M-LEVEL ROOK PLACEMENTS

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ABSTRACT

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Rook theory focuses on placements of non-attacking rooks on boards of various shapes. An important role is played by the rook numbers which count the number of non-attacking placements of a given number of rooks on a board. Ferrers boards, which are boards indexed by integer partitions, are of particular interest. Briggs and Remmel introduced a generalization of rook placements, called m-level rook placements, where a rook is able to attack a subset of the rows.

This manuscript presents generalizations of many of the central results regarding rook placements to the case of m-level rook placements. Goldman, Joichi, and White defined the rook polynomial of a board to be the generating function for the rook numbers of that board in the falling factorial basis. By doing so, they were able to give an elegant factorization of the rook polynomial of a Ferrers board in terms of the various column heights. Briggs and Remmel were able to generalize this factorization to the m-level rook polynomial of a subset of Ferrers boards called singleton boards.

We give two factorization theorems for the m-level rook polynomial of a Ferrers board. The first is a generalization of the factorization theorem of Briggs and Remmel, working from similar principles. The second relies on a generalization of transposition which we present, called the l-operator. We are also able to use the factorization to describe a unique representative in any m-level equivalence class of Ferrers boards and count the number of singleton boards in the class..

When generalizing the factorization from singleton boards to all Ferrers boards, we pre-

serve the definition of the m-level rook polynomial and alter the factorization to apply to all Ferrers boards. We also consider the dual of this problem, applying the factorization of Briggs and Remmel to all Ferrers boards, then trying to determine what is counted by the coefficients of the polynomial in the m-falling factorial basis. It turns out that the coefficients count weighted file placements on a Ferrers board. We also describe a unique representative in each weighted file placement equivalence class of Ferrers boards, as well as count of the number of Ferrers boards in a given weighted file placement equivalence class.

Foata and Schützenberger presented explicit bijections between rook placements on any two rook equivalent Ferrers boards as part of their construction of a unique representative in each equivalence class of Ferrers boards. A key tool in their construction was local transposition. We present analogous bijections between m-level rook placements on any two m-level rook equivalent Ferrers boards using the local l-operator.

The Garsia-Milne Involution Principle was first used in Garsia and Milne's bijective proof of the Rogers-Ramanujan identities. We use it to construct two types of explicit bijections. The first is an explicit bijection between m-level rook placements on any two m-level rook equivalent singleton boards. The second bijection is between the sets counted by the m-level analogue of hit numbers of any two m-level rook equivalent Ferrers boards, providing a bijective proof that m-level equivalent Ferrers boards have the same hit numbers.

To Jenn, whose patience with this project often outstripped my own.

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Similarly, that I am here stands testament to the enormous talent of the mathematics educators from whom I have been fortunate enough to study. In public schools these educators began with Mrs. Hanshumaker and ended with my friend Mr. Stallons. Thank you both, and I'm still not calling you Frank. As an undergraduate, I benefited from the courses and advice I received from my eventual thesis advisor, Dr. Flahive. Finally, as a doctoral student I owe great gratitude to my committee members, Dr. Bell, Dr. Hall, Dr. Magyar, and Dr. Meierfrankenfeld, all of whom have led courses that I greatly enjoyed.

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1.1 Introduction

1.1.1 Background

The study of rook placements began with a 1946 paper of Kaplansky and Riordan [KR46]. They introduced the concept as a way of unifying various results counting the number of permutations which satisfied certain conditions, or violated a specified number of them. To do so, they considered placing non-attacking rooks on "chessboards" of various shapes.

A *board* B is any finite subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$. Geometrically one can think of a board as a finite number of square cells in a square grid, this is illustrated on the left of Figure 1.1. The squares with darkened edges form a board and the lighter lines show the underlying grid. On the right side of the figure there is another board, this time with the underlying grid omitted. The *size* of a board B is the number of squares in the board, this is often denoted $|B|$. For example, both boards in Figure 1.1 have size 7.

A placement of non-attacking rooks on B, henceforth called a *rook placement*, is a subset $P \subseteq B$ such that no two elements of P are in the same row or column. Graphically, one can think of placing rooks in the squares of the board so that no rook can capture another rook in a single move, according to the rules of chess. The board on the left in Figure 1.1 contains a rook placement of 3 rooks. We label each cell by the Cartesian coordinates of its northeast

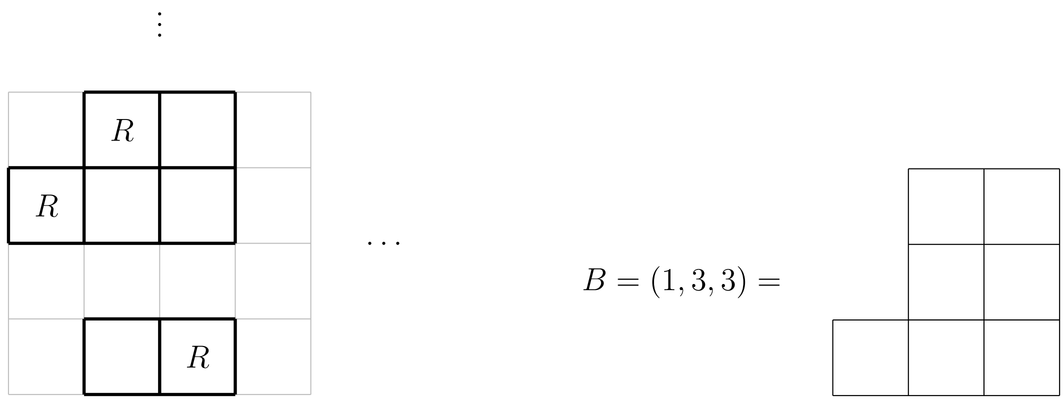

Figure 1.1: On the left, a board of size 7 and a placement of three rooks on the board. On the right, a Ferrers board, B, also of size 7.

corner, so the rooks on the left of Figure 1.1 are in cells $(1, 3)$, $(2, 4)$, and $(3, 1)$ from left to right. Note that this is neither the English nor the French style of writing Ferrers diagrams, but it is the standard convention in modern rook theory literature. It is useful because we usually consider placing rooks on the board from left to right and enumerating the number of such placements is facilitated by our convention.

Kaplansky and Riordan also introduced the generating polynomial of rook placements on a board. For any board B we let

$$r_k(B) = \text{the number of rook placements } P \subseteq B \text{ with } k \text{ rooks.}$$

We call $r_k(B)$ the *kth rook number of* B. Note that for any board $r_0(B) = 1$, as there is a unique way to place no rooks on the board, and $r_1(B) = |B|$, because a single rook can be placed in any square of the board without violating the non-attacking condition. For either board in Figure 1.1 $r_0(B) = 1$, $r_1(B) = 7$, $r_2(B) = 10$, $r_3(B) = 2$, and $r_k(B) = 0$ for $k \geq 4$. Kaplansky and Riordan defined the rook polynomial of B to be the the generating function of these rook numbers in the standard polynomial basis, yielding

$$\sum_{k\geq 0} r_k(B)x^k. \tag{1.1.1}$$

By this definition of the rook polynomial, the rook polynomial of either board in Figure 1.1 is $1 + 7x + 10x^2 + 2x^3$.

In 1970, Foata and Schützenberger [FS70] introduced the term *rook equivalent* to describe two boards which have the same rook polynomial. For example, the two boards pictured in Figure 1.1 are rook equivalent because they have all the same rook numbers. Foata and Schützenberger restricted their consideration to a subset of boards associated with (integer) partitions. A *partition* is a weakly increasing sequence $(b_1, \ldots, b_n)$ of nonnegative integers. We will use the same notation for the corresponding *Ferrers board* $B = (b_1, \ldots, b_n)$ which consists of the the b_i lowest squares in column i for $1 \leq i \leq n$. The board $B = (1, 3, 3)$ is shown on the right in Figure 1.1, while the board on the left side of Figure 1.1 is not a Ferrers board.

A Ferrers board $B = (b_1, \ldots, b_n)$ is *strictly increasing* if $b_i < b_{i+1}$ for all i. Note that the board on the right of Figure 1.1 is not strictly increasing because $b_2 = b_3 = 3$. Restricting attention to Ferrers boards, Foata and Schützenberger were able to characterize the equivalence classes under the rook equivalence relation by showing that there is a unique strictly increasing Ferrers board in each equivalence class. To do so, they constructed explicit bijections between rook placements on any Ferrers board in the class and rook placements on the unique representative.

Theorem 1.1.1 (Theorem 11 [FS70])**.** *Each Ferrers board is rook equivalent to exactly one strictly increasing Ferrers board.*

Goldman, Joichi, and White [GJW75] gave a different definition of the rook polynomial of a board in 1975. Instead of writing it in the standard basis for polynomials, as seen in 1.1.1, they used the basis of *falling factorials*

$$x{\downarrow_n} = x(x-1)(x-2)\dots(x-n+1)$$

for any integer $n \geq 0$. In this basis, the *rook polynomial* of a board B with n columns is

$$\sum_{k\geq 0}^{n} r_k(B) x{\downarrow_{n-k}} .$$

This is the definition of rook polynomial we will be using throughout the rest of this document. Using this definition, the rook polynomial of either board in Figure 1.1, both of which have $n = 3$, is

$$1x{\downarrow_3} + 7x{\downarrow_2} + 10x{\downarrow_1} + 2x{\downarrow_0} = x(x-1)(x-2) + 7x(x-1) + 10x + 2 = x^3 + 4x^2 + 5x + 2.$$

Goldman, Joichi, and White were able to show that the falling factorial rook polynomial of any Ferrers board factored over the integers.

Theorem 1.1.2 (Theorem 2 [GJW75])**.** *Let $B = (b_1, \dots, b_n)$ be a Ferrers board, then*

$$\sum_{k=0}^{n} r_k(B) x{\downarrow_{n-k}} = \prod_{i=0}^{n} (x + b_i - i + 1).$$

Continuing the above example, we determined that the rook polynomial of $B = (1, 3, 3)$

is

$$x^3 + 4x^2 + 5x + 2 = (x+1)(x+2)(x+1) = (x+1-1+1)(x+3-2+1)(x+3-3+1)$$

as stated in Theorem 1.1.2.

In 2009 Loehr and Remmel [LR09] introduced another explicit bijection between rook placements on rook equivalent Ferrers board. Their bijection made use of the Garsia-Milne Involution Principle [GM81], which was developed as part of Garsia and Milne's bijective proof of the Rogers-Ramanujan identities.

Another important set of numbers associated with a board are its hit numbers. Given a permutation $\omega = (\omega(1), \ldots, \omega(N))$ in S_N, the symmetric group on N elements, one can associate a rook placement of N rooks on an $N \times N$ square board with ω in the following way. If $\omega(i) = j$, then place a rook in square $(i, N+1-j)$. We will denote this placement $R(\omega)$. This is equivalent to the permutation matrix of ω, considered as a left action, where the zeroes in the matrix are empty squares and the ones in the matrix are squares containing rooks.

For any board B and any integer N such that B can be considered as a subset of the squares of an $N \times N$ board, we can define the *kth hit set* of B by,

$$H_{k,N}(B) = \{R(\omega) \mid \omega \in S_N \text{ and } |R(\omega) \cap B| = k\}.$$

Then the *kth hit number* of B is

$$h_{k,N}(B) = |H_{k,N}(B)|.$$

In their seminal 1946 paper, Kaplansky and Riordan noted that this definition implied that

$$\sum_{k=0}^{N} h_{k,N}(B)x^k = \sum_{k=0}^{N} r_k(B)(N-k)!(x-1)^k,$$

which implies that two rook equivalent boards will have the same hit numbers for any N large enough for both to be considered as subboards of the $N \times N$ square board. In the paper cited above, Loehr and Remmel also used the Garsia-Milne method to give explicit bijections between the hit sets of any two equivalent Ferrers boards. This provided a bijective proof that any two rook equivalent Ferrers boards have the same hit numbers.

1.1.2 m-level rook placements

In 2008 Briggs and Remmel [BR06] introduced a generalization of rook placements, called m-level rook placements. The new rook placements correspond with elements of the wreath product of the cyclic group of order m with the symmetric group on N elements, $C_m \wr S_N$, in the same way that Kaplansky and Riordan's placements correspond to elements of S_N as detailed at the end of the last subsection. Using m-level rook placements and the concept of flag descents developed by Adin, Brenti, and Roichman [ABR01], Briggs and Remmel were able to generalize a formula of Frobenius to $C_m \wr S_n$. They also provided a factorization, similar to that of Goldman, Joichi, and White, for the m-rook polynomial of a specific type of Ferrers board, called a singleton board.

In order to make the concepts of the previous paragraph more precise, we first define what a level is. Fix a positive integer m. Partition the rows of $\mathbb{Z}^+ \times \mathbb{Z}^+$ into levels where the *pth level* consists of rows $(p-1)m+1, (p-1)m+2, \ldots, pm$. The situation for $m = 2$ is shown on the left in Figure 1.2 where the boundaries between the levels have been thickened.

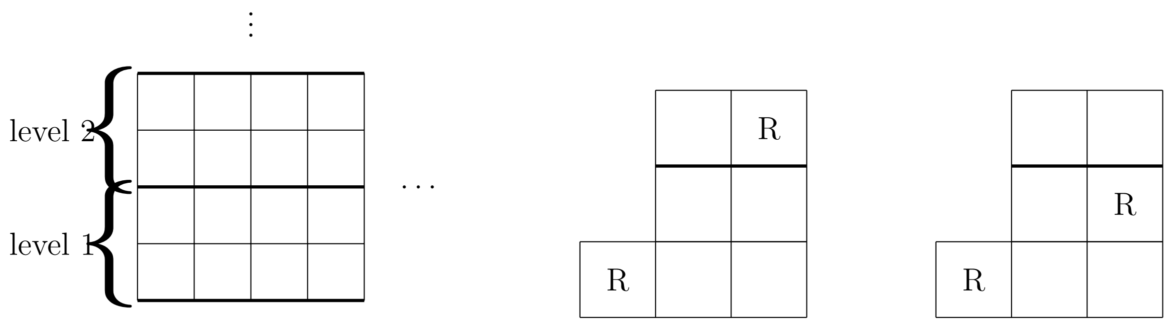

Figure 1.2: Levels and rook placements

Given a board, B, an *m-level rook placement* (called an *m-rook placement* by Briggs and Remmel) is $P \subseteq B$ where no two elements of P are in the same level or the same column. Note that when $m = 1$ we recover the ordinary notion of a rook placement. By way of example, in Figure 1.2, the placement on the middle board is a 2-level rook placement while the one on the right is not since it has two rooks in the first level. We let

$$r_{k,m}(B) = \text{the number of } m\text{-level rook placements } P \subseteq B \text{ with } k \text{ rooks.}$$

In general, we will add a subscript m to quantities when considering their m-level equivalents. For $B = (1, 3, 3)$ we have $r_{0,2}(B) = 1$, $r_{1,2}(B) = 7$, $r_{2,2}(B) = 6$, and $r_{k,2}(B) = 0$ for $k \geq 3$.

To state the Briggs-Remmel generalization of Theorem 1.1.2, we need a few more concepts. One is of an *m-falling factoring*, a generalization of the falling factorial, defined by

$$x{\downarrow_{n,m}} = x(x-m)(x-2m)\dots(x-(n-1)m).$$

Now we define an important subset of Ferrers boards when considering m-level rook placements. Having already fixed a positive integer m, a *singleton board* (called an *m-Ferrers board* by Briggs and Remmel), is a Ferrers board $B = (b_1, \dots, b_n)$ such that there is at most one b_i in each of the intervals $(0, m)$, $(m, 2m)$, $\ldots$ where $(km, (k+1)m) = \{km+1, km+2, \ldots, km+m-1\}$.

Another way of defining singleton boards is to consider the highest level a column has squares in. We say that the ith column of B *terminates in level* p if p is the largest integer such that the ith column has non-empty intersection with $\mathcal{Q}_p$. This gives another characterization of a singleton board, as any Ferrers board such that, for each positive integer p, the set of all columns b_i terminating in level p contains at most one i such that $b_i \not\equiv 0 \bmod m$. Briggs and Remmel gave a factorization theorem, similar to that of Goldman, Joichi, and White, but only for singleton boards.

Theorem 1.1.3 (Theorem 2 [BR06]). *If $B = (b_1, \ldots, b_n)$ is a singleton board then*

$$\sum_{k=0}^{n} r_{k,m}(B) x\downarrow_{n-k,m} = \prod_{i=1}^{n} (x + b_i - (i-1)m).$$

1.2 Organization

We proceed to generalize the above results for m-level rook placements. In Chapter 2 we remove the singleton requirement from Theorem 1.1.3. This facilitates generalizing Theorem 1.1.1 to m-level rook placements on Ferrers boards and enumerating the singleton boards in a given equivalence class, an analogue of a result of Goldman, Joichi, and White. Finally we consider what is counted if we take the product on the right side of Theorem 1.1.3 for any Ferrers board and expand it in the m-falling factorial basis.

In Chapter 3 we begin by extending the bijections detailed by Foata and Schützenberger to m-level rook placements on Ferrers boards. We create a second explicit bijection between m-level rook placements on singleton boards by generalizing that of Loehr and Remmel, again using the Garsia-Milne Involution Theorem. Finally we use similar methods to create bijections between the sets counted by appropriately defined hit numbers of m-level rook equivalent Ferrers boards.

Chapter 2

Factorization for m-level rook polynomials

2.1 Introduction

The material in this chapter is a result of a collaboration with Nicholas Loehr, Jeffrey Remmel, and Bruce Sagan, and has appeared in the paper [BLRS14].

The first goal of this chapter is to remove the singleton board restriction from Theorem 1.1.3 and prove a generalization of this theorem for any Ferrers board. This will be done in the next section. Call boards B, B' *m-level rook equivalent* if $r_{k,m}(B) = r_{k,m}(B')$ for all k. In Section 2.3 we extend to all m Theorem 1.1.1 of Foata and Schützenberger [FS70] giving a distinguished member of each 1-level rook equivalence class. Goldman, Joichi and White used the Factorization Theorem to enumerate the number of Ferrers boards 1-level rook equivalent to a given board. In Section 2.4 we generalize this formula to count m-level rook equivalent singleton boards for arbitrary m. The rest of the chapter is devoted to the following dual problem. Rather than changing the product side of Theorem 1.1.3, keep the same product for all Ferrers boards and expand it in the m-falling factorial basis. What do the coefficients count? We show in Section 2.5 that they are generating functions for certain weighted file placements, where such placements allow more than one rook in a given row. The next sections investigate properties of the corresponding equivalence classes. In

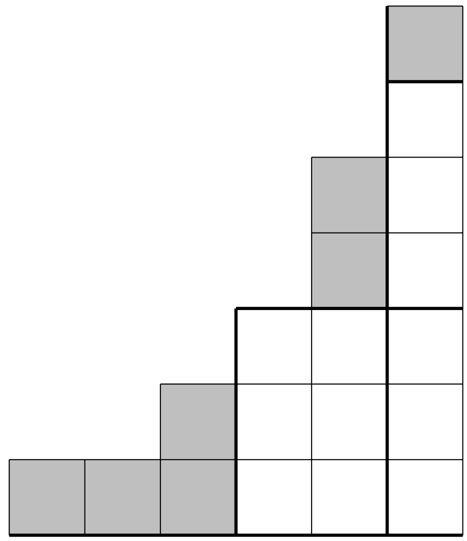

Figure 2.1: The zones of $(1, 1, 2, 3, 5, 7)$ when $m = 3$

particular, in Section 2.7 we count the number of boards in a given class.

2.2 The m-Factorization Theorem

In order to generalize Theorem 1.1.3 to all Ferrers boards, it will be convenient to break a board up into zones depending on the lengths of the columns. Given integers s, t, the interval from s to t will be denoted $[s, t] = \{s, s + 1, \ldots, t\}$. An *m-zone*, z, of a board $B = (b_1, \ldots, b_n)$ is a maximal interval $[s, t]$ such that $\lfloor b_s \rfloor_m = \lfloor b_{s+1} \rfloor_m = \cdots = \lfloor b_t \rfloor_m$. To illustrate this concept, consider $m = 3$ and the board $B = (1, 1, 2, 3, 5, 7)$ shown in Figure 2.1. In this case the zones are $z_1 = [1, 3]$ since $\lfloor b_1 \rfloor_3 = \lfloor b_2 \rfloor_3 = \lfloor b_3 \rfloor_3 = 0$, $z_2 = [4, 5]$ since $\lfloor b_4 \rfloor_3 = \lfloor b_5 \rfloor_3 = 3$, and $z_3 = [6, 6]$ since $\lfloor b_6 \rfloor_3 = 6$. The zones in Figure 2.1 are separated by thick lines (as are the levels). Note that a Ferrers board is a singleton board if and only if each zone contains at most one column whose length is not a multiple of m. This is the reason for our choice of terminology.

The m-floor function of an integer n is defined as follows, let

$$\lfloor n \rfloor_m = \text{the largest multiple of } m \text{ less than or equal to } n.$$

For example, $\lfloor 17 \rfloor_3 = 15$ since $15 \leq 17 < 18$. In addition to taking m-floors, we will have to consider remainders modulo m. Given an integer n, we denote its remainder on division by m by $\rho_m(n) = n - \lfloor n \rfloor_m$. Continuing the previous example, $\rho_m(17) = 17 - \lfloor 17 \rfloor_3 = 2$. If z is a zone of a Ferrers board $B = (b_1, \ldots, b_n)$ then its *m-remainder* is

$$\rho_m(z) = \sum_{i \in z} \rho_m(b_i).$$

In Figure 2.1, the boxes corresponding to the 3-remainders of the zones are shaded. In particular $\rho_3(z_1) = 1 + 1 + 2 = 4$, $\rho_3(z_2) = 0 + 2 = 2$, and $\rho_3(z_3) = 1$. We are now in a position to state and prove our generalization of Theorem 1.1.3.

Theorem 2.2.1 (m-Factorization Theorem)**.** *If $B = (b_1, \ldots, b_n)$ is any Ferrers board then*

$$\sum_{k=0}^{n} r_{k,m}(B) x\downarrow_{n-k,m} = \prod_{i=1}^{n} \begin{cases} x + \lfloor b_i \rfloor_m - (i-1)m + \rho_m(z) & \text{if } i \text{ is the last index in its zone } z, \\ x + \lfloor b_i \rfloor_m - (i-1)m & \text{otherwise.} \end{cases}$$

Proof. Since this is a polynomial identity, it suffices to prove it for an infinite number of values for x. We will do so when x is a nonnegative multiple of m. Consider the board B_x derived from B by adding an $x \times n$ rectangle below B. Figure 2.2 shows a schematic representation of B_x. Note that since x is a multiple of m, the zones and remainders of B and B_x are the same. We will show that both the sum and the product count the number of m-rook placements on B_x consisting of n rooks.

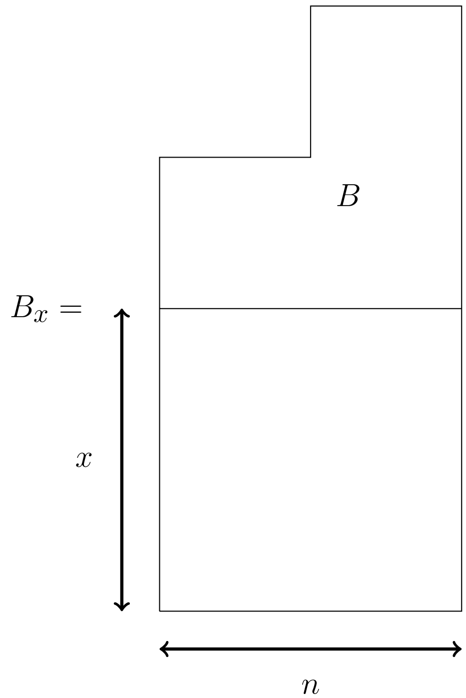

Figure 2.2: The board B_x

For the sum side, note that any placement of n rooks on B_x must have k rooks in B and $n-k$ rooks in the rectangle for some $0 \leq k \leq n$. By definition, $r_{k,m}(B)$ counts the number of placements on B. Once these rooks are placed, one must place the remaining rooks in the $x \times (n-k)$ subrectangle consisting of those columns of the original rectangle not used for the rooks on B. Placing these rooks from left to right, there will be x choices for the position of the first rook, then $x-m$ choices for the next, and so on, for a total of $x{\downarrow_{n-k,m}}$ choices. Thus the sum side is $r_{n,m}(B_x)$ as desired.

On the product side, it will be convenient to consider placing rooks on B_x zone by zone from left to right. So suppose $z = [s,t]$ is a zone and all rooks in zones to its left have been placed. Because z is a zone we have $\lfloor b_s \rfloor_m = \cdots = \lfloor b_t \rfloor_m = cm$ for some constant c. Also, among all the rooks placed in the columns of z, there is at most one which is in the set of squares $\mathcal{R}$ corresponding to $\rho_m(z)$. If there are no rooks in $\mathcal{R}$ then they all go in a rectangle of height $x+cm$. Thus, using the same ideas as in the previous paragraph, the number of placements is

$$(x+cm-(s-1)m)(x+cm-sm)\dots(x+cm-(t-1)m). \tag{2.2.1}$$

When there is one rook in $\mathcal{R}$, say it is in the column with index i. So there are $\rho_m(b_i)$ choices for the placement of this rook and the rest of the rooks go in a rectangle of height $x+cm$. This gives a count of

$$\rho_m(b_i)(x+cm-(s-1)m)(x+cm-sm)\dots(x+cm-(t-2)m). \tag{2.2.2}$$

Adding together the contributions from (2.2.1) and (2.2.2) and factoring, we see that the total number of placements is

$$(x+cm-(s-1)m)\dots(x+cm-(t-2)m)(x+cm-(t-1)m+\rho_m(z)).$$

Remembering that $\lfloor b_s \rfloor_m = \cdots = \lfloor b_t \rfloor_m = cm$, we see that this is exactly the contribution needed for the product. $\square$

We should show why our result implies the theorems of Goldman-Joichi-White and Briggs-Remmel. In both cases, it is clear that the sum sides correspond, so we will concentrate on the products.

For Theorem 1.1.2 we take $m=1$. Since $\lfloor n \rfloor_1 = n$ for any n, $\rho_1(z)=0$ for any zone z and the two cases in Theorem 2.2.1 are the same. So the contribution of the ith column to the product is

$$x + \lfloor b_i \rfloor_1 - (i-1) \cdot 1 = x + b_i - i + 1$$

in agreement with the Factorization Theorem.

As for Theorem 1.1.3, suppose that B is a singleton board and consider any zone $z = [s, t]$. If $s \leq i < t$ then b_i is a multiple of m and

$$x + \lfloor b_i \rfloor_m - (i-1)m = x + b_i - (i-1)m.$$

And if $i = t$ then $\rho_m(z) = \rho_m(b_t)$ so that

$$x + \lfloor b_t \rfloor_m - (t-1)m + \rho_m(z) = x + b_t - (t-1)m.$$

So in either case one gets the same factor as in the Briggs-Remmel result.

2.3 Rook equivalence

Two Ferrers boards B, B' are *m-level rook equivalent* if $r_{k,m}(B) = r_{k,m}(B')$ for all $k \geq 0$. In this case we will write $B \equiv_m B'$. We will drop the "m" and just say "rook equivalent" if $m = 1$. Recall that Foata and Schützenberger [FS70] proved a beautiful theorem giving a distinguished board in each equivalence class.

Theorem 2.3.1 ([FS70]). *Every Ferrers board is rook equivalent to a unique strictly increasing board.* □

The purpose of this section is to extend Theorem 2.3.1 to arbitrary m. The Foata-Schützenberger result was reproved by Goldman-Joichi-White using their Factorization Theorem. To see the connection, suppose that $B = (b_1, \ldots, b_n)$ and $B' = (b'_1, \ldots, b'_n)$ are two Ferrers boards. Although we are writing the boards with the same number of columns, this is no restriction since we can always pad the shorter board with columns of height 0 on the left. So B and B' are rook equivalent if and only if they have the same generating function in the falling factorial basis. By Theorem 1.1.2, this happens if and only if the two vectors

$$\zeta(B) = (-b_1, 1-b_2, 2-b_3, \ldots, (n-1)-b_n) \quad \text{and} \quad \zeta(B') = (-b'_1, 1-b'_2, 2-b'_3, \ldots, (n-1)-b'_n)$$

are rearrangements of each other since these are the zeros of the corresponding products. We call $\zeta(B)$ the *root vector* of B. For example, if $B = (1, 1, 3)$ and $B' = (2, 3)$ then, rewriting $B' = (0, 2, 3)$, we have $\zeta(B) = (-1, 0, -1)$ and $\zeta(B') = (0, -1, -1)$ and so $B \equiv B'$. We should note that padding a board B with zeros will change the entries of $\zeta(B)$. Also, if $\zeta(B)$ is a rearrangement of $\zeta(B')$ then the same will be true when padding both B and B' with any given number of zeros such that both resulting boards have the same number of columns.

We now return to considering general m. Define the *m-level root vector* of $B = (b_1, \ldots, b_n)$ to be $\zeta_m(B) = (a_1, \ldots, a_n)$ where

$$a_i = \begin{cases} (i-1)m - \lfloor b_i \rfloor_m - \rho_m(z) & \text{if } i \text{ is the last index in its zone } z, \\ (i-1)m - \lfloor b_i \rfloor_m & \text{otherwise.} \end{cases}$$

The next result is immediate from Theorem 2.2.1.

Proposition 2.3.2. *Ferrers boards B and B' satisfy $B \equiv_m B'$ if and only if $\zeta_m(B)$ is a rearrangement of $\zeta_m(B')$.* □

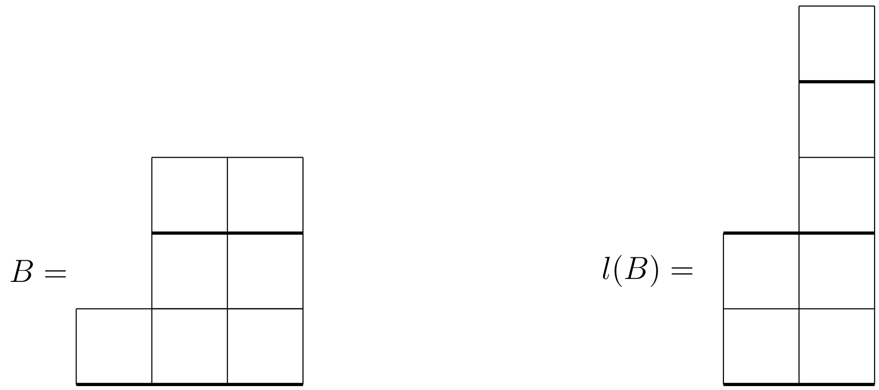

Figure 2.3: A board B and $l(B)$ when $m = 2$

Our first order of business will be to restrict the class representative problem to considering singleton boards $B = (b_1, \ldots, b_n)$ since then the a_i are simpler to compute. Indeed, the argument in the last paragraph of Section 2.2 shows that in this case

$$a_i = (i-1)m - b_i$$

for all i. To describe a singleton board in each equivalence class, let $\mathcal{Q}_p$ denote the set of squares in the pth level of the tiling $\mathcal{Q}$ of the first quadrant and, for any board $B = (b_1, \ldots, b_n)$, let $l_p = |B \cap \mathcal{Q}_p|$. For example, if $m = 2$ and $B = (1, 3, 3)$ as shown on the left in Figure 2.3, then $l_1 = 5$, $l_2 = 2$, and $l_i = 0$ for $i \geq 3$. For any Ferrers board B, if t is the largest index with $l_t \neq 0$ then we let $l(B) = (l_t, l_{t-1}, \ldots, l_1)$. We call this function the *l-operator* on boards. The board on the right in Figure 2.3 shows $l(1, 3, 3) = (2, 5)$.

Lemma 2.3.3. *For any m and any Ferrers board $B = (b_1, \ldots, b_n)$ the sequence $l(B) = (l_t, \ldots, l_1)$ is a partition, the Ferrers board $l(B)$ is singleton, and $B \equiv_m l(B)$.*

Proof. To see that $l(B)$ is a partition first note that, for any $p > 1$, the set of columns of $B \cap \mathcal{Q}_p$ is a subset of the columns of $B \cap \mathcal{Q}_{p-1}$. Furthermore, for each of the former columns we have m squares of that column in $B \cap \mathcal{Q}_{p-1}$. It follows that $l_p(B) \leq l_{p-1}(B)$ as desired.

To show that $l(B)$ is singleton, it suffices to show that if l_p is not a multiple of m then $\lfloor l_p \rfloor_m < \lfloor l_{p-1} \rfloor_m$. Let c be the number of columns of $B \cap \mathcal{Q}_p$ which contain m squares and d be the number of columns containing fewer than m squares. By assumption $d > 0$. Since these $c + d$ columns of $B \cap \mathcal{Q}_{p-1}$ must all contain m cells we have $l_p < (c+d)m \leq l_{p-1}$. Taking floors finishes this part of the proof.

To prove rook equivalence, pick any $L = \{p_1 > \cdots > p_k\} \subseteq \{1, \ldots, t\}$. It suffices to show that the number of ways to place rooks on the levels of B indexed by L is the same as the number of ways to place rooks in the columns of $l(B)$ indexed by L. For the former, if one places the rooks level by level from top to bottom then, since each rook in a higher level rules out m squares in each level below it, we obtain a count of

$$l_{p_1}(l_{p_2} - m)(l_{p_3} - 2m) \ldots (l_{p_k} - (k-1)m).$$

Now consider placing the rooks in $l(B)$ column by column from left to right. Since $l(B)$ is singleton, each rook placed eliminates m squares in each column to its right from consideration. Thus we obtain the same count as before. □

It is worth noting here that this gives us an alternative form of the factorization from Theorem 2.2.1.

Theorem 2.3.4. *If $B = (b_1, \ldots, b_n)$ is any Ferrers board where t is the largest index with $l_t \neq 0$ and N is any integer greater than or equal to both n and t, then*

$$\sum_{k=0}^{N} r_{k,m}(B) x \downarrow_{N-k,m} = \prod_{i=1}^{N} (x + l_{t+1-i} - (i-1)m)$$

Proof. We know that $l(B)$ is a singleton board by Lemma 2.3.3. To facilitate comparing their m-level rook polynomials, pad both B and $l(B)$ on the left with columns of height 0 until both have N columns. This will not alter the singleton property of $l(B)$, so by Theorem 1.1.3, we know that the product side is the factorization of the m-level rook polynomial of $l(B)$. Furthermore, since we know $B \equiv_m l(B)$ from Lemma 2.3.3, we know

$$\sum_{k=0}^{N} r_{k,m}(B)x\downarrow_{N-k,m} = \sum_{k=0}^{N} r_{k,m}(l(B))x\downarrow_{N-k,m} ,$$

which completes the proof. □

The possible ζ_m-vectors of singleton boards are easy to characterize.

Proposition 2.3.5. *Consider a vector $\zeta = (a_1, \ldots, a_n)$ where $a_1 = 0$. Let $B = (b_1, \ldots, b_n)$ where we define $b_i = (i-1)m - a_i$ for all i. We have that $\zeta = \zeta_m(B)$ where B is a singleton board if and only if the following conditions are satisfied:*

(i) $a_{i+1} \leq a_i + m$ for $i \geq 1$, and

(ii) if neither of a_{i+1}, a_i are multiples of m then $\lfloor a_{i+1} \rfloor_m \leq \lfloor a_i \rfloor_m$.

Proof. We claim that (i) and the fact that $a_1 = 0$ are equivalent to B being a weakly increasing sequence of nonnegative integers. Since $a_1 = 0$ we have $b_1 = -a_1 = 0$ which is nonnegative. And for $i \geq 1$

$$b_{i+1} - b_i = (im - a_{i+1}) - ((i-1)m - a_i) = m + a_i - a_{i+1}. \tag{2.3.1}$$

So (i) is equivalent to B being weakly increasing. A similar argument shows that (ii) is equivalent to the singleton condition. □

We are finally in a position to give distinguished representatives for the m-level equivalence classes. A Ferrers board $B = (b_1, \dots, b_n)$ is called *m-increasing* if $b_1 > 0$ and $b_{i+1} \geq b_i + m$ for $i \geq 1$. Note that a 1-increasing board is strictly increasing in the sense of Theorem 2.3.1. Also note that, although most properties of Ferrers boards are not affected by padding with columns of length zero, the m-increasing condition will be destroyed.

Theorem 2.3.6. *Every Ferrers board is m-level rook equivalent to a unique m-increasing board.*

Proof. Clearly any m-increasing board is a singleton board. So, by Lemma 2.3.3, it suffices to show that any singleton board B is m-rook equivalent to a unique m-increasing board. An example of the construction of this board is given after this proof to illustrate the technique. Let $N = |B| + 1$ and pad B with columns of zeros so as to write $B = (b_1, \dots, b_N)$. By the choice of N, any board B' which is m-equivalent to B can be written as $B' = (b'_1, \dots, b'_N)$ and $b_1 = b'_1 = 0$. Letting $\zeta_m(B) = (a_1, \dots, a_N)$ and $\zeta_m(B') = (a'_1, \dots, a'_N)$, the choice of N also ensures that $a_1 = a'_1 = 0$ and $a_i, a'_i \geq 0$ for all i.

We claim that a singleton board $B' = (b'_1, \dots, b'_n)$ will be m-increasing, after discarding any columns of height 0, if and only $\zeta_m(B') = (a'_1, \dots, a'_n)$ is weakly decreasing. Indeed, this follows directly from (2.3.1).

We first show existence. By the previous paragraph, we wish to rearrange $\zeta_m(B)$ in such a way that the portion of the rearrangement corresponding to nonzero entries of the board is weakly decreasing. And the portion of the rearrangement corresponding to zero entries of the board must be of the form $0, m, 2m, \dots, cm$ for some c. So choose cm to be the largest multiple of m in $\zeta = \zeta_m(B)$. We claim that the elements $0, m, \dots, (c-1)m$ also occur in ζ. By definition, $b_1 = 0$ so this multiple of m occurs as does cm by assumption. Suppose, towards a contradiction, that jm does not occur where $0 < j < c$. Then the definition of jm and condition (i) of Proposition 2.3.5 implies there must be an index i such that we have the inequalities $(j-1)m < a_i < jm < a_{i+1} < (j+1)m$. But now $\lfloor a_i \rfloor_m < \lfloor a_{i+1} \rfloor_m$ which contradicts condition (ii) of the same lemma.

We now define $\zeta' = (a'_1, \ldots, a'_N)$ where

$$(a'_1, a'_2, a'_3, \ldots, a'_{c+1}) = (0, m, 2m, \ldots, cm)$$

and $(a'_{c+2}, a'_{c+3}, \ldots, a'_N)$ is the rest of ζ arranged in weakly decreasing order. Since $a'_1 = 0$, we can show that ζ' corresponds to an m-increasing board by checking conditions (i) and (ii) of Proposition 2.3.5. Condition (i) is clearly true for $i \leq c$. For $i = c$, one can show, by using a proof as in the previous paragraph and the choice of cm as the largest multiple of m in ζ, that $a'_{c+2} < (c+1)m$. So (i) also holds in this case. And for $i > c$ the fact that the sequence is weakly decreasing makes (i) a triviality. The same reasoning as for (i) shows that (ii) must also hold. Thus, defining $B' = (b'_1, \ldots, b'_N)$ where $b'_i = (i-1)m - a'_i$ for all i results in a singleton board. Furthermore, by construction, $b'_1 = \cdots = b'_{c+1} = 0$ and $(b'_{c+2}, \ldots, b'_N)$ is m-increasing. Hence removing the zeros from B' leaves the desired m-increasing board.

To show uniqueness, suppose $\zeta' = (a'_1, \ldots, a'_N)$ is a rearrangement of ζ corresponding to a padded m-increasing board. Then ζ' must start with $0, m, \ldots, cm$ for some c and be weakly decreasing thereafter by equation (2.3.1). Without loss of generality, we can assume $a'_{c+2} \neq (c+1)m$, since if the two are equal we can just add a'_{c+2} to the initial run of multiples of m. So ζ' will be the rearrangement of ζ constructed in the existence proof as long as cm is the largest multiple of m in ζ. But if cm is not the largest multiple of m in ζ then $(a'_{c+2}, \ldots, a'_N)$ must contain an element greater than or equal to $(c+1)m$. And since this portion of ζ' is weakly decreasing, this forces $a'_{c+2} > (c+1)m$ since equality was ruled out earlier. But then $a'_{c+1} = cm$ and a'_{c+2} do not satisfy condition (i) of Proposition 2.3.5, contradicting the fact that ζ' corresponds to a singleton board. This finishes the proof of uniqueness and of the theorem. □

To illustrate the construction in the previous proof, consider $m = 2$ and the singleton board $B = (1, 2, 2, 3)$. Now $N = 1 + 2 + 2 + 3 = 8$ and we pad B with zeros to length $8 + 1 = 9$, obtaining $B = (0, 0, 0, 0, 0, 1, 2, 2, 3)$. Thus $\zeta = \zeta_2(B) = (0, 2, 4, 6, 8, 9, 10, 12, 13)$. The largest multiple of 2 in ζ is 12, so we rearrange ζ to begin with the multiples of 2 up through 12 and then decrease. The result is $\zeta' = (0, 2, 4, 6, 8, 10, 12, 13, 9)$ with associated board $B' = (0, 0, 0, 0, 0, 0, 0, 1, 7)$. Removing the initial zeros, we get the 2-increasing board $(1, 7)$ which is easily seen to be 2-level rook equivalent to B.

2.4 Enumeration of singleton boards

Goldman, Joichi, and White [GJW75] used their factorization theorem to give a simple formula for the size of a given rook equivalence class. The basic idea is to count, for any board B, the number of rearrangements of $\zeta_1(B)$ which correspond to a Ferrers board. To state their result, given any finite vector ν of nonnegative integers, we let $n(\nu) = (n_0, n_1, \dots)$ be defined by

$$n_i = \text{the number of copies of } i \text{ in } \nu.$$

So $n_i(\nu) = 0$ if $i < 0$ or i is sufficiently large.

Theorem 2.4.1 ([GJW75]). *Let $B = (b_1, \dots, b_N)$ be a Ferrers board where $N = |B| + 1$, and suppose $n(\zeta_1(B)) = (n_0, n_1, \dots)$. The number of Ferrers boards in the equivalence class* *of B is*

$$\prod_{i\geq 1}\binom{n_i+n_{i-1}-1}{n_i}. \qquad \square$$

Because the entries of $\zeta_m(B)$ are more complicated for $m \geq 2$, we will not be able to count all boards in an m-level equivalence class. But we can at least enumerate the singleton boards. The formula will be in terms of multinomial coefficients.

Theorem 2.4.2. *Let $B = (b_1, \ldots, b_N)$ be a singleton board where $N = |B| + 1$, and suppose $n(\zeta_m(B)) = (n_0, n_1, \ldots)$. Then the number of singleton boards which are m-level rook equivalent to B is*

$$\prod_{i\geq 0}\binom{n_{im}+n_{im+1}+\cdots+n_{im+m}-1}{n_{im}-1, n_{im+1}, n_{im+2}, \ldots, n_{im+m}}.$$

Proof. By the choice of N we have that $\zeta = \zeta_m(B)$ begins with a zero and has all entries nonnegative. So it suffices to compute the number of rearrangements of ζ beginning with 0 and satisfying conditions (i) and (ii) of Proposition 2.3.5. Let d be the maximum entry of ζ. If $d = 0$ then the result is easy to verify, so assume $d > 0$. Let cm be the largest nonnegative multiple of m with $cm < d$. Note that cm exists since $d > 0$ and also that, by the argument given in the proof of Theorem 2.3.6, $n_{cm} > 0$.

Consider the vector $\zeta' = (a'_1, \ldots, a'_n)$ obtained from ζ by removing all entries which are larger than cm. We claim that $\zeta' = \zeta_m(B')$ for some singleton board B'. As usual, we use Propostion 2.3.5. Certainly $a'_1 = a_1 = 0$ since none of the zeros were removed from ζ. Suppose, towards a contradiction, that condition (i) is false in that $a'_{j+1} > a'_j + m$ for some j. Let a_i be the element of ζ corresponding to a'_j. But since we removed the largest elements of ζ we have

$$a_{i+1} \geq a'_{j+1} > a'_j + m = a_i + m$$

which is impossible. A similar contradiction demonstrates (ii), and our claim is proved.

Now, by induction, it suffices to show that the number of rearrangements of ζ which come from a given ζ' by inserting elements larger than cm is

$$\binom{n_{cm} + n_{cm+1} + \cdots + n_{cm+m} - 1}{n_{cm} - 1, n_{cm+1}, n_{cm+2}, \ldots, n_{cm+m}}.$$

Consider elements a_i, a_{i+1} in ζ. First note that if a_i comes from ζ' and $a_{i+1} > cm$ then we must have $a_i = cm$. If this were not the case then, since $a_i < cm < a_{i+1}$, to make condition (i) true neither a_i nor a_{i+1} would be multiplies of m. But by the same pair of inequalities we would have $\lfloor a_{i+1} \rfloor_m \geq cm > \lfloor a_i \rfloor_m$ which contradicts condition (ii). Thus we can insert elements greater than cm only after copies of cm itself. This argument is analogous to that given in the proof of Theorem 2.3.6. In addition, any a_{i+1} with $cm < a_{i+1} \leq cm + m$ can come after $a_i = cm$ as it is easily verified that we always have conditions (i) and, vacuously, (ii) for such a_{i+1}.

We also claim that the elements larger than cm can be arranged in any order with respect to each other. To see this, suppose $cm < a_i, a_{i+1} \leq cm + m$. Condition (i) is immediate because of the given bounds. And if neither is a multiple of m then we have $cm < a_i, a_{i+1} < cm + m$ which implies condition (ii).

Finally, if $a_i > cm$ and a_{i+1} comes from ζ' then $a_{i+1} < a_i$ and conditions (i) and (ii) are trivial. So an element greater than cm can be followed by any element of ζ'.

We now enumerate the number of ζ coming from ζ' using the structural properties from the previous three paragraphs. There are n_{cm} copies of cm and $n_{cm+1}+n_{cm+1}+\cdots+n_{cm+m}$ elements to be inserted after these copies where the space after a copy can be used multiple times. And any element of ζ' can follow the inserted elements. So the total number of choices

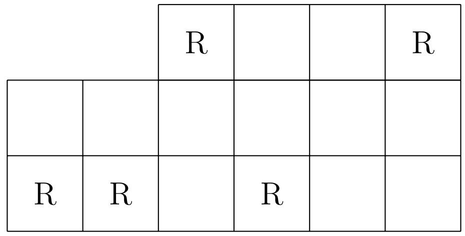

Figure 2.4: A file placement

for this step is the binomial coefficient

$$\binom{n_{cm} + n_{cm+1} + \cdots + n_{cm+m} - 1}{n_{cm+1} + n_{cm+2} + \cdots + n_{cm+m}}.$$

Now we need to arrange the elements greater than cm among themselves. This can be done arbitrarily, so the number of choices is

$$\binom{n_{cm+1} + n_{cm+2} + \cdots + n_{cm+m}}{n_{cm+1}, n_{cm+2}, \ldots, n_{cm+m}}.$$

Multiplying the two displayed expressions and canceling $(n_{cm+1} + n_{cm+2} + \cdots + n_{cm+m})!$ gives the desired result. □

Note that the result just proved does indeed generalize Theorem 2.4.1. This is because when $m = 1$, every board is a singleton board. And the products clearly conicide in this case.

2.5 File placements

Thus far our focus has been to keep the sum side of Theorem 1.1.3 the same and modify the product side to get equality for all Ferrers boards. Another possibility would be to keep the product side the same, expand it in the m-falling factorial basis, and see if the coefficients of the linear combination count anything. This will be our approach in the current section. An equivalent formula in the case where $m = 2$, albeit in a different rook model, can be found in a paper of Haglund and Remmel [HR01, Theorem 8].

It turns out that these coefficients count weighted file placements. A *file placement* on a board B is $F \subseteq B$ such that no two rooks (elements) of F are in the same column. However, we permit rooks to be in the same row. Figure 2.4 displays such a placement on the Ferrers board $(2, 2, 3, 3, 3, 3)$.

We let

$$f_k(B) = \text{the number of file placements } F \subseteq B \text{ with } k \text{ rooks.}$$

It is easy to count such placements. If B has b_i squares in column i for $1 \leq i \leq n$ (B need not be a Ferrers board) then $f_k(B) = e_k(b_1, \ldots, b_n)$ where e_k is the kth elementary symmetric function. So in order to get more interesting results, we will weight file placements.

Fix, as usual, a positive integer m. Given a board B and a file placement $F \subseteq B$, let t be the largest index of a row in B and consider $y_1, \ldots, y_t$ where y_j is the number of rooks of F in row j. Define the *m-weight* of F to be

$$\operatorname{wt}_m F = 1\!\downarrow_{y_1,m} 1\!\downarrow_{y_2,m} \ldots 1\!\downarrow_{y_t,m} .$$

In the example of Figure 2.4 with $m = 3$ we have

$$\operatorname{wt}_m F = 1\!\downarrow_{3,3} \cdot 1\!\downarrow_{0,3} \cdot 1\!\downarrow_{2,3} = (1)(-2)(-5) \cdot (1) \cdot (1)(-2) = -20.$$

Note that if F is actually a rook placement then $\operatorname{wt}_m F = 1$ because $1\!\downarrow_{0,m} = 1\!\downarrow_{1,m} = 1$ for any m.

Given a board, B, we define the associated *m-weighted file placement numbers* to be

$$f_{k,m}(B) = \sum_{F} \mathrm{wt}_m F$$

where the sum is over all file placements $F \subseteq B$ with k rooks. These are the coefficients which we seek.

Theorem 2.5.1 (m-weight Factorization Theorem)**.** *For any Ferrers board $B = (b_1, \ldots, b_n)$*

$$\sum_{k=0}^{n} f_{k,m}(B) x{\downarrow_{n-k,m}} = \prod_{i=1}^{n} (x + b_i - (i-1)m). \tag{2.5.1}$$

Proof. In the manner to which we have become accustomed, we consider the board B_x obtained by attaching an $x \times n$ rectangle R to B where x is a nonnegative multiple of m. Consider *mixed placements* $F \subseteq B_x$ which are file placements when restricted to B, but satisfy the m-level condition when restricted to R. We will compute $S = \sum_F \mathrm{wt}_m F$ where the sum is over the mixed placements F on B_x with n rooks.

We first recover the sum side of equation (2.5.1). The mixed placements with k rooks on B will contribute $f_{k,m}(B)$ to S from these rooks. And $x{\downarrow_{n-k,m}}$ will be the contribution from the $n-k$ rooks on R since any rook placement has m-weight equal to one as noted previously. So

$$f_{k,m}(B) x{\downarrow_{n-k,m}} = \sum_{F^k} \mathrm{wt}_m F^k \tag{2.5.2}$$

where the sum is over all mixed placements $F^k \subseteq B_x$ having k rooks in B. Now summing over k gives us the desired equality.

To obtain the product, let B' and B'_x be B and B_x with their nth columns removed, respectively. By induction on n, it suffices to prove that

$$\sum_{k=0}^{n} f_{k,m}(B)x{\downarrow_{n-k,m}} = (x+b_n-(n-1)m)\sum_{k=0}^{n-1} f_{k,m}(B')x{\downarrow_{n-k-1,m}} \,. \tag{2.5.3}$$

Comparing equations (2.5.2) and (2.5.3), we see it suffices to show that, for any mixed placement $F' \subseteq B'_x$,

$$(x+b_n-(n-1)m)\,\mathrm{wt}_m F' = \sum_F \mathrm{wt}_m F \tag{2.5.4}$$

where the sum is over all mixed placements $F \subseteq B_x$ whose restriction to B'_x is F'. To this end, let y_0 be the number of rooks in F' which are in R. Also let y_j, $1 \le j \le b_n$, be the number of rooks in the jth row of $F' \cap B'$. Since every column of F' has a rook, we have

$$y_0 + y_1 + \cdots + y_{b_n} = n-1. \tag{2.5.5}$$

We now consider two cases depending on whether the rook in column n of F lies in B or in R. If it lies in R then, by the m-level condition, there are $x - y_0 m$ places for the rook. Since these placements are in rows not occupied by rooks of F', each of them contributes a factor of 1 to the weight for a total change in weight of $x - y_0 m$ from this case. Now suppose that the rook lies in B, say in the jth row. Then in passing from F' to F, the weight is changed from $1{\downarrow_{y_j,m}}$ to $1{\downarrow_{y_j+1,m}}$. This means that the weight is multiplied by $1 - y_j m$ when placing a rook in row j of B. Adding together all the contributions and using equation (2.5.5) gives

$$x - y_0 m + \sum_{j=1}^{b_n}(1-y_j m) = x + b_n - (n-1)m.$$

This completes the proof of equation (2.5.4) and of the theorem. □

We note that Theorem 2.5.1 is another generalization of the Factorization Theorem. Indeed, when $m = 1$, then any file placement having a row with $y \geq 2$ rooks will have a factor of $1{\downarrow_{y,1}} = 0$. And any rook placement will have a weight of 1. Thus $f_{k,1}(B) = r_{k,1}(B)$.

2.6 Weight equivalence classes

Given m, define two boards B, B' to be *m-weight file equivalent*, written $B \approx_m B'$, if $f_{k,m}(B) = f_{k,m}(B')$ for all $k \geq 0$. Our goal in this section is to find distinguished representatives for the m-weight file equivalence classes. Interestingly, our result will be dual to the one for m-level rook equivalence in the sense that the inequalities will be reversed. In order to define the representatives, we will have to assume that all our boards start with at least one zero. So for this section we will write our Ferrers boards in the form $B = (b_0, b_1, \ldots, b_n)$ where $b_0 = 0$.

We can use Theorem 2.5.1 to test m-weight file equivalence. The *m-weight root vector* of a Ferrers board $B = (b_0, b_1, \ldots, b_n)$ is

$$\omega_m(B) = (-b_0, m - b_1, 2m - b_2, \ldots, nm - b_n).$$

From the m-weight Factorization Theorem we immediately get the following.

Proposition 2.6.1. *Ferrers boards B and B' satisfy $B \approx_m B'$ if and only if $\omega_m(B)$ is a rearrangement of $\omega_m(B')$.* □

We will also need a characterization of the vectors which can be m-weight root vectors. The proof of the next result is similar to that of Proposition 2.3.5 and so is omitted.

Proposition 2.6.2. *Consider a vector* $\omega = (a_0, a_1, \ldots, a_n)$*. Let* $B = (b_0, b_1, \ldots, b_n)$ *where we define* $b_i = im - a_i$ *for all* i*. We have that* $\omega = \omega_m(B)$ *where* B *is a Ferrers board if and only if the following conditions are satisfied:*

(i) $a_0 = 0$*, and*

(ii) $a_{i+1} \leq a_i + m$ *for* $i \geq 0$*.* □

Now define a Ferrers board $B = (b_0, b_1, \ldots, b_n)$ to be *m-restricted* if $b_{i+1} \leq b_i + m$ for all $i \geq 0$. We now show that the m-weight file equivalence class of any Ferrers board contains a unique m-restricted board. An example of the construction of this board follows the proof.

Theorem 2.6.3. *Every Ferrers board* $B = (b_0, \ldots, b_n)$ *is* m*-weight file equivalent to a unique* m*-restricted board.*

Proof. Similarly to the proof of Theorem 2.3.6, we rewrite $B = (b_0, \ldots, b_N)$ where $N = |B|$. This assures us that any equivalent board $B' = (b'_0, \ldots, b'_N)$ has $\omega_m(B')$ which is nonnegative and starts with zero. Also, using equation (2.3.1), we see that B' is m-restricted if and only if $\omega(B')$ is weakly increasing. So consider $\omega' = (a'_0, \ldots, a'_N)$ which is the unique weakly increasing rearrangement of $\omega = \omega_m(B) = (a_0, \ldots, a_N)$. It suffices to show that $\omega' = \omega_m(B')$ for some board B'. So we just need to check the two conditions of Proposition 2.6.2. Condition (i) follows from the choice of N and the fact that ω is nonnegative. For condition (ii), assume, towards a contradiction, that there is an index i such that $a'_{i+1} > a'_i + m$. Let a_j be the element of ω which was rearranged to become a'_{i+1}. Then $a_j = a'_{i+1} > 0$ and so there must be an element a_k with $k < j$ and $a_k < a_j$. Let k be the largest such index. By the choice of k and the fact that ω satisfies (ii), we must have $a_k \geq a_j - m$. Thus

$$a'_i < a'_{i+1} - m = a_j - m \leq a_k < a_j = a'_{i+1}.$$

But then when rearranging ω in weakly increasing order, a_k should have been placed between a'_i and a'_{i+1}, a contradiction. □

By way of illustration, suppose that we take $m = 2$ and $B = (1, 5)$. This gives $N = |B| = 6$ and $\omega = \omega_2(B) = (0, 2, 4, 6, 8, 10, 12) - (0, 0, 0, 0, 0, 1, 5) = (0, 2, 4, 6, 8, 9, 7)$. The weakly increasing rearrangement of ω is $\omega' = (0, 2, 4, 6, 7, 8, 9)$ and so $B' = (0, 2, 4, 6, 8, 10, 12) - (0, 2, 4, 6, 7, 8, 9) = (0, 0, 0, 0, 1, 2, 3)$.

There is a close relationship between the m-increasing boards introduced in Section 2.3 and m-restricted boards. This is easy to see if $m = 1$. In this case, board B is 1-increasing if and only if its transpose B^t (obtained by interchanging rows and columns) is 1-restricted. Indeed, a Ferrers board is 1-increasing if and only if the northwestern boundary of B contains no horizontal line segment of length at least 2. And a board is 1-restricted if and only if this boundary contains no vertical line segment of length at least 2. Note also that when $m = 1$, the l-operator of Section 2.3 satisfies $l(B) = B^t$. In generalizing these ideas to all m, it is the l-operator which is key as the next result shows.

Proposition 2.6.4. *The l-operator has the following properties.*

(i) If B is m-restricted then $l(B)$ is m-increasing.

(ii) If B is m-increasing then $l(B)$ is m-restricted.

(iii) If B is a singleton board then $l^2(B) = B$, so l is an involution on singleton boards.

Proof. (i) Let $B = (b_1, \ldots, b_n)$ be m-restricted and $l(B) = (l_t, \ldots, l_1)$. Keeping in mind that the subscripts in $l(B)$ are decreasing, we wish to show that $l_p \geq l_{p+1} + m$. Let B_i be the set of cells in column i and let $c_i = |B_i \cap \mathcal{Q}_p|$ and $d_i = |B_i \cap \mathcal{Q}_{p+1}|$ for all i. Now $l_p - l_{p+1} = \sum_i (c_i - d_i)$ and $c_i - d_i \geq 0$ for all i. Since B is m-restricted, there is an index

k such that B_k has its highest cell in $\mathcal{Q}_p$. Let k be the largest such index. Using the fact that B is m-restricted again forces B_{k+1} to have its highest cell in $\mathcal{Q}_{p+1}$. Thus, using the m-restricted condition a third time,

$$l_p - l_{p+1} \geq (c_k - d_k) + (c_{k+1} - d_{k+1}) = (b_k - (p-1)m) - 0 + m - (b_{k+1} - pm) = b_k - b_{k+1} + 2m \geq m.$$

which is what we wished to prove

(ii) This is similar to (i), finding an upper bound for $l_p - l_{p+1}$ using the fact that, for an m-increasing board, there is at most one B_k with its highest square in $\mathcal{Q}_p$ for each p. Details are left to the reader.

(iii) Induct on $|B|$. Given B, let B' be the board obtained by removing its first level. Since B is singleton so is B' and thus, by induction, $l^2(B') = B'$. If $l(B) = (l_t, \ldots, l_1)$ then the definition of the l-operator shows that $l(B') = (l_t, \ldots, l_2)$ and $l_1 = |B \cap \mathcal{Q}_1|$. Applying l to $l(B)$ we see that the column for l_1 in $l(B)$ adds m to every column of $l^2(B')$. Hence every column of $l^2(B)$ which contains cells in $\mathcal{Q}_p$ for $p \geq 2$ agrees with the corresponding column in B. Also, again by definition of l, those columns of $l^2(B)$ which lie wholly in $\mathcal{Q}_1$ are obtained from the column for l_1 in $l(B)$ by breaking it into columns of length m and a column of length $\rho_m(l_1)$. Since this is also the unique way to complete $l^2(B)$ so that it is a singleton board, it must be that $l^2(B) = B$ as desired. □

We now return to considering m-level rook equivalence classes as in Section 2.3. Using the proposition just proved, we obtain a second distinguished representative in each m-level rook equivalence class.

Corollary 2.6.5. *Every Ferrers board is m-level rook equivalent to a unique m-restricted* *singleton board.*

Proof. By Theorem 2.3.6, we know that each class has a representative B which is m-increasing and so also a singleton board. Applying the previous proposition and Lemma 2.3.3, we see that $l(B)$ is an m-restricted singleton board in the class. If there is a second such board $B' \neq l(B)$ then, by Proposition 2.6.4 again, $l(B')$ and $l^2(B) = B$ will be distinct m-increasing singleton boards in the class, contradicting the uniqueness part of Theorem 2.3.6. □

2.7 Weight equivalence class sizes

In this section we will generalize Theorem 2.4.1 to m-weight equivalence classes.

Theorem 2.7.1. *Let $B = (b_0, \ldots, b_N)$ be a Ferrers board where $N = |B|$, and suppose $n(\omega_m(B)) = (n_0, n_1, \ldots)$. The number of Ferrers boards in the m-weight equivalence class of B is*

$$\prod_{i \geq 1} \binom{n_i + n_{i-1} + \cdots + n_{i-m} - 1}{n_i}.$$

Proof. We use Proposition 2.6.1 and count the number of rearrangements of $\omega = \omega(B)$ which correspond to a Ferrers board. Let d be the maximum value of an entry of ω. Our assumptions imply that d is nonnegative and all entries of ω are between between 0 and d inclusive. Consider ω' which is obtained from $\omega_m(B)$ by removing all values equal to d. Using Proposition 2.6.2, it is easy to see that $\omega' = \omega_m(B')$ for some Ferrers board B'. By induction, it suffices to show that the number of rearrangements of ω which come from a given ω' is

$$\binom{n_d + n_{d-1} + \cdots + n_{d-m} - 1}{n_d}. \tag{2.7.1}$$

By condition (ii) in the proposition just cited, we can insert d after any element of ω' which is at least $d - m$. So the number of places for insertion is $n_{d-m} + \cdots + n_{d-1}$. Since we need to insert n_d copies of d and more than one copy can go in a given place, the number of choices is given by (2.7.1). □

Chapter 3

Bijections on m-level rook placements

3.1 Introduction

The material in this chapter is a result of a collaboration with Nicholas Loehr, Jeffrey Remmel, and Bruce Sagan, and will appear in a forthcoming paper.

The purpose of this chapter is to generalize the bijection of Foata and Schützenberger and those of Loehr and Remmel to m-level rook placements. The remainder of this section gives the background terminology necessary to begin this task. In Section 3.2 we generalize the bijection used by Foata and Schützenberger. Although this bijection is the composition of many intermediary bijections, and is therefore not direct, it does provide an explicit bijection between m-level rook placements on arbitrary m-level rook equivalent Ferrers boards. We will need this bijection again in Section 3.4. In Section 3.3 we generalize the construction of Loehr and Remmel. In this case the bijection can only be specified for singleton boards, a subset of all Ferrers boards. However, the construction leads to an explicit calculation of the m-level rook numbers for such boards using elementary symmetric functions and Stirling numbers of the second kind. In Section 3.4, we generalize a second bijection of Loehr and Remmel, and in doing so prove that any two m-level rook equivalent Ferrers boards have the same hit numbers. The last two bijections involve the Garsia-Milne Involution Principle [GM81].

Recall that we say that the ith column of B terminates in level p if p is the largest integer such that the ith column has non-empty intersection with $\mathcal{Q}_p$, which provides a character-

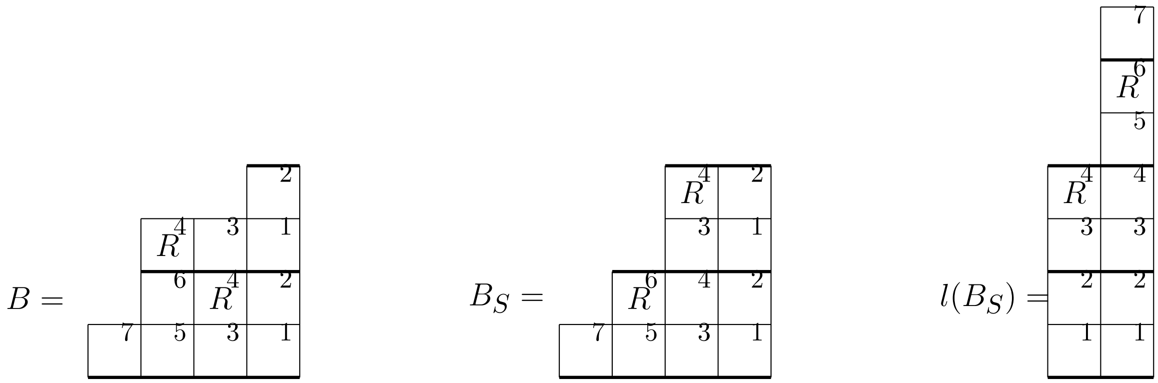

Figure 3.1: On the left, a placement of two 2-level rooks on B. In the middle, the corresponding placement from Lemma 3.2.3 of two 2-level rooks on singleton board B_S. On the right, the placement on $l(B_S)$ from Lemma 3.2.5.

ization of singleton boards, specifically, for any given level, a singleton board contains at most one column that terminates in that level and has a height that is not a multiple of m. The Ferrers board on the left in Figure 3.1 is not a singleton board, as two different columns terminate in the second level without having 2 cells in that level, while the Ferrers boards in the middle and on the right are singleton boards.

3.2 Rook equivalence and bijections

3.2.1 Reduction to singleton boards

In order to produce bijections between m-level rook placements on Ferrers boards, it is convenient to restrict our attention to singleton boards. In order to do this we prove the following two lemmas. First we show that for every Ferrers board there is a unique singleton board which has the same number of cells at each level. Then we prove that there is a bijection between the rook placements on a Ferrers board and those on the singleton board guaranteed in the first lemma. These lemmas together imply that every Ferrers board is

m-level rook equivalent to a singleton board and that there is an explicit bijection between the corresponding rook placements.

Lemma 3.2.1. *Given a Ferrers board B, there exists a unique singleton board B_S which has the same number of cells at each level as B.*

Proof. Let B have l_p cells in the pth level. In order for B_S to be a singleton board with l_p cells in the pth level, the cells of the pth level must be arranged uniquely as follows. If $l_p = cm + r$ with $0 \leq r < m$, then level p of B_S must have one column with r cells followed on the right by c columns with a full m cells in the level. This is because a singleton board may have at most one column which intersects a given level non-trivially in fewer than m cells. Thus B_S must be unique if it exists.

In order to show that B_S exists, we shall construct it. Arrange each level as specified above and line up the furthest right column in each level to create the furthest right column of B_S. This yields a Ferrers board because every column which has any cells in the pth level of B must have a full m cells in the $(p-1)$st level of B. Thus the total number of columns in the pth level of B_S will be less than or equal to the number of columns in the $(p-1)$st level of B_S containing m cells at that level. Hence a singleton board B_S exists and is unique. $\square$

Ignoring the rook placement, Figure 3.1 shows a board B and its corresponding board B_S. Since we know that an arbitrary Ferrers board B has the same number of cells at each level as a unique singleton board B_S, we wish to provide an explicit bijection between rook placements on the two boards. In order to do so we require the following numbering on a Ferrers board.

Definition 3.2.2. *A* level numbering *of board B assigns a number to each cell of B in the following way. Proceeding level by level in B, number the cells in the level by numbering each* *column from bottom to top, starting with the rightmost column and working left. In each level begin the numbering with 1.*

Figure 3.1 presents two examples of this numbering, on the left and middle boards, and also illustrates the bijection of the next lemma.

Lemma 3.2.3. *Given a Ferrers board B, there is an explicit bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on B_S, where B_S is as constructed in Lemma 3.2.1.*

Proof. Give both B and B_S a level numbering as shown in Figure 3.1. Since both boards have the same number of cells in each level, corresponding levels will each be numbered with the same set of numbers. Given any m-level rook placement on B, place rooks on B_S initially so that each rook occupies the same numbered cell in the same level as it does in B. This may not provide an m-level rook placement on B_S since two rooks could end up in the same column, so we will modify it as follows.

Notice that if a rook in column i and level p of B is not in the same cell in B_S, then column i must be to the left of a column of B that intersects $\mathcal{Q}_p$ in less than m cells. Furthermore, if the rook ends up in column i' in B_S, then all columns in the interval $[i, i']$ have a full m rooks in levels below p in B. Thus, if any of the rooks that move create a column with two or more rooks, there will be exactly two rooks in the column and the upper rook will have moved while the lower rook remained stationary. To rectify the situation, whenever a rook is moved from column i in b to column i' in B_S, move all other rooks in columns $(i, i']$ one column to the left, preserving their row. This is possible since these columns must contain m cells in the lower level of both B and B_S in order for the upper rook to have been in that column in B. Rearranging the rooks at each level in this fashion provides a function from

m-level rook placements on B to m-level rook placements on B_S. Figure 3.1 illustrates this map on a rook placement, including moving a lower rook one column to the left.

To see that this is a bijection, use the level numbering to produce a set of rooks on B from those on B_S. All the rooks will return to their initial positions once the appropriate right shift is applied. Similarly one can show that applying the map first to B_S and then to B is the identity. Thus we have a bijection between m-level rook placements on B and m-level rook placements on B_S. □

Lemma 3.2.1 and Lemma 3.2.3 guarantee that every Ferrers board is m-level rook equivalent to a singleton board. Additionally, there is an explicit bijection between m-level rook placements on the two boards. This permits us to restrict our attention to singleton boards henceforth.

3.2.2 The l-operator

Transposition of boards plays a central role in the Foata-Schützenberger construction of bijections between rook-equivalent Ferrers boards when $m = 1$. We will use the l-operator, defined in Section 2.3 of the previous chapter, as a generalization of this operation for arbitrary m.

Definition 3.2.4. *Given a Ferrers board B the* l-operator *applied to B is defined as follows. If t is the largest index of a non-empty level of B and the number of cells in the pth level of B is l_p, then*

$$l(B) = (l_t, l_{t-1}, \ldots, l_1).$$

Figure 3.1 contains an example board B_S as well as $l(B_S)$. The fact that $l(B)$ is a Ferrers board was shown in Lemma 2.3.3 and is analogous to an argument in the proof of

Lemma 3.2.1. In particular, if B is a Ferrers board then its pth level must fit above its $(p-1)$st level which implies

$$\lfloor l_p \rfloor_m \leq \lfloor l_{p-1} \rfloor_m,$$

with strict inequality if $l_p \not\equiv 0 \bmod m$. It follows that $l(B)$ is a weakly increasing sequence and so $l(B)$ is a Ferrers board and, because of the strict inequality for non-multiples of m, a singleton board.

To see that the l-operator is a generalization of transposition, note that if $m = 1$ then the levels of B are individual rows and these become the columns of $l(B)$. Furthermore, when restricted to the set of singleton boards the l-operator is an involution. This is shown in Proposition 2.6.4 of the previous chapter. Thus, the l-operator is a surjection from the set of Ferrers boards onto the set of singleton boards with $B = l(l(B))$ when B is singleton. We now provide a bijection between m-level rook placements on B and m-level rook placements on $l(B)$ to generalize the well-known bijection for transposition.

Lemma 3.2.5. *Given a singleton board B and a non-negative integer k, there is an explicit bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on $l(B)$.*

Proof. Give B a level numbering, then number the columns of $l(B)$ from bottom to top beginning with the number 1 in each column. Note that in this case the numbering of a level of B will consist of the same set of numbers as the numbering of the corresponding column of $l(B)$. Assume that B has t non-empty levels. For a given m-level rook placement of k rooks on B, place rooks on $l(B)$ in the following way. If a rook was in the cell numbered n of level p in B, then place a rook in the cell numbered n in column $t - p + 1$ in $l(B)$. See Figure 3.1 for an example of this map for a 2-level placement.

We must show that this gives a valid m-level rook placement on $l(B)$. If two rooks end up in the same column of $l(B)$ they must have originated in the same level of B, contradicting having an m-level rook placement on B. Similarly, if two rooks end up in the same level of $l(B)$, then they must have originated in the same column of B, since B is a singleton board.

The inverse of this map acts as follows. If a rook is in the cell numbered a of column $t-p+1$ in $l(B)$ then it is placed in the cell numbered a in level p of B. The proof that this gives a rook placement is similar to the one in the previous paragraph and so is omitted. $\square$

Note that Lemma 3.2.3 and Lemma 3.2.5 combine to provide an explicit bijection between m-level rook placements on any Ferrers board B and on its m-transpose, $l(B) = l(B_S)$.

3.2.3 The local l-operator

For any set S, let $\#S$ be the cardinality of S. Given a column i and a level p define the *m-arm length* of column i, level p by

$$\mathrm{arml}_m(i,p) = \#\{c_{i,j'} \in B \mid c_{i,j'} \text{ is strictly above level } p\}.$$

In Figure 3.2 the cells counted by the 2-arm length of column 4, level 1 have a dashed line through them. (Reflecting our boards to put them in English notation will result in the arm being the usual set of squares when $m = 1$.) We let $\mathrm{arml}_m(i,p) = \infty$ if the number of columns in B is less than i for reasons detailed in Lemma 3.2.7.

Similarly, define the *m-leg length* of column i, level p to be

$$\mathrm{legl}_m(i,p) = \#\{c_{i',j'} \in B \mid c_{i',j'} \text{ is in level } p \text{ and } i' < i\}.$$

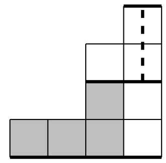

Figure 3.2: The dashed line goes through the cells counted by the 2-arm length of the fourth column and first level, and the shaded cells are counted by the corresponding 2-leg length.

The cells counted by the 2-leg length of column 4, level 1 are shaded in Figure 3.2. As before, this is equivalent to the usual notion of leg length in the $m = 1$ case. We also let $\mathrm{legl}_m(i, 0) = \infty$ by convention.

Since the l operation generalizes the transposition of a Ferrers board, one would expect that some sort of local l operation would be the appropriate generalization of the local transposition introduced by Foata and Schützenberger. This is indeed the case, and we define the local l operation as follows.

Given a Ferrers board B with non-empty intersection of the ith column and pth level, let $B_{i,p}$ denote the subboard of B consisting of all cells in or above the pth level and in or to the left of the ith column: see Figures 3.3 and 3.4 for examples. Note that if B is a singleton board, then $B_{i,p}$ is also, because the set of rows in level p' of $B_{i,p}$ will be the same as the set of rows in level $p + p' - 1$ of B. If B is a Ferrers board then the *local l operation at* (i, p) is the result of applying the l operator to the subboard $B_{i,p}$ and leaving the rest of B fixed. We will denote the resulting board by $l_{i,p}(B)$.

As defined above $l_{i,p}(B)$ may not be a Ferrers board, let alone a singleton board. We now develop a pair of conditions to determine if $l_{i,p}(B)$ will be a singleton board.

Definition 3.2.6. *The operation $l_{i,p}$ is* permissible *for a singleton board B if*

$$\mathrm{arml}_m(i, p) \leq \lfloor \mathrm{legl}_m(i, p-1) \rfloor_m \quad \textit{and} \quad \mathrm{legl}_m(i, p) \leq \lfloor \mathrm{arml}_m(i+1, p) \rfloor_m.$$

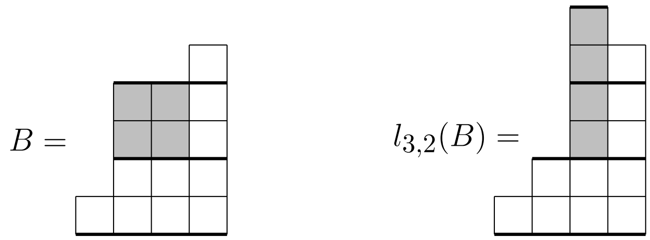

Figure 3.3: On the left, $B_{3,2}$ is shaded within $B = (1, 4, 4, 5)$. Notice that $l_{3,2}$ is not permissible for B since $\lfloor \mathrm{arml}_m(4, 2) \rfloor_m < \mathrm{legl}_m(3, 2)$, which means $l_{3,2}(B)$ will not be a singleton board. On the right the shaded cells in $l(B_{3,2})$ illustrate this; in this case $l_{3,2}(B)$ is not even a Ferrers board.

See Figure 3.3 for an example of a local l-operation not permissible for the given board, and Figure 3.4 for a local l-operation which is permissible.

Lemma 3.2.7. *Let a singleton Ferrers board B have a non-empty intersection of the ith column and pth level. Then $l_{i,p}$ is permissible for B if and only if $l_{i,p}(B)$ is a singleton Ferrers board.*

Proof. If column i, level p in B contains fewer than m cells, then $l_{i,p}(B) = B$ since B is singleton, and there is nothing to prove. Henceforth, assume that column i, level p in B contains m cells. We know that B, $B_{i,p}$, and $l(B_{i,p})$ are all singleton Ferrers boards. It follows that $l_{i,p}(B)$ will be a singleton Ferrers board if and only if these three conditions hold for the board $l_{i,p}(B)$.

(a) The lowest row of level p is weakly shorter than the highest row of level $p - 1$;

(b) column i is weakly lower than column $i + 1$; and

(c) if columns i and $i + 1$ terminate at the same level, then the height of column $i + 1$ is a multiple of m.

Condition (c) is needed to ensure $l_{i,p}(B)$ will be singleton.

To determine when these conditions hold, first note that applying $l_{i,p}$ to B exchanges $\mathrm{arml}_m(i,p)$ and $\mathrm{legl}_m(i,p)$. Because B is singleton, the top row of level $p-1$ in B (and in $l_{i,p}(B)$) extends left of column i by $\lfloor \mathrm{legl}_m(i,p-1)\rfloor_m$ cells. On the other hand, the new bottom row of level p in $l_{i,p}(B)$ extends left of column i by $\lceil \mathrm{arml}_m(i,p)\rceil_m$ cells. Thus, condition (a) holds iff

$$\lceil \mathrm{arml}_m(i,p)\rceil_m \leq \lfloor \mathrm{legl}_m(i,p-1)\rfloor_m.$$

Since both sides are multiples of m, this inequality is equivalent to $\mathrm{arml}_m(i,p) \leq \lfloor \mathrm{legl}_m(i,p-1)\rfloor_m$, which is the first condition in the definition of permissibility.

Now consider the heights of columns i and $i+1$ in $l_{i,p}(B)$. Both column i and column $i+1$ have a full m cells in level p. So, in both B and $l_{i,p}(B)$, column $i+1$ has height $pm+\mathrm{arml}_m(i+1,p)$. On the other hand, the new column i in $l_{i,p}(B)$ has height $pm+\mathrm{legl}_m(i,p)$. So condition (b) will hold iff

$$\mathrm{legl}_m(i,p) \leq \mathrm{arml}_m(i+1,p).$$

To deal with condition (c), consider two cases. First suppose that $\mathrm{arml}_m(i+1,p)$ is a multiple of m. Then condition (c) must hold, and here condition (b) will hold iff $\mathrm{legl}_m(i,p) \leq \lfloor \mathrm{arml}_m(i+1,p)\rfloor_m$. Now suppose that $\mathrm{arml}_m(i+1,p)$ is not a multiple of m. Given that condition (b) holds, the new board $l_{i,p}(B)$ will be singleton iff the strengthened inequality $\mathrm{legl}_m(i,p) \leq \lfloor \mathrm{arml}_m(i+1,p)\rfloor_m$ is true. Thus, this last inequality is equivalent to the truth of (b) and (c) in all cases. □

3.2.4 The Local l-operation on an m-level rook placement

Since there is a bijection between rook placements on B and $l(B)$ when B is singleton, it stands to reason that it would generalize to a bijection between rook placements on B and $l_{i,p}(B)$. The following lemma makes this precise.

Lemma 3.2.8. *For a singleton board B, suppose $l_{i,p}$ is permissible for B. Then there is an explicit bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on $l_{i,p}(B)$.*

Proof. Use the bijection induced by the l operation in Lemma 3.2.5 on the subboard transposed by $l_{i,p}$, not moving the rooks on the part of board B which is fixed. However, this may cause a rook in the transposed subboard to occupy the same column or level of $l_{i,p}(B)$ as one of the rooks which was fixed. We deal with this possibility next.

In order for two rooks to end up in the same column, there must be rooks placed on B beneath $B_{i,p}$, so we can assume $p > 1$ without losing generality. Consider the set of columns of B which do not contain rooks in $B_{i,p}$, and the set of columns of $l_{i,p}(B)$ which do not contain rooks in $l(B_{i,p})$. By our assumption on p, these two sets have the same cardinality and so we can put a canonical bijection on them by pairing the leftmost columns in each set and moving to the right. If there is a rook lower than level p in one of these columns of B, use this bijection on the columns to move it to the cell in the same row of the corresponding column of $l_{i,p}(B)$. After doing so, there must be at most one rook in each column of $l_{i,p}(B)$. For example, in Figure 3.4 the rook in $c_{3,2}$ is in the second column from the left of B which does not contain a rook in $B_{4,2}$. Thus it moves to column 2, which is the second column from the left of $l_{4,2}(B)$ that does not contain a rook in $l(B_{4,2})$.

If two rooks end up in the same level we treat them similarly where we can assume, without loss of generality, that the i-th column is not the rightmost column of B. There is a canonical bijection between the levels of B which do not contain rooks in $B_{i,p}$ and those of $l_{i,p}(B)$ that do not contain rooks in $l(B_{i,p})$. Adjust the levels of all rooks to the right of column i using this bijection, fixing the column of the rook that moves. Furthermore, fix the height of the rook that moves within the level, that is, if the rook was in cell $c_{x,y}$, move the rook to cell $c_{x,y'}$ in the appropriate level with $y \equiv y' \pmod{m}$. Note that since B is a singleton board, columns to the right of column i will contain a full m cells at any level which contained a rook in the subboard $B_{i,p}$.

To see that this is a bijection, we construct its inverse. Recall that the l operator is an involution on singleton boards. Thus, since $B_{i,p}$ is a singleton subboard, $l_{i,p}(l_{i,p}(B)) = B$. Similarly, applying the bijection from Lemma 3.2.5 and then its inverse returns the original placement of rooks on $B_{i,p}$. All that remains to check is that any rooks moved outside of $B_{i,p}$ return to their original cells. Since the rooks return to their original placement on $B_{i,p}$, the set of columns that gain a rook in $B_{i,p}$ after the first application of l will be the same set as those that lose a rook in $B_{i,p}$ after the second application of l. Thus the bijection on the columns induced by the first application of l will be the inverse of the bijection induced by the second application, and any rook required to move in $l_{i,p}(B)$ will move back in $l_{i,p}(l_{i,p}(B))$. A similar argument holds for levels, noting that since $l_{i,p}(B)$ is singleton ensures that any level that gains a rook in $B_{i,p}$ after applying $l_{i,p}$ contains a full m cells in every column to the right of column i. Thus this yields a bijection between rook placements on B and $l_{i,p}(B)$. Figure 3.4 illustrates this bijection. □

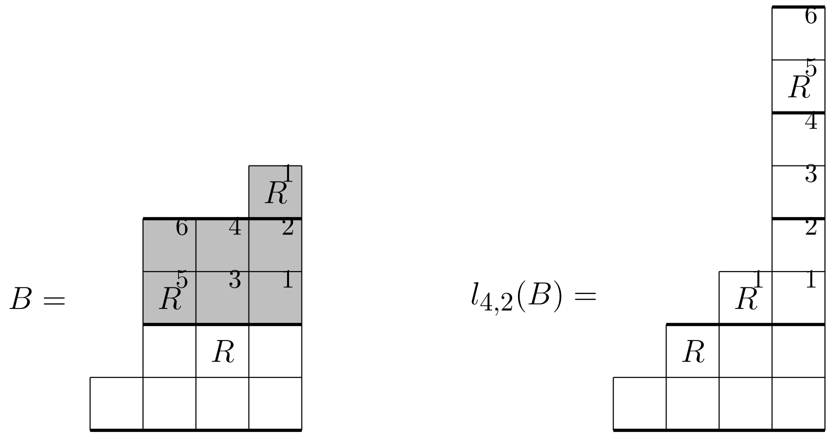

Figure 3.4: On the left, $B_{4,2}$ is shaded. Here $l_{4,2}$ is permissible for B and is shown on the right.

3.2.5 Bijections with m-increasing boards

Foata and Schützenberger proved there is a unique Ferrers board in every rook equivalence class whose column lengths are strictly increasing and used this board as a target for their bijections. To accomplish the same thing, we need the the analogous concepts for m-level rook placements, developed in the previous chapter.

Definition 3.2.9. *A Ferrers board* $B = (b_1, b_2, \ldots, b_n)$ *is called* m-increasing *if* $b_{i+1} \geq b_i + m$ *for all* $1 \leq i \leq n-1$.

Recall that when $m = 1$ strictly increasing and m-increasing are equivalent.

Theorem 3.2.10 (2.3.6)**.** *Every Ferrers board is m-level rook equivalent to a unique m-increasing board.*

We are now almost ready to prove the main result of this section, Theorem 3.2.12 below. However, to do so we must put an order on Ferrers boards. Once we have established this order, we will be able to give an explicit bijection between m-level rook placements on an arbitrary Ferrers board B and on an m-level rook equivalent Ferrers board which is greater than B in this order, if such a board exists. Additionally, the set of all Ferrers boards equivalent to B will have a unique maximum element under this order, namely the m-increasing board guaranteed by [BLRS14].

To define this order, if $B = (b_1, \ldots, b_n)$ then consider the reversal of B, $B^r = (b_n, \ldots, b_1)$. Now let $B < B'$ if B^r is lexicographically smaller than $(B')^r$. It is important to note that when applying Lemma 3.2.3 we will always have

$$B_S \geq B \tag{3.2.1}$$

since in B_S all the cells in each level are as far to the right as possible.

Lemma 3.2.11. *Given a singleton board B containing a column i and a level p with the property that*

$$\text{arml}_m(i, p) < \text{legl}_m(i, p), \tag{3.2.2}$$

there is a singleton board $B' = l_{i',p}(B)$ with $i' \geq i$ and $B' > B$.

Furthermore, if B is not m-increasing then a column i and level p satisfying equation 3.2.2 must exist.

Proof. To prove the first statement, let $i' \geq i$ be the maximum index such that $\text{arml}_m(i', p) < \text{legl}_m(i', p)$. Note that by our convention on arml_m, we must have that i' is at most the number of columns of B. We claim that it suffices to show that $l_{i',p}$ is permissible for B. This is because if $l_{i',p}$ is permissible for B, then the resulting board B' must satisfy $B' > B$. Indeed, $l_{i',p}(B)$ increases the length of column i' by $\text{legl}_m(i', p) - \text{arml}_m(i', p)$, which must be greater than 0, and column i' is the rightmost column affected by $l_{i',p}(B)$. Thus $B' > B$.

If $l_{i',p}$ is not permissible for B, then we claim that we have $\text{arml}_m(i'+1, p) < \text{legl}_m(i'+1, p)$

which will contradict the maximality of i' and complete this part of the proof. Note that

$$\mathrm{arml}_m(i',p) < \mathrm{legl}_m(i',p) \le \lfloor \mathrm{legl}_m(i',p-1) \rfloor_m.$$

So $l_{i',p}$ not being permissible for B implies that $\lfloor \mathrm{arml}_m(i'+1,p) \rfloor_m < \mathrm{legl}_m(i',p) = \mathrm{legl}_m(i'+1,p) - m$ since B is singleton. This implies the desired contradiction that $\mathrm{arml}_m(i'+1,p) < \mathrm{legl}_m(i'+1,p)$.

To prove the second statement of the theorem, note that if B is not m-increasing there are two possible cases, either there are two adjacent columns $i-1, i$ of B which terminate at the same level, or column $i-1$ terminates in level p and B has exactly r_1 cells in the pth level of column $i-1$ and exactly r_2 cells in the $(p+1)$st level of column i where $r_1 > r_2 > 0$.

Case 1: Let columns $i-1$ and i both terminate at level p. Then $\mathrm{arml}_m(i,p) = 0$, by the assumption that column i terminates at level p, but $\mathrm{legl}_m(i,p) \ge 1$ since column $i-1$ also terminates at the pth level. Thus $\mathrm{arml}_m(i,p) < \mathrm{legl}_m(i,p)$ as desired.

Case 2: By assumption $\mathrm{arml}_m(i,p) = r_2 < r_1 \le \mathrm{legl}_m(i,p)$ which completes the proof. □

We are now in a position to prove our main theorem of this section.

Theorem 3.2.12. *Given any two m-level rook equivalent Ferrers boards, there is an explicit bijection between m-level rook placements of k rooks on them.*

Proof. Given any Ferrers board B, let B_m be the unique m-increasing board in the m-level rook equivalence class of B guaranteed by Theorem 3.2.10. It suffices to show that there is an explicit bijection between the m-level rook placements of k rooks on B and those on B_m. This is trivial if $B = B_m$ so assume $B \neq B_m$. By Lemma 3.2.3, we have an explicit bijection

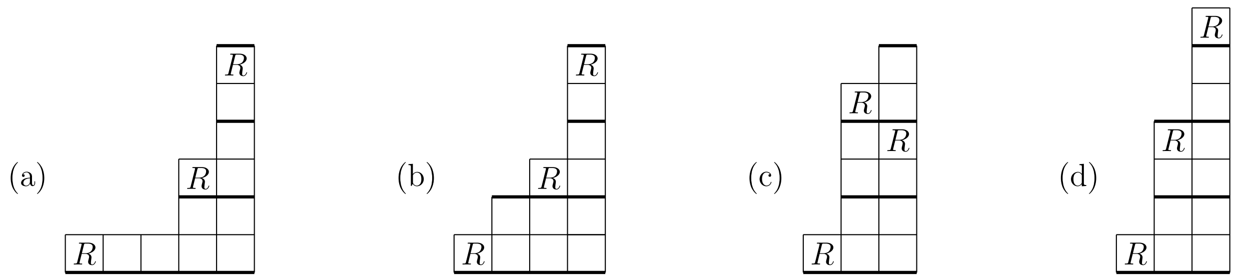

Figure 3.5: (a) A 2-level rook placement on a Ferrers board. (b) The placement on the singleton board obtained after applying Lemma 3.2.3. (c) The placement obtained after applying Lemma 3.2.8 using $l_{3,1}$. (d) The placement obtained on a 2-increasing board after applying Lemma 3.2.8 again using $l_{3,3}$.

between the placements on B and those on B_S where $B_S \geq B$ by equation 3.2.1. If $B_S = B_m$ then we are done. Otherwise, apply the local l operator defined in Lemma 3.2.11 which will give $B' = l_{i,p}(B_S)$ with $B' > B_S$ and, by Lemma 3.2.8, another explicit bijection between rook placements. We now repeat this process if necessary. Since there are only finitely many boards in an m-level rook equivalence class and the lexicographic order increases with every application of Lemma 3.2.8, we must eventually terminate. And, by Lemma 3.2.11 again, termination must occur at B_m. Composing all the bijections finishes the proof. □

See Figure 3.5 for a short example of this process.

3.3 A second bijection on m-level rook placements

Our next two main results will require the Garsia-Milne Involution Principle. The first will use the Involution Principle to construct another explicit bijection between two arbitrary m-level rook placements of k rooks on m-level rook equivalent singleton boards.

Theorem 3.3.1 (Garsia-Milne Involution Principle [GM81])**.** *Consider a triple (S, T, I) where S is a signed set, I is a sign-reversing involution on S, and the set T of fixed points of*

I is required to be a subset of the positive part S^+ of S. Let (S', T', I') be defined similarly. Then, given an explicit sign-preserving bijection f from S to S', one can construct an explicit bijection between T and T'.

The way that Garsia and Milne define the explicit bijection is as follows. Start with an element $t \in T \subseteq S^+$. If $f(t) \notin T'$, then apply $(f \circ I \circ f^{-1} \circ I')$ to $f(t)$. This takes $f(t) \in S'^+$ to S'^-, then to S^-, then S^+, and finally back to S'^+. Iterating this procedure must ultimately yield an element of T' which is considered the image of t under the desired bijection.

3.3.1 A Garsia-Milne bijection for rook placements

We will use the Involution Principle to construct a bijection between m-level rook placements on two m-level rook equivalent singleton boards. We must first construct a signed set and a sign-reversing involution so that the m-level rook placements are the fixed points under the involution. We do this as follows.

Given two Ferrers boards, B and B', we shall say B *fits inside* B' if juxtaposing the two boards with their lower right cells in the same position makes the cells of B a subset of the cells of B'. Figure 3.6 shows that the thick bordered $B = (2, 3)$ fits inside $B' = (0, 2, 4, 6)$. The shading and rook placement may be ignored for now. Let $\Delta_{n,m}$ denote the triangular Ferrers board $(0, m, 2m, \ldots, (n-1)m)$. Given a singleton board B, fix N large enough that B fits inside $\Delta_{N,m}$. If B has fewer than N columns, expand B on the left with columns of height zero so $B = (b_0, b_1, \ldots, b_{N-1})$ has the same number of columns as $\Delta_{N,m}$. Fix a non-negative integer k with $k < N$ and let the integer i vary over $0 \leq i \leq k$. Then S will consist of all configurations C constructed as follows. Take $\Delta_{N,m}$ with B fitting inside and

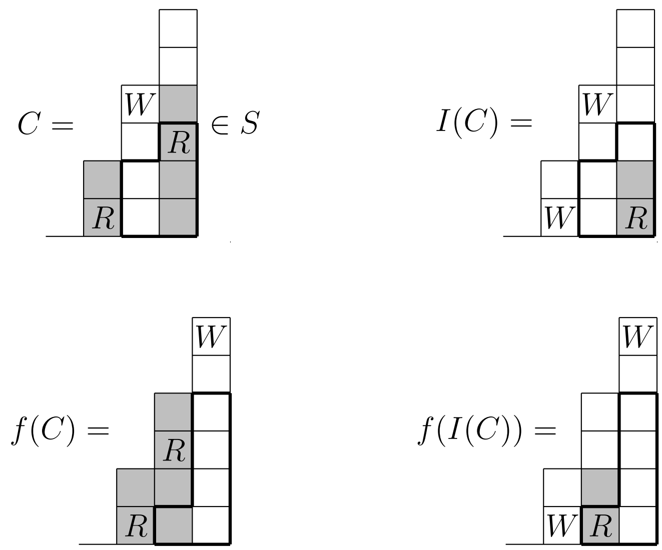

Figure 3.6: On the top left, an element in S with sign -1. On the top right, the image under I which has sign $+1$. Beneath each board is its image under f.

place white rooks W in i cells of $\Delta_{N,m}$ that are outside of B so that no two white rooks are in the same column. Next, place $k-i$ black rooks R forming an m-level rook placement on the subboard $\Delta_{N-i,m}$ which is located in the columns of $\Delta_{N,m}$ which do not contain a white rook. We will call this the *inset* $\Delta_{N-i,m}$ *board*. Note that the columns of the inset $\Delta_{N-i,m}$ may not be contiguous.

See the top left board of Figure 3.6 for an example of such an object C where $m = 2$. The singleton board $B = (0, 0, 2, 3)$ fits inside $\Delta_{4,2}$. Here $k = 3 < 4$ and there is $i = 1$ white rook on the board $\Delta_{4,2} \setminus B$ and $k - i = 2$ black rooks on the board $\Delta_{3,2}$ which is represented by the grey shaded cells inside $\Delta_{4,2}$. The rooks on $\Delta_{3,2}$ form a 2-level rook placement, but there is both a black rook and a white rook in the second level of $\Delta_{4,2}$.

Note that each column of $\Delta_{N,m}$ may contain at most one white rook or black rook. On the other hand, a level of $\Delta_{N,m}$ will contain at most one black rook, but may contain any number of white rooks. Further, define the sign of such a placement to be $(-1)^i$. The sign of the placement on the top left in Figure 3.6 is -1.

To define I on an element $C \in S$, if all rooks of C are in B, and therefore black, then C is a fixed point. Otherwise, examine the columns of C from left to right until coming to a column with a rook outside of B. If that column contains a black rook, change the rook to a white rook, increase i by one, and move every black rook above and to the right of the cell containing the new white rook down m cells. If that column contains a white rook, change it to a black rook, decrease i by one, and move every black rook to the right and at the same level or higher as the new rook up m cells. The placement on the top right in Figure 3.6 illustrates what happens to the board on the left under I. Similarly, I takes the placement on the right to the placement on the left.

We must show that $I(C)$ will be an element of S. Clearly each column has at most one rook. When a black rook becomes white, it is clear that there will be at most one black rook per level on the resulting board, as there was at most one black rook per level on the original board. If a black rook is added, each level will have at most one black rook since when a black rook is added all black rooks at its level or above to the right of the new rook move up one level. Furthermore, there can be no black rooks at the same level or higher to the left of the new black rook if we change a white rook to a black rook. This is because the new black rook was a white rook which, by definition, was above board B. Since B is a singleton board, no columns of B to the left of the white rook in question will terminate in the level of the white rook. Thus if there were a black rook at the same level or higher to the left, it too would be outside of board B, which contradicts the white rook being the leftmost rook outside of board B.

When a black rook is added, the black rooks must be placed on a board $\Delta_{N-i+1,m}$ where the column in which the new black rook is placed is added to the columns in the initial inset

$\Delta_{N-i,m}$. Since there are no white rooks to the left of the new black rook, there will be no omitted columns to the left of the column containing the new black rook, thus all cells of that column will be in the inset $\Delta_{N-i+1,m}$ and the new rook must be inside $\Delta_{N-i+1,m}$. This means that all the columns to the right of the new black rook that do not contain a white rook will contain m more squares as the inset $\Delta_{N-i+1,m}$ than they did as the inset $\Delta_{N-i,m}$. Thus moving black rooks to the right of the new black rook up m cells will keep them within the new $\Delta_{N-i+1,m}$. Similarly, changing a black rook to a white rook will decrease the number of cells in the columns of $\Delta_{N-i-1,m}$ to the right of the new white rook by m, but all black rooks to the right of the new white rook and at a higher level than it are moved down m cells, so they will be in $\Delta_{N-i-1,m}$ because they were in $\Delta_{N-i,m}$ originally. Finally, if there are any black rooks below the level of the new white rook but to its right, they will remain in $\Delta_{N-i-1,m}$ because the first column in $\Delta_{N-i-1,m}$ to the right of the new white rook must go up to at least the level of the new white rook since previously it was a black rook contained in $\Delta_{N-i,m}$.

By construction, I is an involution. The fixed set of I will be denoted T. It is the set of all configurations which only have rooks on the subboard B and, by definition, these rooks must be black. As such, T is equal to the set of m-level rook placements of k rooks on B. Furthermore, if a board is not in T, then I either increases or decreases the number of white rooks on the board by one. Either way I will change the sign of the board. And if a board is in T, then it has positive sign.

Given a singleton board B', define N', S', T', and I' similarly for B' contained in $\Delta_{N',m}$. Without a loss of generality, assume $N = N'$. Let $B' = (b'_0, b'_1, \ldots, b'_{N-1})$. If B and B' are m-level rook equivalent singleton boards we can use I and I' to construct an explicit bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on

B'. We do this by constructing a sign-preserving bijection between S and S'. We will need the following characterization of when two singleton boards are m-level rook equivalent.

The *root vector* of B is

$$\zeta_m = (-b_0, m - b_1, \ldots, (N-1)m - b_{N-1}).$$

The following result of Briggs and Remmel determines when two singleton boards are m-level rook equivalent simply by considering their root vectors.

Theorem 3.3.2 (Briggs-Remmel Theorem 2 [BR06])**.** *Two singleton boards are m-level rook equivalent if and only if they have the same root vector up to rearrangement for a sufficiently large N.*

We are now ready to apply the Garsia-Milne Involution Principle.

Theorem 3.3.3. *Let B and B' be m-level rook equivalent singleton boards. Then there exists an explicit Garsia-Milne bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on B'.*

Proof. By Theorem 3.3.1 and what we have already established, it suffices to find a sign-preserving bijection $f : S \to S'$. We construct f as follows.

For clarity of notation, let B be placed in $\Delta_{N,m}$ and B' be placed in a copy $\Delta'_{N,m}$ of $\Delta_{N,m}$. Notice that the kth element of the root vector of B, $km - b_k$, is the number of cells in the kth column of $\Delta_{N,m}$ which lie outside of board B. Since B and B' are m-level rook equivalent, the root vector for B' is a rearrangement of the root vector for B. Therefore there is a length-preserving bijection between the columns of the set difference $\Delta_{N,m} \setminus B$ and the columns of $\Delta'_{N,m} \setminus B'$ which takes the leftmost column of a given length in $\Delta_{N,m} \setminus B$

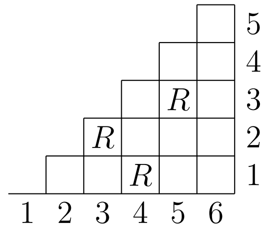

Figure 3.7: The placement corresponding to partition $\{1,4\},\{2,3,5\},\{6\}$.

to the leftmost column with that length in $\Delta'_{N,m} \setminus B'$ and so forth. This bijection induces a bijection on the placement of the white rooks. If a white rook appears in the jth cell above B, place a white rook in the jth cell above B' in the associated column.

Once all the white rooks are placed, create a copy of $\Delta'_{N-i,m}$ inside of $\Delta'_{N,m}$ using the columns which do not contain a white rook. Place the black rooks on the board with relation to the $\Delta'_{N-i,m}$ subboard exactly as they are placed on the original board with relation to the original $\Delta_{N-i,m}$ subboard. Each placement on the bottom of Figure 3.6 is the image under f of the corresponding placement on the top where $B = (0,0,2,3)$ and $B' = (0,0,1,4)$. Notice that in the top left board, the white rook is at the top of the second column from the left which has two cells above B. In the board on the bottom left the white rook is still at the top of the second column from the left which has two cells above B'.

Under this map the white rooks must be placed inside $\Delta'_{N,m}$ but outside B', and the black rooks are placed inside $\Delta'_{N-i,m}$, so f maps S to S'. Further this map preserves the number of white rooks placed on the board, so it is sign preserving. Therefore we may conclude from the Involution Principle that there is an explicit bijection between m-level rook placements of k rooks on B and m-level rook placements of k rooks on B'. □

To obtain a consequence of this construction, we will need some background on symmetric functions and Stirling numbers. For $d \le n$ both non-negative integers, let $e_d(x_1, x_2, \ldots, x_n)$

denote the *elementary symmetric function of degree d in n variables*, that is,

$$e_d(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} x_{i_1} x_{i_2} \ldots x_{i_d}. \tag{3.3.1}$$

Let $S(n, d)$ denote the *Stirling numbers of the second kind.* Recall that $S(n, d)$ can be defined as the number of ways to partition a set of n elements into d subsets called *blocks*.

Further, note that $S(n, d)$ counts the number of rook placements of $n - d$ rooks on $\Delta_{n,1}$. To see this, number the rows of $\Delta_{n,1}$ from 1 to $n - 1$ from bottom to top. Then number the columns, including the zero column, from 1 to n left to right. Given a partition of $\{1, \ldots, n\}$ into d blocks, order the elements of each block increasingly. Now, if i and j are adjacent within a block then place a rook in row i column j. See Figure 3.7 for the rook placement corresponding to $\{1, 4\}, \{2, 3, 5\}, \{6\}$. Thus the number of m-level rook placements of $n - d$ rooks on $\Delta_{n,m}$ is $m^{n-d} S(n, d)$. The extra m^{n-d} counts the number of ways of choosing a placement for each of the $(n - d)$ rooks in the m cells of a level.

It is interesting to note that the construction of I yields the following theorem giving an explicit calculation for the m-level rook numbers of a singleton Ferrers board B.

Theorem 3.3.4. *For any singleton board $B = (b_0, b_1, \ldots, b_{N-1})$ fitting inside $\Delta_{N,m}$,*

$$r_{k,m}(B) = \sum_{i=0}^{k} (-1)^i m^{k-i} S(N - i, N - k) e_i(-b_0, m - b_1, \ldots, (N - 1)m - b_{N-1}).$$

Proof. Since the fixed points of the involution I are counted by $r_{k,m}(B)$, it suffices to show that the sum counts all elements of the set S by sign. First note that the number of ways of putting i white rooks in i different columns of $\Delta_{N,m}$ outside of B is $e_i(-b_0, m - b_1, \ldots, (N - 1)m - b_{N-1})$. Furthermore the number of m-level rook placements of $k - i$ rooks on $\Delta_{N-i,m}$

is $m^{k-i}S(N-i, N-k)$. Putting these two counts together with the appropriate sign gives the sum as desired. □

Note that this theorem implies the backward direction of Theorem 3.3.2.

3.4 A bijection for hit numbers

We will now use the Involution Principle to prove that two boards that are m level rook equivalent have the same hit numbers. We begin with some definitions. As usual, let m be a fixed positive integer. We will sometimes suppress m in the following notation to limit the number of subscripts.

Let B be a Ferrers board and let the integer N be sufficiently large so that B fits inside a rectangular board, $\mathrm{Sq}_{N,m}$ with N columns and mN rows. If α is a generator of the cyclic group C_m, then

$$C_m \wr S_N = \{(\alpha^{s_1}, \alpha^{s_2}, \ldots, \alpha^{s_N}; \sigma) \mid 1 \le s_i \le m \text{ for each } i \text{ and } \sigma \in S_N\}.$$

We associate with $\omega \in C_m \wr S_N$ a placement on $\mathrm{Sq}_{N,m}$ by placing a rook in level $N+1-p$ and column i if $\sigma(i) = p$. Furthermore, the rook in column i will be j cells from the bottom of the level if $s_i = j$. See Figure 3.8 for an example with $m = 2$ and $N = 3$, where the placement corresponds to $(\alpha^1, \alpha^2, \alpha^1; (1, 3, 2))$, and σ is in cycle notation. Let $R(\omega)$ denote the rook placement corresponding to ω. Define the *kth hit set* of B to be

$$H_{k,N}(B) = \{R(\omega) \mid \omega \in C_m \wr S_N \text{ and } \#(R(\omega) \cap B) = k\}. \tag{3.4.1}$$

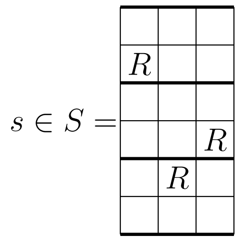

Figure 3.8: The placement on $\mathrm{Sq}_{3,2}$ corresponding to $(\alpha^1, \alpha^2, \alpha^1; (1,3,2))$.

Also define the *kth hit number* of B to be

$$h_{k,N} = \#H_{k,N}. \tag{3.4.2}$$

In order to show that two m-level rook equivalent Ferrers boards have the same hit numbers, we use Garsia and Milne's result again. To do so, we must construct a signed set and a sign reversing involution which has a set counted by $h_{k,N}$ as its fixed set. We do this as follows.

Let N be large enough that B fits inside $\mathrm{Sq}_{N,m}$, fix non-negative integers i and k with $i \le k \le N$. Then the set S will consist of all configurations C constructed as follows. Place k black, m-level rooks R on the board B. Furthermore, circle i of the rooks. Finally, consider the $N-k$ columns and $N-k$ levels which do not contain a black rook as a subboard of shape $\mathrm{Sq}_{N-k,m}$, as in the previous section, we will call this the *inset* $\mathrm{Sq}_{N-k,m}$ *board.* Place $N-k$ white m-level rooks, denoted by W, on the inset $\mathrm{Sq}_{N-k,m}$. Notice that, ignoring the color of the rooks, this is an m-level rook placement of N rooks on $\mathrm{Sq}_{N,m}$. Thus it corresponds to some element of $C_m \wr S_N$. Let the sign of a configuration be $(-1)^i$. See Figure 3.9 for two examples of such configurations. Here $m = 2$ and $B = (1,2,4)$ is placed fitting in $\mathrm{Sq}_{3,2}$. On the left, there are no circled black rooks so $i = 0$ and the white rooks are placed on the shaded inset $\mathrm{Sq}_{2,2}$. On the right there is one circled black rook so $i = 1$ and the white rooks

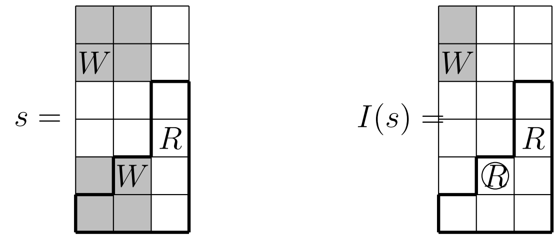

Figure 3.9: On the left, an element in S with sign $+1$. On the right, the image under I which has sign -1.

are on a shaded inset $\mathrm{Sq}_{1,2}$.

In order to produce a sign-reversing involution I on such configurations C, we do the following. If B contains neither a white rook nor a circled black rook, then C is fixed by I. Otherwise, examine the columns of B from left to right until the first white rook or circled black rook is found. If the first rook found is white, exchange it for a circled black rook and increase i by 1. If the first rook found is a circled black rook, exchange it for a white rook and decrease i by 1. See Figure 3.9 again for an illustration. It is easy to see that I is an involution and reverses signs in its 2-cycles. Also note that fixed points have no circled black rooks, so $i = 0$ and the sign of the configuration is $+1$. Furthermore, for a fixed point there are no white rooks placed on B, so the m-level placement intersects B in exactly k black rooks. Thus the fixed points are exactly the elements of $H_{k,N}(B)$ if one just ignores the colors of the rooks.

The reader will find an example illustrating the next proof in Figure 3.10. This example uses the boards from the example of Theorem 3.2.12 found in Figure 3.5.

Theorem 3.4.1. *Let B and B' be two m-level rook equivalent Ferrers boards and N be large enough that B and B' both fit inside $\mathrm{Sq}_{N,m}$. Then for any non-negative integer $k \leq N$, there is an explicit bijection between $H_{k,N}(B)$ and $H_{k,N}(B')$.*

Proof. As in the proof of Theorem 3.3.3, we use the Garsia-Milne Involution Principle.

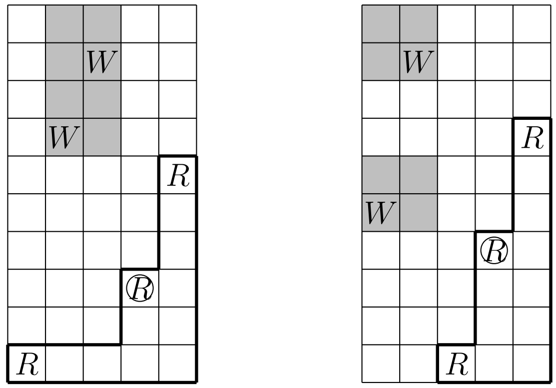

Figure 3.10: On the left, a 2-level placement of white rooks, black rooks, and circled black rooks on $(1,1,1,3,6)$ fit inside $\mathrm{Sq}_{5,2}$. On the right, the corresponding placement under the construction in Theorem 3.4.1.

Construct S for B placed inside $\mathrm{Sq}_{N,m}$ and S' for B' placed inside $\mathrm{Sq}'_{N,m}$. From what we have already done, all the remains is to construct the sign-preserving bijection $f : S \to S'$.

Consider an element $C \in S$. The black rooks, circled and uncircled, form an m-level rook placement of k rooks on B. Map this to an m-level rook placement of k rooks on B' using the explicit bijection guaranteed by Theorem 3.2.12. Furthermore, add circles to the rooks on B' in such a way so that if the rth rook from the right on board B is circled, the rth rook from the right on board B' is circled. Finally, place the white rooks on $\mathrm{Sq}'_{N,m}$ by considering the inset $\mathrm{Sq}'_{N-k,m}$ of columns and levels containing no black rooks. Place the white rooks on this inset board in the exact same arrangement as they are in on the inset $\mathrm{Sq}_{N-k,m}$ of $\mathrm{Sq}_{N,m}$. This is easily seen to be a bijection and so the proof is complete. □

The next corollary follows immediately from the previous theorem.

Corollary 3.4.2. *Let B and B' be two m-level rook equivalent Ferrers boards and N be large enough such that B and B' both fit inside $\mathrm{Sq}_{N,m}$. Then for any non-negative integer $k \le N$, $h_{k,N}(B) = h_{k,N}(B')$.*

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