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TESTING ECONOMIC TIME SERIES FOR STATIONARITY AND NONSTATIONARITY

By

Yongcheol Shin

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ABSTRACT

TESTING ECONOMIC TIME SERIES FOR STATIONARITY AND NONSTATIONARITY

By

Yongcheol Shin

It is a well-established empirical fact that standard unit root tests fail to reject the unit root hypothesis for many economic time series. However, these results do not indicate strong evidence against relevant trend stationarity alternatives, because it is well-known that unit root tests are not very powerful.

Recently, various attempts, including a Bayesian approach, have been made to reconsider the important problem of distinguishing trend stationary and unit root processes. However, there have been very few previous attempts to test the null hypothesis of stationarity directly. Kwiatkowski, Phillips, Schmidt, and Shin (1992, KPSS) propose an LM test of the null hypothesis that an observable series is stationary around a deterministic trend, using the components representation in which the series is decomposed into the sum of deterministic trend, random walk, and stationary error.

This dissertation extends the KPSS test statistic for stationarity in two ways. First, finite sample size and power of the KPSS statistic for stationarity are extensively studied in a Monte Carlo experiment. Next the use of the KPSS statistic as a unit root test is suggested, because the KPSS statistic is consistent and a different limning

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distribution is obtained under the hypothesis that the series is difference stationary.

Both tests are applied to the Nelson-Plosser data, and for many of these series it is not very clear whether they contain a unit root or are trend stationary. These results are quite consistent with recent (inconclusive) empirical findings.

One implication of the above empirical findings is that many economic time series may be in the region of "near stationarity." A lot of Monte Carlo studies have shown that standard unit root tests have severe size distortions when the process is nearly stationary. This dissertation also considers the asymptotics of standard unit root tests in this case using generalized "nearly stationary model." It is found that the above size distortion problem is well predicted by our asymptotics. It is also argued that the superiority of the augmented Dickey-Fuller statistic is not established and that more efficient estimation techniques will be needed to improve the tradeoff between size distortions and low power.

Dedicated to my family

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

Many economic time series are clearly nonstationary, and one important statistical issue is the appropriate representation of this nonstationarity. For simplicity, assume that any deterministic trend is linear, so that we can write

$$(1) \qquad y_t = \psi + \xi t + X_t, \quad t = 1,\cdots,T,$$

where y_t is the observed series, $(\psi + \xi t)$ represents deterministic trend, and X_t is the unobserved stochastic deviation of y_t from deterministic trend. If X_t is stationary, then y_t is often said to be "trend stationary", and the long-run behavior of y_t is essentially determined by its deterministic trend component. However, if X_t is I(1), so that ΔX_t is stationary, y_t is said to be "difference stationary", and $\Delta y_t = \xi + \Delta X_t$ so that changes in y_t contain a component that is fundamentally unpredictable even in the long run. The trend stationary (TS) and difference stationary (DS) time series have very different long run properties, and this has important economic and statistical implications.

There has been extensive interest in the use of the autoregressive integrated moving average (ARIMA) process for modelling nonstationary time series. Ignoring deterministic trend, for the moment, suppose that the time series is represented by the

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(2) $$y_t = \beta y_{t-1} + \varepsilon_t,\ \varepsilon_t = u_t + \theta u_{t-1}.$$

If $\beta = 1$, y_t has a "unit root" in its AR representation, and y_t is difference stationary so long as θ is not equal to -1. In fact y_t is a random walk if $\beta = 1$ and $\theta = 0$. Unit root tests typically test the hypothesis $\beta = 1$, and θ is a nuisance parameter. However, these roles can be reversed. If $\beta = 1$ and $\theta = -1$, y_t is white noise. More generally, we can test the stationarity hypothesis $\theta = -1$, in which case β is a nuisance parameter. If a series is generated by a member of the linear TS class we should fail to reject the hypothesis of a unit MA root in the ARMA model for its first difference, and if it is generated by a member of the DS subclass we should fail to reject the hypothesis of a unit AR root in the ARMA model for its levels.

As noted above, the difference between DS and TS series may be economically important, since a unit AR root implies long run persistence, in the sense that at least part of the effects of random shocks on macroeconomic variables are permanent. Correct treatment of the stationary or nonstationary nature of the data is also necessary for meaningful statistical inference, owing to the spurious regression phenomenon pointed out by Granger and Newbold (1974) and Phillips (1986). Thus, it is important to be able to distinguish a series with a unit root from a stationary series. General surveys of the unit root literature are given by Diebold and Nerlove (1990) and Campbell and Perron (1991).

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1.2 Unit Root Tests and Error Autocorrelation

Attempts to distinguish the DS series from the TS series have generally taken the form of a test of the null hypothesis of a unit AR root against the alternative of stationarity. Most of the existing unit roots tests are variants of the Dickey and Fuller (DF) tests provided by Fuller (1976) and Dickey and Fuller (1979). The DF unit root tests are based on the regressions:

(3) $$y_t = \beta y_{t-1} + \varepsilon_t$$

(4) $$y_t = \alpha + \beta y_{t-1} + \varepsilon_t$$

(5) $$y_t = \alpha + \beta y_{t-1} + \delta t + \varepsilon_t$$

for t = 1,...,T. In each case, the unit root hypothesis is $\beta = 1$. Two types of test statistics are used: one is the normalized coefficient test statistic $T(\hat{\beta} - 1)$, where $\hat{\beta}$ is the OLS estimate of β; this yields the DF statistics $\hat{\rho}$, $\hat{\rho}_\mu$ and $\hat{\rho}_\tau$ from regressions (3), (4) and (5), respectively. The other is the usual t-statistic for testing the hypothesis $\beta = 1$, which yields the DF statistics $\hat{\tau}$, $\hat{\tau}_\mu$ and $\hat{\tau}_\tau$ corresponding to the same three regressions.

The DF regressions differ in the way that they handle level and deterministic trend. The regression (3) does not allow for non-zero level or trend under the alternative. The equation (4) allows for linear deterministic trend under the null, but it allows only for non-zero level under the alternative. Finally, regression (5) allows for non-

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non-zero level and linear and quadratic trends under the null, but implies level and linear trend under the alternative. The $\hat{\rho}_\tau$ and $\hat{\tau}_\tau$ tests based on regression (5) are the most commonly used in econometric work, because inclusion of the trend term (δt) in (5) is necessary for the tests to be consistent against TS alternatives.

Schmidt and Phillips (1991, SP) suggest an alternative set of unit root tests, based on the parameterization:

(6) $$y_t = \psi + \xi \cdot t + X_t, \quad X_t = \beta X_{t-1} + \varepsilon_t,$$

Note that this parameterization is also used by Bhargava (1986), and it mimics the form of equation (1) above. The SP test statistics are based on the LM (score) principle for the null hypothesis $\beta = 1$ in equation (6). The test statistics are derived from the regression:

(7) $$\Delta y_t = \text{constant} + \phi \tilde{S}_{t-1} + \text{error}, \quad t = 2, \cdots, T$$

where $\tilde{S}_{t-1} = y_{t-1} - \tilde{\psi}_X - \tilde{\xi}(t-1)$ and the restricted MLE's of ξ and $\psi_X = \psi + X_0$ are given by: $\tilde{\xi} = (y_T - y_1)/(T-1)$, and $\tilde{\psi}_X = y_1 - \tilde{\xi}$. Let $\tilde{\phi}$ be the OLS estimate of ϕ for (5). The test statistics are given by:

(8) $$\tilde{\rho} = T\tilde{\phi}$$

(9) $$\tilde{\tau} = \text{t-statistic for the hypothesis } \phi = 0.$$

SP show that $\tilde{\rho}$ and $\tilde{\tau}$ are monotonic transformations of each other, so the tests are identical in the absence of corrections for error autocorrelation. Also, $\tilde{\rho}$ is almost identical to the R_2 statistic of Bhargava (1986).

The main difference between the DF and SP parameterizations is that the meanings of the parameters α and δ in (4) and (5) are different under the null and under the alternative, while in the SP parameterization (6), ψ and ξ represent level and linear trend respectively under both the null and the alternative. The distribution of the SP test statistics $\tilde{\rho}$ and $\tilde{\tau}$, and of the DF test statistics $\hat{\rho}_\tau$ and $\hat{t}_\tau$, are independent of ψ and ξ in (6); this is the further evidence of the usefulness of equation (6) to represent the data generating process.

Since the tabulated distributions for the DF and SP test statistics are obtained under the assumption that the errors in the model (i.e, ε_t in equations (3), (4), (5), and (6)) are iid, they are not expected to be robust to more general error structures, in particular to the presence of autocorrelated errors. Furthermore, it is known from Phillips (1987), Phillips and Perron (1988) and Schmidt and Phillips (1991) that error autocorrelation affects the distributions of the test statistics even asymptotically. Therefore, modifications of the basic DF and the SP test statistics have been developed that allow for error autocorrelation. These modifications can be put into four groups.

First, augmented Dickey-Fuller (ADF) tests are proposed to accommodate error autocorrelation by adding lagged differences of y_t to the regressions (3) - (5). Said and Dickey (1984, 1985) show that, if the number of lagged differences is suitably chosen, the ADF test statistics have the same asymptotic distribution as the original DF test statistics would have under iid errors. If the errors are AR(p), the number of lagged differences must be at least as large as p. If the errors have an MA

component, the number of lagged differences is allowed to increase with sample size, though at a slower rate (e.g., at the rate $T^{1/3}$). Lee (1990) proposes an analogous augmented version of the SP tests.

Second, Phillips (1987) and Phillips and Perron (1988, PP) use semiparametric corrections to the DF statistics to develop general tests to allow for a wide class of weakly dependent and heterogeneous errors. The limiting distributions of the corrected test statistics are the same as the original DF test statistics would have under iid errors. SP provide similar semiparametric corrections for their tests.

Third, Hall (1989) has considered unit root tests based on instrumental variables (IV) estimation of the DF regressions. He assumes MA(q) errors, with q treated as known and y_{t-k}, with $k > q$, is used as the instrument for y_{t-1}. Lee and Schmidt (1991) also propose similar IV versions of the SP test statistics, where the instrument for $\tilde{S}_{t-1}$ in (7) is $\tilde{S}_{t-k}$, with $k > q$.

Finally, Choi (1990) develops tests based on GLS estimation. This requires a parametric (ARMA) form for the autocorrelation.

A large body of simulation evidence has shown that these methods of accommodating error autocorrelation can perform poorly in finite samples. Phillips and Perron (1988), Schwert (1989), Kim and Schmidt (1990) and Lee (1990) have shown that the uncorrected DF and SP tests reject the true null hypothesis too often in the presence of negative autocorrelation and too seldom in the presence of positive autocorrelation. These size distortions can be quite considerable. For example, with MA(1) errors with $\theta = -0.8$, the size of the tests (the probability of rejecting the null when it is true) is almost unity, even for T as large as 500 or 1000. The Phillips-Perron corrected DF and SP tests perform somewhat better than the uncorrected tests, but still suffer from considerable size distortions even for surprisingly large sample sizes. The augmented DF and SP tests also perform somewhat better than the uncorrected tests, with the extent of the improvement depending on the number of augmentations. With a sufficiently large number of augmentations, the size of the augmented tests becomes more or less correct, even for cases in which the errors are strongly autocorrelated. However, this result is less optimistic than it might first seem, because the tests with many augmentations have almost no power.

Hall finds that his IV tests are a significant improvement over the uncorrected DF or the PP tests, when the errors are MA(1). Lee and Schmidt (1991) also provide fairly optimistic results for the IV versions of the SP tests, which have surprisingly smaller size distortions and greater power than other tests of similar size. Pantula and Hall (1991) provide some moderately optimistic results for the Hall's IV tests when the errors are ARMA. Similarly, Choi's results for his GLS-based tests are fairly good for most parameter values. However, no testing procedures seem to work well in finite samples with strongly negatively correlated errors.

1.3 Unit Root Tests Under Near Stationarity

To interpret the Monte Carlo evidence summarized in the last section, we return to the ARMA representation of y_t given in equation (2). Suppose that $\beta = 1$ so that there is a unit AR root. As $\theta \rightarrow -1$, y_t approaches stationarity. Correspondingly, the process can be called "nearly stationary" when θ is close to but not equal to minus one; for example, when $\theta = -0.8$.

It is not surprising that most unit root tests perform very poorly when the process is nearly stationary. For example, Blough (1989) argues that there is no discontinuity between unit root and stationary processes. For a given sample size, every stationary process can be arbitrarily well approximated by a unit root process which is nearly stationary. Correspondingly, a true level α test of the unit root null cannot have power greater than α against stationary alternatives. Therefore, the tests with small size distortions in the presence of strongly negatively correlated errors might be expected to be essentially without power. For example, Blough's Monte Carlo simulations show that, for T = 100, the ADF unit root test with 12 lags has power of only 20% against white noise. Power is even lower when the stationary process is serially correlated. Therefore, it seems that no tests may survive in the presence of strongly autocorrelated errors in terms of <u>both</u> size and power of the tests. The tests with correct size have poor power and the tests with high power have serious size distortions.

The preceding discussion reflects the fact that, in the nearly stationary case, the unit root asymptotic distributions are poor guides to the finite sample distributions of the test statistics, even for fairly large sample sizes. Thus it may be useful to find other types of asymptotic distributions to approximate finite sample distributions in this case. The important step in this direction has been taken by Pantula (1991), who investigates the behavior of some unit root test statistics under the null of a unit root when the process is nearly stationary, using the local approximation for $\theta = -1$:

(10) $$\theta = -1 + 1/T^{\delta}, \quad \delta > 0.$$

Combining (2) and (10), y_t becomes a random walk as δ approaches zero; however, for fixed δ, y_t approaches white noise as $T \rightarrow \infty$. Pantula shows that the asymptotic distribution of the unit root test statistics depends on the speed with which θ approaches minus one (i.e., the value of δ) as well as the sample size, T. Pantula uses his asymptotic distributions to infer differences among tests in their sensitivity to near stationarity, and, based on these differences, he suggests the use of the augmented Dickey-Fuller (ADF) test. However, note that he does not consider the power of the tests.

In Chapter 4, we will investigate the behavior of the DF and the SP unit root test statistics under the null of a unit root when the process is nearly stationary by using the more general local approximation for $\theta = -1$:

(11) $$\theta = -1 + C/T^{\delta}, \quad C > 0, \delta > 0.$$

With two parameters (C and δ) instead of only one, we can hope to find more accurate asymptotic approximations to the finite sample distributions of the test statistics. Furthermore, our concern is somewhat different from Pantula's. We will make detailed comparisons of our asymptotic approximations and the true finite sample distributions (calculated by a Monte Carlo simulation), to see under what conditions these asymptotic distributions are accurate enough to be useful. This is actually a logical prior step to Pantula's type of analysis, since there is no point in choosing tests based on inaccurate asymptotic approximations.

1.4 Testing the Null Hypothesis of Stationarity

Nelson and Plosser (1982) failed to reject the hypothesis that long historical time series for the U.S. are difference stationary, using the DF tests. Similar results have been obtained using the SP tests and other types of unit root tests, which have generally failed to reject the null hypothesis of a unit root in many macroeconomic time series.

However, it is important to note that in this empirical work the unit root is set up as the null hypothesis to be tested, and the way in which classical hypothesis testing is carried out ensures that the null hypothesis is accepted unless there is strong evidence against it. Therefore, an alternative explanation for the common failure to reject a unit root is simply that standard unit root tests are not very powerful against relevant alternatives. For example, see Dejong et al. (1989). This point is also discussed in the recent survey paper by Campbell and Perron (1991).

Therefore, in trying to decide whether economic data are stationary or integrated, it would be useful to have available tests of the null hypothesis of stationarity as well as tests of the null hypothesis of a unit root. There have been relatively few previous attempts to test the null hypothesis of stationarity. See Park and Choi (1988), Rudebush (1990), Dejong et al. (1989), for examples. These are reasonable first attempts to test stationarity, but they suffer from the lack of a plausible model in which the null of stationarity is naturally framed as a parametric restriction.

Recently, Kwiatkowski, Phillips, Schmidt, and Shin (1992, KPSS) propose a test of stationarity based on the decomposition of the series into deterministic trend, random walk, and stationary errors:

(12) $$y_t = \xi t + \gamma_t + v_t$$

(13) $$\gamma_t = \gamma_{t-1} + u_t$$

Here ξt represents linear deterministic trend, γ_t represents random walk (so the u_t are iid), and v_t is the stationary error. This parameterization provides a plausible representation of both stationary and nonstationary variables, and leads naturally to a test of the hypothesis of stationarity. Note that the decomposition into stationary and random walk components is a popular way of thinking about the properties of a time series in macroeconomics applications. See Nelson and Plosser (1982), Watson (1986), Clark (1987), and Cochrane (1988). KPSS derive the statistic for stationarity as the LM test of the null hypothesis $\sigma_u^2 = 0$; i.e., the variance of the random walk component of y_t equals zero. Thus the null hypothesis implies that the series is trend stationary.

We can note that the model of (12) and (13) implies

(14) $$\Delta y_t = \xi + u_t + \Delta v_t.$$

Define $w_t = u_t + \Delta v_t$ as the error in this expression for Δy_t. If u_t and v_t are iid and mutually independent, then w_t has a non-zero one period autocovariance, with all other autocovariances equal to zero, and accordingly it can be expressed as an MA(1) process. Thus the KPSS model is equivalent to the ARIMA model

(15) $$y_t = \beta y_{t-1} + w_t, \quad w_t = \varepsilon_t + \theta\varepsilon_{t-1}, \; \beta = 1, \; \varepsilon_t \text{ iid.}$$

This is of the same form as equation (2) above. The connection between θ and the variances of u and v is straightforward. Let $\lambda = \sigma_u^2/\sigma_v^2$. Then we can get the relationship between θ and λ as

(16) $$\theta = -\{(\lambda+2) - [\lambda(\lambda+4)]^{1/2}\}/2, \text{ or } \lambda = -(1 + \theta)^2/\theta,$$

where $\lambda \geq 0$, $|\theta| \leq 1$. Thus $\lambda = 0$ corresponds to $\theta = -1$ (stationarity) while $\lambda = \infty$ corresponds to $\theta = 0$ (a pure random walk). Equation (16) shows an interesting connection between the KPSS tests and the usual DF tests. The DF tests test $\beta = 1$ assuming $\theta = 0$; θ is a nuisance parameter. KPSS effectively test $\theta = -1$ assuming $\beta = 1$; now β is a nuisance parameter.

Since the reduced form of the KPSS model is ARIMA(0,1,1), a test of $\lambda = 0$ corresponds to a test of $\theta = -1$. The model is strictly noninvertible under the null and it follows from the results of Sargan and Bhargava (1983) that classical procedures cannot be applied in this case. An LM test statistic can be constructed but, as noted in Tanaka (1983), its asymptotic distribution is nonstandard. A locally best invariant

(LBI) test of the hypothesis that σ_u^2 is zero can also be constructed and is in fact the same as the LM test. See Nabeya and Tanaka (1988).

KPSS derive the LM statistic as a special case of the statistic developed by Nabeya and Tanaka (1988) to test for random coefficients. Let e_t, t = 1,2,...,T, be the OLS residuals from the regression of y_t on an intercept and time trend. Let $\hat{\sigma}_v^2$ be the usual estimate of the error variance from this regression. The LM statistic for stationarity is derived as:

(17) $$LM = \sum_{t=1}^{T} S_t^2 / \hat{\sigma}_v^2$$

where S_t is the partial sum process of the residuals

(18) $$S_t = \sum_{i=1}^{t} e_i \,, \; t = 1,2,\ldots,T$$

However, the LM derivation assumes that the stationary errors v_t in (12) are normal white noise. If they are not white noise, but satisfy the regularity conditions of Phillips (1987), the asymptotic distribution of the statistic is the same as under white noise errors if we divide by an estimate of the "long run variance" σ^2 rather than the innovation variance σ_v^2. Let $\hat{\sigma}^2$ be any consistent estimate of the long run variance. Then KPSS define the statistic

(19) $$\hat{\eta}_\tau = T^{-2} \sum_{t=1}^{T} S_t^2 / \hat{\sigma}^2$$

Under the null hypothesis $\sigma_u^2 = 0$, they establish the asymptotic distribution

(20) $$\hat{\eta}_\tau \rightarrow \int_0^1 V_2(r)dr$$

where $V_2(r)$ is the second-level Brownian bridge given by

$$V_2(r) = W(r) + (2r - 3r^2)W(1) + (-6r + 6r^2)\int_0^1 W(r)dr \tag{21}$$

and W(r) is Standard Brownian motion.

The same statistic may also arise in other contexts. Saikkonen and Luukkonen (1990) derive the statistic as the locally best unbiased invariant test of the hypothesis $\theta = -1$ in the model $\Delta y_t = \varepsilon_t + \theta\varepsilon_{t-1}$ with the ε_t iid normal. Based on the discussion above, this is not a surprising result.

In Chapter 3 we will consider the finite sample properties of the KPSS stationarity test, which uses semiparametric corrections for error autocorrelation in the presence of the autocorrelated errors. These results will be compared with the results for the Saikkonen and Luukkonen test, which uses parametric corrections based on an assumed ARMA model for the stationary error. We will consider both the size and the power of the tests.

1.5 The KPSS Test As a Unit Root Test

Many economic time series have sample autocorrelations for their first difference that are positive and significant only at lag one or two, but are insignificantly negative at longer lags. Conventional model selection procedures choose a low order ARIMA model in order to parsimoniously capture the short run dynamics. Then only the role of the random walk component is important, and possible trend reversion over long horizons is ignored. Variance ratio tests have been suggested to handle this situation, but unfortunately their confidence bounds are very wide, because there are very few independent observations on the long run behavior for most macroeconomic time series. See Cochrane (1988) and Lo and MacKinlay (1989). More generally, standard unit root tests, such as the DF or the SP tests, propose the null hypothesis that a random walk component exists, whereas the tests of the random walk hypothesis (e.g., variance ratio tests) have as their null hypothesis that stationary components do not exist.

Stock (1990) has recently developed a "generic" class of unit root tests, based on the fact that an I(1) process grows at rate $T^{1/2}$, while an I(0) process does not. If we let e_t be the detrended series, and let S_t be the partial sum process of the e's, as in equation (18), then the S_t process is $O_p(T^{1/2})$ if the original series is stationary, while it is $O_p(T^{3/2})$ if the original series is I(1). In either case, suitably normalized functions of S_t will converge to corresponding functionals of detrended Brownian motion. As a result, many different functions of S_t could be used to test the unit root null. Furthermore, the same statistics (with a different asymptotic distribution) can be used to test the null of stationarity.

A difficulty with such generic tests is that they may have low power compared to tests derived more explicitly as unit root tests or stationarity tests. For example, the KPSS test is derived from the LM (score) principle as a stationarity test, and can therefore be expected to have desirable power properties near the null of stationarity. However, it is a function of the partial sum process and can be fit into the class of

Stock's generic unit root tests, despite the fact that there is no reason to expect it to have good power properties as a unit root test.

In Chapter 3, we will consider the KPSS test as a unit root test. We provide Monte Carlo evidence that shows that it is generally less powerful than traditional unit root tests, like the DF or the SP tests. We also use the KPSS statistic to test the hypothesis of a unit root in the Nelson-Plosser data.

1.6 Plan of the Dissertation

The structure of the dissertation is as follows. Chapter 2 investigates the finite sample performance of the KPSS stationarity tests in the presence of autocorrelation. Chapter 3 considers unit root tests based on the KPSS statistics. Chapter 4 provides asymptotic results for the DF and SP unit root tests when the process is "nearly stationary", and shows the possible source for the size distortion problem in this case. Chapter 5 gives our concluding remarks.

CHAPTER 2

CHAPTER 2

THE FINITE SAMPLE PERFORMANCE OF THE STATIONARITY TEST

2.1 Introduction

There has been considerable interest in the use of autoregressive processes for modelling nonstationary time series. Nonstationarity is implied by the presence of unit roots in the autoregressive polynomial, and therefore the unit root hypothesis has recently attracted a lot of attention. Furthermore, the standard conclusion that is drawn from the empirical evidence is that most aggregate economic time series contain a unit root. See Nelson and Plosser (1982). However, it is important to note that in this empirical work the unit root is set up as the null hypothesis to be tested, and the way in which classical hypothesis testing is carried out ensures that the null hypothesis is accepted unless there is strong evidence against it. Therefore, an alternative explanation for the common failure to reject a unit root is simply that standard unit root tests are not very powerful against relevant alternatives. For further discussion see Dejong et al. (1989).

Therefore, it would be useful to have available tests of the null hypothesis of stationarity as well as tests of the null hypothesis of a unit root. There have been relatively few previous attempts to test the null hypothesis of stationarity. See Park and Choi (1988), Rudebush (1990), and Dejong et al. (1989). However, they all suffer from the lack of a plausible model in which the null of stationarity is naturally framed as a parametric restriction.

Nonstationary series can be decomposed into an integrated part and a weakly stationary part. A well-known decomposition of time series into random walk with drift and weak stationary components is proposed by Beveridge and Nelson (1981). However, we note that, in the absence of some additional theory or assumption on the data generating process, such a decomposition may not be unique. See Aoki (1990).

Recently, Kwiatkowski, Phillips, Schmidt, and Shin (1992) use a parameterization which provides a plausible representation of both stationary and nonstationary variables to derive a test for the null hypothesis of stationarity. They choose an unobserved components representation in which the time series under study is written as the sum of a deterministic trend, a random walk and a stationary error process:

$$y_t = \xi t + \gamma_t + v_t, \quad \gamma_t = \gamma_{t-1} + u_t$$

They wish to test the hypothesis $\sigma_u^2 = 0$, which implies that y_t is stationary around a deterministic trend (trend stationarity). Since the reduced form of the components model is also ARIMA(0,1,1), a test for $\sigma_u^2 = 0$ corresponds to a test for a unit moving average root. The model is therefore strictly noninvertible under the null and it follows from the results of Sargan and Bhargava (1983) that the classical procedures cannot be applied in this case.

A univariate time series model can be regarded as a special case of a time varying parameter regression model. Very little attention has been paid to the way in which time variation should be introduced into the coefficients of explanatory variables. Testing for time variation in the coefficients of the explanatory variables is also subject to the same problems encountered in carrying out tests of $\sigma_u^2 = 0$. The difficulties stem from the random walk nature of the time varying parameters. Testing the null hypothesis $\sigma_u^2 = 0$ has also been considered in context of time varying coefficients model. See Nicholls and Pagan (1985), Nyblom (1986), and Nabeya and Tanaka (1988).

The purpose of this chapter is to examine the finite sample performance of the KPSS stationarity test. We explain and compare various models in section 2.2. The main results of the simulations are given in section 2.3. The results of applying the KPSS statistics for stationarity to the Nelson-Plosser data are briefly discussed in section 2.4. Some suggestions and concluding remarks are given in 2.5.

2.2 The KPSS Test for Stationarity

Suppose that the economic time series y_t can be decomposed into a deterministic trend, random walk, and stationary error process:

(1) $y_t = \xi t + \gamma_t + v_t$

(2) $\gamma_t = \gamma_{t-1} + u_t$

where the u_t are iid $(0,\sigma_u^2)$ and v_t is stationary. KPSS derive the LM statistic for the null hypothesis of stationarity, $\sigma_u^2 = 0$, under the assumption that the v_t are iid $N(0,\sigma_v^2)$. Their model is a special case of the model developed by Nabeya and Tanaka (1988) to test for random regression coefficients. Nabeya and Tanaka consider the regression

$$(3) \qquad y_t = x_t\beta_t + z_t'\gamma + v_t,$$

in which β_t is a normal random walk and the errors v_t are iid $N(0,\sigma_v^2)$. Therefore, the KPSS model is the special case of (3) in which $x_t = 1$ for all t, $z_t = t$, and $\beta_t = \gamma_t$.

Let e_t, t = 1,2,...,T, be the OLS residuals from the regression of y_t on an intercept and time trend (or intercept only for the test of level stationarity). Let $\hat{\sigma}_v^2$ be the usual estimate of the error variance from this regression. Then the LM statistic is given by

$$(4) \qquad LM = \sum_{t=1}^{T} S_t^2 / \hat{\sigma}_v^2$$

where S_t is the partial sum process of the residuals

$$(5) \qquad S_t = \sum_{i=1}^{t} e_i \,, \; t = 1,2,\ldots,T$$

The same statistic may arise in other contexts. Saikkonen and Luukkonen (1990, SL) derive the locally best unbiased invariant (LBUI) test of the hypothesis $\theta = -1$ in the model

$$(6) \qquad \Delta y_t = w_t + \theta w_{t-1}$$

with $E(y_0)$ unknown and playing the role of intercept and w_t iid normal. Note that y_t is stationary under the null hypothesis of $\theta = -1$.

Comparing both parameterizations carefully, we find that they are the same. Consider starting with equations (1) and (2). After some algebra, we get the following equation:

(7) $$y_t = \alpha + \beta y_{t-1} + c_t,$$

with $\beta = 1$, $\alpha = \xi$, $\gamma_0 = y_0$, and

(8) $$c_t = \Delta(\gamma_t + v_t) = u_t + \Delta v_t$$

Therefore, $\sigma_u^2 = 0$ corresponds to $c_t = \Delta v_t$. If we rewrite model (6) as

(9) $$y_t = \alpha + y_{t-1} + \varepsilon_t,$$

(10) $$\varepsilon_t = w_t + \theta w_{t-1}$$

then testing for $\sigma_u^2 = 0$ in equations (1) and (2) is exactly the same as testing for $\theta = -1$ in the equations (9) and (10). We can also derive the exact relationship between $\lambda = \sigma_u^2/\sigma_v^2$ in (1) and (2) and θ in (9) and (10):

(11) $$\theta = -(1/2)[(\lambda + 2) - \{\lambda(\lambda + 4)\}^{1/2}]$$

(12) $$\lambda = -(1 + \theta)^2/\theta$$

for $\lambda \geq 0$, $|\theta| \leq 1$. As $\lambda \to 0$, $\theta \to -1$ while as $\lambda \to \infty$, $\theta \to 0$.

Tanaka (1990) also analyzes testing for a moving average unit root when the data follow a simple MA(1) process given by

(13) $$x_t = \varepsilon_t - \theta\varepsilon_{t-1}$$

and derives the score test statistic for $\theta = 1$:

(14) $$S^T = \frac{\sum_{t=1}^{T} [(t-1)x_1 + (t-2)x_2 + \ldots + x_{t-1} - \{t/(T+1)\}\sum_{s=1}^{T}(T - s + 1)x_s]^2}{T \sum_{t=1}^{T} [(x_1 + 2x_2 + \ldots + tx_t)^2 / \{t(t + 1)\}]}$$

This test can be extended to a more general model such as (9) and (10). Using (4), we obtain the KPSS score test statistic for the level stationarity as follow:

(15) $$LM = \sum_{t=1}^{T} S_{Lt}^2 / T^{-1} \sum_{t=1}^{T} (y_t - \bar{y})^2$$

where $S_{Lt} = \sum_{j=1}^{t}(y_j - \bar{y})$. Define $x_t = \Delta y_t$ and $S_t = \sum_{j=1}^{t} y_j$. Then, Tanaka's score test statistic (14) is the same as the statistic for level stationarity (15). See appendix A.

However, the assumption that the stationary error (v_t in equation (1)) is iid is unrealistic in time series modelling. There are two possibilities to generalize the testing procedures to allow for stationary but not iid v_t.

One possibility is a parametric correction. In the special case of a Gaussian MA(1) model of the form of (6), SL derive the LBUI test statistic (R in their notation), which is the same as the KPSS LM statistic for level stationarity under the assumption that stationary errors are iid. The general ARMA(p,q) model for w_t in (6) is given by

$$(16) \qquad \Delta y_t = w_t + \theta w_{t-1}, \quad \rho(L)w_t = \alpha(L)\varepsilon_t$$

where $\rho(L) = 1 - \rho_1 L - \cdots - \rho_p L^p$ and $\alpha(L) = 1 + \alpha_1 L + \cdots + \alpha_q L^q$, and ε_t are iid normal. The roots of both the lag polynomials $\rho(L)$ and $\alpha(L)$ are outside the unit circle so that the stationarity and invertibility conditions are satisfied. They use a parametric approach to generalize test statistic R, which is based on the appropriate residuals of the general ARMA(p,q) null model first fitted to the original mean corrected series. When $p > 0$, the idea of the test procedures is to replace the original null model by an MA approximation. Define

$$(17) \qquad \rho(L)^{-1}\alpha(L) = \psi(L) = \sum_{j=0}^{\infty} \psi_j L^j$$

Then w_t can be decomposed as

$$(18) \qquad w_t = W_t + \Delta z_t$$

$W_t = \psi(1)\varepsilon_t$; $z_t = \pi(L)\varepsilon_t$; and $\pi(L) = \sum_{j=0}^{\infty} \pi_j L^j$ where $\pi_j = -\sum_{i=j+1}^{\infty} \psi_j$. To derive modified test statistics in the presence of ARMA errors, they approximate the rational transfer function $\psi(L)$ by a polynomial of a finite but sufficiently large order, say m. However, one caution is that unnecessarily large values of m have an adverse effect on the power of the tests. Finally, the modification of test statistic R becomes

$$(19) \qquad R_m = (T - p)^{-2} \sum_{t=p+1}^{T} s_t^2 / \tilde{w}^2, \quad \text{where}$$

(20) $$\tilde{w} = (T - p)^{-1} \sum_{t=p+1}^{T} (h_t - \bar{h})^2$$

(21) $$s_t = -\sum_{j=p+1}^{t} (h_j - \bar{h}),\ t = p+1,\cdots,T-1.$$

$h_t \equiv y_t - \Delta\tilde{z}_t$ and $\tilde{z}_t = \sum_{j=0}^{m-1} \hat{\pi}_j \hat{\varepsilon}_{t-j}$, where $\hat{\pi}_j = -\sum_{i=j+1}^{m} \hat{\psi}_j$ and $\hat{\varepsilon}_t$ are the residuals. Then as T $\to \infty$, the R_m statistic has the same asymptotic distribution as R.

The other possibility, followed by KPSS, is to use a semiparametric correction of the type suggested by Phillips and Perron (1988). An advantage of this approach is that no knowledge of the parametric form of autocorrelation is required. The correction only amounts to replacing the denominator of the LM statistic by an appropriate variance estimator. Define the long run variance as

(22) $$\sigma^2 = \lim_{T\to\infty} T^{-1}E(S_T^2), \quad S_T = \sum_{t=1}^{T} v_t$$

which will enter into the asymptotic distribution of the test statistic, when the v_t are stationary but not iid. In this case the appropriate denominator of the LM statistic is an estimate of σ^2 instead of σ_v^2. A consistent estimator of σ^2, $s^2(\ell)$ can be constructed as

(23) $$s^2(\ell) = T^{-1} \sum_{t=1}^{T} e_t^2 + 2T^{-1}\sum_{s=1}^{\ell} w(s,\ell) \sum_{t=s+1}^{T} e_t e_{t-s}$$

where $w(s,\ell) = 1 - s/(1 + \ell)$ is the Bartlett window which guarantees the nonnegativity of $s^2(\ell)$. For consistency of $s^2(\ell)$, it is necessary that the lag truncation parameter $\ell \to \infty$ at an appropriate rate as $T \to \infty$. The generalized KPSS LM statistics for level stationarity and for trend stationarity are defined as follow:

$$(24) \qquad \hat{\eta}_{\mu} = T^{-2}\Sigma S_{Lt}^{2}/s^{2}(\ell) \rightarrow \int_{0}^{1} V(r)^{2}dr$$

$$(25) \qquad \hat{\eta}_{\tau} = T^{-2}\Sigma S_{Tt}^{2}/s^{2}(\ell) \rightarrow \int_{0}^{1} V_{2}(r)^{2}dr$$

S_{Lt} and S_{Tt} are the partial sum processes of the OLS residuals from the regression of y_t on [1] and on [1,t] respectively. The subscript 'μ' indicates that we have extracted only a mean from y, and the subscript 'τ' indicates that we have extracted mean and trend from y. W(r) is a standard Brownian motion; $V(r) = W(r) - rW(1)$ is a standard Brownian bridge; and $V_2(r) = W(r) + (2r - 3r^2)W(1) + (-6r + 2r^2)\int_0^1 W(r)dr$ is the second-level Brownian bridge.

The critical values of $\int V(r)^2dr$ and of $\int V_2(r)^2dr$ are given in Table 2-0, which are calculated via a direct simulation using a sample size of 2,000 with 5,0000 replications, and the random number generator GASDEV/RAN3 of Press <u>et al.</u> (1986).

2.3 Finite Sample Performance

The finite sample distribution of the test statistics $\hat{\eta}_{\mu}$ and $\hat{\eta}_{\tau}$ will be tabulated by simulation. The distribution of both statistics under the null hypothesis depends only on the sample size T, while the distribution under the alternative depends on λ as well as T. See appendix B. The simulation results using 20,000 replications are given in Tables 2-1 through 2-10. We consider three lag specifications in calculating the denominator of the LM statistics, the long run variance of the residuals, $s^2(\ell)$ in (23): $\ell_0 = 0$, $\ell_4 = \text{int}\{4(T/100)^{1/4}\}$, and $\ell_{12} = \text{int}\{12(T/100)^{1/4}\}$.

2.3.1 Size

1) iid errors

We can see in Table 2-1 that the tests have approximately correct size except when T is small and ℓ is large. For $\ell = \ell_0$, the tests have correct size even for T = 30, so that the asymptotic validity of the tests holds even for fairly small samples. Using $\ell = \ell_4$, the tests are slightly less accurate, and the improvement as T increases is slow. For $\ell = \ell_{12}$, there are considerable size distortions for T = 30, and moderate distortions (too few rejections) even for T = 100 or 200, though the tests are quite accurate for T = 500. Unsurprisingly, the larger ℓ is, the larger is the sample size required for the asymptotic results to be relevant.

2) AR(1) errors

We next consider the size of the tests in the presence of autocorrelated errors. In particular, we will consider AR(1) errors, of the form $v_t = \rho v_{t-1} + \varepsilon_t$ with the ε_t iid. The AR(1) parameter ρ is a convenient parameter to consider, since it naturally measures the distance of the null from the alternative. In particular, under the null that $\sigma_u^2 = 0$, y_t approaches a random walk as $\rho \to 1$. As a result, we expect a problem of over-rejection for $\rho > 0$, with its severity depending on how close ρ is to unity. Table 2-2 presents our simulation results giving the size of the tests for $\rho = 0$, ± 0.2, ± 0.5, and ± 0.8, and for T between 30 and 500. As expected, the tests reject too often for $\rho > 0$ and too seldom for $\rho < 0$. The over-rejection problem is very severe for $\ell = \ell_0$, which is not surprising since the test is not valid even asymptotically in this case. However, the ℓ_4 and ℓ_{12} versions of the tests do not improve very rapidly with the sample size. The tests using ℓ_4 have moderate size distortions for $\rho = 0.5$ and considerable size distortions for $\rho = 0.8$, while the test using ℓ_{12} are fairly good for $T \geq 30$ and $\rho \leq 0.5$, but not so good for $\rho = 0.8$. Unfortunately, $\rho = 0.8$ is a plausible parameter value since, if we take most series to be stationary, their first-order autocorrelations will often be in this range.

3) MA(1) errors

We next consider MA(1) errors, of the form

$$v_t = \varepsilon_t + \alpha\varepsilon_{t-1} \tag{26}$$

These results are given in Table 2-3. In the presence of MA(1) errors, the size distortions of the test are not as severe as in the AR(1) case. Therefore, the use of long lags (large value of ℓ) is not necessary for test statistics to have approximately correct size. For example, for $\alpha = 0.8$, the sizes of the $\hat{\eta}_\mu$ test using ℓ_0, ℓ_4, ℓ_{12} are .206, .057 and .033 and the sizes of the $\hat{\eta}_\tau$ test using ℓ_0, ℓ_4 and ℓ_{12} are .302, .061 and .038, both of which are far less than for the AR(1) case with $\rho = 0.8$.

For positive α, the size of the tests using ℓ_{12} is converging around the nominal level as T increases; e.g., when T = 500, all sizes are very close to .05. However, when T is small, the use of ℓ_{12} still gives unsatisfactory results; e.g., when T = 30 and $\alpha = 0.8$, the size of $\hat{\eta}_\mu(\ell_{12})$ is .004 but the size of $\hat{\eta}_\tau(\ell_{12})$ is .217.

We also note that when T is in the range of 70 to 120, which are plausible sample sizes encountered in economic data, the size performance of the tests using ℓ_4 is slightly better than when $\ell = \ell_{12}$ and better than when $\ell = \ell_0$.

For negative α, the sizes are too low for all cases, and these results are consistent with the asymptotic results that the size $\rightarrow 0$ as $\alpha \rightarrow -1$.

To sum up, in the presence of positively autocorrelated errors, the tests using ℓ_0 have an over-rejection problem and the tests using ℓ_{12} generally have the least size distortion. The size distortions are more severe in the AR(1) case. We also note that the tests using ℓ_4 perform better than the tests using ℓ_{12} in some cases. On the other hand, we have an under-rejection problem for negatively autocorrelated errors.

2.3.2 Power

Results for the power of the tests in the presence of iid errors are given in Table 2-4. We will discuss these results before going on to power in the presence of AR(1) or MA(1) errors. Note that both test statistics, $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$, are consistent. However, for fixed T, the power of the test approaches a limit (as $\lambda \rightarrow \infty$), which is usually less than one. We can represent as the limit power the power of test for $\lambda = \infty$. For example, when T = 100, the limit powers of the $\hat{\eta}_\mu$ test using ℓ_0, ℓ_4, and ℓ_{12} are .998, .827, and .582, and they are .999, .820, and .410 for the $\hat{\eta}_\tau$ test, as can be seen for λ = 10,000 in Table 2-4. Although the limit power of the tests using ℓ_4 or ℓ_{12} is close to one in large samples, it is generally far less than the power of the tests using ℓ_0 in finite samples. This is especially so for the tests using ℓ_{12}. Even for T = 500, the limit power is about .901 for the $\hat{\eta}_\mu(\ell_{12})$ test and .911 for the $\hat{\eta}_\tau(\ell_{12})$ test.

However, the fact that the limit power of test is not equal to one can be explained well by the asymptotics under the alternative. The asymptotic distribution for the stationarity test under the alternative hypothesis $\sigma_u^2 > 0$ is given in KPSS. For the test of level stationarity, we have the following results:

(27) $$(\ell/T)\,\hat{\eta}_\mu \rightarrow \int_0^1 \Big[\int_0^a \bar{W}(s)ds\Big]^2 da \Big/ K\int_0^1 \bar{W}(s)^2 ds$$

where $\bar{W}(s) = W(s) - \int_0^1 W(b)db$ is a demeaned Wiener process, and the constant K is defined by

(28) $$K = \int_{-1}^{1} k(s)ds$$

where k(s) represents the weighting function used in calculating $s(\ell)$in equation (23); $w(s,\ell) = k(s/\ell)$ in the notation above. For the Newey-West estimator, $k(s) = 1 - |s|$ and therefore $K = 1$.

The analysis for the trend stationarity case (i.e., for the statistic $\hat{\eta}_\tau$) is only slightly more complicated. We just need to replace the demeaned Wiener process $\bar{W}(s)$ above with the demeaned and detrended Wiener process $W^*(s)$:

(29) $$W^*(s) = W(s) + (6s - 4)\int_0^1 W(r)dr + (-12s + 6)\int_0^1 rW(r)dr$$

defined by Park and Phillips (1988, equation (16), p.474). The rest of our analysis then follows without further change.

Note that the asymptotic distribution (27) does not depend on the variance of the stationary error. This is so, because, under the alternative hypothesis, the random walk component dominates the stationary component. In that sense these asymptotics correspond to $\sigma_v^2 = 0$, or $\lambda = \infty$, and can be used to predict the limit power of the stationarity test.

Percentiles of the asymptotic distribution in (27), and of the corresponding asymptotic distribution for $\hat{\eta}_\tau$, are calculated in Chapter 3, where we consider these statistics as unit root test statistics. These can be converted to percentiles for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ (under $\lambda = \infty$) by multiplying by (T/ℓ), and we can therefore predict the limit power of both tests. These results are given in Table 2-5. We find that the limit power of the stationarity test in finite samples is generally consistent with the asymptotic results. For example, for $\hat{\eta}_\tau$ with $T = 100$ and $\ell = 12$ the actual power of .410 compares to .417 predicted by the asymptotic distribution under the alternative.

As described in section 2.2, Tanaka (1990) tabulates the limiting power of the S^T test in equation (14) under local alternatives of the form $\theta = -1 + C/T$ and C fixed, and therefore we can use his table for the limiting power as a good approximations to the power against nearly stationary alternatives in finite samples. However, the Tanaka's test is not most powerful at higher values of λ in finite samples. For example, when $T = 100$ and $\theta = -0.8$ ($\lambda = 0.011$), the power of S^T is .863, which is less than the power of the MPI test, .954. We also note that he does not consider cases with $T > 100$ and $0 > \theta > -0.4$ ($\infty > \lambda > 1$).

We compare the actual (simulation) power of the $\hat{\eta}_\mu$ test with Tanaka's limiting power and these results are given in Table 2-6. When $\lambda < 1$, the power of the $\hat{\eta}_\mu(\ell_0)$ test is close to Tanaka's limiting power for all sample sizes, while the use of more lags loses power in finite samples. For example, when $T = 100$ and $\lambda = 0.01$, the powers of the $\hat{\eta}_\mu$ test using ℓ_0, ℓ_4, and ℓ_{12} are .587, .508, and .376, and Tanaka's limiting power is .584. When $\lambda \geq 1$, the powers of the $\hat{\eta}_\mu$ test are generally less than Tanaka's limiting power when $T \leq 50$. For example, when $T = 50$, the powers of the $\hat{\eta}_\mu$ tests using ℓ_0, ℓ_4, and ℓ_{12} are .959, .704, and .343 respectively for $\lambda = 10{,}000$, which is less than Tanaka's limiting power of .991.

The above results show that the power of the $\hat{\eta}_\mu$ test for $\ell = \ell_0$ and for small λ is well predicted by the asymptotics of Tanaka. More generally, its power performance is satisfactory unless both λ is large and T is small. However, note that the $\hat{\eta}_\mu(\ell_{12})$ test is not generally powerful at all even against the pure random walk alternative of $\lambda = \infty$ in finite samples.

For more detailed power investigations, we choose 4 different values of λ (0.0001, 0.01, 1 and 10,000). We first analyze the case of iid errors, for which the results are given in Table 2-4. When $\lambda = 0.0001$, the powers of the tests are not much different from nominal level in finite samples. This implies that it is almost impossible to distinguish between the distribution under the null and under a local alternative in finite samples. When $\lambda = 0.01$, the powers of the tests using ℓ_0 approach one as T increases, but in finite samples they are not very large. As expected, both tests are not very powerful against alternatives with small values of λ, say $\lambda < 1$. Note that the $\hat{\eta}_\tau$ test is generally less powerful than the $\hat{\eta}_\mu$ test. This finding is also consistent with previous studies. For $\lambda \geq 1$, the tests using ℓ_0 and ℓ_4 are reasonably powerful even in finite samples but the tests using ℓ_{12} are not powerful, as mentioned, even against the pure random walk alternative. It should be noted that the loss in power from using a large value of ℓ persists for large sample sizes. This is also expected from the asymptotics under the alternative; (ℓ/T) enters the expression (27) above.

We now consider the power in the presence of AR(1) errors. These results are given in Table 2-7. In this case the power generally depends on the value of ρ as well as the value of λ. As $\rho \to 1$, the test is more powerful, as expected. For $\rho = 0.8$, even when $\lambda = 0.0001$ and T = 100, the powers of the $\hat{\eta}_\mu$ tests using ℓ_0, ℓ_4, and ℓ_{12} are .799, .252 and .083 and the powers of the $\hat{\eta}_\tau$ tests using ℓ_0, ℓ_4, and ℓ_{12} are .953, .340 and .092, all of which are far greater than the powers for iid errors. This is a reflection of the size distortion caused by AR(1) errors with positive ρ. On the other hand, for negative ρ, power is small unless T is very large.

In the presence of AR(1) errors, the use of longer lags (a larger value of ℓ) is needed for the test to avoid size distortions, while the use of shorter lags is needed for the test to be more powerful. This implies an inevitable tradeoff between power and size in finite samples. Therefore, we have to weigh size against power of the test in determining the choice of the number of lags to be used. Based on the above simulation results and considering the fact that $\rho = 0.8$ is a plausible parameter value, the use of $\ell = \ell_8$ may be a compromise between the large size distortions under the null that we would expect for $\ell \leq \ell_4$ and the very low power under the alternative that we would expect for $\ell = \ell_{12}$.

We now consider power in the presence of MA(1) errors. These results are given in Table 2-8. In this case the power also depends on the value of α as well as the value of λ, but the influence of α is not as important as the AR(1) parameter ρ.

Generally, the test using ℓ_0 is more powerful for positive α than for negative α. For $\lambda = 0.0001$, when $\alpha = 0.8$ and $T = 100$, the powers of the $\hat{\eta}_\mu$ test using ℓ_0, ℓ_4, and ℓ_{12} are .218, .062 and .037 and the powers of the $\hat{\eta}_\tau$ test are .307, .063 and .039 respectively. On the other hand, for negative α, the powers are less than nominal level unless T is very large. Once again these levels of power reflect the size distortions caused by $\alpha \neq 0$.

In the presence of MA(1) errors, size distortion is not as severe as in the AR(1) case, so that the use of very long lags is not required for the test statistics to be more accurate. For positive values of the moving average parameter, there is still a tradeoff between power and size in finite samples, but the size distortions with $\ell = \ell_4$ are not as severe as for the AR(1) case. Based on the above simulation results, the use of $\ell = \ell_4$ may be a compromise between some size distortions under the null that we would expect for $\ell = \ell_0$ and decreased power under the alternative that we would expect for $\ell > \ell_4$.

2.3.3 Comparison to The Saikkonen and Luukkonen Test

The simulation results for the $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ tests can be compared with those for the test of Saikkonen and Luukkonen. They suggest the use of their R_1 statistic in the presence of MA(1) errors and of their R_{15} statistic in the presence of AR(1) errors. These comparisons are given in Table 2-9 and 2-10. Note that for iid errors, size is relatively correct in all cases.

With MA(1) errors, the R_1 statistic can be compared with the $\hat{\eta}_\mu(\ell_4)$ statistic.

For positive α, when T = 100, the sizes of both tests are correct and similar but R_1 is always more powerful than $\hat{\eta}_\mu(\ell_4)$. For example, for $\theta = -0.95$ ($\lambda = .0026$), the power of $\hat{\eta}_\mu(\ell_4)$ is .201 and the power of R_1 is .335 when $\alpha = 0.8$. For negative α the sizes of both tests are low but the size of the $\hat{\eta}_\mu(\ell_4)$ test is almost zero, but interestingly, $\hat{\eta}_\mu(\ell_4)$ is always more powerful than R_1.

With AR(1) errors, we compare the $\hat{\eta}_\mu$ test using ℓ_{12} or ℓ_4 with R_{15}. For $\rho < 0$, the sizes of $\hat{\eta}_\mu(\ell_4)$ or $\hat{\eta}_\mu(\ell_{12})$ are too small but the size of R_{15} is relatively correct. On the other hand, for $\rho > 0$, some size distortions occur for $\hat{\eta}_\mu(\ell_4)$, but R_{15} and $\hat{\eta}_\mu(\ell_{12})$ show almost the correct size performance. However, the power of $\hat{\eta}_\mu(\ell_{12})$ is always less than the power of R_{15} except when $\theta = -0.95$ and ρ is negative. For example, for $\theta = -0.90$ ($\lambda = 0.0111$) the power of R_{15} is .447 and the power of $\hat{\eta}_\mu(\ell_{12})$ is .219 when $\rho = 0.8$ and T = 100. However, no general conclusion can be drawn from the comparison of the power of R_{15} with the power of $\hat{\eta}_\mu(\ell_4)$. For example, for $\rho = 0.5$ and T = 100, the powers of $\hat{\eta}_\mu(\ell_4)$ and R_{15} are .209 and .265 for $\theta = -0.95$, but they are .648 and .567 for $\theta = -0.8$. This ordering is reversed when $\rho = -0.5$. However, it is generally true that the power of R_{15} exceeds the power of $\hat{\eta}_\mu(\ell_4)$ except where $\hat{\eta}_\mu(\ell_4)$ suffers from considerable size distortions.

To sum up, it is very difficult to draw any clear conclusion from the above comparison. The Saikkonen and Luukkonen testing procedure is more complicated than KPSS's, because they have to estimate a general ARMA process and then derive the approximate MA(m) representation. There is also the problem of choice of the appropriate number of m. However, interestingly, the size of their test is relatively correct in most cases with autocorrelated errors, especially when the errors are AR(1).

2.4 Applications to the Nelson-Plosser Data

In this section we briefly discuss the results of application of the stationarity tests to actual data. KPSS apply their tests for stationarity to the data analyzed by Nelson and Plosser in order to check whether their approach to testing stationarity corroborates the main findings of Nelson and Plosser. They consider values of the lag truncation parameter ℓ from 0 to 8. The values of the test statistics are fairly sensitive to the choice of ℓ, and in fact for every series the value of the test statistic decreases as ℓ increases. This is a reflection of large and persistent positive autocorrelations in the series. For all series except the unemployment rate and the interest rate, they can reject the hypothesis of level stationarity, but this is not very surprising in light of the obvious deterministic trends present in these series.

Based on the observations that for most of the series the value of the long run variance estimate has settled down reasonably by the time $\ell = 8$ is reached, they use the results for $\ell = 8$ for the trend stationarity test. The choice of $\ell = 8$ is relatively consistent with the above simulation results. Their results have very different implications for many of series considered from Nelson and Plosser. Nelson and Plosser can reject the null hypothesis of a unit root at the 5 % significance level for only two series (unemployment rate and industrial production) of the 14 series considered, while KPSS can reject the null hypothesis of trend stationarity at the 5%

level only for five series (industrial production, consumer prices, real wages, velocity and stock prices). Therefore, KPSS conclude that most economic time series are not very informative about whether or not they contain a unit root.

2.5 Suggestions and Concluding Remarks

We have investigated the finite sample performance of the stationarity tests of Kwiatkowski, Phillips, Schmidt, and Shin (1991). In the process we have shown the close relationship between the KPSS stationarity test using a components parameterization and a test for the moving average unit root using the traditional ARIMA parameterization, and discussed some intrinsic testing problems involved. We summarize the main findings as follows.

First, when the stationary errors are iid, the size of the tests using ℓ_0 is correct. Given positive autocorrelation, the tests using ℓ_0 show some size distortions but the tests using ℓ_{12} show very small size distortions in most cases. The use of ℓ_4 shows the intermediate behavior. The size distortion problem is more severe with AR(1) errors than with MA(1) errors. In particular, when ρ is large and positive, the use of longer lags (e.g., ℓ_{12}) is required to avoid severe size distortions. On the other hand, when the errors are negatively autocorrelated, we have a under-rejection problem.

Second, with iid errors, the tests using ℓ_0 are most powerful. The power performance of the tests in finite samples depends on the value of λ, as expected. Power is low when λ is small and increases as λ increases. Power does not usually approach one as $\lambda \to \infty$ with T fixed, however.

Third, using ℓ_4 or ℓ_{12} instead of ℓ_0 loses power, so that there is an inevitable trade-off between size and power of the test. Our simulation results suggest the use of shorter lags (e.g., ℓ_4) unless it appears that the stationary errors follow an AR(1) process with large positive parameter.

Fourth, $\hat{\eta}_\tau$ is less powerful than $\hat{\eta}_\mu$. This confirms that it is difficult to distinguish between a unit root and trend stationary series in finite samples.

We have compared the finite sample performance of the KPSS stationarity test, which uses semiparametric corrections for error autocorrelation, with that of the Saikkonen and Luukkonen (1990) test, which uses a parametric correction instead. Although it is difficult to draw any clear conclusion about their relative performance in terms of size and power, it is important to mention the relatively good finite sample performance of the Saikkonen and Luukkonen test when errors are the AR(1) with large positive parameter. A possible combination of both approaches to tackle the problem of autocorrelation is a further research topic.

It should be noted that we estimate the long run variance from residuals from a fit of the model with the stationarity hypothesis imposed, and so if the null hypothesis is not true we should expect $s^2(\ell)$ to diverge as ℓ increases. This implies the need for further research to find an estimate of the long run variance that is consistent under the null and that increases the rate of divergence of the LM statistic under the alternative.

We can possibly modify the stationarity test to make it into a cointegration test.

One of advantages of this approach is that we can set up the null of cointegration directly, instead of the null of no-cointegration which is a direct extension of the unit root test and has been mainly used in the literature. The basic idea is simple. Suppose that the $n \times 1$ vector X is I(1). Then variables in X are cointegrated if there exists an $n \times 1$ vector r such that r'X is I(0). r'X is called the long run relationship. If we know r, or estimate it efficiently, we can set $y = r'X$ and apply the stationarity test to y. We expect this test of cointegration to give further light on the true relationships among important economic variables. We are currently working on this topic. In particular, the asymptotic theory for the case that r is estimated must be derived.

Appendix A

We will now demonstrate that the extended version of Tanaka's score test is the same as the $\hat{\eta}_\mu$ test. Tanaka's simple model is given in (13) in the text and the score test statistic for a moving average unit root is given in (14). We extend this to the more general model

(A1) $\Delta y_t = \varepsilon_t - \theta\varepsilon_{t-1}$

where $\Delta y_t = x_t$, and $y_1 = x_1$. As shown in the text, the null hypothesis $\theta = 1$ in (A1) corresponds to the null of $\sigma_u^2 = 0$ in equations (1) and (2) in the text.

Define $S_t = \sum_{j=1}^{t} y_j$ and $S_{Lt} = \sum_{j=1}^{t} (y_j - \bar{y})$. Then it is straightforward to show that

(A2) $S_{Lt} = S_t - t\bar{y} = S_t - (t/T)S_T$

where $S_T = \sum_{t=1}^{T} y_t$.

<u>Lemma A1</u> $(t - 1)x_1 + (t - 2)x_2 + \ldots + x_{t-1} = S_{t-1}$

(proof) $(t - 1)x_1 + (t - 2)x_2 + \ldots + x_{t-1}$

$= (t - 1)y_1 + (t - 2)(y_2 - y_1) + \ldots + (y_{t-1} - y_{t-2})$

$= y_1[(t - 1) - (t - 2)] + y_2[(t - 2) - (t - 3)] + \ldots + y_{t-1}$

$= y_1 + y_2 + \ldots + y_{t-1} = S_{t-1}$

<u>Lemma A2</u> $\sum_{s=1}^{T}(T - s + 1)x_s = \sum_t y_t = T\bar{y}$

(Proof) $\sum_{s=1}^{T}(T - s + 1)x_s$

$= Ty_1 + (T - 1)(y_2 - y_1) + (T - 2)(y_3 - y_2) + \ldots + (y_T - y_{T-1})$

$= y_1[T - (T - 1)] + y_2[(T - 1) - (T - 2)] + \ldots + y_T$

$= \sum_t y_t = T\bar{y}$

Using Lemma A1 and A2 and (A2) we can show that

(A3) $\quad$ Numerator of (14) $= \sum_{t=1}^{T} (S_t - t\bar{y})^2 = \sum_{t=1}^{T} S_{Lt}^2$

We now show that the denominator of(14) is the same as the denominator of the $\hat{\eta}_\mu$ test using Lemma A3. The denominator of (14) can be expressed as (see Tanaka (1990))

(A4) $\quad x'\Omega^{-1}x = [\sum_1^T x_t \sum_2^T x_t \cdots x_T]\,[I - ee'/T]\,[\sum_1^T x_t \sum_2^T x_t \cdots x_T]'$

where e is the unit column vector.

<u>Lemma A3</u> $\; x'\Omega^{-1}x = \sum_{t=1}^{T} (y_t - \bar{y})^2$

(proof) $\; x'\Omega^{-1}x$

$= [y_T\ (y_T - y_1) \cdots (y_T - y_{t-1})][I - ee'/T][y_T\ (y_T - y_1) \cdots (y_T - y_{t-1})]'$

$= y_T^2 + (y_T - y_1)^2 + \cdots + (y_T - y_{T-1})^2 - (T+1)^{-1}[y_T + (y_T - y_1) + \cdots + (y_T - y_{T-1})]^2$

$= \sum_1^T y_t^2 + (T+1)y_T^2 - 2y_T(y_1 + y_2 + \cdots + y_T) - (T+1)^{-1}[(T+1)y_T - \sum_1^T y_T]^2$

$= \sum_1^T y_t^2 - T\bar{y}^2 = \sum_1^T (y_t - \bar{y})^2$

Here we use the fact that $y_t = \sum_t x_t$ with $y_0 = 0$.

Therefore, the extended version of Tanaka's score test should reduce to the KPSS level stationarity test.

Appendix B

We will demonstrate that the distribution of the $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ tests under the null hypothesis of stationarity ($\sigma_u^2 = 0$) depends only on the sample size T, while the distribution under the alternative depends on λ (the ratio of the random walk error variance to the stationary error variance) as well as T.

Consider the level stationarity test first. Under the null ($\sigma_u^2 = 0$), y_t can be expressed as

(B1) $$y_t = \gamma_0 + v_t$$

Then the partial sum process S_{Lt} is given by

(B2) $$S_{Lt} = \sum_{j=1}^{t} (y_j - \bar{y}) = \sum_{j=1}^{t} (v_j - \bar{v})$$

S_{Lt} does not depend on γ_0 and the $\hat{\eta}_\mu$ test is a function of S_{Lt} ($t = 1,\cdots,T$), so that its distribution is invariant to γ_0. The scale factor σ_v cancels out of the expression for the $\hat{\eta}_\mu$ test, therefore, the distribution of the statistic under the null is independent of nuisance parameters (σ_v and γ_0).

Under the alternative ($\sigma_u^2 > 0$), y_t, e_t, and S_{Lt} are expressed as

(B3) $$y_t = \gamma_0 + \gamma_t + v_t, \quad \gamma_t = \sum_{j=1}^{t} u_j$$

(B4) $$e_t = (y_t - \bar{y}) = (\gamma_t - \bar{\gamma}) + (v_t - \bar{v})$$

(B5) $$S_{Lt} = \sum_{j=1}^{t} (y_j - \bar{y}) = \sum_{j=1}^{t} \{(\gamma_t - \bar{\gamma}) + (v_j - \bar{v})\}$$

Since S_{Lt} does not depend on γ_0 and the $\hat{\eta}_\mu$ test can be written as a function of S_{Lt}, $t = 1,\cdots,T$, its distribution under the alternative is still invariant to γ_0. However, the distribution for the $\hat{\eta}_\mu$ test under the alternative depends on $\lambda = \sigma_u^2/\sigma_v^2$ as well as T. To show this, we rescale (B3) by dividing by σ_v. Accordingly, we get the following results

(B6) $$y_t^* = \gamma_0^* + \gamma_t^* + v_t^*, \quad \gamma_t^* = \sum_{j=1}^{t} u_j^*$$

(B7) $$e_t^* = (y_t^* - \bar{y}^*) = (\gamma_t^* - \bar{\gamma}^*) + (v_t^* - \bar{v}^*)$$

(B8) $$S_{Lt}^* = \sum_{j=1}^{t}(y_j^* - \bar{y}^*) = \sum_{j=1}^{t} \{(\gamma_t^* - \bar{\gamma}^*) + (v_j^* - \bar{v}^*)\}$$

Note that v_t^* (v_t/σ_v) now follow iid N(0,1) and that the u_t^* are iid N(0, λ) where $\lambda = \sigma_u^2/\sigma_v^2$. Then the $\hat{\eta}_\mu$ test can be written as a function of e_t^* and S_{Lt}^*, because $e_t^* = e_t/\sigma_v$ and $S_{Lt*} = S_{Lt}/\sigma_v$. Since e_t^* and S_{Lt}^* depend on λ but in a different way, the distribution under the alternative clearly depends on λ as well as T.

Following the same logic as for the case of level stationarity, it is straightforward to show that the distribution for the $\hat{\eta}_\tau$ test under the null is independent of nuisance parameters σ_v, γ_0 and ξ and that its distribution under the alternative depends on λ as well as T.

Table 2-0

Critical Values for Stationarity Test

	$\hat{\eta}_\mu$	$\hat{\eta}_\tau$
0.010	0.0248	0.0174
0.025	0.0302	0.0204
0.050	0.0367	0.0235
0.100	0.0460	0.0280
0.200	0.0624	0.0349
0.300	0.0788	0.0413
0.400	0.0970	0.0481
0.500	0.1193	0.0557
0.600	0.1473	0.0645
0.700	0.1853	0.0757
0.800	0.2435	0.0915
0.900	0.3493	0.1203
0.950	0.4648	0.1488
0.975	0.5826	0.1787
0.990	0.7444	0.2193

Table 2-1

Size for iid Errors (λ=0)

		$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
30	.049	.038	.004	.054	.041	.248
50	.050	.039	.012	.052	.041	.043
80	.049	.045	.029	.049	.042	.032
90	.048	.043	.030	.051	.045	.034
100	.049	.043	.029	.049	.044	.033
120	.051	.045	.034	.052	.046	.038
200	.051	.049	.041	.052	.048	.040
500	.050	.048	.046	.052	.051	.049

Table 2-2 Size for AR(1) Errors ($\lambda = 0$)

ρ	T	$\hat{\eta}_\mu$ ℓ_0	$\hat{\eta}_\mu$ ℓ_4	$\hat{\eta}_\mu$ ℓ_{12}	$\hat{\eta}_\tau$ ℓ_0	$\hat{\eta}_\tau$ ℓ_4	$\hat{\eta}_\tau$ ℓ_{12}
0.8	30	.654	.301	.007	.769	.317	.124
	50	.725	.264	.039	.886	.319	.057
	80	.779	.300	.080	.936	.401	.084
	100	.796	.250	.081	.952	.337	.092
	120	.807	.256	.091	.960	.354	.104
	200	.833	.271	.094	.977	.396	.108
	500	.852	.239	.092	.989	.361	.111
0.5	30	.321	.114	.005	.425	.129	.178
	50	.331	.098	.021	.486	.113	.047
	80	.350	.108	.042	.521	.124	.046
	100	.352	.090	.043	.538	.107	.047
	120	.359	.092	.047	.542	.114	.054
	200	.367	.099	.053	.559	.121	.054
	500	.370	.090	.058	.586	.110	.062
0.2	30	.118	.055	.004	.147	.062	.227
	50	.118	.053	.015	.156	.059	.045
	80	.122	.060	.033	.157	.060	.036
	100	.123	.054	.033	.159	.057	.038
	120	.125	.057	.038	.166	.064	.042
	200	.128	.061	.045	.168	.065	.043
	500	.129	.059	.049	.170	.065	.052
−0.2	30	.017	.025	.003	.016	.027	.268
	50	.015	.029	.011	.012	.029	.039
	80	.014	.034	.024	.011	.031	.028
	100	.014	.033	.026	.010	.031	.029
	120	.014	.036	.029	.013	.035	.034
	200	.014	.038	.037	.011	.036	.036
	500	.013	.042	.045	.010	.039	.042
−0.5	30	.002	.010	.002	.001	.010	.301
	50	.001	.015	.007	.001	.016	.032
	80	.001	.018	.017	.001	.015	.021
	100	.001	.019	.020	.000	.016	.021
	120	.001	.020	.023	.000	.019	.025
	200	.001	.021	.030	.000	.020	.029
	500	.001	.026	.036	.000	.024	.036
−0.8	30	.000	.002	.001	.000	.001	.319
	50	.000	.007	.002	.000	.013	.028
	80	.000	.008	.007	.000	.008	.013
	100	.000	.007	.007	.000	.002	.009
	120	.000	.003	.010	.000	.002	.011
	200	.000	.008	.015	.000	.002	.015
	500	.000	.010	.022	.000	.008	.020

Table 2-3 Size for MA(1) Errors (λ = 0)

		$\hat{\eta}_\mu$			$\hat{\eta}_\tau$		
α	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.202	.062	.004	.287	.068	.217
	50	.201	.057	.016	.295	.064	.046
	80	.205	.063	.033	.299	.064	.037
	100	.206	.057	.033	.302	.061	.038
	120	.208	.061	.039	.305	.068	.043
	200	.209	.065	.046	.306	.070	.044
	500	.208	.060	.050	.316	.067	.053
0.5	30	.169	.058	.004	.236	.065	.222
	50	.169	.055	.016	.242	.061	.046
	80	.174	.062	.033	.244	.062	.036
	100	.176	.056	.033	.251	.059	.038
	120	.176	.058	.038	.025	.066	.043
	200	.179	.063	.045	.254	.067	.044
	500	.181	.059	.049	.263	.066	.053
0.2	30	.102	.050	.004	.129	.055	.233
	50	.102	.049	.014	.133	.052	.045
	80	.106	.055	.031	.131	.053	.035
	100	.105	.051	.031	.132	.052	.036
	120	.011	.052	.036	.138	.058	.041
	200	.110	.057	.044	.139	.059	.042
	500	.110	.055	.048	.141	.060	.051
−0.2	30	.015	.021	.002	.012	.024	.274
	50	.012	.025	.010	.008	.025	.038
	80	.011	.029	.023	.008	.026	.027
	100	.011	.030	.025	.007	.027	.027
	120	.011	.032	.028	.010	.031	.032
	200	.010	.034	.035	.008	.032	.035
	500	.009	.036	.041	.007	.035	.043
−0.5	30	.000	.003	.001	.000	.004	.360
	50	.000	.004	.003	.000	.004	.028
	80	.000	.000	.003	.000	.003	.013
	100	.000	.006	.010	.000	.004	.010
	120	.000	.006	.012	.000	.005	.013
	200	.000	.006	.016	.000	.004	.017
	500	.000	.007	.023	.000	.006	.021
−0.8	30	.000	.000	.000	.000	.000	.481
	50	.000	.000	.000	.000	.000	.019
	80	.000	.000	.000	.000	.000	.000
	100	.000	.000	.000	.000	.000	.000
	120	.000	.000	.000	.000	.000	.000
	200	.000	.000	.000	.000	.000	.000
	500	.000	.000	.000	.000	.000	.000

Table 2-4 Power for iid Errors

			$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
λ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
.0001	30	.050	.038	.004	.053	.040	.243
	50	.051	.041	.013	.054	.041	.045
	80	.056	.052	.032	.052	.043	.034
	100	.063	.055	.038	.054	.047	.036
	120	.066	.060	.044	.059	.052	.041
	200	.097	.092	.078	.065	.060	.051
	500	.307	.295	.275	.137	.132	.118
.001	30	.058	.046	.004	.054	.042	.244
	50	.075	.060	.020	.060	.047	.048
	100	.168	.147	.100	.084	.070	.053
	200	.399	.372	.314	.193	.174	.132
	500	.788	.757	.682	.621	.583	.503
.01	30	.146	.110	.009	.080	.056	.240
	50	.287	.232	.089	.129	.096	.065
	80	.489	.429	.288	.249	.203	.117
	100	.587	.508	.376	.352	.278	.172
	120	.667	.587	.459	.444	.363	.235
	200	.846	.776	.626	.729	.645	.448
	500	.997	.962	.865	.983	.957	.843
0.1	30	.514	.403	.034	.287	.189	.200
	50	.721	.566	.267	.547	.357	.129
	100	.927	.762	.551	.878	.675	.357
	200	.990	.924	.713	.990	.922	.637
	500	1.00	.989	.897	1.00	.996	.903
1.0	30	.806	.600	.053	.725	.431	.152
	50	.924	.683	.332	.914	.579	.171
	80	.977	.810	.532	.982	.794	.344
	100	.989	.818	.579	.993	.810	.411
	120	.994	.865	.633	.996	.859	.492
	200	.999	.943	.725	1.00	.956	.667
	500	1.00	.992	.901	1.00	.998	.911
10000	30	.887	.641	.059	.888	.508	.141
	50	.959	.704	.343	.974	.627	.176
	80	.988	.822	.536	.995	.822	.353
	100	.998	.827	.582	.999	.820	.410
	120	.998	.871	.635	.999	.871	.496
	200	1.00	.947	.725	1.00	.966	.675
	500	1.00	.992	.901	1.00	.998	.911

Table 2-5

Comparison of the Limiting Power

With Predictions Based on Asymptotics

		T	30	50	100	200	500
ℓ_0	$\hat{\eta}_\mu$	Actual	.887	.959	.995	1.00	1.00
		Predicted	.888	.961	.995	1.00	1.00
	$\hat{\eta}_\tau$	Actual	.888	.974	.999	1.00	1.00
		Predicted	.880	.971	.999	1.00	1.00
ℓ_4	$\hat{\eta}_\mu$	Actual	.640	.704	.830	.947	.992
		Predicted	.641	.701	.826	.947	.992
	$\hat{\eta}_\tau$	Actual	.508	.627	.820	.966	.998
		Predicted	.507	.623	.828	.966	.997
ℓ_{12}	$\hat{\eta}_\mu$	Actual	.059	.343	.580	.725	.901
		Predicted	.055	.348	.583	.728	.900
	$\hat{\eta}_\tau$	Actual	.141	.176	.410	.675	.911
		Predicted	.137	.177	.417	.671	.909

[1] Actual power is obtained as the proportion of rejection of the null of stationarity when we apply the 95 % critical value for the stationary test to data generated under the alternative of $\lambda = 10{,}000$.

[2] Predicted power is calculate as the probability (for the asymptotic distribution under the alternative) that the statistic exceeds its 95% critical value.

Table 2-6

Comparison of the Power of the $\hat{\eta}_\mu$ Test to

Tanaka's Limiting Power

T	λ	θ	C	ℓ_0	ℓ_4	ℓ_{12}	Tanaka[1]
30	.0001	−0.990	0.299	.050	.038	.004	
	.001	−0.969	0.934	.058	.046	.004	.061
	.01	−0.905	2.854	.146	.110	.009	.147
	.1	−0.730	8.105	.514	.403	.034	.505
	1.0	−0.382	18.541	.806	.600	.053	.839
	100	−0.010	29.706	.883	.639	.059	.949
	10000	−0.0001	29.997	.887	.641	.059	.949
50	.0001	−0.990	0.498	.051	.041	.013	
	.001	−0.969	1.556	.075	.060	.020	.079
	.01	−0.905	4.756	.287	.232	.089	.293
	.1	−0.730	13.508	.721	.566	.267	.733
	1.0	−0.382	30.902	.924	.683	.332	.952
	100	−0.010	49.510	.958	.703	.342	.990
	10000	−0.0001	49.995	.959	.704	.343	.991
100	.0001	−0.990	0.995	.063	.055	.038	.062
	.001	−0.969	3.113	.168	.147	.100	.172
	.01	−0.905	9.512	.587	.508	.376	.584
	.1	−0.730	27.016	.927	.762	.551	.931
	1.0	−0.382	61.803	.989	.818	.579	.996[2]
	100	−0.010	99.020	.994	.826	.582	
	10000	−0.0001	99.990	.998			
200	.0001	−0.990	1.990	.097	.092	.078	.099
	.001	−0.969	6.225	.399	.372	.314	.400
	.01	−0.905	19.025	.846	.776	.626	.848
	.1	−0.730	54.031	.990	.924	.713	.993
	1.0	−0.382	123.607	.999	.943	.725	.996[2]
	100	−0.010	198.039	1.00	.945	.726	
	10000	−0.0001	199.980	1.00	.947	.725	
500	.0001	−0.990	4.975	.307	.295	.275	.309
	.001	−0.969	15.563	.788	.757	.682	.786
	.01	−0.905	47.562	.997	.962	.865	.988
	.1	−0.730	135.078	1.00	.989	.897	.996[2]
	1.0	−0.382	309.017	1.00	.992	.901	
	100	−0.010	495.098	1.00	.992	.901	
	10000	−0.0001	499.950	1.00	.992	.901	

1. These powers are interpolated from Tanaka's table of the limiting power for different C values.
2. Tanaka's results are available only up to C=60.

Table 2-7 Power for AR(1) Errors (λ = 0.0001)

			$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
ρ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.653	.303	.007	.769	.316	.123
	50	.728	.264	.039	.886	.318	.057
	80	.779	.300	.081	.936	.401	.084
	100	.799	.252	.083	.953	.340	.092
	120	.809	.258	.091	.960	.356	.104
	200	.839	.282	.099	.979	.396	.112
	500	.873	.304	.137	.988	.383	.123
0.5	30	.322	.115	.005	.424	.130	.178
	50	.334	.097	.021	.487	.114	.047
	80	.353	.111	.043	.520	.125	.046
	100	.363	.095	.045	.540	.108	.049
	120	.368	.098	.051	.544	.116	.055
	200	.397	.121	.067	.565	.130	.061
	500	.531	.218	.159	.634	.150	.086
0.2	30	.118	.055	.004	.148	.061	.227
	50	.122	.055	.016	.157	.059	.045
	80	.129	.065	.035	.158	.060	.038
	100	.134	.061	.038	.165	.060	.039
	120	.140	.067	.045	.171	.067	.045
	200	.176	.094	.067	.186	.079	.055
	500	.373	.260	.229	.267	.124	.101
−0.2	30	.018	.026	.003	.016	.028	.267
	50	.016	.031	.012	.013	.031	.039
	80	.020	.042	.029	.012	.032	.030
	100	.021	.045	.035	.011	.034	.031
	120	.027	.054	.045	.015	.041	.038
	200	.050	.097	.089	.019	.054	.054
	500	.280	.365	.351	.073	.151	.149
−0.5	30	.002	.011	.002	.001	.011	.300
	50	.001	.019	.008	.001	.016	.032
	80	.002	.029	.025	.000	.018	.024
	100	.003	.038	.035	.000	.020	.026
	120	.004	.051	.049	.001	.026	.033
	200	.017	.118	.129	.001	.041	.057
	500	.241	.470	.467	.025	.210	.225
−0.8	30	.000	.002	.001	.000	.001	.318
	50	.000	.014	.004	.000	.014	.029
	80	.000	.031	.025	.000	.014	.017
	100	.000	.026	.041	.000	.005	.017
	120	.000	.040	.069	.000	.005	.022
	200	.005	.149	.222	.000	.021	.065
	500	.217	.649	.634	.008	.378	.396

Table 2-7 (Continued) (λ = 0.01)

ρ	T	$\hat{\eta}_\mu$ ℓ_0	$\hat{\eta}_\mu$ ℓ_4	$\hat{\eta}_\mu$ ℓ_{12}	$\hat{\eta}_\tau$ ℓ_0	$\hat{\eta}_\tau$ ℓ_4	$\hat{\eta}_\tau$ ℓ_{12}
0.8	30	.679	.335	.009	.771	.327	.122
	50	.772	.332	.066	.891	.345	.065
	80	.846	.434	.137	.945	.442	.105
	100	.881	.431	.207	.964	.411	.128
	120	.902	.487	.272	.973	.448	.157
	200	.960	.668	.431	.991	.612	.260
	500	.996	.891	.747	.999	.876	.630
0.5	30	.389	.163	.009	.444	.140	.179
	50	.488	.207	.056	.542	.151	.058
	80	.625	.352	.189	.634	.214	.085
	100	.700	.400	.254	.697	.237	.116
	120	.755	.475	.336	.737	.295	.156
	200	.893	.686	.527	.880	.524	.320
	500	.991	.925	.818	.994	.901	.754
0.2	30	.214	.118	.009	.176	.077	.223
	50	.341	.209	.072	.248	.108	.062
	80	.525	.388	.246	.359	.188	.100
	100	.617	.461	.328	.452	.245	.147
	120	.689	.540	.414	.531	.320	.199
	200	.864	.749	.596	.775	.590	.407
	500	.989	.952	.850	.987	.940	.823
−0.2	30	.106	.116	.009	.031	.047	.256
	50	.251	.263	.108	.077	.105	.067
	80	.465	.477	.329	.184	.229	.136
	100	.568	.555	.422	.286	.321	.204
	120	.652	.632	.498	.386	.421	.277
	200	.846	.819	.652	.699	.707	.507
	500	.987	.973	.876	.984	.973	.868
−0.5	30	.065	.131	.011	.006	.032	.279
	50	.215	.334	.148	.033	.126	.076
	80	.440	.553	.393	.127	.298	.177
	100	.551	.629	.478	.225	.405	.259
	120	.637	.699	.548	.328	.514	.341
	200	.837	.863	.684	.660	.792	.565
	500	.986	.981	.888	.982	.986	.889
−0.8	30	.062	.156	.013	.002	.023	.258
	50	.209	.469	.205	.020	.234	.088
	80	.432	.666	.459	.108	.471	.238
	100	.547	.684	.530	.200	.483	.322
	120	.633	.750	.593	.304	.599	.406
	200	.831	.890	.706	.643	.851	.620
	500	.985	.987	.896	.979	.994	.904

Table 2-7 (Continued) ($\lambda = 1$)

			$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
ρ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.863	.609	.049	.864	.472	.135
	50	.949	.674	.318	.964	.592	.162
	80	.983	.802	.517	.993	.789	.329
	100	.992	.807	.569	.998	.796	.394
	120	.996	.858	.624	.999	.848	.475
	200	1.00	.942	.720	1.00	.956	.656
	500	1.00	.989	.894	1.00	.996	.903
0.5	30	.834	.592	.050	.799	.426	.152
	50	.936	.671	.326	.937	.564	.163
	80	.979	.802	.525	.988	.777	.335
	100	.990	.810	.574	.995	.795	.403
	120	.994	.860	.628	.997	.848	.485
	200	1.00	.943	.724	1.00	.956	.663
	500	1.00	.990	.900	1.00	.998	.911
0.2	30	.817	.598	.052	.746	.421	.155
	50	.928	.676	.334	.920	.571	.166
	80	.978	.806	.530	.984	.787	.341
	100	.989	.815	.579	.994	.803	.409
	120	.994	.863	.631	.997	.855	.490
	200	1.00	.944	.725	1.00	.959	.667
	500	1.00	.991	.901	1.00	.998	.913
−0.2	30	.802	.612	.053	.705	.441	.152
	50	.924	.684	.340	.902	.591	.171
	80	.977	.812	.534	.981	.800	.346
	100	.988	.820	.582	.993	.813	.413
	120	.994	.867	.633	.996	.862	.494
	200	.999	.946	.725	1.00	.962	.670
	500	1.00	.991	.901	1.00	.998	.914
−0.5	30	.798	.619	.053	.687	.454	.147
	50	.922	.691	.342	.895	.605	.174
	80	.976	.816	.535	.980	.807	.349
	100	.988	.822	.582	.992	.817	.415
	120	.993	.869	.634	.996	.866	.494
	200	.999	.946	.726	1.00	.963	.670
	500	1.00	.991	.902	1.00	.998	.913
−0.8	30	.800	.624	.053	.686	.458	.140
	50	.922	.698	.345	.892	.617	.174
	80	.976	.819	.535	.980	.817	.350
	100	.988	.823	.584	.992	.820	.416
	120	.993	.870	.634	.996	.868	.496
	200	.999	.946	.725	1.00	.964	.670
	500	1.00	.992	.902	1.00	.998	.914

Table 2-7 (Continued) (λ = 10,000)

ρ	T	$\hat{\eta}_\mu$ ℓ_0	$\hat{\eta}_\mu$ ℓ_4	$\hat{\eta}_\mu$ ℓ_{12}	$\hat{\eta}_\tau$ ℓ_0	$\hat{\eta}_\tau$ ℓ_4	$\hat{\eta}_\tau$ ℓ_{12}
0.8	30	.889	.643	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.996	.822	.353
	100	.995	.825	.583	.999	.826	.417
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
0.5	30	.889	.642	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.353
	100	.994	.825	.584	.999	.825	.417
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
0.2	30	.889	.642	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.353
	100	.994	.825	.584	.999	.826	.417
	120	.996	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.2	30	.889	.643	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.353
	100	.995	.825	.584	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.5	30	.889	.642	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.994	.825	.584	.999	.825	.416
	120	.997	.970	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.8	30	.889	.642	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.353
	100	.994	.825	.583	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914

Table 2-8 Power for MA(1) Errors (λ = 0.0001)

			$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
α	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.203	.061	.004	.287	.068	.217
	50	.202	.058	.016	.296	.064	.046
	80	.213	.067	.035	.299	.066	.037
	100	.218	.062	.037	.307	.063	.039
	120	.223	.068	.044	.309	.070	.046
	200	.256	.088	.062	.320	.082	.054
	500	.426	.223	.191	.400	.113	.089
0.5	30	.170	.058	.004	.236	.065	.223
	50	.172	.056	.016	.244	.061	.046
	80	.181	.065	.035	.245	.063	.037
	100	.187	.061	.037	.253	.061	.040
	120	.193	.067	.045	.256	.067	.045
	200	.226	.088	.063	.271	.079	.054
	500	.409	.232	.203	.351	.116	.092
0.2	30	.103	.050	.004	.128	.054	.234
	50	.105	.050	.015	.135	.053	.045
	80	.112	.059	.034	.133	.054	.036
	100	.117	.057	.036	.138	.055	.038
	120	.124	.064	.044	.144	.061	.044
	200	.159	.090	.068	.156	.072	.054
	500	.360	.265	.238	.237	.122	.104
-0.2	30	.015	.022	.002	.012	.024	.275
	50	.013	.027	.011	.010	.026	.039
	80	.015	.036	.028	.009	.028	.028
	100	.017	.042	.034	.008	.030	.030
	120	.021	.051	.044	.011	.036	.036
	200	.044	.091	.091	.014	.049	.053
	500	.275	.371	.360	.065	.151	.153
-0.5	30	.000	.004	.001	.000	.003	.360
	50	.000	.006	.004	.000	.004	.029
	80	.000	.010	.018	.000	.004	.015
	100	.000	.021	.028	.000	.006	.015
	120	.000	.031	.045	.000	.008	.020
	200	.007	.104	.145	.000	.021	.047
	500	.226	.498	.533	.014	.201	.271
-0.8	30	.000	.000	.000	.000	.000	.480
	50	.000	.000	.000	.000	.000	.019
	80	.000	.000	.006	.000	.000	.001
	100	.000	.005	.019	.000	.000	.002
	120	.000	.013	.049	.000	.000	.003
	200	.002	.117	.252	.000	.003	.037
	500	.204	.602	.696	.003	.260	.462

Table 2-8 (Continued) (λ = 0.01)

α	T	$\hat{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\hat{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.290	.114	.007	.312	.081	.215
	50	.401	.182	.060	.380	.101	.058
	80	.564	.347	.214	.471	.168	.088
	100	.648	.416	.294	.552	.215	.127
	120	.713	.496	.379	.615	.281	.172
	200	.875	.712	.567	.818	.536	.363
	500	.990	.939	.836	.989	.920	.795
0.5	30	.262	.113	.008	.263	.078	.220
	50	.379	.189	.063	.333	.103	.060
	80	.550	.359	.224	.431	.173	.092
	100	.636	.430	.304	.513	.223	.132
	120	.703	.510	.391	.586	.292	.181
	200	.873	.724	.578	.802	.553	.379
	500	.990	.944	.840	.989	.928	.805
0.2	30	.201	.112	.009	.158	.070	.229
	50	.328	.208	.075	.224	.103	.062
	80	.518	.390	.252	.335	.185	.102
	100	.610	.468	.335	.431	.246	.150
	120	.685	.548	.422	.512	.323	.204
	200	.863	.754	.602	.765	.597	.418
	500	.989	.955	.853	.987	.944	.830
−0.2	30	.099	.112	.009	.027	.043	.262
	50	.244	.264	.110	.070	.101	.068
	80	.461	.481	.334	.176	.228	.139
	100	.566	.560	.427	.277	.324	.209
	120	.650	.637	.504	.377	.424	.284
	200	.845	.822	.657	.692	.711	.512
	500	.987	.974	.877	.984	.975	.870
−0.5	30	.050	.113	.010	.002	.019	.318
	50	.198	.323	.158	.020	.100	.078
	80	.428	.554	.414	.108	.279	.190
	100	.543	.641	.498	.204	.408	.278
	120	.631	.713	.566	.306	.527	.360
	200	.834	.869	.694	.650	.807	.586
	500	.986	.982	.892	.980	.988	.896
−0.8	30	.033	.116	.010	.000	.008	.373
	50	.181	.369	.201	.009	.098	.085
	80	.416	.600	.461	.083	.318	.234
	100	.533	.684	.537	.176	.468	.324
	120	.624	.750	.600	.280	.590	.414
	200	.830	.891	.712	.631	.852	.629
	500	.985	.986	.899	.978	.993	.909

Table 2-8 (Continued) ($\lambda = 1$)

α	T	$\hat{\eta}_\mu$ ℓ_0	$\hat{\eta}_\mu$ ℓ_4	$\hat{\eta}_\mu$ ℓ_{12}	$\hat{\eta}_\tau$ ℓ_0	$\hat{\eta}_\tau$ ℓ_4	$\hat{\eta}_\tau$ ℓ_{12}
0.8	30	.825	.588	.051	.777	.408	.158
	50	.931	.671	.330	.930	.560	.165
	80	.978	.802	.528	.985	.778	.338
	100	.989	.812	.578	.994	.797	.406
	120	.994	.861	.630	.997	.851	.488
	200	1.00	.944	.725	1.00	.958	.665
	500	1.00	.991	.901	1.00	.998	.914
0.5	30	.822	.590	.052	.766	.412	.157
	50	.930	.672	.332	.926	.564	.165
	80	.978	.803	.528	.985	.781	.340
	100	.989	.813	.578	.994	.799	.408
	120	.994	.862	.631	.997	.852	.489
	200	1.00	.944	.725	1.00	.959	.666
	500	1.00	.991	.901	1.00	.998	.914
0.2	30	.815	.598	.052	.742	.421	.156
	50	.927	.677	.335	.918	.572	.166
	80	.977	.807	.531	.984	.788	.342
	100	.988	.815	.580	.993	.803	.409
	120	.994	.864	.631	.997	.856	.490
	200	1.00	.945	.725	1.00	.959	.667
	500	1.00	.991	.901	1.00	.998	.914
−0.2	30	.802	.612	.052	.704	.440	.151
	50	.924	.685	.341	.902	.591	.172
	80	.976	.812	.534	.981	.800	.346
	100	.988	.820	.582	.993	.813	.413
	120	.994	.867	.634	.996	.863	.494
	200	.999	.946	.726	1.00	.962	.670
	500	1.00	.991	.902	1.00	.998	.914
−0.5	30	.796	.619	.053	.684	.453	.146
	50	.921	.691	.343	.894	.602	.175
	80	.976	.816	.534	.979	.808	.350
	100	.988	.822	.583	.992	.818	.416
	120	.993	.869	.634	.996	.866	.495
	200	.999	.946	.725	1.00	.963	.670
	500	1.00	.992	.902	1.00	.998	.914
−0.8	30	.795	.621	.052	.678	.458	.147
	50	.920	.695	.344	.889	.606	.175
	80	.976	.818	.535	.979	.811	.352
	100	.988	.823	.584	.991	.820	.415
	120	.993	.869	.635	.996	.868	.495
	200	.999	.946	.726	1.00	.964	.671
	500	1.00	.992	.902	1.00	.998	.914

Table 2-8 (Continued) (λ = 10,000)

			$\hat{\eta}_\mu$			$\hat{\eta}_\tau$	
α	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.889	.643	.057	.889	.512	.141
	50	.961	.701	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
0.5	30	.889	.642	.057	.889	.512	.141
	50	.961	.701	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
0.2	30	.889	.642	.057	.889	.512	.141
	50	.961	.701	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.996	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.2	30	.889	.642	.057	.889	.512	.141
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.5	30	.889	.642	.057	.889	.512	.140
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.997	.970	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914
−0.8	30	.889	.642	.057	.889	.512	.140
	50	.961	.702	.348	.973	.627	.177
	80	.988	.822	.536	.995	.822	.354
	100	.995	.826	.583	.999	.825	.416
	120	.997	.870	.635	.999	.871	.496
	200	1.00	.947	.726	1.00	.965	.671
	500	1.00	.992	.903	1.00	.998	.914

Table 2-9

Size Comparison of the $\hat{\eta}_\mu$ Test to the Saikkonen and Luukkonen Test

MA(1) errors

T	Test	α −0.8	−0.5	0.0	0.5	0.8
100	R_1	.023	.041	.053	.056	.060
	ℓ_0	.000	.000	.049	.176	.206
	ℓ_4	.000	.006	.043	.056	.057
	ℓ_{12}	.000	.010	.029	.033	.033
200	R_1	.030	.047	.052	.057	.060
	ℓ_0	.000	.000	.051	.176	.209
	ℓ_4	.000	.006	.049	.063	.065
	ℓ_{12}	.000	.016	.041	.045	.046

AR(1) errors

T	Test	ρ −0.8	−0.5	0.0	0.5	0.8
100	R_{10}	.028	.051	.053	.052	.081
	R_{15}	.041	.051	.053	.052	.066
	R_{30}	.039	.051	.053	.052	.059
	ℓ_0	.000	.001	.049	.352	.796
	ℓ_4	.007	.019	.043	.090	.250
	ℓ_{12}	.007	.020	.029	.043	.081
200	R_{10}	.027	.046	.047	.045	.074
	R_{15}	.044	.046	.047	.045	.056
	R_{30}	.041	.046	.047	.045	.050
	ℓ_0	.000	.001	.051	.363	.833
	ℓ_4	.008	.021	.049	.099	.271
	ℓ_{12}	.015	.030	.041	.047	.094

Table 2-10

Power Comparison of the $\hat{\eta}_\mu$ Test to the Saikkonen and Luukkonen Test

MA(1) errors

T	θ	Test	α -0.8	-0.5	0.0	0.5	0.8
100	-0.95	R_1	.122	.243	.304	.324	.335
		ℓ_0	.197	.216	.301	.401	.419
		ℓ_4	.483	.416	.268	.209	.201
		ℓ_{12}	.439	.351	.193	.137	.130
	-0.90	R_1	.309	.505	.587	.616	.631
		ℓ_0	.557	.568	.609	.654	.666
		ℓ_4	.694	.652	.525	.450	.436
		ℓ_{12}	.541	.505	.392	.319	.308
	-0.80	R_1	.534	.749	.831	.858	.867
		ℓ_0	.841	.843	.858	.874	.877
		ℓ_4	.785	.770	.715	.673	.665
		ℓ_{12}	.574	.564	.521	.491	.484

AR(1) errors

T	θ	Test	ρ -0.8	-0.5	0.0	0.5	0.8
100	-0.95	R_{15}	.273	.292	.288	.265	.266
		R_{30}	.269	.292	.288	.265	.226
		ℓ_0	.218	.233	.309	.513	.827
		ℓ_4	.492	.401	.268	.209	.312
		ℓ_{12}	.422	.317	.193	.115	.121
	-0.90	R_{15}	.563	.581	.555	.481	.447
		R_{30}	.562	.581	.555	.480	.377
		ℓ_0	.573	.575	.609	.715	.885
		ℓ_4	.695	.643	.525	.417	.444
		ℓ_{12}	.535	.487	.392	.270	.219
	-0.80	R_{15}	.822	.812	.743	.567	.605
		R_{30}	.823	.818	.743	.534	.457
		ℓ_0	.844	.845	.858	.894	.950
		ℓ_4	.785	.766	.715	.648	.635
		ℓ_{12}	.572	.558	.524	.461	.399

Using equation (11) or (12) in text we find that if θ = -0.95, -0.9, and -0.8, then λ = .0026, .0111, and .05.

CHAPTER 3

CHAPTER 3

TESTING FOR A UNIT ROOT: A DUAL APPROACH

3.1 Introduction

This chapter considers using the KPSS test statistic for stationarity to test for integration in time series data. Many of the existing tests for a unit root are motivated by considering the problem of testing whether an autoregressive root equals one against the alternative that it is not equal to one. In contrast, the statistics proposed in this chapter can be derived from a very different specification, but have almost the same implication.

The decomposition into stationary and random walk components is a popular way of thinking about the properties of macroeconomic time series. Since a series with a unit root is equivalent to a series that is composed of a random walk and a stationary component, tests for a unit root are attempts to distinguish between series that have no random walk component and series that have a random walk component.

KPSS propose a statistic ($\hat{\eta}_\tau$, given in equation (25) of the last chapter) to test the null hypothesis of stationarity (no random walk component) against the alternative of a unit root (a non-zero random walk component). In this chapter we reverse this process and consider using the KPSS statistic to test the null hypothesis of a unit root against the alternative of stationarity. The basic idea behind this procedure has been suggested by Stock (1990). Suppose that y_t is the series in question and $S_t = \sum_{j=1}^{t} y_j$ is the partial sum process of y_t. If y_t is I(0), then y_t is $O_P(1)$ and S_t is $O_P(T^{1/2})$; while if y_t is I(1), then y_t is $O_P(T^{1/2})$ and S_t is $O_P(T^{3/2})$. Thus Stock suggests that functions of S_t can be used to test H_1: y_t is I(1) vs H_0: y_t is I(0), or vice versa. The idea is that one statistic can be used to test both hypotheses. The KPSS statistic is of this form (it depends on S_t) and KPSS use it to test H_0 vs H_1. In this chapter we consider using it to test H_1 vs H_0. The question of interest is whether a test designed as a stationarity test will perform well as a unit root test. As will be shown, the answer turns out to be "no."

We will derive the asymptotic distribution of the KPSS statistic as a unit root test and discuss its characteristics in section 3.2. This test is compared to other similar kinds of unit root tests in section 3.3. The finite sample performance of the test is investigated via a Monte Carlo simulation in section 3.4. We apply our unit root test statistics to the Nelson-Plosser data in section 3.5. The concluding remarks are given in section 3.6.

3.2 The KPSS Test As a Unit Root Test

We will use a components representation of an economic time series to derive test statistics for the unit root hypothesis. The series of interest y_t, can be decomposed into the sum of a deterministic trend, a random walk and a stationary error:

(1) $$y_t = \xi t + \gamma_t + v_t\ ,\quad t = 1,2,\cdots,T$$

(2) $$\gamma_t = \gamma_{t-1} + u_t$$

where u_t are iid $(0,\sigma_u^2)$ and v_t are stationary errors.

The null hypothesis is simply that $\sigma_u^2 > 0$, so that y_t is an I(1) process under the null. Under the null (1) can be expressed as:

(3) $$(1 - L)y_t = \xi + u_t + \Delta v_t,\quad \text{or}$$

(4) $$y_t = \gamma_0 + \xi t + \sum_{j=1}^{t} u_j + v_t.$$

On the other hand, (1) can be expressed in the form of an I(0) process under the alternative of $\sigma_u^2 = 0$:

(5) $$y_t = \gamma_0 + \xi t + v_t$$

The fundamental difference is that the deviations from trend in (5) are stationary while in (4) they are an integrated process whose variance increases without bound as t gets large.

Note that (3) or (4) is a generalization of the first order difference stationary process which has been used as the counterpart of (5) in most of the unit root literature:

(6) $$(1 - L)y_t = \xi + u_t,\quad \text{or}$$

(7) $$y_t = \gamma_0 + \xi t + \sum_{j=1}^{t} u_t$$

Comparison shows that (6) or (7) is a special case of (3) or (4); (4) reduces to (7) when $\sigma_v^2 = 0$. Therefore, our model of (1) and (2) is more general than the Dickey-Fuller type model. In fact, as shown in Chapter 1 (equations (15) and (16)), our model (3) or (4) is equivalent to a Dickey-Fuller model with MA(1) errors.

KPSS derive the asymptotic distributions of their test statistics under $\sigma_u^2 > 0$, which is our null hypothesis. First, we consider the statistic used by KPSS to test for level stationarity:

$$\hat{\eta}_\mu = T^{-2} \sum_{t=1}^{T} S_t^2 / s^2(\ell) \tag{8}$$

where $S_t = \sum_{j=1}^{t} e_j$, $e_j = y_j - \bar{y}$, and $s^2(\ell)$ is the Newey-West estimate of the long run variance of v_t.

$$s^2(\ell) = (1/T)[\sum_{t=1}^{T} e_t^2 + 2\sum_{s=1}^{\ell} w(s,\ell) \sum_{t=s+1}^{T} e_t e_{t-s}] \tag{9}$$

where the Bartlett window, $w(s,\ell) = 1 - \ell/(s+1)$ is used for nonnegativeness of $s^2(\ell)$. For consistency of $s^2(\ell)$ under stationarity, it is necessary that the lag truncation parameter $\ell \to \infty$ as $T \to \infty$. The rate $\ell = o_p(T^{1/2})$ will usually be satisfactory (see, e.g., Andrews (1991)).

We now use the invariance principles (10) and (11) to derive the asymptotic results for the unit root test statistics.

$$T^{-1/2}\gamma_{[bT]} = T^{-1/2} \sum_{j=1}^{[bT]} u_j \to \sigma_u W(b) \tag{10}$$

$$T^{-3/2} S_{[aT]} = T^{-3/2} \sum_{j=1}^{[aT]} (\gamma_j - \bar{\gamma}) + T^{-3/2} \sum_{j=1}^{[aT]} (v_j - \bar{v}) \tag{11}$$

$$= T^{-3/2}\sum_{j=1}^{[aT]}(\gamma_j - \bar{\gamma}) + o_P(1) \rightarrow \sigma_u\int_0^a \bar{W}(s)ds$$

where a,b ε [0,1], [aT] and [bT] are integer parts of aT and bT, and $\bar{W}(s) = W(s) - \int_0^1 W(b)db$ is the demeaned Wiener process. Therefore,

(12) $$T^{-4}\sum_{t=1}^{T} S_t^2 = T^{-1}\sum_{t=1}^{T}(T^{-3/2}S_t)^2 \rightarrow \sigma_u^2\int_0^1 [\int_0^a \bar{W}(s)ds]^2\, da$$

From Phillips (1991) we can also show that

(13) $$(\ell T)^{-1}\, s^2(\ell) \rightarrow K\sigma_u^2\int_0^1 \bar{W}(s)^2 ds$$

provided $T^{-1/2}\ell \rightarrow 0$ as $T \rightarrow \infty$. K is defined by $K = \int_{-1}^{1} k(s)ds$ where k(s) represents the weighting function used in calculating $s^2(\ell)$. Note that if $w(s,\ell) = 1 - \ell/(s + 1)$ is used, then $k(s) = 1 - |s|$ and therefore, K = 1. However, if $\ell = 0$ is used, the following holds instead of (13):

(14) $$T^{-1}s^2(\ell) \rightarrow \sigma_u^2\int_0^1 \bar{W}(s)^2 ds$$

Combining (12) with (13), it is straightforward to see that the unit root test with level, defined as $\tilde{\eta}_\mu(\ell)$, has the following asymptotic distribution:

(15) $$\tilde{\eta}_\mu(\ell) = (\ell/T)\,\hat{\eta}_\mu(\ell) \rightarrow \int_0^1 [\int_0^a \bar{W}(s)ds]^2 da \,/\, K\int_0^1 \bar{W}(s)^2 ds$$

If $\ell = 0$ is used,

(16) $$\tilde{\eta}_\mu(0) = (1/T)\,\hat{\eta}_\mu(0) \rightarrow \int_0^1 [\int_0^a \bar{W}(s)ds]^2 da \,/ \int_0^1 \bar{W}(s)^2 ds$$

The analysis for the unit root test statistic in the presence of trend is only slightly more complicated. Now e_t is the residual from an OLS regression of y_t on intercept and trend. Correspondingly, we just need to replace the demeaned Wiener process $\bar{W}(s)$ above with the demeaned and detrended Wiener process $W^*(s)$:

$$(17) \qquad W^*(s) = W(s) + (6s - 4)\int_0^1 W(r)dr + (-12s + 6)\int_0^1 rW(r)dr$$

Therefore, the unit root test statistic with level and trend, defined as $\tilde{\eta}_\tau$, has the following asymptotic distribution:

$$(18) \qquad \tilde{\eta}_\tau(\ell) = (\ell/T)\,\hat{\eta}_\tau(\ell) \rightarrow \int_0^1 [\int_0^a W^*(s)ds]^2 da \,/\, K\int_0^1 W^*(s)^2 ds$$

If $\ell = 0$ is used,

$$(19) \qquad \tilde{\eta}_\tau(0) = (1/T)\,\hat{\eta}_\tau(0) \rightarrow \int_0^1 [\int_0^a W^*(s)ds]^2 da \,/\int_0^1 W^*(s)^2 ds$$

The stationary errors v_t, do not have any effect on the asymptotic distribution of the test statistics under the null hypothesis $\sigma_u^2 > 0$. This implies that under the unit root null the statistic has the same limiting distribution as that for a pure random walk process. Thus, although the unit root null can be stated as $\lambda = \sigma_u^2/\sigma_v^2 > 0$, in fact our null is effectively $\lambda = \infty$ ($\sigma_v^2 = 0$). $\tilde{\eta}_\mu$ and $\tilde{\eta}_\tau$ are free of nuisance parameters because the scale effect from the variance $\sigma_u^2 > 0$ in the numerator and the denominator of the limiting distribution cancels out. It is important to note that this is so regardless of the choice of the lag truncation parameter ℓ. From the point of view of the KPSS statistics $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ as unit root statistics, $\ell = 0$ is the obvious choice. Other choices of ℓ (e.g., ℓ proportional to $T^{1/4}$) are necessary to correct for error autocorrelation in *testi*

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testing the null hypothesis of stationarity, but are not necessary for the unit root test.

In Table 3-0, the critical values of the r.h.s of equations (16) and (19) are given, calculated via a direct simulation, using a sample size of 2,000, 50,000 replications, and the random number generator GASDEV/RAN3 of Press et al. (1986). Note that the unit root test is a lower tail test. Therefore, we compare $\tilde{\eta}_{\mu}(\ell) = (\ell/T)\hat{\eta}_{\mu}(\ell)$ and $\tilde{\eta}_{\tau}(\ell) = (\ell/T)\hat{\eta}_{\tau}(\ell)$ [$\tilde{\eta}_{\mu}(0) = (1/T)\hat{\eta}_{\mu}(0)$ and $\tilde{\eta}_{\tau}(0) = (1/T)\hat{\eta}_{\tau}(0)$ when ℓ = 0] to the lower 5% critical values to test the unit root null at the 5% level.

3.3 Comparison to Other Similar Tests

As mentioned above, Stock (1990) develops a unifying framework for so-called "generic" unit root tests based on the fact that an I(1) process is $O_P(T^{1/2})$ but an I(0) process is $O_P(1)$. Under the null hypothesis that the time series contains deterministic trend plus an integrated process, Stock works with functionals of the detrended series itself and shows that those converge weakly to the corresponding functionals of a detrended Brownian motion. Our unit root tests, $\tilde{\eta}_{\mu}$ and $\tilde{\eta}_{\tau}$, also converge weakly to functionals of a detrended Brownian motion under the null.

According to Stock's simulation results, the modified Sargan and Bhargava tests perform relatively well, and they are similar in form to our tests, $\tilde{\eta}_{\mu}$ and $\tilde{\eta}_{\tau}$. Note that under the null of a unit root, Stock's specification is the same as that of Schmidt and Phillips (1990) or Bhargava (1986):

$$(17) \qquad y_t = d_t(\beta) + \sum_{j=1}^{t} u_j$$

where $d_t(\beta)$ represents deterministic trend and the second term is an I(1) process. Therefore, (17) is basically the same as (7), though (7) assumes linear deterministic trend. Modified Sargan-Bhargava tests are

(18) $$g_{SB}(V_T^{\mu}) = T^{-2}\sum_{t=1}^{T}(y_t^{\mu})^2/\hat{\sigma}^2 \rightarrow \int_0^1 \bar{W}(s)^2$$

(19) $$g_{SB}(V_T^{B}) = T^{-2}\sum_{t=1}^{T}(y_t^{B})^2/\hat{\sigma}^2 \rightarrow \int_0^1 W^B(s)^2$$

where $y_t^{\mu} = y_t - \bar{y}$, $y_t^{B} = y_t - \tilde{\beta}_0 - \tilde{\beta}_1(t/T)$. $W^B = W(r) - (r - 1/2)W(1) - \int W(s)$. $\tilde{\beta}_0$ and $\tilde{\beta}_1$ are the maximum likelihood estimators of β_0 and β_1 assuming $d_t(\beta) = \beta_0 + \beta_1 t$. Stock uses the detrended series itself while we use the partial sum process of detrended series to derive the test statistics.

Furthermore, Stock estimates the long run variance of the residual as follows:

(20) $$\hat{\sigma}^2 = (1/T)[\sum_{t=1}^{T}\hat{u}_t^2 + 2\sum_{j=1}^{\ell}\sum_{t=j+1}^{T}\hat{u}_t\hat{u}_{t-j}]$$

where $\hat{u}_t$ is the residual from the regression of detrended (or demeaned) series on lagged detrended (or demeaned) series, used for consistency. That is, $\hat{\sigma}^2$ replaces $T^{-1}\sum_{t=2}^{T}(\Delta y_t)^2$ in the original Sargan-Bhargava tests to make the test nuisance parameters free. This treatment of the long run variance is different from ours. The form of the denominator of the KPSS statistic is chosen so as to estimate the long run variance of the stationary error in the <u>absence</u> of a unit root. Under the unit root null, the limiting distribution of the denominator of the KPSS statistic involves a functional of Brownian motion, basically because our residual e_t is I(1), while Stock's residual is I(0), under the unit root null. However, the KPSS statistic is free of nuisance parameters under the unit root null without the need to estimate a long run variance consistently. It is reasonable to guess that this treatment of nuisance parameters might minimize size distortions, but also entail some loss of power.

Blough (1989) also represents a time series as a convex combination of a random walk and a general stationary component following the Beveridge-Nelson (1981) decomposition:

$$(21) \qquad y_t = \delta y_{1t} + (1 - \delta)y_{2t}, \ \delta = [0,1]$$

where $y_{1t} = \mu + y_{1t-1} + \varepsilon_{1t}$ and $y_{2t} = m_t + a(L)\varepsilon_{2t}$. Here m_t is deterministic trend, $a(0) = 1$, $a(1) < \infty$ and the ε_{it} are iid. Blough then considers the problem of testing the null of a unit root $\delta = (0,1]$ against the stationary alternative of $\delta = 0$. Note that a sequence of processes $\{y_t(\delta)\}$ with δ approaching zero is equivalent to a sequence of ARIMA process with MA root approaching minus one. For any $\delta > 0$, the asymptotic properties will be dominated by the random walk component $\{y_{1t}\}$. However, for finite samples, $\{y_t(\delta)\}$ will behave like $\{y_{2t}\}$ for small δ and therefore the finite sample distribution of statistics will be dominated by $\{y_{2t}\}$. This implies that some unit root processes behave almost like white noise for a given sample size, which raises questions about both the possibility of and the need for generic unit root tests.

3.4 Finite Sample Behavior

The finite sample distribution of the unit root test statistics $\tilde{\eta}_\mu$ and $\tilde{\eta}_\tau$ will be tabulated by a Monte Carlo simulation. The results of these simulations (using 20,000 replications when $T \leq 100$ and 10,000 replications when $T > 100$) are given in Tables 3-1 through 3-8.

We will use three different specifications of the stationary errors v_t: iid, AR(1) and MA(1) errors.

(22) $$v_t = \rho v_{t-1} + \varepsilon_t \quad \text{for AR(1) errors}$$

(23) $$v_t = \varepsilon_t + \alpha\varepsilon_{t-1} \quad \text{for MA(1) errors}$$

where ε_t are iid and values of ±0.2, ±0.5, and ±0.8 are used for ρ and α. When the errors are iid, the size of the test statistics under the null depends on the variance ratio λ and the sample size T, whereas the power under the alternative depends only on T, since the alternative specifies $\lambda = 0$. We will consider three different choices of the lag truncation parameter ℓ. These are $\ell_0 = 0$, $\ell_4 = \text{integer}[4(T/100)^{1/4}]$, and $\ell_{12} = \text{integer}[12(T/100)^{1/4}]$. As noted above, we expect $\ell = \ell_0$ to be the best choice, but this may depend on how one weighs the tradeoff between size distortions and low power.

3.4.1 Size

For investigation of size we will choose four different values of λ (0.0001, 0.01, 1 and 10,000) and seven values of T (30, 50, 80, 100, 120, 250, 500). We expect better size performance for large λ and worse size performance for small λ, because our null is effectively $\lambda = \infty$ as discussed above. Furthermore, because size distortions disappear asymptotically, we expect better size performance (for any given λ) when T is large than when T is small.

1) <u>iid Errors</u>

The sizes of the tests under iid errors are given in Table 3-1. The tests using ℓ_{12} perform very poorly. They reject too seldom (except when λ is very small), and this problem persists even for rather large values of λ and T, such as $\lambda = 10{,}000$ and $T = 500$.

The tests using ℓ_0 and ℓ_4 perform very poorly when $\lambda = 0.0001$. Since $\lambda = 0.0001$ represents near stationarity, it is not surprising that the tests overreject substantially. When $\lambda = 0.01$ the tests still perform poorly, but there is some evidence of improvement as T increases. The tests using ℓ_0 and ℓ_4 perform reasonably well under the null for $\lambda \geq 1$ and $T \geq 100$.

The $\tilde{\eta}_\tau$ test does not do as well as the $\tilde{\eta}_\mu$ test. This is unfortunate, since most economic time series appear to contain deterministic trends, and thus the $\tilde{\eta}_\tau$ test is the one needed in practice.

2) <u>AR(1) Errors</u>

We next consider the size of the test in the presence of AR(1) errors. Table 3-2 presents our simulation results. It consists of four pages, corresponding to the four values of λ that we consider.

For $\lambda = 10{,}000$, the sizes of the tests are almost independent of the AR(1) coefficient ρ, and therefore almost identical to the results for $\rho = 0$, as presented previously in Table 3-1. This is so because, with $\lambda = 10{,}000$, the stationary error is negligible.

When $\lambda = 1$, the tests using ℓ_0 and ℓ_4 show similar size performance for posit

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positive ρ, especially for $T \geq 100$. For example, when $\rho = 0.8$ and $T = 100$, the sizes of $\tilde{\eta}_\mu(\ell_0)$ and $\tilde{\eta}_\tau(\ell_0)$ are .058, and the sizes of $\tilde{\eta}_\mu(\ell_4)$ and $\tilde{\eta}_\tau(\ell_4)$ are .056 and .047. However, for negative ρ, the tests using ℓ_0 show small size distortions when T is small, but the sizes of the tests using ℓ_4 are relatively correct in most cases. The tests using ℓ_{12} have low size and in fact, size is zero for $\tilde{\eta}_\tau(\ell_{12})$ when $T < 200$.

For $\lambda = 0.01$, size distortion occurs for the tests using ℓ_0 and ℓ_4, but this is less severe as $\rho \to 1$; e.g., when $T = 100$ and $\rho = 0.8$, sizes are .371 for $\tilde{\eta}_\mu(\ell_0)$ and .345 for $\tilde{\eta}_\tau(\ell_0)$, which are far less than the sizes of the tests using ℓ_0 with iid errors. The improvement of the tests as $\rho \to 1$ is expected, since the stationary error approaches a random walk as $\rho \to 1$. The tests using ℓ_4 show mixed and intermediate behavior. Comparing with the results for iid errors, for positive ρ there is a decrease in size distortion for small T but an increase in size distortion for large T, while for negative ρ we find a slight improvement of size performance. As in other cases, the tests using ℓ_{12} do not perform well.

For $\lambda = 0.0001$, the sizes of the tests using ℓ_0 are almost one unless T is small and $\rho \to 1$. This is reasonable because, as $\lambda \to 0$, the series becomes stationary and the test should reject the unit root null.

Comparing the above results with the results for iid errors, we find that size distortions are generally less severe as $\rho \to 1$, as expected. However, this pattern is more clear for tests using ℓ_0 than for tests using ℓ_4 or ℓ_{12}.

3) <u>MA(1) Errors</u>

We now consider the size performance of the test in the presence of MA(1) errors, the results of which are given in Table 3-3.

For $\lambda = 10{,}000$, the results are essentially the same as the results for iid and AR(1) errors. As $\lambda \to \infty$, the size performance of tests does not depend upon the autocorrelation of the stationary errors.

For $\lambda = 1$, the tests using ℓ_4 perform somewhat better than the tests using ℓ_0 and much better than the tests using ℓ_{12}. A similar pattern occurs for $\lambda = 0.01$, except that the size distortions are larger for the tests using ℓ_0 and ℓ_4 (especially ℓ_0). The tests using ℓ_{12} perform well if T is large enough ($T \geq 200$, say).

For $\lambda = 0.0001$, the sizes of the tests using ℓ_0 approach one as T increases, and the sizes of the tests using ℓ_4 are also quite large. Generally, the size distortion of the tests using ℓ_0 is slightly more severe as $\alpha \to -1$. The tests using ℓ_4 show less size distortion for small T, but more size distortion for large T, when α is positive compared to when α is negative. The tests using ℓ_{12} also show size distortions for large T, but their sizes are nearly zero in small samples, especially for the $\tilde{\eta}_\tau$ test.

To sum up, the main determinant of the size performance of the test in finite samples is the relative variance ratio λ. For small values of λ, the tests are not expected to be very exact in finite samples, and therefore the use of longer lags is needed to avoid severe size distortion. However, when λ is large so that random walk components dominate stationary components, the sizes of the tests using ℓ_0 and ℓ_4 are relatively correct, but the sizes of the tests using ℓ_{12} are too low unless T is very large. Therefore, the use of longer lags (e.g., ℓ_{12}) is not necessary or desirable in this case. Comparing the results for MA(1) errors with the results for AR(1) errors, the size distortions are more severe in the MA(1) case, especially when λ is small and the stationary errors are positively autocorrelated.

Generally, the location of the null (the value of λ) is important for the accuracy of inference since our null is simply $\lambda > 0$. As our simulation shows, the test distribution is close to the asymptotic null distribution in finite samples only when λ is sufficiently large.

3.4.2 Power

1) <u>iid Errors</u>

Results for the power performance of the tests in the presence of iid errors are given in Table 3-4. The power of the tests using ℓ_0 is almost 1 for both $\tilde{\eta}_\mu$ and $\tilde{\eta}_\tau$ when T > 50. The power of the tests using ℓ_4 is close to one in large samples, but it is less than that of the tests using ℓ_0 in finite samples. The power of the tests using ℓ_{12} is very small unless T is very large, and, in particular, power is almost zero for $\tilde{\eta}_\mu(\ell_{12})$ when $T \leq 50$ and for $\tilde{\eta}_\tau$ when $T < 200$.

2) <u>AR(1) Errors</u>

We now consider power in the presence of AR(1) errors. These results are given in Table 3-5.

For positive ρ, the tests using ℓ_0, ℓ_4, and ℓ_{12} all suffer from low power. This is as expected, because even under the alternative of stationarity ($\sigma_u^2 = 0$), y_t approaches an I(1) process as $\rho \to 1$. The tests using ℓ_{12} show the poorest power performance. The tests using ℓ_0 show relatively good power performance when ρ is away from 1. For example, when $\rho = 0.5$ and $T = 100$, powers are .904 and .908 for $\tilde{\eta}_\mu(\ell_0)$ and $\tilde{\eta}_\tau(\ell_0)$. The tests using ℓ_4 show reasonable power unless ρ is close to 1, e.g., when $T = 100$ and $\rho = 0.5$, powers are .700 for $\tilde{\eta}_\mu(\ell_4)$ and .551 for $\tilde{\eta}_\tau(\ell_4)$.

For negative ρ, the powers of the tests using ℓ_0 are almost one in most cases, while the tests using ℓ_4 show reasonable power unless T is too small. However, the tests using ℓ_{12} are not powerful unless T is large.

3) MA(1) Errors

We now consider the power performance of the test in the presence of MA(1) errors. These results are given in Table 3-6.

The tests lose power as the MA(1) parameter $\alpha \to 1$, but the power loss is not as large as it is in the AR(1) case. For example, when $T = 100$ and $\alpha = 0.8$, power is .966 for $\tilde{\eta}_\mu(\ell_0)$ and .976 for $\tilde{\eta}_\tau(\ell_0)$, both of which are far greater than the corresponding powers in the presence of AR(1) errors with $\rho = 0.8$. The powers of the tests using ℓ_4 are less than the powers of the tests using ℓ_0, but are reasonable. The tests using ℓ_{12} are the least powerful, and power is nearly zero for small T.

The tests are also more powerful for negative α than for positive α. The powers of the tests using ℓ_0 are almost one in most cases, and powers of using ℓ_4 are above .9 when $T > 50$. However, the tests using ℓ_{12} are not powerful in finite samples.

To sum up, the tests using ℓ_0 are most powerful unless the stationary errors follow an AR(1) process with ρ close to one. The tests using ℓ_4 have reasonable power in most cases, but are not powerful either when ρ is close to one. The use of longer lags (e.g., ℓ_{12}), especially for the $\tilde{\eta}_\tau$ test, leads to a large loss of power.

3.4.3 Comparison with the Dickey-Fuller Unit Root Tests: Size

Consider the components representation (1), and $\lambda = \sigma_u^2/\sigma_v^2$. The series is stationary if $\lambda = 0$, and any $\lambda > 0$ indicates the presence of a unit root. However, it is important to note that the value of σ_v^2 does not appear in the asymptotic distribution theory for the KPSS statistic under the null hypothesis of a unit root. That is, the I(1) component of y_t asymptotically dominates the I(0) component, and asymptotically the distribution is the same as if $\sigma_v^2 = 0$ (i.e., $\lambda = \infty$). Thus, under the unit root null, we should expect finite sample (but not asymptotic) size distortions when $\sigma_v^2 > 0$ (i.e., $\lambda < \infty$).

These size distortions can be related to the size distortions suffered by the Dickey-Fuller tests or other similar unit root tests when the errors in the Dickey-Fuller representation are autocorrelated. KPSS show that the model (1) is equivalent to the ARIMA model (Dickey-Fuller regression with MA(1) errors):

(24) $$y_t = d + \beta y_{t-1} + w_t,\ w_t = \varepsilon_t + \theta\varepsilon_{t-1},\ \beta = 1,\ \varepsilon_t \text{ iid.}$$

Indeed, the connection between θ and λ is straightforward (see Harvey (1989), p. 68):

(25) $$\theta = -\{(\lambda+2) - [\lambda(\lambda+4)]^{1/2}\}/2,\ \lambda = -(1+\theta)^2/\theta;\ \lambda \geq 0,\ |\theta| \leq 1$$

Thus, for a given λ in (1), it is reasonable to compare the size distortions of the KPSS unit root tests to the size distortions of the Dickey-Fuller tests in the presence of MA(1) errors, with parameter θ given by (25) so as to correspond to the given value of λ.

Table 3-7 gives some results on the sizes of the KPSS unit root test, the Dickey-Fuller $\hat{\tau}_\mu$ and $\hat{\tau}_\tau$ tests, and the augmented Dickey-Fuller tests. Other tests that are similar to the $\hat{\tau}_\tau$ test, such as the Dickey-Fuller $\hat{\rho}_\tau$ test or the Schmidt and Phillips test, give similar results. We choose $\lambda = \infty$ ($\theta = 0$), $\lambda = 0.5$ ($\theta = -0.5$), and $\lambda = 0.05$ ($\theta = -0.8$); and we consider T = 25, 50, 100, 250 and 500. The results are based on a simulation using 20,000 replications when $T \leq 100$, and 10,000 replications when $T > 100$.

We analyze first the tests that allow for level but not trend. These are the $\tilde{\eta}_\mu$ and $\hat{t}_\mu$ tests, and their augmented versions.

For $\theta = 0$ ($\lambda = \infty$) most tests have relatively correct size. However, the size of $\tilde{\eta}_\mu(\ell_{12})$ test is almost zero except when T is large. As $\theta \to -1$ ($\lambda \to 0$), positive size distortions occur (except for the $\tilde{\eta}_\mu(\ell_{12})$ test) and $\hat{t}_\mu(\ell_0)$ shows the worst size distortions. The size of the $\hat{t}_\mu(\ell_4)$ test is also quite considerably distorted, but the size of $\tilde{\eta}_\mu(\ell_4)$ is relatively correct. For example, when T = 100 and $\theta = -0.8$, the sizes of $\hat{t}_\mu(\ell_0)$ and $\hat{t}_\mu(\ell_4)$ are .997 and .434, but the sizes of $\tilde{\eta}_\mu(\ell_0)$ and $\tilde{\eta}_\mu(\ell_4)$ are .311 and .118. Finally, the sizes of $\hat{t}_\mu(\ell_{12})$ are relatively correct but the sizes of $\tilde{\eta}_\mu(\ell_{12})$ are too low.

The results for the tests that allow for trend are similar. When $\theta = 0$, most test statistics have relatively correct size, but the sizes of $\tilde{\eta}_\tau(\ell_4)$ and $\tilde{\eta}_\tau(\ell_{12})$ are low in small samples. The problem is much worse for $\tilde{\eta}_\tau(\ell_{12})$ than for $\tilde{\eta}_\tau(\ell_4)$. Again $\hat{t}_\tau(\ell_0)$

shows the worst size distortions as $\theta \rightarrow -1$. When $\theta = -0.8$ and T = 100, for example, the sizes of $\hat{t}_\tau(\ell_0)$ and $\hat{t}_\tau(\ell_4)$ are 1 and .568, but the sizes of $\tilde{\eta}_\tau(\ell_0)$ and $\tilde{\eta}_\tau(\ell_4)$ are .6 and .141. On the other hand, $\hat{t}_\tau(\ell_{12})$ has relatively correct size but the size of $\tilde{\eta}_\tau(\ell_{12})$ is nearly zero unless T is large.

Generally, $\tilde{\eta}_\mu(\ell_0)$, $\tilde{\eta}_\tau(\ell_0)$, $\hat{t}_\mu(\ell_0)$, $\hat{t}_\tau(\ell_0)$, $\hat{t}_\mu(\ell_4)$, and $\hat{t}_\tau(\ell_4)$ are not reliable because of poor size, when θ is close to minus one. This result is consistent with previous empirical findings that the Dickey-Fuller tests show considerable size distortions when the errors are MA(1) with negative MA(1) parameter. See Schwert (1987, 1989) and Lee and Schmidt (1991). The situation where θ is close to minus one is called the "nearly stationary case". The asymptotic distribution of the Dickey-Fuller tests when the process is 'nearly stationary' is derived in Chapter 4, and shows possible sources for the size distortions in this situation. As will be seen in Chapter 4, the above results are consistent with the predictions of the asymptotic theory.

One thing to note is that for $\lambda = .5$ and .05, the size distortions for $\tilde{\eta}_\tau(\ell_0)$ are much smaller than those of the Dickey-Fuller $\hat{\tau}_\tau$ test. The size distortions for $\tilde{\eta}_\tau(\ell_0)$ are also smaller than those of the $\hat{\tau}_\tau(\ell_4)$ test for $T \geq 250$, but not for $T \leq 100$. Asymptotic theory appears to be relevant for smaller values of T for the $\tilde{\eta}_\tau(\ell_0)$ test than for the $\hat{t}_\tau$ test or its augmented versions.

3.4.4 Comparison with the Dickey-Fuller Unit Root Tests: Power

We now turn to the power comparison of the test. Here the null hypothesis of a unit root is false, so that $\lambda = 0$ in (1) and $\beta < 1$ in the Dickey-Fuller model (24). In order to match the data generating process of the KPSS model (1) with that of the Dickey-Fuller model, for a given value $\beta < 1$, we assume that the errors in the Dickey-Fuller regression are iid (so $\theta = 0$ in (24) above), and let the stationary error in (1) be AR(1) with parameter β: $v_t = \beta v_{t-1} + \varepsilon_t$, with ε_t iid. Thus the data generating process implied by both parameterizations is the same; deviations from level and/or linear trend are AR(1) with parameter β. Table 3-8 presents results for the same tests as in Table 3-7, for β = 0, .2, .5, .8, .9, .95 and .99.

We consider first the case of level but no trend. When $\beta = 0$, the power of the $\tilde{\eta}_\mu(\ell_0)$ and $\hat{t}_\mu(\ell_0)$ tests is close to one for most cases. Generally, the KPSS unit root test is less powerful than the Dickey-Fuller test. The difference in power is sometimes substantial. For example, for T = 100 and $\beta = 0.8$, arguably an empirically relevant set of parameter values, the power of $\hat{\tau}_\mu(\ell_0)$ is .875 while the power of $\tilde{\eta}_\mu(\ell_0)$ is .546. The power of $\tilde{\eta}_\mu(\ell_0)$ is roughly comparable to that of $\hat{\tau}_\mu(\ell_4)$. It is generally the case that the KPSS unit root test is slightly more powerful than the augmented $\hat{\tau}_\mu$ test for T $\leq$ 50 and slightly less powerful for T $\geq$ 100. The use of longer lags generally loses power unless T is large, e.g., when T = 100, the powers of $\tilde{\eta}_\mu(\ell_{12})$ and $\hat{t}_\mu(\ell_{12})$ are only .150 and .448.

We now examine the case that allows for liner trend. When $\beta = 0$, the powers of $\tilde{\eta}_\tau(\ell_0)$ and $\hat{t}_\tau(\ell_0)$ are almost one unless T is small. The comparison of the KPSS unit root test to the Dickey-Fuller $\hat{\tau}_\tau$ test is easy to summarize. Again the $\tilde{\eta}_\tau$ test is less powerful than the $\hat{\tau}_\tau$ test. The difference in power is also substantial. For example, for T = 100 and $\beta = .8$, the power of $\hat{\tau}_\tau(\ell_0)$ is .65 while the power of $\tilde{\eta}_\tau(\ell_0)$

is .41. The power of $\tilde{\eta}_\tau(\ell_0)$ is roughly comparable to that of $\hat{\tau}_\tau(\ell_4)$. It is generally the case that $\tilde{\eta}_\tau(\ell_0)$ is slightly more powerful than $\hat{\tau}_\tau(\ell_4)$ for $T \leq 100$ and slightly less powerful for $T \geq 250$.

Interestingly, comparing these results to the results above for size distortions in the presence of autocorrelated errors in the Dickey-Fuller representation, we find support for the general supposition that the tests that are less susceptible to positive size distortions are also less powerful. It is noted that the use of longer lags loses a lot of power, and this is more severe for the KPSS unit root test than for the Dickey-Fuller tests. For example, $\tilde{\eta}_\tau(\ell_{12})$ has no power at all when $T \leq 100$.

3.5 Applications to the Nelson-Plosser Data

In this section we apply our unit root tests, $\tilde{\eta}_\mu$ and $\tilde{\eta}_\tau$, to the data analyzed by Nelson and Plosser (1982). They find that the unit root hypothesis is rejected at the 5% level for only the unemployment rate series, and it is rejected at about the 10% level for the industrial production series. These results are typically interpreted as indicating the presence of a unit root in most of the Nelson-Plosser series.

In Table 3-9 we first present the results for the $\tilde{\eta}_\mu$ test which we use to test the null hypothesis of a unit root with level. We consider values of the lag truncation parameter ℓ from 0 to 8. The values of the test statistics are sensitive to the choice of ℓ, and in fact for every series the value of the test statistic first decreases and then increases as ℓ increases. This is different from the results for the stationarity test, beca

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because in this case test statistic depends on ℓ in a different way. Nevertheless, the test outcome is not in very much doubt: for all series except the unemployment rate, and possibly the nominal wage and the interest rate, we cannot reject the hypothesis of a unit root with level. Because the Nelson-Plosser series contain obvious deterministic trends, and because the $\tilde{\eta}_\mu$ test does not allow for deterministic trend, these results may not be reliable. It is well-known that the DF $\hat{\rho}_\mu$ and $\hat{t}_\mu$ tests are inconsistent against trend stationary alternatives. The $\tilde{\eta}_\mu$ test also suffers from this inconsistency problem. Therefore, for data with linear deterministic trend, the $\tilde{\eta}_\tau$ test should be used instead.

We therefore proceed to test the null hypothesis of a unit root with level and trend, for which $\tilde{\eta}_\tau$ is the appropriate statistic. Once again the test statistics first decline and then increase as ℓ increases. In this case the choice of ℓ is also important to the conclusions. If we do not correct for residual autocorrelation at all, which corresponds to picking $\ell = 0$, we would not reject the null hypothesis of the unit root for any series except for the unemployment rate. Also, if we choose $\ell \geq 4$, then we would not reject the null hypothesis of the unit root in any case, which is very consistent with our simulation findings that the tests using longer lags are not powerful. However, if we choose $\ell = 1$, we find that we can reject the null of a unit root at the 5% level for three series: the unemployment rate, GNP deflator, and money. We cannot reject the null of a unit root at the 5% level for the remaining series, but we can reject a unit root at 10% level for the industrial production series.

We can compare these results to the results from the augmented Dickey-Fuller $\hat{t}_\tau$ test, assuming an AR(p) model with p = 1 to 9. (That is, p is the number of augmentations of the regression leading to the test statistic). Table 3-10 shows that the augmented Dickey-Fuller t-test cannot reject the unit root hypothesis at the 5% level in almost all cases. The rare exceptions are the unemployment rate for the AR(2) and AR(4) specifications and the money stock for the AR(8) specification. These findings are quite consistent with those of Nelson and Plosser (1982), Rudebush (1990), and others. Using the bootstrapping method, Rudebush derives the p-value (the marginal significance level) for rejection of the unit root null hypothesis and finds that, among all of the variables, there is only one for which the unit root hypothesis can be rejected at the 5% level: the unemployment rate.

Combining the results of our unit root tests with the results of the KPSS stationarity tests, the following picture emerges. Three series (unemployment rate, GNP deflator, and money) appear to be trend stationary, since we can reject the unit root hypothesis and cannot reject the trend stationarity hypothesis. Five series (consumer prices, real wages, velocity, and stock prices and possibly industrial production) appear to have unit roots, since we can reject the trend stationarity hypothesis and cannot reject the unit root hypothesis. Three more series (real GNP, nominal GNP, and the interest rate) probably have unit roots, though the evidence against the trend stationarity hypothesis is only marginally significant. Employment and real per capita GNP are probably trend stationary, which is consistent with Cochrane (1988), though the evidence against the unit root is only marginally significant. For the nominal wage we cannot reject either the unit root or the trend stationarity hypothesis, and the appropriate conclusion is presumably just that the data are not sufficiently informative.

There are two interesting cases: industrial production and real GNP. For industrial production we can reject the trend stationarity hypothesis at the 5% level and the unit root hypothesis at the 10% level, while for real GNP we can reject the trend stationarity hypothesis at the 10% level and the unit root hypothesis at about the 20% level. It seems that the data are not sufficiently informative to distinguish clearly between these hypotheses. These results may be indicative of "near stationarity" of the series. These results are also consistent with the Clark's (1987) finding that the data allocate a substantial fraction of the short run variation in real GNP and industrial production to a persistent business cycle, with less variation allocated to a stochastic trend that evolves smoothly over time. However, the results could also indicate the necessity to consider other, different models, such as fractional integration.

Finally, our result indicates that stock prices seem to behave as a pure random walk process. This is contrary to the weak evidence of a slowly mean reverting characteristics of stock prices over a long horizon (probably, 3 - 5 years) suggested in the finance literature. See Fama and French (1988) and Poterba and Summers (1987). One possible reason is that we deal with yearly data only and further research is needed for more general analysis.

3.6 Concluding Remarks

The KPSS stationary test is based on a components model in which an economic time series is expressed as the sum of a deterministic trend, a random walk, and a stationary error. We have considered the use of this statistic to test the null hypothesis of a unit root. We have derived its asymptotic distribution under the unit root null, considered its finite sample performance by a Monte Carlo simulation, and applied it to the Nelson-Plosser data series.

The asymptotic distribution of the test has been shown to be free of nuisance parameters and is the same as the distribution for the pure random walk process. In finite samples, the main determinant of the size performance of the test is the relative variance ratio λ rather than the autocorrelation of the stationary errors. Therefore, since our null is simply $\lambda > 0$, the location of the null is important for the quality of inference in finite samples. Simulation results show that the distribution of the test statistics in finite samples is close to the asymptotic null distribution when λ and T are sufficiently large. In this case the use of shorter lags (ℓ) is preferred. On the other hand, when λ is small, the test is not expected to be exact. In this case, the use of longer lags is needed to avoid size distortions. Comparing the results for AR(1) errors with those for MA(1) errors, the size distortion is more severe in the MA(1) case, especially when λ is small and the errors are positively autocorrelated. This is not surprising because the process with AR(1) errors approaches an I(1) process as $\rho \rightarrow 1$.

Our results on power performance can be summarized as follows: the test using $\ell = 0$ is more powerful than the test using lags ($\ell > 0$) in most cases. Especially, $\tilde{\eta}_\tau(\ell_{12})$ does not have any power even against the white noise alternative ($\lambda = 0$ and stationary error is iid) in finite samples. The test using ℓ_4 has reasonable power in most cases. However, when the stationary errors follow an AR(1) process with AR(1) parameter close to one, even the tests using ℓ_0 and ℓ_4 are not powerful.

To sum up, the use of longer lags is preferred in terms of size performance in the "nearly stationary case" while the use of shorter lags is preferred in terms of power performance in the "nearly integrated case."

We have also compared the size and power performance of our test statistics with those of the Dickey-Fuller test statistics. Our results are not very encouraging for dual-use statistics. The KPSS statistic, designed for use as a test of stationarity, does not make a particularly powerful unit root test. In particular, its power is noticeably less than the power of the Dickey-Fuller $\hat{\tau}_\tau$ test (or other similar tests) against trend stationary alternatives. This is intuitively reasonable. We would expect a similar lack of power if standard unit root test statistics were used to test the null hypothesis of stationarity.

We have applied our unit root tests to the data analyzed by Nelson and Plosser (1982). Based on simulation results on their finite sample performance and based on the sample autocorrelations of the series in first differences, we choose the lag truncation parameter as $\ell = 1$ (or $\ell = 0$) for the $\tilde{\eta}_\tau$ test. Using the results for $\ell = 1$, we find that we can reject the null of a unit root at the 5% level for only three series: the unemployment rate, GNP deflator, and money.

Combining the above results with the results of the KPSS stationarity tests, the following picture emerges. Three series (unemployment rate, GNP deflator, and money) appear to be trend stationary. Five series (consumer prices, real wages, velocity, stock prices, and industrial production) appear to have unit roots. Three more series (real GNP, nominal GNP, and the interest rate) probably have unit roots, while two more series (employment and real per capita GNP) are probably trend stationary. For the nominal wage we have no clear conclusion.

Our results are in broad agreement with the results of Clark (1987), Cochrane (1988), Dejong et al. (1989), and Rudebush (1990), and with the Bayesian analyses of DeJong and Whiteman (1991) and Phillips (1991). It suggests that for many series the existence of a unit root is in doubt, despite the failure of the Dickey-Fuller tests (and other unit root tests) to reject the unit root hypothesis. Presumably other alternatives, such as fractional integration or stationarity around more general non-linear trend, could be considered.

Table 3-0

Critical Values for Unit Root Tests

	$\tilde{\eta}_\mu(0)$	$\tilde{\eta}_\tau(0)$
0.010	0.0053	0.0021
0.025	0.0074	0.0027
0.050	0.0099	0.0033
0.100	0.0141	0.0043
0.200	0.0213	0.0058
0.300	0.0300	0.0072
0.400	0.0405	0.0086
0.500	0.0514	0.0100
0.600	0.0615	0.0116
0.700	0.0708	0.0135
0.800	0.0793	0.0156
0.900	0.0872	0.0183
0.950	0.0915	0.0199
0.975	0.0940	0.0211
0.990	0.0959	0.0221

Table 3-1

Size with iid Errors

			$\tilde{\eta}_\mu$			$\tilde{\eta}_\tau$	
λ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
.0001	30	.854	.528	.000	.814	.126	.000
	50	.958	.610	.000	.967	.289	.000
	80	.992	.811	.054	.997	.754	.000
	100	.996	.787	.140	.999	.707	.000
	120	.998	.821	.222	.999	.789	.000
	200	1.00	.921	.461	1.00	.963	.161
	500	1.00	.904	.558	1.00	.994	.684
.01	30	.715	.406	.000	.768	.114	.000
	50	.734	.344	.000	.901	.209	.000
	80	.676	.374	.016	.937	.495	.000
	100	.647	.282	.030	.930	.372	.000
	120	.612	.274	.050	.921	.412	.000
	200	.509	.231	.059	.868	.416	.023
	500	.327	.141	.057	.653	.238	.053
1.0	30	.097	.066	.000	.131	.016	.000
	50	.083	.049	.000	.120	.019	.000
	80	.072	.065	.004	.104	.045	.000
	100	.067	.052	.007	.089	.034	.000
	200	.057	.063	.021	.068	.059	.004
	500	.050	.062	.037	.053	.063	.022
10000	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.013	.000
	80	.045	.053	.003	.042	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.046	.060	.022	.044	.053	.004
	500	.046	.062	.037	.044	.061	.022

Table 3-2 Size with AR(1) Errors (λ = 10,000)

ρ	T	$\tilde{\eta}_\mu$ ℓ_0	$\tilde{\eta}_\mu$ ℓ_4	$\tilde{\eta}_\mu$ ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	$\tilde{\eta}_\tau$ ℓ_4	$\tilde{\eta}_\tau$ ℓ_{12}
0.8	30	.045	.046	.000	.034	.008	.000
	50	.046	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.048	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.004
	500	.046	.060	.036	.042	.061	.022
0.5	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
0.2	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
−0.2	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.037	.043	.061	.021
−0.5	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
−0.8	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021

Table 3–2 (Continued) (λ = 1)

ρ	T	$\tilde{\eta}_\mu$ ℓ_0	$\tilde{\eta}_\mu$ ℓ_4	$\tilde{\eta}_\mu$ ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	$\tilde{\eta}_\tau$ ℓ_4	$\tilde{\eta}_\tau$ ℓ_{12}
0.8	30	.060	.060	.000	.042	.107	.000
	50	.059	.049	.000	.054	.016	.000
	80	.060	.069	.003	.058	.049	.000
	100	.058	.056	.007	.058	.037	.000
	120	.059	.062	.014	.061	.047	.000
	200	.053	.067	.022	.058	.065	.005
	500	.048	.063	.037	.050	.066	.024
0.5	30	.078	.068	.000	.075	.015	.000
	50	.073	.054	.000	.086	.022	.000
	80	.067	.070	.004	.079	.055	.000
	100	.064	.056	.008	.077	.039	.000
	120	.065	.060	.013	.081	.046	.000
	200	.057	.066	.021	.065	.063	.005
	500	.049	.062	.036	.052	.064	.023
0.2	30	.092	.067	.000	.112	.016	.000
	50	.081	.052	.000	.111	.021	.000
	80	.072	.067	.004	.092	.051	.000
	100	.065	.054	.007	.086	.035	.000
	120	.065	.060	.013	.081	.045	.000
	200	.059	.065	.021	.068	.061	.004
	500	.049	.061	.037	.053	.063	.022
−0.2	30	.103	.061	.000	.146	.014	.000
	50	.086	.048	.000	.127	.017	.000
	80	.074	.064	.004	.101	.045	.000
	100	.068	.051	.007	.091	.033	.000
	120	.067	.058	.013	.085	.040	.000
	200	.059	.063	.021	.070	.059	.005
	500	.049	.060	.036	.053	.062	.022
−0.5	30	.107	.057	.000	.163	.012	.000
	50	.088	.045	.000	.135	.015	.000
	80	.075	.062	.003	.103	.042	.000
	100	.069	.049	.007	.093	.030	.000
	120	.067	.056	.013	.087	.039	.000
	200	.059	.062	.021	.071	.057	.005
	500	.049	.060	.036	.053	.061	.022
−0.8	30	.106	.056	.000	.170	.014	.000
	50	.087	.043	.000	.139	.012	.000
	80	.074	.060	.004	.103	.039	.000
	100	.069	.049	.007	.092	.031	.000
	120	.067	.055	.013	.087	.038	.000
	200	.059	.062	.021	.071	.055	.005
	500	.049	.060	.036	.053	.061	.022

Table 3-2 (Continued) ($\lambda = 0.01$)

ρ	T	$\tilde{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.162	.150	.000	.082	.018	.000
	50	.252	.198	.000	.153	.045	.000
	80	.342	.323	.027	.282	.211	.000
	100	.371	.304	.052	.352	.202	.000
	120	.387	.333	.095	.416	.290	.000
	200	.385	.355	.137	.537	.464	.049
	500	.290	.264	.139	.541	.475	.175
0.5	30	.413	.307	.000	.297	.060	.000
	50	.540	.341	.000	.540	.143	.000
	80	.568	.432	.028	.729	.452	.000
	100	.563	.363	.053	.765	.390	.000
	120	.546	.367	.086	.785	.468	.000
	200	.472	.336	.097	.784	.547	.044
	500	.312	.209	.089	.627	.389	.101
0.2	30	.616	.386	.000	.603	.103	.000
	50	.684	.360	.000	.810	.199	.000
	80	.651	.406	.020	.891	.506	.000
	100	.625	.321	.038	.893	.400	.000
	120	.594	.316	.062	.891	.450	.000
	200	.496	.270	.071	.845	.472	.028
	500	.317	.163	.065	.647	.289	.064
−0.2	30	.776	.406	.000	.882	.115	.000
	50	.772	.321	.000	.947	.208	.000
	80	.702	.335	.012	.957	.466	.000
	100	.662	.241	.023	.950	.334	.000
	120	.622	.234	.040	.937	.362	.000
	200	.512	.196	.049	.876	.351	.015
	500	.319	.117	.049	.659	.189	.042
−0.5	30	.841	.404	.000	.964	.139	.000
	50	.802	.269	.000	.979	.182	.000
	80	.717	.267	.007	.975	.393	.000
	100	.677	.188	.016	.964	.263	.000
	120	.632	.178	.026	.952	.282	.000
	200	.515	.152	.035	.886	.259	.010
	500	.321	.094	.043	.664	.135	.033
−0.8	30	.859	.415	.000	.986	.308	.000
	50	.807	.170	.000	.986	.061	.000
	80	.719	.170	.004	.978	.229	.000
	100	.676	.148	.011	.966	.217	.000
	120	.632	.137	.020	.953	.220	.000
	200	.513	.121	.027	.889	.187	.007
	500	.321	.075	.039	.663	.093	.026

Table 3–2 (Continued) (λ = 0.0001)

		$\tilde{\eta}_\mu$			$\tilde{\eta}_\tau$		
ρ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.178	.165	.000	.083	.018	.000
	50	.300	.238	.000	.161	.049	.000
	80	.458	.436	.038	.317	.234	.000
	100	.544	.455	.089	.410	.242	.000
	120	.612	.535	.177	.504	.358	.000
	200	.790	.743	.382	.764	.683	.096
	500	.952	.919	.710	.982	.958	.656
0.5	30	.471	.347	.000	.313	.064	.000
	50	.695	.464	.000	.601	.165	.000
	80	.846	.693	.056	.837	.563	.000
	100	.898	.690	.129	.907	.548	.000
	120	.929	.763	.250	.944	.682	.000
	200	.982	.896	.463	.992	.909	.148
	500	.997	.953	.687	1.00	.993	.730
0.2	30	.733	.470	.000	.638	.109	.000
	50	.895	.568	.000	.888	.252	.000
	80	.966	.778	.056	.981	.702	.000
	100	.984	.758	.138	.994	.663	.000
	120	.992	.820	.265	.996	.781	.000
	200	.998	.922	.473	1.00	.951	.158
	500	1.00	.932	.608	1.00	.994	.708
−0.2	30	.930	.577	.000	.925	.135	.000
	50	.987	.645	.000	.993	.320	.000
	80	.999	.832	.051	1.00	.795	.000
	100	1.00	.797	.143	1.00	.740	.000
	120	1.00	.844	.269	1.00	.838	.000
	200	1.00	.917	.440	1.00	.968	.158
	500	1.00	.856	.489	1.00	.991	.643
−0.5	30	.986	.674	.000	.990	.186	.000
	50	.999	.697	.000	1.00	.356	.000
	80	1.00	.862	.048	1.00	.851	.000
	100	1.00	.820	.141	1.00	.800	.000
	120	1.00	.853	.266	1.00	.881	.000
	200	1.00	.898	.390	1.00	.974	.155
	500	1.00	.755	.376	1.00	.979	.546
−0.8	30	.999	.845	.000	1.00	.503	.000
	50	1.00	.709	.000	1.00	.203	.000
	80	1.00	.854	.036	1.00	.864	.000
	100	1.00	.869	.141	1.00	.923	.000
	120	1.00	.877	.253	1.00	.956	.000
	200	1.00	.869	.303	1.00	.988	.149
	500	1.00	.578	.235	1.00	.917	.372

Table 3-3 Size with MA(1) Errors (λ = 10,000)

α	T	$\tilde{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.059	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
0.5	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.059	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
0.2	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.059	.003	.043	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
-0.2	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.042	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
-0.5	30	.045	.046	.000	.034	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.042	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021
-0.8	30	.045	.046	.000	.035	.008	.000
	50	.045	.041	.000	.041	.012	.000
	80	.049	.060	.003	.042	.037	.000
	100	.046	.049	.007	.043	.029	.000
	120	.049	.054	.013	.044	.036	.000
	200	.047	.062	.022	.044	.053	.005
	500	.046	.060	.036	.043	.061	.021

Table 3–3 (Continued) ($\lambda = 1$)

α	T	$\tilde{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.085	.072	.000	.088	.019	.000
	50	.078	.055	.000	.097	.023	.000
	80	.070	.069	.004	.086	.054	.000
	100	.065	.055	.008	.082	.038	.000
	120	.065	.061	.013	.079	.047	.000
	200	.057	.066	.022	.067	.063	.004
	500	.049	.062	.036	.053	.063	.023
0.5	30	.087	.070	.000	.097	.018	.000
	50	.079	.054	.000	.102	.022	.000
	80	.070	.068	.004	.087	.053	.000
	100	.065	.055	.008	.084	.038	.000
	120	.065	.061	.013	.080	.047	.000
	200	.058	.066	.022	.067	.062	.004
	500	.049	.062	.036	.053	.063	.023
0.2	30	.081	.051	.000	.115	.017	.000
	50	.083	.052	.000	.112	.021	.000
	80	.072	.066	.004	.093	.050	.000
	100	.065	.054	.007	.086	.036	.000
	120	.065	.059	.013	.082	.045	.000
	200	.059	.065	.021	.068	.061	.004
	500	.049	.060	.037	.053	.063	.022
−0.2	30	.103	.061	.000	.147	.014	.000
	50	.086	.047	.000	.128	.017	.000
	80	.074	.063	.004	.102	.045	.000
	100	.068	.051	.007	.092	.033	.000
	120	.067	.057	.013	.085	.040	.000
	200	.059	.063	.021	.070	.059	.004
	500	.049	.060	.036	.053	.062	.022
−0.5	30	.108	.057	.000	.166	.011	.000
	50	.089	.046	.000	.138	.016	.000
	80	.075	.062	.004	.104	.043	.000
	100	.070	.050	.007	.093	.032	.000
	120	.067	.056	.013	.088	.040	.000
	200	.059	.062	.021	.070	.057	.004
	500	.050	.060	.036	.053	.062	.022
−0.8	30	.109	.057	.000	.174	.010	.000
	50	.091	.044	.000	.143	.016	.000
	80	.075	.061	.003	.105	.042	.000
	100	.069	.050	.007	.094	.031	.000
	120	.067	.055	.013	.088	.038	.000
	200	.059	.062	.021	.070	.055	.004
	500	.050	.060	.036	.054	.062	.022

Table 3-3 (Continued) (λ = 0.01)

		$\tilde{\eta}_\mu$			$\tilde{\eta}_\tau$		
α	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.520	.391	.000	.416	.109	.000
	50	.625	.385	.000	.694	.207	.000
	80	.619	.442	.024	.836	.530	.000
	100	.603	.359	.046	.850	.432	.000
	120	.577	.353	.074	.855	.493	.000
	200	.487	.309	.083	.823	.529	.035
	500	.316	.185	.073	.641	.342	.077
0.5	30	.553	.393	.000	.477	.109	.000
	50	.646	.379	.000	.736	.207	.000
	80	.630	.431	.022	.856	.526	.000
	100	.610	.348	.043	.867	.424	.000
	120	.583	.341	.080	.870	.481	.000
	200	.491	.296	.079	.831	.514	.032
	500	.317	.178	.070	.643	.324	.072
0.2	30	.633	.398	.000	.630	.111	.000
	50	.694	.364	.000	.829	.207	.000
	80	.658	.405	.019	.901	.513	.000
	100	.630	.316	.037	.902	.400	.000
	120	.598	.310	.060	.898	.448	.000
	200	.498	.264	.069	.849	.464	.026
	500	.318	.159	.063	.648	.281	.062
−0.2	30	.784	.414	.000	.893	.120	.000
	50	.777	.323	.000	.953	.214	.000
	80	.705	.333	.012	.960	.470	.000
	100	.664	.237	.022	.953	.333	.000
	120	.624	.230	.038	.940	.358	.000
	200	.513	.192	.048	.878	.343	.015
	500	.319	.115	.048	.660	.183	.041
−0.5	30	.867	.433	.000	.983	.142	.000
	50	.820	.288	.000	.988	.227	.000
	80	.727	.270	.006	.981	.426	.000
	100	.682	.181	.013	.971	.265	.000
	120	.638	.169	.023	.958	.276	.000
	200	.519	.142	.031	.890	.244	.009
	500	.322	.088	.041	.668	.125	.030
−0.8	30	.902	.445	.000	.997	.175	.000
	50	.836	.263	.000	.995	.242	.000
	80	.738	.235	.004	.987	.402	.000
	100	.688	.150	.010	.976	.225	.000
	120	.643	.138	.019	.963	.226	.000
	200	.522	.120	.026	.896	.189	.006
	500	.324	.076	.038	.671	.100	.025

Table 3-3 (Continued) ($\lambda = 0.0001$)

α	T	$\tilde{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.606	.458	.000	.439	.113	.000
	50	.821	.560	.000	.773	.250	.000
	80	.928	.773	.057	.942	.691	.000
	100	.962	.756	.139	.974	.655	.000
	120	.975	.820	.267	.987	.774	.000
	200	.995	.927	.479	.999	.950	.160
	500	.999	.949	.647	1.00	.995	.727
0.5	30	.651	.465	.000	.505	.115	.000
	50	.848	.564	.000	.817	.253	.000
	80	.944	.777	.057	.959	.698	.000
	100	.972	.758	.140	.984	.660	000
	120	.983	.821	.267	.991	.779	.000
	200	.996	.926	.478	1.00	.951	.160
	500	.999	.946	.635	1.00	.995	.723
0.2	30	.757	.489	.000	.668	.118	.000
	50	.909	.583	.000	.907	.265	.000
	80	.973	.789	.056	.986	.719	.000
	100	.988	.768	.140	.996	.679	.000
	120	.994	.827	.267	.998	.792	.000
	200	.999	.926	.474	1.00	.956	.159
	500	1.00	.928	.596	1.00	.995	.706
−0.2	30	.940	.597	.000	.935	.143	.000
	50	.990	.661	.000	.995	.335	.000
	80	.999	.843	.051	1.00	.812	.000
	100	1.00	.807	.144	1.00	.755	.000
	120	1.00	.850	.271	1.00	.845	.000
	200	1.00	.919	.438	1.00	.971	.159
	500	1.00	.850	.479	1.00	.991	.637
−0.5	30	.997	.777	.000	.998	.209	.000
	50	1.00	.803	.000	1.00	.505	.000
	80	1.00	.922	.045	1.00	.941	.000
	100	1.00	.876	.152	1.00	.893	.000
	120	1.00	.894	.282	1.00	.944	.000
	200	1.00	.911	.377	1.00	.989	.167
	500	1.00	.726	.320	1.00	.979	.501
−0.8	30	1.00	.955	.000	1.00	.349	.000
	50	1.00	.963	.000	1.00	.817	.000
	80	1.00	.988	.036	1.00	.999	.000
	100	1.00	.945	.183	1.00	.994	.000
	120	1.00	.941	.316	1.00	.998	.000
	200	1.00	.898	.309	1.00	.999	.193
	500	1.00	.612	.186	1.00	.957	.322

Table 3–4

Power with iid Errors ($\lambda = 0$)

T	$\tilde{\eta}_\mu$ ℓ_0	ℓ_4	ℓ_{12}	$\tilde{\eta}_\tau$ ℓ_0	ℓ_4	ℓ_{12}
30	.854	.528	.000	.814	.126	.000
50	.960	.615	.000	.969	.291	.000
80	.994	.823	.057	.997	.757	.000
90	.996	.860	.140	.999	.825	.000
100	.998	.802	.150	1.00	.712	.000
120	.999	.862	.286	1.00	.822	.000
200	1.00	.961	.535	1.00	.971	.172
500	1.00	.998	.854	1.00	.999	.811

Table 3-5 Power with AR(1) Errors ($\lambda = 0$)

			$\tilde{\eta}_\mu$			$\tilde{\eta}_\tau$	
ρ	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.177	.165	.000	.083	.018	.000
	50	.301	.239	.000	.161	.049	.000
	80	.461	.436	.038	.317	.234	.000
	100	.546	.455	.089	.410	.244	.000
	120	.614	.538	.179	.507	.358	.000
	200	.799	.756	.391	.767	.687	.097
	500	.976	.951	.771	.985	.964	.682
0.5	30	.472	.349	.000	.312	.063	.000
	50	.697	.465	.000	.602	.166	.000
	80	.851	.698	.057	.837	.561	.000
	100	.904	.700	.134	.908	.551	.000
	120	.935	.773	.256	.946	.684	.000
	200	.988	.916	.494	.994	.919	.153
	500	1.00	.993	.835	1.00	.997	.779
0.2	30	.733	.473	.000	.639	.109	.000
	50	.898	.572	.000	.889	.254	.000
	80	.969	.789	.059	.981	.704	.000
	100	.987	.774	.146	.994	.668	.000
	120	.994	.837	.276	.997	.785	.000
	200	.999	.949	.521	1.00	.960	.168
	500	1.00	.997	.849	1.00	.999	.802
−0.2	30	.933	.580	.000	.924	.135	.000
	50	.989	.652	.000	.994	.322	.000
	80	.999	.851	.056	1.00	.801	.000
	100	1.00	.828	.154	1.00	.749	.000
	120	1.00	.883	.296	1.00	.852	.000
	200	1.00	.971	.546	1.00	.978	.178
	500	1.00	.999	.862	1.00	1.00	.822
−0.5	30	.987	.679	.000	.990	.187	.000
	50	.999	.713	.000	1.00	.361	.000
	80	1.00	.891	.054	1.00	.857	.000
	100	1.00	.870	.161	1.00	.819	.000
	120	1.00	.918	.318	1.00	.900	.000
	200	1.00	.984	.572	1.00	.990	.190
	500	1.00	1.00	.874	1.00	1.00	.843
−0.8	30	.999	.856	.000	1.00	.504	.000
	50	1.00	.753	.000	1.00	.704	.000
	80	1.00	.927	.043	1.00	.888	.000
	100	1.00	.956	.203	1.00	.952	.000
	120	1.00	.978	.391	1.00	.978	.000
	200	1.00	.998	.642	1.00	1.00	.240
	500	1.00	1.00	.908	1.00	1.00	.891

Table 3-6 Power with MA(1) Errors ($\lambda = 0$)

		$\tilde{\eta}_\mu$			$\tilde{\eta}_\tau$		
α	T	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
0.8	30	.606	.460	.000	.440	.114	.000
	50	.823	.564	.000	.775	.250	.000
	80	.933	.781	.059	.943	.693	.000
	100	.966	.768	.146	.976	.660	.000
	120	.981	.832	.275	.987	.778	.000
	200	.998	.946	.519	1.00	.956	.166
	500	1.00	.997	.848	1.00	.999	.799
0.5	30	.652	.468	.000	.505	.115	.000
	50	.852	.569	.000	.819	.253	.000
	80	.949	.786	.059	.960	.700	.000
	100	.975	.771	.147	.985	.665	.000
	120	.987	.835	.276	.992	.782	.000
	200	.999	.948	.520	1.00	.958	.167
	500	1.00	.997	.849	1.00	.999	.800
0.2	30	.757	.490	.000	.669	.119	.000
	50	.912	.587	.000	.908	.265	.000
	80	.976	.802	.058	.987	.723	.000
	100	.990	.783	.148	.996	.684	.000
	120	.996	.845	.279	.998	.798	.000
	200	1.00	.954	.526	1.00	.963	.168
	500	1.00	.998	.851	1.00	.999	.804
-0.2	30	.942	.599	.000	.936	.143	.000
	50	.991	.667	.000	.995	.338	.000
	80	1.00	.863	.055	1.00	.816	.000
	100	1.00	.838	.156	1.00	.765	.000
	120	1.00	.891	.301	1.00	.863	.000
	200	1.00	.976	.552	1.00	.982	.179
	500	1.00	.999	.865	1.00	1.00	.827
-0.5	30	.998	.783	.000	.998	.208	.000
	50	1.00	.825	.000	1.00	.513	.000
	80	1.00	.954	.052	1.00	.949	.000
	100	1.00	.935	.188	1.00	.914	.000
	120	1.00	.962	.371	1.00	.962	.000
	200	1.00	.996	.627	1.00	.998	.227
	500	1.00	1.00	.903	1.00	1.00	.888
-0.8	30	1.00	.964	.000	1.00	.351	.000
	50	1.00	.987	.000	1.00	.836	.000
	80	1.00	1.00	.063	1.00	1.00	.000
	100	1.00	1.00	.385	1.00	1.00	.000
	120	1.00	1.00	.688	1.00	1.00	.000
	200	1.00	1.00	.903	1.00	1.00	.528
	500	1.00	1.00	.996	1.00	1.00	.997

Table 3-7

Size Comparison of the KPSS Unit Root Tests to The Dickey-Fuller Tests Under MA(1) Errors

(a) Tests That Allow Level But Not Trend

			$\tilde{\eta}_\mu$			$\hat{\tau}_\mu$	
T	θ	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
25	−0.8	.463	.197	.000	.923	.522	.036
	−0.5	.151	.070	.000	.418	.143	.038
	0.0	.044	.036	.000	.050	.052	.039
50	−0.8	.418	.161	.000	.989	.471	.046
	−0.5	.114	.059	.000	.523	.082	.035
	0.0	.045	.041	.000	.051	.047	.036
100	−0.8	.311	.118	.012	.997	.434	.055
	−0.5	.088	.057	.007	.573	.069	.039
	0.0	.046	.048	.007	.053	.049	.043
250	−0.8	.193	.090	.036	.999	.371	.054
	−0.5	.067	.063	.028	.604	.058	.045
	0.0	.050	.060	.028	.049	.047	.044
500	−0.8	.122	.077	.039	.999	.403	.058
	−0.5	.053	.062	.036	.610	.057	.046
	0.0	.046	.060	.036	.053	.052	.046

(b) Tests That Allow Trend

			$\tilde{\eta}_\tau$			$\hat{\tau}_\tau$	
T	θ	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}
25	−0.8	.549	.203	.000	.900	.466	.033
	−0.5	.195	.011	.000	.514	.166	.034
	0.0	.032	.001	.000	.050	.052	.041
50	−0.8	.673	.185	.000	1.00	.518	.045
	−0.5	.186	.025	.000	.709	.099	.032
	0.0	.041	.012	.000	.052	.045	.034
100	−0.8	.600	.141	.000	1.00	.568	.055
	−0.5	.134	.040	.000	.794	.079	.039
	0.0	.043	.029	.000	.054	.044	.040
250	−0.8	.381	.111	.011	1.00	.551	.055
	−0.5	.082	.053	.006	.841	.064	.039
	0.0	.043	.048	.006	.051	.050	.040
500	−0.8	.237	.095	.028	1.00	.613	.057
	−0.5	.063	.064	.023	.853	.065	.046
	0.0	.043	.061	.021	.052	.049	.048

Table 3–8

Power Comparison of the KPSS Unit Root Tests To The Dickey–Fuller Tests

(a) Power Comparison With Level But No Trend

		$\tilde{\eta}_\mu$			$\hat{\tau}_\mu$			$\hat{\rho}_\mu$
T	β	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}	
25	.00	.798	.414	.000	.990	.794	.064	.989
	.20	.667	.364	.000	.937	.656	.068	.923
	.50	.409	.257	.000	.564	.380	.068	.510
	.80	.151	.115	.000	.145	.135	.075	.111
	.90	.093	.076	.000	.088	.091	.073	.054
	.95	.067	.055	.000	.075	.078	.073	.037
	.99	.048	.041	.000	.067	.069	.070	.026
50	.00	.960	.615	.000	1.00	.955	.135	1.00
	.20	.898	.572	.000	1.00	.906	.127	1.00
	.50	.697	.465	.000	.983	.687	.109	.991
	.80	.301	.239	.000	.348	.224	.077	.404
	.90	.162	.138	.000	.127	.105	.057	.147
	.95	.101	.089	.000	.079	.067	.051	.081
	.99	.056	.052	.000	.063	.060	.049	.044
100	.00	.998	.802	.150	1.00	1.00	.448	1.00
	.20	.987	.774	.146	1.00	.999	.425	1.00
	.50	.904	.700	.134	1.00	.979	.351	1.00
	.80	.546	.456	.089	.875	.605	.199	.940
	.90	.289	.265	.048	.320	.243	.113	.430
	.95	.151	.150	.015	.123	.108	.075	.170
	.99	.066	.068	.009	.064	.060	.050	.063
250	.00	1.00	.961	.617	1.00	1.00	.980	1.00
	.20	1.00	.951	.607	1.00	1.00	.972	1.00
	.50	.995	.924	.581	1.00	1.00	.949	1.00
	.80	.861	.791	.490	1.00	.999	.778	1.00
	.90	.609	.589	.352	.972	.852	.501	.994
	.95	.350	.366	.197	.452	.357	.215	.606
	.99	.095	.113	.054	.066	.061	.053	.096
500	.00	1.00	.998	.855	1.00	1.00	1.00	1.00
	.20	1.00	.997	.849	1.00	1.00	1.00	1.00
	.50	1.00	.993	.779	1.00	1.00	1.00	1.00
	.80	.976	.951	.771	1.00	1.00	.999	1.00
	.90	.849	.842	.656	1.00	1.00	.958	1.00
	.95	.604	.636	.473	.967	.911	.690	.993
	.99	.156	.194	.126	.109	.104	.087	.176

Table 3-8 (Continued)

(b) Power Comparison With Trend

		$\tilde{\eta}_\tau$			$\hat{\tau}_\tau$			$\hat{\rho}_\tau$
T	β	ℓ_0	ℓ_4	ℓ_{12}	ℓ_0	ℓ_4	ℓ_{12}	
25	.00	.713	.004	.000	.954	.600	.057	.956
	.20	.511	.005	.000	.810	.459	.063	.797
	.50	.221	.003	.000	.373	.248	.069	.343
	.80	.061	.001	.000	.113	.112	.077	.080
	.90	.038	.001	.000	.081	.085	.073	.055
	.95	.031	.001	.000	.073	.079	.075	.042
	.99	.030	.001	.000	.073	.075	.076	.040
50	.00	.969	.291	.000	1.00	.831	.072	1.00
	.20	.889	.254	.000	1.00	.721	.068	1.00
	.50	.602	.166	.000	.898	.447	.061	.924
	.80	.161	.049	.000	.215	.136	.052	.221
	.90	.075	.023	.000	.095	.075	.045	.092
	.95	.052	.015	.000	.069	.060	.046	.055
	.99	.042	.012	.000	.059	.052	.043	.045
100	.00	1.00	.712	.000	1.00	.996	.237	1.00
	.20	.994	.668	.000	1.00	.985	.221	1.00
	.50	.908	.551	.000	1.00	.896	.181	1.00
	.80	.410	.244	.000	.646	.377	.106	.727
	.90	.163	.103	.000	.192	.141	.068	.228
	.95	.081	.052	.000	.087	.079	.051	.092
	.99	.044	.030	.000	.051	.050	.042	.051
250	.00	1.00	.972	.338	1.00	1.00	.873	1.00
	.20	1.00	.962	.326	1.00	1.00	.855	1.00
	.50	.998	.931	.299	1.00	1.00	.790	1.00
	.80	.855	.739	.206	1.00	.980	.516	1.00
	.90	.497	.445	.101	.826	.607	.267	.900
	.95	.208	.205	.083	.257	.206	.121	.341
	.99	.052	.055	.037	.057	.053	.043	.058
500	.00	1.00	.999	.812	1.00	1.00	1.00	1.00
	.20	1.00	.999	.802	1.00	1.00	1.00	1.00
	.50	1.00	.997	.779	1.00	1.00	.999	1.00
	.80	.985	.964	.682	1.00	1.00	.980	1.00
	.90	.838	.825	.498	1.00	.997	.810	1.00
	.95	.490	.528	.264	.813	.701	.424	.885
	.99	.080	.110	.039	.077	.070	.058	.091

Table 3-9

The KPSS Unit Root Tests Applied To the Nelson-Plosser Data

The $\tilde{\eta}_\mu$ Test for a Unit Root without Trend

(5% Critical Value is .0099)

Series	Lag Truncation Parameter (ℓ)								
	0	1	2	3	4	5	6	7	8
Real GNP	.0961	.0493	.0671	.0771	.0839	.0892	.0936	.0975	.1011
Nominal GNP	.0937	.0481	.0657	.0755	.0823	.0876	.0921	.0961	.0998
PCR GNP	.0893	.0459	.0627	.0723	.0791	.0844	.0888	.0928	.0965
IP	.0972	.0493	.0666	.0759	.0819	.0863	.0898	.0927	.0953
Employment	.0935	.0478	.0649	.0744	.0807	.0856	.0897	.0933	.0965
Unemployment	.0039	.0022	.0034	.0042	.0050	.0058	.0067	.0076	.0085
GNP deflator	.0916	.0466	.0631	.0720	.0779	.0824	.0860	.0892	.0921
CPI	.0712	.0363	.0492	.0562	.0609	.0644	.0672	.0696	.0717
Nominal Wage	.0086	.0045	.0062	.0073	.0082	.0090	.0098	.0105	.0114
Real Wage	.0980	.0500	.0677	.0773	.0837	.0885	.0924	.0958	.0989
Money Stock	.0976	.0497	.0673	.0768	.0831	.0878	.0917	.0949	.0979
Velocity	.0824	.0421	.0570	.0651	.0705	.0744	.0775	.0801	.0824
Bond	.0110	.0060	.0085	.0101	.0113	.0123	.0133	.0141	.0149
SP500	.0801	.0410	.0558	.0640	.0697	.0740	.0775	.0806	.0835

The $\tilde{\eta}_\tau$ Test for a Unit Root with Trend

(5% Critical Value is .0033)

Series	Lag Truncation Parameter (ℓ)								
	0	1	2	3	4	5	6	7	8
Real GNP	.0102	.0054	.0078	.0096	.0112	.0127	.0143	.0159	.0177
Nominal GNP	.0122	.0063	.0088	.0104	.0117	.0128	.0139	.0149	.0160
PCR GNP	.0085	.0046	.0066	.0081	.0095	.0108	.0122	.0137	.0152
IP	.0074	.0040	.0058	.0069	.0079	.0088	.0097	.0104	.0112
Employment	.0065	.0034	.0049	.0059	.0067	.0075	.0083	.0091	.0100
Unemployment	.0027	.0015	.0023	.0029	.0035	.0041	.0047	.0053	.0060
GNP Deflator	.0060	.0031	.0043	.0051	.0057	.0063	.0068	.0073	.0079
CPI	.0167	.0085	.0115	.0133	.0145	.0154	.0162	.0170	.0177
Nominal Wage	.0946	.0484	.0657	.0752	.0816	.0864	.0904	.0940	.0972
Real Wage	.0135	.0072	.0103	.0124	.0142	.0159	.0176	.0192	.0208
Money Stock	.0054	.0028	.0038	.0045	.0051	.0056	.0061	.0067	.0073
Velocity	.0174	.0091	.0127	.0148	.0164	.0177	.0187	.0197	.0206
Bond	.0119	.0064	.0091	.0108	.0120	.0131	.0140	.0149	.0157
SP500	.0123	.0065	.0091	.0108	.0121	.0132	.0142	.0151	.0159

Table 3-10

Augmented Dickey-Fuller Unit Root ($\hat{\tau}_\tau$) Tests Applied to

the Nelson-Plosser Data

(5% critical value is -3.45)

Number of the AR Order (P)

Series	0	1	2	3	4	5	6	7	8
Real GNP	-2.03	-2.99	-2.94	-2.69	-2.43	-2.12	-2.38	-2.68	-2.23
Nominal GNP	-1.35	-2.32	-2.04	-1.83	-1.53	-1.79	-2.20	-2.17	-2.26
PCR GNP	-2.12	-3.05	-3.00	-2.80	-2.56	-2.22	-2.48	-2.84	-2.39
IP	-3.08	-3.36	-3.19	-3.27	-3.08	-2.53	-2.49	-2.67	-2.68
Employment	-2.17	-3.13	-2.66	-3.23	-3.23	-2.57	-3.36	-3.63	-3.60
Unemployment	-3.36	-3.92	-3.14	-3.55	-3.09	-2.84	-2.98	-3.33	-2.97
GNP Deflator	-1.83	-2.52	-2.57	-2.45	-2.39	-2.47	-2.52	-2.44	-2.65
CPI	-0.65	-1.86	-1.44	-1.97	-2.75	-2.37	-2.28	-2.38	-2.41
Nominal Wage	-1.46	-2.52	-2.24	-2.24	-2.07	-2.12	-2.62	-2.92	-2.64
Real wage	-2.33	-3.05	-2.97	-2.80	-2.54	-2.56	-2.26	-2.33	-1.93
Money Stock	-1.44	-3.08	-2.79	-2.93	-2.91	-3.00	-3.40	-3.68	-3.46
Velocity	-1.66	-1.75	-1.47	-1.40	-1.08	-0.74	-0.79	-0.90	-1.08
Bond	1.86	1.46	0.69	0.49	0.66	0.60	0.55	0.85	0.76
SP500	-1.94	-2.65	-2.12	-2.12	-1.60	-1.06	-0.97	-1.06	-1.02

CHAPTER 4

CHAPTER 4

ASYMPTOTIC DISTRIBUTION OF UNIT ROOT TESTS WHEN THE PROCESS IS NEARLY STATIONARY

4.1 Introduction

The unit root hypothesis has recently attracted a lot of attention in time series econometrics. Dickey and Fuller (1979, 1981, DF) develop several tests of the unit root hypothesis. They use a Monte Carlo simulation to tabulate the sampling distributions of the coefficient and t statistics, assuming iid errors. These distributions are nonstandard; they are skewed to the left and have too many large negative values relative to a normal distribution. Furthermore, it is well-known that many economic time series are mixed processes, in the sense that they contain not only autoregressive but also moving average components. In this case, the DF tests are not expected to be robust because the distribution of test statistics is derived under the assumption of the errors being white noise.

There have been some attempts to derive alternative testing procedures to correct for the presence of autocorrelated errors. Said and Dickey (1984, 1985) have suggested the use of augmented Dickey-Fuller tests (ADF), based on the DF regression augmented with lagged differences of the dependent variable, which should have the correct size asymptotically even in the presence of autocorrelated errors, if the number of augmentations increases with the sample size at an appropriate rate. Phillips (1987) and Phillips and Perron (1988, PP) allow for a wide class of weakly dependent and heterogeneous errors and use semiparametric corrections to derive transformed statistics which have the same limiting distributions that the DF statistics have under iid errors.

However, the finite sample performance of most corrected unit root test statistics is not robust when the data follow an ARIMA process. Schwert (1987, 1989) shows by extensive Monte Carlo simulations that most unit root tests have a problem of size distortions when the errors are MA(1) with negative parameter. For example, when the MA(1) parameter is -0.8, the size distortion of standard unit root tests such as the DF and the PP tests is almost one.

This problem can be examined theoretically by using the "nearly stationary model":

$$y_t = \beta y_{t-1} + \varepsilon_t, \quad \beta = 1, \quad \varepsilon_t = u_t + \theta u_{t-1}$$

$$\theta = -1 + C/T^{\delta}$$

where u_t is iid, and $C > 0$ and $\delta > 0$ are fixed numbers. Note that, even under the maintained hypothesis of a unit root ($\beta = 1$), y_t is white noise when $\theta = -1$, and so y_t approaches white noise as $T \to \infty$. By using this model we will investigate the asymptotic behavior of uncorrected unit root tests when the process is nearly stationary. It will be shown that the order in probability and the asymptotic distributions of standard unit root tests in this case may depend on the value of δ as well a

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well as C. This approach is similar to the approach of Phillips(1988) and Chan and Wei(1988) who consider the "nearly integrated process" in which the AR parameter β approaches unity as $T \rightarrow \infty$, and who therefore investigate power against a sequence of alternatives getting close to the null of a unit root.

Our model is an extension of Pantula (1991), who considers the same model, but sets $C = 1$. He considers corrected versions of the unit root tests such as the ADF and the PP tests, and he is primarily interested in finding and comparing the order in probability of those statistics. However, our concern is different in the sense that we study uncorrected unit root tests, and we are primarily interested in the finite sample adequacy of the asymptotic results. The asymptotic results must be accurate in finite samples if they are to explain the reason for the size distortions of unit root tests when the process is nearly stationary.

The main purposes of this chapter are to derive the asymptotic distribution of the Dickey-Fuller and the Schmidt-Phillips (1991, SP) unit root test statistics using the local approximation of the MA(1) parameter to minus one, to tabulate the distributions of the unit root test statistics predicted by our asymptotic theory, and then to compare the predicted distributions with the actual sampling distributions. Based on our findings we suggest directions for the further research.

In section 2 we discuss the model and derive the main asymptotic results. In section 3 a Monte Carlo simulation is conducted to present the main results. In section 4 we give some concluding remarks and discussion. The proofs and tables are given in the appendix.

4.2 Model and the Asymptotic Results

To derive the asymptotic distribution of the standard unit root test statistics when the process is 'nearly stationary' we consider the following model:

(1) $$y_t = \beta y_{t-1} + \varepsilon_t, \quad \varepsilon_t = u_t + \theta u_{t-1}$$

(2) $$\theta = -1 + C/T^{\delta}$$

where u_t is iid $(0, \sigma_u^2)$, $\beta = 1$, $\delta > 0$, and $C > 0$. As $T \to \infty$, $\theta \to -1$. In fact, y_t becomes stationary when $\theta = -1$.

We consider two types of unit root tests. First, we use the Dickey-Fuller tests based on the regressions:

(3) $$y_t = \beta y_{t-1} + \varepsilon_t$$

(4) $$y_t = \mu + \beta y_{t-1} + \varepsilon_t$$

(5) $$y_t = \mu + \delta t + \beta y_{t-1} + \varepsilon_t$$

The $\hat{\rho}$, $\hat{\rho}_\mu$, and $\hat{\rho}_\tau$ tests are based on the coefficient statistic $T(\hat{\beta} - 1)$, where $\hat{\beta}$, $\hat{\beta}_\mu$, and $\hat{\beta}_\tau$ are the OLS estimators of β in (3), (4), and (5) respectively, while the $\hat{t}$, $\hat{t}_\mu$, and $\hat{t}_\tau$ tests are based on the t statistics for the hypothesis $\beta = 1$ in the same three regressions.

Next we consider the Schmidt-Phillips test which is based on the parameterization:

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(6) $y_t = \psi + \xi t + X_t,\ \ X_t = \beta X_{t-1} + \varepsilon_t$

The SP test of the unit root hypothesis can be derived from the results of OLS applied to the following regression:

(7) $\Delta y_t = \text{intercept} + \phi \tilde{S}_{t-1} + \text{error}$

Here $\tilde{S}_{t-1} = y_{t-1} - \tilde{\psi}_X - \tilde{\xi}(t - 1)$ is a residual, with $\tilde{\xi} = (y_T - y_1)/(T - 1)$ and $\tilde{\psi}_X = y_1 - \tilde{\xi}$, which are the restricted maximum likelihood estimators of ξ and $\psi_X = \psi + X_0$. The SP test statistics are defined as $\tilde{\rho} = T\tilde{\phi}$ and $\tilde{\tau}$ = t statistic for the hypothesis $\tilde{\phi} = 0$. For later use, we define $\tilde{\beta} = \tilde{\phi} + 1$.

<u>Remark 1</u>

The reason for including the SP test is that its parameterization can avoid the awkward interpretation of nuisance parameters that the DF tests have. While the meaning of intercept and coefficient of time trend under the null is different from that under the alternative in the DF regressions (3) - (5), ψ and ξ always represent level and linear trend in the SP model (6). In addition, the detrending method of the SP test is different from that of the DF test. It is well-known from the general regression theory that the inference on β in (5) is the same when we replace (5) by

(5)' $\Delta y_t = \text{intercept} + \phi \hat{S}_{t-1} + \text{error}$

where $\hat{S}_{t-1}$ is the residual from the OLS regression of y_{t-1} on intercept and trend, and $\varnothing = \beta - 1$. While the DF test uses the OLS estimators of coefficients on level and trend in constructing the residual $\hat{S}_{t-1}$ in (5)', the SP test uses restricted maximum likelihood estimators under the null of a unit root in constructing the residual $\tilde{S}_{t-1}$ in (7).

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We now show main our asymptotic results. Theorems 1 through 5 show the limiting distribution of the estimates of the coefficients of the lagged dependent variable ($\hat{\beta}$, $\hat{\beta}_\mu$, $\hat{\beta}_\tau$, and $\tilde{\beta}$) under the maintained hypothesis that y_t is generated by (1) and (2). We use the mixing and moment conditions of Phillips and Perron (1988) to derive the asymptotic results. Let us define functionals of the Brownian motion which are used in deriving the following results: standard Brownian motion, W(r); demeaned Brownian motion, $\bar{W} = W - \int W$; demeaned and detrended Brownian motion, $\bar{\bar{W}} = W + (6\int rW - 4\int W) + r(6\int W - 12\int rW)$; and demeaned Brownian Bridge, $\bar{V} = V - \int V$, where $V = W(r) - rW(1)$ is a Brownian Bridge. All integrals are understood to be taken over the interval [0,1] and with respect to Lebesgue measure.

<u>Theorem 1</u>

For $0 < \delta < 1/2$, the estimates of the coefficient β of the lagged dependent variable have the following asymptotic distributions as $T \to \infty$:

(8-1) $\quad T^{1-2\delta}(\hat{\beta} - 1) \to -1/(C^2\int W^2)$

(8-2) $\quad T^{1-2\delta}(\hat{\beta}_\mu - 1) \to -1/(C^2\int \bar{W}^2)$

(8-3) $\quad T^{1-2\delta}(\hat{\beta}_\tau - 1) \to -1/(C^2\int \bar{\bar{W}}^2)$

(8-4) $\quad T^{1-2\delta}(\tilde{\beta} - 1) \to -1/(C^2\int \bar{V}^2)$

<u>Theorem 2</u>

For $\delta = 1/2$, the estimates of the coefficient β of the lagged dependent variable have the following asymptotic distributions as $T \to \infty$:

(9-1) $\quad (\hat{\beta} - 1) \to -1/(1 + C^2\int W^2)$

(9-2) $\quad (\hat{\beta}_\mu - 1) \to -1/(1 + C^2\int \bar{W}^2)$

(9-3) $\quad (\hat{\beta}_\tau - 1) \to -1/(1 + C^2\int \bar{\bar{W}}^2)$

(9-4) $\quad (\tilde{\beta} - 1) \to -1/(1 + C^2\int \bar{V}^2)$

<u>Theorem 3</u>

For $1/2 < \delta < 3/4$, the estimates of the coefficient β of the lagged dependent variable have the following asymptotic distributions as $T \to \infty$:

(10-1) $\quad T^{2\delta-1}\hat{\beta} \to C^2\int W^2$

(10-2) $\quad T^{2\delta-1}\hat{\beta}_\mu \to C^2\int \bar{W}^2$

(10-3) $\quad T^{2\delta-1}\hat{\beta}_\tau \to C^2\int \bar{\bar{W}}^2$

(10-4) $\quad T^{2\delta-1}\tilde{\beta} \to C^2\int \bar{V}^2$

<u>Theorem 4</u>

For $\delta = 3/4$, the estimates of the coefficient β of the lagged dependent variable have the following asymptotic distributions as $T \to \infty$:

(11-1) $\quad T^{1/2}\hat{\beta} \to W(1) + C^2\int W^2$

(11-2) $\quad T^{1/2}\hat{\beta}_\mu \to W(1) + C^2\int \bar{W}^2$

(11-3) $\quad T^{1/2}\hat{\beta}_\tau \to W(1) + C^2\int \bar{\bar{W}}^2$

(11-4) $\quad T^{1/2}\tilde{\beta} \to W(1) + C^2\int \bar{V}^2$

<u>Theorem 5</u>

For $\delta > 3/4$, the estimates of the coefficient β of the lagged dependent variable have the following asymptotic distributions as $T \to \infty$:

(12-1) $\quad T^{1/2}\hat{\beta} \to W(1)$

(12-2) $\quad T^{1/2}\hat{\beta}_\mu \to W(1)$

(12-3) $\quad T^{1/2}\hat{\beta}_\tau \to W(1)$

(12-4) $\quad T^{1/2}\,\tilde{\beta} \;\rightarrow\; W(1)$

Basically the same results are obtained for each test, with the difference being only the functional form of Brownian motion used.

Remark 2

For $0 < \delta < 1/2$, $(\hat{\beta}_j - 1) \rightarrow 0$, but $T^{1-2\delta}(\hat{\beta}_j - 1)$ converges to a random limit, where $\hat{\beta}_j = \hat{\beta}, \hat{\beta}_\mu, \hat{\beta}_\tau$, and $\tilde{\beta}$. The speed of convergence is slower than in the case of standard asymptotics (for fixed θ), where $T(\hat{\beta}_j - 1) \rightarrow$ the limit. For $\delta = 1/2$, $(\hat{\beta}_j - 1)$ converges to a random limit and the speed of convergence is even slower.

When $\delta > 1/2$, $(\hat{\beta}_j - 1) \rightarrow -1$ so that $\hat{\beta}_j \rightarrow 0$. Therefore, we get limiting distributions for $\hat{\beta}_j$ instead of $(\hat{\beta}_j - 1)$. For $1/2 < \delta < 3/4$, $T^{2\delta-1}\,\hat{\beta}_j$ has a limiting distribution and the speed of convergence is between 0 and 1/2; i.e., $0 < 2\delta - 1 < 1/2$. For $\delta = 3/4$, $T^{1/2}\,\hat{\beta}_j$ has a limiting distribution which is a mixture of a functional of Brownian motion and a standard normal process.

Finally, for $\delta > 3/4$, $T^{1/2}\,\hat{\beta}_j$ always converges to the standard normal process $W(1)$, as in the usual stationary case. Fuller (1976) also shows that the process can be approximated by the standard normal process when $\delta > 3/4$.

Next we use the results in Theorems 1 through 5 to find what happens to standard unit root tests when the process is nearly stationary. Corollaries 1 through 5 show that the standard unit root tests have different orders in probability, depending mainly on the value of δ and that their asymptotic distributions depend on the functionals of Brownian motion and C.

Corollary 1

For $0 < \delta < 1/2$, the coefficient and the t statistics of the DF tests and of the SP test have the following asymptotic distributions as $T \to \infty$:

(13-1) $T^{-2\delta}\hat{\rho} \to -1/(C^2\int W^2)$ and $T^{-\delta}\hat{t} \to -1/(2C^2\int W^2)^{1/2}$

(13-2) $T^{-2\delta}\hat{\rho}_\mu \to -1/(C^2\int \bar{W}^2)$ and $T^{-\delta}\hat{t}_\mu \to -1/(2C^2\int \bar{W}^2)^{1/2}$

(13-3) $T^{-2\delta}\hat{\rho}_\tau \to -1/(C^2\int \bar{\bar{W}}^2)$ and $T^{-\delta}\hat{t}_\tau \to -1/(2C^2\int \bar{\bar{W}}^2)^{1/2}$

(13-4) $T^{-2\delta}\tilde{\rho} \to -1/(C^2\int \bar{V}^2)$ and $T^{-\delta}\tilde{\tau} \to -1/(2C^2\int \bar{V}^2)^{1/2}$

Corollary 2

For $\delta = 1/2$, the coefficient and the t statistics of the DF tests and of the SP test have the following asymptotic distributions as $T \to \infty$:

(14-1) $T^{-1}\hat{\rho} \to -1/(1 + C^2\int W^2)$ and $T^{-1/2}\hat{t} \to -1/(1 + 2C^2\int W^2)^{1/2}$

(14-2) $T^{-1}\hat{\rho}_\mu \to -1/(1 + C^2\int \bar{W}^2)$ and $T^{-1/2}\hat{t}_\mu \to -1/(1 + 2C^2\int \bar{W}^2)^{1/2}$

(14-3) $T^{-1}\hat{\rho}_\tau \to -1/(1 + C^2\int \bar{\bar{W}}^2)$ and $T^{-1/2}\hat{t}_\tau \to -1/(1 + 2C^2\int \bar{\bar{W}}^2)^{1/2}$

(14-4) $T^{-1}\tilde{\rho} \to -1/(1 + C^2\int \bar{V}^2)$ and $T^{-1/2}\tilde{\tau} \to -1/(1 + 2C^2\int \bar{V}^2)^{1/2}$

Corollary 3

For $1/2 < \delta < 3/4$, the coefficient and the t statistics of the DF tests and of the SP test have the following asymptotic distributions as $T \to \infty$:

(15-1) $T^{-2(1-\delta)}(\hat{\rho} + T) \to C^2\int W^2$ and $T^{-(3/2)+2\delta}(\hat{t} + T^{1/2}) \to C^2\int W^2$

(15-2) $T^{-2(1-\delta)}(\hat{\rho}_\mu + T) \to C^2\int \bar{W}^2$ and $T^{-(3/2)+2\delta}(\hat{t}_\mu + T^{1/2}) \to C^2\int \bar{W}^2$

(15-3) $T^{-2(1-\delta)}(\hat{\rho}_\tau + T) \to C^2\int \bar{\bar{W}}^2$ and $T^{-(3/2)+2\delta}(\hat{t}_\tau + T^{1/2}) \to C^2\int \bar{\bar{W}}^2$

(15-4) $T^{-2(1-\delta)}(\tilde{\rho} + T) \to C^2\int \bar{V}^2$ and $T^{-(3/2)+2\delta}(\tilde{\tau} + T^{1/2}) \to C^2\int \bar{V}^2$

Corollary 4

For $\delta = 3/4$, the coefficient and the t statistics of the DF tests and of the SP test have the following asymptotic distributions as $T \rightarrow \infty$:

(16-1) $T^{-1/2}(\hat{\rho} + T) \rightarrow W(1) + C^2\int W^2$ and $(\hat{t} + T^{1/2}) \rightarrow W(1) + C^2\int W^2$

(16-2) $T^{-1/2}(\hat{\rho}_\mu + T) \rightarrow W(1) + C^2\int \bar{W}^2$ and $(\hat{t}_\mu + T^{1/2}) \rightarrow W(1) + C^2\int \bar{W}^2$

(16-3) $T^{-1/2}(\hat{\rho}_\tau + T) \rightarrow W(1) + C^2\int \bar{\bar{W}}^2$ and $(\hat{t}_\tau + T^{1/2}) \rightarrow W(1) + C^2\int \bar{\bar{W}}^2$

(16-4) $T^{-1/2}(\tilde{\rho} + T) \rightarrow W(1) + C^2\int \bar{V}^2$ and $(\tilde{\tau} + T^{1/2}) \rightarrow W(1) + C^2\int \bar{V}^2$

Corollary 5

For $\delta > 3/4$, the coefficient and the t statistics of the DF tests and of the SP test have the following asymptotic distributions as $T \rightarrow \infty$:

(17-1) $T^{-1/2}(\hat{\rho} + T) \rightarrow W(1)$ and $(\hat{t} + T^{1/2}) \rightarrow W(1)$

(17-2) $T^{-1/2}(\hat{\rho}_\mu + T) \rightarrow W(1)$ and $(\hat{t}_\mu + T^{1/2}) \rightarrow W(1)$

(17-3) $T^{-1/2}(\hat{\rho}_\tau + T) \rightarrow W(1)$ and $(\hat{t}_\tau + T^{1/2}) \rightarrow W(1)$

(17-4) $T^{-1/2}(\tilde{\rho} + T) \rightarrow W(1)$ and $(\tilde{\tau} + T^{1/2}) \rightarrow W(1)$

Remark 3

For $0 \leq \delta < 1/2$ the coefficient and t statistics have order in probability $O_P(T^{2\delta})$ and $O_P(T^{\delta})$, respectively. When $\delta \geq 1/2$, the coefficient statistic is of order $O_P(T)$ and the t statistic is of order $O_P(T^{1/2})$. (Note that the $\delta = 1/2$ and $3/4$ are the discontinuity points dividing the limiting distributions which have different behaviors.) This implies that all the standard unit root test statistics diverge to negative infinity as $T \rightarrow \infty$, when the process is nearly stationary.

Our results can be directly compared to Pantula (1991), in which he studies the performance of the various unit root test statistics when the process is nearly stationary. He uses the same model as (1) and (2) but restricts the value of C to 1. However, there are some differences between Pantula's analysis and ours.

First, Pantula analyzes only the case of random walk without drift, while we extend it into the more general cases of random walk with drift and random walk with drift and time trend. We also consider the SP unit root test.

Second, he divides δ into three regions: $0 < \delta < 1/4$ (nonstationary region), $1/4 < \delta < 1/2$ (grey region), $1/2 < \delta$ (stationary region), but we unify first two regions and consider additional cases of $\delta = 1/2$ and $\delta = 3/4$. As will be shown, the choice of $\delta = 1/2$ is important because the predicted distribution of standard unit root tests in this case approximates the actual sampling distribution relatively well unless θ is very close to minus one.

Third, Pantula fixes C to one, but C is unrestricted in our model. In both our model and Pantula's, $\theta \to -1$ as $T \to \infty$. From the point of view of using asymptotics to approximate finite sample distributions, our model is obviously more flexible. For example, if we pick $\theta = -0.8$ and $T = 100$, we pick $\delta = 0.35$ in Pantula's model, whereas in our model we can have $\theta = -0.8$ for $\delta = 1/4$ and $C = 0.632$, or $\delta = 1/2$ and $C = 2$, or $\delta = 5/8$ and $C = 3.557$, or $\delta = 3/4$ and $C = 6.325$, or $\delta = 7/8$ and $C = 11.247$. In our calculations we typically set δ (e.g., $\delta = 1/2$) and let C be the value that is chosen to yield (θ,T) pair.

Finally, Pantula compares the performance of various unit root tests such as the DF tests, ADF tests, PP tests and Hall's (1989) IV test, based on the criteria that good unit root tests should accept the null of unit root when the process is in the 'nonstationary region' ($0 < \delta < 1/4$) and reject the null when the process is in the 'stationary region' ($\delta > 1/2$). Based on limited simulations he suggest the use of the ADF unit root test as best when the process is nearly stationary. We are rather interested in examining the behavior of uncorrected DF and SP tests when the process is nearly stationary in more detail to see how accurate asymptotic approximations of this type are in finite samples.

4.3 Simulation Results

In this section we use extensive Monte Carlo simulations to study the finite sample performance of standard unit root tests when the process is nearly stationary. The basic data generating process we use is the ARIMA(0,1,1) process:

(18) $$y_t = y_{t-1} + \varepsilon_t, \quad \varepsilon_t = u_t + \theta u_{t-1}$$

where u_t's are serially uncorrelated standard normal random

variables. The data are generated by choosing the initial value of u_t after discarding the first 20 observations. The standard normal random numbers are selected using the Fortran subroutine GASDEV/RAN3 of Press et al. (1986). The actual sampling distributions are tabulated by applying the standard DF and SP tests directly to data generated according to the basic DGP (18), using 50,000 replications. Values of θ = -0.5, -0.8, -0.9, -0.95, -0.99, -1.0 and values of T = 25, 50, 100, 250, 500, 1000 are used. The results for the 5 % and 95 % fractiles are given in Table 4-1.

Since the unit root test is a lower tail test, we are mainly interested in the behavior of the lower 5 % fractiles. The 5 % critical values of the coefficient statistics are diverging around -T, as $\theta \to -1$. This is expected, because if $\theta = -1$ then y_t reduces to the iid process and therefore $\beta \to 0$. In this case $T(\hat{\beta} - 1)$ almost acts like -T; t statistics almost act like $-T^{1/2}$. The speed of divergence depends on the sample size. These critical values are far less than those of the original DF statistics. This implies that standard unit root (coefficient) tests are expected to reject almost always the null hypothesis of the unit root when $\theta \leq -0.8$ and T is large.

It is well-known that the more regressors such as constant and time trend we include, the more negative critical values we get. When we use the $\hat{\rho}_\tau$ test, the speed of divergence of the 5 % critical value to -T is very fast in finite samples. For example, when $\theta = -0.8$ and T = 100, the 5 % critical value of $\hat{\rho}_\tau$ is -101.9. This indicates that there is a very strong bias of $\hat{\beta}$ toward 0 when the process is nearly stationary with trend. Even though we increase the sample size to 1,000, this bias is still quite large. Therefore, the direct application of uncorrected DF and SP tests to a nearly stationary process is dangerous way as is well known.

We obtain basically the same results for the t statistics. The 5% critical values of the t statistic become more negative as $\theta \to -1$, given sample size, and these critical values are still far less than those of the DF and SP t statistics.

To tabulate the predicted distributions of the unit root test statistics ($\hat{\rho}$, $\hat{t}$, $\hat{\rho}_\mu$, $\hat{t}_\mu$, $\hat{\rho}_\tau$, $\hat{t}_\tau$, $\tilde{\rho}$, $\tilde{\tau}$) when the process is nearly stationary, a sample size of T = 2,000 is used to find the limiting fractiles of standard Brownian motion, demeaned Brownian motion, demeaned and detrended Brownian motion, and demeaned Brownian bridge. Each experiment is replicated 50,000 times. We then use the formulas given in Corollaries 1 - 5 to convert these fractiles into predicted fractiles for the unit root test statistics. We use values of δ = 1/4, 1/2, 5/8, 3/4, and 7/8, and the same values of T and θ as above. For given values of θ and T, we use the following formula for C:

(19) $$C = (1 + \theta)T^{\delta}$$

The results for the 5 % and 95 % critical values are given in Tables 4-2 through 4-6.

We now compare the predicted distributions with the actual sampling distribution to see how closely they are. The results are based on the comparison of Table 4-1 to Tables 4-2 through 4-6.

1) $0 < \delta < 1/2$ ($\delta = 1/4$)

Table 4-2 shows that the predicted critical values of the unit root tests for δ = 1/4 are independent of the sample size, given a value of θ. This is so, because (19) implies that C is proportional to $T^{1/4}$, in which case T cancels from the expression in Corollary 1 above. These critical values are generally far more negative than the actual sampling values. Furthermore, the discrepancy is wider as θ is closer to -1. Even though we increase the sample size to 1,000, the actual sampling fractiles never catch up with the predicted critical values. Therefore, the predicted distributions of the unit root test statistics in this case are not a good approximation of the actual ones.

2) $\delta = 1/2$

The results for δ = 1/2 are given in Table 4-3. The predicted distributions of the test statistics in this case approximate the actual ones relatively well, especially when θ is in the range of -0.8 to -0.9. For example, for θ = -0.8 and T = 100, the actual 5 % critical values of $\hat{\rho}$ is -81.7 and the predicted value is -81.1. The discrepancies between actual and predicted critical values is larger but not very great for the other test statistics and for other values of T and θ. The actual critical values change more rapidly than the predicted ones as $\theta \rightarrow -1$; that is, the speed of divergence of the actual distribution is relatively faster than that expected from the predicted distribution for $\delta = 1/2$ as $\theta \rightarrow -1$.

3) $1/2 < \delta < 3/4$ ($\delta = 5/8$)

These results are given in Table 4-4. The predicted distributions for $\delta = 5/8$ show quite different behavior from the previous ones. The right tail critical values are explosively positive except when θ = -0.99, while the left tail critical values are dependent upon the sample size and the value of θ chosen. When θ is -0.5 or -0.8, the left tail predicted critical values are getting positive as the sample size increases. This behavior is not present in the actual sampling distribution. More generally, the quality of the approximations is fairly good in the left tail for θ = -0.9 and -0.95, but not usually as good as it is with $\delta = 1/2$.

4) $\delta = 3/4$

The predicted critical values for $\delta = 3/4$ given in Table 4-5 show the intermediate behavior between those for $\delta = 5/8$ and for $\delta = 7/8$. When $\theta \geq -0.9$, the predicted critical values are close to those for $\delta = 5/8$, and therefore they have the same problem of explosive positive right tail fractiles as before. However, when θ =

-0.95 or -0.99, the predicted distributions are close to those for $\delta = 7/8$, whose behavior will be explained below.

5) $\delta > 3/4$ ($\delta = 7/8$)

For $\delta > 3/4$, Corollary 5 indicates that all of the statistics (appropriately normalized) converge to a standard normal distribution. Thus the predicted distribution is the same for all test statistics and all values of θ.

We conclude that, as a general statement, $\delta = 1/2$ is the choice that leads to the best match between predicted and actual sampling distributions. Pantula does not consider this value, and thus this is an important and novel result. We proceed to compare the predicted distribution for $\delta = 1/2$ with the actual sampling distribution in more detail than above. For simplicity we do so only for the $\hat{\rho}$ and $\hat{t}$ statistics; similar results are obtained for the other statistics.

For convenience we use A_D to represent the actual distribution and P_D to represent the predicted distribution. We also denote A_C as the actual critical value and P_C as the predicted critical value. We consider critical values of .01, .025, .05, .95, .975, and .99. These results are given in Table 4-7. Note that Table 4-7 contains results also for $\delta = 1/4$, 5/8, 3/4, and 7/8, but our discussion applies only to the results for $\delta = 1/2$. Also, while Table 4-7 contains results for the $\hat{\rho}$ and $\hat{t}$ statistics, we will only discuss the results for the $\hat{\rho}$ test.

1) $\theta = -0.5$

When $T \leq 50$, A_D and P_D have similar dispersion with P_D a little more concentrated, and for lower tail critical values A_C is more negative than P_C. However, when $T \geq 100$, P_D becomes more negatively skewed than A_D. Therefore, P_C is more negative than A_C and the difference becomes larger as T increases.

2) $\theta = -0.8$

When $T \leq 100$, P_D is more concentrated than A_D. A_C is more negative than P_C in the left tail, while A_C is less negative than P_C in the right tail. When $T = 250$, P_D is similar in dispersion to A_D, but P_D begins to skew negatively. When $T \geq 500$, P_C becomes more negative than A_C.

3) $\theta = -0.9$

When $T \leq 100$, P_D is more concentrated than A_D. The smaller T is, the more concentrated P_D will be. A_C is more negative than P_C in the left tail, while P_C is more negative than A_C in the right tail. When $T = 250$, P_D is similar in dispersion to A_D, but A_C is still more negative than P_C in the left tail. When $T = 500$, P_D becomes less concentrated and negatively skewed, but still similar in dispersion to A_D. When $T = 1{,}000$, P_D becomes more negatively skewed than A_D, and P_C is more negative than A_C.

4) $\theta = -0.95$

When $T \leq 500$, P_D is more concentrated than A_D. A_C is more negative than P_C in the left tail, while P_C is more negative than A_C in the right tail. However, when T is small, the predicted distribution is too narrowly clustering around -T. For example, when $T = 25$, P_D ranges only from -24.95 to -21.33, while A_D ranges from -34.12 to -10.69. When $T = 1{,}000$, P_D is finally similar in dispersion to A_D, but A_C is more negative than P_C in the left tail.

5) $\theta = -0.99$

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For all sample sizes, P_D is more concentrated than A_D. Therefore, A_C is more negative than P_C in the left tail, and P_C is more negative than A_C in the right tail. Except when T = 1,000, the problem of converging around the negative sample size exists. Especially when $T \leq 100$, all the critical values of the predicted distribution are almost the same as -T. For example, when T = 100, P_D ranges from -99.97 to -97.32, but A_D ranges from -120.76 to -73.68.

4.4 Discussions and Concluding Remarks

We have examined the asymptotic and finite sample behavior of the standard Dickey-Fuller and Schmidt-Phillips unit root tests when the process is nearly stationary, using a local approximation of the MA(1) parameter around minus one. Pantula (1991) has studied the same problem. However, some implications of these studies are different, mainly due to the different treatment of the parameter C in the local approximation of the MA(1) parameter around minus one and our inclusion of $\delta = 1/2$. The main findings we have obtained are as follow:

First, when $\theta \to -1$ as $T \to \infty$, the coefficient statistics $\hat{\rho}$, $\hat{\rho}_\mu$, $\hat{\rho}_\tau$, $\tilde{\rho}$ and the t statistics $\hat{t}$, $\hat{t}_\mu$, $\hat{t}_\tau$, $\tilde{\tau}$ diverge to negative infinity, but have different orders in probability depending on the value of δ.

Second, we find by Monte Carlo simulations that the choice of $\delta = 1/2$ is generally best, in the sense that its predicted distributions are closer than for other choices of δ to the actual sampling distribution, at least unless θ is very close to -1.

When $0 < \delta < 1/2$, the predicted distributions of the unit root test statistics are the same for all the statistics we consider and for all T, which is clearly unsatisfactory. When $1/2 < \delta \leq 3/4$, the predicted distributions have a problem of critical values being explosively positive, which again is not consistent with the actual sampling distribution.

To sum up, the tendency of most unit root tests such as the Dickey-Fuller test and the Phillips-Perron test and their modifications, to have considerable size distortions in finite samples when the process is nearly stationary is not surprising, and is well predicted by our asymptotics. It reflects the fact that a nearly stationary process can be expected to behave approximately as a stationary process in finite samples. This point has been made before by others, including Wichern (1973) and Blough (1989). For example, Wichern shows that information about the sample autocorrelations does not differentiate appreciably between stationary and nonstationary ARIMA(0,1,1) processes, and argues that, in practice, it is not possible to prove that the series is stationary or nonstationary in finite samples. Furthermore, he argues that the use of either model will lead to very similar results in testing and other applications for reasonable sample sizes.

Schwert (1989) concludes from his Monte Carlo evidence that the augmented Dickey-Fuller statistic provides the most accurate unit root test in the presence of strongly autocorrelated errors. Pantula (1991) also recommends the augmented Dickey-Fuller tests, based on his asymptotics. We do not consider asymptotics for the augmented Dickey-Fuller test in this chapter. However, we note that in our view we cannot take granted for the superiority of the augmented Dickey-Fuller statistic over any other unit root test statistic, as Pantula and Schwert have suggested. The augmented Dickey-Fuller test requires a very large number of augmentations to have correct size in finite samples when the process is nearly stationary, because otherwise the OLS estimator of $\hat{\beta}$ is seriously biased toward zero when θ is close to -1. It is well-known that there should be a finite sample trade-off between size and power of the test when the null is close to the alternative. That is, the cost of the good size performance of the augmented Dickey-Fuller test should be very poor power in finite samples. This has also been argued by Blough (1989), and the Monte Carlo evidence to this effect is given by a number of authors, including Lee and Schmidt (1991).

One of the possible ways to overcome the above problems is the use of the IV unit root test suggested by Hall (1989). He uses an instrumental variable approach to handle the size distortion problem caused by moving average errors. The instrument for y_{t-1} is y_{t-k} , where $k > q$ for MA(q). Hall applies the IV method to the Dickey-Fuller regression and shows a significant improvement. Lee and Schmidt (1991) also derive the IV versions of the Schmidt-Phillips test and show much more improvement. Although the IV unit root tests still have some size distortions when the process is nearly stationary and it is difficult to know the exact order of q, more research in this direction would probably give further insights.

Another possibility is the construction of a unit root test by using a more efficient estimator of β, which can also possibly reduce the bias in the estimation of β when the process is nearly stationary. Basically, this would involve estimation of the

Dickey-Fuller or Schmidt-Phillips regression by GLS, given an assumption on the order of the ARMA process of the errors. Choi (1990) considers a partial GLS estimator, in which the AR portion of the process is handled by data transformation while the MA portion is handled by IV estimation, and he gives some optimistic results on the finite sample properties of the resulting test. How this compares to a full GLS treatment is not clear. Again, future research is needed.

Appendix

In this appendix we prove the main asymptotic results given in section 4.2. The data generating process is given in equations (1) and (2) in the main text. First, set $u_0 = y_0 = 0$ without loss of generality. Then equation (1) is solved for y_t:

$$\text{(A1)} \qquad y_t = u_t + (1 + \theta)S_{t-1}$$

where $S_t = \sum_{j=1}^{t} u_t$ is a partial sum process. Substituting for $(1 + \theta)$ from (2) into (A1) we get

$$\text{(A2)} \qquad y_t = u_t + (C/T^{\delta})S_{t-1}$$

Note that this is the solution for the nearly stationary process using our local approximation for the MA(1) parameter to minus one. That is, as $T \to \infty$, $y_t \to u_t$. This transformation will be intensively used in the proof. Using (A2) we will derive the asymptotic results for the statistics:

$$\text{(A3)} \qquad \hat{\rho}_j = T(\hat{\beta}_j - 1)$$

$$\text{(A4)} \qquad \hat{t}_j = (\hat{\beta}_j - 1)/(s^2 \cdot XX_{jj}^{-1})^{1/2}$$

$\hat{\beta}_j$ is the estimate of the coefficient of the lagged dependent variable in (3), (4), (5), and (6) so that $\hat{\beta}_j = \hat{\beta}, \hat{\beta}_\mu$, and $\tilde{\beta}$; $s^2 = (1/T)\sum_{j=1}^{T} \hat{e}_j^2$, where $\hat{e}_j$ are the appropriate residuals; and $XX_{jj} = \Sigma y_{t-1}^2$, $\Sigma(y_{t-1} - \bar{y}_{-1})^2$, $\Sigma(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})^2$, and $\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2$, respectively for regressions (3), (4), (5)' and (7). $\hat{S}_{t-1}$ and $\tilde{S}_{t-1}$ are defined in text. As shown in Theorems 1 throug

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through 5 in the main text, the asymptotic results depend mainly on C, δ and functionals of the Brownian motion. Note that we use the basic preliminaries of Phillips and Perron (1988) (given on p. 377) extensively in the proofs.

[1] $\hat{\rho}$ and $\hat{t}$ tests

It is useful to write

(A5) $$\hat{\beta} = (\Sigma y_{t-1}^2)^{-1} \Sigma y_{t-1} y_t \quad \text{or} \quad (\hat{\beta} - 1) = (\Sigma y_{t-1}^2)^{-1} \Sigma y_{t-1} \varepsilon_t$$

(A6) $$s^2 = (1/T)\Sigma(y_t - \hat{\beta} y_{t-1})^2$$

Case 1: $0 < \delta < 1/2$

Lemma 1.1 $(1/T^{2-2\delta})\Sigma y_t^2 \to \sigma_u^2 C^2 \int W^2$

Lemma 1.2 $(1/T)\Sigma y_{t-1}\varepsilon_t \to -\sigma_u^2$ for all $\delta > 0$.
(proof)

$$(1/T)\Sigma y_{t-1}\varepsilon_t$$

$$= (1/T)\Sigma u_{t-1}u_t + (C/T^\delta)(1/T)\Sigma S_{t-2}u_t + (\theta/T^\delta)(1/T)\Sigma S_{t-1}u_t + (\theta/T)\Sigma u_{t-1}^2$$

$\to -\sigma_u^2$ since first three terms $\to 0$ and $\theta \to -1$ as $T \to \infty$. □

Theorem 1.3 $T^{1-2\delta}(\hat{\beta} - 1) \to -1/(C^2\int W^2)$

Lemma 1.4 $s^2 \to 2\sigma_u^2$
(proof)

$$s^2 = (1/T)\Sigma\varepsilon_t^2 + (\hat{\beta} - 1)^2(1/T)\Sigma y_{t-1}^2 - 2(\hat{\beta} - 1)(1/T)\Sigma y_{t-1}\varepsilon_t$$

$\to 2\sigma_u^2$ □

Corollary 1.5 $T^{\delta}\, \hat{t} \to -1/(2C^2\int W^2)^{1/2}$

Case 2: $\delta = 1/2$

<u>Lemma 1.6</u> $(1/T)\Sigma y_t^2 \to \sigma_u^2(1 + C^2\int W^2)$

<u>Theorem 1.7</u> $\hat{\beta} - 1 \to -1/(1 + C^2\int W^2)$

<u>Lemma 1.8</u> $s^2 \to \sigma_u^2\{2 - 1/(1 + C^2\int W^2)\}$
(proof)

$$s^2 = (1/T)\Sigma\varepsilon_t^2 + (\hat{\beta} - 1)^2(1/T)\Sigma y_{t-1}^2 - 2(\hat{\beta} - 1)(1/T)\Sigma y_{t-1}\varepsilon_t$$

(1) $(\hat{\beta} - 1)^2(1/T)\Sigma y_{t-1}^2 \to \sigma_u^2/(1 + C^2\int W^2)$

(2) $-2(\hat{\beta} - 1)(1/T)\Sigma y_{t-1}\varepsilon_t \to -2\sigma_u^2/(1 + C^2\int W^2)$

Using (1), (2), and $(1/T)\Sigma\varepsilon_t^2 \to 2\sigma_u^2$, we get the result. □

<u>Corollary 1.9</u> $T^{-1/2}\,\hat{t} \to -1/(1 + 2C^2\int W^2)^{1/2}$

Case 3: $1/2 < \delta < 3/4$

<u>Lemma 1.10</u> $(1/T)\Sigma y_t^2 \to \sigma_u^2$
(proof)

$$T^{-1}\Sigma y_t^2 = T^{-1}\Sigma u_t^2 + (C^2/T^{2\delta})T^{-1}\Sigma S_{t-1}^2 + (2C/T^{\delta})T^{-1}\Sigma S_{t-1}u_t$$

$$\to \sigma_u^2 \quad \square$$

<u>Lemma 1.11</u> $\hat{\beta} - 1 \to -1.$

Since the above lemma implies that $\hat{\beta} \to 0$ when $\delta > 1/2$, we derive asymptotics for $\hat{\beta}$ instead of $(\hat{\beta} - 1)$.

<u>Lemma 1.12</u> $(1/T^{2-2\delta})\Sigma y_{t-1}y_t \to \sigma_u^2C^2\int W^2$
(proof)

$$(1/T^{2-2\delta})\Sigma y_{t-1}y_t$$

$$= (1/T^{2-2\delta})\Sigma\{u_{t-1}u_t + (C/T^{\delta})S_{t-2}u_t + (C/T^{\delta})u_{t-1}S_{t-1} + (C^2/T^{2\delta})S_{t-1}S_{t-2}\}$$

$$\to \sigma_u^2C^2\int W^2$$

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since $T^{-2}\Sigma S_t S_{t-1} \to T^{-2}\Sigma S_t^2 \to \sigma_u^2 \int W^2$ and the remaining terms $\to 0$. □

<u>Theorem 1.13</u> $T^{2\delta-1}\hat{\beta} \to C^2\int W^2$

<u>Lemma 1.14</u> $s^2 \to \sigma_u^2$ for $\delta > 1/2$
(proof)

$$s^2 = T^{-1}\Sigma\varepsilon_t^2 + (\hat{\beta} - 1)^2(1/T)\Sigma y_{t-1}^2 - 2(\hat{\beta} - 1)(1/T)\Sigma y_{t-1}\varepsilon_t$$

$$\to 2\sigma_u^2 + \sigma_u^2 - 2\sigma_u^2 = \sigma_u^2 \quad \square$$

<u>Corollary 1.15</u> $\hat{t} + T^{1/2} \to C^2\int W^2$

Case 4: $\delta = 3/4$

<u>Lemma 1.16</u> $T^{-1/2}\Sigma y_{t-1}y_t \to \sigma_u^2\{W(1) + C^2\int W^2\}$
(proof)

$$T^{-1/2}\Sigma y_{t-1}y_t$$

$$= T^{-1/2}\{\Sigma u_{t-1}u_t + (C/T^{3/4})S_{t-2}u_t + (C/T^{3/4})S_{t-1}u_{t-1} + (C^2/T^{3/2})S_{t-1}S_{t-2}\}$$

$$\to \sigma_u^2 W(1) + \sigma_u^2 C^2\int W^2 \quad \square$$

<u>Theorem 1.17</u> $T^{1/2}\hat{\beta} \to W(1) + C^2\int W^2$

<u>Corollary 1.18</u> $\hat{t} + T^{1/2} \to W(1) + C^2\int W^2$

Case 5: $\delta > 3/4$

<u>Lemma 1.19</u> $T^{-1/2}\Sigma y_{t-1}y_t \to \sigma_u^2 W(1)$
(proof)

$$T^{-1/2}\Sigma y_{t-1}y_t$$

$$= T^{-1/2}\Sigma\{u_{t-1}u_t + (C/T^{\delta})S_{t-2}u_t + (C/T^{\delta})u_{t-1}S_{t-1} + (C^2/T^{2\delta})S_{t-1}S_{t-2}\}$$

$$\to \sigma_u^2 W(1) \quad \square$$

<u>Theorem 1.20</u> $T^{1/2}\hat{\beta} \to W(1)$

Corollary 1.21 $\hat{t} + T^{1/2} \rightarrow W(1)$

[2] $\hat{\rho}_\mu$ and $\hat{t}_\mu$ tests

In this case it is useful to write

(A7) $$\hat{\beta}_\mu = \Sigma(y_{t-1} - \bar{y}_{-1})y_t \,/\, \Sigma(y_{t-1} - \bar{y}_{-1})^2 \quad \text{or}$$
$$\hat{\beta}_\mu - 1 = \Sigma(y_{t-1} - \bar{y}_{-1})\varepsilon_t \,/\, \Sigma(y_{t-1} - \bar{y}_{-1})^2$$

(A8) $$s^2 = (1/T)\Sigma\{(y_t - \bar{y}) - \hat{\beta}_\mu(y_{t-1} - \bar{y}_{-1})\}^2$$

Case 1: $0 < \delta < 1/2$

Lemma 2.1 $T^{-1/2+\delta}\bar{y} \rightarrow C\sigma_u \int W$ for all $\delta > 0$.

Lemma 2.2 $(1/T^{2-2\delta})\Sigma(y_{t-1}{}^2 - \bar{y}_{-1})^2 \rightarrow \sigma_u{}^2 C^2 \int \bar{W}^2$
(proof)
$$(1/T^{2-2\delta})\Sigma(y_{t-1} - \bar{y}_{-1})^2 = (1/T^{2-2\delta})\Sigma y^2 - (T^{-1/2+\delta}\bar{y}_{-1})^2$$

Using Lemma 1.1 and 2.1 we get the result. □

Lemma 2.3 $(1/T)\Sigma(y_{t-1} - \bar{y}_{-1})\varepsilon_t \rightarrow -\sigma_u{}^2$ for all δ.
(proof)
$$(1/T)\Sigma y_{t-1}\varepsilon_t = (1/T)\Sigma\{u_{t-1} + (1+\theta)S_{t-2}\}(u_t + \theta u_{t-1})$$
$$= (1/T)\Sigma u_{t-1}u_t + (C/T^\delta)(1/T)\Sigma S_{t-2}u_t + (\theta/T^\delta)(1/T)\Sigma S_{t-1}u_t + (\theta/T)\Sigma u_{t-1}{}^2$$

Since first 3 terms $\rightarrow 0$ and $\theta \rightarrow -1$ as $T \rightarrow \infty$, we get the result. □

Theorem 2.4 $T^{1-2\delta}(\hat{\beta}_\mu - 1) \rightarrow -1/C^2 \int \bar{W}^2$

Lemma 2.5 $s^2 \rightarrow 2\sigma_u{}^2$
(proof)
$$s^2 = T^{-1}\Sigma\varepsilon_t{}^2 + (\hat{\beta} - 1)^2 T^{-1}\Sigma(y_{t-1} - \bar{y}_{-1})^2 - 2(\hat{\beta} - 1)T^{-1}\Sigma(y_{t-1} - \bar{y}_{-1})\varepsilon_t$$

Since $T^{-1}\Sigma\varepsilon_t{}^2 \rightarrow 2\sigma^2$ and the last two terms $\rightarrow 0$, we get the result. □

<u>Corollary 2.6</u> $T^{-\delta}\hat{t}_\mu \rightarrow -1/(2C^2\int\bar{W}^2)^{1/2}$

Case 2: $\delta = 1/2$

<u>Lemma 2.7</u> $T^{-1}\Sigma(y_{t-1} - \bar{y}_{-1})^2 \rightarrow \sigma_u^2(1 + C^2\int\bar{W}^2)$

<u>Theorem 2.8</u> $\hat{\beta}_\mu - 1 \rightarrow -1/(1 + C^2\int\bar{W}^2)$

<u>Lemma 2.9</u> $s^2 \rightarrow \sigma_u^2\{2 - 1/(1 + C^2\int\bar{W}^2)\}$

<u>Corollary 2.10</u> $T^{-1/2}\hat{t}_\mu \rightarrow -1/(1 + 2C^2\int\bar{W}^2)^{1/2}$

Case 3: $1/2 < \delta < 3/4$

<u>Lemma 2.11</u> $T^{-1}\Sigma(y_{t-1} - \bar{y}_{-1})^2 \rightarrow \sigma_u^2$

<u>Theorem 2.12</u> $\hat{\beta}_\mu - 1 \rightarrow -1$

<u>Lemma 2.13</u> $(1/T^{2-2\delta})\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y}) \rightarrow \sigma_u^2C^2\int\bar{W}^2$
(proof)

$$(1/T^{2-2\delta})\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y})$$

$$= (1/T^{2-2\delta})\Sigma\{y_{t-1}y_t - \bar{y}_{-1}y_t - y_{t-1}\bar{y} + \bar{y}_{-1}\bar{y}\}$$

(1) $(1/T^{2-2\delta})\Sigma y_{t-1}y_t \rightarrow \sigma_u^2C^2\int W^2$

(2) $(-1/T^{2-2\delta})\bar{y}_{-1}\Sigma y_t \rightarrow -(C\sigma_u\int W)^2$

(3) $(-1/T^{2-2\delta})\Sigma\bar{y}y_{t-1} \rightarrow -(C\sigma_u\int W)^2$

(4) $(1/T^{2-2d})\Sigma\bar{y}_{-1}\bar{y} \rightarrow (C\sigma_u\int W)^2$

Using (1), (2), (3), and (4) we get the result. □

<u>Theorem 2.14</u> $T^{2\delta-1}\hat{\beta} \rightarrow C^2\int\bar{W}^2$.

<u>Lemma 2.15</u> $s^2 \rightarrow \sigma_u^2$ for $\delta > 1/2$

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<u>Corollary 2.16</u> $\hat{t}_\mu + T^{1/2} \rightarrow C^2\int\bar{W}^2$

Case 4: $\delta = 3/4$

<u>Lemma 2.17</u> $T^{-1/2}\,\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y}) \rightarrow \sigma_u^2\{W(1) + C^2\int\bar{W}^2\}$
(proof)

$$T^{-1/2}\,\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y}) = T^{-1/2}\Sigma\{y_{t-1}y_t - \bar{y}_{-1}y_t - \bar{y}_{t-1}y + \bar{y}_{-1}\bar{y})$$

(1) $T^{-1/2}\,\Sigma y_{t-1}y_t \rightarrow \sigma_u^2\{W(1) + C^2\int W^2\}$

(2) $-T^{-1/2}\bar{y}_{-1}\Sigma y_t \rightarrow -(C\sigma_u\int W)^2$

(3) $-T^{-1/2}\,\Sigma\bar{y}y_{t-1} \rightarrow -(C\sigma_u\int W)^2$

(4) $T^{-1/2}\,\Sigma\bar{y}_{-1}\bar{y} \rightarrow (C\sigma_u\int W)^2$

Using (1), (2), (3), and 4), we get the result. □

<u>Theorem 2.18</u> $T^{1/2}\hat{\beta}_\mu \rightarrow W(1) + C^2\int\bar{W}^2$

<u>Corollary 2.19</u> $\hat{t}_\mu + T^{1/2} \rightarrow W(1) + C^2\int\bar{W}^2$

Case 5: $\delta > 3/4$

<u>Lemma 2.20</u> $T^{-1/2}\,\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y}) \rightarrow \sigma_u^2 W(1)$
(proof)

$$T^{-1/2}\Sigma(y_{t-1} - \bar{y}_{-1})(y_t - \bar{y}) = T^{-1/2}\Sigma\{y_t y_{t-1} - \bar{y}y_{t-1} - y_t\bar{y}_{-1} + \bar{y}\bar{y}_{-1}\}$$

Since $T^{-1/2}\Sigma y_t y_{t-1} \rightarrow \sigma_u^2 W(1)$ and the remaining terms $\rightarrow 0$ we get the result.

<u>Theorem 2.21</u> $T^{1/2}\hat{\beta}_\mu \rightarrow W(1)$

<u>Corollary 2.22</u> $\hat{t}_\mu + T^{1/2} \rightarrow W(1)$

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[3] $\hat{\rho}_\tau$ and $\hat{t}_\tau$ tests

The derivation is simpler when we use the transformation of regression equation (5):

(A9) $$\Delta y_t = \alpha + \delta t + (\beta - 1)y_{t-1} + \varepsilon_t$$

First, consider the regression of y_{t-1} on $X = [1,t]$ to get the residuals $\hat{S}_{t-1}$ and define the matrix D and Q as follow:

(A10) $$D = \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{bmatrix}$$

(A11) $$D'X'XD \rightarrow \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} = Q$$

(A12) $$Q^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

It is straightforward to obtain the asymptotic results for the OLS estimator of the coefficients of level and trend as follow:

(A13) $$\begin{bmatrix} \hat{a} \\ T\hat{b} \end{bmatrix} = (1/\sqrt{T})(D'X'XD)D'X'Y \rightarrow Q^{-1}\begin{bmatrix} (1/T)\,\Sigma y_{t-1} \\ (1/T^2)\Sigma t y_{t-1} \end{bmatrix}$$

$$\rightarrow Q^{-1}\begin{bmatrix} C\sigma_u \int W \\ C\sigma_u \int rW \end{bmatrix} \rightarrow C\sigma_u \begin{bmatrix} 4\int W - 6\int rW \\ -6\int W + 12\int rW \end{bmatrix}$$

Using (A13) we get the asymptotics for the residual from the regression of y_t on $[1, t]$:

(A14) $$\hat{S}_{[Tr]} = y_{[Tr]} - \hat{a} - \hat{b}[Tr]$$

$$= u_{[Tr]} + (C/\sqrt{T})S_{[Tr]} - \hat{a} - r(T\hat{b}) \rightarrow u_{[Tr]} + C\sigma_u\bar{\bar{W}}$$

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where $\bar{\bar{W}}(r) = W(r) - 4\int W + 6\int rW + 6r\int W - 12r\int rW$ is a detrended Brownian motion as defined in the main text.

Next consider the regression of Δy_{t-1} on $[1, \hat{S}_{t-1}]$ to draw an inference on the coefficient $\phi = \beta - 1$.

$$\text{(A15)} \qquad \Delta y_t = \text{intercept} + \phi\hat{S}_{t-1} + \text{error}$$

which is the same as (5)' in the text. We use (A15) to derive the asymptotics for the $\hat{\rho}_\tau$ and $\hat{t}_\tau$ statistics. Note that we use the same symbol ε_t as the error in (A15) without loss of generality. Therefore, the $\hat{\rho}_\tau$ statistic is defined as $T\hat{\phi}_\tau$, where

$$\text{(A16)} \qquad \hat{\phi}_\tau = \Sigma(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})\Delta y_t \,/\, \Sigma(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})^2$$
$$= \Sigma(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})\varepsilon_t \,/\, \Sigma(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})^2$$

Accordingly, s^2 is defined as

$$\text{(A17)} \qquad s^2 = (1/T)\Sigma\{(\Delta y_t - \Delta\bar{y}) - \hat{\phi}_\tau(\hat{S}_{t-1} - \bar{\hat{S}}_{-1})\}^2$$

Case 1: $0 < \delta < 1/2$

<u>Lemma 3.1</u> $T^{-1/2+\delta}\begin{bmatrix} \hat{a} \\ T\hat{b} \end{bmatrix} \rightarrow C\sigma_u \begin{bmatrix} 4\int W - 6\int rW \\ -6\int W + 12\int rW \end{bmatrix}$

This is true for all $\delta > 0$.

Using Lemma 3.1 we get the following result:

$$(1/T^{1/2-\delta})\,\hat{S}_{[Tr]} = (1/T^{1/2-\delta})u_{[Tr]} + (C/\sqrt{T})S_{[Tr]} - (1/T^{1/2-\delta})\hat{a} - (r/T^{1/2-\delta})T\hat{b}$$
$$\rightarrow C\sigma_u\bar{\bar{W}}(r)$$

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Lemma 3.2 $(1/T^{2-2\delta})\Sigma\hat{S}_t^2 \to C^2\sigma_u^2\int\bar{\bar{W}}^2$

Lemma 3.3 $(1/T^{1/2-\delta})\hat{S} \to 0$

Lemma 3.4 $(1/T^{2-2\delta})\Sigma(\hat{S}_{t-1} - \hat{S}_{-1})^2 \to \sigma_u^2C^2\int\bar{\bar{W}}^2$

Lemma 3.5 $T^{-1}\,\Sigma(\hat{S}_{t-1} - \hat{S}_{-1})\varepsilon_t \to -\sigma_u^2$ for all δ.

(proof)

(1) $T^{-1}\Sigma y_{t-1}\varepsilon_t \to -\sigma_u^2$

(2) $T^{-1}\,\hat{a}\Sigma\varepsilon_t = (T^{-1/2+\delta}\hat{a})(T^{-1/2-\delta})\Sigma\varepsilon_t \to 0$

(3) $(1/T)\,\hat{b}\Sigma t\varepsilon_t = (T^{-1/2+\delta})(t\hat{b})(1/T^{3/2+\delta})\Sigma t\varepsilon_t \to 0$

Using (1), (2), and (3) we get $(1/T)\Sigma\hat{S}_{t-1}\varepsilon_t \to -\sigma_u^2$.

Since $T^{-1}\hat{S}_{-1}\Sigma\varepsilon_t = (T^{-1/2+\delta})\hat{S}_{-1}(1/T^{1/2+\delta})\Sigma\varepsilon_t \to 0$, the result comes. □

Theorem 3.6 $T^{1-2\delta}(\hat{\beta}_\tau - 1) \to -1/C^2\int\bar{\bar{W}}^2$

Lemma 3.7 $s^2 \to 2\sigma_u^2$

Corollary 3.8 $T^{\delta}\hat{t}_\tau \to -1/(2C^2\int\bar{\bar{W}}^2)^{1/2}$

Case 2: $\delta = 1/2$

Lemma 3.9 $\hat{S}_{-1} \to C\sigma_u(\int W - 4\int W + 6\int rW + 3\int W - 6\int rW) = 0$

Lemma 3.10 $T^{-1}\,\Sigma(\hat{S}_{t-1} - \hat{S}_{-1})^2 \to \sigma_u^2(1 + C^2\int\bar{\bar{W}}^2)$

Theorem 3.11 $\hat{\phi}_\tau = (\hat{\beta}_\tau - 1) \to -1/(1 + C^2\int\bar{\bar{W}}^2)$

Lemma 3.12 $s^2 \to \sigma_u^2\{2 - 1/(1 + C^2\int\bar{\bar{W}}^2)\}$

Corollary 3.13 $T^{-1/2}\hat{t}_\tau \to -1/(1 + 2C^2\int\bar{\bar{W}}^2)^{1/2}$

Case 3: $1/2 < \delta < 3/4$

<u>Lemma 3.14</u> $\hat{S}_{[Tr]} \to u_{[Tr]}$

(proof)

$$\hat{S}_{[Tr]} = u_{[Tr]} + (C/T^{\delta})S_{[Tr]} - \hat{a} - rT\hat{b}$$

$$= u_{[Tr]} + (C/T^{\delta-1/2})T^{-1/2}S_{[Tr]} - T^{-\delta+1/2}(T^{\delta-1/2}\hat{a}) - T^{-\delta+1/2}(T^{\delta-1/2}rT\hat{b})$$

$$\to u_{[Tr]} \quad \square$$

<u>Lemma 3.15</u> $T^{-1}\Sigma(\hat{S}_{t-1} - \hat{\bar{S}}_{-1})^2 \to \sigma_u^2$

<u>Lemma 3.16</u> $\hat{\phi}_\tau = (\hat{\beta}_\tau - 1) \to -1$

Since the above lemma implies that $\hat{\beta} \to 0$ when $\delta > 1/2$, we use (A18) to derive the asymptotics for $\hat{\beta}_\tau$.

(A18) $$\hat{\beta}_\tau = \Sigma(\hat{S}_{t-1} - \hat{\bar{S}}_{-1})y_t \,/\, \Sigma(\hat{S}_{t-1} - \hat{\bar{S}}_{-1})^2$$

<u>Lemma 3.17</u> $(1/T^{2-2\delta})\Sigma(\hat{S}_{t-1} - \hat{\bar{S}}_{-1})y_t \to \sigma_u^2 C^2 \int \bar{\bar{W}}^2$

(proof)

$$\Sigma\hat{S}_{t-1}y_t = \Sigma\{y_{t-1} - \hat{a} - \hat{b}(t-1)\}y_t = \Sigma y_{t-1}y_t - \hat{a}\Sigma y_t - \hat{b}\Sigma(t-1)y_t$$

(1) $(1/T^{2-2\delta})\Sigma y_{t-1}y_t \to \sigma_u^2 C^2 \int W^2$

(2) $(-1/T^{2-2\delta})\hat{a}\Sigma y_{t-1} \to -C^2\sigma_u^2(4\int W - 6\int rW) \int W$

(3) $(-1/T^{2-2\delta})\, \hat{b}\Sigma(t-1)y_t \to -C^2\sigma_u^2(-6\int W + 12\int rW) \int rW$

Using (1), (2), and (3) we show that $(1/T^{2-2\delta})\Sigma \hat{S}_{t-1}y_t \to C^2\sigma_u^2 \int \bar{\bar{W}}^2$

Since $(1/T^{2-2\delta})\, \hat{\bar{S}}_{t-1}\Sigma y_t \to 0$, we get the result. $\square$

<u>Theorem 3.18</u> $T^{2\delta-1}\, \hat{\beta}_\tau \to C^2 \int \bar{\bar{W}}^2$

<u>Lemma 3.19</u> $s^2 \to \sigma_u^2$ for $\delta > 1/2$

<u>Corollary 3.20</u> $\hat{t}_\tau + T^{1/2} \rightarrow C^2\int \bar{\bar{W}}^2$

Case 4: $\delta = 3/4$

<u>Lemma 3.21</u> $T^{-1/2}\Sigma(\hat{S}_{t-1} - \hat{S}_{-1})y_t \rightarrow \sigma_u^2\{W(1) + C^2\int\bar{W}^2\}$
(proof)

(1) $T^{-1/2}\Sigma y_{t-1}y_t = T^{-1/2}\Sigma y_{t-1}y_t \rightarrow \sigma_u^2\{W(1) + C^2\int W^2\}$

(2) $-T^{-1/2}\hat{a}\Sigma y_t \rightarrow -C^2\sigma_u^2(4\int W - 6\int rW)\int W$

(3) $-T^{-1/2}\hat{b}\Sigma(t - 1)y_t \rightarrow -C^2\sigma_u^2(-6\int W + 12\int rW)\int rW$

Using (1), (2), and (3) we can show that $T^{-1/2}\Sigma\hat{S}_{t-1}y_t \rightarrow \sigma_u^2\{W(1) + C^2\int \bar{\bar{W}}^2\}$. Since $-T^{-1/2}\hat{S}_{-1}\Sigma y_t \rightarrow 0$ the results comes. □

<u>Theorem 3.22</u> $T^{1/2}\hat{\beta}_\tau \rightarrow W(1) + C^2\int \bar{\bar{W}}^2$

<u>Corollary 3.23</u> $\hat{t}_\tau + T^{1/2} \rightarrow W(1) + C^2\int \bar{\bar{W}}^2$

Case 5: $\delta > 3/4$

<u>Lemma 3.24</u> $T^{-1/2}\Sigma(\hat{S}_{t-1} - \hat{S}_{-1})y_t \rightarrow \sigma_u^2 W(1)$

<u>Theorem 3.25</u> $T^{1/2}\hat{\beta}_\tau \rightarrow W(1)$

<u>Corollary 3.26</u> $\hat{t}_\tau + T^{1/2} \rightarrow W(1)$

[4] $\tilde{\rho}$ and $\tilde{\tau}$ tests

Regression equation (7) can be transformed into:

(A19) $$\Delta y_t = (\beta - 1)y_{t-1} + \psi(1 - \beta) + \xi(t + \beta - t\beta) + \varepsilon_t$$

$$\approx \text{intercept} + \phi\tilde{S}_{t-1} + \text{error}$$

where $\tilde{S}_{t-1} = y_{t-1} - \tilde{\psi}_X - \tilde{\xi}(t - 1)$, $t \geq 2$, with $\psi_X = \psi + X_0$. As before, we use the same symbol ε_t as the error in (A19) without loss of generality. Using the DGP of (1) and (2) we get

(A20) $$X_t = X_0 + u_t + (1 + \theta)S_{t-1}$$

(A21) $$y_t = \psi_X + \xi t + (1 + \theta)S_{t-1},$$

The restricted MLE's of $\tilde{\psi}$ and $\tilde{\xi}$ are derived under the null as follow:

(A22) $$\tilde{\xi} = \text{mean } \Delta y = (y_T - y_1)/(T - 1)$$

(A23) $$\tilde{\psi}_X = y_1 - \tilde{\xi}$$

Substituting y_t from (A21) into (A22) and (A23) we get

(A22)' $$\tilde{\xi} = \xi + (1 + \theta)\bar{u} + o_p(1)$$

(A23)' $$\tilde{\psi}_X = \psi_X - (1 + \theta)\bar{u} + u_1$$

By using the above results we show that

(A24) $$\tilde{S}_{t-1} = y_{t-1} - \tilde{\psi}_x - \tilde{\xi}(t - 1)$$

$$= (1 + \theta)\sum_{j=1}^{t-2} (u_j - \bar{u}) + (u_{t-1} - u_1)$$

or

$$\tilde{S}_{[Tr]} = (1 + \theta)S_{[Tr]} - [Tr](1 + \theta)(1/T)S_T + u_{[Tr]} - u_1$$

$$= (C/T^{\delta})S_{[Tr]} - r(C/T^{\delta})S_T + u_{[Tr]} - u_1$$

We use (A19) to derive the asymptotics for the $\tilde{\rho}$ and $\tilde{\tau}$ statistics. Note that $\tilde{\rho}$ statistic is defined as $T\tilde{\phi}$, where

(A25) $\tilde{\phi} = \tilde{\beta} - 1 = \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\Delta y_t \,/\, \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2$

$$= \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\varepsilon_t \,/\, \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2$$

Now, s^2 is defined as

(A26) $s^2 = (1/T)\Sigma\{(\Delta y_t - \Delta\bar{y}) - \tilde{\phi}(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\}^2$

Case 1: $0 < \delta < 1/2$

Lemma 4.1 $(1/T^{2-2\delta})\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2 \to \sigma_u^2 C^2 \int \bar{V}^2$
(proof)

$$(1/T^{2-2\delta})\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2 = (1/T^{2-2\delta})\Sigma\tilde{S}_{t-1}^2 - (T^{-1/2+\delta}\bar{\tilde{S}}_{-1}\}^2$$

Using (A22)' we can show that

$$T^{-1/2+\delta}S_{[Tr]} = (C/\sqrt{T})S - r(C/\sqrt{T})S_T + o_p(1)$$

$$\to \sigma_u W(r) - r\sigma_u W(1) = \sigma_u V(r)$$

$$\bar{\tilde{S}}_{-1} = \bar{y}_{-1} - \tilde{\psi}_x - \xi(\bar{t}-1)$$

$$= \bar{u}_{-1} - u_1 + (C/T^\delta)(1/T)\Sigma S_{t-2} + (C/T^\delta)\bar{u} - (C/T^\delta)S_T(1/T)(\bar{t}-1)$$

Therefore,

$$T^{-1/2+\delta}\bar{\tilde{S}} \to (C/T^{3/2})\Sigma S_{t-2} - (C/\sqrt{T})S_T\,(1/2) + o_p(1)$$

$$\to C\sigma_u\int W - (1/2)C\sigma_u W(1)$$

Combining the above results we get

$$T^{-1/2+\delta}(\tilde{S}_{[Tr]} - \bar{\tilde{S}}) \to C\sigma_u\{W(r) + (1/2 - r)W(1) - \int W\} = C\sigma_u \bar{V}(r)$$

Therefore, the result follows. □

Lemma 4.2 $T^{-1}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\varepsilon_t \to -\sigma_u^2$ for all δ.

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(proof)

$$T^{-1}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\varepsilon_t$$

$$= T^{-1}\Sigma\{(C/T^{\delta})S_{t-1} - r(C/T^{\delta})S_T + u_{t-1} - u_1 - (C/T^{\delta})T^{-1}\Sigma S_{t-2}$$

$$+ (1/2)\cdot(C/T^{\delta})S_T\}\cdot(u_t + \theta u_{t-1})$$

$$\rightarrow (1/T)\theta\Sigma u_{t-1}^2 \rightarrow -\sigma_u^2 \quad \square$$

<u>Theorem 4.3</u> $T^{1-2\delta}(\tilde{\beta} - 1) \rightarrow -1/C^2\int\bar{V}^2$

<u>Lemma 4.4</u> $s^2 \rightarrow 2\sigma_u^2$

<u>Corollary 4.5</u> $T^{\delta}\tilde{\tau} \rightarrow -1/(2C^2\int\bar{V}^2)^{1/2}$

Case 2: $\delta = 1/2$

<u>Lemma 4.6</u> $T^{-1}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2 \rightarrow \sigma_u^2(1 + C^2\int\bar{V}^2)$

<u>Theorem 4.7</u> $\tilde{\beta} - 1 \rightarrow -1/(1 + C^2\int\bar{V}^2)$

<u>Lemma 4.8</u> $s^2 \rightarrow \sigma_u^2\{2 - 1/(1 + C^2\int\bar{V}^2)\}$

<u>Corollary 4.9</u> $T^{1/2}\,\tilde{\tau} \rightarrow -1/(1 + 2C^2\int\bar{V}^2)^{1/2}$

Case 3: $1/2 < \delta < 3/4$

<u>Lemma 4.10</u> $\tilde{S}_{[Tr]} \rightarrow u_{[Tr]} - u_1$

<u>Lemma 4.11</u> $T^{-1}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2 \rightarrow \sigma_u^2$

(proof)

Using Lemma 4.10 and the following result

$$\bar{\tilde{S}} = \bar{u}_{-1} - u_1 + (C/T^{\delta})(1/T)\Sigma S_{t-2} + (C/T^{\delta})\bar{u} - (1/2)\cdot(C/T^{\delta})S_T \rightarrow -u_1,$$

then we can show that $\tilde{S}_{[Tr]} - \bar{\tilde{S}} \rightarrow u_{[Tr]}$. Therefore,

$$T^{-1}\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2 = (1/T)\Sigma u_{t-1}^2 \rightarrow \sigma_u^2 \quad \square$$

Since $\tilde{\phi} \to -1$ ($\tilde{\beta} \to 0$) when $\delta > 1/2$, we use (A27) to derive the relevant asymptotics.

(A27) $$\tilde{\beta} = \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\tilde{S}_t \,/\, \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2$$
$$= \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}}) \,/\, \Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})^2$$

where we use the facts that $\tilde{S}_t = y_t - \xi$ and $\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})\xi = 0$.

<u>Lemma 4.12</u> $(1/T^{2-2\delta})\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}}) \to \sigma_u^2 C^2 \int \bar{V}^2$

(proof)

$$(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}})$$
$$= (1+\theta)^2(S_{t-2} - \bar{S}) + (1+\theta)^2\,\bar{u}^2(t - \bar{t}) - 2(1+\theta)^2\bar{u}(S_{t-2} - \bar{S})(t - \bar{t})$$
$$+ (1+\theta)(S_{t-2} - \bar{S})(u_t - \bar{u}) - (1+\theta)\,\bar{u}(t - \bar{t})(u_t - \bar{u}) + (1+\theta)(S_{t-2} - \bar{S})(u_{t-1} - \bar{u})$$
$$- (1+\theta)\,\bar{u}(t - \bar{t})(u_{t-1} - \bar{u}) + (u_{t-1} - \bar{u})(u_t - \bar{u}) + (u_{t-1} - \bar{u})(1+\theta)u_{t-1}$$
$$+ (1+\theta)^2(S_{t-2} - \bar{S})u_{t-1} - (1+\theta)^2\bar{u}(t - \bar{t})u_{t-1}$$

Therefore,

$$(1/T^{2-2\delta})\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}})$$
$$\to C^2/T^2\Sigma(S_{t-2} - \bar{S})^2 + (1/T)S_T^2(1/T^3)\Sigma(t - \bar{t})^2$$
$$- (2C^2/\sqrt{T})S_T(1/T^{5/2})\Sigma(S_{t-2} - \bar{S})(t - \bar{t})$$
$$\to C^2\sigma_u^2\int \bar{V}^2 \quad \square$$

<u>Theorem 4.13</u> $T^{2\delta-1}\tilde{\beta} \to C^2\int \bar{V}^2$

<u>Lemma 4.14</u> $s^2 \to \sigma_u^2$ for $\delta > 1/2$.

<u>Corollary 4.15</u> $\tilde{\tau} + T^{1/2} \to C^2\int \bar{V}^2$

Case 4: $\delta = 3/4$

Lemma 4.16 $T^{-1/2}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}}) \rightarrow \sigma_u^2\{W(1) + C^2\int \bar{V}^2\}$

Theorem 4.17 $T^{1/2}\tilde{\beta} \rightarrow W(1) + C^2\int \bar{V}^2$

Corollary 4.18 $\tilde{\tau} + T^{1/2} \rightarrow W(1) + C^2\int \bar{V}^2$

Case 5: $3/4 < \delta < 1$

Lemma 4.19 $T^{-1/2}\,\Sigma(\tilde{S}_{t-1} - \bar{\tilde{S}}_{-1})(\tilde{S}_t - \bar{\tilde{S}}) \rightarrow \sigma_u^2 W(1)$

Theorem 4.20 $T^{1/2}\tilde{\beta} \rightarrow W(1)$

Corollary 4.21 $\tilde{\tau} + T^{1/2} \rightarrow W(1)$

Table 4-1 (a)

5 % Fractiles of the Actual Sampling Distribution

T	$\hat{\rho}$	$\hat{\rho}_{\mu}$	$\hat{\rho}_{\tau}$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_{\mu}$	$\hat{\tau}_{\tau}$	$\tilde{\tau}$
$\theta = -0.5$								
25	−18.37	−23.87	−25.87	−25.85	−3.86	−4.83	−5.64	−5.07
50	−25.50	−33.72	−42.87	−38.79	−4.12	−5.14	−6.07	−5.54
100	−31.78	−45.10	−59.73	−53.05	−4.32	−5.40	−6.49	−5.97
250	−38.10	−57.30	−80.60	−70.60	−4.50	−5.70	−7.00	−6.39
500	−40.05	−62.20	−92.20	−79.30	−4.60	−5.80	−7.20	−6.55
1000	−41.91	−65.30	−98.10	−84.90	−4.70	−5.90	−7.20	−6.69
$\theta = -0.8$								
25	−27.36	−30.76	−32.76	−30.53	−5.67	−6.44	−6.86	−6.27
50	−47.83	−55.07	−58.00	−54.37	−6.84	−7.68	−8.26	−7.65
100	−81.10	−92.80	−101.90	−95.60	−8.28	−9.31	−10.10	−9.51
250	−150.44	−180.71	−206.04	−191.13	−10.37	−11.90	−13.21	−12.45
500	−210.61	−270.20	−328.80	−301.20	−11.60	−13.60	−15.70	−14.66
1000	−264.55	−362.10	−473.40	−425.70	−11.60	−14.90	−17.60	−16.36
$\theta = -0.9$								
25	−29.81	−32.14	−33.48	−31.31	−6.31	−6.87	−7.10	−6.38
50	−54.70	−58.47	−61.04	−57.58	−7.88	−8.52	−8.82	−8.16
100	−100.40	−107.60	−112.43	−106.65	−10.10	−10.80	−11.30	−10.65
250	−220.12	−238.50	−252.06	−239.63	−14.04	−15.10	−15.91	−15.15
500	−382.37	−425.00	−457.80	−435.10	−17.60	−19.30	−20.60	−19.59
1000	−606.31	−718.50	−806.10	−755.00	−21.10	−23.70	−26.00	−24.62
$\theta = -0.95$								
25	−30.94	−32.60	−33.64	−31.49	−6.63	−7.01	−7.15	−6.45
50	−57.68	−59.82	−61.04	−58.44	−8.39	−8.83	−8.98	−8.35
100	−108.43	−113.50	−115.85	−110.19	−10.96	−11.47	−11.71	−11.00
250	−250.96	−262.11	−269.05	−258.91	−15.91	−16.61	−17.04	−16.33
500	−471.70	−494.00	−510.10	−491.10	−21.20	−22.10	−22.80	−21.95
1000	−872.40	−925.00	−965.40	−928.50	−27.90	−29.40	−30.60	−29.47
$\theta = -0.99$								
25	−31.67	−32.76	−33.68	−31.52	−6.86	−7.05	−7.17	−6.46
50	−59.81	−60.88	−62.16	−58.71	−8.78	−8.94	−9.04	−8.40
100	−114.12	−116.04	−117.21	−111.46	−11.62	−11.80	−11.86	−11.17
250	−271.30	−275.20	−276.70	−268.64	−17.30	−17.58	−17.64	−16.79
500	−525.61	−533.60	−536.30	−519.70	−23.70	−24.00	−24.10	−23.21
1000	−1025.2	−1040.0	−1048.0	−1020.0	−32.50	−33.00	−33.20	−32.23

Table 4-1 (b)

95 % Fractiles of the Actual Sampling Distribution

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
θ = −0.5								
25	−0.08	−3.71	−9.03	−7.28	−0.05	−1.28	−2.15	−1.98
50	−0.20	−4.36	−12.07	−9.57	−0.12	−1.43	−2.50	−2.25
100	−0.29	−4.77	−14.10	−11.30	−0.17	−1.48	−2.69	−2.43
250	−0.33	−5.00	−15.60	−12.50	−0.30	−1.50	−2.80	−2.52
500	−0.37	−5.20	−16.10	−13.00	−0.30	−1.60	−2.90	−2.57
1000	−0.40	−5.20	−16.40	−13.10	−0.30	−1.60	−2.90	−2.59
θ = −0.8								
25	−4.90	−12.61	−16.34	−12.27	−1.54	−2.82	−3.30	−2.76
50	−7.10	−21.19	−31.89	−23.50	−1.85	−3.62	−4.73	−3.84
100	−8.83	−29.34	−54.61	−40.30	−2.05	−4.12	−6.07	−5.00
250	−10.33	−37.44	−87.75	−66.70	−2.20	−4.49	−7.27	−6.19
500	−11.09	−40.40	−107.60	−83.90	−2.30	−4.60	−7.80	−6.75
1000	−11.20	−42.40	−121.80	−95.20	−2.30	−4.70	−8.10	−7.08
θ = −0.9								
25	−10.43	−16.04	−17.76	−13.19	−2.53	−3.35	−3.53	−2.90
50	−19.13	−33.56	−37.73	−27.40	−3.39	−4.98	−5.40	−4.25
100	−29.05	−61.62	−76.58	−53.66	−4.08	−6.65	−7.82	−6.04
250	−40.79	−108.13	−171.20	−120.80	−4.67	−8.30	−11.38	−8.89
500	−45.96	−139.50	−271.70	−198.20	−4.90	−9.00	−13.54	−11.13
1000	−49.56	−163.00	−378.80	−285.70	−5.10	−9.50	−15.30	−12.88
θ = −0.95								
25	−13.92	−16.93	−18.10	−13.39	−3.09	−3.50	−3.59	−2.91
50	−30.24	−37.51	−39.24	−28.46	−4.65	−5.44	−5.57	−4.35
100	−57.27	−78.39	−82.82	−57.46	−6.32	−8.01	−8.34	−6.33
250	−107.86	−185.25	−212.07	−143.41	−8.28	−12.12	−13.64	−9.96
500	−145.26	−306.70	−406.50	−272.30	−9.20	−14.90	−18.50	−13.71
1000	−179.04	−449.40	−711.90	−494.00	−9.95	−17.10	−23.60	−18.09
θ = −0.99								
25	−15.94	−17.20	−18.23	−13.45	−3.42	−3.55	−3.60	−2.91
50	−37.00	−38.70	−39.68	−28.75	−5.44	−5.58	−5.62	−4.40
100	−80.77	−83.53	−84.64	−58.68	−8.26	−8.46	−8.51	−6.40
250	−215.30	−222.90	−224.70	−149.60	−13.80	−14.28	−14.30	−10.31
500	−433.05	−458.80	−462.70	−301.00	−19.60	−20.60	−20.80	−14.64
1000	−821.89	−923.80	−940.80	−606.10	−26.50	−29.30	−29.80	−20.95

Table 4-2

Fractiles of the Predicted Distribution for $\delta = 1/4$

5 % fractile

θ	−0.5	−0.8	−0.9	−0.95	−0.99
$\hat{\rho}$	−71.43	−446.43	−1785.0	−7143.0	−178571
$\hat{\rho}_\mu$	−111.11	−694.44	−2777.8	−11111.0	−277778
$\hat{\rho}_\tau$	−173.91	−1087.00	−4347.8	−17391.0	−434783
$\tilde{\rho}$	−148.15	−925.93	−3703.7	−14815.0	−370371
$\hat{\tau}$	−5.98	−14.94	−29.88	−59.76	−298.81
$\hat{\tau}_\mu$	−7.45	−18.63	−37.27	−74.54	−372.68
$\hat{\tau}_\tau$	−9.33	−23.31	−46.63	−93.25	−466.25
$\tilde{\tau}$	−8.61	−21.52	−43.03	−86.07	−430.33

95 % fractile

θ	−0.5	−0.8	−0.9	−0.95	−0.99
$\hat{\rho}$	−2.42	−15.14	−60.57	−242.28	−6056.9
$\hat{\rho}_\mu$	−8.73	−54.59	−218.34	−873.36	−21834.0
$\hat{\rho}_\tau$	−27.21	−170.07	−680.27	−2721.10	−68027.0
$\tilde{\rho}$	−21.51	−134.41	−537.63	−2150.50	−53764.0
$\hat{\tau}$	−1.10	−2.75	−5.50	−11.01	−55.03
$\hat{\tau}_\mu$	−2.09	−5.22	−10.45	−20.90	−92.69
$\hat{\tau}_\tau$	−3.69	−9.22	−18.44	−36.89	−184.43
$\tilde{\tau}$	−3.28	−8.20	−16.40	−32.79	−163.96

Table 4-3 (a)

5 % Fractiles of the Predicted Distribution for $\delta = 1/2$

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
$\theta = -0.5$								
25	-18.52	-20.41	-21.86	-21.39	-3.83	-4.15	-4.41	-4.32
50	-29.41	-34.48	-38.83	-37.38	-4.56	-5.13	-5.63	-5.46
100	-41.67	-52.63	-63.49	-59.70	-5.13	-5.98	-6.82	-6.52
250	-55.56	-76.92	-102.56	-93.02	-5.59	-6.74	-8.03	-7.56
500	-62.50	-90.91	-129.03	-114.29	-5.77	-7.07	-8.61	-8.03
1000	-66.67	-100.00	-148.15	-129.03	-5.87	-7.26	-8.94	-8.31
$\theta = -0.8$								
25	-23.67	-24.13	-24.44	-24.34	-4.74	-4.83	-4.89	-4.87
50	-44.96	-46.64	-47.80	-47.44	-6.39	-6.61	-6.77	-6.72
100	-81.69	-87.41	-91.58	-90.25	-8.31	-8.81	-9.19	-9.07
250	-160.26	-183.82	-203.25	-196.85	-10.86	-12.06	-13.09	-12.74
500	-235.85	-290.70	-342.47	-324.68	-12.42	-14.32	-16.14	-15.50
1000	-308.64	-409.84	-520.83	-480.77	-13.51	-16.05	-18.77	-17.79
$\theta = -0.9$								
25	-24.65	-24.78	-24.86	-24.83	-4.93	-4.96	-4.97	-4.97
50	-48.64	-49.12	-49.43	-49.33	-6.88	-6.95	-6.99	-6.98
100	-94.70	-96.53	-97.75	-97.37	-9.48	-9.67	-9.78	-9.74
250	-219.30	-229.36	-236.41	-234.19	-13.98	-14.56	-14.97	-14.84
500	-390.60	-423.73	-448.43	-440.53	-17.90	-19.17	-20.16	-19.84
1000	-641.00	-735.30	-813.01	-787.40	-21.72	-24.11	-26.17	-25.48
$\theta = -0.95$								
25	-24.91	-24.94	-24.96	-24.96	-4.98	-4.99	-4.99	-4.99
50	-49.65	-49.78	-49.86	-49.83	-7.02	-7.04	-7.05	-7.05
100	-98.62	-99.11	-99.43	-99.33	-9.86	-9.91	-9.94	-9.93
250	-241.55	-244.50	-246.46	-245.85	-15.26	-15.47	-15.59	-15.55
500	-467.29	-478.47	-486.03	-483.68	-20.80	-21.42	-21.74	-21.64
1000	-877.19	-917.43	-945.63	-936.77	-27.20	-29.11	-29.95	-29.68
$\theta = -0.99$								
25	-25.00	-25.00	-25.00	-25.00	-5.00	-5.00	-5.00	-5.00
50	-49.99	-49.99	-49.99	-49.99	-7.07	-7.07	-7.07	-7.07
100	-99.94	-99.96	-99.98	-99.97	-9.99	-10.00	-10.00	-10.00
250	-249.65	-249.78	-249.86	-249.83	-15.79	-15.80	-15.80	-15.80
500	-498.60	-499.10	-499.43	-499.33	-22.30	-22.32	-22.34	-22.33
1000	-994.40	-996.41	-997.71	-997.31	-31.45	-31.51	-31.55	-31.54

Table 4–3 (b)

95 % Fractiles of the Predicted Distribution for δ = 1/2

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
θ = −0.5								
25	−2.21	−6.47	−13.03	−11.56	−1.07	−1.93	−2.97	−2.74
50	−2.31	−7.43	−17.62	−15.04	−1.09	−2.00	−3.27	−2.97
100	−2.37	−8.03	−21.39	−17.70	−1.09	−2.05	−3.46	−3.12
250	−2.40	−8.44	−24.54	−19.80	−1.10	−2.07	−3.59	−3.21
500	−2.41	−8.58	−25.81	−20.62	−1.10	−2.08	−3.64	−3.24
1000	−2.42	−8.66	−26.49	−21.05	−1.10	−2.09	−3.66	−3.26
θ = −0.8								
25	−9.43	−17.15	−21.80	−21.08	−2.41	−3.61	−4.40	−4.27
50	−11.62	−26.10	−38.64	−36.44	−2.56	−4.20	−5.61	−5.35
100	−13.15	−35.31	−62.97	−57.34	−2.65	−4.63	−6.78	−6.34
250	−14.28	−44.80	−101.22	−87.41	−2.71	−4.96	−7.97	−7.28
500	−14.70	−49.21	−126.90	−105.93	−2.73	−5.09	−8.53	−7.70
1000	−14.92	−51.76	−145.35	−118.48	−2.74	−5.15	−8.85	−7.94
θ = −0.9								
25	−17.70	−22.43	−24.11	−23.89	−3.70	−4.51	−4.83	−4.78
50	−27.39	−40.68	−46.58	−45.75	−4.34	−5.86	−6.60	−6.49
100	−37.72	−68.59	−87.18	−84.32	−4.82	−7.22	−8.79	−8.54
250	−48.76	−116.55	−182.82	−170.65	−5.20	−8.72	−12.00	−11.38
500	−54.03	−151.98	−288.18	−259.07	−5.34	−9.47	−14.23	−13.22
1000	−57.11	−179.21	−404.86	−349.65	−5.42	−9.92	−15.93	−14.56
θ = −0.95								
25	−22.66	−24.30	−24.77	−24.71	−4.55	−4.86	−4.95	−4.94
50	−41.45	−47.29	−49.10	−48.86	−5.95	−6.70	−6.94	−6.91
100	−70.78	−89.73	−96.46	−95.56	−7.40	−9.02	−9.65	−9.57
250	−123.04	−194.36	−228.96	−223.96	−9.03	−12.61	−14.53	−14.24
500	−163.20	−317.97	−422.39	−405.68	−9.88	−15.27	−19.12	−18.47
1000	−195.03	−466.20	−731.26	−682.59	−10.40	−17.43	−24.01	−22.76
θ = −0.99								
25	−24.90	−24.97	−24.99	−24.99	−4.98	−4.99	−5.00	−5.00
50	−49.59	−49.89	−49.96	−49.95	−7.01	−7.05	−7.07	−7.07
100	−98.38	−99.54	−99.85	−99.81	−9.84	−9.95	−9.99	−9.98
250	−240.09	−247.17	−249.09	−248.84	−15.20	−15.63	−15.75	−15.74
500	−461.87	−488.81	−496.35	−495.39	−20.72	−21.85	−22.20	−22.15
1000	−858.30	−956.21	−985.51	−981.74	−27.42	−30.17	−31.17	−31.03

Table 4-4 (a)

5 % Fractiles of the Predicted Distribution for $\delta = 5/8$

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
$\theta = -0.5$								
25	-16.25	-19.38	-21.41	-20.78	-3.25	-3.88	-4.28	-4.16
50	-15.00	-27.50	-35.63	-33.13	-2.12	-3.89	-5.04	-4.68
100	40	-10.00	-42.50	-32.50	4.00	-1.00	-4.25	-3.25
250	625	312.5	109.4	171.9	39.53	19.76	6.92	10.87
500	3000	1750.0	937.5	1187.5	134.16	78.26	41.93	53.11
1000	13000	8000.0	4750.0	5750.0	411.10	252.98	150.21	181.83
$\theta = -0.8$								
25	-23.60	-24.10	-24.43	-24.33	-4.72	-4.82	-4.89	-4.86
50	-44.40	-46.40	-47.70	-47.30	-6.28	-6.56	-6.75	-6.69
100	-77.60	-85.60	-90.80	-89.20	-7.76	-8.56	-9.08	-8.92
250	-110.00	-160.00	-192.50	-182.50	-6.96	-10.12	-12.18	-11.54
500	60	-140.00	-270.00	-230.00	2.68	-6.26	-12.08	-10.29
1000	1240	440.00	-80.00	80.00	39.21	13.91	-2.53	2.53
$\theta = -0.9$								
25	-24.65	-24.78	-24.86	-24.83	-4.93	-4.96	-4.97	-4.97
50	-48.60	-49.10	-49.43	-49.33	-6.87	-6.94	-6.99	-6.98
100	-94.40	-96.40	-97.70	-97.30	-9.44	-9.64	-9.77	-9.73
250	-215.00	-227.50	-235.63	-233.13	-13.60	-14.39	-14.90	-14.74
500	-360.00	-410.00	-442.50	-432.50	-16.10	-18.34	-19.79	-19.34
1000	-440.00	-640.00	-770.00	-730.00	-13.91	-20.24	-24.35	-23.09
$\theta = -0.95$								
25	-24.91	-24.94	-24.96	-24.96	-4.98	-4.99	-4.99	-4.99
50	-49.65	-49.78	-49.86	-49.83	-7.02	-7.04	-7.05	-7.05
100	-98.60	-99.10	-99.43	-99.33	-9.86	-9.91	-9.94	-9.93
250	-241.25	-244.38	-246.41	-245.78	-15.26	-15.46	-15.58	-15.55
500	-465.00	-477.50	-485.63	-483.13	-20.80	-21.35	-21.72	-21.61
1000	-860.00	-910.00	-942.50	-932.50	-27.20	-28.77	-29.80	-29.49
$\theta = -0.99$								
25	-25.00	-25.00	-25.00	-25.00	-5.00	-5.00	-5.00	-5.00
50	-49.99	-49.99	-49.99	-49.99	-7.07	-7.07	-7.07	-7.07
100	-99.94	-99.96	-99.98	-99.97	-9.99	-10.00	-10.00	-10.00
250	-249.65	-249.78	-249.86	-249.83	-15.79	-15.80	-15.80	-15.80
500	-498.60	-499.10	-499.43	-499.33	-22.30	-22.32	-22.34	-22.33
1000	-994.40	-996.40	-997.70	-997.30	-31.45	-31.51	-31.55	-31.54

Table 4-4 (b)

95 % Fractiles of the Predicted Distribution for $\delta = 5/8$

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
$\theta = -0.5$								
25	233	47	-2	7	47	9	-0.4	0.8
50	982	236	42	66	139	33	5.9	9.4
100	4028	1045	268	365	403	105	26.8	36.5
250	25547	6906	2047	2656	1616	437	129.5	168.0
500	102687	28125	8688	11125	4592	1258	388.5	497.5
1000	411750	113500	35750	45500	13021	3590	1130.5	1438.8
$\theta = -0.8$								
25	16.3	-13.6	-21.3	-20.4	3.3	-2.7	-4.27	-4.07
50	115.1	-4.2	-35.3	-31.4	16.3	-0.6	-4.99	-4.44
100	560.4	83.2	-41.2	-25.6	56.0	8.3	-4.12	-2.56
250	3877.5	895.0	117.5	215.0	245.2	56.6	7.43	13.60
500	16010.0	4080.0	970.0	1360.0	716.0	182.5	43.38	60.82
1000	65034.0	17320.0	4880.0	6440.0	2056.8	547.7	154.48	203.65
$\theta = -0.9$								
25	-14.7	-22.1	-24.08	-23.84	-2.94	-4.43	-4.82	-4.77
50	-8.7	-38.6	-46.33	-45.35	-1.23	-5.45	-6.55	-6.41
100	65.1	-54.2	-85.30	-81.42	6.5	-5.42	-8.53	-8.14
250	781.9	36.3	-158.13	-133.75	49.5	2.29	-10.00	-8.46
500	3627.5	645.0	-132.50	-35.00	162.2	28.85	-5.93	-1.57
1000	15510.0	3580.0	470.00	860.00	490.5	113.21	14.86	27.20
$\theta = -0.95$								
25	-22.42	-24.28	-24.77	-24.71	-4.48	-4.86	-4.95	-4.94
50	-39.68	-47.14	-49.08	-48.84	-5.61	-6.67	-6.94	-6.91
100	-58.73	-88.55	-96.33	-95.30	-5.87	-8.86	-9.63	-9.54
250	8.0	-178.44	-227.03	-220.94	0.50	-11.29	-14.36	-13.97
500	531.9	-213.75	-408.13	-387.75	23.79	-9.56	-18.25	-17.16
1000	3127.5	145.00	-632.50	-535.00	98.90	4.59	-20.00	-16.92
$\theta = -0.99$								
25	-24.90	-24.97	-24.99	-24.99	-4.98	-4.99	-5.00	-5.00
50	-49.59	-49.89	-49.96	-49.95	-7.01	-7.05	-7.07	-7.06
100	-98.35	-99.54	-99.85	-99.81	-9.84	-9.95	-9.99	-9.98
250	-239.68	-247.14	-249.08	-248.84	-15.16	-15.63	-15.75	-15.74
500	-458.73	-488.55	-496.33	-495.35	-20.52	-21.85	-22.20	-22.15
1000	-834.90	-954.20	-985.30	-981.40	-26.40	-30.17	-31.16	-31.03

Table 4-5 (a)

5 % Fractiles of the Predicted Distribution for $\delta = 3/4$

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
$\theta = -0.5$								
25	-16.50	-19.5	-24.5	-23.0	-3.30	-3.90	-4.90	-4.60
50	-14.64	-27.4	-37.3	-34.4	-2.07	-3.87	-5.27	-4.87
100	42.00	-9.0	-42.0	-32.0	4.20	-0.90	-4.20	-3.20
250	-25.48	305.0	116.8	180.1	40.29	19.29	7.39	11.39
500	-182.48	1555.0	971.3	1259.8	136.44	69.54	43.44	56.34
1000	-550.96	13205.0	4888.2	6045.6	417.58	417.58	154.58	191.18
$\theta = -0.8$								
25	-23.5	-26.00	-29.50	-29.00	-4.70	-5.20	-5.90	-5.80
50	-44.3	-47.17	-53.54	-52.83	-6.27	-6.67	-7.57	-7.47
100	-78.0	-86.00	-95.00	-93.00	-7.80	-8.60	-9.50	-9.30
250	-107.7	-158.29	-193.08	-183.59	-6.81	-10.01	-12.21	-11.61
500	72.4	-128.81	-265.21	-224.96	3.14	-5.76	-11.86	-10.06
1000	1280.0	483.11	-57.64	106.80	40.47	15.28	-1.82	3.38
$\theta = -0.9$								
25	-27.00	-29.50	-30.00	-30.00	-5.40	-5.90	-6.00	-6.00
50	-49.29	-53.54	-56.36	-56.36	-6.97	-7.57	-7.97	-7.97
100	-95.00	-97.00	-107.00	-106.00	-9.50	-9.70	-10.70	-10.60
250	-215.22	-227.86	-242.09	-238.93	-13.61	-14.41	-15.31	-15.11
500	-359.13	-408.32	-446.33	-435.15	-16.06	-18.26	-19.96	-19.46
1000	-433.95	-630.01	-765.99	-728.04	-13.72	-19.92	-24.22	-23.02
$\theta = -0.95$								
25	-29.40	-30.00	-29.95	-30.00	-5.88	-6.00	-5.99	-5.99
50	-54.03	-56.29	-56.86	-56.86	-7.64	-7.96	-8.04	-8.04
100	-100.00	-107.00	-110.00	-109.00	-10.00	-10.70	-11.00	-10.90
250	-242.09	-246.84	-261.07	-259.49	-15.31	-15.61	-16.51	-16.41
500	-466.46	-477.64	-500.00	-493.29	-20.86	-21.36	-22.36	-22.06
1000	-860.86	-908.29	-949.40	-939.92	-27.22	-28.72	-30.02	-29.72
$\theta = -0.99$								
25	-29.50	-30.00	-30.00	-30.00	-5.90	-6.00	-6.00	-6.00
50	-57.07	-57.07	-57.07	-57.07	-8.07	-8.07	-8.07	-8.07
100	-109.70	-109.90	-110.00	-110.00	-10.97	-10.99	-11.00	-11.00
250	-264.23	-265.02	-265.65	-265.81	-16.66	-16.76	-16.80	-16.81
500	-511.28	-519.01	-521.69	-522.36	-22.82	-23.21	-23.33	-23.36
1000	-996.84	-1015.81	-1028.46	-1025.30	-31.49	-32.12	-32.52	-32.52

Table 4–5 (b)

95 % Fractiles of the Predicted Distribution for δ = 3/4

T	$\hat{\rho}$	$\hat{\rho}_\mu$	$\hat{\rho}_\tau$	$\tilde{\rho}$	$\hat{\tau}$	$\hat{\tau}_\mu$	$\hat{\tau}_\tau$	$\tilde{\tau}$
θ = −0.5								
25	233	47	−1	5	47	9	−0.2	0.9
50	982	236	42	67	139	33	6.0	9.4
100	3998	1043	269	366	400	104	26.9	36.6
250	23388	6984	2055	2662	1479	410	130.0	168.4
500	107635	22307	8723	11047	4636	997	390.0	494.0
1000	414130	112370	35907	45195	13063	3324	1135.5	1429.2
θ = −0.8								
25	16.5	−11.0	−18.0	−17.0	3.3	−2.2	−3.60	−3.40
50	113.3	−2.6	−32.3	−28.8	16.0	−0.4	−4.57	−4.07
100	555.0	84.0	−40.0	−25.0	55.8	8.4	−4.00	−2.50
250	3840.4	897.9	120.0	216.4	243.4	56.8	7.59	13.69
500	15885.9	4090.7	973.6	1358.2	711.3	182.9	43.54	60.74
1000	64020.0	17385.5	4885.0	6431.4	2111.3	549.8	154.48	203.38
θ = −0.9								
25	−12.5	−18.0	−19.50	−19.50	−2.5	−3.60	−3.90	−3.90
50	−7.6	−34.4	−40.81	−40.10	−1.1	−4.87	−5.77	−5.67
100	65.0	−51.0	−80.00	−77.00	6.5	−5.10	−8.00	−7.70
250	780.9	37.8	−155.13	−133.00	49.4	2.39	−9.81	−8.41
500	3625.6	647.1	−128.81	−32.66	162.1	28.94	−5.76	−1.46
1000	15506.0	3588.5	473.62	865.74	490.3	113.48	14.98	27.38
θ = −0.95								
25	−18.2	−19.58	−19.90	−19.85	−3.63	−3.89	−3.98	−3.97
50	−35.4	−40.74	−42.43	−42.22	−5.00	−5.76	−6.00	−5.97
100	−57.0	−82.00	−88.00	−88.00	−5.70	−8.20	−8.80	−8.80
250	7.7	−174.11	−218.38	−213.63	0.49	−11.01	−13.81	−13.51
500	533.1	−211.55	−403.85	−381.49	23.84	−9.46	−18.06	−17.06
1000	3126.8	151.07	−630.01	−531.98	98.88	4.78	−19.92	−16.82
θ = −0.99								
25	−19.50	−20.00	−20.00	−20.00	−3.90	−4.00	−4.00	−4.00
50	−42.93	−42.93	−42.93	−42.93	−6.07	−6.07	−6.07	−6.07
100	−89.00	−89.70	−90.00	−89.90	−8.88	−8.97	−9.00	−8.99
250	−227.86	−231.98	−233.71	−234.19	−14.33	−14.67	−14.78	−14.81
500	−448.57	−468.92	−475.40	−475.40	−19.98	−20.97	−21.26	−21.26
1000	−829.24	−932.96	−959.52	−958.89	−26.21	−29.50	−30.34	−30.32

Table 4-6

Fractiles of the Predicted Distribution for $\delta = 7/8$

T		25	50	100	250	500	1000
ρ tests	5%	-33.15	-61.53	-116.30	-275.77	-536.45	-1051.55
	95%	-16.80	-38.40	-83.60	-224.07	-463.33	-948.14
τ tests	5%	-6.63	-8.70	-11.63	-17.44	-23.99	-33.25
	95%	-3.36	-5.43	-8.36	-14.17	-20.72	-29.98

Table 4-7 (a)

Comparison of the Predicted Distribution with

The Actual Sampling Distribution ($\hat{\rho}$ Test)

$\theta = -0.5$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-23.96	-117.65	-20.62	-19.69	-20.00	-36.55
	.025	-20.84	-90.91	-19.61	-18.12	-18.50	-34.80
	.050	-18.37	-71.43	-18.52	-16.25	-16.50	-33.15
	.950	-0.08	-2.42	-2.21	232.97	233.00	-16.80
	.975	0.29	-1.88	-1.75	306.72	308.00	-15.25
	.990	0.68	-1.45	-1.37	405.16	409.00	-13.35
50	.010	-34.90	-117.65	-35.09	-28.75	-28.79	-66.33
	.025	-29.85	-90.91	-32.26	-22.50	-22.42	-63.86
	.050	-25.50	-71.43	-29.41	-15.00	-14.64	-61.53
	.950	-0.20	-2.42	-2.31	981.88	982.38	-38.40
	.975	0.18	-1.88	-1.82	1276.88	1227.95	-36.21
	.990	0.55	-1.45	-1.41	1670.63	1623.01	-33.52
100	.010	-46.17	-117.65	-54.05	-15.00	-13.00	-123.10
	.025	-38.04	-90.91	-47.62	10.00	11.00	-119.60
	.050	-31.78	-71.43	-41.67	40.00	42.00	-116.30
	.950	-0.29	-2.42	-2.37	4027.50	3998.00	-83.60
	.975	0.10	-1.88	-1.85	5207.50	5106.00	-80.50
	.990	0.47	-1.45	-1.43	6782.50	6555.00	-76.70
250	.010	-58.13	-117.65	-80.00	281.25	293.91	-286.52
	.025	-46.60	-90.91	-66.67	437.50	452.03	-280.99
	.050	-37.51	-71.43	-55.56	625.00	637.02	-275.77
	.950	-0.33	-2.42	-2.40	25546.87	23388.03	-224.07
	.975	0.07	-1.88	-1.87	32921.87	32368.50	-219.17
	.990	0.40	-1.45	-1.44	42765.62	44123.71	-213.16
500	.010	-65.16	-117.65	-95.24	1625.0	1680.27	-551.65
	.025	-50.62	-90.91	-76.92	2250.0	2283.90	-543.83
	.050	-40.05	-71.43	-62.50	3000.0	3050.88	-536.45
	.950	-0.37	-2.42	-2.41	102687.5	107635.26	-463.33
	.975	0.02	-1.88	-1.88	132187.5	139912.36	-456.40
	.990	0.37	-1.45	-1.45	171562.5	183725.30	-447.90
1000	.010	-67.39	-117.65	-105.26	7500	7721.5	-1073.05
	.025	-53.70	-90.91	-83.33	10000	10137.5	-1061.98
	.050	-41.91	-71.43	-66.67	13000	13204.9	-1051.55
	.950	-0.40	-2.42	-2.42	411750	414130.0	-948.14
	.975	0.00	-1.88	-1.88	529750	536124.0	-938.34
	.990	0.32	-1.45	-1.45	687250	696323.5	-926.32

Table 4-7 (a) (Continued)

$\theta = -0.8$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	−31.37	−735.29	−24.18	−24.15	−24.50	−36.55
	.025	−29.19	−568.18	−23.95	−23.90	−24.00	−34.80
	.050	−27.36	−446.43	−23.67	−23.60	−23.50	−33.15
	.950	−4.90	−15.14	−9.43	16.27	16.50	−16.80
	.975	−3.69	−11.78	−8.01	28.07	29.00	−15.25
	.990	−2.61	−9.08	−6.66	43.82	44.00	−13.35
50	.010	−54.66	−735.29	−46.82	−46.60	−47.17	−66.33
	.025	−50.96	−568.18	−45.96	−45.60	−45.76	−63.86
	.050	−47.83	−446.43	−44.96	−44.40	−44.34	−61.53
	.950	−7.10	−15.14	−11.62	115.10	113.34	−38.40
	.975	−5.45	−11.78	−9.53	162.30	162.13	−36.21
	.990	−3.88	−9.08	−7.69	225.30	225.06	−33.52
100	.010	−93.88	−735.29	−88.03	−86.40	−87.00	−123.10
	.025	−87.42	−568.18	−85.03	−82.40	−82.70	−119.60
	.050	−81.12	−446.43	−81.70	−77.60	−77.70	−116.30
	.950	−8.83	−15.14	−13.15	560.40	560.90	−83.60
	.975	−6.71	−11.78	−10.54	749.20	753.00	−80.50
	.990	−4.95	−9.08	− 8.33	1001.20	1009.10	−76.70
250	.010	−183.01	−735.29	−186.57	−165.00	−163.04	−286.52
	.025	−166.15	−568.18	−173.61	−140.00	−139.32	−280.99
	.050	−150.44	−446.43	−160.26	−110.00	−107.70	−275.77
	.950	−10.33	−15.14	−14.28	3877.50	3840.41	−224.07
	.975	−7.00	−11.78	−11.25	5057.50	5007.29	−219.17
	.990	−5.88	−9.08	−8.76	6632.50	6574.20	−213.16
500	.010	−276.29	−735.29	−297.62	−160.0	−151.17	−551.65
	.025	−241.72	−568.18	−265.96	−60.0	−55.02	−543.83
	.050	−210.68	−446.43	−235.85	60.0	72.43	−536.45
	.950	−11.09	−15.14	−14.70	16010.0	15885.91	−463.33
	.975	−8.39	−11.78	−11.51	20730.0	20523.51	−456.40
	.990	−6.16	−9.08	−8.92	27030.0	26795.68	−447.90
1000	.010	−368.93	−735.29	−423.73	360.0	372.4	−1073.05
	.025	−312.54	−568.18	−362.32	760.0	777.2	−1061.98
	.050	−264.55	−446.43	−308.64	1240.0	1280.0	−1051.55
	.950	−11.20	−15.14	−14.92	65040.0	64020.6	−948.14
	.975	−8.36	−11.78	−11.64	83920.0	84110.2	−938.34
	.990	−6.24	−9.08	−9.00	109120.0	110821.0	−926.32

Table 4-7 (a) (Continued)

$\theta = -0.9$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-33.28	-2941.2	-24.79	-24.79	-29.00	-36.55
	.025	-31.44	-2272.7	-24.73	-24.73	-28.00	-34.80
	.050	-29.81	-1785.7	-24.65	-24.65	-27.00	-33.15
	.950	-10.43	-60.6	-17.70	-14.68	-12.50	-16.80
	.975	-8.71	-47.1	-16.33	-11.73	-8.00	-15.25
	.990	-7.04	-36.3	-14.81	-7.80	-2.00	-13.35
50	.010	-60.17	-2941.2	-49.16	-49.15	-51.41	-66.33
	.025	-57.28	-2272.7	-48.92	-48.90	-49.29	-63.86
	.050	-54.70	-1785.7	-48.64	-48.60	-49.29	-61.53
	.950	-19.13	-60.6	-27.39	-8.72	-7.57	-38.40
	.975	-16.06	-47.1	-24.25	3.08	5.86	-36.21
	.990	-12.74	-36.3	-21.04	18.80	22.83	-33.52
100	.010	-109.41	-2941.2	-96.71	-96.60	-97.00	-123.10
	.025	-104.42	-2272.7	-95.79	-95.60	-96.00	-119.60
	.050	-100.43	-1785.7	-94.70	-94.40	-95.00	-116.30
	.950	-29.05	-60.6	-37.72	65.10	65.00	-83.60
	.975	-23.73	-47.1	-32.02	112.30	114.00	-80.50
	.990	-18.97	-36.3	-26.65	175.30	179.00	-76.70
250	.010	-240.37	-2941.2	-230.42	-228.75	-229.45	-286.52
	.025	-230.28	-2272.7	-225.23	-222.50	-223.12	-280.99
	.050	-220.12	-1785.7	-219.30	-215.00	-215.22	-275.77
	.950	-40.79	-60.6	-48.76	781.88	780.90	-224.07
	.975	-32.47	-47.1	-39.64	1076.88	1081.32	-219.17
	.990	-25.80	-36.3	-31.72	1470.63	1481.35	-213.16
500	.010	-431.02	-2941.2	-427.35	-415.0	-412.79	-551.65
	.025	-407.41	-2272.7	-409.84	-390.0	-390.43	-543.83
	.050	-382.37	-1785.7	-390.63	-360.0	-359.13	-536.45
	.950	-45.96	-60.6	-54.03	3627.5	3625.55	-463.33
	.975	-36.07	-47.1	-43.05	4807.5	4812.90	-456.40
	.990	-26.90	-36.3	-33.86	6382.5	6407.21	-447.90
1000	.010	-727.44	-2941.2	-746.27	-660.0	-652.15	-1073.05
	.025	-667.21	-2272.7	-694.44	-560.0	-557.28	-1061.98
	.050	-606.31	-1785.7	-641.03	-440.0	-433.95	-1051.55
	.950	-49.56	-60.6	-57.11	15510.0	15506.00	-948.14
	.975	-39.42	-47.1	-44.98	20230.0	20360.72	-938.34
	.990	-30.86	-36.3	-35.05	26530.0	26642.80	-926.32

Table 4-7 (a) (Continued)

$\theta = -0.95$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-34.12	-11764.0	-24.95	-24.95	-32.45	-36.55
	.025	-32.42	-9090.9	-24.93	-24.93	-29.75	-34.80
	.050	-30.94	-7142.8	-24.91	-24.91	-29.40	-33.15
	.950	-13.92	-242.2	-22.66	-22.42	-18.15	-16.80
	.975	-12.35	-188.4	-22.07	-21.68	-16.50	-15.25
	.990	-10.69	-145.3	-21.33	-20.70	-11.75	-13.35
50	.010	-62.53	-11764.0	-49.79	-49.79	-55.87	-66.33
	.025	-60.06	-9090.9	-49.73	-49.73	-55.16	-63.86
	.050	-57.68	-7142.8	-49.65	-49.65	-54.03	-61.53
	.950	-30.24	-242.2	-41.45	-39.68	-35.36	-38.40
	.975	-27.26	-188.4	-39.51	-36.73	-30.06	-36.21
	.990	-23.64	-145.3	-37.20	-32.80	-22.71	-33.52
100	.010	-116.12	-11764.0	-99.13	-99.15	-104.40	-123.10
	.025	-111.93	-9090.9	-98.91	-98.90	-101.00	-119.60
	.050	-108.43	-7142.9	-98.62	-98.60	-100.00	-116.30
	.950	-57.27	-242.2	-70.78	-58.73	-57.00	-83.60
	.975	-50.99	-188.4	-65.33	-46.93	-42.00	-80.50
	.990	-43.91	-145.3	-59.23	-31.18	-24.00	-76.70
250	.010	-265.13	-11764.0	-244.80	-244.69	-245.26	-286.52
	.025	-257.74	-9090.9	-243.31	-243.13	-243.68	-280.99
	.050	-250.96	-7142.9	-241.55	-241.25	-242.09	-275.77
	.950	-107.86	-242.2	-123.04	7.97	7.73	-224.07
	.975	- 92.43	-188.4	-107.44	81.72	83.62	-219.17
	.990	- 77.91	-145.3	- 91.89	180.16	186.39	-213.16
500	.010	-497.73	-11764.0	-479.62	-478.75	-479.88	-551.65
	.025	-484.06	-9090.9	-473.94	-472.50	-473.17	-543.83
	.050	-471.70	-7142.9	-467.29	-465.00	-466.46	-536.45
	.950	-145.26	-242.3	-163.20	531.88	533.06	-463.33
	.975	-121.22	-188.4	-136.85	826.88	830.46	-456.40
	.990	- 98.83	-145.3	-112.58	1220.63	1237.42	-447.90
1000	.010	-932.60	-11764.0	-921.66	-915.00	-914.62	-1073.05
	.025	-903.10	-9090.9	-900.90	-890.00	-889.32	-1061.98
	.050	-872.40	-7142.9	-877.19	-860.00	-860.86	-1051.55
	.950	-179.04	-242.3	-195.03	3127.50	3126.77	-948.14
	.975	-145.15	-188.4	-158.54	4307.50	4213.63	-938.34
	.990	-110.93	-145.3	-126.86	5882.50	5912.74	-926.32

Table 4-7 (a) (Continued)

$\theta = -0.99$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	−34.75	−294118	−25.00	−25.00	−35.00	−36.55
	.025	−33.17	−227273	−25.00	−25.00	−30.00	−34.80
	.050	−31.67	−178571	−25.00	−25.00	−29.50	−33.15
	.950	−15.94	−6057	−24.90	−24.90	−19.50	−16.80
	.975	−14.57	−4710	−24.87	−24.87	−19.00	−15.25
	.990	−13.04	−3632	−24.83	−24.80	−15.00	−13.35
50	.010	−64.37	−294118	−49.99	−49.99	−64.14	−66.33
	.025	−61.83	−227273	−49.99	−49.99	−57.07	−63.86
	.050	−59.81	−178571	−49.99	−49.99	−57.07	−61.53
	.950	−37.00	−6057	−49.59	−49.59	−42.93	−38.40
	.975	−34.88	−4710	−49.47	−49.47	−42.93	−36.21
	.990	−32.46	−3632	−49.32	−49.30	−35.86	−33.52
100	.010	−120.76	−294118	−99.97	−99.97	−119.00	−123.10
	.025	−117.24	−227273	−99.96	−99.96	−110.00	−119.60
	.050	−114.12	−178571	−99.94	−99.94	−109.70	−116.30
	.950	−80.77	−6057	−98.38	−98.35	−89.00	−83.60
	.975	−77.50	−4710	−97.92	−97.88	−88.00	−80.50
	.990	−73.68	−3632	−97.32	−97.25	−78.00	−76.70
250	.010	−281.9	−294118	−249.79	−249.79	−272.14	−286.52
	.025	−276.3	−227273	−249.73	−249.73	−265.81	−280.99
	.050	−271.3	−178571	−249.65	−249.65	−264.23	−275.77
	.950	−215.3	−6057	−240.09	−239.68	−227.86	−224.07
	.975	−209.3	−4710	−237.40	−236.73	−221.54	−219.17
	.990	−201.9	−3632	−233.90	−232.79	−205.73	−213.16
500	.010	−542.3	−294118	−499.15	−499.15	−517.89	−551.65
	.025	−534.1	−227273	−498.90	−498.90	−515.65	−543.83
	.050	−526.5	−178571	−498.60	−498.60	−511.18	−536.45
	.950	−432.2	−6057	−461.87	−458.73	−448.57	−463.33
	.975	−419.2	−4710	−452.02	−446.93	−428.45	−456.40
	.990	−401.7	−3632	−439.50	−431.18	−403.85	−447.90
1000	.010	−1049.3	−294118	−996.61	−996.60	−1009.49	−1073.05
	.025	−1036.1	−227273	−995.62	−995.60	−1000.00	−1061.98
	.050	−1025.2	−178571	−994.43	−994.40	−996.84	−1051.55
	.950	−821.9	−6057	−858.30	−834.90	−829.24	−948.14
	.975	−785.5	−4710	−824.88	−787.70	−775.48	−938.34
	.990	−733.9	−3632	−784.13	−724.70	−705.91	−926.32

Table 4-7 (b)

Comparison of the Predicted Distribution with

The Actual Sampling Distribution ($\hat{\tau}$ Test)

$\theta = -0.5$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-4.81	-7.67	-4.19	-3.94	-4.00	-7.31
	.025	-4.23	-6.74	-4.02	-3.63	-3.70	-6.96
	.050	-3.78	-5.98	-3.83	-3.25	-3.30	-6.63
	.950	-0.03	-1.10	-1.07	46.59	46.60	-3.36
	.975	0.21	-0.97	-0.95	61.34	61.60	-3.05
	.990	0.50	-0.85	-0.84	81.03	81.80	-2.67
50	.010	-5.19	-7.67	-5.20	-4.07	-4.07	-9.38
	.025	-4.61	-6.74	-4.88	-3.18	-3.17	-9.03
	.050	-4.12	-5.98	-4.56	-2.12	-2.07	-8.70
	.950	-0.12	-1.10	-1.09	138.86	138.93	-5.43
	.975	0.11	-0.97	-0.96	180.58	180.73	-5.12
	.990	0.36	-0.85	-0.85	236.26	229.53	-4.74
100	.010	-5.47	-7.67	-6.09	-1.50	-1.30	-12.31
	.025	-4.82	-6.74	-5.59	1.00	1.10	-11.96
	.050	-4.32	-5.98	-5.13	4.00	4.20	-11.63
	.950	-0.17	-1.10	-1.09	402.75	399.80	-8.36
	.975	0.03	-0.97	-0.97	520.75	510.60	-8.05
	.990	0.29	-0.85	-0.85	678.25	655.50	-7.67
250	.010	-5.68	-7.67	-6.90	17.79	18.59	-18.12
	.025	-5.02	-6.74	-6.20	27.67	28.59	-17.77
	.050	-4.46	-5.98	-5.59	39.53	40.29	-17.44
	.950	-0.19	-1.10	-1.10	1615.73	1479.19	-14.17
	.975	0.04	-0.97	-0.97	2082.16	2106.29	-13.86
	.990	0.28	-0.85	-0.85	2704.74	2784.59	-13.48
500	.010	-5.88	-7.67	-7.25	72.67	75.14	-24.67
	.025	-5.11	-6.74	-6.45	100.62	102.14	-24.32
	.050	-4.52	-5.98	-5.77	134.16	136.44	-23.99
	.950	-0.21	-1.10	-1.10	4592.33	4636.14	-20.72
	.975	0.01	-0.97	-0.97	5911.60	5981.84	-20.41
	.990	0.25	-0.85	-0.85	7672.51	7767.94	-20.03
1000	.010	-5.88	-7.67	-7.45	237.17	244.18	-33.93
	.025	-5.21	-6.74	-6.59	316.23	320.58	-33.58
	.050	-4.58	-5.98	-5.87	411.10	417.58	-33.25
	.950	-0.23	-1.10	-1.10	13020.68	13662.98	-29.98
	.975	0.00	-0.97	-0.97	16752.17	16883.98	-29.67
	.990	0.22	-0.85	-0.85	21732.75	22741.18	-29.29

Table 4-7 (b) (Continued)

$\theta = -0.8$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	−6.77	−19.17	−4.84	−4.83	−4.90	−7.31
	.025	−6.15	−16.85	−4.79	−4.78	−4.80	−6.96
	.050	−5.67	−14.94	−4.74	−4.72	−4.70	−6.63
	.950	−1.54	−2.75	−2.41	3.25	3.30	−3.36
	.975	−1.28	−2.43	−2.18	5.61	5.80	−3.05
	.990	−1.00	−2.13	−1.96	8.76	8.80	−2.67
50	.010	−8.62	−19.17	−6.63	−6.59	−6.67	−9.38
	.025	−8.12	−16.85	−6.52	−6.45	−6.47	−9.03
	.050	−7.68	−14.94	−6.39	−6.28	−6.27	−8.70
	.950	−3.62	−2.75	−2.56	16.28	16.03	−5.43
	.975	−3.29	−2.43	−2.30	22.95	22.93	−5.12
	.990	−2.93	−2.13	−2.04	31.86	31.83	−4.74
100	.010	−9.45	−19.17	−8.87	−8.64	−8.70	−12.31
	.025	−8.85	−16.85	−8.60	−8.24	−8.30	−11.96
	.050	−8.28	−14.94	−8.31	−7.76	−7.80	−11.63
	.950	−2.05	−2.75	−2.65	56.04	55.80	−8.36
	.975	−1.75	−2.43	−2.36	74.92	72.90	−8.05
	.990	−1.45	−2.13	−2.08	100.12	96.80	−7.67
250	.010	−12.04	−19.17	−12.20	−10.44	−10.41	−18.12
	.025	−11.16	−16.85	−11.53	−8.85	−8.81	−17.77
	.050	−10.37	−14.94	−10.86	−6.96	−6.81	−17.44
	.950	−2.20	−2.75	−2.71	245.24	243.39	−14.17
	.975	−1.88	−2.43	−2.40	319.86	313.79	−13.86
	.990	−1.59	−2.13	−2.11	419.48	369.19	−13.48
500	.010	−13.77	−19.17	−14.56	−7.16	−7.61	−24.67
	.025	−12.62	−16.85	−13.46	−2.68	−2.56	−24.32
	.050	−11.53	−14.94	−12.42	2.68	3.14	−23.99
	.950	−2.27	−2.75	−2.73	715.99	711.34	−20.72
	.975	−1.94	−2.43	−2.41	927.07	862.64	−20.41
	.990	−1.63	−2.13	−2.12	1208.82	1210.14	−20.03
1000	.010	−15.03	−19.17	−16.40	11.38	11.78	−33.93
	.025	−13.58	−16.85	−14.87	24.03	24.58	−33.58
	.050	−12.33	−14.94	−13.51	39.21	40.47	−33.25
	.950	−2.26	−2.75	−2.74	2056.75	2111.28	−29.98
	.975	−1.93	−2.43	−2.42	2653.78	2667.48	−29.67
	.990	−1.64	−2.13	−2.13	3450.68	3542.58	−29.29

Table 4-7 (b) (Continued)

$\theta = -0.9$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-7.43	-38.35	-4.96	-4.96	-5.80	-7.31
	.025	-6.79	-33.71	-4.95	-4.95	-5.60	-6.96
	.050	-6.31	-29.88	-4.93	-4.93	-5.40	-6.63
	.950	-2.53	-5.50	-3.70	-2.94	-2.50	-3.36
	.975	-2.24	-4.85	-3.48	-2.35	-1.60	-3.05
	.990	-1.93	-4.26	-3.24	-1.56	-0.40	-2.67
50	.010	-8.84	-38.35	-6.95	-6.95	-7.27	-9.38
	.025	-8.32	-33.71	-6.92	-6.92	-6.97	-9.03
	.050	-7.88	-29.88	-6.88	-6.87	-6.97	-8.70
	.950	-3.39	-5.50	-4.34	-1.23	-1.07	-5.43
	.975	-3.03	-4.85	-4.00	0.43	-0.83	-5.12
	.990	-2.63	-4.26	-3.65	2.66	3.23	-4.74
100	.010	-11.02	-38.35	-9.68	-9.66	-9.70	-12.31
	.025	-10.49	-33.71	-9.59	-9.56	-9.60	-11.96
	.050	-10.03	-29.88	-9.48	-9.44	-9.50	-11.63
	.950	-4.08	-5.50	-4.82	6.51	6.50	-8.36
	.975	-3.63	-4.85	-4.37	11.23	11.40	-8.05
	.990	-3.18	-4.26	-3.92	17.53	17.90	-7.67
250	.010	-15.25	-38.35	-14.62	-14.47	-14.51	-18.12
	.025	-14.64	-33.71	-14.32	-14.07	-14.11	-17.77
	.050	-14.04	-29.88	-13.98	-13.60	-13.61	-17.44
	.950	-4.67	-5.50	-5.20	49.45	49.39	-14.17
	.975	-4.11	-4.85	-4.64	68.11	68.39	-13.86
	.990	-3.63	-4.26	-4.12	93.01	93.69	-13.48
500	.010	-19.46	-38.35	-19.32	-18.56	-18.46	-24.67
	.025	-18.54	-33.71	-18.64	-17.44	-17.46	-24.32
	.050	-17.63	-29.88	-17.90	-16.10	-16.06	-23.99
	.950	-4.86	-5.50	-5.34	162.23	162.14	-20.72
	.975	-4.27	-4.85	-4.74	215.00	215.24	-20.41
	.990	-3.66	-4.26	-4.19	285.43	286.54	-20.03
1000	.010	-23.91	-38.35	-24.40	-20.87	-20.62	-33.93
	.025	-22.37	-33.71	-23.06	-17.71	-17.62	-33.58
	.050	-20.85	-29.88	-21.72	-13.91	-13.72	-33.25
	.950	-5.00	-5.50	-5.42	490.47	490.30	-29.98
	.975	-4.43	-4.85	-4.80	639.73	650.72	-29.67
	.990	-3.89	-4.26	-4.22	838.95	877.72	-29.29

Table 4-7 (b) (Continued)

$\theta = -0.95$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-7.71	-76.70	-4.99	-4.99	-6.49	-7.31
	.025	-7.10	-67.42	-4.99	-4.99	-5.95	-6.96
	.050	-6.63	-59.76	-4.98	-4.98	-5.88	-6.63
	.950	-3.09	-11.01	-4.55	-4.48	-3.63	-3.36
	.975	-2.84	-9.71	-4.44	-4.34	-3.30	-3.05
	.990	-2.55	-8.52	-4.31	-4.14	-2.35	-2.67
50	.010	-9.32	-76.70	-7.04	-7.04	-7.90	-9.38
	.025	-8.83	-67.42	-7.03	-7.03	-7.80	-9.03
	.050	-8.39	-59.76	-7.02	-7.02	-7.64	-8.70
	.950	-4.65	-11.01	-5.95	-5.61	-5.00	-5.43
	.975	-4.32	-9.71	-5.72	-5.19	-4.25	-5.12
	.990	-3.90	-8.52	-5.44	-4.64	-3.21	-4.74
100	.010	-11.85	-76.70	-9.92	-9.92	-10.40	-12.31
	.025	-11.35	-67.42	-9.89	-9.89	-10.10	-11.96
	.050	-10.96	-59.76	-9.86	-9.86	-10.00	-11.63
	.950	-6.32	-11.01	-7.40	-5.87	-5.70	-8.36
	.975	-5.84	-9.71	-6.97	-4.69	-4.20	-8.05
	.990	-5.28	-8.52	-6.49	-3.12	-2.40	-7.67
250	.010	-16.84	-76.70	-15.49	-15.48	-15.51	-18.12
	.025	-16.35	-67.42	-15.39	-15.38	-15.41	-17.77
	.050	-15.91	-59.76	-15.29	-15.26	-15.31	-17.44
	.950	-8.28	-11.01	-9.03	0.50	0.49	-14.17
	.975	-7.51	-9.71	-8.27	5.17	5.29	-13.86
	.990	-6.77	-8.52	-7.50	11.39	11.79	-13.48
500	.010	-22.27	-76.70	-21.47	-21.41	-21.46	-24.67
	.025	-21.66	-67.42	-21.22	-21.13	-21.16	-24.32
	.050	-21.13	-59.76	-20.94	-20.80	-20.86	-23.99
	.950	-9.19	-11.01	-9.88	23.79	23.84	-20.72
	.975	-8.27	-9.71	-8.90	36.98	37.14	-20.41
	.990	-7.37	-8.52	-7.96	54.59	55.34	-20.03
1000	.010	-29.56	-76.70	-29.24	-28.94	-28.92	-33.93
	.025	-28.70	-67.42	-28.63	-28.14	-28.12	-33.58
	.050	-27.81	-59.76	-27.95	-27.20	-27.22	-33.25
	.950	-9.90	-11.01	-10.40	98.90	98.88	-29.98
	.975	-8.75	-9.71	-9.28	136.22	136.38	-29.67
	.990	-7.63	-8.52	-8.23	186.02	186.98	-29.29

Table 4-7 (b) (Continued)

$\theta = -0.99$

T		Actual	δ=1/4	δ=1/2	δ=5/8	δ=3/4	δ=7/8
25	.010	-7.92	-383.48	-5.00	-5.00	-7.00	-7.31
	.025	-7.35	-337.10	-5.00	-5.00	-6.00	-6.96
	.050	-6.86	-298.81	-5.00	-5.00	-5.90	-6.63
	.950	-3.42	-55.03	-4.98	-4.98	-3.90	-3.36
	.975	-3.18	-48.53	-4.97	-4.97	-3.80	-3.05
	.990	-2.92	-42.62	-4.97	-4.97	-3.00	-2.67
50	.010	-9.69	-383.48	-7.07	-7.07	-9.07	-9.38
	.025	-9.18	-337.10	-7.07	-7.07	-8.07	-9.03
	.050	-8.78	-298.81	-7.07	-7.07	-8.07	-8.70
	.950	-5.44	-55.03	-7.01	-7.01	-6.07	-5.43
	.975	-5.18	-48.53	-7.00	-7.00	-6.07	-5.12
	.990	-4.90	-42.62	-6.98	-6.97	-5.07	-4.74
100	.010	-12.43	-383.48	-10.00	-10.00	-11.85	-12.31
	.025	-12.93	-337.10	-10.00	-10.00	-10.99	-11.96
	.050	-11.56	-298.81	-9.99	-9.99	-10.97	-11.63
	.950	-8.23	-55.03	-9.84	-9.84	-8.88	-8.36
	.975	-7.96	-48.53	-9.79	-9.79	-7.96	-8.05
	.990	-7.68	-42.62	-9.74	-9.73	-7.80	-7.67
250	.010	-17.98	-383.48	-15.80	-15.80	-17.20	-18.17
	.025	-17.58	-337.10	-15.79	-15.79	-16.74	-17.77
	.050	-17.21	-298.81	-15.79	-15.79	-16.66	-17.44
	.950	-13.74	-55.03	-15.20	-15.16	-14.33	-14.17
	.975	-13.40	-48.53	-15.03	-14.97	-13.55	-13.86
	.990	-12.95	-42.62	-14.82	-14.72	-12.99	-13.48
500	.010	-24.33	-383.48	-22.32	-22.32	-23.14	-24.67
	.025	-24.90	-337.10	-22.31	-22.31	-23.01	-24.32
	.050	-23.55	-298.81	-22.30	-22.29	-22.82	-23.99
	.950	-19.54	-55.03	-20.72	-20.52	-19.98	-20.72
	.975	-19.03	-48.53	-20.31	-19.99	-19.04	-20.41
	.990	-18.38	-42.62	-19.80	-19.28	-17.92	-20.03
1000	.010	-33.24	-383.48	-31.52	-31.52	-31.91	-33.93
	.025	-32.80	-337.10	-31.49	-31.48	-31.55	-33.58
	.050	-32.43	-298.81	-31.45	-31.45	-31.49	-33.25
	.950	-26.40	-55.03	-27.42	-26.40	-26.21	-29.98
	.975	-25.43	-48.53	-26.49	-24.91	-24.58	-29.67
	.990	-24.06	-42.62	-25.40	-22.92	-22.47	-29.29

CHAPTER 5

CHAPTER 5

CONCLUSION

This dissertation has developed two main topics. The first topic is the question of testing the univariate properties of economic time series; that is, testing whether they have a unit root or are trend stationary. Economically, the distinction between a trend stationary process and a unit root process is important, because the latter implies long run persistence (current shocks have permanent effects), while the former implies trend reversion (current shocks have temporary effects). This distinction is also important statistically, due to the 'spurious regression phenomena' described by Granger and Newbold (1974) and Phillips (1986). In other words, if there are variables that are I(1) processes and no cointegrating relationship exists among the variables, standard regression inferences, based on the assumption that all variables are (trend) stationary, are not valid any longer. For a further discussion see Park and Phillips (1988, 1989).

There have been a lot of attempts to test the unit root hypothesis, including the Dickey-Fuller (1979, 1981), the Phillips-Perron (1988), and the Schmidt-Phillips (1991) unit root tests. Furthermore, the null hypothesis of a unit root has not been rejected for many economic time series. Since the seminal paper by Nelson and Plosser (1982), in which the null of a unit root is rejected for only two out of fourteen series, many applied economists have tended to take views that most economic time series are I(1) processes.

However, it is well-known from a large body of Monte Carlo studies that standard unit root tests are not very powerful against plausible trend stationarity alternatives. Therefore, it is sensible to check (from the different direction) whether or not the data are trend stationary in order to have a complete view of the properties of economic time series. In Chapter 2 we derive an LM test for the null of trend stationarity under very general assumptions about the stationary errors; that is, the mixing and moment conditions of Phillips and Perron (1988). We use a components model in which the series of interest is decomposed into the sum of the deterministic trend, the random walk, and the stationary error. This test has a nonstandard limiting distribution, which depends on a functional of a Brownian bridge. By a Monte Carlo simulation we derive the critical values for the LM statistic for both level stationarity and trend stationarity, and we consider finite sample (size and power) performance of the test statistic in the presence of autocorrelated errors. We consider both AR(1) and MA(1) errors. Generally, we find that there is a finite sample trade-off between size and power of the stationarity test statistic, and that the choice of lags used in calculating the long run variance has a major impact on the outcome of the test for all reasonable sample sizes. We find that the use of shorter lags (e.g., $\ell = 4$ when T =100) is suggested unless it appears that the stationary errors follow an AR(1) process with a large positive parameter. Empirically, however, $\rho \geq 0.8$ is very plausible since, if we take most series to be trend stationary (which is the null), their first order autocorrelations will often be in this range. According to our simulation results, the use of longer lags (e.g., $\ell = 12$ when T =100) is needed for the test to avoid severe size distortions in this case. This substantially reduces the power of the test. In our empirical work, we attempt to compromise between the possible size distortions from using $\ell = 4$ and the power loss from using $\ell = 12$ by picking $\ell = 8$. We find that the null of trend stationarity is rejected only for five series (industrial production, consumer prices, real wages, velocity, and stock prices) of the 14 series considered by Nelson and Plosser.

Next we consider the consistency of the stationarity test under the unit root alternative. In doing so, we derive the different limiting distribution of the test statistic under the alternative of a unit root (the random walk component has positive variance), which depends on detrended (or demeaned) Brownian motion. Therefore, in Chapter 3 we consider the use of the KPSS statistic as a unit root test statistic. Simulation results show that the main determinant of the finite sample size performance of our unit root tests is the relative variance ratio λ (variance of the stationary error divided by variance of the random walk innovation) rather than the autocorrelation of the stationary errors. Since the null is simply $\lambda > 0$, the exact location of the null is important for the quality of inference. Generally, the use of longer lags is preferred in terms of correct size when the process is nearly stationary, but the use of no lags is preferred in terms of good power when the process is nearly integrated, which again confirms the finite sample trade-off of the unit root test. We have compared the finite sample performance of the KPSS unit root test with that of the Dickey-Fuller test and found that the Dickey-Fuller test is more powerful, but also has more size distortion. Since our main interest lies in the search for a more powerful unit root test, we conclude that dual-use statistics are not very promising. Finally, since the values of the long run variance under the null of a unit root (normalized by ℓ/T) are almost constant at all lags, we use the results for $\ell = 0$ (or $\ell = 1$) for the unit root test in empirical applications. The main finding is that a unit root is strongly rejected only for the unemployment series, which is almost the same as Nelson and Plosser's results.

Combining the empirical results for the Nelson-Plosser data from both stationarity and unit root tests, we reach the following conclusions. Three series (unemployment rate, GNP deflator, and money) appear to be trend stationary. Four series (consumer prices, real wages, velocity, and stock prices) appear to have a unit root. Two series (nominal GNP and bond yield) probably have unit roots, while two more series (employment and real per capita GNP) are probably trend stationary. For the nominal wage we cannot reject either the unit root or the trend stationarity hypothesis. There are two interesting cases: industrial production and real GNP series. For industrial production we can reject the trend stationarity hypothesi at the 5 % level and the unit root hypothesis at the 10 % level. For real GNP we can reject the trend stationarity hypothesis at the 10 % level and the unit root hypothesis at the 20 % level. It seems that for many economic time series it is not very clear whether they have a unit root or are trend stationary. Probably other alternatives such as fractional integration or a nonlinear trend model are needed in further research.

The above empirical results may indicate that many economic time series could be in the region of 'near stationarity', in which the series have combinations of a random walk component and a stationary component. Schwert (1989) also gives some empirical evidence of near stationarity. The second topic of my thesis (Chapter 4) is the study of the asymptotic and finite sample behavior of standard unit root tests such as the Dickey-Fuller and the Schmidt-Phillips tests when the process is nearly stationary. A lot of Monte Carlo studies including Schwert (1987, 1989) show that the size distortion of the Dickey-Fuller and the Phillips-Perron tests is almost one when the process is nearly stationary. By using a local approximation of the MA(1) parameter to minus one, we derive the asymptotic distribution of the Dickey-Fuller and the Schmidt-Phillips unit root tests. We then examine their finite sample performance when the process is nearly stationary by a Monte Carlo simulation. We find that standard unit root tests have considerable size distortions when the process is nearly stationary, because OLS estimation strongly biases the coefficient of the dependent variable towards zero when the MA(1) parameter is near minus one. Our simulation results show that this bias could be more severe in small samples, but also considerable even for large sample sizes. Furthermore, our asymptotic results predict the extent of the finite sample size distortions quite accurately.

Recently, various attempts have been made to reconsider the important problem of distinguishing trend stationary and unit root processes. In particular, Perron (1989) has suggested that a time series structure with very infrequent changes in slope or intercept can be a useful approximation in empirical applications and argued that the conclusion of Nelson and Plosser can be reversed if one time structural break is allowed within the Dickey-Fuller test framework. On the other hand, Amsler and Lee (1991) apply the same logic to the Schmidt-Phillips test and find that a suitably modified Schmidt-Phillips test allowing for a structural change reverses the results of Perron (1989). However, the implicit assumption in these studies is that there is only one such "big shock", which is given exogenously. Banerjee et al. (1990) and Zivot and Andrews (1990) criticize this assumption and develop a framework that endogenizes the structural change. Even more general models can be entertained if we adopt nonlinear structural models suggested by Hamilton (1989) and Lam (1990). However, if we want to test for a more general type of trend specification, the testing procedure for trend stationarity (proposed in this dissertation) is also relevant, and the extension of the analyses of Banerjee et al. and Zivot and Andrews to the stationarity test is an interesting path for future research.

Bayesian methods can also be used to distinguish trend stationary and unit root processes. Dejong and Whiteman (1991) use a Monte Carlo based Bayesian methodology and study the posterior distributions of dominant roots corresponding to an AR(3) representation of the time series using flat (uninformative) priors. They find that eleven of fourteen Nelson-Plosser series support trend stationarity over integration. Similar results are obtained by Sims and Uhlig (1991). Phillips (1991) criticizes this inference, claiming that the results are sensitive to the model and prior distribution. Generally, he challenges the assertions (made by Bayesian econometricians such as Sims (1988) and Sims and Uhlig (1991)) about the impropriety of classical methods and the superiority of flat prior Bayesian methods. He employs ignorance (objective) priors rather than flat priors, because under flat priors the bias towards the trend stationary model is shown to be substantial, and especially in models with a fitted deterministic trend. He finds under ignorance priors that Bayesian inference accords more closely with the results of classical methods and that seven of the 14 Nelson-Plosser series show evidence of stochastic trends.

Generally, the conclusion that many economic time series are not very informative about stationarity is quite consistent with the above (inconclusive) findings. Therefore, this leaves a lot of rooms for the further research.

First, it is interesting to see what happens to the empirical results of the stationarity test if allowance is made for the presence of structural change. A simple extension is straightforward, but adjusting for endogeneity of the structural break, considering Hamilton's nonlinear trend function, or combining our analysis with Bayesian methods is more complicated.

Another second promising extension of this thesis is to a test of cointegration. This would involve extending the test of stationarity to the error in a cointegrating regression, instead of an observed series. This would be one of the first attempts of a direct test of cointegration rather than a test of the hypothesis of no cointegration (which is a direct extension of the unit root test). See Engle and Granger (1987), Phillips and Ouliaris (1990), Stock and Watson (1988), and Johansen (1991). We expect such a test of cointegration to give further light on the true relationships among important economic relationship such as the consumption function, purchasing power parity, the term structure of interest rates, the quantity theory of money, and so forth. The asymptotic theory for the case that the generalized cointegrating vector is efficiently estimated must be derived and we are currently working on this topic.

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