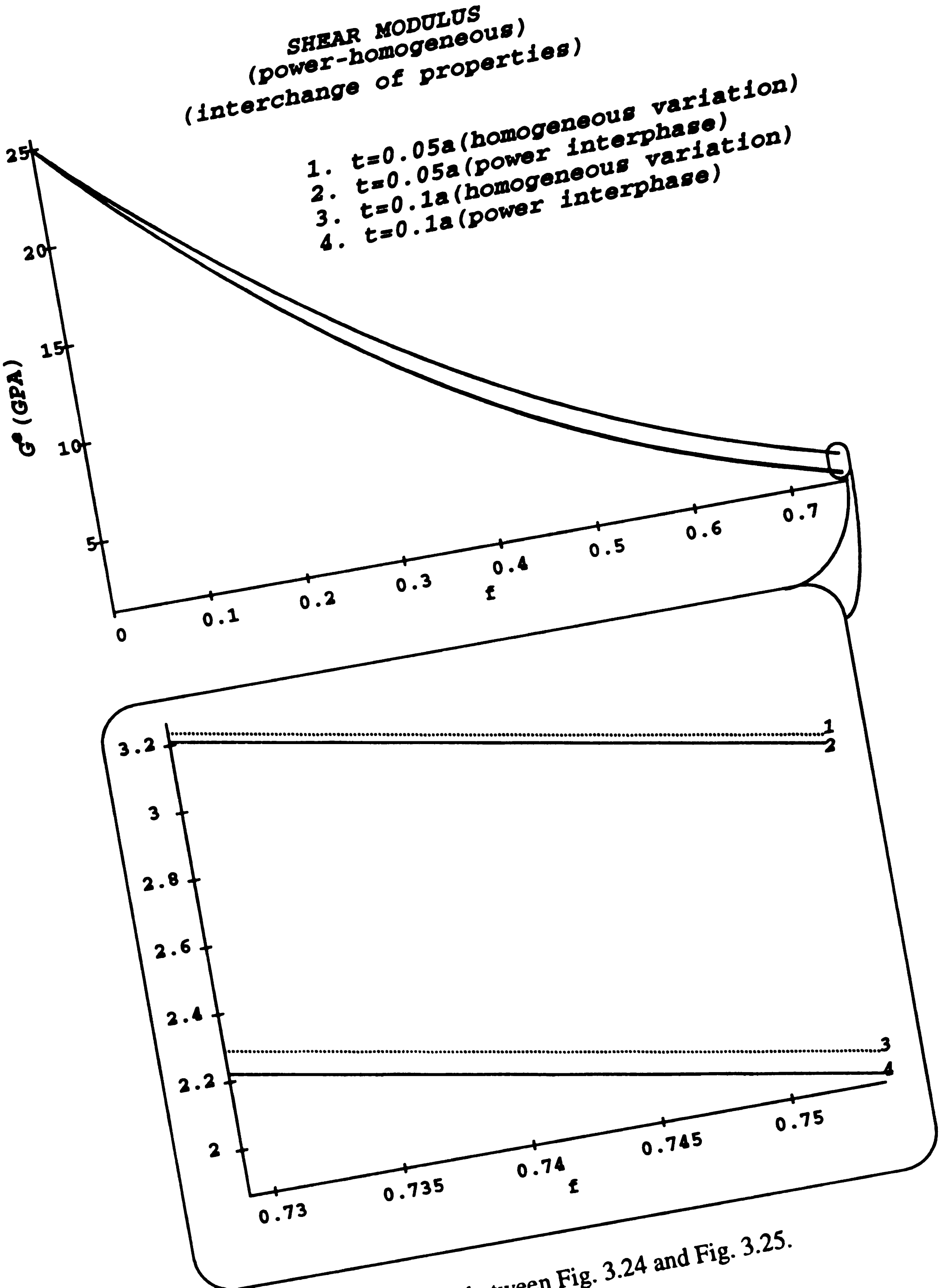

Fig. 3.26 The comparison between Fig. 3.24 and Fig. 3.25.

3.3. THERMAL EXPANSION COEFFICIENT ANALYSIS

We assume that the thermal expansion coefficients in the inclusion, matrix and interphase are follows

$$\begin{aligned} \alpha^p &= 5 \times 10^{-6}/^oC \\ \alpha^m &= 66 \times 10^{-6}/^oC \\ \alpha^I &= M\left(\frac{r}{a}\right)^N \end{aligned} \tag{3.20}$$

where M, N are constants. We can sketch eqn.(3.20) in Fig. 3.27.

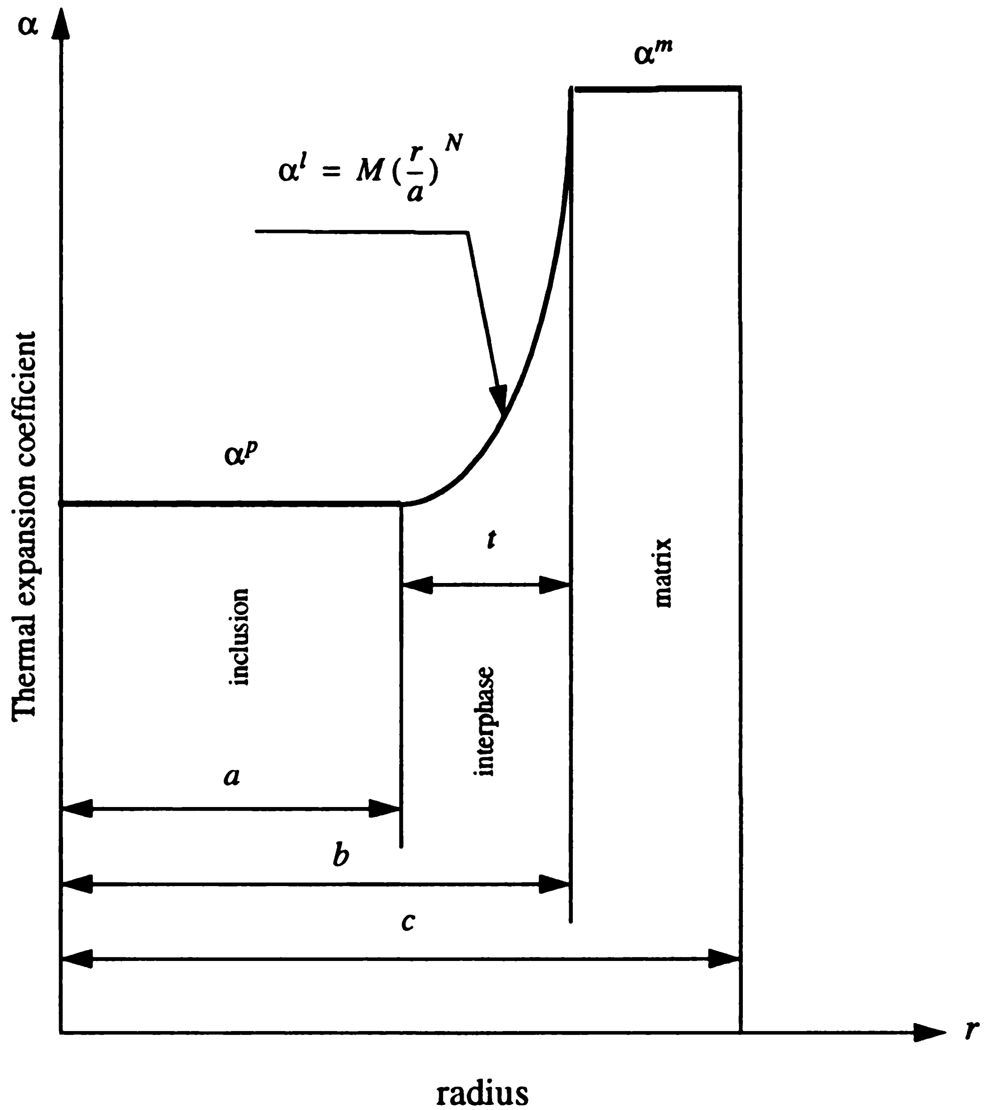

Fig. 3.27 Variation of a thermal expansion coefficient in the interphase.

where

$$M = \alpha^p \tag{3.21}$$

$$N = \frac{ln\left(\frac{\alpha^m}{\alpha^p}\right)}{ln\left(\frac{b}{a}\right)} \tag{3.22}$$

We substitute eqn. (3.20) and (3.4) into the MAPLE computer program (Appendix H), which gives thermal expansion coefficients for the inhomogeneous interphase with a power variation α^l. The results are given in Fig. 3.29.

When we replace $\alpha^l = M(r/a)^N$ with $\alpha^l = [\int_a^b M(r/a)^N dr]/(b-a)$ in eqn. (3.20), we obtain Fig. 3.30. The comparison of the results between Fig. 3.29 and Fig. 3.30 are given in Fig. 3.31. Then we assume that the interphase is homogeneous, and we replace $E^l = P_1(r/a)^{Q_1}$ by $E^l = 34\,(GPA)$ in eqn. (3.4). For power variation α^l and constant α^l cases are illustrated in Fig. 3.32 and Fig. 3.33 respectively. We also compare above two cases in Fig. 3.34, and then, we set $b = a$ and calculate no interphase case in Fig. 3.28. Finally, we assume that E^l has a power variation and ν^l is constant, compare those cases of different variations of α^l in Fig. 3.35. By studying Fig. 3.28 - Fig. 3.33, we find that again the effect of interphase is pronounced and it is higher for the larger volume fraction of particles and for the thicker interphase, and no interphase assumption produces highest values of effective thermal expansion coefficient. The comparison shows that there is not a big difference among different variations of α^l. But, we can see that these results have the same tendency as elastic moduli study with E^l having different variations. The tendency is that the smoother variation of α^l gives the lower estimation of the effective thermal expansion coefficient. In other words, the uniform variation of α^l overestimate the thermal expansion coefficient. We may conclude that the influence of the thickness of interphase is greater than the different variations of thermal expansion coefficient in interphase, which have the same volumetric average value.

We also check the relationship between the thermal expansion coefficient and bulk modulus that is given in Christensen and Lo (1979) for the two phase model.

$$\alpha^e = \alpha^m + \frac{(\alpha^p - \alpha^m)\left(\frac{1}{K^c} - \frac{1}{K^m}\right)}{\frac{1}{K^p} - \frac{1}{K^m}} \tag{3.23}$$

We find that our result for no interphase cases is identical to the eqn. (3.23) [Appendix I]. So we prove that eqn. (3.23) is valid in CSA model. This infers that the similar relation may be obtained to our interphase problem. Here, we introduce two numerical consequences to illustrate the possibility that we may obtain the similar relation. The first consequence is the comparison of effective thermal expansion coefficient from different variations of Young's modulus in the interphase (Fig. 3.36). We compare this result (Fig. 3.36) to the similar result (Fig. 3.14) in bulk modulus analysis, we find that they have the same tendency which is that uniform variation of E^I overestimates both K^e and α^e, and the smoother variation of E^I gives lower estimations. The second consequence is the Comparison of effective thermal expansion coefficient from different variations of Poisson's ratio in the interphase (Fig. 3.37). We also compare Fig3.37 with Fig. 3.15 in bulk modulus study. We note that the same tendency is achieved for both effective bulk modulus and thermal expansion coefficient. When the Poisson's ratio in the interphase has uniform variation, we obtain highest value in K^e and α^e, and the sharper variation of ν^I gives the lower estimation of K^e and α^e. These two numerical consequences strongly support our conjecture that there may be a relation similar to eqn. (3.23) in our interphase problem. But for the mathematical proof, again we leave for the future study.

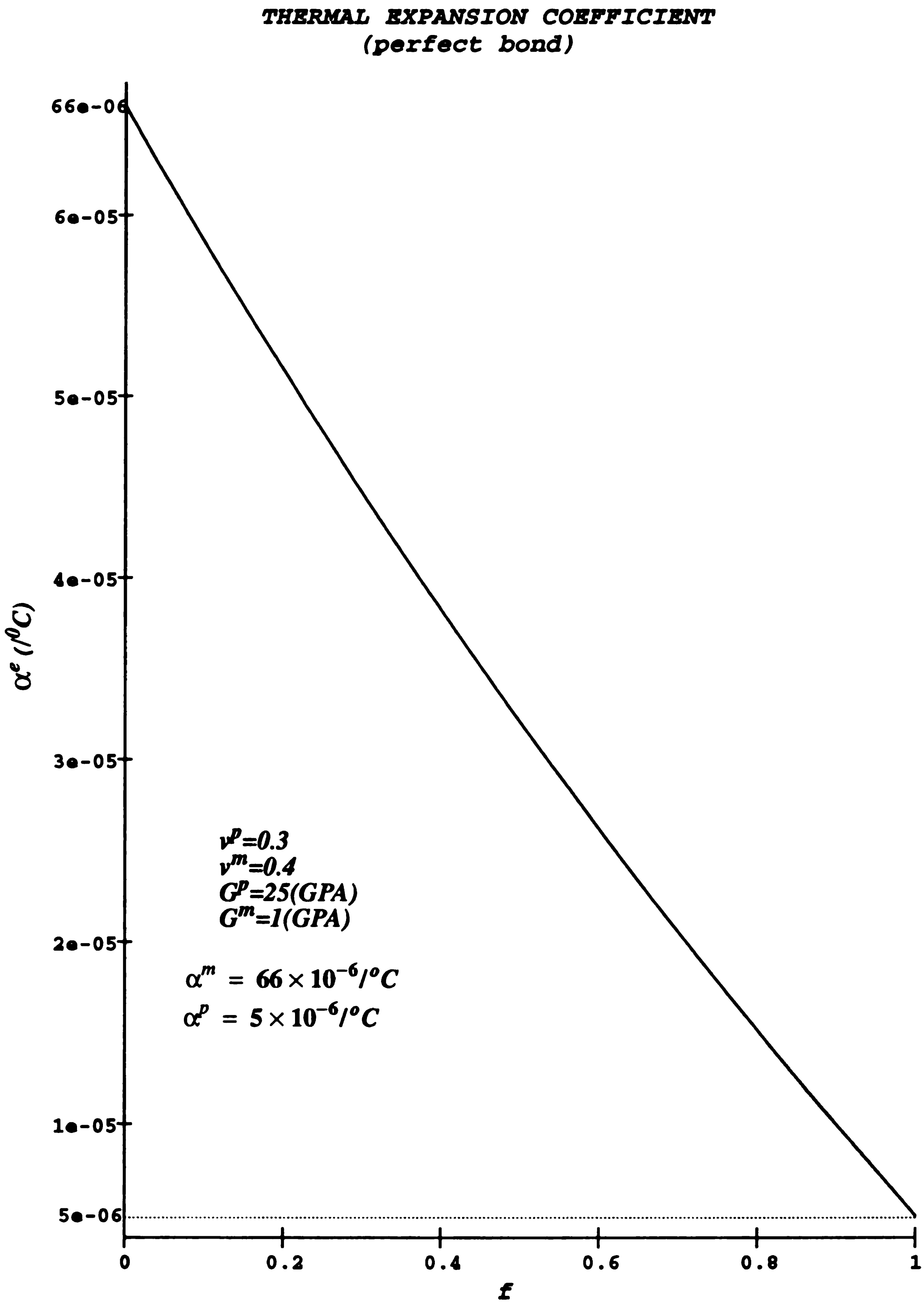

Fig. 3.28 Effective thermal expansion coefficient vs. f for the no interphase case.

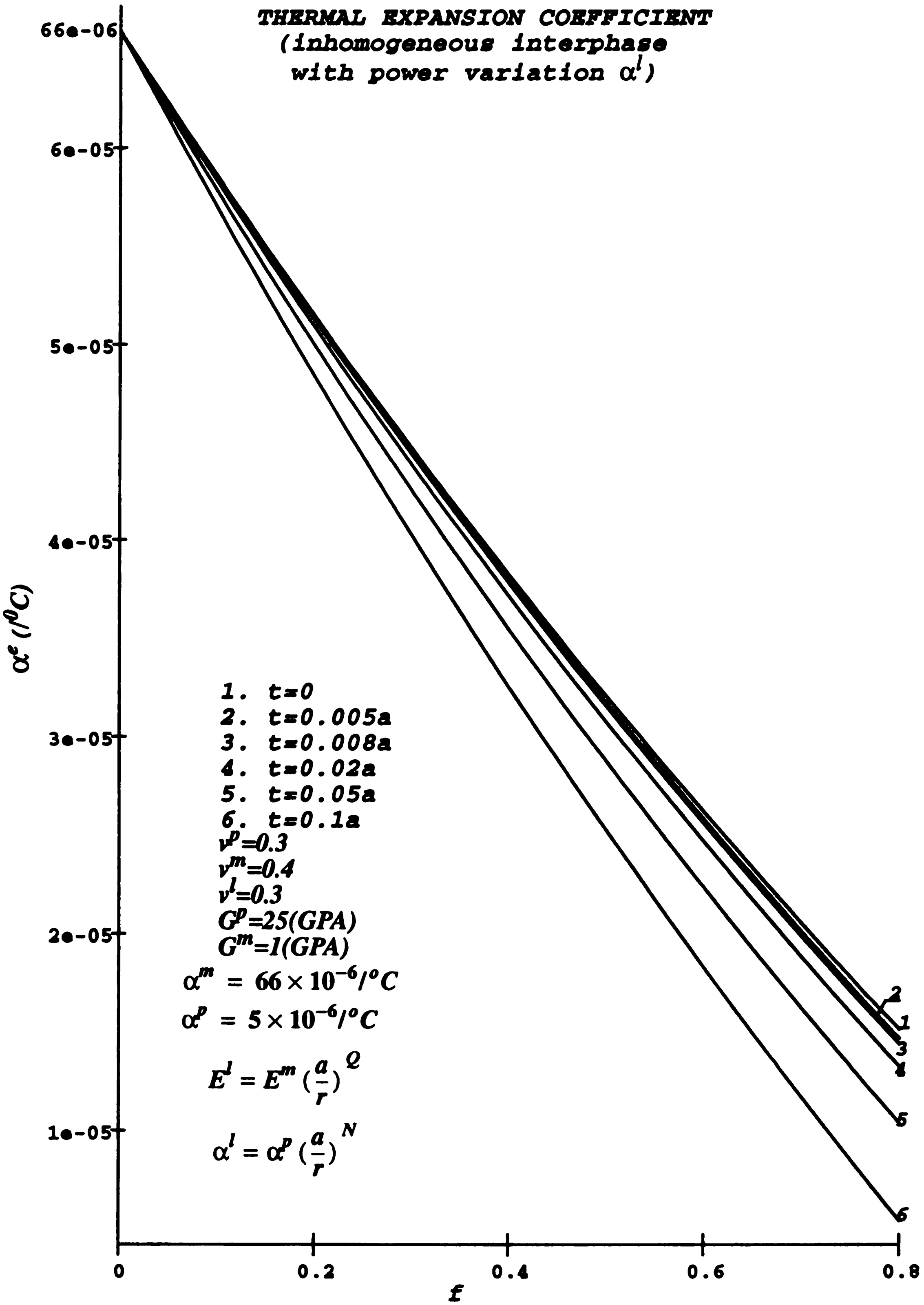

Fig. 3.29 Effective thermal expansion coefficient vs. f for an inhomogeneous interphase with a power variation in the thermal expansion coefficient

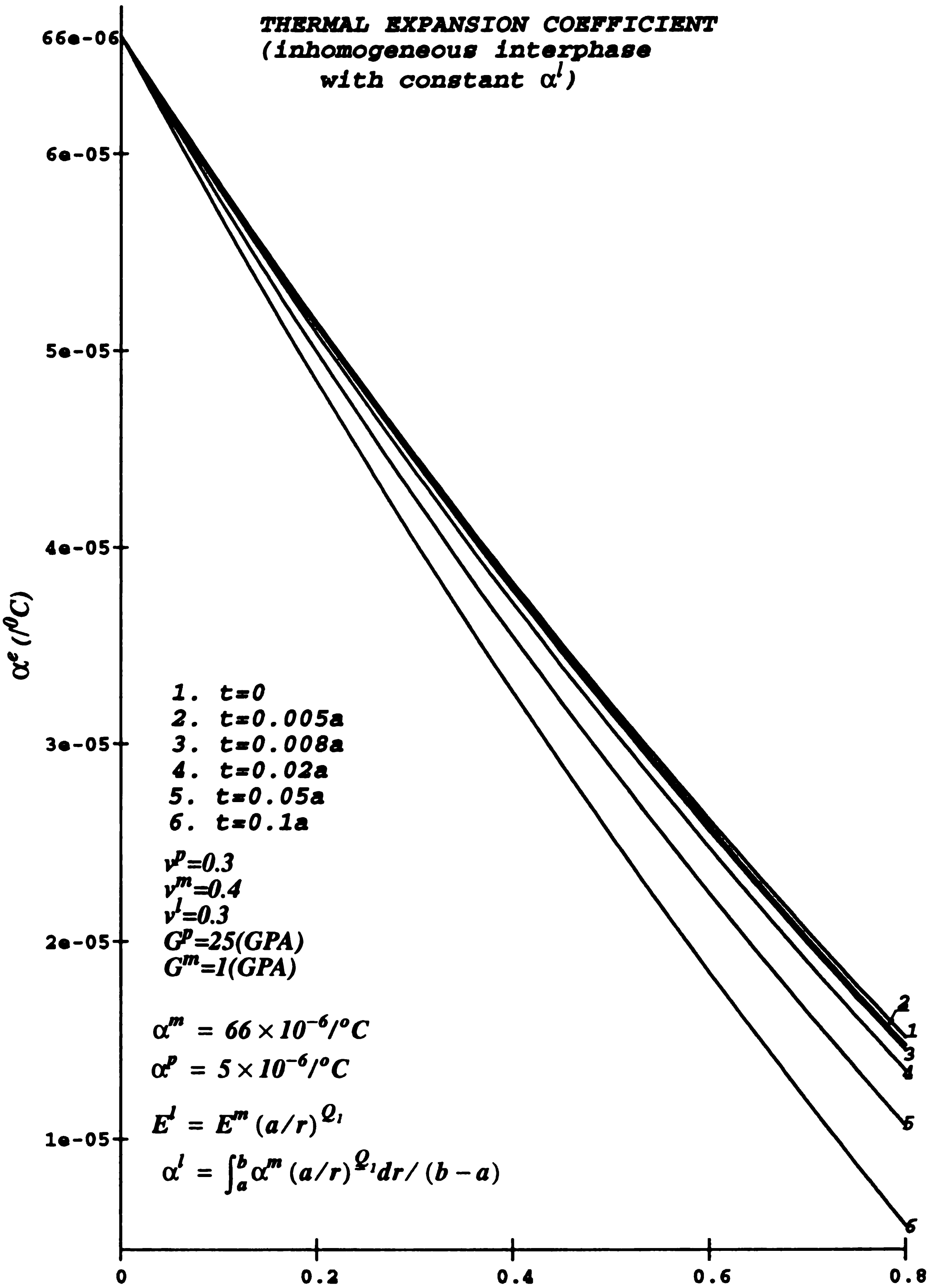

Fig. 3.30 Effective thermal expansion coefficient vs. f for an inhomogeneous interphase with the constant thermal expansion coefficient.

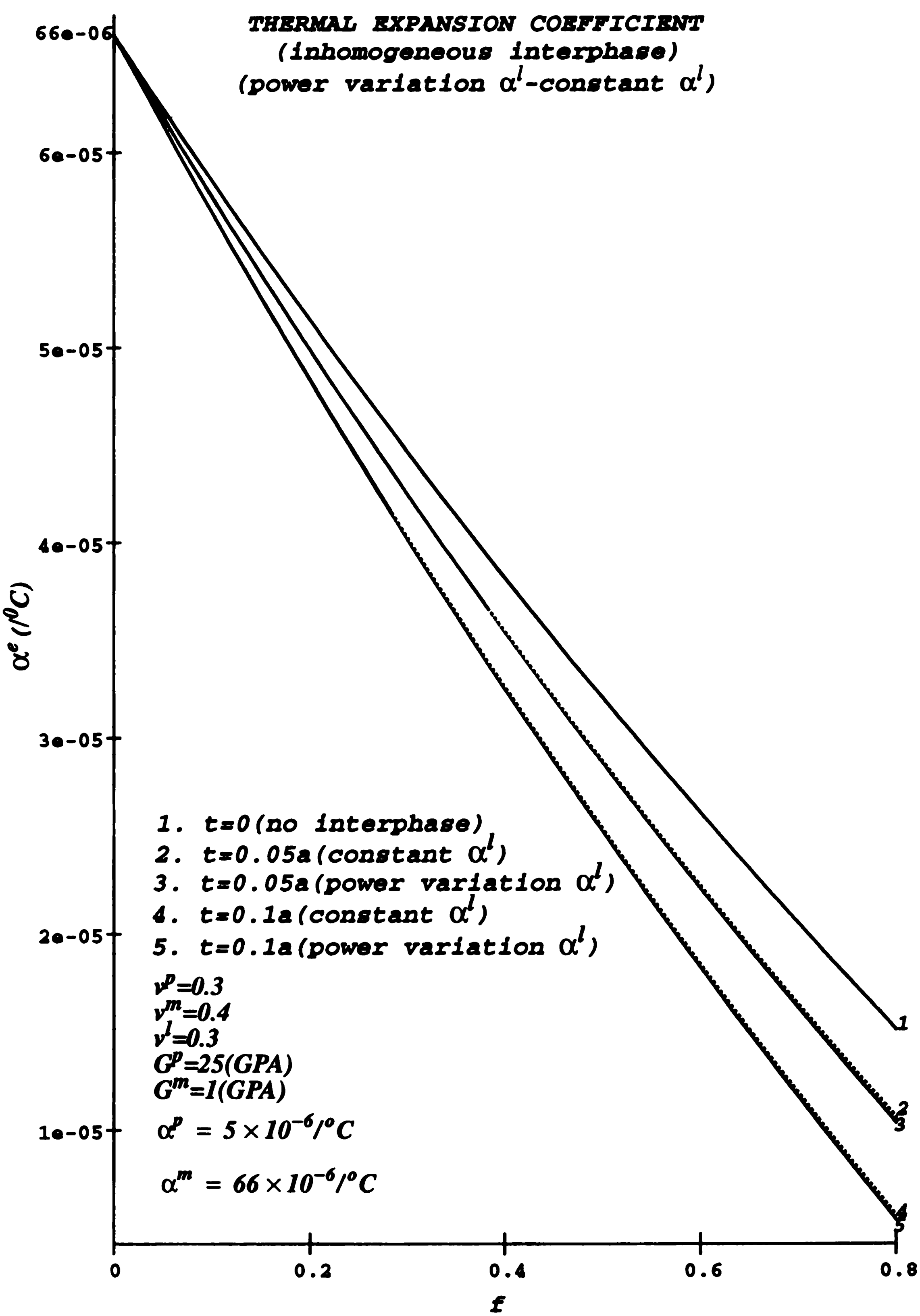

Fig. 3.31 Comparison between Fig. 3.29 and Fig. 3.30.

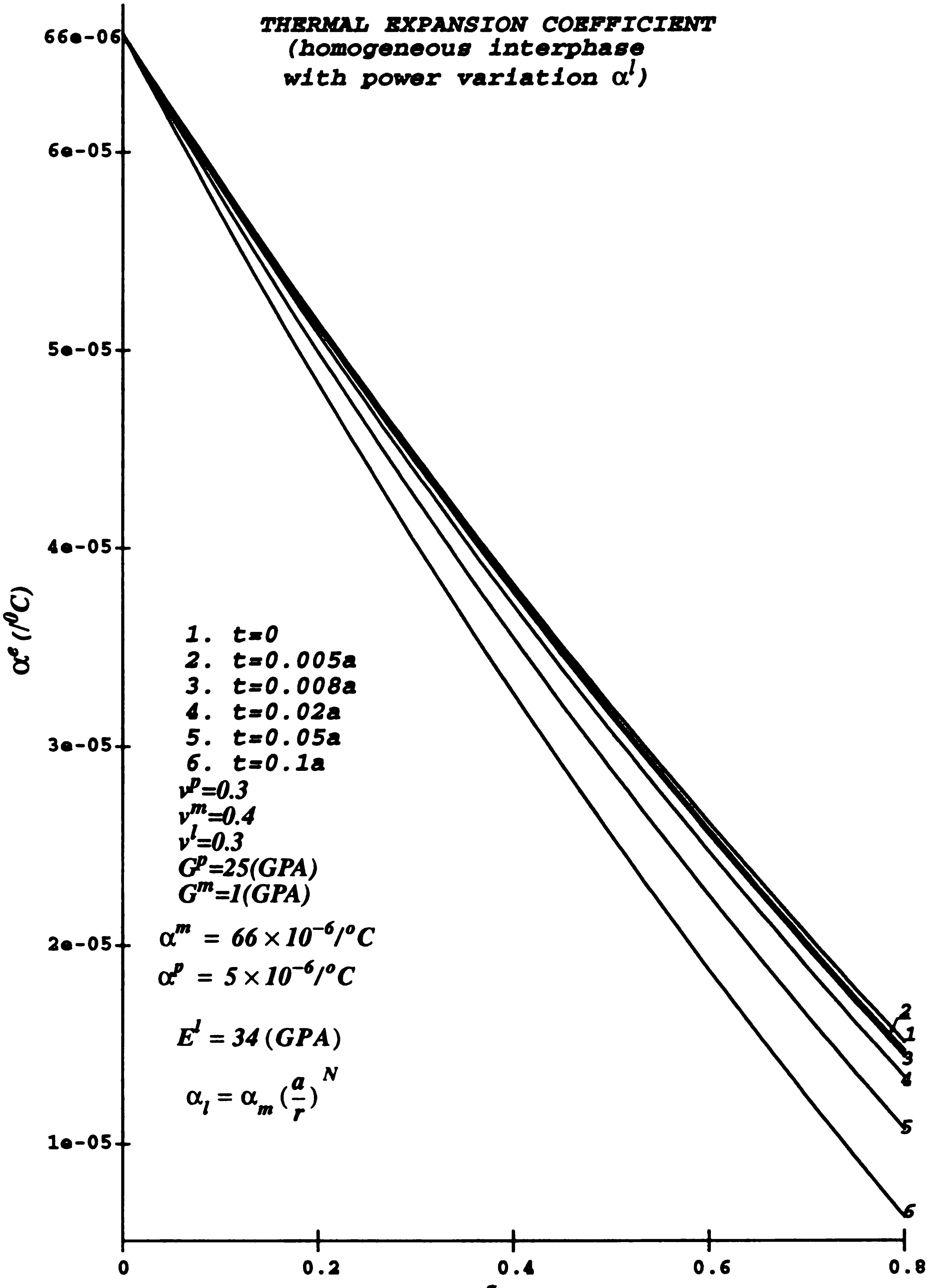

Fig.3.32 Effective thermal expansion coefficient vs. f for a homogeneous interphase having a power variation in the thermal expansion coefficient.

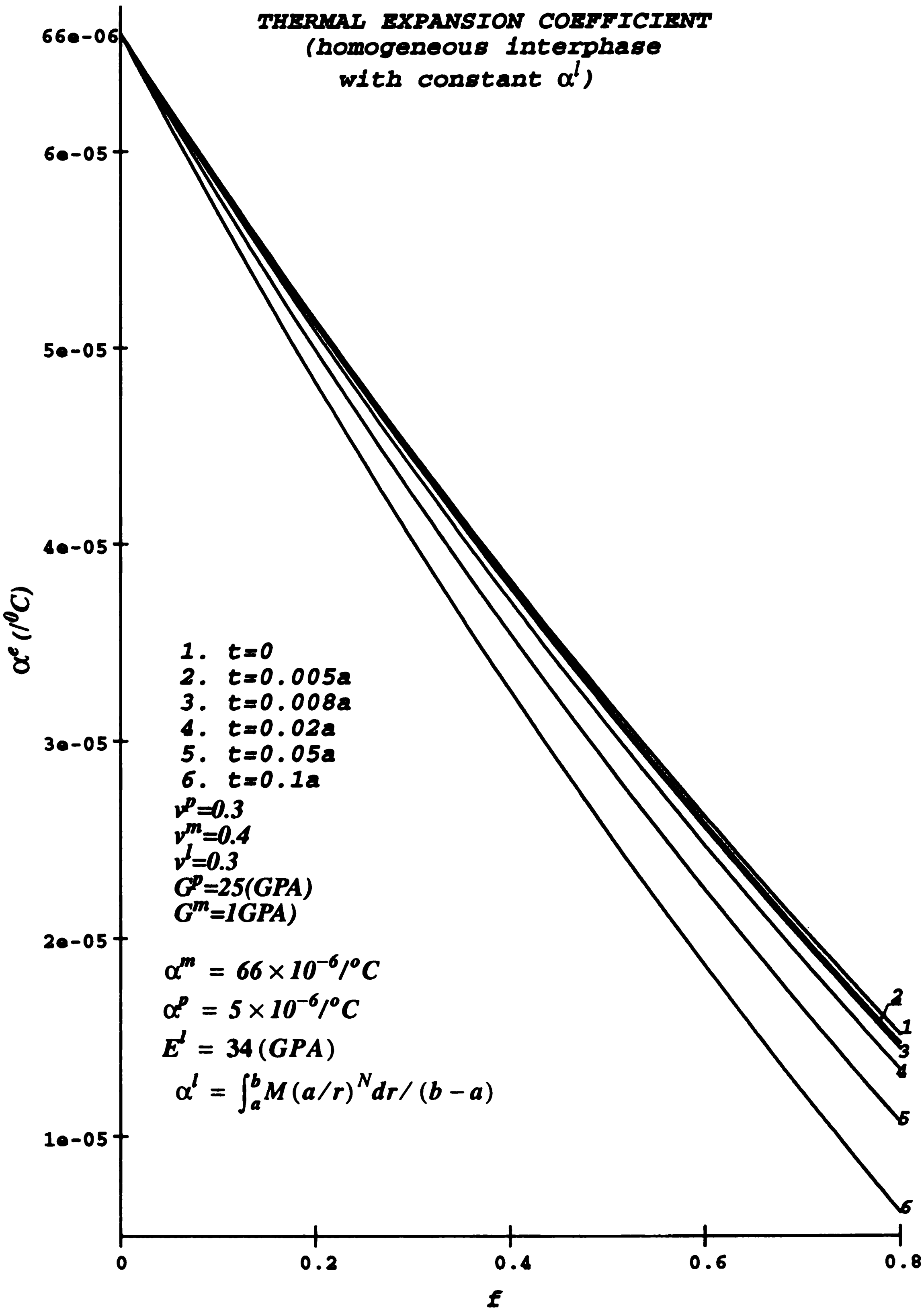

Fig. 3.33 Effective thermal expansion coefficient vs. *f* for a homogeneous interphase having a constant thermal expansion coefficient.

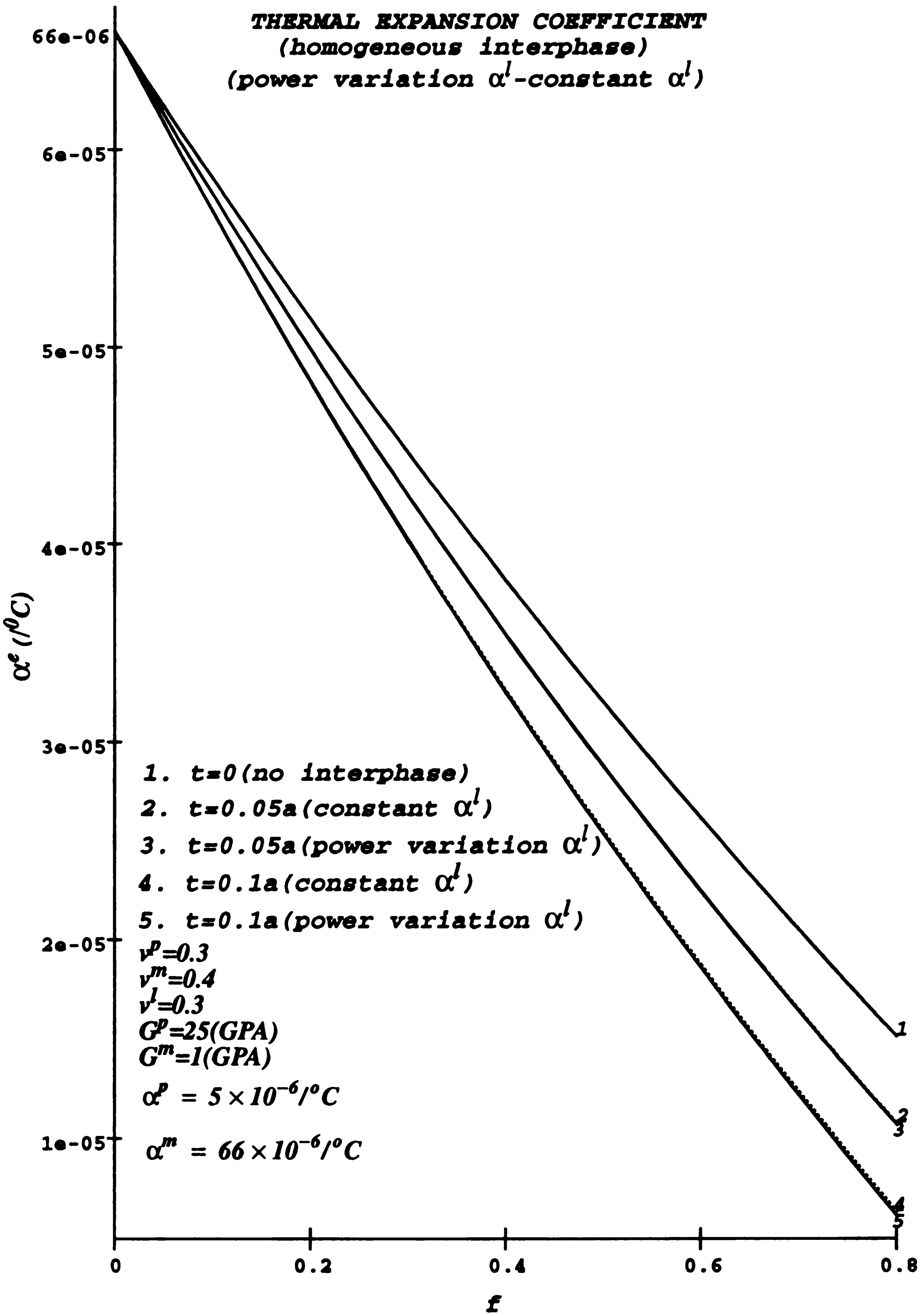

Fig. 3.34 Comparison between Fig. 3.32 and Fig. 3.33

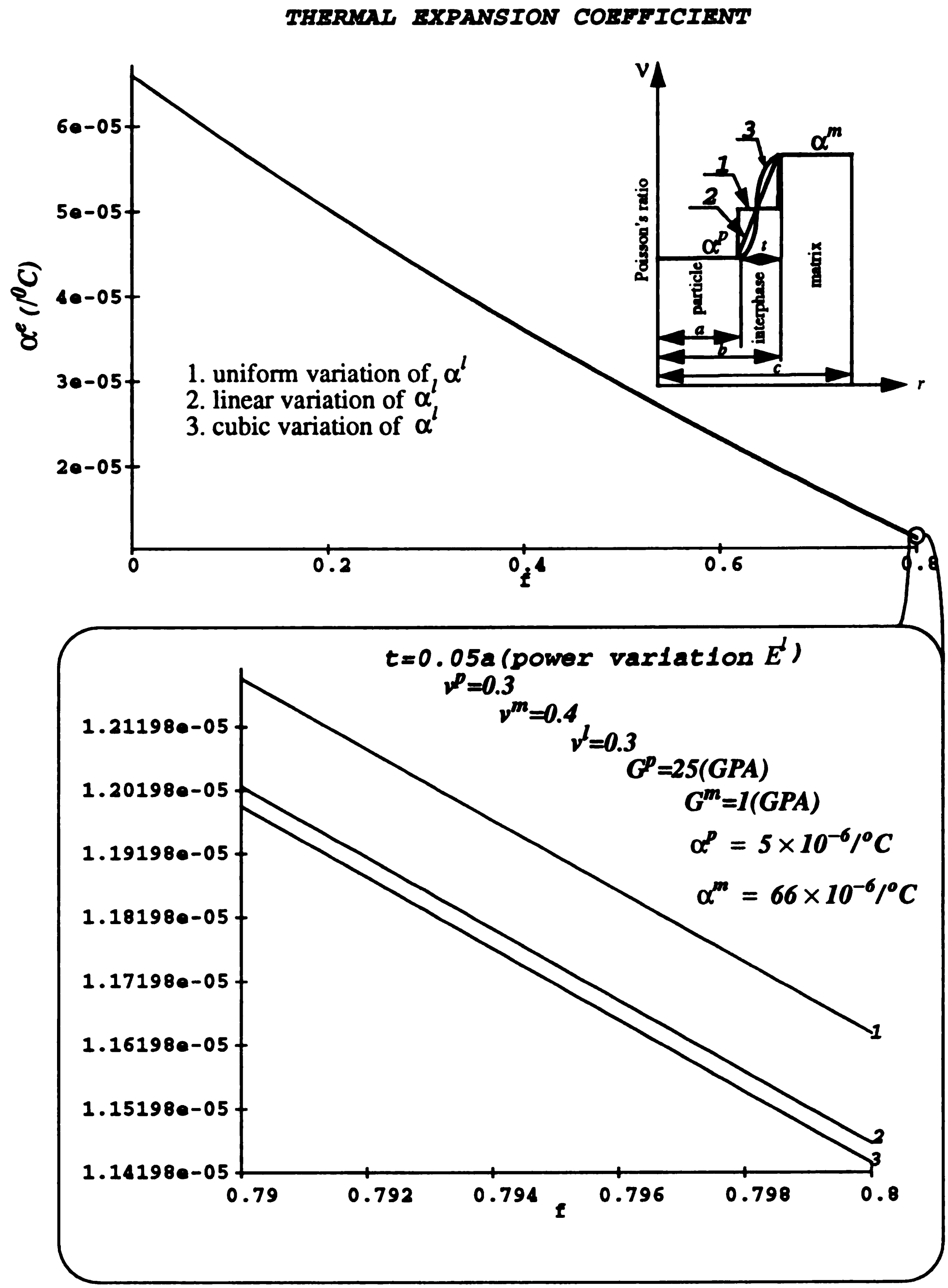

Fig. 3.35 Comparison of effective thermal expansion coefficient for the case of different variations of thermal expansion coefficients in the interphase.

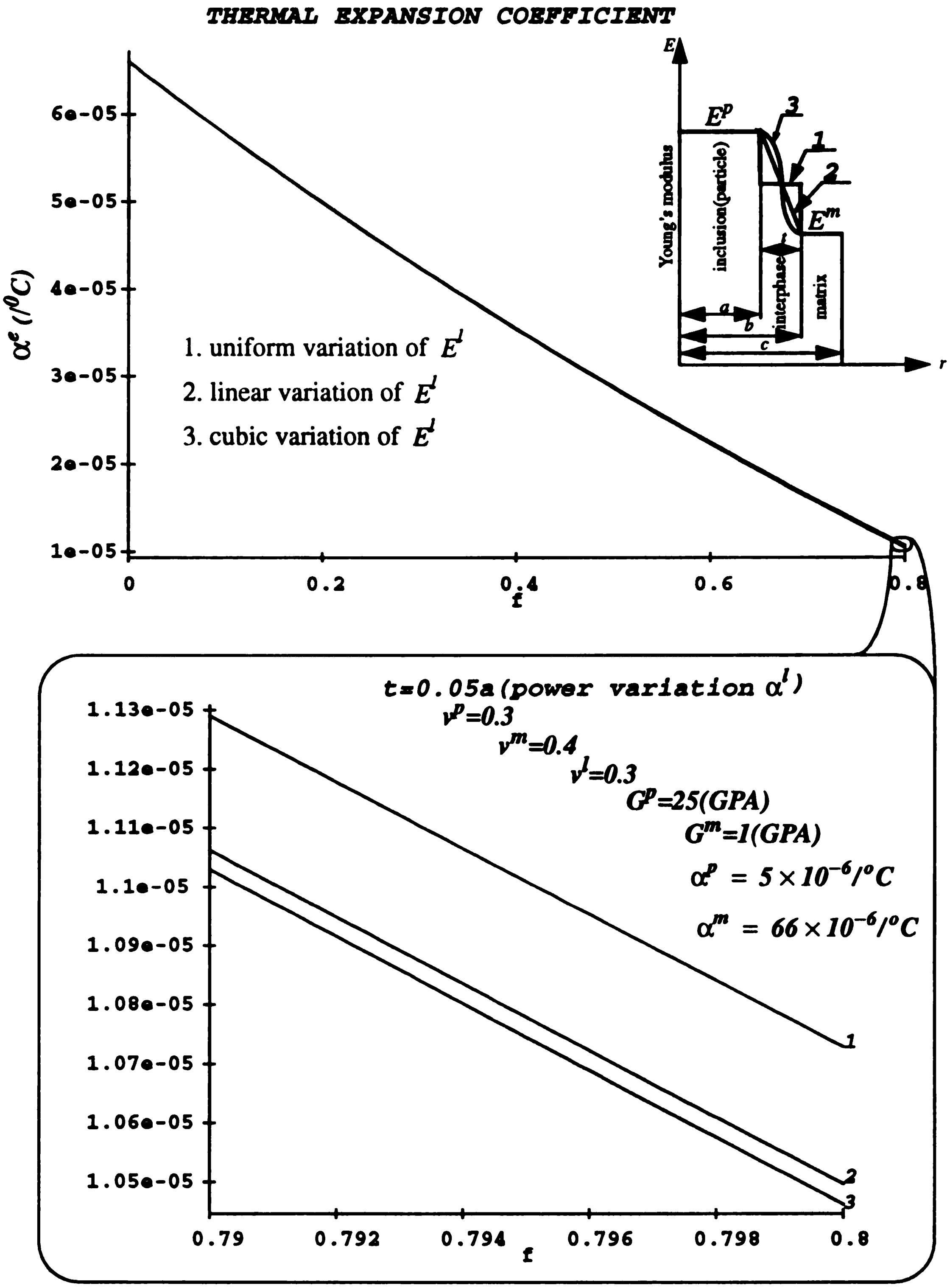

Fig. 3.36 Comparison of effective thermal expansion coefficient for the case of different variations of Young's modulus in the interphase.

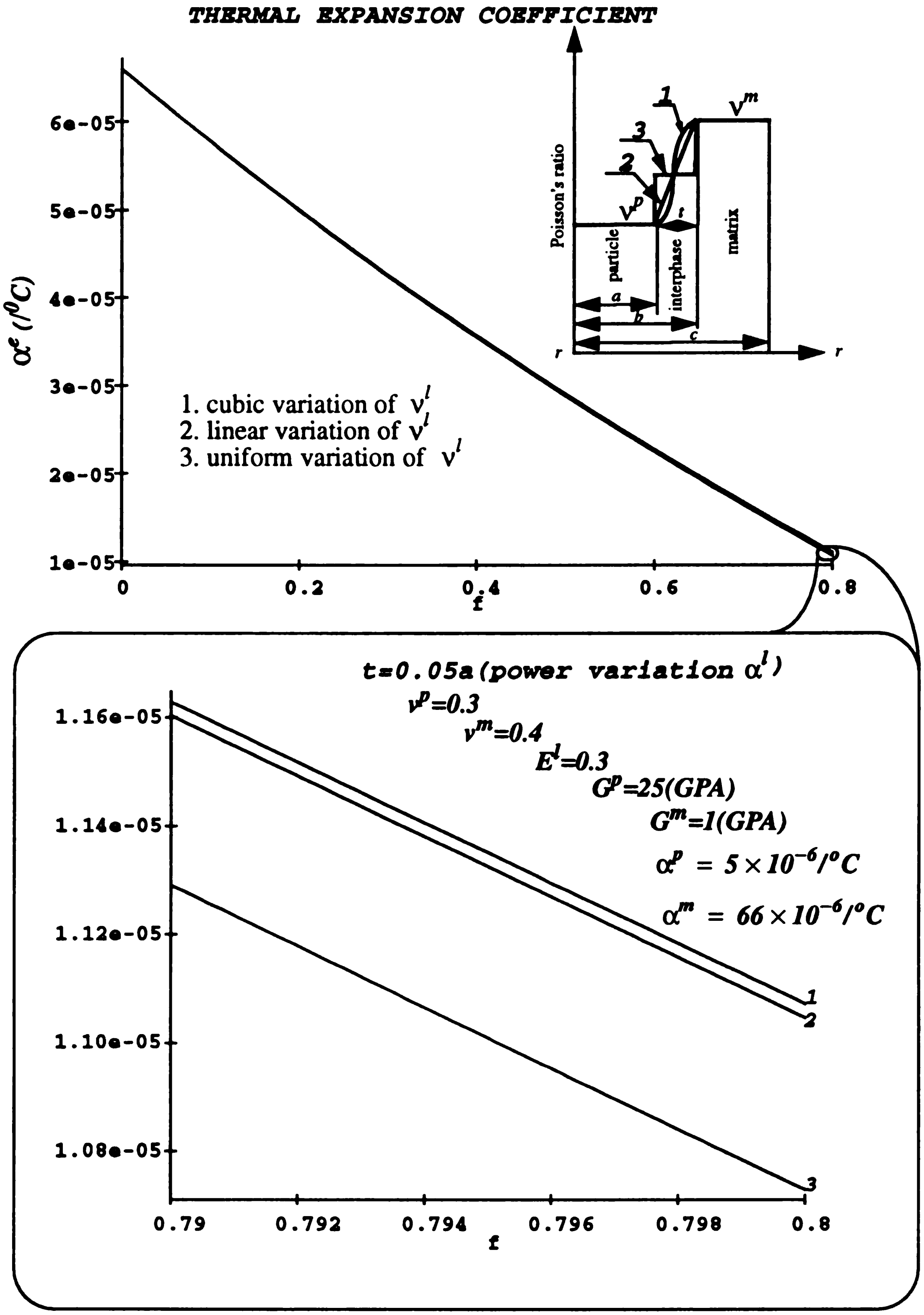

Fig. 3.37 Comparison of effective thermal expansion coefficient for the case of different variations of Poisson's ratio in the interphase.

4.CONCLUSION

We obtained the effective bulk modulus, shear modulus and thermal expansion coefficient of particle reinforced composites with inhomogeneous interphases. We explored the effects of interphases, both inhomogeneous and homogeneous, on the overall elastic properties of composites as well as thermal properties and showed that the study of interphase in composite materials is important. The elastic and thermal properties of interphase strongly influence the overall elastic and thermal properties of composite. Controlling the properties of interphase in order to improve the characteristics of composite is an important topic in composite manufacturing.

In this thesis we consider an idealized interfacial model, in which the interphase is inhomogeneous but isotropic. A more realistic model is the one that accounts for the local anisotropy and randomness of the constitutive law of interphase. This is being studied by Ostoja-Starzewski and Jasiuk (1992), Jasiuk and Ostoja-Starzewski (1993) and Ostoja-Starzewski et al. (1994).

APPENDICES

5.APPENDICES

NOMENCLATURE

u[r]=$u_{[r]}$ .... u_ru

[theta]=$u_{[\theta]}$ .... u_θ

u[phi]=$u_{[\phi]}$ .... u_φ

U[r]=$U_{[r]}$ .... U_r

U[theta]=$U_{[\theta]}$ .... U_θ

U[phi]=$U_{[\phi]}$ .... U_φ

epsilon[rr]=$\varepsilon_{[rr]}$ .... ε_{rr}

epsilon[theta,theta]=$\varepsilon_{[\theta,\theta]}$ .... $\varepsilon_{\theta\theta}$

epsilon[phi,phi]=$\varepsilon_{[\phi,\phi]}$ .... $\varepsilon_{\varphi\varphi}$

epsilon[r,theta]=$\varepsilon_{[r,\theta]}$ .... $\varepsilon_{r\theta}$

epsilon[theta,phi]=$\varepsilon_{[\theta,\phi]}$ .... $\varepsilon_{\theta\varphi}$

epsilon[r,phi]=$\varepsilon_{[r,\phi]}$ .... $\varepsilon_{r\varphi}$

sigma[rr]=$\sigma_{[rr]}$ .... σ_{rr}

sigma[theta,theta]=$\sigma_{[\theta,\theta]}$ .... $\sigma_{\theta\theta}$

sigma[phi,phi]=$\sigma_{[\phi,\phi]}$ .... $\sigma_{\varphi\varphi}$

tau[r,theta]=$\tau_{[r,\theta]}$ .... $\tau_{r\theta}$

tau[theta,phi]=$\tau_{[\theta,\phi]}$ .... $\tau_{\theta\varphi}$

tau[r,phi]=$\tau_{[r,\phi]}$ .... $\tau_{r\varphi}$

b[1],b[2],b[3],b[4]=$b_{[1]}, b_{[2]}, b_{[3]}, b_{[4]}$ .... b_1, b_2, b_3, b_4

D[1],D[2],D[3],D[4]=$D_{[1]}, D_{[2]}, D_{[3]}, D_{[4]}$ .... $D_1^l, D_2^l, D_3^l, D_4^l$

m[1],m[2],m[3],m[4]=$m_{[1]}, m_{[2]}, m_{[3]}, m_{[4]}$	m_1, m_2, m_3, m_4
n[1],n[2],n[3],n[4]=$n_{[1]}, n_{[2]}, n_{[3]}, n_{[4]}$	n_1, n_2, n_3, n_4
P,Q	P_1, Q_1
nu(p),nu(l),nu(m)	ν^p, ν^l, ν^m
G(p),G(m)	G^p, G^m
D[p1],D[p2]	D_1^p, D_2^p
D[l1],D[l2],D[l3],D[l4]	$D_1^l, D_2^l, D_3^l, D_4^l$
D[m1],D[m2],D[m3],D[m4]	$D_1^m, D_2^m, D_3^m, D_4^m$
D[e1],D[e3],D[e4]	D_1^e, D_3^e, D_4^e
E(p),E(m)	E^p, E^m
alpha(p),alpha(m)	α^p, α^m
C[pt1]	C_{t1}^p
C[lt1],C[lt2]	C_{t1}^l, C_{t2}^l
C[mt1],C[mt2]	C_{t1}^m, C_{t2}^m

APPENDIX A

This computer program generates the eqn. (2.11) in bulk modulus the study.

```
> u[r](r,theta,phi):=u[r](r);
```

$$u_{[r]}(r, \theta, \phi) := u_{[r]}(r)$$

```
> u[theta](r,theta,phi):=0;
```

$$u_{[\theta]}(r, \theta, \phi) := 0$$

```
> u[phi](r,theta,phi):=0;
```

$$u_{[\phi]}(r, \theta, \phi) := 0$$

```
> epsilon[rr]:=diff(u[r](r,theta,phi),r);
```

$$\varepsilon_{[rr]} := \frac{\partial}{\partial r} u_{[r]}(r)$$

```
> epsilon[theta,theta]:=1/(r*sin(phi))*diff(u[theta](r,theta,phi),theta)+u[r](r,theta,p
  hi)/r+(cot(phi)/r)*u[phi](r,theta,phi);
```

$$\varepsilon_{[\theta,\theta]} := \frac{u_{[r]}(r)}{r}$$

```
> epsilon[phi,phi]:=diff(u[phi](r,theta,phi),phi)/r+u[r](r,theta,phi)/r;
```

$$\varepsilon_{[\phi,\phi]} := \frac{u_{[r]}(r)}{r}$$

```
> gamma[r, theta]:=1/(r*sin(phi))*diff(u[r](r,theta,phi),theta)+diff(u[theta](r,theta,
  phi),r)-u[theta](r,theta,phi)/r;
```

$$\gamma_{[r,\theta]} := 0$$

```
>  gamma[theta,phi]:=diff(u[theta](r,theta,phi),phi)/r-cot(phi)*u[theta](r,theta,phi)/
  r+1/(r*sin(phi))*diff(u[phi](r,theta,phi),theta);
```

$$\gamma_{[\theta,\phi]} := 0$$

```
> gamma[r,phi]:=diff(u[phi](r,theta,phi),r)-u[phi](r,theta,phi)/r+diff(u[r](r,theta,phi)
  ,phi)/r;
```

$$\gamma_{[r,\phi]} := 0$$

```
> e:=epsilon[rr]+epsilon[theta,theta]+epsilon[phi,phi];
```

$$e := \left(\frac{\partial}{\partial r} u_{[r]}(r)\right) + 2\,\frac{u_{[r]}(r)}{r}$$

```
> sigma[rr]:=E(r)/(1+nu(r))*(epsilon[rr]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[rr]} := \frac{\mathrm{E}(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+\frac{\nu(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+2\,\frac{u_{[r]}(r)}{r}\right)}{1-2\,\nu(r)}\right)}{1+\nu(r)}$$

```
> sigma[theta,theta]:=E(r)/(1+nu(r))*(epsilon[theta,theta]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[\theta,\theta]} := \frac{\mathrm{E}(r)\left(\frac{u_{[r]}(r)}{r}+\frac{\nu(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+2\,\frac{u_{[r]}(r)}{r}\right)}{1-2\,\nu(r)}\right)}{1+\nu(r)}$$

```
> sigma[phi,phi]:=E(r)/(1+nu(r))*(epsilon[phi,phi]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[\phi,\phi]} := \frac{\mathrm{E}(r)\left(\frac{u_{[r]}(r)}{r}+\frac{\nu(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+2\,\frac{u_{[r]}(r)}{r}\right)}{1-2\,\nu(r)}\right)}{1+\nu(r)}$$

```
> tau[r,theta]:=E(r)/2/(1+nu(r))*gamma[r,theta];
```

$$\tau_{[r,\theta]} := 0$$

```
> tau[theta,phi]:=E(r)/2/(1+nu(r))*gamma[theta,phi];
```

$$\tau_{[\theta,\phi]} := 0$$

```
> tau[r,phi]:=E(r)/2/(1+nu(r))*gamma[r,phi];
```

$$\tau_{[r,\phi]} := 0$$

```
> equi1:=diff(sigma[rr],r)*sin(phi)+1/r*(2*sigma[rr]*sin(phi)-sigma[theta,theta]*si
  n(phi)-sigma[phi,phi]*sin(phi)+tau[r,phi]*cos(phi)+diff(tau[r,theta],theta)+diff(ta
  u[r,phi],phi)*sin(phi))=0;
```

$$equi1 := \left(\frac{\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+\frac{\nu(r)\,\%1}{1-2\,\nu(r)}\right)}{1+\nu(r)}\right.$$

$$-\frac{\mathrm{E}(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+\frac{\nu(r)\,\%1}{1-2\,\nu(r)}\right)\left(\frac{\partial}{\partial r}\nu(r)\right)}{(1+\nu(r))^2}+\mathrm{E}(r)\left(\left(\frac{\partial^2}{\partial r^2}u_{[r]}(r)\right)\right.$$

$$+\frac{\left(\frac{\partial}{\partial r}\nu(r)\right)\%1}{1-2\nu(r)}+2\frac{\nu(r)\,\%1\left(\frac{\partial}{\partial r}\nu(r)\right)}{(1-2\nu(r))^2}$$

$$+\frac{\nu(r)\left(\left(\frac{\partial^2}{\partial r^2}u_{[r]}(r)\right)+2\frac{\frac{\partial}{\partial r}u_{[r]}(r)}{r}-2\frac{u_{[r]}(r)}{r^2}\right)}{1-2\nu(r)}\Bigg)\Bigg/(1+\nu(r))\Bigg)\sin(\phi)+\Bigg($$

$$2\frac{\mathrm{E}(r)\left(\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+\frac{\nu(r)\,\%1}{1-2\nu(r)}\right)\sin(\phi)}{1+\nu(r)}$$

$$-2\frac{\mathrm{E}(r)\left(\frac{u_{[r]}(r)}{r}+\frac{\nu(r)\,\%1}{1-2\nu(r)}\right)\sin(\phi)}{1+\nu(r)}\Bigg)\Bigg/r=0$$

$$\%1:=\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+2\frac{u_{[r]}(r)}{r}$$

```
> eqn1:=collect(equi1,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor):
> eqn2:=normal(eqn1*(-1+2*nu(r))/sin(phi)):
> eqn3:=collect(eqn2,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor):
> eqn4:=normal(eqn3*(-1+2*nu(r))*(1+nu(r))^2*r^2):
> eqn5:=collect(eqn4,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor);
```

$$eqn5:=\mathrm{E}(r)\,r^2\,(-1+\nu(r))\,(-1+2\,\nu(r))\,(1+\nu(r))\left(\frac{\partial^2}{\partial r^2}u_{[r]}(r)\right)+r\Bigg(\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r$$

$$-2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)-\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^2+2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^3$$

$$+4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r)-2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r)^2-4\,\mathrm{E}(r)\,\nu(r)$$

$$-2\,\mathrm{E}(r)\,\nu(r)^2+4\,\mathrm{E}(r)\,\nu(r)^3+2\,\mathrm{E}(r)\Bigg)\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)+\Bigg(2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)$$

$$-2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^2-4\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^3+4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r)^2$$

$$+2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r+4\,\mathrm{E}(r)\,\nu(r)+2\,\mathrm{E}(r)\,\nu(r)^2-4\,\mathrm{E}(r)\,\nu(r)^3-2\,\mathrm{E}(r)\Bigg)$$

$$u_{[r]}(r)=0$$

APPENDIX B

Here, we discuss the condition of existence of convergent series solution. The first concept we shall need is that of an analytic function: A function f(r) is said to be analytic at a point $r = r_0$ if and only if it has a Taylor series at $r = r_0$ which represents the function in some neighborhood of $r = r_0$. In particular, polynomial functions are analytic everywhere, and rational functions are analytic at all points where they are defined, i.e., at all points except the zeros in denominators. Using the concept of an analytic function, we can classify the point r_0 about which we seek a solution of eqn. (2.16), here we just consider the general form of (2.16) and neglect the physical meaning of $g(r)$ and $f(r)$. If the coefficient functions $f(r)$ and $g(r)$ in eqn. (2.16) are both analytic at $r = r_0$, the r_0 is said to be an ordinary point of the equation.

If at least one of the functions $f(r)$ and $g(r)$ is not analytic at $r = r_0$ but if the functions defined by the products $(r-r_0)f(r)$ and $(r-r_0)^2 g(r)$ are analytic at $r = r_0$, then r_0 is said to be a regular singular point of the equation.

If at least one of the products $(r-r_0)f(r)$ and $(r-r_0)^2 g(r)$ is not analytic at $r = r_0$ then r_0 is said to be an irregular singular point of equation.

If r_0 is an ordinary point of (2.16), then a convergent series solution exists of the form $u_r = \sum_{n=0}^{\infty} a_n (r-r_0)^n$, if r_0 is regular singular point, then convergent series solution exists of the form

$$u_r = \sum_{n=0}^{\infty} a_n (r-r_0)^{n+k} \tag{B.1}$$

If r_0 is an irregular point, there are in general no solutions with expansion consisting solely of powers of $r-r_0$. The method is known as the Frobenius Method and the conditions of

existence of the series solution are called the Fuchs' theorem.

Even though the conditions of Fuchs' theorem are satisfied, it may not be possible to obtain two in dependent solutions.

Let us suppose now that r_0 is a regular singular point of eqn. (2.16). This means that both $(r-r_0)f(r)$ and $(r-r_0)^2 g(r)$ are analytic at r_0 and can be therefore written in the form

$$\begin{aligned} (r-r_0)f(r) &= p_0 + p_1(r-r_0) + p_2(r-r_0)^2 + \ldots \\ (r-r_0)\,g(r) &= q_0 + q_1(r-r_0) + q_2(r-r_0)^2 + \ldots \end{aligned} \qquad \text{(B.2)}$$

By substituting eqn. (B.1) and eqn. (B.2) into eqn. (2.16), we obtain the equation $k^2 + (p_0 - 1)\,k + q_0 = 0$. This quadratic equation is known as the indicial equation of eqn. (2.16). If the roots of this indicial equation differ by an integer, this process yields only one solution [Johnson and Johnson (1982), p. 47). However, a second independent solution can be found by assuming $u_r = \phi(r)\,u_{r1}(r)$, where u_{r1} is the first series solution, and then determining $\phi(r)$ so that the product $\phi(r)\,u_{r1}(r)$ will satisfy the given differential equation. By using MAPLE (1992) symbolic manipulation program, we can solve eqn (2.16) in the series solution using Fuchs' method.

APPENDIX C

This computer program generates the eqns. (2.38), (2.39) and (2.40) in the shear modulus study.

```
> u[r](r,theta,phi):=U[r](r)*(sin(phi))^2*sin(2*theta);
```

$$u_{[r]}(r,\theta,\phi) := U_{[r]}(r)\sin(\phi)^2\sin(2\theta)$$

```
> u[theta](r,theta,phi):=U[theta](r)*sin(phi)*cos(2*theta);
```

$$u_{[\theta]}(r,\theta,\phi) := U_{[\theta]}(r)\sin(\phi)\cos(2\theta)$$

```
> u[phi](r,theta,phi):=U[phi](r)*sin(2*phi)*sin(2*theta);
```

$$u_{[\phi]}(r,\theta,\phi) := U_{[\phi]}(r)\sin(2\phi)\sin(2\theta)$$

```
> epsilon[rr]:=diff(u[r](r,theta,phi),r);
```

$$\varepsilon_{[rr]} := \left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\theta)$$

```
> epsilon[theta,theta]:=1/(r*sin(phi))*diff(u[theta](r,theta,phi),theta)+u[r](r,theta,p
  hi)/r+(cot(phi)/r)*u[phi](r,theta,phi);
```

$$\varepsilon_{[\theta,\theta]} := -2\frac{U_{[\theta]}(r)\sin(2\theta)}{r} + \frac{U_{[r]}(r)\sin(\phi)^2\sin(2\theta)}{r} + \frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\phi)\sin(2\theta)}{r}$$

```
> epsilon[phi,phi]:=diff(u[phi](r,theta,phi),phi)/r+u[r](r,theta,phi)/r;
```

$$\varepsilon_{[\phi,\phi]} := 2\frac{U_{[\phi]}(r)\cos(2\phi)\sin(2\theta)}{r} + \frac{U_{[r]}(r)\sin(\phi)^2\sin(2\theta)}{r}$$

```
> gamma[r, theta]:=1/(r*sin(phi))*diff(u[r](r,theta,phi),theta)+diff(u[theta](r,theta,
  phi),r)-u[theta](r,theta,phi)/r;
```

$$\gamma_{[r,\theta]} := 2\frac{\sin(\phi)\,U_{[r]}(r)\cos(2\theta)}{r} + \left(\frac{\partial}{\partial r}U_{[\theta]}(r)\right)\sin(\phi)\cos(2\theta) - \frac{U_{[\theta]}(r)\sin(\phi)\cos(2\theta)}{r}$$

```
> gamma[theta,phi]:=diff(u[theta](r,theta,phi),phi)/r-cot(phi)*u[theta](r,theta,phi)/
  r+1/(r*sin(phi))*diff(u[phi](r,theta,phi),theta);
```

$$\gamma_{[\theta,\phi]} := \frac{U_{[\theta]}(r)\cos(\phi)\cos(2\theta)}{r} - \frac{\cot(\phi)\,U_{[\theta]}(r)\sin(\phi)\cos(2\theta)}{r}$$

$$+2\,\frac{U_{[\phi]}(r)\sin(2\,\phi)\cos(2\,\theta)}{r\sin(\phi)}$$

```
> gamma[r,phi]:=diff(u[phi](r,theta,phi),r)-u[phi](r,theta,phi)/r+diff(u[r](r,theta,phi)
  ,phi)/r;
```

$$\gamma_{[r,\phi]} := \left(\frac{\partial}{\partial r}U_{[\phi]}(r)\right)\sin(2\,\phi)\sin(2\,\theta) - \frac{U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r} + 2\,\frac{U_{[r]}(r)\sin(\phi)\sin(2\,\theta)\cos(\phi)}{r}$$

```
> e:=epsilon[rr]+epsilon[theta,theta]+epsilon[phi,phi];
```

$$e := \left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\,\theta) - 2\,\frac{U_{[\theta]}(r)\sin(2\,\theta)}{r} + 2\,\frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r} + \frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r} + 2\,\frac{U_{[\phi]}(r)\cos(2\,\phi)\sin(2\,\theta)}{r}$$

```
> sigma[rr]:=E(r)/(1+nu(r))*(epsilon[rr]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[rr]} := \mathrm{E}(r)\left(\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\,\theta) + \nu(r)\left(\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\,\theta) - 2\,\frac{U_{[\theta]}(r)\sin(2\,\theta)}{r} + 2\,\frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r} + \frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r} + 2\,\frac{U_{[\phi]}(r)\cos(2\,\phi)\sin(2\,\theta)}{r}\right)\Big/(1-2\,\nu(r))\right)\Big/(1+\nu(r))$$

```
> sigma[theta,theta]:=E(r)/(1+nu(r))*(epsilon[theta,theta]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[\theta,\theta]} := \mathrm{E}(r)\left(-2\,\frac{U_{[\theta]}(r)\sin(2\,\theta)}{r} + \frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r} + \frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r} + \nu(r)\left(\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\,\theta) - 2\,\frac{U_{[\theta]}(r)\sin(2\,\theta)}{r} + 2\,\frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r}\right.\right.$$

$$+\frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r}+2\,\frac{U_{[\phi]}(r)\cos(2\,\phi)\sin(2\,\theta)}{r}\Bigg)\Bigg/(1-2\,\nu(r)$$

$$)\Bigg)\Bigg/(1+\nu(r))$$

```
> sigma[phi,phi]:=E(r)/(1+nu(r))*(epsilon[phi,phi]+(nu(r)/(1-2*nu(r)))*(e));
```

$$\sigma_{[\phi,\phi]}:=\mathrm{E}(r)\Bigg(2\,\frac{U_{[\phi]}(r)\cos(2\,\phi)\sin(2\,\theta)}{r}+\frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r}+\nu(r)\Bigg($$

$$\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\sin(\phi)^2\sin(2\,\theta)-2\,\frac{U_{[\theta]}(r)\sin(2\,\theta)}{r}+2\,\frac{U_{[r]}(r)\sin(\phi)^2\sin(2\,\theta)}{r}$$

$$+\frac{\cot(\phi)\,U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r}+2\,\frac{U_{[\phi]}(r)\cos(2\,\phi)\sin(2\,\theta)}{r}\Bigg)\Bigg/(1-2\,\nu(r)$$

$$)\Bigg)\Bigg/(1+\nu(r))$$

```
> tau[r,theta]:=E(r)/2/(1+nu(r))*gamma[r,theta];
```

$$\tau_{[r,\theta]}:=\frac{1}{2}\,\mathrm{E}(r)\Bigg(2\,\frac{\sin(\phi)\,U_{[r]}(r)\cos(2\,\theta)}{r}+\left(\frac{\partial}{\partial r}U_{[\theta]}(r)\right)\sin(\phi)\cos(2\,\theta)$$

$$-\frac{U_{[\theta]}(r)\sin(\phi)\cos(2\,\theta)}{r}\Bigg)\Bigg/(1+\nu(r))$$

```
> tau[theta,phi]:=E(r)/2/(1+nu(r))*gamma[theta,phi];
```

$$\tau_{[\theta,\phi]}:=\frac{1}{2}\,\mathrm{E}(r)\Bigg(\frac{U_{[\theta]}(r)\cos(\phi)\cos(2\,\theta)}{r}-\frac{\cot(\phi)\,U_{[\theta]}(r)\sin(\phi)\cos(2\,\theta)}{r}$$

$$+2\,\frac{U_{[\phi]}(r)\sin(2\,\phi)\cos(2\,\theta)}{r\sin(\phi)}\Bigg)\Bigg/(1+\nu(r))$$

```
> tau[r,phi]:=E(r)/2/(1+nu(r))*gamma[r,phi];
```

$$\tau_{[r,\phi]}:=\frac{1}{2}\,\mathrm{E}(r)\Bigg(\left(\frac{\partial}{\partial r}U_{[\phi]}(r)\right)\sin(2\,\phi)\sin(2\,\theta)-\frac{U_{[\phi]}(r)\sin(2\,\phi)\sin(2\,\theta)}{r}$$

$$+2\,\frac{U_{[r]}(r)\sin(\phi)\sin(2\,\theta)\cos(\phi)}{r}\Bigg)\Bigg/(1+\nu(r))$$

Since computer confused U[theta] and U[phi] with U(theta) and U(phi), when it take derivative with them. We replace U[theta] and U[phi] by U[t] and U[p].

```
> U[phi]:=U[p]:
> U[theta]:=U[t]:
> equi1:=diff(sigma[rr],r)*sin(phi)+1/r*(2*sigma[rr]*sin(phi)-sigma[theta,theta]*sin(phi)-sigma[phi,phi]*sin(phi)+tau[r,phi]*cos(phi)+diff(tau[r,theta],theta)+diff(tau[r,phi],phi)*sin(phi))=0:
> eqn1:=simplify(equi1,trig):
> eqn2:=expand(eqn1/(-1/2*sin(2*theta)*sin(phi))):
>  eqn3:=subs(cos(phi)^2=1-sin(phi)^2,eqn2):
> eqn4:=expand(eqn3):
> eqn5:=sort(eqn4,[sin(phi)],plex):
> eqn6:=op(1,eqn5):
> eqn7:=coeff(eqn6,sin(phi)^2):
> eqn8:=collect(eqn7,[diff(U[r](r),r,r),diff(U[r](r),r),U[r](r),diff(U[phi](r),r),U[phi](r)],factor)=0:
> eqn9:=normal(eqn8*(1+nu(r))*(-1+2*nu(r))/2):
> eqn10:=collect(eqn9,[diff(U[r](r),r,r),diff(U[r](r),r),Ur(r),diff(U[phi](r),r),U[phi](r)],factor):
> eqn11:=normal(eqn10/(-1+2*nu(r))):
> eqn12:=collect(eqn11,[diff(U[r](r),r,r),diff(U[r](r),r),U[r](r),diff(U[phi](r),r),U[phi](r)],factor);
```

$$eqn12 := -\frac{\mathrm{E}(r)\,(\nu(r)-1)\left(\frac{\partial^2}{\partial r^2}U_{[r]}(r)\right)}{-1+2\,\nu(r)} - \Bigg(\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)^2\,r$$
$$+ 4\,\mathrm{E}(r)\,\nu(r)^3 - \left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2\,r - 2\,\mathrm{E}(r)\,\nu(r)^2 - 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r$$
$$+ 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^3\,r + 4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)\,r - 4\,\mathrm{E}(r)\,\nu(r) + 2\,\mathrm{E}(r)\Bigg)$$
$$\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)\Big/\left(r\,(-1+2\,\nu(r))^2\,(1+\nu(r))\right) + \Bigg(5\,\mathrm{E}(r) + 4\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^3\,r$$
$$+ 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2\,r - 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r + 16\,\mathrm{E}(r)\,\nu(r)^3 - 2\,\mathrm{E}(r)\,\nu(r)^2$$
$$- 13\,\mathrm{E}(r)\,\nu(r) - 4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)^2\,r - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\Bigg)U_{[r]}(r)\Big/\Big($$
$$(-1+2\,\nu(r))^2\,(1+\nu(r))\,r^2\Big) - 3\,\frac{\mathrm{E}(r)\left(\frac{\partial}{\partial r}U_{[p]}(r)\right)}{r\,(-1+2\,\nu(r))} - 3\Big(-7\,\mathrm{E}(r)\,\nu(r)$$
$$- 2\,\mathrm{E}(r)\,\nu(r)^2 + 3\,\mathrm{E}(r) - 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r + 8\,\mathrm{E}(r)\,\nu(r)^3$$

$$+4\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^3 r - 4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)^2 r + 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2 r\Bigg)U_{[p]}(r)$$
$$\Big/\left((-1+2\,\nu(r))^2\,(1+\nu(r))\,r^2\right)=0$$

This is one of six equations from equilibrium equations. We choose it as our governing eqn. (2.39).

```
> eqn13:=expand(eqn6-eqn7*sin(phi)^2):
```

```
> eqn14:=collect(eqn13,[diff(U[phi](r),r),diff(U[theta](r),r),U[phi](r),U[theta](r)],factor)=0:
```

```
> eqn15:=normal(eqn14*(1+nu(r))*(-1+2*nu(r))/2):
```

```
> eqn16:=collect(eqn15,[diff(U[phi](r),r),diff(U[theta](r),r),U[phi](r),U[theta](r)],factor);
```

$$eqn16 := 2\frac{\mathrm{E}(r)\left(\frac{\partial}{\partial r}U_{[p]}(r)\right)}{r} - \frac{\mathrm{E}(r)\left(\frac{\partial}{\partial r}U_{[t]}(r)\right)}{r} + 2\Bigg(-7\,\mathrm{E}(r)\,\nu(r) - 2\,\mathrm{E}(r)\,\nu(r)^2$$
$$+3\,\mathrm{E}(r) - 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r + 8\,\mathrm{E}(r)\,\nu(r)^3$$
$$+4\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^3 r - 4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)^2 r + 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2 r\Bigg)U_{[p]}(r)$$
$$\Big/\left((1+\nu(r))\,(-1+2\,\nu(r))\,r^2\right) - \Bigg(-7\,\mathrm{E}(r)\,\nu(r) - 2\,\mathrm{E}(r)\,\nu(r)^2 + 3\,\mathrm{E}(r)$$
$$-2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r + 8\,\mathrm{E}(r)\,\nu(r)^3 + 4\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^3 r$$
$$-4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)^2 r + 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2 r\Bigg)U_{[t]}(r)\Big/\big((1+\nu(r))$$
$$(-1+2\,\nu(r))\,r^2\big)=0$$

This is one of six equations from equilibrium equations.

```
> equi2:=diff(tau[r,theta],r)*sin(phi)+1/r*(3*tau[r,theta]*sin(phi)+diff(tau[theta,phi],phi)*sin(phi)+diff(sigma[theta,theta],theta)+2*tau[theta,phi]*cos(phi))=0:
```

```
>  eqn21:=simplify(equi2,trig):
```

```
> eqn22:=expand(eqn21/(cos(2*theta)/2)):
```

```
> eqn23:=subs(cos(phi)^2=1-sin(phi)^2,eqn22):
```

```
> eqn24:=expand(eqn23):
```

```
> eqn25:=sort(eqn24,[sin(phi)],plex):
```

```
> eqn26:=op(1,eqn25):
```

```
> eqn27:=coeff(eqn26,sin(phi)^2):
```

```
> eqn28:=collect(eqn27,[diff(U[r](r),r),U[r](r),diff(U[theta](r),r,r),diff(U[theta](r),r),
U[theta](r),U[phi](r)],factor)=0:
```

```
> eqn29:=normal(eqn28*(1+nu(r))/2):
> eqn210:=collect(eqn29,[diff(U[r](r),r),U[r](r),diff(U[theta](r),r,r),diff(U[theta](r),r)
  ,U[theta](r),U[phi](r)],factor);
```

$$eqn210 := -\frac{\mathrm{E}(r)\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)}{r\,(-1+2\,\nu(r))}+\frac{1}{2}\mathrm{E}(r)\left(\frac{\partial^2}{\partial r^2}U_{[t]}(r)\right)+\frac{1}{2}$$
$$\left(\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r+2\,\mathrm{E}(r)\,\nu(r)+\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r+2\,\mathrm{E}(r)-\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\right)$$
$$\left(\frac{\partial}{\partial r}U_{[t]}(r)\right)\Big/(r\,(1+\nu(r)))-\frac{1}{2}$$
$$\left(\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r+2\,\mathrm{E}(r)\,\nu(r)+\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r+2\,\mathrm{E}(r)-\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\right)$$
$$U_{[t]}(r)\Big/((1+\nu(r))\,r^2)-2\,\frac{(4\,\nu(r)-5)\,\mathrm{E}(r)\,U_{[p]}(r)}{(-1+2\,\nu(r))\,r^2}+U_{[r]}(r)\Bigg(-4\,\mathrm{E}(r)$$
$$+4\,\mathrm{E}(r)\,\nu(r)^2+\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r-2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)\,r+\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)\,r$$
$$+2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\nu(r)^2\,r-\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\Bigg)\Big/((1+\nu(r))\,(-1+2\,\nu(r))\,r^2)=0$$

This is one of six equations from equilibrium equations.

```
> eqn211:=expand(eqn26-eqn27*sin(phi)^2)=0:
> eqn212:=factor(eqn211);
```

$$eqn212 := -8\,\frac{\left(U_{[t]}(r)-2\,U_{[p]}(r)\right)(\nu(r)-1)\,\mathrm{E}(r)}{(1+\nu(r))\,(-1+2\,\nu(r))\,r^2}=0$$

This is one of six equations from equilibrium equations.

```
> equi3:=diff(tau[r,phi],r)*sin(phi)+1/r*(3*tau[r,phi]*sin(phi)+diff(tau[theta,phi],the
  ta)+(sigma[phi,phi]-sigma[theta,theta])*cos(phi)+diff(sigma[phi,phi],phi)*sin(ph
  i))=0:
> eqn31:=simplify(equi3,trig):
> eqn32:=expand(eqn31/(sin(2*theta)*cos(phi))):
> eqn33:=subs(cos(phi)^2=1-sin(phi)^2,eqn32):
> eqn34:=expand(eqn33):
> eqn35:=sort(eqn34,[sin(phi)],plex):
> eqn36:=op(1,eqn35):
> eqn37:=coeff(eqn36,sin(phi)^2):
> eqn38:=collect(eqn37,[diff(U[r](r),r),U[r](r),diff(U[phi](r),r,r),diff(U[phi](r),r),U[ph
  i](r)],factor):
> eqn39:=normal(eqn38*(1+nu(r))):
```

```
> eqn310:=collect(eqn39,[diff(U[r](r),r),U[r](r),diff(U[phi](r),r,r),diff(U[phi](r),r),U[phi](r)],factor)=0;
```

$$eqn310 := -\frac{E(r)\left(\frac{\partial}{\partial r}U_{[r]}(r)\right)}{r\,(-1+2\,\nu(r))} + U_{[r]}(r)\left(-4\,E(r)+4\,E(r)\,\nu(r)^2+E(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r - 2\,E(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)\,r+\left(\frac{\partial}{\partial r}E(r)\right)\nu(r)\,r+2\left(\frac{\partial}{\partial r}E(r)\right)\nu(r)^2\,r - \left(\frac{\partial}{\partial r}E(r)\right)r\right)\Big/((1+\nu(r))\,(-1+2\,\nu(r))\,r^2)+E(r)\left(\frac{\partial^2}{\partial r^2}U_{[p]}(r)\right)+ \left(\left(\frac{\partial}{\partial r}E(r)\right)\nu(r)\,r+2\,E(r)\,\nu(r)+\left(\frac{\partial}{\partial r}E(r)\right)r+2\,E(r)-E(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\right)\left(\frac{\partial}{\partial r}U_{[p]}(r)\right)\Big/(r\,(1+\nu(r))) - \left(12\,E(r)\,\nu(r)^2-12\,E(r)+E(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r + \left(\frac{\partial}{\partial r}E(r)\right)\nu(r)\,r-\left(\frac{\partial}{\partial r}E(r)\right)r+2\left(\frac{\partial}{\partial r}E(r)\right)\nu(r)^2\,r - 2\,E(r)\left(\frac{\partial}{\partial r}\nu(r)\right)\nu(r)\,r\right)U_{[p]}(r)\Big/((1+\nu(r))\,(-1+2\,\nu(r))\,r^2)=0$$

This is one of six equations from equilibrium. We choose it as our governing equation (2.38).

```
> eqn311:=expand(eqn36-eqn37*sin(phi)^2):
> eqn312:=factor(eqn311)=0:
> eqn313:=eqn312*r^2*(1+nu(r))/2/E(r);
```

$$eqn313 := U_{[r]}(r) - 2\,U_{[p]}(r) = 0$$

This is one of six equations from equilibrium equations. We choose it as our governing eqn. (2.40).

APPENDIX D

This computer program generates the eqns. (2.52), (2.53), (2.55), (2.56) and (2.57) in the shear modulus study.

```
> gov1:= -1/r*E(r)/(-1+2*nu(r))*diff((U[r])(r),r)+(U[r])(r)*(-4*E(r)+4*E(r)*nu(r)^2+E
  (r)*diff(nu(r),r)*r-2*E(r)*diff(nu(r),r)*nu(r)*r+diff(E(r),r)*nu(r)*r+2*diff(E(r),r)*nu(r
  )^2*r-diff(E(r),r)*r)/(1+nu(r))/(-1+2*nu(r))/r^2+E(r)*diff(diff((U[phi])(r),r),r)+(diff(
  E(r),r)*nu(r)*r+2*E(r)*nu(r)+diff(E(r),r)*r+2*E(r)-E(r)*diff(nu(r),r)*r)/r/(1+nu(r))*d
  iff((U[phi])(r),r)-(12*E(r)*nu(r)^2-12*E(r)+E(r)*diff(nu(r),r)*r+diff(E(r),r)*nu(r)*r-d
  iff(E(r),r)*r+2*diff(E(r),r)*nu(r)^2*r-2*E(r)*diff(nu(r),r)*nu(r)*r)/(1+nu(r))/(-1+2*n
  u(r))/r^2*(U[phi])(r) = 0:
> gov2:= (U[theta])(r)-2*(U[phi])(r) = 0:
> gov3:= -E(r)*(nu(r)-1)/(-1+2*nu(r))*diff(diff((U[r])(r),r),r)-1/r*(diff(E(r),r)*r-2*E(r)*
  diff(nu(r),r)*nu(r)^2*r+4*E(r)*nu(r)^3-diff(E(r),r)*nu(r)^2*r-2*E(r)*nu(r)^2-2*diff(
  E(r),r)*nu(r)*r+2*diff(E(r),r)*nu(r)^3*r+4*E(r)*diff(nu(r),r)*nu(r)*r-4*E(r)*nu(r)+2
  *E(r))/(-1+2*nu(r))^2/(1+nu(r))*diff((U[r])(r),r)+(5*E(r)+4*diff(E(r),r)*nu(r)^3*r+2
  *diff(E(r),r)*nu(r)^2*r-2*diff(E(r),r)*nu(r)*r+16*E(r)*nu(r)^3-2*E(r)*nu(r)^2-13*E(
  r)*nu(r)-4*E(r)*diff(nu(r),r)*nu(r)^2*r-2*E(r)*diff(nu(r),r)*r)/(-1+2*nu(r))^2/(1+nu(
  r))/r^2*(U[r])(r)-3*E(r)/r/(-1+2*nu(r))*diff((U[phi])(r),r)-3*(-7*E(r)*nu(r)-2*E(r)*nu
  (r)^2+3*E(r)-2*diff(E(r),r)*nu(r)*r-2*E(r)*diff(nu(r),r)*r+8*E(r)*nu(r)^3+4*diff(E(r
  ),r)*nu(r)^3*r-4*E(r)*diff(nu(r),r)*nu(r)^2*r+2*diff(E(r),r)*nu(r)^2*r)/(-1+2*nu(r))^
  2/(1+nu(r))/r^2*(U[phi])(r) = 0:
> E(r):=P*(r/a)^Q:
> nu(r):=nu:
> gov1:
> gov3:
> eqn1:=normal(gov1):
> eqn2:=normal(gov3):
> eqn3:=sort(eqn1,[diff(U[r](r),r),U[r](r),diff(U[phi](r),r,r),diff(U[phi](r),r),U[phi](r)])
  ;
```

$$eqn3 := -\left(r\left(\frac{\partial}{\partial r}U_{[r]}(r)\right) - 2\,Q\,\nu\,U_{[r]}(r) - 4\,\nu\,U_{[r]}(r) + Q\,U_{[r]}(r) + 4\,U_{[r]}(r) + r^2\left(\frac{\partial^2}{\partial r^2}U_{[\phi]}(r)\right) - 2\,r^2\,\nu\left(\frac{\partial^2}{\partial r^2}U_{[\phi]}(r)\right) - 4\,r\,\nu\,\%1 - 2\,r\,Q\,\nu\,\%1 + r\,Q\,\%1 + 2\,r\,\%1 + 12\,\nu\,U_{[\phi]}(r) + 2\,Q\,\nu\,U_{[\phi]}(r) - 12\,U_{[\phi]}(r) - Q\,U_{[\phi]}(r)\right)P\left(\frac{r}{a}\right)^Q\Big/\left(r^2\,(-1+2\,\nu)\right) = 0$$

$$\%1 := \frac{\partial}{\partial r}U_{[\phi]}(r)$$

This equation is equivalent to eqn. (2.52)

```
> eqn4:=sort(eqn2,[diff(U[r](r),r,r),diff(U[r](r),r),Ur(r),diff(U[phi](r),r),U[phi](r)]);
```

$$eqn4 := -\left(\frac{r}{a}\right)^{Q} P \left(r^2 \nu \left(\frac{\partial^2}{\partial r^2} U_{[r]}(r)\right) - r^2 \left(\frac{\partial^2}{\partial r^2} U_{[r]}(r)\right) - r\,Q \left(\frac{\partial}{\partial r} U_{[r]}(r)\right) + 2\,r\,\nu \left(\frac{\partial}{\partial r} U_{[r]}(r)\right) + r\,Q\,\nu \left(\frac{\partial}{\partial r} U_{[r]}(r)\right) - 2\,r \left(\frac{\partial}{\partial r} U_{[r]}(r)\right) + 3\,r \left(\frac{\partial}{\partial r} U_{[\phi]}(r)\right) - 9\,U_{[\phi]}(r) + 6\,Q\,\nu\,U_{[\phi]}(r) + 12\,\nu\,U_{[\phi]}(r) - 2\,Q\,\nu\,U_{[r]}(r) + 5\,U_{[r]}(r) - 8\,\nu\,U_{[r]}(r) \right) \Big/ (r^2\,(-1 + 2\,\nu)) = 0$$

This equation is equivalent to equation (2.53)

```
> eqn5:=-d*U[r]+(4*nu+2*Q*nu-4-Q)*U[r]+(2*nu-1)*d*(d-1)*U[phi]+(-Q+4*nu+2*
  Q*nu-2)*d*U[phi]+(-2*Q*nu+Q+12-12*nu)*U[phi]=0:
> eqn6:=(-1+nu)*d*(d-1)*U[r]+(2*nu-2+Q*nu-Q)*d*U[r]+(-8*nu+5-2*Q*nu)*U[r]+
  3*d*U[phi]+(12*nu-9+6*Q*nu)*U[phi]=0:
> eqn7:=collect(eqn5,[U[r],U[phi]],distributed,factor);
```

$$eqn7 := (-d + 4\,\nu + 2\,Q\,\nu - 4 - Q)\,U_{[r]} + (-d^2 - d + 2\,\nu\,d^2 + 2\,\nu\,d - d\,Q + 2\,d\,Q\,\nu - 2\,Q\,\nu + Q + 12 - 12\,\nu)\,U_{[\phi]} = 0$$

This equation is equivalent to eqn. (2.55)

```
> eqn8:=collect(eqn6,[U[r],U[phi]],distributed,factor);
```

$$eqn8 := (\nu\,d^2 + \nu\,d - d^2 - d + d\,Q\,\nu - d\,Q - 8\,\nu + 5 - 2\,Q\,\nu)\,U_{[r]} + (3\,d + 6\,Q\,\nu + 12\,\nu - 9)\,U_{[\phi]} = 0$$

This equation is equivalent to eqn. (2.56)

```
> eqn9:=eqn7*(12*nu-9+6*Q*nu+3*d):
> eqn10:=eqn8*(-2*Q*nu+Q+12-12*nu+2*nu*d^2+2*nu*d-d^2-d-d*Q+2*d*Q*nu)
  :
> eqn11:=eqn10-eqn9:
> eqn12:=simplify(eqn11):
> eqn13:=collect(eqn12,[U[r]],distributed,factor):
> eqn14:=normal(eqn13/(2*nu-1)/U[r]):
> eqn15:=collect(eqn14,[d]);
```

$$eqn15 := (\nu - 1)\,d^4 + (2\,Q\,\nu - 2\,Q - 2 + 2\,\nu)\,d^3 + (Q^2\,\nu - Q\,\nu + 13 - Q^2 - 13\,\nu - Q)\,d^2 + (-17\,Q\,\nu + 15\,Q - 3\,Q^2\,\nu + 14 - 14\,\nu + Q^2)\,d - 4\,Q\,\nu + 4\,Q + 24\,\nu - 24 - 4\,Q^2\,\nu = 0$$

This equation is equivalent to eqn. (2.57)

```
> eqn16:=solve(eqn15,{d});
```

$$eqn16 := \left\{ d = -\frac{1}{4}\,\%2 + \frac{1}{2}\,\frac{\sqrt{Q^2\,\nu - Q^2 - 4\,Q + 8\,Q\,\nu - 29 + 29\,\nu + 2\,\%1}}{\sqrt{\nu - 1}} \right\},$$

$$\left\{ d = -\frac{1}{4}\,\%2 - \frac{1}{2}\,\frac{\sqrt{Q^2\,\nu - Q^2 - 4\,Q + 8\,Q\,\nu - 29 + 29\,\nu + 2\,\%1}}{\sqrt{\nu - 1}} \right\},$$

$$\left\{ d = -\frac{1}{4}\,\%2 + \frac{1}{2}\,\frac{\sqrt{Q^2\,\nu - Q^2 - 4\,Q + 8\,Q\,\nu - 29 + 29\,\nu - 2\,\%1}}{\sqrt{\nu - 1}} \right\},$$

$$\left\{ d = -\frac{1}{4}\,\%2 - \frac{1}{2}\,\frac{\sqrt{Q^2\,\nu - Q^2 - 4\,Q + 8\,Q\,\nu - 29 + 29\,\nu - 2\,\%1}}{\sqrt{\nu - 1}} \right\}$$

$$\%1 := \left(25\,Q^2\,\nu^2 - 22\,Q^2\,\nu + Q^2 + 100\,Q\,\nu^2 - 144\,Q\,\nu + 44\,Q + 100\,\nu^2 - 200\,\nu + 100\right)^{1/2}$$

$$\%2 := \frac{2\,Q\,\nu - 2\,Q - 2 + 2\,\nu}{\nu - 1}$$

Four roots of eqn. (2.57).
Note: Poisson's ratio in this appendix represents Poisson's ratio in interphase.

APPENDIX E

This computer program determines unknown constants of eqns (2.58), (2.61) and (2.62) in the shear modulus study.

```
> U[r](r):=D[1]*r^d[1]+D[2]*r^d[2]+D[3]*r^d[3]+D[4]*r^d[4];
```

$$U_{[r]}(r) := D_{[1]}\, r^{d_{[1]}} + D_{[2]}\, r^{d_{[2]}} + D_{[3]}\, r^{d_{[3]}} + D_{[4]}\, r^{d_{[4]}}$$

```
> U[phi](r):=b[1]*r^d[1]+b[2]*r^d[2]+b[3]*r^d[3]+b[4]*r^d[4];
```

$$U_{[\phi]}(r) := b_{[1]}\, r^{d_{[1]}} + b_{[2]}\, r^{d_{[2]}} + b_{[3]}\, r^{d_{[3]}} + b_{[4]}\, r^{d_{[4]}}$$

```
> eqn1:=(r^2*nu*diff(diff((U[r])(r),r),r)-r^2*diff(diff((U[r])(r),r),r)+r*Q*nu*diff((U[r])(r
),r)+2*r*nu*diff((U[r])(r),r)-2*r*diff((U[r])(r),r)-r*Q*diff((U[r])(r),r)+3*r*diff((U[phi])
(r),r)+6*Q*nu*(U[phi])(r)+12*nu*(U[phi])(r)-9*(U[phi])(r)-8*nu*(U[r])(r)+5*(U[r])(
r)-2*Q*nu*(U[r])(r))=0:
> eqn2:=collect(eqn1,[r],distributed,factor):
> eqn3:=sort(eqn2,[r^d[1],r^d[2],r^d[3],r^d[4]],plex):
> eqn4:=op(1,eqn3):
> eqn5:=coeff(eqn4,r^d[1])=0:
> eqn6:=coeff(eqn4,r^d[2])=0:
> eqn7:=coeff(eqn4,r^d[3])=0:
> eqn8:=coeff(eqn4,r^d[4])=0:
> b[1]:=solve(eqn5,b[1]);
```

$$b_{[1]} := -\left(5\, D_{[1]} + Q\, \nu\, D_{[1]}\, d_{[1]} - D_{[1]}\, d_{[1]} - 2\, Q\, \nu\, D_{[1]} + \nu\, D_{[1]}\, d_{[1]} - D_{[1]}\, d_{[1]}^{2} + \nu\, D_{[1]}\, d_{[1]}^{2} - 8\, \nu\, D_{[1]} - Q\, D_{[1]}\, d_{[1]}\right) \Big/ \left(6\, Q\, \nu + 3\, d_{[1]} - 9 + 12\, \nu\right)$$

```
> b[2]:=solve(eqn6,b[2]);
```

$$b_{[2]} := -\left(-2\, Q\, \nu\, D_{[2]} + 5\, D_{[2]} + Q\, \nu\, D_{[2]}\, d_{[2]} - 8\, \nu\, D_{[2]} - D_{[2]}\, d_{[2]}^{2} - D_{[2]}\, d_{[2]} + \nu\, D_{[2]}\, d_{[2]} - Q\, D_{[2]}\, d_{[2]} + \nu\, D_{[2]}\, d_{[2]}^{2}\right) \Big/ \left(6\, Q\, \nu - 9 + 3\, d_{[2]} + 12\, \nu\right)$$

```
> b[3]:=solve(eqn7,b[3]);
```

$$b_{[3]} := -\left(-8\, \nu\, D_{[3]} + 5\, D_{[3]} + Q\, \nu\, D_{[3]}\, d_{[3]} - Q\, D_{[3]}\, d_{[3]} + \nu\, D_{[3]}\, d_{[3]}^{2} + \nu\, D_{[3]}\, d_{[3]} - 2\, Q\, \nu\, D_{[3]} - D_{[3]}\, d_{[3]}^{2} - D_{[3]}\, d_{[3]}\right) \Big/ \left(6\, Q\, \nu + 12\, \nu + 3\, d_{[3]} - 9\right)$$

```
> b[4]:=solve(eqn8,b[4]);
```

$$b_{[4]} := -\left(5\, D_{[4]} - D_{[4]}\, d_{[4]}^{2} + \nu\, D_{[4]}\, d_{[4]}^{2} + \nu\, D_{[4]}\, d_{[4]} - D_{[4]}\, d_{[4]} - 8\, \nu\, D_{[4]} + Q\, \nu\, D_{[4]}\, d_{[4]} - 2\, Q\, \nu\, D_{[4]} - Q\, D_{[4]}\, d_{[4]}\right) \Big/ \left(-9 + 6\, Q\, \nu + 3\, d_{[4]} + 12\, \nu\right)$$

When the roots of the characteristic functions are complex,

```
> d[1]:=m[1]+I*n[1];d[2]:=m[1]-I*n[1];d[3]:=m[2]+I*n[2];d[4]:=m[2]-I*n[2];
```

$$d_{[1]} := m_{[1]} + I\, n_{[1]}$$

$$d_{[2]} := m_{[1]} - I\, n_{[1]}$$

$$d_{[3]} := m_{[2]} + I\, n_{[2]}$$

$$d_{[4]} := m_{[2]} - I\, n_{[2]}$$

```
> eqn1:=(r^2*nu*diff(diff((U[r])(r),r),r)-r^2*diff(diff((U[r])(r),r),r)+r*Q*nu*diff((U[r])(r
  ),r)+2*r*nu*diff((U[r])(r),r)-2*r*diff((U[r])(r),r)-r*Q*diff((U[r])(r),r)+3*r*diff((U[phi])
  (r),r)+6*Q*nu*(U[phi])(r)+12*nu*(U[p])(r)-9*(U[phi])(r)-8*nu*(U[r])(r)+5*(U[r])(r)-
  2*Q*nu*(U[r])(r))=0:
> U[r](r):=D[1]*cos(n[1]*ln(r))*r^m[1]+D[2]*sin(n[1]*ln(r))*r^m[1]+D[3]*cos(n[2]*ln
  (r))*r^m[2]+D[4]*sin(n[2]*ln(r))*r^m[2];
```

$$U_{[r]}(r) := D_{[1]} \cos\left(n_{[1]} \ln(r)\right) r^{m_{[1]}} + D_{[2]} \sin\left(n_{[1]} \ln(r)\right) r^{m_{[1]}} + D_{[3]} \cos\left(n_{[2]} \ln(r)\right) r^{m_{[2]}} + D_{[4]} \sin\left(n_{[2]} \ln(r)\right) r^{m_{[2]}}$$

```
> U[phi](r):=b[1]*cos(n[1]*ln(r))*r^m[1]+b[2]*sin(n[1]*ln(r))*r^m[1]+b[3]*cos(n[2]*l
  n(r))*r^m[2]+b[4]*sin(n[2]*ln(r))*r^m[2];
```

$$U_{[\phi]}(r) := b_{[1]} \cos\left(n_{[1]} \ln(r)\right) r^{m_{[1]}} + b_{[2]} \sin\left(n_{[1]} \ln(r)\right) r^{m_{[1]}} + b_{[3]} \cos\left(n_{[2]} \ln(r)\right) r^{m_{[2]}} + b_{[4]} \sin\left(n_{[2]} \ln(r)\right) r^{m_{[2]}}$$

```
> eqn2:=collect(eqn1,[r],distributed,factor):
> eqn3:=sort(eqn2,[r^m[1],r^m[2]]):
> eqn4:=op(1,eqn3):
> eqn5:=coeff(eqn4,r^m[1])=0:
> eqn6:=subs({cos(n[1]*ln(r))=x,sin(n[1]*ln(r))=y},eqn5):
> eqn7:=collect(eqn6,[x,y],factor):
> eqn8:=op(1,eqn7):
> eqn81:=coeff(eqn8,x)=0:
> eqn82:=coeff(eqn8,y)=0:
> eqn83:=solve({eqn81,eqn82},{b[1],b[2]}):
> eqn9:=coeff(eqn4,r^m[2]):
> eqn10:=subs({cos(n[2]*ln(r))=u,sin(n[2]*ln(r))=v},eqn9):
> eqn11:=collect(eqn10,[u,v],factor):
> eqn111:=coeff(eqn11,u)=0:
> eqn112:=coeff(eqn11,v)=0:
> eqn113:=solve({eqn111,eqn112},{b[3],b[4]}):
> assign({eqn83,eqn113});
> b[1]:=b[1];
```

$$b_{[1]} := -\frac{1}{3}\Big(-4 Q^2 \nu^2 D_{[1]} - Q D_{[1]} m_{[1]}^2 - 15 D_{[1]} - Q D_{[1]} n_{[1]}^2 + 6 D_{[2]} n_{[1]} m_{[1]}$$
$$- n_{[1]} D_{[2]} m_{[1]}^2 + 4 \nu D_{[1]} n_{[1]}^2 - 6 \nu D_{[2]} n_{[1]} m_{[1]} - 3 Q \nu D_{[2]} n_{[1]}$$
$$- 7 Q \nu D_{[1]} m_{[1]} - 11 \nu D_{[1]} m_{[1]} - 2 \nu D_{[1]} m_{[1]}^2 - 16 Q \nu^2 D_{[1]}$$
$$- D_{[1]} n_{[1]}^2 m_{[1]} + 3 Q D_{[1]} m_{[1]} + \nu D_{[1]} m_{[1]}^3 + 8 D_{[1]} m_{[1]} + 2 D_{[1]} m_{[1]}^2$$
$$+ 24 \nu D_{[1]} - 4 D_{[1]} n_{[1]}^2 - 2 D_{[2]} n_{[1]} - D_{[2]} n_{[1]}^3 - D_{[1]} m_{[1]}^3$$
$$+ \nu D_{[1]} n_{[1]}^2 m_{[1]} + 3 Q \nu D_{[1]} n_{[1]}^2 - 4 n_{[1]} Q \nu D_{[2]} m_{[1]} + n_{[1]} \nu D_{[2]} m_{[1]}^2$$
$$+ 2 Q \nu^2 D_{[1]} m_{[1]}^2 + \nu D_{[2]} n_{[1]}^3 + 5 \nu D_{[2]} n_{[1]} + 16 Q \nu D_{[1]}$$
$$+ 2 Q \nu^2 D_{[1]} m_{[1]} + 4 Q \nu^2 D_{[2]} n_{[1]} m_{[1]} - 2 Q^2 \nu D_{[1]} m_{[1]} - 2 Q^2 \nu D_{[2]} n_{[1]}$$
$$+ 2 Q \nu^2 D_{[2]} n_{[1]} - 2 Q \nu^2 D_{[1]} n_{[1]}^2 - Q \nu D_{[1]} m_{[1]}^2 + 2 Q^2 \nu^2 D_{[2]} n_{[1]}$$
$$+ 2 Q^2 \nu^2 D_{[1]} m_{[1]} + 3 Q D_{[2]} n_{[1]}\Big) \Big/ \Big($$
$$9 + 4 Q^2 \nu^2 - 6 m_{[1]} + n_{[1]}^2 + m_{[1]}^2 + 4 Q \nu m_{[1]} - 12 Q \nu\Big)$$

```
> b[2]:=b[2];
```

$$b_{[2]} := \frac{1}{3}\Big(15 D_{[2]} + m_{[1]} D_{[2]} n_{[1]}^2 + 7 Q \nu D_{[2]} m_{[1]} - 3 Q \nu D_{[1]} n_{[1]}$$
$$- 6 \nu D_{[1]} n_{[1]} m_{[1]} - 3 Q D_{[2]} m_{[1]} - 2 D_{[2]} m_{[1]}^2 - 8 D_{[2]} m_{[1]} - 2 D_{[1]} n_{[1]}$$
$$- 24 \nu D_{[2]} + 4 D_{[2]} n_{[1]}^2 + 6 D_{[1]} n_{[1]} m_{[1]} - 4 \nu D_{[2]} n_{[1]}^2 - 16 Q \nu D_{[2]}$$
$$+ 3 Q D_{[1]} n_{[1]} + 5 \nu D_{[1]} n_{[1]} + 2 \nu D_{[2]} m_{[1]}^2 + 11 \nu D_{[2]} m_{[1]} + 16 Q \nu^2 D_{[2]}$$
$$+ 4 Q^2 \nu^2 D_{[2]} + Q D_{[2]} n_{[1]}^2 - D_{[1]} n_{[1]} m_{[1]}^2 + \nu D_{[1]} n_{[1]}^3 + D_{[2]} m_{[1]}^3$$
$$- 2 Q \nu^2 D_{[2]} m_{[1]} + 2 Q \nu^2 D_{[1]} n_{[1]} + 2 Q \nu^2 D_{[2]} n_{[1]}^2 + \nu D_{[1]} n_{[1]} m_{[1]}^2$$
$$- \nu D_{[2]} m_{[1]}^3 - m_{[1]} \nu D_{[2]} n_{[1]}^2 + 2 m_{[1]} Q^2 \nu D_{[2]} + Q D_{[2]} m_{[1]}^2$$
$$- 2 Q^2 \nu D_{[1]} n_{[1]} + Q \nu D_{[2]} m_{[1]}^2 - 4 Q \nu D_{[1]} n_{[1]} m_{[1]} - 3 Q \nu D_{[2]} n_{[1]}^2$$
$$- D_{[1]} n_{[1]}^3 + 4 Q \nu^2 D_{[1]} n_{[1]} m_{[1]} + 2 Q^2 \nu^2 D_{[1]} n_{[1]} - 2 Q^2 \nu^2 D_{[2]} m_{[1]}$$
$$- 2 Q \nu^2 D_{[2]} m_{[1]}^2\Big) \Big/ \Big(9 + 4 Q^2 \nu^2 - 6 m_{[1]} + n_{[1]}^2 + m_{[1]}^2 + 4 Q \nu m_{[1]} - 12 Q \nu$$
$$\Big)$$

> b[3]:=b[3];

$$b_{[3]} := -\frac{1}{3}\Big(-Q D_{[3]} m_{[2]}^2 - n_{[2]} D_{[4]} m_{[2]}^2 - D_{[3]} n_{[2]}^2 m_{[2]} + 6 D_{[4]} n_{[2]} m_{[2]} - 15 D_{[3]} + 5 \nu D_{[4]} n_{[2]} + \nu D_{[4]} n_{[2]}^3 + 3 Q \nu D_{[3]} n_{[2]}^2 - 11 \nu D_{[3]} m_{[2]} - 16 Q \nu^2 D_{[3]} + \nu D_{[3]} m_{[2]}^3 + 3 Q D_{[3]} m_{[2]} + 4 \nu D_{[3]} n_{[2]}^2 - 2 \nu D_{[3]} m_{[2]}^2 - 3 Q \nu D_{[4]} n_{[2]} + 2 D_{[3]} m_{[2]}^2 - Q D_{[3]} n_{[2]}^2 + 16 Q \nu D_{[3]} + 3 Q D_{[4]} n_{[2]} - 4 n_{[2]} Q \nu D_{[4]} m_{[2]} - 4 Q^2 \nu^2 D_{[3]} + 8 D_{[3]} m_{[2]} + 24 \nu D_{[3]} - 4 D_{[3]} n_{[2]}^2 - 2 D_{[4]} n_{[2]} - 7 Q \nu D_{[3]} m_{[2]} - 6 \nu D_{[4]} n_{[2]} m_{[2]} - D_{[4]} n_{[2]}^3 - D_{[3]} m_{[2]}^3 + n_{[2]} \nu D_{[4]} m_{[2]}^2 + \nu D_{[3]} n_{[2]}^2 m_{[2]} - Q \nu D_{[3]} m_{[2]}^2 - 2 Q^2 \nu D_{[4]} n_{[2]} - 2 Q^2 \nu D_{[3]} m_{[2]} + 2 Q^2 \nu^2 D_{[4]} n_{[2]} + 2 Q \nu^2 D_{[3]} m_{[2]} + 2 Q \nu^2 D_{[3]} m_{[2]}^2 - 2 Q \nu^2 D_{[3]} n_{[2]}^2 + 2 Q^2 \nu^2 D_{[3]} m_{[2]} + 2 Q \nu^2 D_{[4]} n_{[2]} + 4 Q \nu^2 D_{[4]} n_{[2]} m_{[2]}\Big) \Big/ \Big(9 + 4 Q^2 \nu^2 - 12 Q \nu - 6 m_{[2]} + n_{[2]}^2 + m_{[2]}^2 + 4 Q \nu m_{[2]}\Big)$$

> b[4]:=b[4];

$$b_{[4]} := \frac{1}{3}\Big(3 Q D_{[3]} n_{[2]} + 15 D_{[4]} - 4 \nu D_{[4]} n_{[2]}^2 + 2 \nu D_{[4]} m_{[2]}^2 + 6 D_{[3]} n_{[2]} m_{[2]} - 3 Q D_{[4]} m_{[2]} + 11 \nu D_{[4]} m_{[2]} - 2 D_{[3]} n_{[2]} - 24 \nu D_{[4]} - 16 Q \nu D_{[4]} + D_{[4]} m_{[2]}^3 - D_{[3]} n_{[2]} m_{[2]}^2 + m_{[2]} D_{[4]} n_{[2]}^2 + Q D_{[4]} n_{[2]}^2 + \nu D_{[3]} n_{[2]}^3 + 2 m_{[2]} Q^2 \nu D_{[4]} + Q \nu D_{[4]} m_{[2]}^2 - \nu D_{[4]} m_{[2]}^3 + \nu D_{[3]} n_{[2]} m_{[2]}^2 - 2 Q \nu^2 D_{[4]} m_{[2]}^2 + 2 Q \nu^2 D_{[4]} n_{[2]}^2 + 2 Q^2 \nu^2 D_{[3]} n_{[2]} - 2 Q^2 \nu D_{[3]} n_{[2]} - 3 Q \nu D_{[4]} n_{[2]}^2 - m_{[2]} \nu D_{[4]} n_{[2]}^2 - 4 m_{[2]} Q \nu D_{[3]} n_{[2]} - 2 m_{[2]} Q^2 \nu^2 D_{[4]} + 4 m_{[2]} Q \nu^2 D_{[3]} n_{[2]} + Q D_{[4]} m_{[2]}^2 + 2 Q \nu^2 D_{[3]} n_{[2]} + 4 Q^2 \nu^2 D_{[4]} + 16 Q \nu^2 D_{[4]} + 5 \nu D_{[3]} n_{[2]} + 7 Q \nu D_{[4]} m_{[2]} - 6 \nu D_{[3]} n_{[2]} m_{[2]} - 2 D_{[4]} m_{[2]}^2 - 8 D_{[4]} m_{[2]} + 4 D_{[4]} n_{[2]}^2 - 3 Q \nu D_{[3]} n_{[2]} - 2 Q \nu^2 D_{[4]} m_{[2]} - D_{[3]} n_{[2]}^3\Big) \Big/ \Big(9 + 4 Q^2 \nu^2 - 12 Q \nu - 6 m_{[2]} + n_{[2]}^2 + m_{[2]}^2 + 4 Q \nu m_{[2]}\Big)$$

Note: Poisson's ratio in this appendix represents Poisson's ratio in interphase.

APPENDIX F

This computer program determines the shear modulus of the composite with a power variation Young's modulus in the interphase.

```
> for a from 10^(-6) by 0.01 to 0.9 do
> nu(p):=3/10;
> G(p):=25;
> nu(I):=3/10;
> P:=65;
> G(m):=1;
> b := (?/?)*a;(Input thickness value of interphase.)
> c:=1;
> nu(m) :=4/10;
> Q :=(ln(E(m)/E(p))/(ln(b/a));
> m[1]:=?;(Input one real component of eqn(2.59).)
> m[2]:=?;(Input another real component of eqn(2.59).)
> n[1]:=?;(Input one imaginary component of eqn(2.59).)
> n[2]:=?;(Input another imaginary component of eqn(2.59).)
> C:=3/10;
```

```
> b[1] :=-1/3*(-D[1]*n[1]^2*m[1]+8*nu(I)*D[1]*n[1]^2-24*Q*nu(I)^2*D[1]-15*D[1]-
D[2]*n[1]^3-D[1]*m[1]^3+2*D[1]*m[1]^2-4*D[1]*n[1]^2+8*D[1]*m[1]-2*D[2]*n[1]
-32*nu(I)^2*D[1]+44*nu(I)*D[1]+4*nu(I)^2*D[1]*m[1]^2+nu(I)*D[2]*n[1]^3-Q*D[
1]*m[1]^2+4*nu(I)^2*D[2]*n[1]-6*nu(I)*D[1]*m[1]^2+3*Q*D[2]*n[1]-Q*D[1]*n[1]^
2-n[1]*D[2]*m[1]^2+3*Q*D[1]*m[1]+16*Q*nu(I)*D[1]+nu(I)*D[1]*m[1]^3-4*nu(I)
^2*D[1]*n[1]^2+6*D[2]*n[1]*m[1]+n[1]*nu(I)*D[2]*m[1]^2+8*nu(I)^2*D[2]*n[1]*
m[1]-Q*nu(I)*D[1]*m[1]^2-4*Q^2*nu(I)^2*D[1]+4*nu(I)^2*D[1]*m[1]+nu(I)*D[2]*
n[1]-15*nu(I)*D[1]*m[1]+nu(I)*D[1]*n[1]^2*m[1]+6*Q*nu(I)^2*D[1]*m[1]-2*Q^2*
nu(I)*D[2]*n[1]-2*Q*nu(I)^2*D[1]*n[1]^2+4*Q*nu(I)^2*D[2]*n[1]*m[1]-2*Q^2*nu(
I)*D[1]*m[1]+2*Q^2*nu(I)^2*D[2]*n[1]+2*Q*nu(I)^2*D[1]*m[1]^2+2*Q^2*nu(I)^2
*D[1]*m[1]-4*n[1]*Q*nu(I)*D[2]*m[1]+6*Q*nu(I)^2*D[2]*n[1]+3*Q*nu(I)*D[1]*n[1
]^2-11*Q*nu(I)*D[1]*m[1]-7*Q*nu(I)*D[2]*n[1]-14*nu(I)*D[2]*n[1]*m[1])/(8*nu(I)*
m[1]+16*Q*nu(I)^2+n[1]^2-12*Q*nu(I)+9+16*nu(I)^2+m[1]^2+4*Q*nu(I)*m[1]+
4*Q^2*nu(I)^2-6*m[1]-24*nu(I));
```

```
> b[2] :=1/3*(Q*D[2]*m[1]^2+15*D[2]-D[1]*n[1]^3+D[2]*m[1]^3-8*D[2]*m[1]-2*D[
2]*m[1]^2-2*D[1]*n[1]+4*D[2]*n[1]^2+32*nu(I)^2*D[2]-44*nu(I)*D[2]-D[1]*n[1]*
m[1]^2+Q*D[2]*n[1]^2+m[1]*D[2]*n[1]^2+3*Q*D[1]*n[1]+6*D[1]*n[1]*m[1]-3*Q*
D[2]*m[1]+nu(I)*D[1]*n[1]+6*nu(I)*D[2]*m[1]^2+nu(I)*D[1]*n[1]^3-nu(I)*D[2]*m[
1]^3+4*nu(I)^2*D[1]*n[1]-4*nu(I)^2*D[2]*m[1]^2-4*nu(I)^2*D[2]*m[1]+4*Q^2*nu
(I)^2*D[2]+24*Q*nu(I)^2*D[2]-8*nu(I)*D[2]*n[1]^2+15*nu(I)*D[2]*m[1]-16*Q*nu(
I)*D[2]+4*nu(I)^2*D[2]*n[1]^2+Q*nu(I)*D[2]*m[1]^2+nu(I)*D[1]*n[1]*m[1]^2-m[1
]*nu(I)*D[2]*n[1]^2-2*Q^2*nu(I)*D[1]*n[1]-3*Q*nu(I)*D[2]*n[1]^2+4*Q*nu(I)^2*D
[1]*n[1]*m[1]+2*Q*nu(I)^2*D[2]*n[1]^2-4*Q*nu(I)*D[1]*n[1]*m[1]-2*Q^2*nu(I)^2
*D[2]*m[1]+2*Q^2*nu(I)^2*D[1]*n[1]-2*Q*nu(I)^2*D[2]*m[1]^2+8*m[1]*nu(I)^2*
```

```
D[1]*n[1]+2*m[1]*Q^2*nu(l)*D[2]+6*Q*nu(l)^2*D[1]*n[1]-6*Q*nu(l)^2*D[2]*m[1]
-7*Q*nu(l)*D[1]*n[1]-14*nu(l)*D[1]*n[1]*m[1]+11*Q*nu(l)*D[2]*m[1])/(8*nu(l)*m
[1]+16*Q*nu(l)^2+n[1]^2-12*Q*nu(l)+9+16*nu(l)^2+m[1]^2+4*Q*nu(l)*m[1]+4*
Q^2*nu(l)^2-6*m[1]-24*nu(l));
```

```
> b[3] :=-1/3*(-D[4]*n[2]^3-D[3]*m[2]^3+2*D[3]*m[2]^2-2*D[4]*n[2]-4*D[3]*n[2]^2
+8*D[3]*m[2]-32*nu(l)^2*D[3]+44*nu(l)*D[3]-15*D[3]-Q*D[3]*n[2]^2-D[3]*n[2]^
2*m[2]-Q*D[3]*m[2]^2-n[2]*D[4]*m[2]^2+6*D[4]*n[2]*m[2]+3*Q*D[3]*m[2]+3*Q
*D[4]*n[2]-15*nu(l)*D[3]*m[2]+8*nu(l)*D[3]*n[2]^2+nu(l)*D[4]*n[2]+nu(l)*D[4]*n
[2]^3+nu(l)*D[3]*m[2]^3+4*nu(l)^2*D[4]*n[2]-6*nu(l)*D[3]*m[2]^2+4*nu(l)^2*D[
3]*m[2]^2-4*Q^2*nu(l)^2*D[3]-24*Q*nu(l)^2*D[3]+16*Q*nu(l)*D[3]-4*nu(l)^2*D[
3]*n[2]^2+4*nu(l)^2*D[3]*m[2]+nu(l)*D[3]*n[2]^2*m[2]+n[2]*nu(l)*D[4]*m[2]^2-
Q*nu(l)*D[3]*m[2]^2+3*Q*nu(l)*D[3]*n[2]^2-4*n[2]*Q*nu(l)*D[4]*m[2]-2*Q^2*n
u(l)*D[4]*n[2]-2*Q*nu(l)^2*D[3]*n[2]^2+2*Q*nu(l)^2*D[3]*m[2]^2-2*Q^2*nu(l)*
D[3]*m[2]+4*Q*nu(l)^2*D[4]*n[2]*m[2]+2*Q^2*nu(l)^2*D[3]*m[2]+2*Q^2*nu(l)^
2*D[4]*n[2]+6*Q*nu(l)^2*D[3]*m[2]+6*Q*nu(l)^2*D[4]*n[2]+8*nu(l)^2*D[4]*n[2]*
m[2]-14*nu(l)*D[4]*n[2]*m[2]-11*Q*nu(l)*D[3]*m[2]-7*Q*nu(l)*D[4]*n[2])/(n[2]^2
+m[2]^2+16*nu(l)^2+4*Q*nu(l)*m[2]-24*nu(l)-6*m[2]-12*Q*nu(l)+8*nu(l)*m[2]+
9+4*Q^2*nu(l)^2+16*Q*nu(l)^2);
```

```
> b[4] :=1/3*(3*Q*D[3]*n[2]-D[3]*n[2]^3+D[4]*m[2]^3-8*D[4]*m[2]-2*D[4]*m[2]^2-
2*D[3]*n[2]+4*D[4]*n[2]^2-44*nu(l)*D[4]+32*nu(l)^2*D[4]+15*D[4]+Q*D[4]*m[2
]^2-D[3]*n[2]*m[2]^2+m[2]*D[4]*n[2]^2+Q*D[4]*n[2]^2-3*Q*D[4]*m[2]+6*D[3]*n
[2]*m[2]-16*Q*nu(l)*D[4]-4*nu(l)^2*D[4]*m[2]^2+nu(l)*D[3]*n[2]-nu(l)*D[4]*m[2]
^3+nu(l)*D[3]*n[2]^3-8*nu(l)*D[4]*n[2]^2+6*nu(l)*D[4]*m[2]^2+4*Q^2*nu(l)^2*
D[4]+24*Q*nu(l)^2*D[4]-4*nu(l)^2*D[4]*m[2]+4*nu(l)^2*D[4]*n[2]^2+4*nu(l)^2*
D[3]*n[2]+15*nu(l)*D[4]*m[2]+nu(l)*D[3]*n[2]*m[2]^2+Q*nu(l)*D[4]*m[2]^2-m[2
]*nu(l)*D[4]*n[2]^2+6*Q*nu(l)^2*D[3]*n[2]-6*Q*nu(l)^2*D[4]*m[2]-2*Q*nu(l)^2*
D[4]*m[2]^2-2*Q^2*nu(l)*D[3]*n[2]+2*Q^2*nu(l)^2*D[3]*n[2]-3*Q*nu(l)*D[4]*n[
2]^2+2*Q*nu(l)^2*D[4]*n[2]^2-4*m[2]*Q*nu(l)*D[3]*n[2]+2*m[2]*Q^2*nu(l)*D[4]
+4*m[2]*Q*nu(l)^2*D[3]*n[2]-2*m[2]*Q^2*nu(l)^2*D[4]+8*m[2]*nu(l)^2*D[3]*n[2
]-7*Q*nu(l)*D[3]*n[2]+11*Q*nu(l)*D[4]*m[2]-14*nu(l)*D[3]*n[2]*m[2])/(n[2]^2+m
[2]^2+16*nu(l)^2+4*Q*nu(l)*m[2]-24*nu(l)-6*m[2]-12*Q*nu(l)+8*nu(l)*m[2]+9+
4*Q^2*nu(l)^2+16*Q*nu(l)^2);
```

```
> eqn1 :=D[p1]*a-6*nu(p)*D[p2]*a^3/(1-2*nu(p)) = D[l1]*cos(n[1]*ln(a))*a^m[1]+
D[l2]*sin(n[1]*ln(a))*a^m[1]+D[l3]*cos(n[2]*ln(a))*a^m[2]+D[l4]*sin(n[2]*ln(a))*
a^m[2];
```

```
> eqn2 :=1/2*D[p1]*a-(7-4*nu(p))*D[p2]*a^3/(2-4*nu(p)) = b1*cos(n[1]*ln(a))*a^
m[1]+b2*sin(n[1]*ln(a))*a^m[1]+b3*cos(n[2]*ln(a))*a^m[2]+b4*sin(n[2]*ln(a))*a
^m[2];
```

```
> eqn3:=2*G(p)*D[p1]-6*nu(p)*D[p2]*a^2*G(p)/(-1+2*nu(p)) = -P*a^m[1]*(-nu(l)*
cos(n[1]*ln(a))*m[1]+cos(n[1]*ln(a))*m[1]+nu(l)*sin(n[1]*ln(a))*n[1]-sin(n[1]*ln(
a))*n[1]+2*nu(l)*cos(n[1]*ln(a)))/(1+nu(l))/(-1+2*nu(l))/a*D[l1]+P*a^m[1]*(nu(l)*
sin(n[1]*ln(a))*m[1]-2*nu(l)*sin(n[1]*ln(a))-cos(n[1]*ln(a))*n[1]+nu(l)*cos(n[1]*l
n(a))*n[1]-sin(n[1]*ln(a))*m[1])/(1+nu(l))/(-1+2*nu(l))/a*D[l2]-P*a^m[2]*(cos(n[2
]*ln(a))*m[2]-nu(l)*cos(n[2]*ln(a))*m[2]+nu(l)*sin(n[2]*ln(a))*n[2]-sin(n[2]*ln(a))
*n[2]+2*nu(l)*cos(n[2]*ln(a)))/(1+nu(l))/(-1+2*nu(l))/a*D[l3]+P*a^m[2]*(nu(l)*sin
```

```
(n[2]*ln(a))*m[2]+nu(l)*cos(n[2]*ln(a))*n[2]-sin(n[2]*ln(a))*m[2]-2*nu(l)*sin(n[2]
*ln(a))-cos(n[2]*ln(a))*n[2])/(1+nu(l))/(-1+2*nu(l))/a*D[l4]+6*nu(l)*(b1*cos(n[1]*
ln(a))*a^m[1]+b2*sin(n[1]*ln(a))*a^m[1]+b3*cos(n[2]*ln(a))*a^m[2]+b4*sin(n[2]
*ln(a))*a^m[2])*P/(1+nu(l))/(-1+2*nu(l))/a;
```

```
> eqn4:=2*G(p)*D[p1]+2*G(p)*a^2*(2*nu(p)+7)/(-1+2*nu(p))*D[p2] = P/(1+nu(l))
/a*D[l1]*cos(n[1]*ln(a))*a^m[1]+P/(1+nu(l))/a*D[l2]*sin(n[1]*ln(a))*a^m[1]+P/(1
+nu(l))/a*D[l3]*cos(n[2]*ln(a))*a^m[2]+P/(1+nu(l))/a*D[l4]*sin(n[2]*ln(a))*a^m[2
]+P*(-b2*sin(n[1]*ln(a))*a^m[1]-b4*sin(n[2]*ln(a))*a^m[2]+b2*cos(n[1]*ln(a))*n[
1]*a^m[1]+b2*sin(n[1]*ln(a))*a^m[1]*m[1]-b1*sin(n[1]*ln(a))*n[1]*a^m[1]+b1*c
os(n[1]*ln(a))*a^m[1]*m[1]+b4*cos(n[2]*ln(a))*n[2]*a^m[2]-b1*cos(n[1]*ln(a))*a
^m[1]-b3*sin(n[2]*ln(a))*n[2]*a^m[2]+b3*cos(n[2]*ln(a))*a^m[2]*m[2]+b4*sin(n[
2]*ln(a))*a^m[2]*m[2]-b3*cos(n[2]*ln(a))*a^m[2])/(1+nu(l))/a;
```

```
> eqn5 :=D[l1]*cos(n[1]*ln(b))*b^m[1]+D[l2]*sin(n[1]*ln(b))*b^m[1]+D[l3]*cos(n[2
]*ln(b))*b^m[2]+D[l4]*sin(n[2]*ln(b))*b^m[2] = D[m1]*b-6*nu(m)*D[m2]*b^3/(1-
2*nu(m))+3*D[m3]/b^4+(5-4*nu(m))*D[m4]/(1-2*nu(m))/b^2;
```

```
> eqn6 :=b1*cos(n[1]*ln(b))*b^m[1]+b2*sin(n[1]*ln(b))*b^m[1]+b3*cos(n[2]*ln(b))
*b^m[2]+b4*sin(n[2]*ln(b))*b^m[2] = 1/2*D[m1]*b-1/2*(7-4*nu(m))*D[m2]*b^3/(
1-2*nu(m))-D[m3]/b^4+D[m4]/b^2;
```

```
> eqn7 :=-P*b^Q*b^m[1]*(2*nu(l)*cos(n[1]*ln(b))+nu(l)*sin(n[1]*ln(b))*n[1]+cos(n
[1]*ln(b))*m[1]-nu(l)*cos(n[1]*ln(b))*m[1]-sin(n[1]*ln(b))*n[1])/(a^Q)/(1+nu(l))/(-
1+2*nu(l))/b*D[l1]+P*b^Q*b^m[1]*(-cos(n[1]*ln(b))*n[1]+nu(l)*cos(n[1]*ln(b))*n
[1]-sin(n[1]*ln(b))*m[1]+nu(l)*sin(n[1]*ln(b))*m[1]-2*nu(l)*sin(n[1]*ln(b)))/(a^Q)/
(1+nu(l))/(-1+2*nu(l))/b*D[l2]-P*b^Q*b^m[2]*(2*nu(l)*cos(n[2]*ln(b))+nu(l)*sin(
n[2]*ln(b))*n[2]-nu(l)*cos(n[2]*ln(b))*m[2]+cos(n[2]*ln(b))*m[2]-sin(n[2]*ln(b))*n
[2])/(a^Q)/(1+nu(l))/(-1+2*nu(l))/b*D[l3]+P*b^Q*b^m[2]*(-sin(n[2]*ln(b))*m[2]+n
u(l)*sin(n[2]*ln(b))*m[2]-cos(n[2]*ln(b))*n[2]+nu(l)*cos(n[2]*ln(b))*n[2]-2*nu(l)*s
in(n[2]*ln(b)))/(a^Q)/(1+nu(l))/(-1+2*nu(l))/b*D[l4]+6*nu(l)*(b1*cos(n[1]*ln(b))*b
^m[1]+b2*sin(n[1]*ln(b))*b^m[1]+b3*cos(n[2]*ln(b))*b^m[2]+b4*sin(n[2]*ln(b))*
b^m[2])*P*b^Q/(1+nu(l))/(-1+2*nu(l))/(a^Q)/b = 2*G(m)*D[m1]-6*nu(m)*b^2*G
(m)/(-1+2*nu(m))*D[m2]-24*G(m)*D[m3]/b^5-4/b^3*(nu(m)-5)*G(m)/(-1+2*nu(
m))*D[m4];
```

```
> eqn8 :=P*b^Q/(a^Q)/(1+nu(l))/b*D[l1]*cos(n[1]*ln(b))*b^m[1]+P*b^Q/(a^Q)/(1+
nu(l))/b*D[l2]*sin(n[1]*ln(b))*b^m[1]+P*b^Q/(a^Q)/(1+nu(l))/b*D[l3]*cos(n[2]*ln
(b))*b^m[2]+P*b^Q/(a^Q)/(1+nu(l))/b*D[l4]*sin(n[2]*ln(b))*b^m[2]-P*b^Q*(b2*s
in(n[1]*ln(b))*b^m[1]+b4*sin(n[2]*ln(b))*b^m[2]-b2*cos(n[1]*ln(b))*n[1]*b^m[1]-
b2*sin(n[1]*ln(b))*b^m[1]*m[1]+b1*sin(n[1]*ln(b))*n[1]*b^m[1]-b1*cos(n[1]*ln(b
))*b^m[1]*m[1]-b4*cos(n[2]*ln(b))*n[2]*b^m[2]+b1*cos(n[1]*ln(b))*b^m[1]+b3*s
in(n[2]*ln(b))*n[2]*b^m[2]-b3*cos(n[2]*ln(b))*b^m[2]*m[2]-b4*sin(n[2]*ln(b))*b^
m[2]*m[2]+b3*cos(n[2]*ln(b))*b^m[2])/(a^Q)/(1+nu(l))/b = 2*G(m)*D[m1]+2*G(
m)*b^2*(2*nu(m)+7)/(-1+2*nu(m))*D[m2]+16*G(m)*D[m3]/b^5-4*G(m)*(1+nu(
m))/b^3/(-1+2*nu(m))*D[m4];
```

```
> eqn9 :=D[m1]*c+6*nu(m)*c^3/(-1+2*nu(m))*D[m2]+3*D[m3]/c^4+(-5+4*nu(m))
/(-1+2*nu(m))/c^2*D[m4] = D[e1]*c+3*D[e3]/c^4+(-5+4*nu(e))/(-1+2*nu(e))/c^
2*D[e4];
```

```
> eqn10 := 1/2*D[m1]*c-1/2*(-7+4*nu(m))*c^3/(-1+2*nu(m))*D[m2]-D[m3]/c^4+
```

```
  D[m4]/c^2 = 1/2*D[e1]*c-D[e3]/c^4+D[e4]/c^2;
> eqn11 := G(m)*D[m1]-3*nu(m)*c^2*G(m)/(-1+2*nu(m))*D[m2]-12*G(m)/c^5*D[
  m3]-2/c^3*(nu(m)-5)*G(m)/(-1+2*nu(m))*D[m4] = G(e)*D[e1]-12*G(e)*D[e3]/c
  ^5-2/c^3*(nu(e)-5)*G(e)/(-1+2*nu(e))*D[e4];
> eqn12 :=2*G(m)*D[m1]+2*G(m)*c^2*(2*nu(m)+7)/(-1+2*nu(m))*D[m2]+16*G(
  m)/c^5*D[m3]-4*G(m)/c^3*(1+nu(m))/(-1+2*nu(m))*D[m4] = 2*G(e)*D[e1]+16*
  G(e)*D[e3]/c^5-4*G(e)/c^3*(1+nu(e))/(-1+2*nu(e))*D[e4];
> x:=solve({eqn1,eqn2,eqn3,eqn4,eqn5,eqn6,eqn7,eqn8,eqn9,eqn10,eqn11,eq
  n12},{D[p1],D[p2],D[l1],D[l2],D[l3],D[l4],D[m1],D[m2],D[m3],D[m4],D[e3],D[e4]
  });
> assign(x);
> G1:=D[e4];
> G2:=G1=0;
> n:=solve(G2,G(e));
> print(n);
> readlib(unassign):
> unassign('D[p1],D[p2],D[l1],D[l2],D[l3],D[l4],D[m1],D[m2],D[m3],D[m4],D[e3],
  D[e4]');
> od:
> end
```

APPENDIX G

This computer program generates the eqns. (2.75), (2.78) and (2.79) in thermal expansion coefficient study.

```
> u[r](r,theta,phi):=u[r](r):
> u[theta](r,theta,phi):=0:
> u[phi](r,theta,phi):=0:
> epsilon[rr]:=diff(u[r](r,theta,phi),r):
> epsilon[theta,theta]:=1/(r*sin(phi))*diff(u[theta](r,theta,phi),theta)+u[r](r,theta,p
  hi)/r+(cot(phi)/r)*u[phi](r,theta,phi):
> epsilon[phi,phi]:=diff(u[phi](r,theta,phi),phi)/r+u[r](r,theta,phi)/r:
> gamma[r, theta]:=1/(r*sin(phi))*diff(u[r](r,theta,phi),theta)+diff(u[theta](r,theta,
  phi),r)-u[theta](r,theta,phi)/r:
>  gamma[theta,phi]:=diff(u[theta](r,theta,phi),phi)/r-cot(phi)*u[theta](r,theta,phi)/
  r+1/(r*sin(phi))*diff(u[phi](r,theta,phi),theta):
> gamma[r,phi]:=diff(u[phi](r,theta,phi),r)-u[phi](r,theta,phi)/r+diff(u[r](r,theta,phi)
  ,phi)/r:
> e:=epsilon[rr]+epsilon[theta,theta]+epsilon[phi,phi]:
> sigma[rr]:=E(r)/(1+nu(r))*(epsilon[rr]-alpha(r)*T+(nu(r)/(1-2*nu(r)))*(e-3*alpha(
  r)*T)):
> sigma[theta,theta]:=E(r)/(1+nu(r))*(epsilon[theta,theta]-alpha(r)*T+(nu(r)/(1-2*
  nu(r)))*(e-3*alpha(r)*T)):
> sigma[phi,phi]:=E(r)/(1+nu(r))*(epsilon[phi,phi]-alpha(r)*T+(nu(r)/(1-2*nu(r)))*(
  e-3*alpha(r)*T)):
> tau[r,theta]:=E(r)/2/(1+nu(r))*gamma[r,theta]:
> tau[theta,phi]:=E(r)/2/(1+nu(r))*gamma[theta,phi]:
> tau[r,phi]:=E(r)/2/(1+nu(r))*gamma[r,phi]:
> equi1:=diff(sigma[rr],r)*sin(phi)+1/r*(2*sigma[rr]*sin(phi)-sigma[theta,theta]*si
  n(phi)-sigma[phi,phi]*sin(phi)+tau[r,phi]*cos(phi)+diff(tau[r,theta],theta)+diff(ta
  u[r,phi],phi)*sin(phi))=0:
> eqn1:=collect(equi1,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor):
> eqn2:=normal(eqn1*(-1+2*nu(r))/sin(phi)):
> eqn3:=collect(eqn2,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor);
```

$$eqn3 := \frac{(-1+\nu(r))\,\mathrm{E}(r)\left(\frac{\partial^2}{\partial r^2}u_{[r]}(r)\right)}{1+\nu(r)} + \left(\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r - 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r) - \left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^2 + 2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^3 + 4\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r) - 2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r)^2 - 4\,\mathrm{E}(r)\,\nu(r) - 2\,\mathrm{E}(r)\,\nu(r)^2 + 4\,\mathrm{E}(r)\,\nu(r)^3 + 2\,\mathrm{E}(r)\right)$$

$$\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)\Big/\left(r\,(-1+2\,\nu(r))\,(1+\nu(r))^2\right)-2\left(-\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)\right.$$
$$+\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^2+2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)r\,\nu(r)^3-2\,\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r\,\nu(r)^2$$
$$\left.-2\,\mathrm{E}(r)\,\nu(r)-\mathrm{E}(r)\,\nu(r)^2+2\,\mathrm{E}(r)\,\nu(r)^3-\mathrm{E}(r)\left(\frac{\partial}{\partial r}\nu(r)\right)r+\mathrm{E}(r)\right)u_{[r]}(r)\Big/\Big($$
$$(-1+2\,\nu(r))\,(1+\nu(r))^2\,r^2\Big)+\left(2\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\alpha(r)\,\nu(r)+2\,\mathrm{E}(r)\,\nu(r)\left(\frac{\partial}{\partial r}\alpha(r)\right)\right.$$
$$\left.-\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right)\alpha(r)-2\,\mathrm{E}(r)\,\alpha(r)\left(\frac{\partial}{\partial r}\nu(r)\right)-\mathrm{E}(r)\left(\frac{\partial}{\partial r}\alpha(r)\right)\right)T\Big/(-1+2\,\nu(r))=$$
$$0$$

This equation is equivalent to eqn. (2.75), and Poisson's ratio in this equation represents Poisson's ratio in interphase.

```
> E(r):=P*(r/a)^Q:
> alpha(r):=M*(r/a)^N:
> nu(r):=nu:
> eqn4:=eqn3:
> eqn5:=collect(eqn4,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor):
> eqn6:=normal(eqn5/(P*(r/a)^Q)):
> eqn7:=collect(eqn6,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor):
> eqn8:=normal(eqn7*(1+nu)*r^2/(-1+nu)):
> eqn9:=collect(eqn8,[diff(u[r](r),r,r),diff(u[r](r),r),u[r](r)],factor);
```

$$eqn9:=\left(\frac{\partial^2}{\partial r^2}u_{[r]}(r)\right)r^2+(Q+2)\,r\left(\frac{\partial}{\partial r}u_{[r]}(r)\right)-2\,\frac{(Q\,\nu+\nu-1)\,u_{[r]}(r)}{-1+\nu}$$
$$+\frac{T\,M\left(\dfrac{r}{a}\right)^N r\,(Q+N)\,(1+\nu)}{-1+\nu}=0$$

This equation is equivalent to eqn. (2.78)

```
> x:=dsolve(eqn9,u[r](r)):
> assign(x);
> Ur1:=u[r](r):
> u[r](r):=collect(Ur1,[_C1,_C2],factor);
```

$$u_{[r]}(r):=-4\,r^{\left(1/2\,\frac{-\sqrt{-1+\nu}\,Q-\sqrt{-1+\nu}+\%1}{\sqrt{-1+\nu}}\right)}(N^2\,\nu-3\,N-Q-Q\,\nu-Q\,N+Q\,N\,\nu+3\,\nu\,N-N^2)$$
$$\_C1\big/(\%3\,\%2)-4\,r^{\left(-1/2\,\frac{\sqrt{-1+\nu}\,Q+\sqrt{-1+\nu}+\%1}{\sqrt{-1+\nu}}\right)}$$

$$(N^2 \nu - 3N - Q - Q\nu - QN + QN\nu + 3\nu N - N^2)\ \_C2/(\%3\ \%2)$$
$$+ 4\frac{T M r^{(N+1)} a^{(-N)} (Q+N)(1+\nu)}{\%3\ \%2}$$

$$\%1 := \sqrt{Q^2 \nu + 10 Q \nu + 9\nu - 9 - 2Q - Q^2}$$

$$\%2 := \sqrt{-1+\nu}\, Q + 3\sqrt{-1+\nu} + \%1 + 2N\sqrt{-1+\nu}$$

$$\%3 := -\sqrt{-1+\nu}\, Q - 3\sqrt{-1+\nu} + \%1 - 2N\sqrt{-1+\nu}$$

This equation is equivalent to eqn. (2.79), and Poisson's ratio in this equation represents Poisson's ratio in interphase.

```
> xi:=coeff(u[r](r),_C1);
```

$$\xi := -4\, r^{\left(1/2 \frac{-\sqrt{-1+\nu}\, Q - \sqrt{-1+\nu} + \%1}{\sqrt{-1+\nu}}\right)} (N^2 \nu - 3N - Q - Q\nu - QN + QN\nu + 3\nu N - N^2)\Big/\Big(\left(-\sqrt{-1+\nu}\, Q - 3\sqrt{-1+\nu} + \%1 - 2N\sqrt{-1+\nu}\right) \left(\sqrt{-1+\nu}\, Q + 3\sqrt{-1+\nu} + \%1 + 2N\sqrt{-1+\nu}\right)\Big)$$

$$\%1 := \sqrt{Q^2 \nu + 10 Q \nu + 9\nu - 9 - 2Q - Q^2}$$

```
> beta:=coeff(u[r](r),_C2);
```

$$\beta := -4\, r^{\left(-1/2 \frac{\sqrt{-1+\nu}\, Q + \sqrt{-1+\nu} + \%1}{\sqrt{-1+\nu}}\right)} (N^2 \nu - 3N - Q - Q\nu - QN + QN\nu + 3\nu N - N^2)\Big/\Big(\left(-\sqrt{-1+\nu}\, Q - 3\sqrt{-1+\nu} + \%1 - 2N\sqrt{-1+\nu}\right) \left(\sqrt{-1+\nu}\, Q + 3\sqrt{-1+\nu} + \%1 + 2N\sqrt{-1+\nu}\right)\Big)$$

$$\%1 := \sqrt{Q^2 \nu + 10 Q \nu + 9\nu - 9 - 2Q - Q^2}$$

```
> eta:=u[r](r)-alpha*_C1-beta*_C2;
```

$$\eta := 4\, T M r^{(N+1)} a^{(-N)} (Q+N)(1+\nu)\Big/\Big(\left(-\sqrt{-1+\nu}\, Q - 3\sqrt{-1+\nu} + \sqrt{Q^2 \nu + 10 Q \nu + 9\nu - 9 - 2Q - Q^2} - 2N\sqrt{-1+\nu}\right) \left(\sqrt{-1+\nu}\, Q + 3\sqrt{-1+\nu} + \sqrt{Q^2 \nu + 10 Q \nu + 9\nu - 9 - 2Q - Q^2} + 2N\sqrt{-1+\nu}\right)\Big)$$

```
>
```

APPENDIX H

This computer program evaluates the thermal expansion coefficient of the composite with power variation Young's modulus and thermal expansion coefficient in interphase.

```
> P:=65:
> Q:=(ln(E(m)/E(p))/(ln(b/a)):
> N:=:(ln(alpha(m)/alpha(p))/(ln(b/a)):
> alpha(p):=5*10^(-6):
> E(p):=65:
> nu(p):=3/10:
> nu(l):=3/10:
> E(m):=28/10:
> nu(m):=4/10:
> alpha(m):=66*10^(-6):
> M:=5*10^(-6):
> b:=(?/?)*a:(Input thickness value of interphase.)
> c:=1:
> eqn1 :=C[pt1]*a = 4*a^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+
9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q
*N*nu(l)-N^2+N^2*nu(l)-Q*nu(l)-Q)/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*
Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l
))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+
2*N*(-1+nu(l))^(1/2))*C[lt1]+4*a^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10
*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*
N-Q*N+Q*N*nu(l)-N^2+N^2*nu(l)-Q*nu(l)-Q)/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/
2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(
3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q
-9)^(1/2)+2*N*(-1+nu(l))^(1/2))*C[lt2]-4*T*M*a^(N+1)*a^(-N)*(1+nu(l))*(Q+N)/(
3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-
9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*n
u(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2)):
> eqn2 := -E(p)/(-1+2*nu(p))*C[pt1]+T*alpha(p)*E(p)/(-1+2*nu(p)) = -2*P*a^(-1/
2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q
-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q*N*nu(l)-N^2+N^2*nu(l)-Q*
nu(l)-Q)*(-(-1+nu(l))^(1/2)+5*(-1+nu(l))^(1/2)*nu(l)-(-1+nu(l))^(1/2)*Q+(-1+nu(l)
)^(1/2)*Q*nu(l)+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)-(10*Q*nu(l)+
9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)*nu(l))/(1+nu(l))/(-1+2*nu(l))/(-1+nu(l))^(1/
2)/a/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2
-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10
*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))*C[lt1]-2*P
*a^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q
```

```
^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q*N*nu(l)-N^2+N^2*n
u(l)-Q*nu(l)-Q)*(5*(-1+nu(l))^(1/2)*nu(l)+(-1+nu(l))^(1/2)*Q*nu(l)+(10*Q*nu(l)+
9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)*nu(l)-(-1+nu(l))^(1/2)-(-1+nu(l))^(1/2)*Q-(
10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(1+nu(l))/(-1+2*nu(l))/(-1+nu(
l))^(1/2)/a/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l
)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*
Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))*C[lt
2]+4*(-3*a*N+a^(N+1)*a^(-N)*Q-a*Q+a^(N+1)*a^(-N)*N^2+a^(N+1)*a^(-N)*N
+a^(N+1)*a^(-N)*Q*N-a^(N+1)*a^(-N)*Q*N*nu(l)-a^(N+1)*a^(-N)*N^2*nu(l)+a
^(N+1)*a^(-N)*Q*nu(l)+a^(N+1)*a^(-N)*nu(l)*N-a*Q*nu(l)+3*a*N*nu(l)-a*Q*N+
a*Q*N*nu(l)-a*N^2+a*N^2*nu(l))*T*M*P/(-1+2*nu(l))/(3*(-1+nu(l))^(1/2)+(-1+n
u(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^
(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q
^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/a:
```

```
> eqn3 :=4*b^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^
2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q*N*nu(l)-N
^2+N^2*nu(l)-Q*nu(l)-Q)/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*
nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-
1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+n
u(l))^(1/2))*C[lt1]+4*b^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+
9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q
*N*nu(l)-N^2+N^2*nu(l)-Q*nu(l)-Q)/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*
Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l
))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+
2*N*(-1+nu(l))^(1/2))*C[lt2]-4*T*M*b^(N+1)*a^(-N)*(1+nu(l))*(Q+N)/(3*(-1+nu(l
))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2
*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)
+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2)) = C[mt1]*b+C[mt2]/b^2:
```

```
> eqn4 :=-2*P*(b/a)^Q*b^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+
9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*nu(l)*N-Q*N+Q
*N*nu(l)-N^2+N^2*nu(l)-Q*nu(l)-Q)*(-(-1+nu(l))^(1/2)+5*(-1+nu(l))^(1/2)*nu(l)-(
-1+nu(l))^(1/2)*Q+(-1+nu(l))^(1/2)*Q*nu(l)+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2
-2*Q-9)^(1/2)-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)*nu(l))/(1+nu(l))
/(-1+2*nu(l))/(-1+nu(l))^(1/2)/b/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu
(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/
2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(
-1+nu(l))^(1/2))*C[lt1]-2*P*(b/a)^Q*b^(-1/2*((-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q
+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2))/(-1+nu(l))^(1/2))*(-3*N+3*n
u(l)*N-Q*N+Q*N*nu(l)-N^2+N^2*nu(l)-Q*nu(l)-Q)*(5*(-1+nu(l))^(1/2)*nu(l)+(-1
+nu(l))^(1/2)*Q*nu(l)+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)*nu(l)-(-
1+nu(l))^(1/2)-(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(
1/2))/(1+nu(l))/(-1+2*nu(l))/(-1+nu(l))^(1/2)/b/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/
2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(
3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q
-9)^(1/2)+2*N*(-1+nu(l))^(1/2))*C[lt2]+4*(-3*(b/a)^N*b*N+b^(N+1)*a^(-N)*Q-(b
/a)^N*b*Q+b^(N+1)*a^(-N)*N^2+b^(N+1)*a^(-N)*N+b^(N+1)*a^(-N)*Q*N-b^(N
```

```
+1)*a^(-N)*Q*N*nu(l)-b^(N+1)*a^(-N)*N^2*nu(l)+b^(N+1)*a^(-N)*Q*nu(l)+b^(N
+1)*a^(-N)*nu(l)*N-(b/a)^N*b*Q*nu(l)+3*(b/a)^N*b*N*nu(l)-(b/a)^N*b*Q*N+(b/
a)^N*b*Q*N*nu(l)-(b/a)^N*b*N^2+(b/a)^N*b*N^2*nu(l))*T*M*P*(b/a)^Q/(-1+2*
nu(l))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q-(10*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^
2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/(3*(-1+nu(l))^(1/2)+(-1+nu(l))^(1/2)*Q+(1
0*Q*nu(l)+9*nu(l)+Q^2*nu(l)-Q^2-2*Q-9)^(1/2)+2*N*(-1+nu(l))^(1/2))/b = -E(m
)/(-1+2*nu(m))*C[mt1]-2*E(m)/(1+nu(m))/b^3*C[mt2]+T*alpha(m)*E(m)/(-1+2*
nu(m)):
> eqn5 := -E(m)/(-1+2*nu(m))*m1-2*E(m)/(1+nu(m))/c^3*m2+T*
  alpha(m)*E(m)/(-1+2*nu(m)) = 0:
> x:=solve({eqn1,eqn2,eqn3,eqn4,eqn5},{p1,l1,l2,m1,m2}):
> assign(x);
> alpha:=(m1+m2/c^3)/T:
> eq1:=evalf(alpha):
> eq2:=normal(eq1):
> eq3:=subs(a=f^(1/3),eq2):
```

APPENDIX I

This computer program proves that the relationship between thermal expansion coefficient and bulk modulus for two phases composite is valid in the CSA model for particle reinfored composite.

For no interphase case:

```
> K[c]:=-(-4*K[m]*G[m]-3*K[p]*K[m]+4*f*G[m]*K[m]-4*f*G[m]*K[p])/(3*K[m]*f+4*
  G[m]-3*f*K[p]+3*K[p]);
```

$$K_{[c]} := -\frac{-4\,K_{[m]}\,G_{[m]} - 3\,K_{[p]}\,K_{[m]} + 4\,f\,G_{[m]}\,K_{[m]} - 4\,f\,G_{[m]}\,K_{[p]}}{3\,K_{[m]}\,f + 4\,G_{[m]} - 3\,f\,K_{[p]} + 3\,K_{[p]}} \tag{I.1}$$

```
> alpha[c]:= (3*f*E[p]*alpha[p]-3*f*E[p]*alpha[p]*nu[m]-2*alpha[m]*f*E[m]+4*alp
  ha[m]*f*E[m]*nu[p]+alpha[m]*E[p]+alpha[m]*E[p]*nu[m]+2*alpha[m]*E[m]-4*al
  pha[m]*E[m]*nu[p]-f*E[p]*alpha[m]-f*E[p]*alpha[m]*nu[m])/(2*f*E[p]-4*f*E[p]*n
  u[m]-2*f*E[m]+4*f*E[m]*nu[p]+E[p]+E[p]*nu[m]+2*E[m]-4*E[m]*nu[p]);
```

$$\alpha_{[c]} := \Big(3\,f\,E_{[p]}\,\alpha_{[p]} - 3\,f\,E_{[p]}\,\alpha_{[p]}\,\nu_{[m]} - 2\,\alpha_{[m]}\,f\,E_{[m]} + 4\,\alpha_{[m]}\,f\,E_{[m]}\,\nu_{[p]} + \alpha_{[m]}\,E_{[p]} + \alpha_{[m]}\,E_{[p]}\,\nu_{[m]} + 2\,\alpha_{[m]}\,E_{[m]} - 4\,\alpha_{[m]}\,E_{[m]}\,\nu_{[p]} - f\,E_{[p]}\,\alpha_{[m]} - f\,E_{[p]}\,\alpha_{[m]}\,\nu_{[m]}\Big) \Big/ \Big(2\,f\,E_{[p]} - 4\,f\,E_{[p]}\,\nu_{[m]} - 2\,f\,E_{[m]} + 4\,f\,E_{[m]}\,\nu_{[p]} + E_{[p]} + E_{[p]}\,\nu_{[m]} + 2\,E_{[m]} - 4\,E_{[m]}\,\nu_{[p]}\Big) \tag{I.2}$$

From Christensen (1979), we have

```
> alpha[c]:=alpha[m]+(alpha[p]-alpha[m])/(1/K[p]-1/K[m])*(1/K[c]-1/K[m]);
```

$$\alpha_{[c]} := \alpha_{[m]} + \frac{\left(\alpha_{[p]} - \alpha_{[m]}\right)\left(\frac{1}{K_{[c]}} - \frac{1}{K_{[m]}}\right)}{\frac{1}{K_{[p]}} - \frac{1}{K_{[m]}}} \tag{I.3}$$

Subbstite K[c][eqn. (I.1)] into above eqn. (I.3), we have

```
> alpha[c] := alpha[m]+(alpha[p]-alpha[m])/(1/K[p]-1/K[m])*(-1/(-4*K[m]*G[m]-3*
  K[p]*K[m]+4*f*G[m]*K[m]-4*f*G[m]*K[p])*(3*K[m]*f+4*G[m]-3*f*K[p]+3*K[p])-1/
  K[m]);
```

$$\alpha_{[c]} := \alpha_{[m]} + \left(\alpha_{[p]} - \alpha_{[m]}\right)$$

$$\left(-\frac{3K_{[m]}f+4G_{[m]}-3fK_{[p]}+3K_{[p]}}{-4K_{[m]}G_{[m]}-3K_{[p]}K_{[m]}+4fG_{[m]}K_{[m]}-4fG_{[m]}K_{[p]}}-\frac{1}{K_{[m]}}\right)\Bigg/\left(\frac{1}{K_{[p]}}-\frac{1}{K_{[m]}}\right)$$

> a1:=simplify(alpha[c]);

$$a1 := -\left(-3\,\alpha_{[m]}K_{[p]}K_{[m]}-3fK_{[p]}\alpha_{[p]}K_{[m]}-4K_{[p]}f\alpha_{[p]}G_{[m]}+3\,\alpha_{[m]}K_{[m]}fK_{[p]}+4\,\alpha_{[m]}K_{[m]}fG_{[m]}-4\,\alpha_{[m]}K_{[m]}G_{[m]}\right)\Big/\left(4K_{[m]}G_{[m]}+3K_{[p]}K_{[m]}-4fG_{[m]}K_{[m]}+4fG_{[m]}K_{[p]}\right)$$

(I.4)

Note, we have following relations

> nu:=1/2*(3*K-2*G)/(3*K+G); E:=9*G*K/(3*K+G);

$$\nu := \frac{1}{2}\frac{3K-2G}{3K+G}$$

$$E = 9\frac{GK}{3K+G}$$

(I.5)

Substitue eqn. (I.5) into eqn. (I.2)

> nu[p]:=1/2*(3*K[p]-2*G[p])/(3*K[p]+G[p]); E[p]:=9*G[p]*K[p]/(3*K[p]+G[p]);nu[m]:=1/2*(3*K[m]-2*G[m])/(3*K[m]+G[m]); E[m]:=9*G[m]*K[m]/(3*K[m]+G[m]);

$$\nu_{[p]} := \frac{1}{2}\frac{3K_{[p]}-2G_{[p]}}{3K_{[p]}+G_{[p]}}$$

$$E_{[p]} := 9\frac{G_{[p]}K_{[p]}}{3K_{[p]}+G_{[p]}}$$

$$\nu_{[m]} := \frac{1}{2}\frac{3K_{[m]}-2G_{[m]}}{3K_{[m]}+G_{[m]}}$$

$$E_{[m]} := 9\frac{G_{[m]}K_{[m]}}{3K_{[m]}+G_{[m]}}$$

> alpha[c]:= (3*f*E[p]*alpha[p]-3*f*E[p]*alpha[p]*nu[m]-2*alpha[m]*f*E[m]+4*alpha[m]*f*E[m]*nu[p]+alpha[m]*E[p]+alpha[m]*E[p]*nu[m]+2*alpha[m]*E[m]-4*alpha[m]*E[m]*nu[p]-f*E[p]*alpha[m]-f*E[p]*alpha[m]*nu[m])/(2*f*E[p]-4*f*E[p]*nu[m]-2*f*E[m]+4*f*E[m]*nu[p]+E[p]+E[p]*nu[m]+2*E[m]-4*E[m]*nu[p]):

```
> al1:=simplify(alpha[c]);
```

$$al1 := \left(3 f K_{[p]} \alpha_{[p]} K_{[m]} + 4 f K_{[p]} \alpha_{[p]} G_{[m]} - 4 \alpha_{[m]} f G_{[m]} K_{[m]} + 3 \alpha_{[m]} K_{[p]} K_{[m]} + 4 \alpha_{[m]} G_{[m]} K_{[m]} - 3 f K_{[p]} \alpha_{[m]} K_{[m]}\right) \Big/ \left(4 f K_{[p]} G_{[m]} - 4 f G_{[m]} K_{[m]} + 3 K_{[p]} K_{[m]} + 4 G_{[m]} K_{[m]}\right)$$

(I.6)

eqn. (I.2) can be rearranged as eqn. (I.6)

Comparing eqn. (I.6) and eqn. (I.4)

```
> a1 := -(-3*alpha[m]*K[p]*K[m]-3*f*K[p]*alpha[p]*K[m]-4*K[p]*f*alpha[p]*G[m]+
3*alpha[m]*K[m]*f*K[p]+4*alpha[m]*K[m]*f*G[m]-4*alpha[m]*K[m]*G[m])/(4*K[
m]*G[m]+3*K[p]*K[m]-4*f*G[m]*K[m]+4*f*G[m]*K[p]):
> al1 := (3*f*K[p]*alpha[p]*K[m]+4*f*K[p]*alpha[p]*G[m]-4*alpha[m]*f*G[m]*K[m]
+3*alpha[m]*K[p]*K[m]+4*alpha[m]*G[m]*K[m]-3*f*K[p]*alpha[m]*K[m])/(4*f*K[
p]*G[m]-4*f*G[m]*K[m]+3*K[p]*K[m]+4*G[m]*K[m]):
> eqn1:=a1-al1:
> eqn2:=simplify(eqn1);
```

$$eqn2 := 0$$

That implies eqn. (I.4) and eqn. (I.6) are identical. So, we prove the relationship btween bulk modulus and thermal expansion coefficient is valid in CCA model.

```
>
```

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