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Xiangfei Zeng

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Sheldon Axler
Major professor

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By

Xiangfei Zeng

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ABSTRACT

TOEPLITZ OPERATORS ON BERGMAN SPACES

By

Xiangfei Zeng

In this thesis, We study Toeplitz operators T_f on the Bergman space $L_a^p(D)$, where D is the open unit disc and f is in $C(\bar{D})$. We give a necessary and sufficient condition for T_f to be compact. We also prove that the commutator $T_fT_g - T_gT_f$ is compact, and show that the commutator ideal of the closed algebra generated by $\{T_f : f \in C(\bar{D})\}$ is equal to the ideal of compact operators. We show that the abelianization of $\{T_f : f \in C(\bar{D})\}$ is isomorphic to $C(\partial D)$, and then use that result to determine the essential spectrum of T_f. Although the above results were known for $p = 2$, the proofs in that case depend heavily on Hilbert space properties that do not work for other values of p. Thus a number of new techniques must be introduced for $1 < p < \infty$.

To my mother.

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1 Introduction

Let D be the open unit disc of the complex plane C. Let dA be the normalized two dimensional (Lebesgue) area measure on D. We will consider the Banach space $L^p(D)\,(1 < p < \infty)$ that consists of the complex-valued, measurable functions defined on D that satisfies

$$\int_D |f|^p \, dA < \infty .$$

For $1 < p < \infty$, the Bergman space $L^p_a(D)$ is the set of functions in $L^p(D)$ that are analytic on D. It is known that the projection P with kernel $(1 - \bar{w}z)^{-2}$ from $L^p(D, dA)$ to $L^p_a(D, dA)$ is bounded (See [1]). When $p = 2$, P is the orthogonal projection. For $f \in L^\infty(D, dA)$, define the Toeplitz operator T_f by $T_f(h) = P(fh)$.

In this thesis, we study the Toeplitz operator T_f with the symbol f in $C(\bar{D})$. Theorem 3.4 gives the necessary and sufficient condition for T_f to be compact. Theorem 4.1 describes a condition on the Taylor coefficients of the function f in $L^p_a(D)$, which is used to prove that the commutator $T_fT_g - T_gT_f$ is compact (Proposition 5.3).

Let $\mathcal{B}$ be the set of bounded linear operators on $L^p_a(D)$. Let $\mathcal{K}$ be the set of compact operators in $\mathcal{B}$. Let $\mathcal{T}$ denote the norm closed subalgebra of $\mathcal{B}$ generated by $\{T_f : \ f \in C(\bar{D})\}$. The commutator ideal $\mathcal{J}$ of $\mathcal{T}$ is defined to be the smallest norm closed two sided ideal containing $\{TS - ST : \ T, \ S \in \mathcal{T}\}$. Proposition 6.1 shows that $\mathcal{J} = \mathcal{K}$.

The last part of the thesis discusses the spectral properties of T_f. An operator $T \in \mathcal{B}$ is called Fredholm if the kernel of T has finite dimension and the range of T has finite codimension. The essential spectrum of T, denoted by $\sigma_e(T)$, is defined to be the set of complex numbers λ such that $T - \lambda$ is not Fredholm. Theorem 7.9 gives a Banach algebra isomorphism between $C(\partial D)$ and $\mathcal{T}/\mathcal{K}$. This result is used to prove that $\sigma_e(T_f) = f(\partial D)$ (Theorem 7.10).

Although the above results were known for $p = 2$ ([2]), the proofs in that case depend heavily on Hilbert space properties that do not work for other values of p. Thus a number of new techniques must be introduced for $1 < p < \infty$.

2 Preliminary Results

For $w \in D$, define the reproducing kernel by

$$k_w(z) = \frac{1}{(1 - \bar{w}z)^2}.$$

The following Proposition and Theorem are proved in [1].

Proposition 2.1 *Let $1 \leq p < \infty$, $f \in L^p_a(D)$, and $w \in D$. Then*

$$f(w) = \int_D \frac{f(z)}{(1 - w\bar{z})^2}\, dA(z). \tag{1}$$

Proof. See Lemma 1.7, [1].

Corollary 2.2 *Let $1 \leq p < \infty$. Then the point evaluation : $f \longrightarrow f(w)$ is bounded on $L^p_a(D)$ for $w \in D$ and uniformly bounded on compact subsets of D.*

Corollary 2.3 *If $1 \leq p < \infty$, then $L^p_a(D)$ is a Banach space.*

For functions f, g measurable such that $fg \in L^1(D)$, $< f, g >$ is defined by

$$< f, g >= \int_D f\bar{g}\, dA. \tag{2}$$

Definition 2.4 *Let $1 \leq p < \infty$, $f \in L^p_a(D)$. Define Pf by*

$$(Pf)(w) = < f, k_w >= \int_D \frac{f(z)}{(1 - w\bar{z})^2}\, dA(z). \tag{3}$$

Theorem 2.5 *Let $1 < p < \infty$. Then P is a bounded projection of $L^p(D)$ onto $L^p_a(D)$.*

Proof. See Theorem 1.10, [1].

Theorem 2.6 *Let* $1 < p < \infty$. *Then the dual of* $L^p(D)$ *can be identified with* $L^{p'}(D)$. *More precisely, every bounded linear functional on* $L^p(D)$ *is of the form*

$$f \longrightarrow \int_D f\bar{g}\,dA$$

for some unique $g \in L^{p'}(D)$. *Furthermore, the norm of the linear functional on* $L^p(D)$ *induced by* $g \in L^{p'}(D)$ *is equivalent to* $\|g\|_{p'}$.

Proof. See Theorem 1.16, [1].

Definition 2.7 *For* $f \in L^\infty(D)$, *the Toeplitz operator with symbol* f *is the operator from* $L^p_a(D)$ *to* $L^p_a(D)$ *defined by*

$$T_f(g) = P(fg). \tag{4}$$

We denote the adjoint of T_f by T_f^*, which is defined such that

$$< T_f h, g > = < h, T_f^* g >, \tag{5}$$

where $h \in L^p_a(D)$, $g \in L^{p'}_a(D)$.

Proposition 2.8 *Let* f, f_1, $f_2 \in L^\infty(D)$. *Then*

$$T_{f_1+f_2} = T_{f_1} + T_{f_2} \tag{6}$$

and

$$T_f^* = T_{\bar{f}}. \tag{7}$$

Proof. (6) is obvious.

Let $h \in L^p_a(D)$ and $g \in L^{p'}_a(D)$. Then

$$< h, T_f^* g > \quad = \quad < T_f h, g > = \int_D (T_f h)\bar{g}\,dA \tag{8}$$

$$= \int_D \overline{g(w)} \left(\int_D \frac{f(z)h(z)}{(1 - w\bar{z})^2} \, dA(z) \right) dA(w)$$

$$= \int_D f(z)h(z) \overline{\int_D \frac{g(w)}{(1 - z\bar{w})^2} \, dA(w)} \, dA(z) \tag{9}$$

$$= \int_D f(z)h(z)\overline{g(z)} \, dA(z) \tag{10}$$

$$= \int_D h\overline{T_{\bar{f}}g} \, dA. \tag{11}$$

The equation (9) is from Fubini's Theorem, (10) follows from Proposition (2.1), (11) is the same argument as (8) through (10).

Therefore $T_f^* g = T_{\bar{f}} g$. Hence $T_f^* = T_{\bar{f}}$. QED

3 Compact Toeplitz Operators

Lemma 3.1 *Let K be a compact subset of D. Let $f \in L^\infty(D, dA)$ be such that $f \equiv 0$ on $D \setminus K$. Then T_f is a compact operator on $L^p_a(D)(1 < p < \infty)$.*

Proof. Let $\{g_n\}$ be a bounded sequence in $L^p_a(D)$. By Corollary (2.2) $\{g_n\}$ is bounded on each compact subset of D. So $\{g_n\}$ is a normal family. Hence there exists an analytic function g on D such that some subsequence $\{g_{n_j}\}$ of $\{g_n\}$ converges uniformly on K to g. Thus $\{g_{n_j} f\}$ converges in $L^p_a(D)$ to gf. Hence

$$T_f(g_{n_j}) = P(fg_{n_j}) \longrightarrow P(fg).$$

Therefore T_f is compact. QED

Lemma 3.2 *Let $1 < p < \infty$. Then $\|k_w\|_p \cong (1 - |w|^2)^{-2/p'}$, that is, there exist positive constants C_1, C_2, such that*

$$C_1(1 - |w|^2)^{-2/p'} \le \|k_w\|_p \le C_2(1 - |w|^2)^{-2/p'}$$

for every $w \in D$.

Proof. See Lemma 3.10, [1]. QED

Lemma 3.3 *$k_w/\|k_w\|_p$ converges to 0 weakly in $L_a^p(D)$ as $|w| \longrightarrow 1$.*

Proof. If $f \in L_a^{p'}(D)$, then by the reproducing property of k_w and the estimate of $\|k_w\|_p$ (See Lemma 3.1, [1]), we have

$$< f, k_w/\|k_w\|_p > \cong (1-|w|^2)^{2/p'} f(w).$$

Thus if f is a bounded function in $L_a^{p'}(D)$, then $< f, k_w/\|k_w\|_p > \longrightarrow 0$ as $|w| \longrightarrow 1$. Since polynomials are dense in $L_a^{p'}(D)$, this implies that $< f, k_w/\|k_w\|_p > \longrightarrow 0$ as $|w| \longrightarrow 1$ for all $f \in L_a^{p'}(D)$, which means that $k_w/\|k_w\|_p$ converges to 0 weakly in $L_a^p(D, dA)$ as $|w| \longrightarrow 1$. QED

Theorem 3.4 *Let $f \in C(\bar{D})$. Then T_f is a compact operator on $L_a^p(D)(1 < p < \infty)$ if and only if $f|_{\partial D} = 0$.*

Proof. $\Leftarrow$ Suppose $f|_{\partial D} = 0$. Then f can be uniformly approximated by functions with compact support in D. By the Lemma, T_f is compact.

$\Rightarrow$ Suppose that T_f is compact on $L_a^p(D)$. Let $w_0 \in \partial D$. By Lemma (3.3), $k_w/\|k_w\|_p$ converges to 0 weakly in $L_a^p(D)$ as $w \longrightarrow w_0$. Hence

$$< T_f \frac{k_w}{\|k_w\|_p}, \frac{k_w}{\|k_w\|_{p'}} > \longrightarrow 0 \qquad \text{as} \qquad w \longrightarrow w_0.$$

On the other hand, by Lemma 3.2 we have

$$\begin{aligned} |< T_f \frac{k_w}{\|k_w\|_p}, \frac{k_w}{\|k_w\|_{p'}} >| &= |\int_D \frac{f|k_w|^2}{\|k_w\|_p \|k_w\|_{p'}} dA| \\ &\cong |\int_D f|k_w|^2 (1-|w|^2)^{2/p'+2/p} dA| \\ &= |\int_D f|k_w|^2(1-|w|^2)^2 dA| \\ &= |\int_D (f \circ \phi_w) dA| \longrightarrow |f(w_0)|, \end{aligned}$$

as $w \longrightarrow w_0$.

This implies that $f(w_0) = 0$. Therefore $f|_{\partial D} = 0$.

4 Hausdorff-Young Theorem

The following theorem gives a necessary condition on the Taylor series of f in $L^p_a(D)$.

Theorem 4.1 (Hausdorff-Young Theorem) *For $1 < p \le 2$, let $f \in L^p_a(D)$ have Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then*

$$\left\{ \sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'-1}} \right\}^{1/p'} \le \|f\|_p \, . \tag{12}$$

Proof. Without loss of generality we assume that $\|f\|_p = 1$.

We first suppose that f is a polynomial with degree n. Let μ denote the measure on the set of non-negative integers such that $\mu(k) = k+1$. Let $l^p(\mu)$ denote the usual space of sequences.

Suppose that $b = (b_0, b_1, \ldots, b_n, 0, 0, \ldots)$ is such that

$$\|b\|^p_{l^p(\mu)} = \sum_{k=0}^{n} |b_k|^p (k+1) = 1.$$

We will show that

$$\left| \sum_{k=0}^{n} b_k \cdot \frac{a_k}{k+1} \cdot \mu(k) \right| = \left| \sum_{k=0}^{n} b_k a_k \right| \le 1. \tag{13}$$

Put $F = |f|^p$, $B_k = |b_k|^p$, $k = 0, 1, 2, \ldots, n$. There exist a function ϕ and complex numbers $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ such that

$$f = F^{1/p} \cdot \phi, \quad |\phi| = 1, \quad \int_D F \, dA = 1 \tag{14}$$

and

$$b_k = B_k^{1/p} \beta_k, \quad |\beta_k| = 1, \quad \sum_{k=0}^{n} B_k (k+1) = 1. \tag{15}$$

Let $\psi_k(w) = w^k$. Then

$$a_k = (k+1) \int_D f \bar{\psi}_k \, dA = (k+1) \int_D F^{1/p} \phi \bar{\psi}_k \, dA \, .$$

So

$$\sum_{k=0}^{n} b_k a_k = \sum_{k=0}^{n} B_k^{1/p} \beta_k (k+1) \int_D F^{1/p} \phi \bar{\psi}_k \, dA \, .$$

Replace $1/p$ by z, and define

$$\Phi(z) = \sum_{k=0}^{n} B_k^z \beta_k (k+1) \int_D F^z \phi \bar{\psi}_k \, dA \,. \tag{16}$$

for any complex number z. Then Φ is a entire function that is bounded on $\{z : a \leq Re(z) \leq b\}$ for any finite a and b. We shall take $a = \frac{1}{2}$ and $b = 1$, shall estimate Φ on the edges of this strip, and then apply Phragmen-Lindelöf Theorem (See page 256, [6]) to estimate $\Phi(1/p)$.

For $-\infty < y < \infty$, define

$$c_k(y) = \int_D F^{1/2} F^{iy} \phi \bar{\psi}_k \, dA \,.$$

Since $\{\psi_k \sqrt{k+1}\}$ is a orthonormal set in $L^2(D)$, by Bessel's inequality we have

$$\begin{aligned}
\sum_{k=0}^{n} |c_k(y)|^2 (k+1) &= \sum_{k=0}^{n} \left| \int_D F^{1/2} F^{iy} \phi \;\; \bar{\psi}_k \sqrt{k+1} \, dA \right|^2 \\
&\leq \int_D |F^{1/2} F^{iy} \phi|^2 \, dA \\
&= \int_D F \, dA = 1
\end{aligned}$$

and then the Schwarz inequality shows that

$$\begin{aligned}
|\Phi(\frac{1}{2} + iy)| &= \left| \sum_{k=0}^{n} B_k^{1/2} B_k^{iy} \beta_k (k+1) c_k(y) \right| \\
&= \left| \sum_{k=0}^{n} (B_k^{1/2} B_k^{iy} \beta_k \sqrt{k+1})(\sqrt{k+1} c_k(y)) \right| \\
&\leq \left\{ \sum_{k=0}^{n} B_k (k+1) \cdot \sum_{k=0}^{n} |c_k(y)|^2 (k+1) \right\}^{1/2} \leq 1 \,.
\end{aligned} \tag{17}$$

The estimate

$$\begin{aligned}
|\Phi(1 + iy)| &= \left| \sum_{k=0}^{n} B_k B_k^{iy} \beta_k (k+1) \int_D F F^{iy} \phi \bar{\psi}_k \, dA \right| \\
&\leq \sum_{k=0}^{n} B_k (k+1) = 1 \,, \quad -\infty < y < \infty \,.
\end{aligned} \tag{18}$$

follows trivially from (14), (15), and (16).

We now conclude from (17), (18) and Phragmen-Lindelöf Theorem that

$$|\Phi(x+iy)| \le 1 \quad (\frac{1}{2} \le x \le 1\,,\ -\infty \le y \le \infty).$$

Let $x = 1/p$ and $y = 0$. We have

$$|\Phi(\frac{1}{p})| = \left|\sum_{k=0}^{n} b_k \cdot \frac{a_k}{k+1} \cdot \mu(k)\right| = \left|\sum_{k=0}^{n} b_k a_k\right| \le 1.$$

Thus for each polynomial $f = \sum_{k=0}^{n} a_k z^k$ with $\|f\|_p = 1$, (13) holds for every $b \in l^p(\mu)$ with $\|b\| = 1$.

By the Hahn-Banach Theorem,

$$\left\{\sum_{k=0}^{n} \frac{|a_k|^{p'}}{(k+1)^{p'}} \cdot \mu(k)\right\}^{1/p'} = \left\{\sum_{k=0}^{n} \frac{|a_k|^{p'}}{(k+1)^{p'-1}}\right\}^{1/p'} \le 1\,.$$

As the Taylor series of any f converges in $L_a^p(D)$ ([8]), we have

$$\left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'-1}}\right\}^{1/p'} \le \|f\|_p$$

for all $f \in L_a^p(D)$. QED

Corollary 4.2 *The operator T on $L_a^p(D)$ defined by*

$$(Tf)(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^k\,, \tag{19}$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in L_a^p(D)$$

is compact.

Proof. We first assume that $1 < p \le 2$. Define T_n by

$$(T_n f)(z) = \sum_{k=0}^{n} \frac{a_k}{k+1} z^k\,.$$

Let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be a function of $L_a^{p'}(D)$ with $\|g\|_{p'} = 1$. Then

$$\begin{aligned}
\left|\int_D (T_n f)\, \bar{g}\, dA\right| &\leq \sum_{k=0}^{\infty} \frac{|a_k b_k|}{(k+1)^2} \leq \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} \left\{\sum_{k=0}^{\infty} \frac{|b_k|^p}{(k+1)^p}\right\}^{1/p} \\
&\leq \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} \left\{\sum_{k=0}^{\infty} \frac{|b_k|^2}{k+1} + \sum_{k=0}^{\infty} \frac{1}{(k+1)^p}\right\}^{1/p} \\
&\leq \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} \{\|g\|_2^2 + C_p\}^{1/p} \\
&\leq \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} \left\{\left(\int_D |g|^{p'}\, dA + 1\right)^2 + C_p\right\}^{1/p} \\
&= \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} M_p \leq M_p \|f\|_p
\end{aligned}$$

where C_p, M_p are constants.

Since $(L_a^p)^* \cong L_a^{p'}(D)$ (Theorem 1.16, [1]) and g is an arbitrary unit vector in $L_a^{p'}(D)$, the Hahn-Banach Theorem gives

$$\|T_n f\|_p \leq M_p \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'},$$

for each n.

By Fatou's Lemma, we have

$$\|Tf\|_p \leq M_p \left\{\sum_{k=0}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'}. \tag{20}$$

T_n is a finite rank operator for each n and by (20) applied to

$$f - \sum_{k=0}^{n} a_k z^k,$$

we have

$$\begin{aligned}
\|(T - T_n)f\|_p &\leq M_p \left\{\sum_{k=n+1}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'}}\right\}^{1/p'} \\
&\leq \frac{M_p}{(n+1)^{1/p'}} \left\{\sum_{k=n+1}^{\infty} \frac{|a_k|^{p'}}{(k+1)^{p'-1}}\right\}^{1/p'} \\
&\leq \frac{M_p}{(n+1)^{1/p'}} \|f\|_p
\end{aligned} \tag{21}$$

where (21) follows from Theorem 4.1.

Thus

$$\|(T - T_n)\| \le \frac{M_p}{(n+1)^{1/p'}} \longrightarrow 0.$$

Therefore T is compact.

For $2 < p < \infty$, consider T^* on $L_a^{p'}(D)$. We have

$$< f, T^*g > = < Tf, g > = \sum_{k=0}^{\infty} \frac{a_k \bar{b}_k}{(k+1)^2}$$

for all $f \in L_a^p(D)$, $g \in L_a^{p'}(D)$. Hence $T^* = T$.

By the above proof, T^* is compact on $L_a^{p'}(D)$, which implies that T is compact. QED

5 Compact Commutators

Definition 5.1 *Let $f \in C(\bar{D})$. An operator $H_f : L_a^p(D) \longrightarrow L^p(D)$ is called the Hankel operator with symbol f if*

$$H_f h = (1 - P)(fh)$$

for all $h \in L_a^p(D)$.

Clearly H_f is a bounded linear operator from $L_a^p(D)$ to $L^p(D)$.

Suppose that $\phi(z) = \bar{z}$. We will show that H_ϕ is a compact operator. We need the following calculation first. We assume that

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \in L_a^p(D).$$

Then

$$\begin{aligned} P(\phi h)(w) &= \int_D \bar{z} h(z)(1 - w\bar{z})^{-2}\, dA \\ &= \lim_{t \to 1} \int_{D_t} \bar{z} h(z) \sum_{k=0}^{\infty} (k+1)(w\bar{z})^k\, dA(z) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 1}\sum_{k=0}^{\infty}(k+1)w^k\int_{D_t}\bar{z}^{k+1}h(z)\,dA(z)\\
&= \lim_{t\to 1}\sum_{k=0}^{\infty}(k+1)w^k\int_0^t\int_0^{2\pi}\left(\sum_{k=0}^{\infty}a_kr^ke^{iks}\right)r^{k+2}e^{-i(k+1)s}\,dsdr\\
&= \sum_{k=0}^{\infty}\frac{k+1}{k+2}a_{k+1}w^k\\
&= \sum_{k=1}^{\infty}a_kw^{k-1}-\sum_{k=0}^{\infty}\frac{a_{k+1}}{k+2}w^k\,. \qquad (22)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(H_\phi h)(z) &= (1-P)(\phi h)\,(z) = \bar{z}h(z) - P(\phi h)(z)\\
&= \bar{z}h(z) - \sum_{k=1}^{\infty}a_kz^{k-1} + \sum_{k=0}^{\infty}\frac{a_{k+1}}{k+2}z^k\\
&= \frac{|z|^2}{z}h(z) + \frac{h(0)-h(z)}{z} + \sum_{k=0}^{\infty}\frac{a_{k+1}}{k+2}z^k\\
&= \frac{|z|^2-1}{z}h(z) + \frac{h(0)}{z} + \sum_{k=0}^{\infty}\frac{a_{k+1}}{k+2}z^k\\
&= (Qh)(z) + (Th)(z)\,, \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
(Qh)(z) &= \frac{|z|^2-1}{z}h(z) + \frac{h(0)}{z}\,,\\
(Th)(z) &= \sum_{k=0}^{\infty}\frac{a_{k+1}}{k+2}z^k\,.
\end{aligned}
$$

By the proof of Corollary 4.2, T is a compact operator on $L_a^p(D)$. Hence $Q = H_\phi - T$ is a bounded linear operator from $L_a^p(D)$ to $L^p(D)$.

Proposition 5.2 *Let $\phi(z) = \bar{z}$, then H_ϕ is compact.*

Proof. By (23) we only need to show that Q is compact. It is sufficient to show that if $\{h_k\}$ is a sequence of $L_a^p(D)$ and $h_n \longrightarrow 0$ weakly, then

$$\|Qh_n\|_p \longrightarrow 0.$$

Since $h_n \longrightarrow 0$ weakly in $L^p_a(D)$, by Cauchy's formula $h_n \longrightarrow 0$ uniformly on compact subsets of D. Let $\epsilon > 0$ and

$$f(z) = \frac{|z|^2 - 1}{z}.$$

We can pick $g \in C(\bar{D} \setminus 0)$ such that $g = 0$ on a neighborhood of ∂D, $g = f$ on a neighborhood O of 0, and

$$\|f - g\|_\infty < \epsilon. \tag{24}$$

Let O_r be a small open disc in O centered at 0 and

$$K = \overline{\{z \in \bar{D} : g(z) \neq 0\} \setminus O_r}.$$

Then K is compact and bounded away from 0.

$$\|(Qh_n)(z)\|_p = \|(fh_n)(z) + \frac{h_n(0)}{z}\|_p \leq \|(f-g)h_n\|_p + \|(gh_n)(z) + \frac{h_n(0)}{z}\|_p. \tag{25}$$

The Principle of Uniform Boundedness gives us that $\{h_n\}$ is norm bounded by some M. So by (24)

$$\|(f-g)h_n\|_p \leq \|f-g\|_\infty \|h_n\|_p < \epsilon M. \tag{26}$$

Now we want to show

$$\|(gh_n)(z) + \frac{h_n(0)}{z}\|_p \longrightarrow 0. \tag{27}$$

The partition of D gives

$$\begin{aligned}
\|(gh_n)(z) + \frac{h_n(0)}{z}\|_p^p &= \int_D \left|(gh_n)(z) + \frac{h_n(0)}{z}\right|^p dA(z) \\
&= \int_{O_r} \left|(gh_n)(z) + \frac{h_n(0)}{z}\right|^p dA(z) \\
&\quad + \int_K \left|(gh_n)(z) + \frac{h_n(0)}{z}\right|^p dA(z) \\
&\quad + \int_{D\setminus(K\cup O)} \left|(gh_n)(z) + \frac{h_n(0)}{z}\right|^p dA(z) \\
&= A_n + B_n + C_n.
\end{aligned}$$

Since $h_n \longrightarrow 0$ uniformly on compact subsets of D, it is clear that

$$B_n \longrightarrow 0,$$

and

$$C_n = \int_{D\setminus(K\cup O)} \left|\frac{h_n(0)}{z}\right|^p dA(z) \longrightarrow 0.$$

For r sufficiently small,

$$\begin{aligned} A_n &= \int_{O_r} \left|\frac{|z|^2-1}{z} h_n(z) + \frac{h_n(0)}{z}\right|^p dA(z) \\ &= \int_{O_r} \left|\bar{z} + \frac{h_n(z)-h_n(0)}{z}\right|^p dA(z) \\ &\leq \int_{O_r} [|h_n'(0)| + O(|z|)]^p \, dA(z) < \epsilon \end{aligned}$$

since $\{h_n'(0)\}$ is bounded.

Thus we proved (27). By (25), (26) and (27) we have $\|Qh_n\|_p \longrightarrow 0$. Therefore Q is a compact operator. This completes the proof that $H_\phi = Q + T$ is compact. QED

Proposition 5.3 *Let f and g be functions in $C(\bar{D})$. Then $T_{fg} - T_fT_g$ and $T_fT_g - T_gT_f$ are compact operators on $L^p_a(D)$.*

Proof. For $f \in C(\bar{D})$, define

$$H_f', \; S_f \; : L^p(D) \longrightarrow L^p(D)$$

by

$$H_f'h = P(fh), \; S_fh = (1-P)(fh),$$

where P is the bounded projection from $L^p(D)$ onto $L^p_a(D)$.

The operators defined above are bounded linear operators. Straightforward calculations show that

$$T_{fg} - T_fT_g = H_f'H_g \tag{28}$$

$$H_{fg} = S_fH_g + H_fT_g. \tag{29}$$

Let

$$B = \{f \in C(\bar{D}) : H_f \textit{ is compact}\}.$$

Clearly B is a closed subspace of $C(\bar{D})$. (28) and (29) shows that B is a closed subalgebra of $C(\bar{D})$.

We know that $H_1 = H_z = 0$, and by Proposition 5.2, $\bar{z} \in B$. Thus $1,\ z,\ \bar{z} \in B$. By Stone-Weierstrass Theorem, $B = C(\bar{D})$. Therefore H_f is compact for every $f \in C(\bar{D})$.

If $f, g \in C(\bar{D})$, then $T_{fg} - T_f T_g = H'_f H_g$ is compact. Consequently,

$$T_f T_g - T_g T_f = (T_f T_g - T_{fg}) + (T_{fg} - T_g T_f)$$

is also compact. QED

6 Commutator Ideal

In this section, we will show that the commutator ideal $\mathcal{J}$ of $\mathcal{T}$ (defined in Introduction) is equal to the set of compact operators $\mathcal{K}$.

Proposition 6.1 $\mathcal{J} = \mathcal{K}$.

Proof. By Proposition 5.3, $T_f T_g - T_g T_f \in \mathcal{K}$ for all $f,\ g \in C(\bar{D})$. Hence $\mathcal{J} \subset \mathcal{K}$.

We need to show that $\mathcal{K} \subset \mathcal{J}$. Since the Taylor series of functions in $L^p_a(D)$ converge in norm (see [8]), $L^p_a(D)$ has Schauder basis. Therefore the set of finite rank operators is dense in $\mathcal{K}$ by the similar argument to the proof in Hilbert space case (Theorem 4.4, Chapter 2, [3]). It is sufficient to show that $\mathcal{J}$ contains the set of finite rank operators.

The following three lemmas will finish the proof of Proposition 6.1.

Lemma 6.2 *The operator A_0 defined by $A_0 f = a_0$, where*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \ \in L^p_a(D).$$

is in $\mathcal{J}$.

Proof. By the calculation in (22), we have

$$(T_{\bar z}T_z f)(w) = \sum_{k=0}^{\infty} a_k w^k - \sum_{k=1}^{\infty} \frac{a_k}{k+2} w^k ,$$

$$(T_z T_{\bar z} f)(w) = \sum_{k=1}^{\infty} a_k w^k - \sum_{k=0}^{\infty} \frac{a_k}{k+1} w^k .$$

Hence

$$(T_{\bar z}T_z - T_z T_{\bar z})h\ (w) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \frac{a_k}{(k+1)(k+2)} w^k . \qquad (30)$$

Let $A = 2(T_{\bar z}T_z - T_z T_{\bar z})$. Then $A \in \mathcal{J}$. By (30),

$$(A^n f)(w) = a_0 + \sum_{k=1}^{\infty} \frac{2^n\, a_k}{(k+1)^n (k+2)^n} w^k . \qquad (31)$$

We want to show that $\|A_0 - A^n\| \longrightarrow 0$. Assume $1 < p \leq 2$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in L_a^p(D),\ \|f\|_p = 1$$

and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \in L_a^{p'}(D),\ \|g\|_{p'} = 1 .$$

By the proof of Corollary 4.2 and (31),

$$\begin{aligned} \left|\int_D (A_0 - A^n) f\, \bar g\, dA\right| &\leq \sum_{k=1}^{\infty} \frac{|a_k b_k|}{k+1} \cdot \frac{2^n}{(k+1)^n (k+2)^n} \\ &\leq \left(\frac{2}{3}\right)^n \left(\sum_{k=1}^{\infty} \frac{|a_k|}{k+1} \cdot \frac{|b_k|}{k+1}\right) \\ &\leq \left(\frac{2}{3}\right)^n (1 + C) \longrightarrow 0 . \end{aligned}$$

By the Hahn-Banach Theorem,

$$\|A_0 - A^n\| \leq (1 + C) \left(\frac{2}{3}\right)^n \longrightarrow 0 .$$

Since $\mathcal{J}$ is a closed ideal, $A_0 \in \mathcal{J}$ for $1 < p \leq 2$. For $p > 2$, we have the same result by interchanging f and g in above proof. QED

Lemma 6.3 *Let $\phi_n(z) = z^n$, $n = 0, 1, 2, \cdots$. Then the operator*

$$A_n f = 2\int_D f\,\bar{\phi}_n\,dA = \frac{a_n}{n+1}\,,\ f(z) = \sum_{k=0}^{\infty} a_k z^k \ \in L_a^p(D)$$

is in $\mathcal{J}$.

Proof. We use induction on n.

For $n = 0$, $A_0 \in \mathcal{J}$ by Lemma 6.2.

For $n = 1$, we will show that $A_1 = A_0 T_{\bar{z}} \in \mathcal{J}$. By (22),

$$(T_{\bar{z}} f)(w) = \sum_{k=1}^{\infty} a_k w^{k-1} - \sum_{k=0}^{\infty} \frac{a_{k+1}}{k+2} w^k\,. \tag{32}$$

Hence

$$A_0(T_{\bar{z}} f) = \frac{a_1}{2} = A_1 f\,.$$

Assume $A_n \in \mathcal{J}$, we want to show that $A_{n+1} \in \mathcal{J}$. By (32), we have

$$A_n T_{\bar{z}} f = \frac{1}{n+2} a_{n+1} = A_{n+1} f\,.$$

Thus

$$A_{n+1} = A_n T_{\bar{z}} \in \mathcal{J}$$

by the assumption. QED

Corollary 6.4 *Let p_n be a polynomial of degree n. Then the operator defined by*

$$B_n f = \int_D f\,\bar{p}_n\,dA\,,\ f \in L_a^p(D)$$

is in $\mathcal{J}$.

Proof. Since $\mathcal{J}$ is an ideal and B_n is a linear combination of operators in Lemma 6.3, $B_n \in \mathcal{J}$ by Lemma 6.3. QED

Lemma 6.5 *$\mathcal{J}$ contains all finite rank operators.*

Proof. We first show that $\mathcal{J}$ contains rank one operators.

Let T be a rank one operator on $L^p_a(D)$. Then there exist $g \in L^p_a(D)$ and $\phi \in L^{p'}_a(D)$ such that

$$Tf = \phi(f)\,g\,, \quad \phi(f) = \int_D f\,\bar{\phi}\,dA\,.$$

Since the set of polynomials is dense in $L^{p'}_a(D)$, there exists a polynomial sequence $\{p_n\}$ convergent to ϕ in $L^{p'}_a(D)$. Let

$$\phi_n(f) = \int_D f\,\bar{p_n}\,dA\,, \quad T_n f = \phi_n(f)g\,.$$

Then $T_n = T_g\phi_n \in \mathcal{J}$ as a consequence of $\phi_n \in \mathcal{J}$ by Corollary 6.4.

We want to show that $T_n \longrightarrow T$ in norm.

$$\begin{aligned} \|T - T_n\| &= \sup_{\|f\|_p \leq 1} \|g\,(\phi - \phi_n)(f)\| \\ &\leq \|g\|_p\,\|\phi - \phi_n\|_{p'} \\ &= \|g\|_p\,\|\phi - p_n\|_{p'} \longrightarrow 0. \end{aligned}$$

Thus $T \in \mathcal{J}$. So $\mathcal{J}$ contains all rank one operators. Therefore $\mathcal{J}$ contains all finite rank operators since a finite rank operator is a linear combination of rank one operators. QED

Now Proposition 6.1 follows easily from Lemma 6.5.

7 Spectral Properties

Consider the map $\alpha\ :\ C(\bar{D}) \longrightarrow \mathcal{T}/\mathcal{K}$ defined by $\alpha(f) = T_f + \mathcal{K}$. By Proposition 5.3, α is a homomorphism, and hence its range is a subalgebra of $\mathcal{T}/\mathcal{K}$. The definition of $\mathcal{T}$ implies that $\alpha(C(\bar{D}))$ is dense in $\mathcal{T}/\mathcal{K}$.

Let $\mathcal{Z}$ denote the set of functions in $C(\bar{D})$ that are zero on ∂D. By Theorem 3.4, the kernel of α is precisely $\mathcal{Z}$. Thus there is a homomorphism $\tilde{\alpha}$ from $C(\bar{D})/\mathcal{Z}$ into $\mathcal{T}/\mathcal{K}$ defined by

$$\tilde{\alpha}(f + \mathcal{Z}) = T_f + \mathcal{K}\,.$$

Now $\tilde{\alpha}$ is an injective homomorphism. We will show that it is bounded below.

Theorem 7.1 *The map* $\tilde{\alpha} : C(\bar{D})/\mathcal{Z} \longrightarrow \mathcal{T}/\mathcal{K}$ *is bounded below. That is, there exists a* $C > 0$ *such that*

$$\|\tilde{\alpha}(f + \mathcal{Z})\| = \|T_f + \mathcal{K}\| \geq C\,\|f + \mathcal{Z}\| = C\,\|f\|_{\partial D}\,, \tag{33}$$

for every $f \in C(\bar{D})$.

The proof of Theorem 7.1 will require several lemmas.

Lemma 7.2 *If for every* $f \in C(\partial D)$ *there-is a continuous extension of* f *to* D *that satisfies* (33), *then Theorem* 7.1 *holds.*

Proof. For each $g \in C(\bar{D})$, consider the function $g|_{\partial D}$ on ∂D. Let $\tilde{g}$ be the continuous extension of $g|_{\partial D}$ to D such that (33) holds.

Since $g - \tilde{g} \in \mathcal{Z}$. Thus $g = \tilde{g}$ in $C(\bar{D})/\mathcal{Z}$.

Therefore that (33) holds for $\tilde{g}$ implies that (33) holds for g. QED

Let $\{w_n\} \subset D$ such that $w_n \longrightarrow w_0 \in \partial D$. Let $k_n = k_{w_n}$, and

$$g_n = \frac{k_n}{\|k_n\|_p}\,.$$

Then $\|g_n\|_p = 1$ and $g_n \longrightarrow 0$ weakly in $L^p_a(D)$ by Lemma 3.3.

Lemma 7.3 *Let* C *be as in* (33). *If for every* $f \in C(\bar{D})$, *there exists a sequence* $\{w_n\}$ *of* D *with* $w_n \longrightarrow w_0 \in \partial D$, *such that*

$$\|T_f g_n\|_p \geq C\|f\|_{\partial D}\,, \tag{34}$$

then Theorem 7.1 *holds.*

Proof. For every $K \in \mathcal{K}$, we have

$$\begin{aligned} \|T_f + K\| &\geq \|(T_f + K)g_n\|_p \\ &\geq \|T_f g_n\|_p - \|K g_n\|_p \\ &\geq C\|f\|_{\partial D} - \|K g_n\|_p. \end{aligned}$$

Since K is compact, $\|Kg_n\|_p \longrightarrow 0$. Thus

$$\|T_f + K\| \geq C\|f\|_{\partial D}.$$

So we have (33). QED

Lemma 7.4 *Let C be as in (33). If for every $f \in C(\bar{D})$, there exists a sequence $\{w_n\}$ of D with $w_n \longrightarrow w_0 \in \partial D$, such that*

$$|\int_D \frac{f(z)}{|1 - w_n\bar{z}|^4}\, dA(z)| \geq \frac{C\,\|f\|_{\partial D}}{(1 - |w_n|^2)^2}, \tag{35}$$

then Theorem 7.1 holds.

Proof. By Lemma 3.2 and (35), we have

$$\begin{aligned}
\|T_f g_n\|_p &\geq \left| < T_f \frac{k_n}{\|k_n\|_p}, \frac{k_n}{\|k_n\|_{p'}} > \right| \\
&= \frac{1}{\|k_n\|_p\,\|k_n\|_{p'}} \,|< T_f k_n , k_n >| \\
&\cong (1 - |w_n|^2)^{2/p + 2/p'}\,|(T_f k_n)(w_n)| \\
&= (1 - |w_n|^2)^2\,|P(f k_n)| \\
&= (1 - |w_n|^2)^2 \left| \int_D \frac{f(z) k_n(z)}{(1 - w_n\bar{z})^2}\, dA(z) \right| \\
&= (1 - |w_n|^2)^2 \left| \int_D \frac{f(z)}{|1 - w_n\bar{z}|^4}\, dA(z) \right| \\
&\geq C\,\|f\|_{\partial D}.
\end{aligned}$$

Hence (34) holds. By Lemma 7.3, Theorem 7.1 follows. QED

For $w \in D$, let ϕ_w be the function defined by

$$\phi_w(z) = \frac{w - z}{1 - \bar{w}z}$$

For w and z in D, the pseudo-hyperbolic distance $d(w, z)$ between w and z is defined by

$$d(w, z) = |\phi_w(z)|.$$

For $w \in D$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(w,r)$ with center w and radius r is defined by

$$D(w,r) = \{z \in D : \ d(w,z) < r\}.$$

Since ϕ_w is a fractional linear transformation, the pseudo-hyperbolic disk $D(w,r)$ is also a Euclidean disk.

Simple calculation shows that ϕ_w preserves pseudo-hyperbolic distance

$$d(\lambda, z) = d(\phi_w(\lambda), \phi_w(z))$$

for all $\lambda, \ z \in D$.

Lemma 7.5 *Let $w \in D$ and let $0 < r < 1$. Then*

$$\int_{D(w,r)} \frac{dA(z)}{|1 - z\bar{w}|^4} = \frac{r^2}{(1 - |w|^2)^2}.$$

Proof. See Lemma 4.7, [1].

Lemma 7.6 *Let u be a harmonic function in D such that $\|u\| \le 1$ and $u(0) = 0$. Then*

$$|u(z)| \le \frac{4}{\pi} \arctan |z|.$$

Proof. See Lemma in [5].

Corollary 7.7 *Let w and z in D. Let $\lambda > 0$. Then there exists a $\delta > 0$ such that for any f harmonic in D, we have*

$$|f(w) - f(z)| \le \lambda \, \|f\|_{\partial D}$$

whenever $d(w,z) < \delta$.

Proof. Suppose that $\beta = \phi_w^{-1}(z)$. Then by the harmonicity of $f \circ \phi_w$ and Lemma 7.6 ,

$$\begin{aligned} |f(w) - f(z)| &= |f(0) - f(\phi_w(\beta))| \\ &\leq \|f \circ \phi_w\|_{\partial D} \frac{8}{\pi} \arctan |\phi_w^{-1}(z)| \\ &= \|f\|_{\partial D} \frac{8}{\pi} \arctan d(w, z) \\ &\leq \|f\|_{\partial D} C_1 d(w, z) , \end{aligned}$$

where $C_1 > 0$.

Let $\delta = \lambda / C_1$. The proof is completed. QED

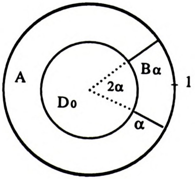
Lemma 7.8 *There exists a constant $C > 0$ such that if $f_1 \in C(\partial D)$, then there is a continuous extension f of f_1 to D that satisfies*

$$\left| \int_D \frac{f(z)}{|1 - w_n \bar{z}|^4} \, dA(z) \right| \geq \frac{C \, \|f\|_{\partial D}}{(1 - |w_n|^2)^2} \tag{36}$$

for some sequence $\{w_n\}$ of D with $w_n \longrightarrow w_0 \in \partial D$.

Proof . Let $f_1 \in C(\partial D)$. We first assume that f_1 is real valued and $f_1(1) = \|f_1\|_{\partial D}$.

We first consider the harmonic extension of f_1 to D, still denoted by f_1. Then $|f_1| \leq \|f_1\|_{\partial D}$ in D. Since $f_1(1) > 0$, there exists an $\alpha > 0$ such that $f_1 \geq 0$ in the region B_α shown in the Figure.



Let $\{w_n\} \subset (0,1) \cap B_\alpha$ be such that $w_n \longrightarrow 1$, and

$$|f_1(w_n) - f_1(1)| \le \frac{1}{4} \|f_1\|_{\partial D}$$

for each n.

By Corollary 7.7, there is a $\delta > 0$ (independent of f_1) such that

$$|f_1(w) - f_1(w_n)| \le \frac{1}{4} \|f_1\|_{\partial D}$$

for each $w \in D(w_n, \delta)$, where $D(w_n, \delta)$ is the pseudo-hyperbolic disk centered at w_n with radius δ.

Thus for $w \in D(w_n, \delta)$, we have

$$\begin{aligned} |f_1(w) - f_1(1)| &\le |f_1(w) - f_1(w_n)| + |f_1(w_n) - f_1(1)| \\ &\le \frac{1}{2} \|f_1\|_{\partial D} . \end{aligned} \tag{37}$$

Since $\|f_1\|_{\partial D} = f_1(1)$, equation (37) gives

$$f_1(w) \ge \frac{1}{2} \|f_1\|_{\partial D} \tag{38}$$

for $w \in D(w_n, \delta)$.

By (38) we know there is some room along the border of B_α and D_0 (See the Figure), so that we can define a continuous function f on $\bar{D}$ such that $f = 0$ on D_0, $f = f_1$ on ∂D, $|f| \le \|f_1\|_{\partial D}$ on D, $f \ge 0$ on B_α, and $f = f_1$ on all $D(w_n, \delta)$.

By the definition of f, equation (38) holds for f. The definition of f also implies that

$$|\int_D \frac{f(z)}{|1 - w_n \bar{z}|^4} \, dA(z)| \ge \int_{D(w_n,\delta)} \frac{f(z)}{|1 - w_n \bar{z}|^4} \, dA(z) - \int_A \frac{|f(z)|}{|1 - w_n \bar{z}|^4} \, dA(z) . \tag{39}$$

We will estimate the two terms separately.

By (38) and Lemma 7.5, the first term becomes

$$\begin{aligned} \int_{D(w_n,\delta)} \frac{f(z)}{|1 - w_n \bar{z}|^4} \, dA(z) &\ge \frac{1}{2} \int_{D(w_n,\delta)} \frac{\|f\|_{\partial D}}{|1 - w_n \bar{z}|^4} \, dA(z) \\ &= \frac{1}{2} \frac{\|f\|_{\partial D}}{(1 - |w_n|^2)^2} \, \delta^2 . \end{aligned} \tag{40}$$

The elementary inequality

$$1 - 2w_n r \cos\alpha + r^2 w_n^2 \geq 1 - \cos^2\alpha\,, \quad 0 < r < 1$$

gives us a boundary for the second term in the following calculation:

$$\begin{aligned}
\int_A \frac{|f(z)|}{|1 - w_n\bar{z}|^4}\,dA(z) &\leq \|f\|_{\partial D}\,\pi^{-1}\int_{1-\alpha}^{1}\int_{\alpha}^{2\pi-\alpha}\frac{rdtdr}{|1 - w_n r e^{it}|^4} \\
&\leq \|f\|_{\partial D}\,2\int_{1-\alpha}^{1}\frac{rdr}{|1 - w_n r e^{i\alpha}|^4} \\
&= 2\|f\|_{\partial D}\int_{1-\alpha}^{1}\frac{rdr}{(1 - 2w_n r\cos\alpha + r^2 w_n^2)^2} \\
&\leq 2\|f\|_{\partial D}\int_{1-\alpha}^{1}\frac{rdr}{(1 - \cos^2\alpha)^2} \qquad (41) \\
&= \|f\|_{\partial D}\,\frac{2\alpha}{(1 - \cos^2\alpha)^2} \\
&= C_1\,\|f\|_{\partial D}\,, \qquad (42)
\end{aligned}$$

where $C_1 > 0$.

Now by (39), (40) and (42), we have

$$\begin{aligned}
\left|\int_D \frac{f(z)}{|1 - w_n\bar{z}|^4}\,dA(z)\right| &\geq \int_{D(w_n,\delta)}\frac{f(z)}{|1 - w_n\bar{z}|^4}\,dA(z) - \int_A \frac{|f(z)|}{|1 - w_n\bar{z}|^4}\,dA(z) \\
&\geq \frac{1}{2}\delta^2\,\frac{\|f\|_{\partial D}}{(1 - |w_n|^2)^2} - C_1\,\|f\|_{\partial D} \\
&= \frac{\|f\|_{\partial D}}{(1 - |w_n|^2)^2}\,(\frac{1}{2}\delta^2 - C_1\,(1 - |w_n|^2)^2)\,.
\end{aligned}$$

Since $C_1\,(1 - |w_n|^2)^2 \longrightarrow 0$,

$$\frac{1}{2}\delta^2 - C_1\,(1 - |w_n|^2)^2 \geq \frac{1}{3}\delta^2\,,$$

for n large.

Let $C = \frac{1}{3}\delta^2$. Then (36) holds.

If f_1 is real and $f_1(w_0) = \|f_1\|_{\partial D}$, then replacing $\{w_n\}$ by $\{w_n\bar{w}_0\}$ and applying (36) to $g(w) = f_1(\bar{w}_0 w)$ gives us the desired result.

Now we consider complex function $f_1 = u_1 + iv_1 \in C(\partial D)$. Suppose that $\|u_1\|_{\partial D} > \|f_1\|_{\partial D}/2$. Then there is an extension u of u_1 satisfying (36) for some $\{w_n\} \subset D$ with $w_n \longrightarrow w_0 \in D$.

Let v be an extension of v_1 and let $f = u + iv$. We have

$$\begin{aligned}\left|\int_D \frac{f(z)}{|1 - w_n\bar{z}|^4}\, dA(z)\right|^2 &\geq \left|\int_D \frac{u(z)}{|1 - w_n\bar{z}|^4}\, dA(z)\right|^2 \\ &\geq \frac{(C\|u\|_{\partial D})^2}{(1 - |w_n|^2)^4} \\ &\geq \frac{C^2\|f\|_{\partial D}^2}{2(1 - |w_n|^2)^4}.\end{aligned}$$

If $\|v_1\|_{\partial D} > \|f_1\|_{\partial D}/2$, a similar argument shows that (43) still holds. QED

Since $\tilde{\alpha}$ bounded below implies that $\tilde{\alpha}$ has closed range, we have proved that $\tilde{\alpha}$ is an isomorphism from $C(\bar{D})/\mathcal{Z}$ onto $\mathcal{T}/\mathcal{K}$. Furthermore, there is an obvious isomorphism between $C(\partial D)$ and $C(\bar{D})/\mathcal{Z}$. Therefore we have the following theorem

Theorem 7.9 *There is a Banach algebra isomorphism between $C(\partial D)$ and $\mathcal{T}/\mathcal{K}$.*

Now we are ready to study the essential spectrum of T_f for $f \in C(\bar{D})$. Let $\sigma_e(T_f)$ denote the essential spectrum of T_f. Then

$$\sigma_e(T_f) = \{\lambda \in C \ : \ T_f - \lambda \ is\ not\ Fredholm\} \tag{43}$$

Theorem 7.10 *Let $f \in C(\bar{D})$. Then $\sigma_e(T_f) = f(\partial D)$.*

Proof. If $\lambda \notin f(\partial D)$, then $f - \lambda$ is invertible in $C(\partial D)$. By Theorem 7.9, $T_f - \lambda + \mathcal{K} = T_{f-\lambda} + \mathcal{K}$ is invertible in $\mathcal{T}/\mathcal{K}$. Thus $T_f - \lambda + \mathcal{K}$ is invertible in $\mathcal{B}/\mathcal{K}$. By Atkinson's Theorem $\lambda \notin \sigma_e(T_f)$. Therefore $\sigma_e(T_f) \subset f(\partial D)$.

If $\lambda \in f(\partial D)$, then for some $w_0 \in \partial D$, $f(w_0) = \lambda$. We want to show that $T_f - \lambda$ is not Fredholm operator.

Without loss of generality we can assume that $f(1) = 0$. We will show that T_f is not Fredholm operator.

Define the function sequence $\{g_n\}$ by

$$g_n(z) = (\frac{z+1}{2})^n\,, \quad n = 1, 2, 3, \cdots .$$

Then $fg_n \longrightarrow 0$ uniformly since $f(1) = 0$. Thus

$$T_f T_{g_n} = T_{fg_n} \longrightarrow 0\,. \tag{44}$$

Suppose that T_f is Fredholm. Then there exist an operator $S \in \mathcal{B}$ and a compact operator K such that

$$ST_f = 1 + K\,. \tag{45}$$

By (44), $ST_fT_{g_n} \longrightarrow 0$. Equation (45) gives

$$ST_fT_{g_n} = T_{g_n} + KT_{g_n} \longrightarrow 0\,.$$

So $T_{g_n} + \mathcal{K} \longrightarrow 0$ in $\mathcal{T}/\mathcal{K}$.

By Theorem 7.9, there is an Banach algebra isomorphism $\Phi\ : \mathcal{T}/\mathcal{K} \longrightarrow C(\partial D)$ such that

$$\Phi(T_f + \mathcal{K}) = f|_{\partial D}\,.$$

Therefore

$$\Phi(T_{g_n} + \mathcal{K}) = g_n|_{\partial D} \longrightarrow 0\,,$$

which contradicts $g_n(1) = 1$. QED

In the second part of the proof above, we showed that if $0 \in f(\partial D)$, then T_f is not Fredholm. We will further show that the same condition also implies that T_f is not bounded below.

Recall that $D(w, r)$ is the pseudo-hyperbolic disk with center $w \in D$ and radius r.

Lemma 7.11 *Let $w \in D$ and $r \in (0,1]$. Then*

$$\int_{D(w,r)} |k_w|^p \, dA = (1-|w|^2)^{2-2p} \int_{D(0,r)} |1-\bar{w}z|^{2p-4} \, dA(z)$$

Proof. Recall that

$$\phi_w(z) = \frac{w-z}{1-\bar{w}z}.$$

Hence

$$\phi_w'(z) = \frac{(|w|^2-1)}{(1-\bar{w}z)^2}.$$

Thus we have

$$\begin{aligned}
\int_{D(w,r)} |k_w|^p \, dA &= \int_{D(w,r)} \frac{1}{|1-\bar{w}z|^{2p}} \, dA(z) \\
&= (1-|w|^2)^{-p} \int_{D(w,r)} |\phi_w'(z)|^{p-2} \, |\phi_w'(z)|^2 \, dA(z) \\
&= (1-|w|^2)^{-p} \int_{D(0,r)} |\phi_w'(\phi_w(z))|^{p-2} \, dA(z),
\end{aligned}$$

where the last equality uses the change of variables formula. But

$$\phi_w'(\phi_w(z)) = \frac{(1-\bar{w}z)^2}{(|w|^2-1)},$$

so we have

$$\int_{D(w,r)} |k_w|^p \, dA = (1-|w|^2)^{2-2p} \int_{D(0,r)} |1-\bar{w}z|^{2p-4} \, dA(z).$$

QED

Proposition 7.12 *Let $f \in C(\bar{D})$. If $0 \in f(\partial D)$, then T_f is not bounded below.*

Proof. Without loss of generality we can assume that $f(1) = 0$.

For $w \in D$, let $k_w(z) = 1/(1-\bar{w}z)^2$. Then $k_w/\|k_w\|_p$ has norm 1.

Since

$$\|T_f(\frac{k_w}{\|k_w\|_p})\|_p = \|P(\frac{fk_w}{\|k_w\|_p})\|_p \le \|\frac{fk_w}{\|k_w\|_p}\|_p,$$

it is sufficient to show that

$$\|\frac{fk_w}{\|k_w\|_p}\|_p \longrightarrow 0,$$

as $w \longrightarrow 1$.

$D(w,r)$ still denote the pseudo-hyperbolic disk $D(w,r)$ with center w and radius r $(0 < r < 1)$. Then by Lemma 7.11,

$$\int_D \frac{|fk_w|^p}{\|k_w\|_p^p}\, dA = \int_{D(w,r)} \frac{|fk_w|^p}{\|k_w\|_p^p}\, dA + \int_{D\backslash D(w,r)} \frac{|fk_w|^p}{\|k_w\|_p^p}\, dA$$

$$\leq (\sup_{D(w,r)} |f|) \cdot \int_{D(w,r)} \frac{|k_w|^p}{\|k_w\|_p^p}\, dA + \frac{\|f\|_\infty^p}{\|k_w\|_p^p} \int_{D\backslash D(w,r)} |k_w|^p\, dA$$

$$\leq \sup_{D(w,r)} |f| + \frac{\|f\|_\infty^p}{\|k_w\|_p^p} \Big(\int_D |k_w|^p\, dA - \int_{D(w,r)} |k_w|^p\, dA\Big). \tag{46}$$

By Lemma 3.2 and Lemma 7.11, the second term of (46) becomes

$$\frac{\|f\|_\infty^p}{\|k_w\|_p^p} \Big(\int_D |k_w|^p\, dA(z) - \int_{D(w,r)} |k_w|^p\, dA\Big)$$

$$= \frac{\|f\|_\infty^p}{\|k_w\|_p^p} (1-|w|^2)^{2-2p} \Big(\int_D |1-\bar{w}z|^{2p-4}\, dA(z) - \int_{D(0,r)} |1-\bar{w}z|^{2p-4}\, dA(z)\Big)$$

$$\cong \|f\|_\infty^p \Big(\int_D |1-\bar{w}z|^{2p-4}\, dA(z) - \int_{D(0,r)} |1-\bar{w}z|^{2p-4}\, dA(z)\Big).$$

Let $\epsilon > 0$. If $p \geq 2$, there exists a $\delta \in (0,1)$ (independent of w) such that for $r \in (\delta, 1)$, the above quantity is less than ϵ. Let $w \longrightarrow 1$. The first term of (46) goes to 0:

$$\sup_{D(w,r)} |f(z)| \longrightarrow f(1) = 0.$$

Therefore T_f is not bounded below on $L_a^p(D)$ for $p \geq 2$.

For $1 < p < 2$, we want to prove that

$$\int_D |1-\bar{w}z|^{2p-4}\, dA(z) - \int_{D(0,r)} |1-\bar{w}z|^{2p-4}\, dA(z) \longrightarrow 0$$

uniformly with respect to w as $r \longrightarrow 1$.

Let $g(z) = (1-z)^{p-2}$, so $g \in L_a^2(D)$. Suppose that

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$\|g\|_2^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$$

and

$$\int_{D(0,r)} |g|^2 \, dA = \int_D |f(rz)|^2 r^2 \, dA = r^2 \sum_{n=0}^{\infty} \frac{|a_n|^2 r^{2n}}{n+1} \,.$$

Therefore

$$\begin{aligned} & \int_D |1 - \bar{w}z|^{2p-4} \, dA(z) - \int_{D(0,r)} |1 - \bar{w}z|^{2p-4} \, dA(z) \\ = & \int_D |g(wz)|^2 \, dA(z) - \int_D |g(wrz)|^2 r^2 \, dA(z) \\ = & \sum_{n=0}^{\infty} \frac{|a_n|^2 w^{2n}}{n+1} - r^2 \sum_{n=0}^{\infty} \frac{|a_n|^2 w^{2n} r^{2n}}{n+1} \longrightarrow 0 \end{aligned}$$

uniformly as $r \longrightarrow 1$. QED

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